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ON MULTIPLICITIES OF INTERPOINT DISTANCE

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Abstract. Given a set $X \subseteq \mathbb{R}^2$ of n points and a distance $d > 0$, the multiplicity of d is the number of times the distance d appears between points in X. Let $a_1(X) \geq a_2(X) \geq \cdots \geq a_m(X)$ denote the multiplicities of the m distances determined by X and let $a(X) = (a_1(X), \ldots, a_m(X))$. In this paper, we study several questions from Erdős's time regarding distance multiplicities. Among other results, we show that:

(1) If X is convex or "not too convex", then there exists a distance other than the diameter that has multiplicity at most n.

(2) There exists a set $X \subseteq \mathbb{R}^2$ of n points, such that many distances occur with high multiplicity. In particular, at least $n^{\Omega(1/\log\log n)}$ distances have superlinear multiplicity in n.

(3) For any (not necessarily fixed) integer $1 \leq k \leq \log n$, there exists $X \subseteq \mathbb{R}^2$ of n points, such that the difference between the k^{th} and $(k+1)^{\text{th}}$ largest multiplicities is at least $\Omega(\frac{n \log n}{k})$. Moreover, the distances in X with the largest k multiplicities can be prescribed.

(4) For every $n \in \mathbb{N}$, there exists $X \subseteq \mathbb{R}^2$ of n points, not all collinear or cocircular, such that $a(X) = (n-1, n-2, \ldots, 1)$. There also exists $Y \subseteq \mathbb{R}^2$ of n points with pairwise distinct distance multiplicities and $a(Y) \neq (n-1, n-2, \ldots, 1)$.

1. Introduction

Let $\operatorname{dist}(x, y)$ denote the Euclidean distance between points x and y in the plane. Given a finite planar point set $X = \{x_1, \ldots, x_n\}$, let $d_1, \ldots, d_m$

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denote the distinct distances between points in X, where $m = m(X) \leq \binom{n}{2}$. The *multiplicity* of d_k in X is defined as

$$a_k(X) = \left|\{(i,j)\colon 1 \leq i < j \leq n,\ \mathrm{dist}(x_i, x_j) = d_k\}\right|.$$

We arrange the m multiplicities as $a_1(X) \geq a_2(X) \geq \cdots \geq a_m(X)$, irrespective to relative values of the d_k, and let $a(X) = (a_1(X), \ldots, a_m(X))$. In this paper, we revisit several questions from the time of Erdős regarding distance multiplicities:

(1) Is it possible that all distances except the diameter have multiplicity larger than n? See [6] and [9, Conjecture 4].

(2) Can it happen that there are many distances of multiplicity at least cn, where $c > 1$ is a constant, or even superlinear in n? See [10] and [8, Problem 11].

(3) Estimate $\max_{X \subseteq \mathbb{R}^2, |X|=n}(a_1(X) - a_2(X))$, and more generally,

$$\max_{X \subseteq \mathbb{R}^2, |X|=n}(a_k(X) - a_{k+1}(X))$$

as well as possible. See [5, Section 3].

(4) For sufficiently large $n \in \mathbb{N}$, is it true that $a(X) = (n-1, n-2, \ldots, 1)$ if and only if X consists of equidistant points on a line or on a circle? See [6, p. 135].

We answer Questions (2) and (4), and give partial answers to the other two.

1.1. Another distance with multiplicity at most n besides the diameter The *diameter* of X, denoted $\Delta = \Delta(X)$, is the maximum distance between points in X. Further, denote by $\Delta_2 = \Delta_2(X)$ and $\delta = \delta(X)$ the second largest and the smallest distances in X, respectively, and by $\mu(X, d)$ the multiplicity of the distance d in X.

Hopf and Pannwitz [16] proved that the multiplicity of the diameter among any n points in the plane is at most n. Erdős [6] further conjectured that for any n-element point set $X \subseteq \mathbb{R}^2$, there must be a second distance besides the diameter that has multiplicity at most n.

Conjecture 1.1 (Erdős [6], see also [9, Conjecture 4]). *Let $n \geq 5$. For any $X \subseteq \mathbb{R}^2$ with $|X| = n$, it is not possible that every distance except the diameter occurs more than n times.*

The condition $n \geq 5$ is necessary, since for $n = 4$ we can glue two equilateral triangles of the same side length together as a rhombus and this gives a counterexample. Erdős and Fishburn [9] proved the Conjecture for $n = 5, 6$ and the case of $n \geq 7$ is still open. Here we confirm Conjecture 1.1 in two special cases. A point set $X \subseteq \mathbb{R}^2$ is said to be *convex*, or in *convex position* if no point lies inside the convex hull of other points.

THEOREM 1.2. *Let $n \ge 5$. For any convex point set $X \subseteq \mathbb{R}^2$ with $|X| = n$, it cannot happen that all distances except the diameter occur more than n times.*

Given a point set $X \subseteq \mathbb{R}^2$, let $L_1 = L_1(X)$ be the set of vertices of the convex hull of X, called the *first (outer) convex layer* of X. Similarly, the *second convex layer* $L_2 = L_2(X)$ of X is the set of vertices of the convex hull of $X \setminus L_1$. Note that X is convex if and only if L_2 is empty. It follows from definition that X is convex if and only if $|L_1| = |X|$. Next we confirm Conjecture 1.1 for "not too convex" point sets, namely for point sets whose first and second convex layers are not too large.

THEOREM 1.3. *Let $X \subseteq \mathbb{R}^2$ be a set of $n \ge 2$ points. If*

$$\min\Big\{\frac{3}{2}(|L_1| + |L_2|),\, \frac{4}{3}|L_1| + 2|L_2|,\, 2|L_1| + |L_2|\Big\} \le n,$$

then the second largest distance in X can occur at most n times.

Theorem 1.3 directly implies that, if the ratio $\Delta(X)/\delta(X)$ of a set $X \subseteq \mathbb{R}^2$ is small enough, then the second largest distance in X occurs at most n times.

COROLLARY 1.4. *If $X \subseteq \mathbb{R}^2$ is a set of $n \in \mathbb{N}$ points with $\Delta(X) \le \frac{n}{3\pi}\delta(X)$, then $\mu(X, \Delta_2) \le n$.*

PROOF. Assume without loss of generality that $\delta(X) = 1$, namely, $\Delta(X) \le \frac{n}{3\pi}$. Then, $|L_1|$ and $|L_2|$ are upper bounded by the perimeter of the convex polygons formed by L_1 and L_2, respectively. Since the perimeter of a convex polygon is at most π times its diameter, see [25, p. 76], we have $|L_1| + |L_2| \le 2n/3$ and thus $\mu(X, \Delta_2) \le n$ by Theorem 1.3. $\square$

Note that the multiplicity of the second largest distance can be larger than n in some planar point sets, see [23,24]. Thus, to fully resolve Conjecture 1.1, one could perhaps consider multiplicities of different distances simultaneously and show that one of them must be at most n. For instance, one could consider the smallest and the second largest distance; however, as we demonstrate below, their multiplicities can both be large.

PROPOSITION 1.5. *Let $m, n \in \mathbb{N}$ with $m \le \lfloor n/2 \rfloor$. There exists a planar point set X with $|X| = n$, such that $\mu(X, \Delta_2) \ge 3m$ and $\mu(X, \delta) \ge 3n - 5m + o(m)$.*

By letting $m = \lfloor 3n/8 \rfloor$, we obtain that $\min\{\mu(X, \Delta_2), \mu(X, \delta)\} \ge 9n/8 + o(n)$. This also motivates the following problem.

PROBLEM 1.6. *Determine*

$$\limsup_{n\to\infty} \sup_{X\subseteq\mathbb{R}^2,\, |X|=n} \frac{\min\{\mu(X, \Delta_2), \mu(X, \delta)\}}{n}.$$

1.2. Point sets with many large distance multiplicities. Erdős and Pach [10], see also [8, Problem 11], asked the following question: Given a set $X \subseteq \mathbb{R}^2$ of n points, can it happen that there are $c_1 n$ distances with multiplicities at least $c_2 n$, for some constant $c_1, c_2 > 0$? Bhowmick [2] recently answered their question in the positive: There exist arbitrary large planar point sets X, $|X| = n$, such that there are $\lfloor n/4 \rfloor$ distances which occur at least $n+1$ times. Bhowmick [2] also considered higher multiplicities, distances that occur at least $n+m$ times, $m \geq 1$. He showed that there are sets with at least $\left\lfloor \frac{n}{2(m+1)} \right\rfloor$ distances that occur at least $n+m$ times. Observe that for m linear in n, this lower bound is only $\Omega(1)$. Here we give a substantial improvement by showing that there exist $X \subseteq \mathbb{R}^2$ of n points, such that at least $n^{c/\log\log n}$ distances have superlinear multiplicity in n. For comparison purposes, note that $n^{c/\log\log n} = \Omega((\log n)^\alpha)$, for any fixed $\alpha > 0$.

THEOREM 1.7. *There exists some constant $c > 0$ such that for sufficiently large $n \in \mathbb{N}$, at least $n^{c/\log\log n}$ distances occur at least $n^{1+c/\log\log n}$ times in the $\sqrt{n} \times \sqrt{n}$ grid.*

As mentioned earlier, Bhowmick [2] answered the question of Erdős and Pach [10] with constants $c_1 = 1/4$ and $c_2 = 1$. One may ask whether the constant $c_1 = 1/4$ resulting from his construction is the best possible. We extend the above investigation for the range $c_2 > 1$. More precisely, we show that there exist n-element planar point sets with m distances so that $c_1 m$ distances occur at least $c_2 n$ times, for suitable constants $c_1 > 0$, $c_2 > 1$. Proposition 1.8 below gives three sample combinations; these combinations are not exhaustive.

PROPOSITION 1.8. *For every $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that if $n \geq n_0(\varepsilon)$, then out of the $m = \Theta(n/\sqrt{\log n})$ distances presented in the $\sqrt{n} \times \sqrt{n}$ grid:*

(i) *at least $(1-\varepsilon)\, m/9$ distances occur at least $16n/9$ times;*

(ii) *at least $(1-\varepsilon)\, m/16$ distances occur at least $9n/4$ times;*

(iii) *at least $(1-\varepsilon)\, m/25$ distances occur at least $64n/25$ times.*

1.3. On the differences $a_k(X) - a_{k+1}(X)$. Let $f(n)$ denote the maximum value of $a_1(X)$ over all $X \subseteq \mathbb{R}^2$ with $|X| = n$. Erdős [5, Section 3] asked whether $f(n) - a_2(X)$ tends to infinity as $n \to \infty$. However, this question is somewhat ambiguous since it leaves the ground set X unspecified. Here we reformulate the question and ask for

$$\max_{X \subseteq \mathbb{R}^2,\, |X| = n} (a_1(X) - a_2(X)).$$

We show that the maximum difference $a_1(X) - a_2(X)$ is at least $\Omega(n \log n)$. This is implied by the following more general result.

THEOREM 1.9. *Let $n \in \mathbb{N}$ be sufficiently large and $1 \le k \le \log n$. There exists a point set $X \subseteq \mathbb{R}^2$ with $|X| = n$, such that $a_k(X) - a_{k+1}(X) = \Omega\big(\frac{n}{k}\log n\big)$. Moreover, the distances with the largest k multiplicities can be prescribed.*

In particular, $a_k(X) - a_{k+1}(X)$ can be superlinear in n for $k \to \infty$.

COROLLARY 1.10. $\max_{X\subseteq\mathbb{R}^2,\,|X|=n}(a_1(X) - a_2(X)) = \Omega(n \log n)$.

Theorem 1.7 suggests that the following stronger lower bound might be true:

PROBLEM 1.11. *Does there exist a constant $c > 0$ such that for sufficiently large $n \in \mathbb{N}$, we have*

$$\max_{X\subseteq\mathbb{R}^2,\,|X|=n}(a_1(X) - a_2(X)) \ge n^{1+c/\log\log n}\,?$$

1.4. Related work. Recall that $f(n)$ denotes the largest possible value of $a_1(X)$ among all subsets $X \subseteq \mathbb{R}^2$ of n points. Determining $f(n)$, also known as the unit distance problem, is notoriously difficult. The current best upper bound is $O(n^{4/3})$ established by Spencer, Szemerédi, and Trotter [19]. A simple and elegant argument based on crossing numbers is due to Székely [20]. From the other direction, it is conjectured by Erdős that a $\sqrt{n} \times \sqrt{n}$ section of the integer lattice gives the correct order magnitude, $n^{1+c/\log\log n}$, and so the current best upper bound seems far off. See also a recent survey by Szemerédi [21] for more on this topic.

Let $A(n)$ be the maximum value of $\sum a_k(X)^2$ over all $X \subseteq \mathbb{R}^2$ with $|X| = n$. Erdős [7] asked whether $A(n) = O(n^3(\log n)^\alpha)$ holds for some positive constant $\alpha > 0$. This question received a complete answer via the work of Guth and Katz [15] on the problem of distinct distances. Specifically, the authors proved that the inequality holds with $\alpha = 1$, i.e., $A(n) = O(n^3 \log n)$, and is tight in the $\sqrt{n} \times \sqrt{n}$ integer grid. Lefmann and Thiele [17] proved that the sharper inequality $A(n) = O(n^3)$ holds for convex point sets; this bound is tight, e.g., for a regular n-gon.

The rest of the paper is organized as follows. We will prove Theorems 1.2, 1.3 and Proposition 1.5 in Section 2. In Section 3 we will prove Theorem 1.7 and Proposition 1.8. Section 4 is devoted to proving Theorem 1.9. Finally, in Section 5 we give a simple answer to Question (4).

2. A second multiplicity at most n in planar point sets

Given a finite point set $X \subseteq \mathbb{R}^2$, recall that $\delta(X) < \Delta_2(X) < \Delta(X)$ denote the smallest, the second largest, and the largest distances in X, respectively (assuming they exist and are different).

2.1. The convex case.

Proof of Theorem 1.2. Let $X \subseteq \mathbb{R}^2$ be an arbitrary convex set of n points. Let R_n denote a regular n-gon and R_n^- denote a regular n-gon minus one vertex. A classical result of Altman [1] states that X determines at least $\lfloor n/2 \rfloor$ distinct distances; and this bound is attained by R_n. Moreover, Altman proved that if n is odd and X determines exactly $\lfloor n/2 \rfloor$ distances, then $X = R_n$; in particular, $a(X) = (n, n, \dots, n)$. See also [9,13].

The complementary result for even n is due to Fishburn [12]. Suppose that $n \geq 6$ is even and X determines exactly $\lfloor n/2 \rfloor$ distances. Then

(i) for $n = 6$, there exist exactly two possibilities, $a(X) = (6, 6, 3)$, or $a(X) = (5, 5, 5)$;

(ii) for $n \geq 8$, either $X = R_n$ or $X = R_{n+1}^-$.

In particular, for the second case we have $a(X) = (n, n, \dots, n, n/2)$ or $a(X) = (n-1, n-1, \dots, n-1)$.

We can now finalize the proof. If X determines strictly more than $\lfloor n/2 \rfloor$ distinct distances and all distances smaller than Δ occur more than n times, then the number of point pairs is at least

$$\left\lfloor \frac{n}{2} \right\rfloor (n+1) + 1 > \binom{n}{2},$$

a contradiction. Otherwise, X determines exactly $\lfloor n/2 \rfloor$ distinct distances, and it is easy to check that all possible cases listed previously satisfy the requirements. $\square$

2.2. The "not too convex" case.

Proof of Theorem 1.3. Let $L_1 = L_1(X)$ and $L_2 = L_2(X)$ be the first and second convex layers of X. It suffices to show that

$$\mu(X, \Delta_2) \leq \min\left\{ \frac{3}{2}(|L_1| + |L_2|),\ \frac{4}{3}|L_1| + 2|L_2|,\ 2|L_1| + |L_2| \right\}.$$

We have the following observations:

(1) [24, Proposition 1] Let $p, q \in X$. If $\mathrm{dist}(p, q) = \Delta$ then $\{p, q\} \subseteq L_1$. If $\mathrm{dist}(p, q) = \Delta_2$ then $\{p, q\} \cap L_1 \neq \emptyset$.

(2) If $\mathrm{dist}(p, q) = \Delta_2$ and $p \in L_1$, then $q \in L_1 \cup L_2$. Indeed, the points of X in the exterior of the circle of radius Δ_2 centered at p, if any, have distance Δ from p and by observation (1) are in L_1. Hence, q is at least in the second convex layer of X.

Combining observations (1) and (2) we have that

$$\mu(X, \Delta_2) = \mu(L_1 \cup L_2, \Delta_2),$$

moreover, Δ_2 is still the second largest distance in $L_1 \cup L_2$. Vesztergombi [23] showed that the multiplicity of the second largest distance among any n points in the plane is at most $3n/2$. Namely, we have

$$\mu(X, \Delta_2) \leq \frac{3}{2}(|L_1| + |L_2|).$$

We proceed to prove $\mu(X, \Delta_2) \leq \min\{\frac{4}{3}|L_1| + 2|L_2|, 2|L_1| + |L_2|\}$. Let G be a graph on $L_1 \dot{\cup} L_2$, where pq is an edge if and only if $\operatorname{dist}(p, q) = \Delta_2$. Then it holds that $\mu(X, \Delta_2) = e(G)$. Iteratively remove vertices of degree less than 2 from G, and let G' be the remaining graph whose vertex set is $L_1' \dot{\cup} L_2'$ with $L_1' \subseteq L_1$ and $L_2' \subseteq L_2$. Then we have

$$e(G) \leq |L_1 \setminus L_1'| + |L_2 \setminus L_2'| + e(G').$$

To bound $e(G')$, we record several observations about the graph G' by Vesztergombi [24]:

(3) L_2' is an independent set (follows from (1) and (2)).

(4) [24, Proposition 3] Every $q \in L_2'$ has degree exactly 2.

(5) [24, Proposition 5] Every $p \in L_1'$ has at most 2 neighbors in L_2'.

(6) [24, Proposition 6] If $p \in L_1'$ has 3 neighbors in L_1', then it has at most 1 neighbor in L_2'.

(7) [24, Proposition 7] If $p \in L_1'$ has 4 neighbors in L_1', then it has no neighbor in L_2'.

(8) [24, Proposition 8] Every $p \in L_1'$ has at most 4 neighbors in L_1'.

Let $e(L_1')$ denote the number of edges in L_1', and let $e(L_1', L_2')$ denote the number of edges between L_1' and L_2'. Due to observations (3)–(5),

$$e(G') = e(L_1') + e(L_1', L_2') = e(L_1') + 2|L_2'|.$$

Vesztergombi [22] showed that the multiplicity of the second largest distance among any n points in convex position in the plane is at most $4n/3$. Since L_1' is convex and Δ_2 is either the largest or second largest distance in L_1',

$$e(L_1') = \mu(L_1', \Delta_2) \leq \frac{4}{3}|L_1'|.$$

On the other hand, let $\deg(p)$, $\deg_1(p)$, and $\deg_2(p)$ denote the number of neighbors of p in G', L_1', and L_2', respectively. For each $p \in L_1'$, by observations (4)–(8) we have $\deg(p) = \deg_1(p) + \deg_2(p) \leq 4$. Therefore,

$$e(L_1') = \frac{1}{2}\sum_{p \in L_1'} \deg_1(p) \leq \frac{1}{2}\sum_{p \in L_1'} (4 - \deg_2(p))$$

$$= 2|L_1'| - \frac{1}{2}e(L_1', L_2') = 2|L_1'| - 2|L_2'|.$$

We conclude

$$
\begin{aligned}
e(G) &\le |L_1 \setminus L_1'| + |L_2 \setminus L_2'| + e(G') \\
&= |L_1 \setminus L_1'| + |L_2 \setminus L_2'| + e(L_1') + e(L_1', L_2') \\
&\le |L_1 \setminus L_1'| + |L_2 \setminus L_2'| + \min\Big\{\frac{4}{3}|L_1'|,\ 2|L_1'| - |L_2'|\Big\} + 2|L_2'| \\
&= |L_1 \setminus L_1'| + |L_2 \setminus L_2'| + \min\Big\{\frac{4}{3}|L_1'| + 2|L_2'|,\ 2|L_1'| + |L_2'|\Big\} \\
&\le \min\Big\{\frac{4}{3}|L_1| + 2|L_2|,\ 2|L_1| + |L_2|\Big\}. \quad \square
\end{aligned}
$$

2.3. The case where the multiplicities of $\boldsymbol{\Delta_2}$ and $\boldsymbol{\delta}$ are both large.

Proof of Proposition 1.5. Our construction is inspired by that of Vesztergombi [23,24], see also [3, Ch. 5.8]. Let $m_1 = m_2 = m$ and $m_3 = n - 2m$. The construction comprises three groups, each containing m_1, m_2 and m_3 points, respectively. Note that $n = m_1 + m_2 + m_3$.

Group I: Place the first m_1 points $v_1, \dots, v_{m_1}$ as the vertices of a regular m_1-gon inscribed in a circle C of radius n. Let Δ and Δ_2 be the largest and second largest distances in this m_1-gon. Note that $\Delta = O(n)$ and the number of occurrences of Δ_2 in Group I is exactly m_1.

Group II: The next m_2 points $u_1, \dots, u_{m_2}$ are positioned inside the circle C such that $\text{dist}(v_i, u_i) = \text{dist}(u_i, v_{i+1}) = \Delta_2$, where the indices are modulo m_2. Notably, the u_i's lie on a circle. Let $\delta = \Theta(1)$ be the smallest distance in the construction so far, representing the distance between consecutive u_i's. This distance appears m_2 times in Group II.

Group III: The final m_3 points form an equilateral triangular lattice with mesh width δ contained in a disk of radius $\Theta(\sqrt{n})$. The lattice is centered at the origin, coinciding with the center of the circle C. Note that the distance δ appears $3m_3 + o(n)$ times inside Group III.

Let X denote the final construction, see Figure 1 for an illustration. Since $L_1(X)$ and $L_2(X)$ correspond to the points in Group I and Group II, respectively, we have $|L_1| = |L_2| = m$. Overall, the distance δ appears $3m_3 + m_2 + o(n) = 3n - 5m + o(n)$ times, and the distance Δ_2 appears $m_1 + 2m_2 = 3m$ times. We complete the proof by noting that $\delta(X) = \delta$ and $\Delta_2(X) = \Delta_2$. $\square$

3. Many large distance multiplicities among planar points

3.1. Proof of Theorem 1.7. For $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, 2 \dots, n\}$. We first prove the following lemma by adapting a well-known

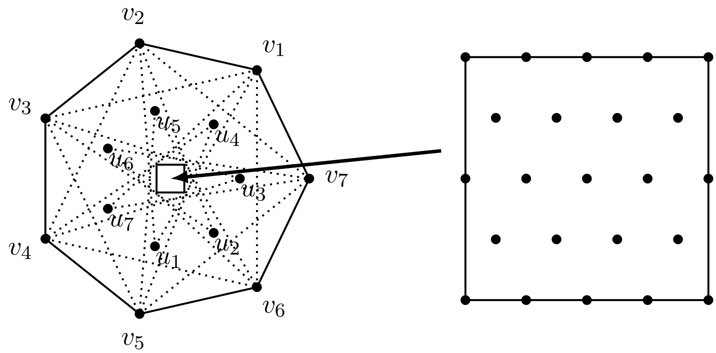


Figure 1: The construction in Proposition 1.5, when $m = 7$. Dotted edges represent the second largest distance Δ_2.

argument for counting representations of a natural number as the sum of two squares; see, e.g., [18, Ch. 3] and [14, Ch. 2].

LEMMA 3.1. *Let $r(n)$ denote the number of distinct ways in which $n \in \mathbb{N}$ can be represented as the sum of two squares. Then there exists a constant $c > 0$ such that for infinitely many $n \in \mathbb{N}$, at least $n^{c/\log\log n}$ distinct elements $n' \in [n]$ have*

$$r(n') \geq n^{c/\log\log n}.$$

PROOF. Let $n = p_1 p_2 \cdots p_k$, where p_j is the j^{th} smallest prime of the form $4m + 1$. Since p_k satisfies

$$c_1 k \log k \leq p_k \leq c_2 k \log k,$$

for suitable constants $c_1, c_2 > 0$, this implies $k \geq 2c \log n / \log\log n$ for a suitable constant $c > 0$. It is well-known that any such prime can be represented (uniquely) as the sum of two squares, i.e.,

$$p_j = a_j^2 + b_j^2 = (a_j + b_j \mathrm{i})(a_j - b_j \mathrm{i}),$$

where $\mathrm{i} = \sqrt{-1}$. There are 2^k subsets of $K = \{1, 2, \ldots, k\}$, and out of these, exactly 2^{k-1} subsets have cardinality at least $k/2$.

Fix any subset $K' \subseteq K$ of cardinality $|K'| \geq k/2$. Let $n' = \prod_{j \in K'} p_j$. For each subset $J \subseteq K'$,

$$\prod_{j \in J} (a_j + b_j \mathrm{i}) \prod_{j \in K' \setminus J} (a_j - b_j \mathrm{i}) = A_J + B_J \mathrm{i},$$

$$\prod_{j \in J} (a_j - b_j \mathrm{i}) \prod_{j \in K' \setminus J} (a_j + b_j \mathrm{i}) = A_J - B_J \mathrm{i},$$

where A_J and B_J satisfy

$$A_J^2 + B_J^2 = (A_J + B_J\mathrm{i})(A_J - B_J\mathrm{i}) = \prod_{j \in K'} p_j = n' \leq n.$$

By the unique factorization theorem for complex integers, $A_J + B_J\mathrm{i}$ is different for different choices of J, so we obtain

$$r(n') \geq 2^{k/2} \geq n^{c/\log\log n}.$$

Since there are $2^{k-1} \geq n^{c/\log\log n}$ distinct values $n' \in [n]$ that have been considered, the lemma is implied. □

PROOF OF THEOREM 1.7. The proof follows a (now standard) argument of Erdős [4] using the estimate in Lemma 3.1. Let $n_0 \leq n$ be the largest integer such that $n_0 = p_1 p_2 \dots p_k$, where p_j is the j^{th} smallest prime of the form $4m+1$. Since $k = \Theta(\log n/\log\log n)$, we have

$$p_{k+1} = \Theta((k+1)\log(k+1)) = \Theta(\log n),$$

namely, $n_0 = \Omega(n/\log n)$. By Lemma 3.1 there exist $n_0^{\Omega(1/\log\log n_0)} = n^{\Omega(1/\log\log n)}$ different values of $n' \in [n_0]$ that can be represented as the sum of two squares in $n^{\Omega(1/\log\log n)}$ ways. For every such value of n' there are $\Omega(n)$ points in the $\sqrt{n} \times \sqrt{n}$ grid, each of which has $n^{\Omega(1/\log\log n)}$ neighbors at distance $\sqrt{n'}$. This completes the proof. □

3.2. Proof of Proposition 1.8. We prove the second estimate; the proofs of the other two estimates are analogous. Let X be a $\sqrt{n} \times \sqrt{n}$ section of the integer grid, where $n = 16k^2$. Then X determines

$$m = (1 \pm o(1))\frac{cn}{\sqrt{\log n}}$$

distinct distances, for some $c > 0$; see [4] or [18, Ch. 12]. X consists of 16 smaller $k \times k$ sections Y, each determining

$$(1 \pm o(1))\frac{cn}{16\sqrt{\log{(n/16)}}} = (1 \pm o(1))\frac{m}{16}$$

distinct distances. See Figure 2.

Take any distance determined by a non-vertical and non-horizontal segment $s = ab$ in Y, where a and b are its left and right endpoint, respectively. Observe that s occurs at least

$$2k \cdot 3k \cdot 4 + 2k \cdot 3k \cdot 2 = 36k^2 = \frac{9}{4}n$$

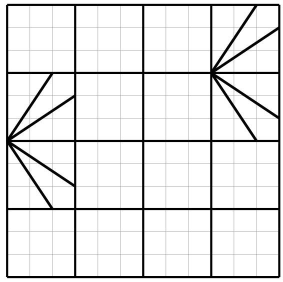


Figure 2: Multiplicities of distances in the grid.

times in X. Indeed, the left degree of every point in the 6 central left smaller sections is at least 4 whereas the left degree of every point in the remaining 6 left smaller sections is at least 2. Note that the number of distances determined by a vertical or horizontal segment in Y is at most $k = o(m)$. This justifies the second estimate.

The first and the third estimate are obtained analogously by subdividing X into 9 and 25 smaller sections, respectively. $\square$

4. On the differences $a_k(X) - a_{k+1}(X)$

Using an inductive construction, Erdős and Purdy [11] showed that the maximum number of times the unit distance occurs among n points in the plane, no three of which are collinear, is at least $\Omega(n \log n)$. Our proof of Theorem 1.9 can be viewed as a refinement of their argument.

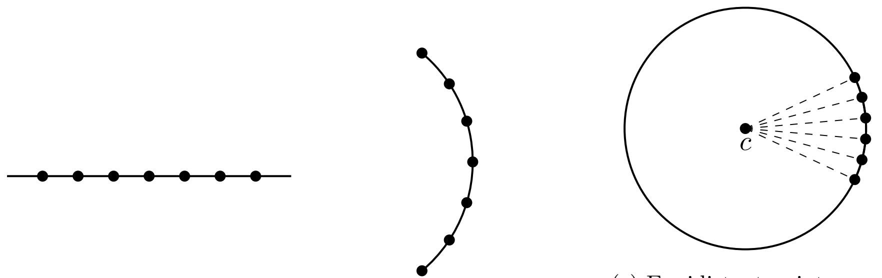
Proof of Theorem 1.9. Let $d_1, \ldots, d_k > 0$ be arbitrary pairwise distinct distances. Let $X_0(m)$ denote a configuration of m points and let $X_i(2m)$ denote the union of $X_{i-1}(m)$ and a translate of $X_{i-1}(m)$ by distance d_i in some generic direction, so that none of the segments connecting the two copies duplicates a distance other than d_i. This is feasible since it amounts to excluding a set of directions of measure zero. We start from a single point and apply translates by $d_1, \ldots, d_k$ in a cyclic fashion. The resulting set after k steps has $2^k \leq n$ points.

For any $1 \leq i \leq k$, the multiplicity $T_i(n)$ of d_i in a set of n points constructed in the above way satisfies the recurrence

$$T_i(n) = 2^k T_i(n/2^k) + n/2, \quad T_i(1) = 0.$$

Its solution satisfies

$$T_i(n) \geq \frac{n}{2} \log_{2^k} n = \frac{n}{2k} \log n.$$



(a) Equidistant points on a line.

(b) Equidistant points on a circular segment.

(c) Equidistant points on a circle together with its center.

Figure 3: Three configurations X satisfying $a(X) = (n-1, n-2, \ldots, 1)$ for $n = 7$.

In the inductive step corresponding to d_i, we ensure that none of the segments connecting the two copies duplicate a distance other than d_i. Consequently, any distance other than $d_1, \ldots, d_k$ occurs at most n times. In conclusion, we have

$$a_1(X), \ldots, a_k(X) = \Omega\Big(\frac{n}{k} \log n\Big), \quad \text{and} \quad a_{k+1}(X), a_{k+2}(X), \ldots \leq n,$$

as required. □

5. Point sets with distinct distance multiplicities

Given a set $X \subseteq \mathbb{R}^2$ of $n \geq 2$ points, which contains $m = m(X) \leq \binom{n}{2}$ distinct distances, recall that $a(X) = (a_1(X), \ldots, a_m(X))$ consists of the multiplicities of all distances ordered by $a_1(X) \geq a_2(X) \geq \ldots \geq a_m(X)$. How many distinct values can $a(X)$ contain? At most $n-1$, this follows easily from $\sum_{k=1}^{m} a_k(X) = \binom{n}{2}$. Moreover, when $a(X)$ contains $n-1$ distinct values, then $a(X) = (n-1, n-2, \ldots, 1)$. One can observe that if X consists of equidistant points on a line or on a circle, see Figure 3 (a) and (b) for an illustration, then $a(X) = (n-1, n-2, \ldots, 1)$. Are there other constructions of X that achieve $a(X) = (n-1, n-2, \ldots, 1)$? Erdős [6, p. 135] conjectured the answer to be negative, when n is large. Here we give a simple counterexample to this conjecture.

Observation 5.1. Let γ be a circular arc subtending a center angle $< \pi/3$ on the circle C of unit radius centered at c. Let X consist of c together with a set of $n-1$ equidistant points on γ. Then $a(X) = (n-1, n-2, \ldots, 1)$. See Figure 3 (c) for an illustration.

Proof. The multiplicities of the $n-1$ points on γ are $1, 2, \ldots, n-2$. Since the multiplicity of the unit distance is $n-1$, we have $a(X) = (n-1, n-2, \ldots, 1)$. Finally, X is not contained in any line or circle. □

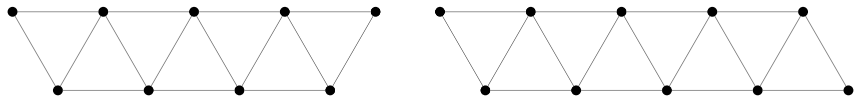


Figure 4: Point sets ($n = 9$ and $n = 10$) with pairwise distinct distance multiplicities and $a(X) \neq (n-1, n-2, \dots, 1)$.

One may further ask:

Problem 5.2. *For sufficiently large $n \in \mathbb{N}$, are the examples in Figure 3 the only point sets with $a(X) = (n-1, n-2, \dots, 1)$? Are these the only ones with pairwise distinct distance multiplicities?*

We answer the latter question in the negative. We also show that an integer grid is not a valid candidate.

Proposition 5.3. *For every $n \in \mathbb{N}$, there is a set $X \subseteq \mathbb{R}^2$ of n points with pairwise distinct distance multiplicities and $a(X) \neq (n-1, n-2, \dots, 1)$.*

Proof. We present the proof for odd n; the case of even n is analogous and left to the reader. Let X be a piece of the hexagonal lattice of side length 1 with $n = 2k+1$ points placed on two adjacent horizontal lines ℓ_1, ℓ_2, so that $|X \cap \ell_1| = k+1$ and $|X \cap \ell_2| = k$. See Figure 4 (left) for an illustration.

There are two types of distances in X, integer and irrational. The integer distances are $\{1, \dots, k\}$, determined by points on the same horizontal line, or by points with consecutive x-coordinates on different horizontal lines. The irrational distances occur between nonconsecutive points on different horizontal lines. Let these be $d_1 < d_2 < \dots < d_{k-1}$, where $d_j = \sqrt{j^2 + j + 1}$ for $j = 1, \dots, k-1$. It is not difficult to verify that

(i) $\mu(X, 1) = 4k - 1$.
(ii) $\mu(X, j) = 2(k-j) + 1$, for $j = 2, \dots, k$.
(iii) $\mu(X, d_j) = 2(k-j)$, for $j = 1, \dots, k-1$.

The multiplicities are clearly distinct and this completes the proof. □

Observation 5.4. Let $k \geq 4$. In the $k \times k$ grid there are two distances which appear exactly 8 times each.

Proof. The distance $d_1 = \sqrt{(k-1)^2 + (k-2)^2}$ appears among the pairs:

$$\{(0,0), (k-1, k-2)\},\ \{(0,0), (k-2, k-1)\},\ \{(1,0), (k-1, k-1)\},$$
$$\{(0,1), (k-1, k-1)\},\ \{(k-1, 0), (0, k-2)\},\ \{(k-1, 0), (1, k-1)\},$$
$$\{(k-2, 0), (0, k-1)\},\ \{(k-1, 1), (0, k-1)\}.$$

The distance d_1 can only appear between point pairs (x_1, y_1) and (x_2, y_2) where $(|x_1 - x_2|, |y_1 - y_2|) \in \{(k-1, k-2), (k-2, k-1)\}$, and thus cannot appear more than eight times, as shown above. The distance $d_2 = \sqrt{2(k-2)^2}$ appears among the pairs:

$$\{(0,0),(k-2,k-2)\},\ \{(1,0),(k-1,k-2)\},\ \{(0,1),(k-2,k-1)\},$$
$$\{(1,1),(k-2,k-2)\},\ \{(k-1,0),(1,k-2)\},\ \{(k-1,1),(1,k-1)\},$$
$$\{(k-2,0),(0,k-2)\},\ \{(k-2,1),(0,k-1)\}.$$

Note that this is an exhaustive list of all point pairs $(|x_1-x_2|,|y_1-y_2|) = (k-2,k-2)$. It is not possible that $|x_1-x_2| = k-1$ (respectively $|y_1-y_2| = k-1$), since the equation

$$(k-1)^2 + (x)^2 = 2(k-2)^2$$

does not have a solution. Indeed, if

$$x^2 = 2(k-2)^2 - (k-1)^2 = k^2 - 6k + 7,$$

then

$$k = 3 \pm \sqrt{3^2 - (7-x^2)} = 3 \pm \sqrt{2+x^2},$$

but x^2+2 is not square for $x \in \mathbb{N}$. $\square$

References

[1] E. Altman, On a problem of P. Erdős, *Amer. Math. Monthly*, **70** (1963), 148–157.

[2] K. Bhowmick, A note on a problem of Erdős about rich distances, *Studia Sci. Math. Hungar.*, **62** (2025), 89–94.

[3] P. Braß, W. Moser, and J. Pach, *Research Problems in Discrete Geometry*, Springer (New york, 2005).

[4] P. Erdős, On sets of distances of n points, *Amer. Math. Monthly*, **53** (1946), 248–250.

[5] P. Erdős, Extremal problems in number theory, combinatorics and geometry, in: *Proc. Int. Congr. Math.* (Warsaw, 1983), PWN-Polish Scientific Publishers, North-Holland (Amsterdam–New York–Oxford, 984), pp. 51–70.

[6] P. Erdős, Some old and new problems in combinatorial geometry, *Ann. Discrete Math.*, **20** (1984), 129–136.

[7] P. Erdős, Some of my favourite unsolved problems, in: *A Tribute to Paul Erdős*, A. Baker, B. Bollobás, A. Hajnal, eds., Cambridge University Press (1990).

[8] P. Erdős, Some old and new problems in various branches of combinatorics, *Discrete Math.*, **165/166** (1997), 227–231.

[9] P. Erdős and P. Fishburn, Multiplicities of interpoint distances in finite planar sets, *Discrete Appl. Math.*, **60** (1995), 141–147.

[10] P. Erdős and J. Pach, Variations on the theme of repeated distances, *Combinatorica*, **10** (1990), 261–269.

[11] P. Erdős and G. Purdy, Some extremal problems in geometry. IV, in: *Proc. 7th southeast. Conf. Comb., Graph Theory, Comput.*, Congressus Numerantium, Utilitas Mathematica Publishing Inc. (Winnipeg, Manitoba, 1976), pp. 307-322.

[12] P. Fishburn, Convex polygons with few intervertex distances, *Comput. Geom.*, **5** (1995), 65–93.

[13] P. Fishburn, Distances in convex polygons, in: *The Mathematics of Paul Erdős*, R. L. Graham, J. Nešetřil, S. Butler, eds., Algorithms Comb. 14, Springer Verlag (New York, 2013), pp. 284–293.
[14] E. Großwald, *Representations of Integers as Sums of Squares*, Springer Verlag (New York, 1985).
[15] L. Guth and N. H. Katz, On the Erdős distinct distances problem in the plane, *Ann. of Math.*, **181** (2015), 155–190.
[16] H. Hopf and E. Pannwitz, Aufgabe 167, *Jahresber. Dtsch. Math.-Ver.*, **43** (1934), 114.
[17] H. Lefmann and T. Thiele, Point sets with distinct distances, *Combinatorica*, **15** (1995), 379–408.
[18] J. Pach and P. K. Agarwal, *Combinatorial Geometry*, John Wiley (New York, 1995).
[19] J. Spencer, E. Szemerédi, and W. T. Trotter, Unit distances in the Euclidean plane, in: *Graph Theory and Combinaorics*, Proc. Conf. Hon. P. Erdős, Academic Press (London, 1984), pp. 293–303.
[20] L. Székely, Crossing numbers and hard Erdős problems in discrete geometry, *Combin. Probab. Comput.*, **6** (1997), 353–358.
[21] E. Szemerédi, Erdős's unit distance problem, in: *Open Problems in Mathematics*, J. F. Nash Jr. and M. Th. Rassias, eds., Springer (Cham, 2016), pp. 459–477.
[22] K. Vesztergombi, On the distribution of distances in finite sets in the plane, *Discrete Math.*, **57** (1985), 129–145.
[23] K. Vesztergombi, On large distances in planar sets, *Discrete Math.*, **67** (1987), 191–198.
[24] K. Vesztergombi, The two largest distances in finite planar sets, *Discrete Math.*, **150** (1996), 379–386.
[25] I. M. Yaglom and V. G. Boltyanskii, *Convex Figures*, Holt, Rinehart and Winston (New York, 1961).

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