Journal of Fourier Analysis and Applications (2025) 31:75
https://doi.org/10.1007/s00041-025-10207-1

Optimal Weighted Fourier Restriction Estimates for the Sphere in 2D

Rainer Mandel[1]

Received: 26 December 2024 / Revised: 17 September 2025 / Accepted: 26 October 2025

Abstract
We prove weighted versions of the 2D Restriction Conjecture for the unit sphere in $\mathbb{R}^2$. Our results involve the weight functions $(1+|x|)^\alpha(1+|y|)^\beta$ and $(1+|x|+|y|)^\gamma$ with $\alpha, \beta, \gamma \geq 0$.

Keywords Fourier Restriction Theory · Weighted Inequalities · Weighted Estimates in Harmonic Analysis

Mathematics Subject Classification 42B10

1 Introduction

In this paper we investigate weighted L^p-L^q-estimates for the Fourier Restriction operator

$$(\mathcal{R}g)(\xi) := \hat{g}(\xi) := \frac{1}{2\pi}\int_{\mathbb{R}^2} g(x,y)e^{-i(x,y)\cdot\xi}\,d(x,y) \quad \text{for } \xi \in \mathbb{S}^1$$

of the two-dimensional unit sphere $\mathbb{S}^1 := \{\xi \in \mathbb{R}^2 : |\xi| = 1\}$. Its adjoint $\mathcal{R}^*$ is the Fourier Extension operator given by the formula

$$(\mathcal{R}^*F)(x,y) = \frac{1}{2\pi}\int_{\mathbb{S}^1} e^{i(x,y)\cdot\omega}F(\omega)\,d\sigma(\omega) \qquad \text{for } (x,y) \in \mathbb{R}^2.$$

Here, σ denotes the canonical surface measure on $\mathbb{S}^1$. The mapping properties of $\mathcal{R}$ induce the mapping properties of $\mathcal{R}^*$ and vice versa, so no generality is lost by

Communicated by Andreas Seeger.

✉ Rainer Mandel
Rainer.Mandel@gmx.de

1 Karlsruhe Institute of Technology, Institute for Analysis, Englerstraße 2, 76131 Karlsruhe, Germany

Published online: 21 November 2025 Birkhäuser investigating $\mathcal{R}^*$. The following fact, which is the celebrated Restriction Conjecture in two spatial dimensions, was proved by Fefferman [5] and Zygmund [12] about 50 years ago.

Theorem 1 (Fefferman, Zygmund) *$\mathcal{R} : L^p(\mathbb{R}^2) \to L^\mu(\mathbb{S}^1)$ is bounded if and only if $1 \le p < \frac{4}{3}$ and $3\mu \le p'$. Equivalently, $\mathcal{R}^* : L^r(\mathbb{S}^1) \to L^q(\mathbb{R}^2)$ is bounded if and only if $q \ge 3r'$ and $q > 4$.*

Here, the whole difficulty is to prove the endpoint estimates $q = 3r'$ in view of the trivial embeddings of Lebesgue spaces on the unit sphere. Our aim is to generalize this result to a weighted setting that contains another result from the literature. Bloom and Sampson [2, Theorem 2.11] proved optimal L^2-bounds for the operator

$$(\mathcal{R}^*_{\alpha,\beta}F)(x, y) := (\mathcal{R}^*F)(x, y)(1 + |x|)^{-\alpha}(1 + |y|)^{-\beta}, \qquad x, y \in \mathbb{R},$$

in the special case $\alpha = \beta$. Their result reads as follows.

Theorem 2 (Bloom, Sampson) *$\mathcal{R}^*_{\alpha,\alpha} : L^2(\mathbb{S}^1) \to L^2(\mathbb{R}^2)$ is bounded if and only if $\alpha \ge \frac{1}{3}$.*

We merge these two results in an optimal weighted version of the Restriction Conjecture.

Theorem 3 *Assume $0 \le \alpha, \beta < \infty$ and $1 \le r \le \infty$, $0 < q < \infty$. Then $\mathcal{R}^*_{\alpha,\beta} : L^r(\mathbb{S}^1) \to L^q(\mathbb{R}^2)$ is bounded provided that $\alpha + \beta > \frac{2}{q} - \frac{1}{2}$ as well as*

(i) $\max\{\alpha, \beta\} \ge \frac{1}{q}$ *and* $2\min\{\alpha, \beta\} > \frac{2}{q} - \frac{1}{r'}$ *or*
(ii) $\max\{\alpha, \beta\} < \frac{1}{q}$ *and* $\alpha + \beta + \min\{\alpha, \beta\} > \frac{3}{q} - \frac{1}{r'}$

holds. If additionally $1 < r \le q$ holds, then the operator is also bounded for

(iii) $\max\{\alpha, \beta\} > \frac{1}{q}$ *and* $2\min\{\alpha, \beta\} = \frac{2}{q} - \frac{1}{r'}$ *or*
(iv) $\max\{\alpha, \beta\} < \frac{1}{q}$ *and* $\alpha + \beta + \min\{\alpha, \beta\} = \frac{3}{q} - \frac{1}{r'}$.

In the case $q = \infty$ the operator is bounded for all $r \in [1, \infty]$, $\alpha, \beta \in [0, \infty)$.
These conditions are optimal.

Note that the special case $\alpha = \beta = 0$ reproduces Theorem 1 whereas the ansatz $\alpha = \beta$ and $q = r = 2$ leads to Theorem 2. The estimates (i),(ii) will be called nonendpoint estimates whereas (iii),(iv) are referred to as endpoint estimates. (See Figure 1).

Remark 4

(a) The conditions on q, r reflect the simple observation that more restrictive assumptions on the input data (larger r) and stronger weights (larger α, β) yield a wider range of admissible exponents q. Moreover, since the estimate for $q = \infty$ holds for all $\alpha, \beta \in [0, \infty)$ and $r \in [1, \infty]$, an interpolation argument shows that the estimates for the lower exponents q imply the ones for larger q. Formally, if $\mathcal{R}^*_{\alpha,\beta} : L^r(\mathbb{S}^1) \to L^q(\mathbb{R}^2)$ is bounded, then it is automatically bounded for parameters (α, β, r, q) replaced by $(\alpha + \varepsilon_\alpha, \beta + \varepsilon_\beta, r + \varepsilon_r, q + \varepsilon_q)$ whenever $\varepsilon_\alpha, \varepsilon_\beta, \varepsilon_r, \varepsilon_q \ge 0$.

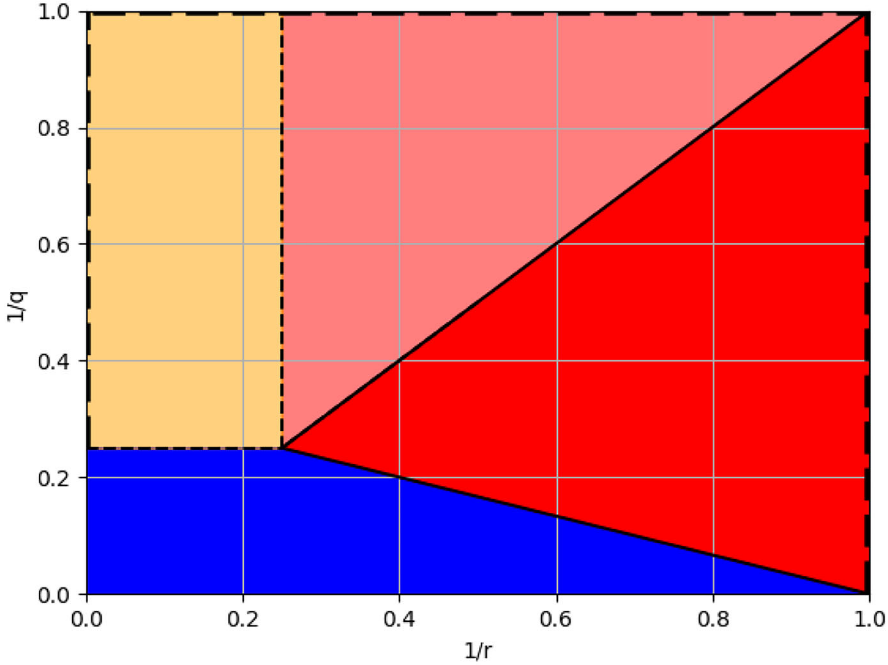

Fig. 1 Riesz diagram for Theorem 3 in the special case $\alpha = \beta$: the indicated $(\frac{1}{r}, \frac{1}{q})$-regions indicate different optimal conditions on α. Lower left: $\alpha \geq 0$, upper left: $\alpha > \frac{1}{q} - \frac{1}{4}$, right: $\alpha \geq \frac{1}{q} - \frac{1}{3r'}$, upper right: $\alpha > \frac{1}{q} - \frac{1}{3r'}$, dashed lines: strict inequalities, solid lines: non-strict inequalities

(b) The sufficiency part of Theorem 3 carries over to weight functions of the form $w_1(x)w_2(y)$ where $w_1 \in X^\alpha$, $w_2 \in X^\beta$ and $X^\gamma := L^{1/\gamma,\infty}(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Even more generally, this is true for weights $w(x, y)$ belonging to the intersection of mixed norm spaces $X_x^\alpha(X_y^\beta) \cap X_y^\beta(X_x^\alpha)$ where x, y indicate one-dimensional integration variables. To see this it suffices to adapt the proofs of Proposition 13 and Lemma 14. The subsequent interpolation procedure based on these two auxiliary results is the same.

(c) The a priori assumption $\alpha, \beta \geq 0$ has been added for the sake of clarity. In fact, it is a necessary condition for the estimates to hold. This follows from the equivariance property $\mathcal{R}^*(Fe^{ih\cdot})(x, y) = (\mathcal{R}^*F)(x + h_1, y + h_2)$ where $h = (h_1, h_2) \in \mathbb{R}^2$, see [2, p.88].

We extend our analysis to the weighted Fourier Extension operator

$$\mathfrak{R}_\gamma^* : L^r(\mathbb{S}^1) \to L^q(\mathbb{R}^2), \quad (\mathfrak{R}_\gamma^* F)(x, y) := (\mathcal{R}^*F)(1 + |x| + |y|)^{-\gamma}.$$

In [2, Theorem 4.8] Bloom and Sampson provided an almost complete study of its mapping properties.

Theorem 5 (Bloom, Sampson) *Assume* $2 \leq q < \infty$, $1 \leq r \leq \infty$. *Then the following assumptions are sufficient for* $\mathfrak{R}_\gamma^* : L^r(\mathbb{S}^1) \to L^q(\mathbb{R}^2)$ *to be bounded:*

(a) if $\frac{3}{q} - \frac{1}{r'} \leq 0, 4 < q < \infty$, *then* $\gamma \geq 0$,
(b) if $2 \leq q \leq 4, q \leq r$, *then* $\gamma > \frac{2}{q} - \frac{1}{2}$,
(c) if $q = 2 > r$, *then* $\gamma \geq \frac{1}{r}$,
(d) if $q = r' > 2$, *then* $\gamma > \frac{1}{q}$,
(e) if $2 < q < r'$, *then* $\gamma \geq \frac{2}{q} - \frac{1}{r'}$,
(f) else $\gamma > \frac{3}{2q} - \frac{1}{2r'}$.

Given the assumptions on q, r, *the range for* γ *is optimal in (a),(b),(c),(e) and optimal possibly up to the endpoint cases* $\gamma = \frac{1}{q}$ *in (d) and* $\gamma = \frac{3}{2q} - \frac{1}{2r'}$ *in (f).*

Our intention is to include exponents $0 < q < 2$ and close the gap about the endpoint cases in (d),(f). It turns out that this can be done along the lines of the proof of Theorem 3.

Theorem 6 *Assume* $0 \leq \gamma < \infty$ *and* $1 \leq r \leq \infty, 0 < q < \infty$. *Then* $\mathfrak{R}_\gamma^* : L^r(\mathbb{S}^1) \to L^q(\mathbb{R}^2)$ *is bounded provided that* $\gamma > \frac{2}{q} - \frac{1}{2}$ *as well as*

(i) $\gamma > \max\{\frac{3}{2q} - \frac{1}{2r'}, \frac{2}{q} - \frac{1}{r'}\}$.

If additionally $1 < r \leq q$ *holds, then the operator is also bounded for*

(ii) $\gamma = \max\{\frac{3}{2q} - \frac{1}{2r'}, \frac{2}{q} - \frac{1}{r'}\}$ *provided that* $q \neq r'$.

In the case $q = \infty$ *the operator is bounded for all* $r \in [1, \infty], \gamma \in [0, \infty)$.
These conditions are optimal.

Case dinstinctions show that Theorem 6 implies Theorem 5 and fill the aforementioned gap: the equality case in (d) does not belong to the boundedness range whereas the case $\gamma = \frac{3}{2q} - \frac{1}{2r'}$ with $q \neq r'$ in (f) does. This is visualized in Figure 2 below.

Remark 7 Our counterexamples show that the conditions $\gamma > \frac{2}{q} - \frac{1}{2}$ respectively $\alpha + \beta > \frac{2}{q} - \frac{1}{2}$ originate from the constant density on the sphere whereas all other conditions on α, β, γ are sharp in view of Knapp-type counterexamples. For the same reason as in Remark 4(c) the a priori assumption $\gamma \geq 0$ is actually a necessary condition for the estimates to hold.

2 Preliminaries

In this section we collect some known facts about real and complex interpolation theory for Lorentz spaces and mixed Lorentz spaces.

To put this into the abstract framework used in the paper [6] by Grafakos-Mastyło, we start with quasi-Banach (function) lattices X on a given measure space. Such a lattice is a vector space of measurable functions that are finite almost everywhere. By definition, it is complete with respect to a quasi-norm $\|\cdot\|_X$ and solid, i.e., $|f| \leq |g|$ almost everywhere and $g \in X$ implies $f \in X$ with $\|f\|_X \leq \|g\|_X$. A quasi-Banach lattice X is said to be maximal whenever $0 \leq f_n \uparrow f$ almost everywhere with $f_n \in X$ and $\sup_{n \geq 1} \|f_n\|_X < \infty$ implies that $f \in X$ and $\|f_n\|_X \to \|f\|_X$. In other words,

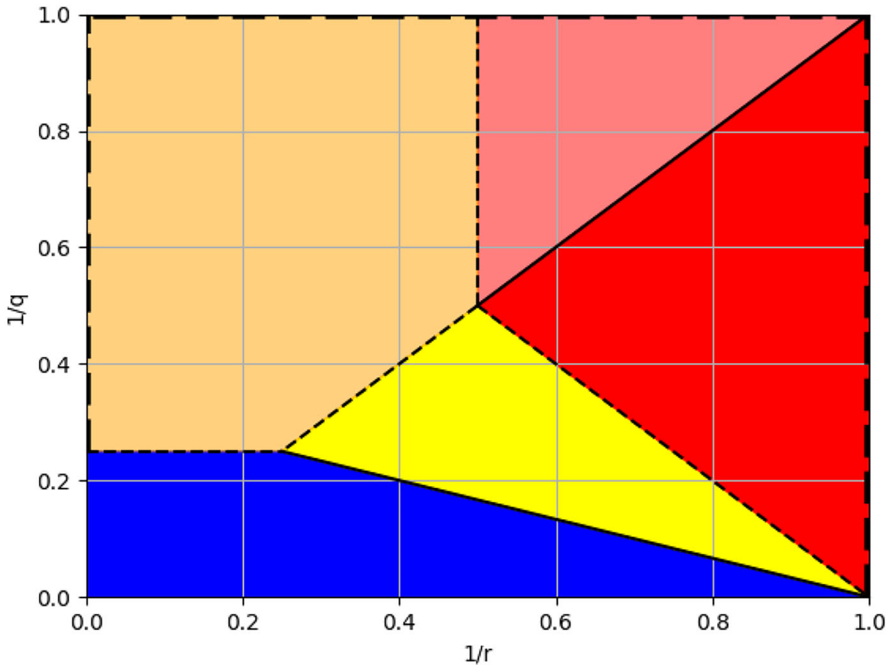

Fig. 2 Riesz diagram for Theorem 6: lower left: $\gamma \geq 0$, orange: $\gamma > \frac{2}{q} - \frac{1}{2}$, lower right: $\gamma \geq \frac{3}{2q} - \frac{1}{2r'}$, right: $\gamma \geq \frac{2}{q} - \frac{1}{r'}$, upper right: $\gamma > \frac{2}{q} - \frac{1}{r'}$, dashed lines: strict inequalities, solid lines: non-strict inequalities. Our refinements with regard to Theorem 5 concern the upper half, the endpoint case $\gamma = \frac{3}{2q} - \frac{1}{2r'}$ in the lower right region and the endpoint case $\gamma = \frac{1}{q}$ on the diagonal bottom right

we have some sort of Monotone Convergence Theorem in X. It is called p-convex for $0 < p < \infty$ if there exists a constant $C > 0$ such that for any $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in X$ we have

$$\left\| \left(\sum_{k=1}^{n} |f_k|^p \right)^{\frac{1}{p}} \right\|_X \leq C \left(\sum_{k=1}^{n} \|f_k\|_X^p \right)^{\frac{1}{p}}.$$

A quasi-Banach lattice is said to have nontrivial convexity whenever it is p-convex for some $0 < p < \infty$. For us the important fact is that Lorentz spaces and mixed Lorentz spaces belong to this class of spaces. To state this more precisely, we say that a tuple of exponents (p, r) is Lorentz-admissible if $0 < p < \infty, 0 < r \leq \infty$ or $p = r = \infty$. We then write $L^{p,r} := L^{p,r}(\mathbb{R}^d)$ for the standard Lorentz spaces and $L^{\vec{p},\vec{r}} := L^{p,r}(\mathbb{R}^{d-k}; L^{p,r}(\mathbb{R}^k))$ for mixed Lorentz spaces where $d \in \mathbb{N}$ and $k \in \{1, \ldots, d-1\}$. Recall that the latter is a quasi-Banach lattice equipped with the norm

$$f \mapsto \left\| \|f(x, y)\|_{L_y^{p,r}(\mathbb{R}^k)} \right\|_{L_x^{p,r}(\mathbb{R}^{d-k})}$$

where the subscripts indicate the integration variable.

Proposition 8 *Let $d \in \mathbb{N}$ and $k \in \{1, \dots, d-1\}$ and let (p, r) be Lorentz-admissible. Then the Lorentz space $L^{p,r}$ and the mixed Lorentz space $L^{\vec{p},\vec{r}}$ are maximal quasi-Banach lattices with nontrivial convexity.*

Note that in the case $r < \infty$ the maximality of Lorentz spaces follows from Fatou's Lemma and in the case of mixed Lorentz spaces one uses that a mixed quasi-Banach lattice $X(Y)$ is maximal if both X, Y are maximal. We omit the details of the proof and turn towards its relevance for interpolation theoretical applications. Real interpolation theory for abstract quasi-Banach spaces is explained in [1, Section 3.11] and the important special case of Lorentz spaces can be found in the sections 5.2 and 5.3 in this book. An analogous theory for mixed Lorentz spaces, however, appears to be missing in the literature. For that reason we prove one basic result in Proposition 12 below. Moreover, we will use (complex) interpolation for analytic families of linear operators, which we call "Stein interpolation" in view of the fundamental contribution by Stein [10] on this matter. Instead of the original version dealing with Banach spaces we use the nontrivial extension to quasi-Banach lattices due to Grafakos and Mastyło [6].

For a given couple of Banach spaces (X_0, X_1) [1, Section 2.3] the symbol $[X_0, X_1]_\theta$ denotes the standard complex interpolation space as defined in the book of Bergh-Löfström [1, p.88]. It is a Banach space equipped with the norm

$$\|f\|_{[X_0,X_1]_\theta} = \inf_{F(\theta)=f,\, F\in\mathcal{F}} \max\{\|F\|_{L^\infty(i\mathbb{R};X_0)}, \|F\|_{L^\infty(1+i\mathbb{R};X_1)}\}.$$

Here, $\mathcal{F}$ denotes the set of functions $F : \overline{S} \to X_0 + X_1$ that are bounded and continuous on $\overline{S}$, with $F : i\mathbb{R} \to X_0$, $F : 1 + i\mathbb{R} \to X_1$ vanishing at $\pm i\infty$, $1 \pm i\infty$ and being analytic in the strip $S := \{s \in \mathbb{C} : 0 < \Re(s) < 1\}$.

For our application, all we need to know is the embedding $[X_0, X_1]_\theta \supset (X_0, X_1)_{\theta,1}$ from [1, Theorem 4.7.1].

We shall moreover need another sort of interpolation space.

If Y_0 and Y_1 are quasi-Banach lattices over the same measure space and $0 < \theta < 1$, then their Calderón product $Y_0^{1-\theta} Y_1^\theta$ is defined as the vector space of all measurable functions g such that $|g| \le |g_0|^{1-\theta} |g_1|^\theta$ holds almost everywhere for some $g_0 \in Y_0$, $g_1 \in Y_1$. It is again a quasi-Banach lattice equipped with the quasi-norm

$$\|g\|_{Y_0^{1-\theta} Y_1^\theta} = \inf \left\{ \|g_0\|_{Y_0}^{1-\theta} \|g_1\|_{Y_1}^\theta : |g| \le |g_0|^{1-\theta} |g_1|^\theta \text{ a.e., } g_0 \in Y_0, g_1 \in Y_1 \right\}.$$

Following [6] we finally introduce the notion of an admissible analytic family of linear operators between couples (X_0, X_1) and (Y_0, Y_1) as introduced above. Let $\mathcal{X}$ be a subspace of $X_0 \cap X_1$. We assume that for every $z \in \bar{S}$ there is a linear map $T_z : \mathcal{X} \to Y_0 \cap Y_1$ such that $T_z f$ is a complex-valued measurable function that is finite almost everywhere for all $f \in \mathcal{X}$. The family $(T_z)_{z\in\bar{S}}$ is then said to be analytic if for any $f \in \mathcal{X}$ and for almost all x the function $z \mapsto T_z f(x)$ is analytic in S and continuous on $\bar{S}$. If additionally $(z, x) \mapsto T_z f(x)$ is measurable for every $f \in \mathcal{X}$, then

$(T_z)_{z\in\bar{S}}$ is called an admissible analytic family. We shall need the following simplified version of Theorem 1.1 in [6].

Theorem 9 (Grafakos-Mastyło) *Let (X_0, X_1) be a couple of Banach spaces and let (Y_0, Y_1) be a couple of maximal quasi-Banach lattices on a measure space such that Y_0, Y_1 have nontrivial convexity. Assume that $\mathcal{X}$ is a dense linear subspace of $X_0 \cap X_1$ and that $(T_z)_{z\in\bar{S}}$ is an admissible analytic family of linear operators $T_z : \mathcal{X} \to Y_0 \cap Y_1$. Suppose that for every $f \in \mathcal{X}$, $t \in \mathbb{R}$ and $j = 0, 1$,*

$$\left\| T_{j+it} f \right\|_{Y_j} \leq C_j \left\| f \right\|_{X_j} .$$

Then there is $C > 0$ such that for all $f \in \mathcal{X}$, $s \in \mathbb{R}$ and $0 < \theta < 1$ we have

$$\| T_{\theta+is} f \|_{Y_0^{1-\theta} Y_1^{\theta}} \leq C \| f \|_{[X_0, X_1]_\theta} .$$

Remark 10 (a) Interpolation theorems in special quasi-Banach spaces can be found in [4, 9]. However, none of those applies to mixed Lorentz spaces $L^{\vec{p},\vec{r}}$ with $0 < p < 1$. This is why we need the rather recent result from [6].

(b) In the context of Lorentz spaces Y_0, Y_1 the Calderón product $Y_0^{1-\theta} Y_1^{\theta}$ is a Lorentz space for most Lorentz-admissible exponents, see [6, Corollary 4.1]. Moreover, under quite general assumptions on Banach spaces Y_0, Y_1 we have $Y_0^{1-\theta} Y_1^{\theta} = [Y_0, Y_1]_\theta$ and suitable analogues are known for quasi-Banach lattices, see [11, pp. 2-3]. In that sense Theorem 9 generalizes the original result by Stein [10] (under slightly stronger assumptions on T_z).

In our application, it will turn out convenient to embed $[X_0, X_1]_\theta$ and $Y_0^{1-\theta} Y_1^{\theta}$ into real interpolation spaces. The embedding

$$(X_0, X_1)_{\theta,1} \subset [X_0, X_1]_\theta \subset (X_0, X_1)_{\theta,\infty}$$

is well-known and turns out to be helpful. We now show the analogous statement for $Y_0^{1-\theta} Y_1^{\theta}$.

Proposition 11 *Let (Y_0, Y_1) be a couple of maximal quasi-Banach lattices on a measure space. Then $(Y_0, Y_1)_{\theta,1} \subset Y_0^{1-\theta} Y_1^{\theta} \subset (Y_0, Y_1)_{\theta,\infty}$ for $0 < \theta < 1$.*

Proof Set $Y := Y_0^{1-\theta} Y_1^{\theta}$. For all $g \in Y_0 \cap Y_1$ we have by definition of the norm

$$\|g\|_Y = \inf_{\substack{|g|\leq|g_0|^{1-\theta}|g_1|^{\theta} \\ g_0\in Y_0, g_1\in Y_1}} \|g_0\|_{Y_0}^{1-\theta} \|g_1\|_{Y_1}^{\theta} \leq \|g\|_{Y_0}^{1-\theta} \|g\|_{Y_1}^{\theta}$$

by choosing $g_0 := g_1 := g$. So Theorem 3.11.4 (b) in [1] gives $(Y_0, Y_1)_{\theta,1} \subset Y$. To prove the other embedding let $g \in Y$ and fix $t > 0$. Choose $g_0 \in Y_0$, $g_1 \in Y_1$ such that $|g| \leq |g_0|^{1-\theta} |g_1|^{\theta}$ almost everywhere and $\|g_0\|_{Y_0}^{1-\theta} \|g_1\|_{Y_1}^{\theta} \leq 2\|g\|_Y$. Then we equally have $|g| \le |\tilde{g}_0|^{1-\theta}|\tilde{g}_1|^{\theta}$ with $\tilde{g}_0 = s^{-\theta} g_0 \in Y_0$, $\tilde{g}_1 = s^{1-\theta} g_1 \in Y_1$ for any given $s > 0$. We may choose $s := \|g_0\|_{Y_0}(t\|g_1\|_{Y_1})^{-1}$ and obtain

$$\begin{aligned}\max\left\{t^{-\theta}\|\tilde{g}_0\|_{Y_0}, t^{1-\theta}\|\tilde{g}_1\|_{Y_1}\right\} &= \max\left\{(st)^{-\theta}\|g_0\|_{Y_0}, (st)^{1-\theta}\|g_1\|_{Y_1}\right\} \\ &= \|g_0\|_{Y_0}^{1-\theta}\|g_1\|_{Y_1}^{\theta} \\ &\le 2\|g\|_Y.\end{aligned}$$

So we conclude that for all $t > 0$ there are $\tilde{g}_0 \in Y_0$, $\tilde{g}_1 \in Y_1$ satisfying the above inequality. Then Theorem 3.11.4 (a) in [1] gives $Y \subset (Y_0, Y_1)_{\theta,\infty}$, which is all we had to show. □

In the case of Lebesgue or Lorentz spaces X_0, X_1, Y_0, Y_1 standard real interpolation theory [1, Theorem 5.3.1] allows to compute the associated real interpolation spaces. In the case of mixed Lorentz spaces, however, we need some partial extension of a result from [7].

Proposition 12 *Assume* $d \in \mathbb{N}$, $k \in \{1, \ldots, d-1\}$, $0 < r \le \infty$ *and* $0 < q_0 \ne q_1 \le \infty$. *Then, for* $0 < \theta < 1$ *and* $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$,

$$\text{(i)}\quad L^{q_\theta,r}(\mathbb{R}^d) = \left(L^{q_0,\infty}(\mathbb{R}^{d-k}; L^{q_0,\infty}(\mathbb{R}^k)), L^{q_1,\infty}(\mathbb{R}^{d-k}; L^{q_1,\infty}(\mathbb{R}^k))\right)_{\theta,r},$$

$$\text{(ii)}\quad L^{q_\theta,r}(\mathbb{R}^d) = \left(L^{q_0}(\mathbb{R}^d), L^{q_1,\infty}(\mathbb{R}^{d-k}; L^{q_1,\infty}(\mathbb{R}^k))\right)_{\theta,r},$$

$$\text{(iii)}\quad L^{q_\theta,\infty}(\mathbb{R}^d) \supset L^{q_0}(\mathbb{R}^d)^{1-\theta}\left(L^{q_1,\infty}(\mathbb{R}^{d-k}; L^{q_1,\infty}(\mathbb{R}^k))\right)^{\theta}.$$

Proof Theorem 2.22 in [3] provides the inequality $\|g\|_{L^{q_\theta}} \le \|g\|_{L^{\vec{q}_0,\vec{\infty}}}^{1-\theta}\|g\|_{L^{\vec{q}_1,\vec{\infty}}}^{\theta}$ for all $q_0, q_1 \in (0, \infty]$ with $q_0 \ne q_1$. As in the previous Proposition this can be combined with [1, Theorem 3.11.4] to deduce, for all $\mu \in [q_0, \infty]$,

$$L^{q_\theta} \supset (L^{\vec{q}_0,\vec{\infty}}, L^{\vec{q}_1,\vec{\infty}})_{\theta,1} \supset (L^{\vec{q}_0,\vec{\mu}}, L^{\vec{q}_1,\vec{\infty}})_{\theta,1}.$$

Exploiting this identity for $\theta \in \{\theta_0, \theta_1\}$ with $0 < \theta_0 < \theta_1 < 1$ as well as the Reiteration Theorem [1, Theorem 3.11.5] we deduce the embedding

$$\begin{aligned}L^{q_\theta,r} &= (L^{q_{\theta_0}}, L^{q_{\theta_1}})_{\vartheta,r} \\ &\supset \left((L^{\vec{q}_0,\vec{\mu}}, L^{\vec{q}_1,\vec{\infty}})_{\theta_0,1}, (L^{\vec{q}_0,\vec{\mu}}, L^{\vec{q}_1,\vec{\infty}})_{\theta_1,1}\right)_{\vartheta,r} \\ &\supset (L^{\vec{q}_0,\vec{\mu}}, L^{\vec{q}_1,\vec{\infty}})_{\theta,r} \\ &\quad \text{whenever}\quad \theta = (1-\vartheta)\theta_0 + \vartheta\theta_1,\ 0 < \vartheta, \theta_0 \ne \theta_1 < 1,\ r \in (0, \infty].\end{aligned}$$

On the other hand, the embeddings $L^{\vec{q}_0,\vec{\mu}} \supset L^{q_0}$ and $L^{\vec{q}_1,\vec{\infty}} \supset L^{q_1}$ imply

$$(L^{\vec{q}_0,\vec{\mu}}, L^{\vec{q}_1,\vec{\infty}})_{\theta,r} \supset (L^{q_0}, L^{q_1})_{\theta,r} = L^{q_\theta,r}.$$

Combining both inclusions we infer (i) by choosing $\mu = \infty$ and (ii) by choosing $\mu = q_0$. Finally, (iii) follows from (ii) and the embedding $(Y_0, Y_1)_{\theta,\infty} \supset Y_0^{1-\theta} Y_1^{\theta}$ that we established in Proposition 11. □

3 Proof of Theorem 3 – Sufficient Conditions

In this section we analyze the mapping properties of $\mathcal{R}^*_{\alpha,\beta} : L^r(\mathbb{S}^1) \to L^q(\mathbb{R}^2)$. Since the claim is trivial for $q = \infty$, we shall assume $0 < q < \infty$. Moreover, without loss of generality, we assume $\alpha \geq \beta \geq 0$. We shall need the following auxiliary result.

Proposition 13 *Assume* $1 \leq p \leq 2, 0 < q < \infty$ *and* $0 \leq \frac{1}{q} - \frac{1}{p'} \leq \beta < \infty$. *Then*

$$\|(1 + |\cdot|)^{-\beta} \hat{f}\|_{L^{q,\infty}(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}.$$

Proof Define r via $\frac{1}{q} = \frac{1}{r} + \frac{1}{p'}$, so $r \in [\frac{1}{\beta}, \infty]$. Then the claim follows from the Lorentz space version of Hölder's Inequality and the Hausdorff-Young Inequality via

$$\|(1 + |\cdot|)^{-\beta} \hat{f}\|_{L^{q,\infty}(\mathbb{R})} \leq \|(1 + |\cdot|)^{-\beta}\|_{L^{r,\infty}(\mathbb{R})} \|\hat{f}\|_{L^{p',\infty}(\mathbb{R})} \leq \|\hat{f}\|_{L^{p'}(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}.$$

□

We deduce some mixed restricted weak-type bounds for $\mathcal{R}^*_{\alpha,\beta}$ for $\alpha \geq \frac{1}{q}$.

Lemma 14 *Assume* $0 \leq \beta \leq \alpha < \infty$ *and* $0 < q < \infty$, $1 \leq r \leq \infty$.

(i) If $r < \infty$, $\alpha \geq \frac{1}{q}$ *and* $2\beta \geq \frac{2}{q} - \frac{1}{r'}$, *then* $\mathcal{R}^*_{\alpha,\beta} : L^{r,1}(\mathbb{S}^1) \to L^{q,\infty}(\mathbb{R}; L^{q,\infty}(\mathbb{R}))$ *is bounded.*
(ii) If $r = \infty$, $\alpha \geq \frac{1}{q}$ *and* $2\beta > \frac{2}{q} - \frac{1}{r'}$, *then* $\mathcal{R}^*_{\alpha,\beta} : L^r(\mathbb{S}^1) \to L^{q,\infty}(\mathbb{R}; L^{q,\infty}(\mathbb{R}))$ *is bounded.*

Proof Given the embeddings $L^\infty(\mathbb{S}^1) \subset L^{r,1}(\mathbb{S}^1)$ for all $r \in [1, \infty)$ it suffices to prove (i). We may w.l.o.g. assume $\beta \leq \frac{1}{q}$ and that F vanishes in the left half of the sphere. Set $F_\star(\phi) := F(\cos(\phi), \sin(\phi))$ and

$$G_x(s) := \sqrt{2\pi} F_\star(-\arcsin(s))(1 - s^2)^{-\frac{1}{2}} e^{ix(1-s^2)^{\frac{1}{2}}} \mathbb{1}_{[-1,1]}(s).$$

Then

$$
\begin{aligned}
(\mathcal{R}^* F)(x,y) &= \int_{\mathbb{S}^1} F(z) e^{i(x,y)\cdot(z_1,z_2)}\, d\sigma(z) \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F_\star(\phi) e^{i(x,y)\cdot(\cos(\phi),\sin(\phi))}\, d\phi \\
&= \int_{-1}^{1} F_\star(\arcsin(s))(1-s^2)^{-\frac{1}{2}} e^{ix(1-s^2)^{\frac{1}{2}}} e^{isy}\, ds \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} G_x(s) e^{-isy}\, ds \\
&= \widehat{G_x}(y).
\end{aligned}
$$

Define the exponents p, μ via $\frac{1}{p} := \frac{1}{q'} + \beta$ and $\frac{1}{\mu} := \frac{1}{p} - \frac{1}{p'}$. Our assumptions on β imply $1 \leq p \leq 2$ and $1 \leq \mu \leq r$. From $\alpha \geq \frac{1}{q}$ and Proposition 13 we infer

$$
\begin{aligned}
\|\mathcal{R}^*_{\alpha,\beta} F\|_{L^{q,\infty}(\mathbb{R};L^{q,\infty}(\mathbb{R}))} &= \left\| (1+|x|)^{-\alpha} \cdot \|(1+|y|)^{-\beta} \hat{G}_x(y)\|_{L^{q,\infty}_y(\mathbb{R})} \right\|_{L^{q,\infty}_x(\mathbb{R})} \\
&\leq \|(1+|x|)^{-\alpha}\|_{L^{q,\infty}_x(\mathbb{R})} \cdot \sup_{x\in\mathbb{R}} \|(1+|y|)^{-\beta} \hat{G}_x(y)\|_{L^{q,\infty}_y(\mathbb{R})} \\
&\leq \sup_{x\in\mathbb{R}} \|G_x\|_{L^p(\mathbb{R})} \\
&\simeq \|F_\star(-\arcsin(\cdot))(1-(\cdot)^2)^{-\frac{1}{2}}\|_{L^p([-1,1])} \\
&\simeq \|F_\star |\cos|^{\frac{1-p}{p}}\|_{L^p([-\frac{\pi}{2},\frac{\pi}{2}])} \\
&\leq \|F_\star\|_{L^{\mu,p}([-\frac{\pi}{2},\frac{\pi}{2}])} \| |\cos|^{\frac{1-p}{p}} \|_{L^{\frac{p}{p-1},\infty}([0,\frac{\pi}{2}])} \\
&\leq \|F_\star\|_{L^{r,1}([-\frac{\pi}{2},\frac{\pi}{2}])} \\
&\leq \|F\|_{L^{r,1}(\mathbb{S}^1)}.
\end{aligned}
$$

Here the third last estimate is justified due to $1 \leq \mu < \infty$, which finishes the proof. □

We shall partially upgrade these estimates via interpolation. Before doing this, we extend the range of restricted weak-type estimates to parameters $\alpha \in (0, \frac{1}{q})$. This is done via Stein interpolation in the setting of quasi-Banach spaces as presented earlier. In contrast to the previous result the range will be $L^{q,\infty}(\mathbb{R}^2)$ instead of $L^{q,\infty}(\mathbb{R}; L^{q,\infty}(\mathbb{R}))$.

Lemma 15 *Assume* $0 \leq \beta \leq \alpha < \frac{1}{q}$ *and* $1 \leq r \leq \infty, 0 < q < \infty$.

(i) If $r < \infty$ *then* $\mathcal{R}^*_{\alpha,\beta} : L^{r,1}(\mathbb{S}^1) \to L^{q,\infty}(\mathbb{R}^2)$ *is bounded if* $\alpha + \beta > \frac{2}{q} - \frac{1}{2}$, $\alpha + 2\beta \geq \frac{3}{q} - \frac{1}{r'}$.

(ii) If $r = \infty$ *then* $\mathcal{R}^*_{\alpha,\beta} : L^r(\mathbb{S}^1) \to L^{q,\infty}(\mathbb{R}^2)$ *is bounded if* $\alpha+\beta > \frac{2}{q}-\frac{1}{2}$, $\alpha+2\beta > \frac{3}{q} - \frac{1}{r'}$.

Proof As in Lemma 14 it suffices to prove (i). We interpolate the estimates from Lemma 14 with the unweighted ones from Theorem 1 and concentrate on the case $r < \infty$ in (i). We show in the Appendix (Proposition 29) that our assumptions on α, β, r, q allow to find $r_0 \in [1, \infty], r_1 \in [1, \infty), q_0 \in [1, \infty], q_1 \in (0, \infty)$ and $\theta \in (0, 1)$ such that

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1} \quad \text{and}$$
$$\frac{1}{q_0} \le \frac{1}{3r_0'}, \quad \frac{1}{q_0} < \frac{1}{4}, \quad \frac{\alpha}{\theta} = \frac{1}{q_1}, \quad \frac{\beta}{\theta} \ge \frac{1}{q_1} - \frac{1}{2r_1'}, \quad q_0 \neq q_1.$$

Then the family of linear operators $\mathcal{E}_s := \mathcal{R}^*_{s\alpha/\theta, s\beta/\theta}$ has the following properties:

(a) $\mathcal{E}_s : L^{r_0}(\mathbb{S}^1) \to L^{q_0}(\mathbb{R}^2)$ is bounded for $\Re(s) = 0$.
This follows from Theorem 1 and $\frac{1}{q_0} \le \frac{1}{3r_0'}, \frac{1}{q_0} < \frac{1}{4}$.

(b) $\mathcal{E}_s : L^{r_1,1}(\mathbb{S}^1) \to L^{q_1,\infty}_x(\mathbb{R}; L^{q_1,\infty}_y(\mathbb{R}))$ is bounded for $\Re(s) = 1$.
This follows from Lemma 14 (i) and $\frac{\alpha}{\theta} = \frac{1}{q_1}, \frac{\beta}{\theta} \ge \frac{1}{q_1} - \frac{1}{2r_1'}$.

Stein interpolation (Theorem 9 for $\mathcal{X} := L^\infty(\mathbb{S}^1)$) then implies the boundedness of

$$\mathcal{E}_s : [L^{r_0}(\mathbb{S}^1), L^{r_1,1}(\mathbb{S}^1)]_\theta \to L^{q_0}(\mathbb{R}^2)^{1-\theta}\big(L^{q_1,\infty}(\mathbb{R}; L^{q_1,\infty}(\mathbb{R}))\big)^\theta \quad \text{whenever } \Re(s) = \theta.$$

Plugging in $s = \theta$ and using the embeddings

$$L^{r,1}(\mathbb{S}^1) \subset [L^{r_0}(\mathbb{S}^1), L^{r_1,1}(\mathbb{S}^1)]_\theta \quad \text{and} \quad L^{q,\infty}(\mathbb{R}^2) \supset L^{q_0}(\mathbb{R}^2)^{1-\theta}\big(L^{q_1,\infty}(\mathbb{R}; L^{q_1,\infty}(\mathbb{R}))\big)^\theta$$

from Proposition 12 (note $q_0 \neq q_1$) we deduce the boundedness of $\mathcal{R}^*_{\alpha,\beta} = \mathcal{E}_\theta : L^{r,1}(\mathbb{S}^1) \to L^{q,\infty}(\mathbb{R}^2)$ for the claimed range of exponents q, r. □

Corollary 16 *Assume $0 \le \beta \le \alpha < \infty$ and $1 < r, q < \infty$. Then $\mathcal{R}^*_{\alpha,\beta} : L^{r,s}(\mathbb{S}^1) \to L^{q,s}(\mathbb{R}^2)$ is bounded for all $s \in [1, \infty]$ if one of the following conditions holds:*

(i) $\alpha > \frac{1}{q}$ *and* $2\beta \ge \frac{2}{q} - \frac{1}{r'}$,
(ii) $\alpha = \frac{1}{q}$ *and* $2\beta > \frac{2}{q} - \frac{1}{r'}$,
(iii) $\alpha < \frac{1}{q}$ *and* $\alpha + \beta > \frac{2}{q} - \frac{1}{2}$ *as well as* $\alpha + 2\beta \ge \frac{3}{q} - \frac{1}{r'}$.

Proof We first consider the range $\alpha > \frac{1}{q}$ and it suffices to prove the endpoint estimate where $2\beta = \frac{2}{q} - \frac{1}{r'}$. In view of $1 < r, q < \infty$ we may choose $\varepsilon > 0$ sufficiently small and define $q_0, q_1, r_0, r_1 \in (1, \infty)$ via

$$\frac{1}{q_0} := \frac{1}{q} - \varepsilon, \quad \frac{1}{q_1} := \frac{1}{q} + \varepsilon, \quad \frac{1}{r_0} := \frac{1}{r} + 2\varepsilon, \quad \frac{1}{r_1} := \frac{1}{r} - 2\varepsilon.$$

We then have $\alpha > \frac{1}{q_i}$ and $2\beta = \frac{2}{q_i} - \frac{1}{r_i'}$ for $i = 0, 1$, so Lemma 14 implies that $\mathcal{R}^*_{\alpha,\beta} : L^{r_i,1}(\mathbb{S}^1) \to L^{q_i,\infty}(\mathbb{R}; L^{q_i,\infty}(\mathbb{R}))$ is bounded. Hence, real interpolation yields the boundedness of

$$\mathcal{R}^*_{\alpha,\beta} : \big(L^{r_0,1}(\mathbb{S}^1), L^{r_1,1}(\mathbb{S}^1)\big)_{\theta,s} \to \big(L^{q_0,\infty}(\mathbb{R}; L^{q_0,\infty}(\mathbb{R})), L^{q_1,\infty}(\mathbb{R}; L^{q_1,\infty}(\mathbb{R}))\big)_{\theta,s}$$

for all $s \in [0, 1]$. Since $q_0 > q > q_1, r_0 < r < r_1$ and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$ for $\theta := \frac{1}{2}$, real interpolation [1, Theorem 3.11.8] and Proposition 12 (i) give the boundedness of $\mathcal{R}^*_{\alpha,\beta} : L^{r,s}(\mathbb{S}^1) \to L^{q,s}(\mathbb{R}^2)$ for all $s \in [1, \infty]$, which implies claim (i).

In order to prove (iii) assume $0 < \alpha < \frac{1}{q}$ and choose $q_0, q_1, r_0, r_1 \in (1, \infty)$ such that

$$\frac{1}{q_0} := \frac{1}{q} - \varepsilon, \quad \frac{1}{q_1} := \frac{1}{q} + \varepsilon, \quad \frac{1}{r_0} := \frac{1}{r} + 3\varepsilon, \quad \frac{1}{r_1} := \frac{1}{r} - 3\varepsilon$$

for some sufficiently small $\varepsilon > 0$. Then one may check $0 < \alpha < \frac{1}{q_i}, \alpha + \beta > \frac{2}{q_i} - \frac{1}{2}$ as well as $\alpha + 2\beta \geq \frac{3}{q_i} - \frac{1}{r_i'}$. Real interpolation theory and Lemma 15 yield the boundedness of $\mathcal{R}^*_{\alpha,\beta} : L^{r,s}(\mathbb{S}^1) \to L^{q,s}(\mathbb{R}^2)$ for all $s \in [1, \infty]$. The case (ii) may be deduced from (iii). □

Proof of Theorem 3 – sufficient conditions: We prove the sufficiency part of the Theorem assuming $0 < q < \infty$ and w.l.o.g. $\alpha \geq \beta \geq 0$. So we have to show that $\mathcal{R}^*_{\alpha,\beta} : L^r(\mathbb{S}^1) \to L^q(\mathbb{R}^2)$ is bounded for q, r as stated in the theorem.

Endpoint estimates (iii),(iv): Claim (iii) is about $\alpha > \frac{1}{q}, 2\beta = \frac{2}{q} - \frac{1}{r'}$ and $1 < r \leq q$. Here, the bound results from Corollary 16 (i) thanks to the embeddings $L^r(\mathbb{S}^1) \subset L^{r,s}(\mathbb{S}^1)$ and $L^{q,s}(\mathbb{R}^2) \subset L^q(\mathbb{R}^2)$ whenever $s \in [r, q]$. Such s exists due to $r \leq q$. Analogously, Corollary 16 (iii) yields the endpoint estimate (iv) dealing with the case $\alpha < \frac{1}{q}, \alpha + 2\beta = \frac{3}{q} - \frac{1}{r'}$ and $1 < r \leq q$.

Nonendpoint estimates (i),(ii): The nonendpoint estimates concern the parameter region

$$(i) \quad \alpha \geq \frac{1}{q},\ 2\beta > \frac{2}{q} - \frac{1}{r'} \qquad \text{or} \qquad (ii) \quad \alpha < \frac{1}{q},\ \alpha + 2\beta > \frac{3}{q} - \frac{1}{r'}.$$

In the case $1 < r \leq \infty$, the boundedness of $R^*_{\alpha,\beta} : L^r(\mathbb{S}^1) \to L^q(\mathbb{R}^2)$ is a consequence of Corollary 16. Indeed, in that case we can find a sufficiently small $\varepsilon > 0$ such that the Corollary implies the boundedness from $L^{r_\varepsilon,q}(\mathbb{S}^1)$ to $L^q(\mathbb{R}^2)$ where $\frac{1}{r_\varepsilon} := \frac{1}{r} + \varepsilon$. So the embedding $L^r(\mathbb{S}^1) \hookrightarrow L^{r_\varepsilon,q}(\mathbb{S}^1)$ yields the claim. So it remains to deal with the nonendpoint estimates for $r = 1$. Since (ii) is void for $r = 1$ (recall $\alpha \geq \beta$), we have to prove the boundedness for $\alpha \geq \beta > \frac{1}{q}$.

Lemma 14 (i) implies the boundedness $L^1(\mathbb{S}^1) \to L^{q-\varepsilon,\infty}(\mathbb{R}; L^{q-\varepsilon,\infty}(\mathbb{R}))$ for sufficiently small $\varepsilon > 0$. On the other hand, the boundedness $L^1(\mathbb{S}^1) \to L^\infty(\mathbb{R}^2)$

is trivial. So interpolation and Proposition 12 (ii) implies the boundedness of $R^*_{\alpha,\beta}$: $L^1(\mathbb{S}^1) \to L^q(\mathbb{R}^2)$, which is all we had to show. □

4 Proof of Theorem 3 – Necessary Conditions

We now show that our result is sharp in the scale of Lebesgue spaces. This is mainly achieved by a detailed analysis of the constant density $F \equiv 1$, which turns out to be responsible for the necessary condition $\alpha + \beta > \frac{2}{q} - \frac{1}{2}$, and Knapp-type counterexamples. In the following we shall always assume at least

$$0 \le \beta \le \alpha < \infty, \qquad 1 \le r \le \infty, \qquad 0 < q < \infty.$$

The necessity of the conditions in Theorem 3 follows from four Lemmas in this section. We shall prove:

(i) The condition $\alpha + \beta > \frac{2}{q} - \frac{1}{2}$ is necessary by Lemma 17.
(ii) Endpoint estimates for $\alpha > \frac{1}{q}$ can only hold for $q \ge r > 1$ by Lemma 18 and 19.
(iii) Endpoint estimates for $\alpha = \frac{1}{q}$ are impossible by Lemma 18.
(iv) Endpoint estimates for $\alpha < \frac{1}{q}$ can only hold for $q \ge r$ by Lemma 21.

From (ii)-(iv) we infer that endpoint estimates can only hold if $1 < r \le q$ and $\alpha \ne \frac{1}{q}$. Note that the endpoint estimates for $\alpha < \frac{1}{q}$ do not occur for $r = 1$. Combining this with Remark 4(a) we obtain the necessity of all sufficient conditions stated in Theorem 3.

The first counterexample deals with the constant density on the unit sphere. The Fourier transform of the surface measure of the unit sphere in $\mathbb{R}^2$ is given by $\widehat{\sigma}(x) = cJ_0(|x|)$ where c is some absolute constant and J_0 denotes a Bessel function of the second kind.

The properties of such Bessel functions are well-known; we shall only need the lower bound

$$|J_0(r)| \gtrsim \sum_{j=1}^{\infty} j^{-\frac{1}{2}} \mathbb{1}_{[z_j-\delta, z_j+\delta]}(r) \tag{1}$$

where $\delta > 0$ is some small fixed number and $\{z_j : j \in \mathbb{N}\}$ denotes the set of positive local extrema of J_0. It is known that $c\, j \le z_j \le C\, j$ and that $z_j - z_{j-1}$ converges to a positive constant as $j \to \infty$. So $\delta > 0$ can be assumed so small that the intervals $[z_j - \delta, z_j + \delta]$ are mutually disjoint.

Lemma 17 *Assume* $0 \le \beta \le \alpha < \infty, 1 \le r \le \infty, 0 < q < \infty$. *If the operator* $\mathcal{R}^*_{\alpha,\beta} : L^r(\mathbb{S}^1) \to L^q(\mathbb{R}^2)$ *is bounded, then* $\alpha + \beta > \frac{2}{q} - \frac{1}{2}$.

Proof For $F := 1 \in L^r(\mathbb{S}^1)$ we have

$$
\begin{aligned}
\|\mathcal{R}^*_{\alpha,\beta} F\|^q_{L^q(\mathbb{R}^2)} &= \|J_0(|(x,y)|)(1+|x|)^{-\alpha}(1+|y|)^{-\beta}\|^q_{L^q(\mathbb{R}^2)} \\
&\overset{(1)}{\gtrsim} \sum_{j=1}^{\infty} j^{-\frac{q}{2}} \|(1+|x|)^{-\alpha}(1+|y|)^{-\beta}\mathbb{1}_{[z_j-\delta, z_j+\delta]}(|(x,y)|)\|^q_{L^q(\mathbb{R}^2)} \\
&\gtrsim \sum_{j=1}^{\infty} j^{-\frac{q}{2}} \int_{z_j/4}^{z_j/2} (1+|x|)^{-q\alpha} \left(\int_{\sqrt{(z_j-\delta)^2-|x|^2}}^{\sqrt{(z_j+\delta)^2-|x|^2}} (1+|y|)^{-q\beta}\, dy \right) dx \\
&\gtrsim \sum_{j=1}^{\infty} j^{-\frac{q}{2}} \int_{z_j/4}^{z_j/2} (1+|x|)^{-q\alpha} \cdot (1+|z_j|)^{-q\beta} \left(\int_{\sqrt{(z_j-\delta)^2-|x|^2}}^{\sqrt{(z_j+\delta)^2-|x|^2}} 1\, dy \right) dx \\
&\gtrsim \sum_{j=1}^{\infty} j^{-\frac{q}{2}} \int_{z_j/4}^{z_j/2} (1+|x|)^{-q\alpha} \cdot \delta (1+|z_j|)^{-q\beta}\, dx \\
&\gtrsim \sum_{j=1}^{\infty} j^{-\frac{q}{2}} z_j (1+|z_j|)^{-q\alpha-q\beta} \\
&\gtrsim \sum_{j=1}^{\infty} j^{1-q(\frac{1}{2}+\alpha+\beta)}.
\end{aligned}
$$

So $\mathcal{R}^*_{\alpha,\beta} F \in L^q(\mathbb{R}^2)$ implies $1 - q(\frac{1}{2} + \alpha + \beta) < -1$, which is equivalent to $\alpha + \beta > \frac{2}{q} - \frac{1}{2}$. ☐

Next we discuss the Knapp example following the ideas from [8].

Lemma 18 *Assume* $0 \le \beta \le \alpha < \infty$, $1 \le r \le \infty$, $0 < q < \infty$. *Then the following conditions are necessary for* $\mathcal{R}^*_{\alpha,\beta} : L^r(\mathbb{S}^1) \to L^q(\mathbb{R}^2)$ *to be bounded:*

(i) if $\alpha > \frac{1}{q}$, *then* $2\beta \ge \frac{2}{q} - \frac{1}{r'}$, $r > 1$ *or* $2\beta > \frac{2}{q}$, $r = 1$,
(ii) if $\alpha = \frac{1}{q}$, *then* $2\beta > \frac{2}{q} - \frac{1}{r'}$,
(iii) if $\alpha < \frac{1}{q}$, *then* $\alpha + 2\beta \ge \frac{3}{q} - \frac{1}{r'}$.

Proof For $\delta > 0$ consider $F := \mathbb{1}_{\mathcal{C}_\delta}$ where $\mathcal{C}_\delta := \{\xi = (\xi_1, \xi_2) \in \mathbb{S}^1 : |\xi_1| < \delta\}$ denotes a spherical cap. Then, for $1 \le r < \infty$,

$$
\|F\|_{L^r(\mathbb{S}^1)} = \left(\int_{\mathcal{C}_\delta} d\sigma(\xi) \right)^{\frac{1}{r}} = \left(\int_0^{2\pi} \mathbb{1}_{|\sin(\phi)|<\delta}\, d\phi \right)^{\frac{1}{r}} \simeq \delta^{\frac{1}{r}}
$$

and the same bound holds for $r = \infty$. To estimate $\|\mathcal{R}^*_{\alpha,\beta} F\|_{L^q(\mathbb{R}^2)}$ from below, define

$$
E_j := \left\{ (x,y) \in \mathbb{R}^2 : 0 < |x| \le c\delta^{-1}, \frac{2\pi j - c}{\sqrt{1-\delta^2}} \le y \le 2\pi j + c \right\}
$$

for some small enough constant $c > 0$, say $c := \frac{\pi}{8}$. We then have for $(x, y) \in E_j$,

$$
\begin{aligned}
|(\mathcal{R}^* F)(x, y)| &= \left| \int_{\mathcal{C}_\delta} e^{i\xi \cdot (x,y)} \, d\sigma(\xi) \right| \\
&= \left| \int_0^{2\pi} \mathbb{1}_{|\sin(\phi)| < \delta} e^{i(\sin(\phi)x + \cos(\phi)y - 2\pi j)} \, d\phi \right| \\
&\geq \int_0^{2\pi} \mathbb{1}_{|\sin(\phi)| < \delta} \cos\big(\sin(\phi)x + \cos(\phi)y - 2\pi j\big) \, d\phi
\end{aligned}
$$

The argument of cosine has absolute value at most $\frac{\pi}{4}$ by choice of $c > 0$ and $(x, y) \in E_j$. Indeed,

$$
\begin{aligned}
&\sin(\phi)x + \cos(\phi)y \quad - 2\pi j \\
&\quad \geq -\delta|x| + \sqrt{1-\delta^2}y - 2\pi j \geq -\delta \cdot c\delta^{-1} - c = -2c = -\frac{\pi}{4}, \\
&\quad \sin(\phi)x + \cos(\phi)y - 2\pi j \\
&\quad \leq \delta|x| + y - 2\pi j \leq \delta \cdot c\delta^{-1} + c = 2c = \frac{\pi}{4}.
\end{aligned}
$$

So we obtain

$$
|(\mathcal{R}^* F)(x, y)| \geq \int_0^{2\pi} \mathbb{1}_{|\sin(\phi)| < \delta} \cos\left(\frac{\pi}{4}\right) d\phi \gtrsim \delta \quad \text{for } (x, y) \in E_j, \, j \in \mathbb{N}.
$$

Moreover, the sets E_j are mutually disjoint and satisfy

$$
2\pi j + c - \frac{2\pi j - c}{\sqrt{1-\delta^2}} \geq c \qquad \text{for } j = 1, \ldots, \lfloor c_0 \delta^{-2} \rfloor
$$

provided that $c_0 > 0$ is small enough. This implies

$$
\begin{aligned}
\frac{\|\mathcal{R}^*_{\alpha,\beta} F\|_{L^q(\mathbb{R}^2)}}{\|F\|_{L^r(\mathbb{S}^1)}} &\gtrsim \delta^{-\frac{1}{r}} \|(\mathcal{R}^* F)(x, y)(1+|x|)^{-\alpha}(1+|y|)^{-\beta}\|_{L^q(\mathbb{R}^2)} \\
&\gtrsim \delta^{-\frac{1}{r}} \cdot \left(\sum_{j=1}^{\lfloor c_0 \delta^{-2} \rfloor} \|\mathbb{1}_{E_j} \delta (1+|x|)^{-\alpha}(1+|y|)^{-\beta}\|^q_{L^q(\mathbb{R}^2)} \right)^{\frac{1}{q}} \\
&\simeq \delta^{-\frac{1}{r}} \left(\sum_{j=1}^{\lfloor c_0 \delta^{-2} \rfloor} \delta^q \int_0^{c\delta^{-1}} (1+|x|)^{-\alpha q} \, dx \int_{\frac{2\pi j - c}{\sqrt{1-\delta^2}}}^{2\pi j + c} (1+|y|)^{-\beta q} \, dy \right)^{\frac{1}{q}} \\
&\simeq \delta^{\frac{1}{r'}} \left(\int_0^{c\delta^{-1}} (1+|x|)^{-\alpha q} \, dx \cdot \sum_{j=1}^{\lfloor c_0 \delta^{-2} \rfloor} j^{-\beta q} \right)^{\frac{1}{q}}
\end{aligned}
$$

$$\gtrsim \delta^{\frac{1}{r'}} \left(\begin{cases} 1 & ,\alpha > \frac{1}{q} \\ |\log(\delta)| & ,\alpha = \frac{1}{q} \\ \delta^{-1+\alpha q} & ,\alpha < \frac{1}{q} \end{cases} \cdot \begin{cases} 1 & ,\beta > \frac{1}{q} \\ |\log(\delta)| & ,\beta = \frac{1}{q} \\ \delta^{-2+2\beta q} & ,\beta < \frac{1}{q} \end{cases} \right)^{\frac{1}{q}}.$$

So a case distinction gives the result. □

We start with proving the necessity of $1 < r \le q$ in the case of endpoint estimates in the parameter range $\alpha > \frac{1}{q}$.

Lemma 19 *Assume $0 \le \beta \le \alpha < \infty$ and $0 < q < \infty$, $1 < r \le \infty$ where $\alpha > \frac{1}{q}$, $\beta = \frac{1}{q} - \frac{1}{2r'}$. Then $\mathcal{R}^*_{\alpha,\beta} : L^r(\mathbb{S}^1) \to L^q(\mathbb{R}^2)$ can only be bounded if $q \ge r$.*

Proof By the monotonicity considerations from Remark 4 (a) with respect to q, it suffices to derive the necessity of $q \ge r$ assuming a priori $q > 1$. We actually prove the equivalent statement that the dual bound

$$\|\hat{f}\|_{L^{r'}(\mathbb{S}^1)} \lesssim \|(1+|x|)^\alpha (1+|y|)^\beta f\|_{L^{q'}(\mathbb{R}^2)}$$

can only hold for $q \ge r$. We consider functions of the form

$$f_\varepsilon(x, y) := \chi(x)(1+|y|)^{-\beta - \frac{1+\varepsilon}{q'}} e^{iy}$$

where $\varepsilon > 0$ is small and χ is a nontrivial nonnegative Schwartz function on $\mathbb{R}$. Then

$$\|(1+|x|)^\alpha (1+|y|)^\beta f_\varepsilon\|_{L^{q'}(\mathbb{R}^2)} \le \left(\int_0^\infty (1+y)^{-1-\varepsilon}\, dy \right)^{\frac{1}{q'}} \le \varepsilon^{-\frac{1}{q'}}.$$

Since $r > 1$, the exponents $\beta + \frac{1+\varepsilon}{q'} = 1 - \frac{1}{2r'} + \frac{\varepsilon}{q'}$ are contained in $(0, 1)$ provided that $\varepsilon > 0$ is sufficiently small. So Proposition 28 in the Appendix applies and yields as $(\xi_1, \xi_2) \to (0, 1)$

$$\begin{aligned} |\hat{f}_\varepsilon(\xi)| &= \left| \int_{\mathbb{R}} \chi(x) e^{-i\xi_1 \cdot x}\, dx \cdot \int_{\mathbb{R}} (1+|y|)^{-1+\frac{1}{2r'}-\frac{\varepsilon}{q'}} e^{-i(\xi_2-1)\cdot y}\, dy \right| \\ &= \left| \int_{\mathbb{R}} \chi(x) e^{-i\xi_1 \cdot x}\, dx \right| \cdot 2 \left| \int_0^\infty (1+r)^{-1+\frac{1}{2r'}-\frac{\varepsilon}{q'}} \cos((\xi_2-1)r)\, dr \right| \\ &\gtrsim \left(\int_{\mathbb{R}} \chi(x)\, dx \right) |\xi_2 - 1|^{-\frac{1}{2r'}+\frac{\varepsilon}{q'}} \\ &\gtrsim |\xi_2 - 1|^{-\frac{1}{2r'}+\frac{\varepsilon}{q'}} \end{aligned}$$

uniformly with respect to small $\varepsilon > 0$. So the assumptions implies, in view of $r' < \infty$, for some small enough $c > 0$

$$\begin{aligned}
\varepsilon^{-\frac{1}{q'}} \gtrsim \|\hat{f}_\varepsilon\|_{L^{r'}(\mathbb{S}^1)} &= \left(\int_0^{2\pi} |\hat{f}_\varepsilon(\sin(t),\cos(t))|^{r'}\,dt\right)^{\frac{1}{r'}} \\
&\gtrsim \left(\int_0^c |\cos(t)-1|^{r'(-\frac{1}{2r'}+\frac{\varepsilon}{q'})}\,dt\right)^{\frac{1}{r'}} \gtrsim \left(\int_0^c t^{-1+\frac{2r'\varepsilon}{q'}}\,dt\right)^{\frac{1}{r'}} \\
&\simeq \varepsilon^{-\frac{1}{r'}}.
\end{aligned}$$

From this we infer $r' \geq q'$, hence $q \geq r$. □

Next we extend this result to the range $\alpha < \frac{1}{q}$ and prove the necessity of $q \geq r$ in the special case $q = 2$.

Lemma 20 *Assume $0 \leq \beta \leq \alpha < \frac{1}{2}$, $1 \leq r \leq \infty$ and $\alpha+\beta > \frac{1}{2}$ as well as $\alpha + 2\beta = \frac{3}{2} - \frac{1}{r'}$. Then $\mathcal{R}^*_{\alpha,\beta} : L^r(\mathbb{S}^1) \to L^2(\mathbb{R}^2)$ can only be bounded if $2 \geq r$.*

Proof First we observe that the assumptions imply $1 < r < \infty$. Indeed, if $r = 1$, then $3\alpha \geq \alpha + 2\beta = \frac{3}{2} > 3\alpha$ and if $r = \infty$, then $\alpha + 2\beta \geq \alpha + \beta > \frac{1}{2} = \alpha + 2\beta$. Both statements are absurd, so $1 < r < \infty$. Moreover, we necessarily have $\beta > 0$ in view of $\alpha < \frac{1}{2} < \alpha + \beta$. To prove the claim set

$$F_\delta(\sin\phi, \cos\phi) := \phi^{-\mu}\mathbb{1}_{[0,\delta]}(\phi) \quad \text{for } \phi \in [0, 2\pi]$$

and $\mu \in (0, \frac{1}{r})$ that we will send to $\frac{1}{r} < 1$ from below. The parameter $\delta > 0$ will be chosen sufficiently small but fixed. Then

$$\|F_\delta\|^2_{L^r(\mathbb{S}^1)} \simeq \left(\int_0^\delta \phi^{-\mu r}\,d\phi\right)^{\frac{2}{r}} = \left(\frac{1}{1-\mu r}\delta^{1-\mu r}\right)^{\frac{2}{r}} \simeq (1-\mu r)^{-\frac{2}{r}}.$$

On the other hand, we have

$$\begin{aligned}
&\|\mathcal{R}^*_{\alpha,\beta}F_\delta\|^2_{L^2(\mathbb{R}^2)} \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} (1+|x|)^{-2\alpha}(1+|y|)^{-2\beta}\left|\int_{\mathbb{S}^1} F_\delta(\xi)e^{i\xi\cdot(x,y)}\,d\sigma(\xi)\right|^2 dy\,dx \\
&= \int_0^\delta\int_0^\delta \phi^{-\mu}\varphi^{-\mu}\left(\int_{\mathbb{R}}(1+|x|)^{-2\alpha}e^{i\lambda_{\phi,\varphi}x}\,dx\right)\left(\int_{\mathbb{R}}(1+|y|)^{-2\beta}e^{-i\mu_{\phi,\varphi}y}\,dy\right)d\varphi\,d\phi \\
&= 4\int_0^\delta\int_0^\delta \phi^{-\mu}\varphi^{-\mu}\left(\int_0^\infty (1+r)^{-2\alpha}\cos(\lambda_{\phi,\varphi}r)\,dr\right)\left(\int_0^\infty (1+r)^{-2\beta}\cos(\mu_{\phi,\varphi}r)\,dr\right)d\varphi\,d\phi \\
&= 8\int_0^\delta \phi^{-\mu}\int_0^\phi \varphi^{-\mu}\left(\int_0^\infty (1+r)^{-2\alpha}\cos(\lambda_{\phi,\varphi}r)\,dr\right)\left(\int_0^\infty (1+r)^{-2\beta}\cos(\mu_{\phi,\varphi}r)\,dr\right)d\varphi\,d\phi
\end{aligned}$$

where $\lambda_{\phi,\varphi} = \sin(\phi) - \sin(\varphi)$ and $\mu_{\phi,\varphi} = \cos(\varphi) - \cos(\phi)$. We observe $\lambda_{\phi,\varphi}, \mu_{\phi,\varphi} \to 0^+$ where $0 \le \varphi \le \phi \le \delta \to 0^+$. Hence, for small $\delta > 0$ we obtain from Proposition 28, in view of $0 < 2\beta \le 2\alpha < 1$,

$$\begin{aligned}\|\mathcal{R}^*_{\alpha,\beta} F_\delta\|^2_{L^2(\mathbb{R}^2)} &\gtrsim \int_0^\delta \phi^{-\mu} \int_0^\phi \varphi^{-\mu} \lambda^{2\alpha-1}_{\phi,\varphi} \mu^{2\beta-1}_{\phi,\varphi} \, d\varphi \, d\phi \\ &= \int_0^\delta \phi^{-\mu} \int_0^\phi \varphi^{-\mu} (\sin(\phi) - \sin(\varphi))^{2\alpha-1} (\cos(\varphi) - \cos(\phi))^{2\beta-1} \, d\varphi \, d\phi.\end{aligned}$$

From the Mean Value Theorem and $\alpha + 2\beta = \frac{3}{2} - \frac{1}{r'}$ we get

$$\begin{aligned}\|\mathcal{R}^*_{\alpha,\beta} F_\delta\|^2_{L^2(\mathbb{R}^2)} &\gtrsim \int_0^\delta \phi^{-\mu} \int_0^\phi \varphi^{-\mu} (\phi - \varphi)^{2\alpha-1} (\phi(\phi - \varphi))^{2\beta-1} \, d\varphi \, d\phi \\ &= \int_0^\delta \phi^{2\beta-1-\mu} \int_0^\phi (\phi - \varphi)^{2\alpha+2\beta-2} \varphi^{-\mu} \, d\varphi \, d\phi \\ &= \int_0^\delta \phi^{2\beta-1-\mu} \int_0^1 \phi(\phi - s\phi)^{2\alpha+2\beta-2} (s\phi)^{-\mu} \, ds \, d\phi \\ &= \int_0^\delta \phi^{2\alpha+4\beta-2-2\mu} \, d\phi \cdot \int_0^1 (1-s)^{2\alpha+2\beta-2} s^{-\mu} \, ds \\ &\gtrsim \int_0^\delta \phi^{-1+\frac{2}{r}-2\mu} \, d\phi \\ &\simeq (\frac{2}{r} - 2\mu)^{-1}.\end{aligned}$$

Here we used $\mu < \frac{1}{r} < 1$. Choose $\mu = \frac{1}{r} - \varepsilon$ and send $\varepsilon \to 0^+$. Then

$$\frac{\|\mathcal{R}^*_{\alpha,\beta} F_\delta\|^2_{L^2(\mathbb{R}^2)}}{\|F_\delta\|^2_{L^r(\mathbb{S}^1)}} \gtrsim \frac{(1-\mu r)^{2/r}}{\frac{2}{r} - 2\mu} \gtrsim \frac{(\varepsilon r)^{2/r}}{2\varepsilon} \simeq \varepsilon^{\frac{2}{r}-1},$$

so $r \le 2$ is necessary for the boundedness of $\mathcal{R}^*_{\alpha,\beta} : L^r(\mathbb{S}^1) \to L^2(\mathbb{R}^2)$. □

We remark that the rather explicit expression for $\|\mathcal{R}^*_{\alpha,\beta} F\|_{L^2(\mathbb{R}^2)}$ from Lemma 20 may be used to give an alternative proof of $\|\mathcal{R}^*_{\alpha,\beta} F\|_{L^2(\mathbb{R}^2)} \le \|F\|_{L^r(\mathbb{S}^1)}$ under optimal conditions on α, β, r. In a very similar context this has been done in [8, Proposition 3.1]. To conclude we now know that $q \ge r$ is necessary for endpoint estimates in the special case $q = 2$. We use this fact to derive the necessity of $q \ge r$ for all $q \in (0, \infty)$ using interpolation.

Lemma 21 *Assume* $0 \le \beta \le \alpha < \frac{1}{q}$ *and* $1 < r \le \infty, 0 < q \le \infty$ *as well as* $\alpha + \beta > \frac{2}{q} - \frac{1}{2}, \alpha + 2\beta = \frac{3}{q} - \frac{1}{r'}$. *Then* $\mathcal{R}^*_{\alpha,\beta} : L^r(\mathbb{S}^1) \to L^q(\mathbb{R}^2)$ *can only be bounded if* $q \ge r$.

Proof The case $q = 2$ is already covered by Lemma 20. To cover $q \in (0, 2)$ and $q \in (2, \infty)$ we argue by contradicition. Assume first the claim is false for some

$q > 2$, i.e., that there are $2 < q_1 < r_1$ and $\mathcal{R}^*_{\alpha_1,\beta_1}$ is bounded with $0 \le \beta_1 \le \alpha_1 < \frac{1}{q_1}$ satisfying $\alpha_1 + \beta_1 > \frac{2}{q_1} - \frac{1}{2}$ and $\alpha_1 + 2\beta_1 = \frac{3}{q_1} - \frac{1}{r_1'}$. Then choose $r_0 = q_0 \in (1, 2)$ and $\alpha_0 \ge \beta_0 > 0$ such that

$$0 < \alpha_0 < \frac{1}{q_0}, \quad \alpha_0 + \beta_0 > \frac{2}{q_0} - \frac{1}{2}, \quad \alpha_0 + 2\beta_0 = \frac{3}{q_0} - \frac{1}{r_0'}. \tag{2}$$

For instance, $\alpha_0 := \beta_0 := \frac{1}{3}(\frac{4}{q_0} - 1)$ and $r_0 := q_0 \in (1, 2)$. Theorem 3 (iii) then implies that $\mathcal{R}^*_{\alpha_\theta,\beta_\theta} : L^{r_\theta}(\mathbb{S}^1) \to L^{q_\theta}(\mathbb{R}^2)$ is bounded for $\theta \in \{0, 1\}$ and Stein interpolation (Theorem 9) allows to deduce that the same is true for $\theta \in (0, 1)$. Here,

$$\alpha_\theta = (1-\theta)\alpha_0 + \theta\alpha_1, \quad \beta_\theta = (1-\theta)\beta_0 + \theta\beta_1, \quad \frac{1}{r_\theta} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

This, however, contradicts our optimal result about the case $q = 2$. To see why choose $\theta \in (0, 1)$ such that $q_\theta = 2$, which is possible due to $q_0 < 2 < q_1$. Then one checks

$$0 < \beta_\theta \le \alpha_\theta < \frac{1}{2}, \quad \alpha_\theta + \beta_\theta > \frac{1}{2}, \quad \alpha_\theta + 2\beta_\theta = \frac{3}{2} - \frac{1}{r_\theta'} \quad \text{and} \quad \frac{1}{r_\theta} < \frac{1}{q_\theta} = \frac{1}{2}.$$

But Lemma 20 implies that $\mathcal{R}^*_{\alpha_\theta,\beta_\theta} : L^{r_\theta}(\mathbb{S}^1) \to L^2(\mathbb{R}^2) = L^{q_\theta}(\mathbb{R}^2)$ is not bounded under these conditions, so the assumption was false and the claim is proved for $q > 2$.

In order to cover the case $0 < q < 2$ one similarly assumes for contradiction that the estimate holds for some exponents $r_1 > q_1$ with $q_1 \in (0, 2)$ and α_1, β_1 as above. Then one chooses $r_0, q_0 \in (2, \infty)$ and α_0, β_0 satisfying (2), say $\alpha_0 := \beta_0 := \frac{1}{3}(\frac{4}{q_0} - 1)$ and $r_0 := q_0 \in (2, 4)$. The same interpolation argument allows to derive a contradiction. After all, the assumption was false and the claim is proved. □

5 Proof of Theorem 6 – Sufficient Conditions

In this section we determine the mapping properties of $\mathfrak{R}^*_\gamma : L^r(\mathbb{S}^1) \to L^q(\mathbb{R}^2)$. We introduce the Banach space $(Z_q, \|\cdot\|_{Z_q})$ as follows:

$$Z_q := L_x^{q,\infty}(\mathbb{R}; L_y^{q,\infty}(\mathbb{R})) + L_y^{q,\infty}(\mathbb{R}; L_x^{q,\infty}(\mathbb{R})),$$
$$\|z\|_{Z_q} := \inf \left\{ \|z_1\|_{L_x^{q,\infty}(\mathbb{R}; L_y^{q,\infty}(\mathbb{R}))} + \|z_2\|_{L_y^{q,\infty}(\mathbb{R}; L_x^{q,\infty}(\mathbb{R}))} : z = z_1 + z_2 \in Z_q \right\}.$$

The counterpart of Lemma 14 reads as follows.

Lemma 22 *Assume $0 \le \gamma < \infty$ and $0 < q \le \infty$, $1 \le r \le \infty$. Then $\mathfrak{R}^*_\gamma : L^r(\mathbb{S}^1) \to Z_q$ is bounded provided that*

$$\gamma \ge \max\left\{\frac{1}{q}, \frac{2}{q} - \frac{1}{r'}, \frac{2}{q} - \frac{1}{2}\right\}.$$

Proof We first assume that $F:\mathbb{S}^1\to\mathbb{R}$ is supported away from the north and south pole, i.e., $(0,1),(0,-1)\notin\operatorname{supp}(F)$. Then set $F_\star(\phi):=F(\cos(\phi),\sin(\phi))$. As in the proof of Lemma 14, with identical notation, we have

$$(\mathcal{R}^*F)(x,y)=\int_{-1}^{1}F_\star(\arcsin(s))(1-s^2)^{-\frac{1}{2}}e^{ix(1-s^2)^{\frac{1}{2}}}e^{isy}\,ds=\widehat{G_x}(y).$$

Our assumption on the support of F makes sure that the factor $(1-s^2)^{-1/2}$ may be treated as a bounded function, set $H_x(z):=(1+|x|)^{-1}G_x((1+|x|)^{-1}z)$. By assumption on γ,q,r we may choose $p\in[1,\infty]$ such that

$$0\le\frac{1}{p}-\frac{1}{q'}\le\gamma,\quad \max\left\{\frac{1}{2},\frac{1}{r}\right\}\le\frac{1}{p}\le 1,\quad \frac{1}{q}-\gamma-1+\frac{1}{p}\le-\frac{1}{q}.$$

Then Proposition 13 yields

$$\begin{aligned}
\|\mathfrak{R}_\gamma^*F\|_{Z_q}&\le\|\mathfrak{R}_\gamma^*F\|_{L_x^{q,\infty}(\mathbb{R};L_y^{q,\infty}(\mathbb{R}))}\\
&=\left\|\|(1+|x|+|y|)^{-\gamma}\hat{G}_x(y)\|_{L_y^{q,\infty}(\mathbb{R})}\right\|_{L_x^{q,\infty}(\mathbb{R})}\\
&=\left\|\|(1+|x|+|y|)^{-\gamma}\hat{H}_x\big((1+|x|)^{-1}y\big)\|_{L_y^{q,\infty}(\mathbb{R})}\right\|_{L_x^{q,\infty}(\mathbb{R})}\\
&=\left\|(1+|x|)^{\frac{1}{q}-\gamma}\|(1+|z|)^{-\gamma}\hat{H}_x(z)\|_{L_z^{q,\infty}(\mathbb{R})}\right\|_{L_x^{q,\infty}(\mathbb{R})}\\
&\le\left\|(1+|x|)^{\frac{1}{q}-\gamma}\|H_x\|_{L^p(\mathbb{R})}\right\|_{L_x^{q,\infty}(\mathbb{R})}\\
&\le\left\|(1+|x|)^{\frac{1}{q}-\gamma-1+\frac{1}{p}}\right\|_{L_x^{q,\infty}(\mathbb{R})}\cdot\sup_{x\in\mathbb{R}}\|G_x\|_{L^p(\mathbb{R})}\\
&\le\|F_\star\|_{L^p([-1,1])}\\
&\le\|F\|_{L^p(\mathbb{S}^1)}\\
&\le\|F\|_{L^r(\mathbb{S}^1)}.
\end{aligned}$$

The reasoning for $F\in L^r(\mathbb{S}^1)$ with $(1,0),(-1,0)\notin\operatorname{supp}(F)$ is analogous since it suffices to repeat the analysis with the roles of x and y interchanged. Since we can write $F=F_1+F_2$ and thus $\mathfrak{R}_\gamma^*F=\mathfrak{R}_\gamma^*F_1+\mathfrak{R}_\gamma^*F_2$ with $(0,\pm1)\notin\operatorname{supp}(F_1)$, $(\pm1,0)\notin\operatorname{supp}(F_2)$, the claim follows. □

As in the proof of our first result, this weak-type estimate needs to be interpolated with the unweighted bound from Theorem 1. We use again Theorem 9 as well as the embedding of interpolation spaces

$$\begin{aligned}
&L^{q_0}(\mathbb{R}^2)^{1-\theta}(Z_{q_1})^\theta\subset(L^{q_0}(\mathbb{R}^2),Z_{q_1})_{\theta,\infty}=L^{q_\theta,\infty}(\mathbb{R}^2)\\
&\text{where}\quad\frac{1-\theta}{q_0}+\frac{\theta}{q_1}=\frac{1}{q_\theta},\quad 0<\theta<1,\quad q_0\ne q_1
\end{aligned}\tag{3}$$

which is proved just like Proposition 12. We deduce the following:

Lemma 23 *Let $\gamma > 0$. Then the operator $\mathfrak{R}_\gamma^* : L^r(\mathbb{S}^1) \to L^{q,\infty}(\mathbb{R}^2)$ is bounded provided that $r \in [1,\infty]$, $q \in (0,\infty]$ satisfy*

$$\gamma \geq \max\left\{\frac{3}{2q} - \frac{1}{2r'}, \frac{2}{q} - \frac{1}{r'}\right\} \quad \text{and} \quad \gamma > \frac{2}{q} - \frac{1}{2}. \tag{4}$$

Proof The Fefferman-Zygmund result (Theorem 1) yields the boundedness $\mathfrak{R}_0^* : L^{r_0}(\mathbb{S}^1) \to L^{q_0}(\mathbb{R}^2)$ and $\mathfrak{R}_{\gamma_1}^* : L^{r_1}(\mathbb{S}^1) \to Z^{q_1}(\mathbb{R}^2)$ follows from Lemma 22. Here, $r_0, r_1 \in [1,\infty]$, $q_0, q_1 \in (0,\infty)$ are such that

$$\frac{3}{q_0} \leq \frac{1}{r_0'}, \quad \frac{1}{q_0} < \frac{1}{4}, \quad \gamma_1 \geq \max\left\{\frac{1}{q_1}, \frac{2}{q_1} - \frac{1}{r_1'}, \frac{2}{q_1} - \frac{1}{2}\right\}.$$

Using interpolation (Theorem 9) and $[L^{r_0}(\mathbb{S}^1), L^{r_1}(\mathbb{S}^1)]_\theta = L^{r_\theta}(\mathbb{S}^1)$ as well as (3) we find that $\mathfrak{R}_\gamma^* : L^r(\mathbb{S}^1) \to L^{q,\infty}(\mathbb{R}^2)$ is bounded provided that

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad \gamma = \theta\gamma_1, \quad 0 < \theta < 1, \quad q_0 \neq q_1$$

In the Appendix (Proposition 30) we show that such a choice can be made if and only if (4) holds and the claim is proved. □

Next we use real interpolation to upgrade these weak estimates via interpolation just as in the proof of Corollary 16. In the non-endpoint case this is possible without additional constraint, but in the endpoint case we have to exclude the case $q = r'$ where $\gamma = \frac{3}{2q} - \frac{1}{2r'} = \frac{2}{q} - \frac{1}{r'}$. One may check that precisely in this case it is impossible to choose $\varepsilon > 0$ as in the proof of Corollary 16.

Corollary 24 *Let $\gamma > 0$ and $0 < q < \infty$, $1 < r < \infty$. Then $\mathfrak{R}_\gamma^* : L^{r,s}(\mathbb{S}^1) \to L^{q,s}(\mathbb{R}^2)$ is bounded for all $s \in [1,\infty]$ provided that $\gamma > \frac{2}{q} - \frac{1}{2}$ and*

$$\gamma > \max\left\{\frac{3}{2q} - \frac{1}{2r'}, \frac{2}{q} - \frac{1}{r'}\right\} \quad \text{or} \quad \gamma = \max\left\{\frac{3}{2q} - \frac{1}{2r'}, \frac{2}{q} - \frac{1}{r'}\right\}, \ q \neq r'. \tag{5}$$

Proof of Theorem 6 – sufficient conditions: The result for $q = \infty$ is trivial and the one for $\gamma = 0$ is already covered by Theorem 1, so assume $0 < q < \infty$ and $\gamma > 0$. Choosing $s = r$ in Corollary 24 and exploiting the embeddings of Lorentz spaces yields that for any given $\gamma > 0$ the operator $\mathfrak{R}_\gamma^* : L^r(\mathbb{S}^1) \to L^q(\mathbb{S}^1)$ is bounded provided that $\gamma > \frac{2}{q} - \frac{1}{2}$ and

$$\begin{aligned}
\gamma &> \max\left\{\frac{3}{2q} - \frac{1}{2r'}, \frac{2}{q} - \frac{1}{r'}\right\}, \ 1 < r < \infty \qquad \text{or} \\
\gamma &= \max\left\{\frac{3}{2q} - \frac{1}{2r'}, \frac{2}{q} - \frac{1}{r'}\right\}, \ q \neq r', \ 1 < r \leq q.
\end{aligned}$$

The non-endpoint estimates actually also hold for $r \in \{1, \infty\}$ as can be deduced from Lemma 23 and a short interpolation argument. So the conditions stated in Theorem 6 are sufficient. $\square$

6 Proof of Theorem 6 – Necessary Conditions

Next we prove that the conditions in Theorem 6 cannot be improved by providing suitable counterexamples. As before, the constant density and Knapp-type examples are all we need. For given exponents $1 \le r \le \infty$, $0 < q < \infty$ and $\gamma > 0$ we shall deduce the necessity of (5) as follows:

(i) $\gamma > \frac{2}{q} - \frac{1}{2}$ is necessary by Lemma 25,
(ii) $\gamma \ge \max\{\frac{3}{2q} - \frac{1}{2r'}, \frac{2}{q} - \frac{1}{r'}\}$ is necessary by Lemma 26,
(iii) $\gamma = \max\{\frac{3}{2q} - \frac{1}{2r'}, \frac{2}{q} - \frac{1}{r'}\}$ implies $q \ne r'$ by Lemma 26 (iii).
(iv) $\gamma = \max\{\frac{3}{2q} - \frac{1}{2r'}, \frac{2}{q} - \frac{1}{r'}\}$ implies $1 < r \le q$ by Lemma 26 (i) and Lemma 27.

The computations are similar to the ones carried out before, so we keep the presentation as short as possible.

Lemma 25 *Assume $0 \le \gamma < \infty, 1 \le r \le \infty, 0 < q < \infty$. If the operator $\mathfrak{R}_\gamma^*$: $L^r(\mathbb{S}^1) \to L^q(\mathbb{R}^2)$ is bounded, then $\gamma > \frac{2}{q} - \frac{1}{2}$.*

Proof As in Lemma 17 the claim follows from taking $F := 1 \in L^r(\mathbb{S}^1)$ and

$$\begin{aligned}\|\mathfrak{R}_\gamma^* F\|_{L^q(\mathbb{R}^2)}^q &= \|J_0(|(x,y)|)(1+|x|+|y|)^{-\gamma}\|_{L^q(\mathbb{R}^2)}^q \\ &\overset{(1)}{\gtrsim} \sum_{j=1}^{\infty} j^{-\frac{q}{2}}(1+z_j)^{-q\gamma}\left|\left\{(x,y)\in\mathbb{R}^2 : z_j-\delta \le |(x,y)| \le z_j+\delta\right\}\right| \\ &\gtrsim \sum_{j=1}^{\infty} j^{-\frac{q}{2}} \cdot z_j^{1-q\gamma} \\ &\gtrsim \sum_{j=1}^{\infty} j^{1-q(\frac{1}{2}+\gamma)}.\end{aligned}$$

$\square$

Next we discuss the Knapp example.

Lemma 26 *Assume $0 \le \gamma < \infty, 1 \le r \le \infty, 0 < q < \infty$. Then the following conditions are necessary for $\mathfrak{R}_\gamma^* : L^r(\mathbb{S}^1) \to L^q(\mathbb{R}^2)$ to be bounded:*

(i) if $\gamma = \frac{2}{q}$, then $r > 1$,
(ii) if $\frac{2}{q} > \gamma > \frac{1}{q}$, then $\gamma \ge \frac{2}{q} - \frac{1}{r'}$,
(iii) if $\gamma = \frac{1}{q}$, then $0 > \frac{1}{q} - \frac{1}{r'}$,
(iv) if $\gamma < \frac{1}{q}$, then $\gamma \ge \frac{3}{2q} - \frac{1}{2r'}$.

Proof We mimick the proof of Lemma 18. With the same notation, the function $F := \mathbb{1}_{\mathcal{C}_\delta}$ satisfies $\|F\|_{L^r(\mathbb{S}^1)} \simeq \delta^{\frac{1}{r}}$ and for $(x,y) \in E_j$ we have $|(\mathcal{R}^* F)(x,y)| \gtrsim \delta$ as $\delta \to 0^+$ provided that $j \le c_0 \delta^{-2}$ for some small constant $c_0 > 0$. This implies, for fixed γ, q, r and small $\delta > 0$,

$$
\begin{aligned}
\frac{\|\mathfrak{R}^*_\gamma F\|_{L^q(\mathbb{R}^2)}}{\|F\|_{L^r(\mathbb{S}^1)}} &\gtrsim \delta^{-\frac{1}{r}} \|(\mathcal{R}^* F)(x,y)(1+|x|+|y|)^{-\gamma}\|_{L^q(\mathbb{R}^2)} \\
&\gtrsim \delta^{-\frac{1}{r}} \left(\sum_{j=1}^{\lfloor c_0\delta^{-2}\rfloor} \|\mathbb{1}_{E_j} \delta (1+|x|+|y|)^{-\gamma}\|^q_{L^q(\mathbb{R}^2)} \right)^{\frac{1}{q}} \\
&\simeq \delta^{-\frac{1}{r}} \left(\sum_{j=1}^{\lfloor c_0\delta^{-2}\rfloor} \delta^q \int_0^{c\delta^{-1}} (x+j)^{-\gamma q}\, dx \right)^{\frac{1}{q}} \\
&\simeq \delta^{\frac{1}{r'}} \left(\sum_{j=1}^{\lfloor c_0\delta^{-2}\rfloor} \begin{cases} j^{1-\gamma q} - (j+\delta^{-1})^{1-\gamma q} & \text{, if } \gamma > \frac{1}{q} \\ \log(j+\delta^{-1}) - \log(j) & \text{, if } \gamma = \frac{1}{q} \\ (j+\delta^{-1})^{1-\gamma q} - j^{1-\gamma q} & \text{, if } \gamma < \frac{1}{q} \end{cases} \right)^{\frac{1}{q}} \\
&\gtrsim \delta^{\frac{1}{r'}} \left(\sum_{j=1}^{\lfloor \delta^{-1}\rfloor} \begin{cases} j^{1-\gamma q} & \text{, if } \gamma > \frac{1}{q} \\ \log(\delta^{-1} j^{-1}) & \text{, if } \gamma = \frac{1}{q} \\ \delta^{-1+\gamma q} & \text{, if } \gamma < \frac{1}{q} \end{cases} + \sum_{j=\lceil \delta^{-1}\rceil}^{\lfloor c_0\delta^{-2}\rfloor} \delta^{-1} j^{-\gamma q} \right)^{\frac{1}{q}} \\
&\simeq \delta^{\frac{1}{r'}} \left(\begin{cases} 1 & \text{, if } \gamma > \frac{2}{q} \\ |\log(\delta)| & \text{, if } \gamma = \frac{2}{q} \\ \delta^{-2+\gamma q} & \text{, if } \gamma < \frac{2}{q} \end{cases} + \begin{cases} \delta^{-2+\gamma q} & \text{, if } \gamma > \frac{1}{q} \\ \delta^{-1}|\log(\delta^{-1})| & \text{, if } \gamma = \frac{1}{q} \\ \delta^{-3+2\gamma q} & \text{, if } \gamma < \frac{1}{q} \end{cases} \right)^{\frac{1}{q}} \\
&\simeq \delta^{\frac{1}{r'}} \left(\begin{cases} 1 & \text{, if } \gamma > \frac{2}{q} \\ |\log(\delta)| & \text{, if } \gamma = \frac{2}{q} \\ \delta^{-2+\gamma q} & \text{, if } \frac{2}{q} > \gamma > \frac{1}{q} \\ \delta^{-1}|\log(\delta)| & \text{, if } \gamma = \frac{1}{q} \\ \delta^{-3+2\gamma q} & \text{, if } \gamma < \frac{1}{q} \end{cases} \right)^{\frac{1}{q}} .
\end{aligned}
$$

This implies the claim as $\delta \to 0^+$. Note that in the second last estimate for $\gamma = \frac{1}{q}$ we used

$$
\sum_{j=1}^{\lfloor \delta^{-1}\rfloor} \log(\delta^{-1} j^{-1}) \simeq \int_1^{\delta^{-1}} \log(\delta^{-1} x^{-1})\, dx \simeq \delta^{-1}.
$$

□

Lemma 27 *Assume* $0 < \gamma < \infty$ *where* $\gamma = \max\{\frac{3}{2q} - \frac{1}{2r'}, \frac{2}{q} - \frac{1}{r'}\} > \frac{2}{q} - \frac{1}{2}$ *and* $0 < q < \infty, 1 < r \leq \infty$. *Then* $\mathfrak{R}^*_\gamma : L^r(\mathbb{S}^1) \to L^q(\mathbb{R}^2)$ *can only be bounded if* $q \geq r$.

Proof The claim is trivial for $\gamma = \frac{3}{2q} - \frac{1}{2r'}$ because $\gamma > \frac{2}{q} - \frac{1}{2}$ already implies $q > r$. So assume $\gamma = \frac{2}{q} - \frac{1}{r'}$. It suffices to prove the claim assuming a priori $q \geq 1$. We then prove the equivalent statement that the dual bound

$$\|\hat{f}\|_{L^{r'}(\mathbb{S}^1)} \lesssim \|(1+|x|+|y|)^\gamma f\|_{L^{q'}(\mathbb{R}^2)}$$

can only hold for $q \geq r$. Set $f_\varepsilon(x, y) := (1+|x|^2+|y|^2)^{-\delta/2} e^{iy}$ for $\delta := 2 - \frac{1}{r'} + \varepsilon = \gamma + \frac{2}{q'} + \varepsilon$. Note that $1 < \delta < 2$. Then

$$\begin{aligned} \|(1+|x|+|y|)^\gamma f_\varepsilon\|_{L^{q'}(\mathbb{R}^2)} &\leq \|(1+|x|+|y|)^{\gamma-\delta}\|_{L^{q'}(\mathbb{R}^2)} \\ &\leq \left(\int_0^\infty r(1+r)^{q'(\gamma-\delta)}\,dr\right)^{\frac{1}{q'}} \\ &\leq \left(\frac{1}{-2-q'(\gamma-\delta)}\right)^{\frac{1}{q'}} \\ &\leq \varepsilon^{-\frac{1}{q'}} \end{aligned}$$

and the same bound holds in the case $q' = \infty$. Proposition 28 yields as $(\xi_1, \xi_2) \to (0, 1)$

$$\begin{aligned} |\hat{f}_\varepsilon(\xi)| &= \left|\int_{\mathbb{R}^2} (1+|x|^2+|y|^2)^{-\delta/2} e^{-i(\xi_1,\xi_2-1)\cdot(x,y)}\,dx\,dy\right| \\ &= \left|\int_{\mathbb{S}^1}\int_0^\infty r(1+r^2)^{-\delta/2} e^{-ir\omega\cdot(\xi_1,\xi_2-1)}\,dr\,d\sigma(\omega)\right| \\ &\gtrsim \int_{\mathbb{S}^1} |\omega\cdot(\xi_1,\xi_2-1)|^{\delta-2}\,d\sigma(\omega) \\ &\simeq |(\xi_1,\xi_2-1)|^{\delta-2} \\ &\simeq (|\xi_1|+|\xi_2-1|)^{-\frac{1}{r'}+\varepsilon} \end{aligned}$$

So the validity of the dual estimate stated above and $r' < \infty$ implies for some small $c > 0$

$$\begin{aligned} \varepsilon^{-\frac{1}{q'}} \gtrsim \|\hat{f}_\varepsilon\|_{L^{r'}(\mathbb{S}^1)} &= \left(\int_0^{2\pi} |\hat{f}_\varepsilon(\sin(t),\cos(t))|^{r'}\,dt\right)^{\frac{1}{r'}} \\ &\gtrsim \left(\int_0^c (|\sin(t)|+|\cos(t)-1|)^{r'(-\frac{1}{r'}+\varepsilon)}\,dt\right)^{\frac{1}{r'}} \gtrsim \left(\int_0^c t^{-1+r'\varepsilon}\,dt\right)^{\frac{1}{r'}} \\ &\simeq \varepsilon^{-\frac{1}{r'}}. \end{aligned}$$

From this we infer $r' \geq q'$, hence $q \geq r$. □

Appendix 1 – An oscillatory intgral

We first analyze the asymptotics of oscillatory integrals

$$\int_0^\infty \chi(r) r^{-\kappa} \cos(\lambda r)\, dr$$

as $\lambda \to 0^+$ under reasonable assumptions on χ and κ.

Proposition 28 *Let $\chi : [0,\infty) \to \mathbb{R}$ be measurable with $|\chi(z) - \chi_\infty| \leq C(1+z)^{-\sigma}$ where $\chi_\infty > 0$ and $0 < \kappa < 1 < \kappa + \sigma$. Then*

$$\lim_{\lambda \to 0^+} \lambda^{1-\kappa} \int_0^\infty \chi(r) r^{-\kappa} \cos(\lambda r)\, dr = \chi_\infty \int_0^\infty \rho^{-\kappa} \cos(\rho)\, d\rho \in (0,\infty)$$

locally uniformly with respect to κ. In particular,

$$\int_0^\infty \chi(r) r^{-\kappa} \cos(\lambda r)\, dr \gtrsim \lambda^{\kappa-1} \qquad \text{as } \lambda \to 0^+.$$

Proof We have for all $\lambda > 0$

$$\begin{aligned}
&\left| \lambda^{1-\kappa} \int_0^\infty \chi(r) r^{-\kappa} \cos(\lambda r)\, dr - \chi_\infty \int_0^\infty \rho^{-\kappa} \cos(\rho)\, d\rho \right| \\
&= \left| \int_0^\infty \rho^{-\kappa} \big(\chi(\rho\lambda^{-1}) - \chi_\infty\big) \cos(\rho)\, d\rho \right| \\
&\leq C \int_0^\infty \rho^{-\kappa} (1+\rho\lambda^{-1})^{-\sigma}\, d\rho \\
&= C\lambda^{1-\kappa} \int_0^\infty s^{-\kappa}(1+s)^{-\sigma}\, ds.
\end{aligned}$$

This term converges zu zero as $\lambda \to 0^+$. So it remains to show

$$\int_0^\infty \rho^{-\kappa} \cos(\rho)\, d\rho > 0.$$

Roughly speaking, it is sufficient to prove that the positivity regions of the integrand dominate the negativity regions. Formally, we prove for all $k \in \mathbb{N}_0$

$$\int_{[2k\pi,(2k+\frac{1}{2})\pi] \cup [(2k+\frac{3}{2})\pi,(2k+2)\pi]} s^{-\kappa} \cos(s)\, ds > \left| \int_{[(2k+\frac{1}{2})\pi,(2k+\frac{3}{2})\pi]} s^{-\kappa} \cos(s)\, ds \right|.$$

Write $\mu := 2\pi k$. Given the properties of cosine this is equivalent to

$$\int_0^{\frac{\pi}{2}} \Big((\mu+s)^{-\kappa} + (\mu+2\pi-s)^{-\kappa}\Big)\cos(s)\,ds$$
$$> \int_0^{\frac{\pi}{2}} \Big((\mu+\pi+s)^{-\kappa} + (\mu+\pi-s)^{-\kappa}\Big)\cos(s)\,ds$$

and the latter statement follows from

$$\cos(s)\Big((\mu+s)^{-\kappa} + (\mu+2\pi-s)^{-\kappa} - (\mu+\pi+s)^{-\kappa} - (\mu+\pi-s)^{-\kappa}\Big) > 0 \quad \text{for } 0 < s < \frac{\pi}{2}.$$

Indeed, the first factor is positive on $(0, \pi/2)$ and second factor decreases to zero on the interval $(0, \frac{\pi}{2}]$. □

7 Appendix 2 – Interpolation Arithmetic

The following result was used in the proof of Lemma 15.

Proposition 29 *Assume* $1 \le r < \infty$ *and* $0 < q < \infty$*. Then there are* $r_0 \in [1,\infty]$, $r_1 \in [1,\infty)$, $q_0 \in [1,\infty]$, $q_1 \in (0,\infty]$ *and* $\theta \in (0,1)$ *such that*

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1} \qquad \text{and}$$
$$\frac{1}{q_0} \le \frac{1}{3r_0'}, \quad \frac{1}{q_0} < \frac{1}{4}, \quad \frac{\alpha}{\theta} = \frac{1}{q_1}, \quad \frac{\beta}{\theta} \ge \frac{1}{q_1} - \frac{1}{2r_1'}, \quad q_0 \ne q_1$$

if and only if

$$\alpha + \beta > \frac{2}{q} - \frac{1}{2} \quad \text{and} \quad \alpha + 2\beta \ge \frac{3}{q} - \frac{1}{r'}.$$

Proof Setting $q_1 := \frac{\theta}{\alpha}$ the given conditions are equivalent to

$$\frac{1}{q} - \alpha = \frac{1-\theta}{q_0}, \ \frac{1}{r'} = \frac{1-\theta}{r_0'} + \frac{\theta}{r_1'}, \quad \frac{1}{q_0} \le \frac{1}{3r_0'}, \ \frac{1}{q_0} < \frac{1}{4}, \ \beta \ge \alpha - \frac{\theta}{2r_1'}, \ q_0 \ne \frac{\theta}{\alpha}.$$

We may choose $r_0 \in [1,\infty]$ according to the second equation if and only if

$$\frac{1}{q} - \alpha = \frac{1-\theta}{q_0}, \ 1-\theta \ge \frac{1}{r'} - \frac{\theta}{r_1'} \ge \frac{3(1-\theta)}{q_0}, \ \frac{1}{q_0} < \frac{1}{4}, \ \beta \ge \alpha - \frac{\theta}{2r_1'}, \ q_0 \ne \frac{\theta}{\alpha}.$$

Choosing $q_0 \in [1,\infty]$ according to these conditions is possible if and only if

$$\frac{1}{q} - \alpha < \frac{1-\theta}{4}, \quad 1-\theta \ge \frac{1}{r'} - \frac{\theta}{r_1'} \ge \frac{3}{q} - 3\alpha, \quad \beta \ge \alpha - \frac{\theta}{2r_1'}, \quad \theta \ne \alpha q.$$

This is equivalent to

$$\theta < 1 - \frac{4}{q} + 4\alpha, \quad \frac{1}{r'} - \frac{3}{q} + 3\alpha \geq \frac{\theta}{r_1'} \geq \max\left\{2\alpha - 2\beta, \theta - \frac{1}{r}\right\}, \quad \theta \neq \alpha q.$$

A choice of $r_1 \in [1, \infty)$ is possible if and only if

$$1 - \frac{4}{q} + 4\alpha > \theta > \max\left\{2\alpha - 2\beta, \theta - \frac{1}{r}\right\},$$
$$\frac{1}{r'} - \frac{3}{q} + 3\alpha \geq \max\left\{2\alpha - 2\beta, \theta - \frac{1}{r}\right\}, \quad \theta \neq \alpha q.$$

This is equivalent to

$$2\alpha - 2\beta < \theta < 1 - \frac{4}{q} + 4\alpha, \quad \theta \leq 1 - \frac{3}{q} + 3\alpha, \quad \alpha + 2\beta \geq \frac{3}{q} - \frac{1}{r'}, \quad \theta \neq \alpha q$$

and hence, in view of $\alpha < \frac{1}{q}$, equivalent to

$$\alpha + \beta > \frac{2}{q} - \frac{1}{2}, \quad \alpha + 2\beta \geq \frac{3}{q} - \frac{1}{r'}.$$

□

Proposition 30 *Assume* $1 \leq r \leq \infty$ *and* $0 < q < \infty$. *Then there are* $r_0, r_1 \in [1, \infty]$, $q_0 \in [1, \infty]$, $q_1 \in (0, \infty)$ *such that*

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad \frac{3}{q_0} \leq \frac{1}{r_0'}, \quad \frac{1}{q_0} < \frac{1}{4}, \quad q_0 \neq q_1$$
$$\gamma = \theta\gamma_1, \quad \gamma_1 \geq \max\left\{\frac{1}{q_1}, \frac{2}{q_1} - \frac{1}{r_1'}, \frac{2}{q_1} - \frac{1}{2}\right\}, \quad 0 < \theta < 1$$

if and only if

$$\gamma \geq \max\left\{\frac{3}{2q} - \frac{1}{2r'}, \frac{2}{q} - \frac{1}{r'}\right\}, \quad \gamma > \frac{2}{q} - \frac{1}{2}.$$

Proof The conditions may be recast as

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1}{r'} = \frac{1-\theta}{r_0'} + \frac{\theta}{r_1'}, \quad \frac{3}{q_0} \leq \frac{1}{r_0'}, \quad \frac{1}{q_0} < \frac{1}{4},$$
$$\gamma \geq \max\left\{\frac{\theta}{q_1}, \frac{2\theta}{q_1} - \frac{\theta}{r_1'}, \frac{2\theta}{q_1} - \frac{\theta}{2}\right\}, \quad 0 < \theta < 1.$$

Substituting r_1 we obtain the equivalent conditions

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad 0 \le \frac{1}{r'} - \frac{1-\theta}{r_0'} \le \theta, \quad \frac{3}{q_0} \le \frac{1}{r_0'}, \quad \frac{1}{q_0} < \frac{1}{4},$$
$$\gamma \ge \max\left\{\frac{\theta}{q_1}, \frac{2\theta}{q_1} - \frac{1}{r'} + \frac{1-\theta}{r_0'}, \frac{2\theta}{q_1} - \frac{\theta}{2}\right\}, \quad 0 < \theta < 1$$

and hence

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \max\left\{\frac{3(1-\theta)}{q_0}, \frac{1}{r'} - \theta\right\} \le \frac{1-\theta}{r_0'} \le \frac{1}{r'}, \quad \frac{1}{q_0} < \frac{1}{4},$$
$$\gamma \ge \max\left\{\frac{\theta}{q_1}, \frac{2\theta}{q_1} - \frac{1}{r'} + \frac{1-\theta}{r_0'}, \frac{2\theta}{q_1} - \frac{\theta}{2}\right\}, \quad 0 < \theta < 1.$$

Choosing r_0 as small as possible leads to the equivalent set of conditions

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \frac{1-\theta}{q_0} \le \frac{1}{3r'}, \quad \frac{1-\theta}{q_0} < \frac{1-\theta}{4}, \quad 0 < \theta < 1,$$
$$\gamma \ge \max\left\{\frac{\theta}{q_1}, \frac{2\theta}{q_1} - \theta, \frac{2\theta}{q_1} - \frac{1}{r'} + \frac{3(1-\theta)}{q_0}, \frac{2\theta}{q_1} - \frac{\theta}{2}\right\}.$$

We now choose $q_1 \in (0, \infty]$ according to the first equation and obtain

$$\frac{1-\theta}{q_0} \le \min\left\{\frac{1}{q}, \frac{1}{3r'}\right\}, \quad \frac{1-\theta}{q_0} < \frac{1-\theta}{4}, \quad 0 < \theta < 1,$$
$$\gamma \ge \max\left\{\frac{1}{q} - \frac{1-\theta}{q_0}, \frac{2}{q} - \frac{1}{r'} + \frac{1-\theta}{q_0}, \frac{2}{q} - \frac{2(1-\theta)}{q_0} - \frac{\theta}{2}\right\},$$

which is equivalent to

$$\max\left\{\frac{1}{q} - \gamma, \frac{1}{q} - \frac{\gamma}{2} - \frac{\theta}{4}\right\}$$
$$\le \frac{1-\theta}{q_0} \le \min\left\{\frac{1}{q}, \frac{1}{3r'}, \gamma - \frac{2}{q} + \frac{1}{r'}\right\}, \quad \frac{1-\theta}{q_0} < \frac{1-\theta}{4}, \quad 0 < \theta < 1.$$

A choice of q_0 according to these conditions is possible if and only if

$$\max\left\{\frac{1}{q} - \gamma, \frac{1}{q} - \frac{\gamma}{2} - \frac{\theta}{4}, 0\right\} \le \min\left\{\frac{1}{q}, \frac{1}{3r'}, \gamma - \frac{2}{q} + \frac{1}{r'}\right\},$$
$$\max\left\{\frac{1}{q} - \gamma, \frac{1}{q} - \frac{\gamma}{2} - \frac{\theta}{4}, 0\right\} < \frac{1-\theta}{4}, \quad 0 < \theta < 1,$$

which is equivalent to

$$\max\left\{\frac{4}{q}-2\gamma-\frac{4}{3r'},\frac{12}{q}-6\gamma-\frac{4}{r'}\right\}\leq\theta<1-\frac{4}{q}+4\gamma,\quad 0<\theta<1$$
$$\gamma\geq\max\left\{\frac{2}{q}-\frac{1}{r'},\frac{3}{2q}-\frac{1}{2r'}\right\},\quad \gamma>\frac{2}{q}-\frac{1}{2}.$$

This simplifies to

$$\gamma\geq\max\left\{\frac{2}{q}-\frac{1}{r'},\frac{3}{2q}-\frac{1}{2r'}\right\},\quad \gamma>\frac{2}{q}-\frac{1}{2},$$

which is all we had to show. □

Acknowledgements I wish to thank L. Grafakos and M. Mastyło for helpful clarifications regarding [6].

Funding Open Access funding enabled and organized by Projekt DEAL.

References

1. Bergh, J., Löfström, J.: Interpolation spaces. An introduction. Grundlehren der Mathematischen Wissenschaften, No. 223. Springer-Verlag, Berlin-New York (1976)
2. Bloom, S., Sampson, G.: Weighted spherical restriction theorems for the Fourier transform. Illinois J. Math. **36**(1), 73–101 (1992)
3. Chen, T., Sun, W.: Iterated weak and weak mixed-norm spaces with applications to geometric inequalities. J. Geom. Anal. **30**(4), 4268–4323 (2020)
4. Cwikel, M., Sagher, Y.: Analytic families of operators on some quasi-Banach spaces. Proc. Amer. Math. Soc. **102**(4), 979–984 (1988)
5. Fefferman, C.: Inequalities for strongly singular convolution operators. Acta Math. **124**, 9–36 (1970)
6. Grafakos, L., Mastyło, M.: Analytic families of multilinear operators. Nonlinear Anal. **107**, 47–62 (2014)
7. Mandel, R.: Real interpolation for mixed Lorentz spaces and Minkowski's inequality. Z. Anal. Anwend. **42**(3/4), 457–469 (2023)
8. Mandel, R., Oliveira, D., Silva, E.: The Stein-Tomas inequality under the effect of symmetries. Journal d'Analyse Mathématique **150**, 547–582 (2023)
9. Sagher, Y.: On analytic families of operators. Israel J. Math. **7**, 350–356 (1969)
10. Stein, E.M.: Interpolation of linear operators. Trans. Amer. Math. Soc. **83**, 482–492 (1956)
11. Yuan, W.: Complex interpolation and Calderón product of quasi-Banach spaces. Front. Math. China **11**(3), 753–762 (2016)
12. Zygmund, A.: On Fourier coefficients and transforms of functions of two variables. Studia Math. **50**, 189–201 (1974)