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Recognition Complexity of Subgraphs of k-Connected Planar Cubic Graphs

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Abstract
We study the recognition complexity of subgraphs of k-connected planar cubic graphs where $k \in \{0, 1, 2, 3\}$. We present polynomial-time algorithms to recognize subgraphs of 1- and 2-connected planar cubic graphs, both in the variable and fixed embedding setting. The main tools involve the GENERALIZED (ANTI)FACTOR-problem for the fixed embedding case, and SPQR-trees for the variable embedding case. Secondly, we prove NP-hardness of recognizing subgraphs of 3-connected planar cubic graphs in the variable embedding setting.

Keywords Planar cubic graphs · k-connectedness · Generalized factors · Recognition problem · NP-hardness

Mathematics Subject Classification Theory of computation · Design and analysis of algorithms · Graph algorithms analysis

1 Introduction

Whether or not the 3-EDGE-COLORABILITY-problem is solvable in polynomial time for planar graphs is one of the most fundamental open problems in algorithmic graph theory:

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Question 1.1 Can we decide in polynomial time, whether the edges of a given planar graph can be colored in three colors such that any two adjacent edges receive distinct colors?

In other words, can we decide for a planar graph G in polynomial time whether $\chi'(G) \leq 3$, where $\chi'(G)$ denotes the chromatic index of G? Clearly, it is enough to consider connected planar graphs G of maximum degree $\Delta(G) = 3$. If G is connected, planar and 3-regular, then by the Four-Color-Theorem [1, 2] and the work of Tait [31] we know that G is 3-edge-colorable if and only if G is 2-connected. As we can check 2-connectivity of subcubic graphs[1] in linear time [33], we hence can decide in polynomial time whether a given 3-regular planar graph is 3-edge-colorable.

In particular, subgraphs of bridgeless 3-regular planar graphs are 3-edge-colorable. However, this does not answer Question 1.1 yet (as sometimes wrongly claimed, e.g., in [8]), because it is for example not clear which planar graphs of maximum degree 3 are subgraphs of 2-connected 3-regular planar graphs, and whether these can be recognized efficiently.

In this paper we consider the corresponding decision problem: Given a subcubic planar graph G, is there a 2-connected 3-regular planar graph H, such that $G \subseteq H$? In other words, can G be augmented, by adding edges and (possibly) vertices, to a supergraph H of G that is planar, 3-regular, and 2-connected? For brevity, we call a planar, 3-regular supergraph H a 3-*augmentation* of G. Motivated by Question 1.1, we are interested in 2-connected 3-augmentations of G.

In fact, we shall consider the decision problems whether a given subcubic planar graph G admits a k-connected 3-augmentation for each $k \in \{0, 1, 2, 3\}$. Equivalently, we study the recognition of subgraphs of k-connected cubic planar graphs. We consider several variants where the input graph G is given with a fixed embedding $\mathcal{E}$ and the desired 3-augmentation H must extend $\mathcal{E}$, and/or where the input graph G is already k'-connected for some $k' \in \{0, 1, 2\}$. Note that if G is 3-connected, then $H = G$ is the only connected 3-augmentation.

Our Results. We resolve the complexity of finding a k-connected 3-augmentation for a given subcubic planar graph G (with or without a given embedding), except when $k = 3$ and the embedding of G is given. See also Fig. 1 for an overview.

Theorem 1.2 *Let G be a planar graph with maximum degree $\Delta(G) \leq 3$, let n be the number of vertices of G, and let $\mathcal{E}$ be an embedding of G.*

1. *We can compute, in time $\mathcal{O}(n^{3/2})$, a connected 3-augmentation H extending $\mathcal{E}$, or conclude that none exists.*
2. *We can compute, in time $\mathcal{O}(n)$, a connected 3-augmentation H or conclude that none exists.*
3. *We can compute, in time $\mathcal{O}(n^4)$, a 2-connected 3-augmentation H extending $\mathcal{E}$, or conclude that none exists. If G is connected, $\mathcal{O}(n^2)$ time suffices.*
4. *We can compute, in time $\mathcal{O}(n^2)$, a 2-connected 3-augmentation H or conclude*

[1] A graph G is *subcubic* if its maximum degree $\Delta(G)$ is at most 3, and for such graphs 2-vertex-connectivity and 2-edge-connectivity are equivalent.

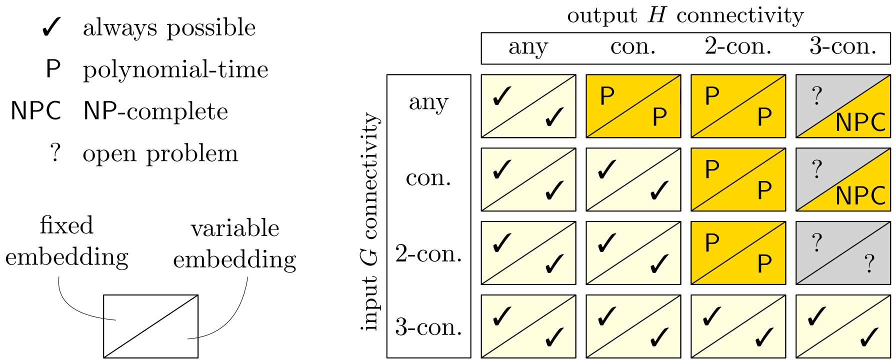
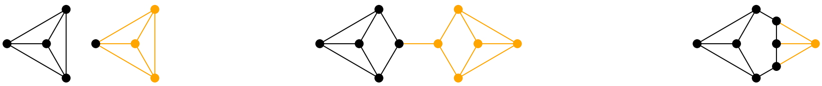


Fig. 1 Complexity of finding k-connected 3-augmentations (output connectivity $k \in \{0, 1, 2, 3\}$) of k'-connected subcubic planar graphs (input connectivity $k' \in \{0, 1, 2\}$)



(a) $G_1 \subseteq H_1$ **(b)** $G_2 \subseteq H_2$ **(c)** $G_3 \subseteq H_3$

Fig. 2 For $k = 1, 2, 3$, the planar subcubic graph G_k (in black) admits a $(k-1)$-connected 3-augmentation H_k (new vertices and edges in orange), but no k-connected 3-augmentation

that none exists.

5. *It is NP-complete to decide whether G admits a 3-connected 3-augmentation, even if G is connected.*

Note that Statements 1 and 3 concern the fixed embedding setting, while Statements 2,4 and 5 concern the variable embedding setting.

Statement 4 can be considered the main result of the paper. Still, we emphasize that this does not answer Question 1.1 yet. In fact, admitting a 2-connected 3-augmentation is a sufficient condition for 3-edge-colorability; but it is in general not necessary. For example, $K_{2,3}$ admits a proper 3-edge-coloring but no 2-connected 3-augmentation. Indeed, every 2-connected 3-regular supergraph of $K_{2,3}$ contains $K_{3,3}$ as a minor and is thus not planar. Question 1.1 remains open and we discuss it and its connection to 3-augmentations in more detail in Sect. 7.

In order to decide whether a given graph G admits a 3-augmentation, we may of course assume that G itself is planar and of maximum degree at most 3. Observe that it is always possible to find a (not necessarily connected) 3-augmentation of G, for example[2] by adding the small gadget $K_4^{(1)}$ consisting of K_4 with one subdivided edge to each vertex that has not degree 3 yet, as illustrated in Fig. 2b.

Observation 1.3 *Every subcubic planar graph G has a 3-augmentation H extending its embedding. If G is connected, then so is H.*

[2] We discuss such augmentation gadgets in more detail below.

However, this becomes non-trivial if we require the 3-augmentation H to be k-connected for some $k \in \{1, 2, 3\}$. See Fig. 2 for some examples.

Previous Results. Hartmann, Rollin and Rutter [20] studied, for each $k, r \in \mathbb{N}$, whether a planar graph G can be augmented by adding edges (but no vertices!), to a k-connected r-regular planar graph H. In particular, for $r = 3$, they show that the problem is NP-complete in the variable embedding setting for all $k \in \{0, 1, 2, 3\}$, as well as in the fixed embedding setting when $k = 3$. For the remaining cases of fixed embedding and $k \in \{0, 1, 2\}$ they present a polynomial-time algorithm.

Remark 1.4 In fact, several of their concepts and techniques [20] are very similar to ours. In case of G having a fixed embedding $\mathcal{E}$, any 3-augmentation H (with or without new vertices) extending $\mathcal{E}$ induces an assignment of each new edge e that is incident to an "old" vertex of G to the face of $\mathcal{E}$ that contains e. New edges at vertices of G are called *free valencies* in their paper [20].

The authors [20] present conditions of this assignment that are necessary and sufficient for a connected or 2-connected 3-augmentation *without new vertices*. (These conditions also allow for a polynomial-time algorithm to find such an assignment.) Their *matching condition* and *planarity condition* become obsolete in our setting. However, their *connectivity condition* and *biconnectivity condition* demand roughly twice as many free valencies to be assigned to a face with several connected components. (Intuitively, these components must be strung together in [20], while we can do a star-like connection.) Most crucially, their *parity condition*, which requires the number of free valencies assigned to each face to be even, is no longer necessary nor sufficient in our setting. It is for example violated in every example in Fig. 2.

In Sects. 3 and 4 we also find connected and 2-connected 3-augmentations of a graph G based on an appropriate assignment of free valencies to faces of G. But our conditions for a feasible assignment are different. Given the assignment, one could do some local modifications (such as adding new vertices) to obtain a new graph G' with a new assignment that then satisfies all the conditions in their paper [20]. An augmentation H of G' (now only adding edges) would then be a desired 3-augmentation of G. However to be selfcontained, we give an explicit construction here; see Claim 3.5.

Finally, for $k \in \{0, 1, 2\}$, finding a k-connected 3-augmentation in the variable embedding setting is in P[16], while the version without new vertices is NP-complete [20]. So to summarize, there is probably no direct reduction between the decision problem of augmenting by only adding edges and the decision problem of augmenting by adding vertices and edges.

Let us mention a few more examples from the rich and diverse area of augmentation problems. Eswaran and Tarjan [14] pioneered the systematic investigation of augmentation problems. They presented algorithms to find in $\mathcal{O}(|V(G)| + |E(G)|)$ time a smallest number of edges whose addition to a given (not necessarily planar) graph G results in a 2-connected respectively 2-edge-connected graph, while the weighted versions of either problem are NP-complete. If we additionally require the result to be planar, already both unweighted problems are NP-complete, the same holds for the fixed embedding setting [23, 27]. Other problems of augmenting to a planar graph consider augmenting to a grid graph [3], or triangulating while minimizing the maximum degree [11, 24], avoiding separating triangles [4], creating a Hamiltonian cycle [13], or resulting in a chordal graph [25], just to name a few.

2 Preliminaries

All graphs considered here are finite, undirected, and contain no loops but possibly multi-edges. We denote the degree of a vertex v in a graph G by $\deg_G(v)$, the minimum degree in G by $\delta(G)$, and the maximum degree by $\Delta(G)$. A graph G is *d-regular*, for some non-negative integer d, if we have $\delta(G) = \Delta(G) = d$.

Planar Embeddings. A *planar embedding* $\mathcal{E}$ of a (planar) graph G is (in a sense that we need not make precise here) an equivalence class of crossing-free drawings of G in the plane. In particular, a planar embedding determines the set F of all *faces*, the distinguished *outer face*, the clockwise ordering of incident edges around each vertex and the *boundary* of each face as a set of *facial walks*, each being a clockwise ordering of vertices and edges (with repetitions allowed). The edges and vertices incident to the outer face are called *outer edges* and *outer vertices*, while all others are *inner edges* and *inner vertices*. For every embedding $\mathcal{E}$ of G we define the *flipped embedding* $\mathcal{E}'$ to be the embedding obtained from $\mathcal{E}$ by reversing the clockwise order of incident edges at each vertex. This operation changes neither the set of faces nor the outer face. Whitney's Theorem [35] states that a 3-connected planar graph G has a unique embedding (up to the choice of the outer face and flipping).

Connectivity. A graph G is *k-connected* if $G - S$ is connected for every set $S \subseteq V(G)$ of at most $k-1$ vertices in G. Similarly, G is *k-edge-connected* if $G - S$ is connected for every set $S \subseteq E(G)$ of at most $k-1$ edges in G. We denote by $\theta(G)$ the largest k for which G is k-edge-connected. If G has maximum degree at most 3, then G is k-connected for $k \leq 3$ if and only if $\theta(G) \geq k$. A *bridge* in a graph G is an edge e whose removal increases the number of connected components, i.e., $G - e$ has strictly more components than G. Equivalently, e is a bridge if e is not contained in any cycle of G. A *bridgeless* graph is one that contains no bridge. Note that a bridgeless graph may be disconnected. Yet, for connected graphs of maximum degree 3, being bridgeless and being 2-connected is equivalent.

Wheel-Extensions. When computing 3-augmentations, we sometimes encounter k-connected planar supergraphs with vertices of degree greater than 3 in intermediate steps. As we are interested in k-connected 3-augmentations, we need a transformation that replaces such a vertex by a small gadget in order to obtain a maximum degree of at most 3 while preserving k-connectivity. This is realized by *wheel-extensions*. We denote by D_ℓ the graph obtained from two copies of the cycle C_ℓ of length $\ell \geq 3$ by connecting for every vertex of C_ℓ its two copies in D_ℓ with an edge (in fact D_ℓ is the *product* of C_ℓ and a path on two vertices). For an integer $\ell \geq 3$, let W_ℓ be the graph obtained from D_ℓ by subdividing each edge in one cycle C_ℓ exactly once. See Fig. 3a for an illustration. Consider a planar graph G with an embedding $\mathcal{E}$, and a vertex $v \in V(G)$ with $\deg_G(v) = \ell \geq 3$. A *wheel-extension at v* is the graph and embedding obtained by replacing v with W_ℓ, and by attaching v's incident edges to the subdivision vertices of W_ℓ in a one-to-one non-crossing way. See Fig. 3b.

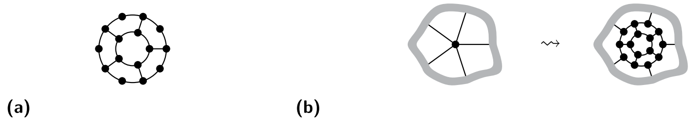


Fig. 3 **a** The graph W_5 obtained from D_5 through subdivision. **b** A Wheel-extension

Observation 2.1 *Let G be a graph (possibly with multi-edges, but no loops), let $v \in V(G)$ be a vertex with $\deg_G(v) \geq 3$, and let G' be obtained from G by a wheel-extension at v. Then $\theta(G') \geq \min\{\theta(G), 3\}$.*

Proof If $\theta(G') \leq 2$ (otherwise there is nothing to show), let S be an edge-cut of size $\theta(G') \leq 2$ in G'. As S is minimal, S does not consist of the two edges at a subdivision vertex of W_ℓ. Thus, as $C_\ell \square P_2$ is 3-connected, it follows that $S \cap E(W_\ell) = \emptyset$. But then, S is also an edge-cut in G and hence $\theta(G) \leq |S| = \theta(G')$, as desired. □

Augmentation Gadgets. When constructing a 3-augmentation of a planar graph G, we add new vertices and edges to some faces of G in such a way that the resulting graph H is planar, 3-regular, and has the desired connectivity. The difficult part is to choose the faces of G (given an embedding) and to choose the vertices of G that receive new edges in that face. Once these choices are made, there are numerous ways how to insert the new vertices and edges. Say, for a face f of G, we decide to add new vertices and edges so that exactly ℓ of the new edges have an endpoint in G. An *augmentation gadget with ℓ attachments* is a planar embedded 2-connected graph A with exactly ℓ vertices of degree 2 that are all incident to the outer face of A. Then we can insert A into the face f of G and draw in a non-crossing way an edge from each of the ℓ degree-2 vertices to designated vertices of G incident to f.

We can take the graph $K_4^{(\ell)}$ that is obtained from K_4 by subdividing one edge ℓ times as an augmentation gadget, as we do in Fig. 2b. We can also take a cycle of length ℓ, as we do in Fig. 2a, which however works only in case $\ell \geq 3$. And Fig. 2c shows that we do not need to use augmentation gadgets at all.

Generalized Factors. Let H be a graph with a set $B(v) \subseteq \{0, \ldots, \deg_H(v)\}$ assigned to each vertex $v \in V(H)$. Following Lovász, a spanning subgraph $G \subseteq H$ is called a *B-factor* of H if and only if $\deg_G(v) \in B(v)$ for every vertex $v \in V(H)$ [26]. Deciding whether a graph H admits a B-factor is known as the Generalized Factor problem. In general, the Generalized Factor problem is NP-complete [26]. Still, for certain well-behaved sets $B(\cdot)$, the problem becomes polynomial-time solvable. A set $B(v)$ is said to have a *gap of length* $\ell \geq 1$ if there is an integer $i \in B(v)$ such that $i+1, \ldots, i+\ell \notin B(v)$, and $i+\ell+1 \in B(v)$.

Theorem 2.2 *(Cornuéjols [9, Section 3]) Let H be a graph with a set $B(v) \subseteq \{0, \ldots, \deg_H(v)\}$ assigned to each vertex $v \in V(H)$. If all gaps of each B(v) have length 1, then a B-factor can be computed in time $\mathcal{O}(|V(H)|^4)$.*

We say that there are *no two consecutive forbidden degrees* for a vertex $v \in V(H)$ if for all $i, i+1 \in \{0, \ldots, \deg_H(v)\}$ we have $i \in B(v)$ or $i+1 \in B(v)$. Under this slightly stronger condition, an algorithm by Sebő yields a better runtime for the Generalized Factor problem.

Theorem 2.3 *(Sebő [29, Section 3]) Let H be a graph with a set $B(v) \subseteq \{0, \ldots, \deg_H(v)\}$ assigned to each vertex $v \in V(H)$. If no two consecutive degrees are forbidden for any vertex, then we can compute a B-factor in time $\mathcal{O}(|V(H)| \cdot |E(H)|)$, or conclude that no B-factor exists.*

If no set $B(v)$ contains a gap, the Generalized Factor problem is also known as the Degree Constrained Subgraph problem [15].

Theorem 2.4 *(Gabow [15]) Let H be a graph with a lower bound $\operatorname{low}(v)$ and an upper bound $\operatorname{up}(v)$ assigned to each vertex $v \in V(H)$ and let $c := \sum_{v \in V(H)} \operatorname{up}(v)$. Then we can compute a spanning subgraph $G \subseteq H$ with $\operatorname{low}(v) \leq \deg_G(v) \leq \operatorname{up}(v)$ in time $\mathcal{O}(\sqrt{c} \cdot |E(H)|)$, or conclude that no such subgraph exists.*

SPQR-Tree. The *SPQR-tree* is a tree-like data structure that compactly encodes all planar embeddings of a 2-connected planar graph. It was introduced by Di Battista and Tamassia [12] and can be computed in linear time [19]. Its precise definition includes quite a number of technical terms, of which we define the crucial ones below. This makes our exposition self-contained, while also ensuring the established terminology for experienced readers. We give an illustrating example in Fig. 4.

The SPQR-tree of a 2-connected planar graph G is a rooted tree T, where each vertex μ of T is associated with a multigraph $\operatorname{skel}(\mu)$ that is called the *skeleton of* μ. This multigraph $\operatorname{skel}(\mu)$ must be of one of four types determining whether μ is an S-, a P-, a Q- or an R-vertex:

- S-vertex: $\operatorname{skel}(\mu)$ is a simple cycle.
- P-vertex: $\operatorname{skel}(\mu)$ consists of two vertices and at least three parallel edges.

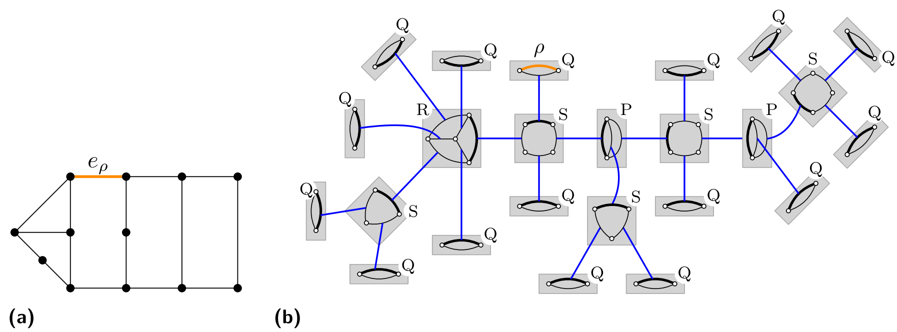


Fig. 4 A graph with an edge e_ρ (**a**) and its SPQR-tree rooted at the Q-vertex ρ corresponding to e_ρ (**b**). Each tree node μ shows the skeleton $\operatorname{skel}(\mu)$ in which the virtual edge to its parent is shown thicker. The (blue) tree edges indicate the associated pairs of virtual edges

- Q-vertex: $\operatorname{skel}(\mu)$ consists of two vertices with two parallel edges.
- R-vertex: $\operatorname{skel}(\mu)$ is 3-connected.

Some of the edges of the skeletons can be marked as *virtual* edges. An edge $e = \mu\nu$ of the SPQR-tree *T* corresponds to two virtual edges, exactly one in $\operatorname{skel}(\mu)$ and one in $\operatorname{skel}(\nu)$. Conversely, each virtual edge corresponds to exactly one tree edge of *T* in this way. We refer again to Fig. 4 for an example.

Under above conditions, the defining property of the SPQR-tree *T* is that *G* can be obtained by *gluing* along the virtual edges: For each tree edge $e = \mu\nu$, the skeletons $\operatorname{skel}(\mu)$ and $\operatorname{skel}(\nu)$ are identified at the corresponding endpoints of the two virtual edges associated with *e*, afterwards the virtual edges are removed.

We additionally require that no two S-vertices and no two P-vertices are adjacent in *T*, as otherwise the skeletons of two such vertices can be merged into the skeleton of a new vertex of the same type. Further, exactly one of the two parallel edges in a Q-vertex is a virtual edge while S-, P- and R-vertices contain only virtual edges. Under these conditions the SPQR-tree of *G* is unique. There is exactly one Q-vertex per edge in *G* and these form the leaves of the SPQR-tree. The inner S-, P- and R-vertices correspond more or less.[3] to the separation pairs (that is, pairs of vertices forming a cut set) of *G* [12].

Assume that an arbitrary vertex ρ of *T* is fixed as the root. For some vertex μ in *T* let π be its parent. Further, let *u*, *v* be the endpoints of the virtual edge in $\operatorname{skel}(\mu)$ associated with the tree edge $\mu\pi$ in *T*. Then the graph obtained by gluing $\operatorname{skel}(\mu)$ with all skeletons in its subtree and without the virtual edge *uv* is called the *pertinent graph* of μ and denoted by $\operatorname{pert}(\mu)$. Note that $\operatorname{pert}(\mu)$ is always connected.

SPQR-Tree and Planar Embeddings. If the SPQR-tree *T* is rooted at a Q-vertex ρ corresponding to an edge e_ρ of *G*, then *T* represents all planar embeddings of *G* in which e_ρ is an outer edge [12]. When *G* is constructed by gluing corresponding virtual edges, one has the following choices on the planar embedding:

- Whenever the corresponding virtual edges of an S-, P- or R-vertex μ and its parent are glued together, this leaves two choices for the planar embedding: Having decided for an embedding $\mathcal{E}_\mu$ of $\operatorname{pert}(\mu)$ already, we can insert $\mathcal{E}_\mu$ or the flipped embedding $\mathcal{E}'_\mu$.
- The parallel virtual edges of a P-vertex μ associated with virtual edges of children can be permuted arbitrarily. Every permutation leads to a different planar embedding of $\operatorname{skel}(\mu)$.
- Gluing at the virtual edge of a Q-vertex μ replaces the virtual edge *uv* by the "real" edge *uv* in *G*. This has no effect on the embedding.

Let $\mathcal{E}$ be a planar embedding of *G* having e_ρ as an outer edge. Further, let μ be an inner vertex of the SPQR-tree and u_μ, v_μ be the endpoints of the virtual edge in $\operatorname{skel}(\mu)$ corresponding to the parent edge of μ in *T*. Lastly, let $\mathcal{E}_\mu$ be the restriction of $\mathcal{E}$ to $\operatorname{pert}(\mu)$ and let f^o_μ be the outer face of $\mathcal{E}_\mu$. As e_ρ is an outer edge of $\mathcal{E}$, it fol-

[3] In fact they correspond to so-called *split pairs* However, we omit their formal discussion, as it is not needed here.

lows that u_μ and v_μ are outer vertices in $\mathcal{E}_\mu$. The $u_\mu v_\mu$-path in $\mathrm{pert}(\mu)$ having f^o_μ to its left (right) is the *left (right) outer path* of $\mathcal{E}_\mu$. Lastly, we define the *left (right) outer face* of $\mathcal{E}_\mu$ inside $\mathcal{E}$ to be the face of $\mathcal{E}$ left (right) of the left (right) outer path of $\mathcal{E}_\mu$.

3 2-Connected 3-Augmentations for a Fixed Embedding

We consider the 3-augmentation problem for arbitrary input graphs G and 2-connected output graphs H, corresponding to the third column of the table in Fig. 1. For the fixed embedding setting here, we present a quartic-time algorithm which yields the following.

Theorem 3.1 *Let G be a planar n-vertex graph with $\Delta(G) \leq 3$ and an embedding $\mathcal{E}$. We can compute, in time $\mathcal{O}(n^4)$, a 2-connected 3-augmentation H of G extending $\mathcal{E}$, or conclude that none exists. If G is connected, then time $\mathcal{O}(n^2)$ suffices.*

This corresponds to Statement 3 of Theorem 1.2. We start with a reduction to graphs G with $\delta(G) \geq 2$.

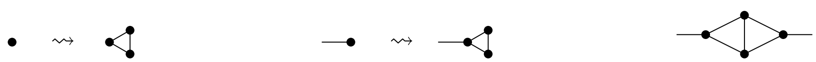
Lemma 3.2 *Let G be a planar graph with embedding $\mathcal{E}$. There is a planar supergraph $G' \supseteq G$ with $\delta(G') \geq 2$ whose embedding $\mathcal{E}'$ extends $\mathcal{E}$, such that G has a 2-connected 3-augmentation extending $\mathcal{E}$ if and only if G' has one extending $\mathcal{E}'$.*

Proof Consider the following two replacement rules, also shown in Fig. 5a–b: Each isolated vertex is replaced by a copy of K_3, and each vertex *v* of degree 1 is replaced by a copy of K_3 with one vertex connected to the single neighbor of *v* in *G*. Let G' be the obtained graph such that its planar embedding $\mathcal{E}'$ extends $\mathcal{E}$.

Let *H* be a 2-connected 3-augmentation of *G*. We obtain a 2-connected 3-augmentation of G' as follows: For each vertex *v* of degree 0 (or 1) in *G*, let *N*(*v*) be its three (two) new neighbors in *H*. In *H*, replace *v* by its corresponding copy of K_3. Connect its three (two) degree-2-vertices with one vertex of *N*(*v*) such that the embedding remains planar.

The other direction works similar: In a 2-connected 3-augmentation of G', contract each copy of K_3 that was introduced for a vertex *v* of *G* into a single vertex. If this creates multi-edges, replace each duplicated edge by the gadget shown in Fig. 5c to obtain a simple graph. □

Lemma 3.3 *Let G be a planar n-vertex graph with an embedding $\mathcal{E}$, $\delta(G) \geq 2$, and $\Delta(G) \leq 3$. Then we can compute, in time $\mathcal{O}(n^4)$, a 2-connected 3-augmentation H*



(a) **(b)** **(c)**

Fig. 5 **a**–**b** Replacement rules. **c** Gadget to avoid parallel edges *of G extending $\mathcal{E}$, or conclude that none exists. If G is connected, then time $\mathcal{O}(n^2)$ suffices.*

Proof The proof is by a linear-time reduction to an equivalent instance A of the Generalized Factor problem, such that A fulfills the necessary condition to apply an $\mathcal{O}(n^4)$-time algorithm by Cornuéjols (Theorem 2.2), or even an $\mathcal{O}(n^2)$-time algorithm by Sebő (Theorem 2.3).

We construct the 2-connected 3-augmentation H of G by adding new edges and vertices into the faces of $\mathcal{E}$. Therefore, the obtained embedding of H extends $\mathcal{E}$.

Some faces of $\mathcal{E}$ stand out, as these *must* contain new edges (and possibly vertices) to reach 2-connectedness. We call these the *connecting faces* and denote the set of connecting faces by F_{conn}. Obviously, all faces incident to at least two connected components are connecting faces. Further, for each bridge *e* of *G*, the unique face *f* incident to both sides of *e* is a connecting face because the only way to add new connections between the components separated by *e* is through *f*. Recall that a 3-regular graph is 2-connected if and only if it is connected and bridgeless, so these are the only two types of connecting faces. All other faces are considered to be *normal* faces, denoted by F_{norm}.

For a connecting face $f \in F_{\mathrm{conn}}$, let G_f be the subgraph of G on the vertices and edges incident to *f*, let B_f be the set of its blocks (i.e., maximal 2-connected components or bridges), and let T_f be its block-cut-forest. We partition B_f into $S_f \cup I_f \cup L_f$, where we call the elements of S_f the *singleton blocks*, the elements of I_f the *inner blocks*, and the elements of L_f the *leaf blocks*:

$$\begin{aligned} S_f &:= \{b \in B_f \mid b \text{ forms a trivial (i.e., single-vertex) tree in } T_f\} \\ I_f &:= \{b \in B_f \mid b \text{ is an inner vertex of a non-trivial tree in } T_f\} \\ L_f &:= \{b \in B_f \mid b \text{ is a leaf in a non-trivial tree in } T_f\} \end{aligned}$$

In fact, we distinguish these types of blocks, since in order to obtain a 2-connected 3-augmentation, every singleton block must be incident to at least two new edges, and every leaf block has to be incident to at least one new edge.

The Generalized Factor instance A is a bipartite graph with bipartition classes $\mathcal{V}$ and $\mathcal{F}$. Here, $\mathcal{V} := \{v \in V(G) \mid \deg_G(v) = 2\}$ contains all vertices of G having degree lower than 3. Vertices in $\mathcal{F}$ represent the faces of $\mathcal{E}$. Edges of a *B*-factor of A will determine the faces of $\mathcal{E}$ containing the new edges. In particular, $\mathcal{F}$ contains one vertex corresponding to each normal face in F_{norm}. Additional vertices in $\mathcal{F}$ are needed to handle the connecting faces. For each connecting face $f \in F_{\mathrm{conn}}$, we add all blocks in B_f as vertices to $\mathcal{F}$. (If there are two faces *f*, *g* in $\mathcal{E}$ such that B_f and B_g contain blocks corresponding to the same subgraph of *G*, then $\mathcal{F}$ contains two such vertices: one corresponding to the block in B_f, and another to the block in B_g. This is for example the case when *G* corresponds to three disjoint cycles, see Fig. 6.)

In A, each $x \in \mathcal{F}$ is incident to exactly the following $v \in \mathcal{V}$: If *x* represents a normal face $f \in F_{\mathrm{norm}}$, then *x* is connected to every $v \in \mathcal{V}$ that is incident to *f* in $\mathcal{E}$. Otherwise, if *x* represents a block $b \in B_f$ for some connecting face $f \in F_{\mathrm{conn}}$, then *x* is connected to every $v \in \mathcal{V}$ that is contained in *b*. See Fig. 7 for an example.

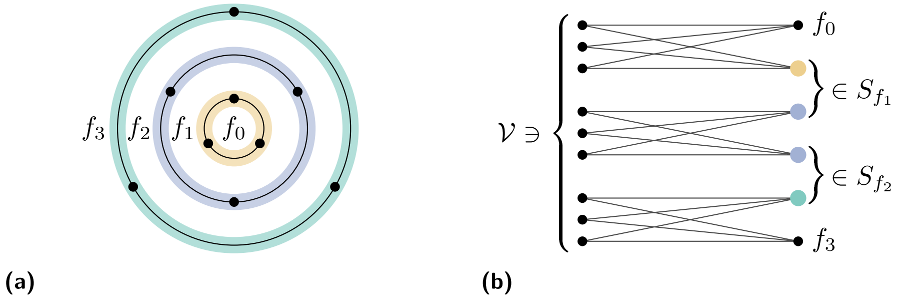


Fig. 6 **a** A planar subcubic graph G where the connecting faces f_1, f_2 share a block (blue). **b** In its Generalized Factor instance the shared block (blue) is represented by two distinct vertices

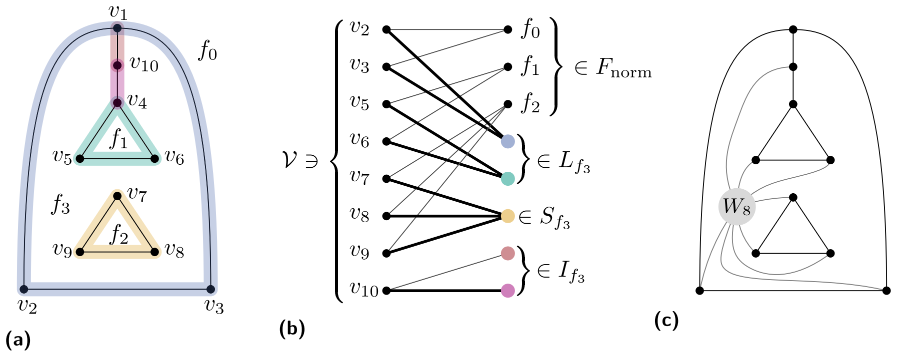


Fig. 7 **a** A planar subcubic graph G. **b** Its corresponding Generalized Factor instance. Thick edges denote a possible solution. **c** A 2-connected 3-augmentation of G (with an indicated wheel-extension)

Lastly, we need to assign a set $B(x) \subseteq \{0, 1, \ldots, \deg_A(x)\}$ of possible degrees to each vertex $x \in \mathcal{V} \cup \mathcal{F}$:

$$B(x) := \begin{cases} \{1\}, & \text{if } x \in \mathcal{V} \\ \{0, 2, 3, \ldots, \deg_A(x)\}, & \text{if } x \in F_{\mathrm{norm}} \\ \{0, 1, 2, \ldots, \deg_A(x)\}, & \text{if } x \in I_f \text{ for some connecting face } f \in F_{\mathrm{conn}} \\ \{1, 2, 3, \ldots, \deg_A(x)\}, & \text{if } x \in L_f \text{ for some connecting face } f \in F_{\mathrm{conn}} \\ \{2, 3, 4, \ldots, \deg_A(x)\}, & \text{if } x \in S_f \text{ for some connecting face } f \in F_{\mathrm{conn}} \end{cases}$$

By the following claim, the above reduction is linear.

Claim 3.4 *The order and size of A is linear in n. Moreover, A can be computed in linear time.*

Proof If $v \in V(G)$ is a vertex incident to a face f of $\mathcal{E}$, then it lies in at most three blocks of G_f, since its degree in G_f is at most 3. Further, every vertex is incident to at most three different faces of $\mathcal{E}$. Thus, there are at most nine blocks in $\mathcal{B} := \bigcup_{f \in F_{\mathrm{conn}}} B_f$ containing v and $|\mathcal{B}| \leq 9n$ follows. As the number of faces of a planar embedding is linear in n, so is $|F_{\mathrm{norm}}|$. Therefore, the order $|\mathcal{V} \cup \mathcal{F}| = |\mathcal{V}| + |\mathcal{B}| + |F_{\mathrm{norm}}|$ of A is linear in n.

A vertex $v \in \mathcal{V}$ is incident to (at most) two faces of $\mathcal{E}$, and therefore it is contained in at most six distinct blocks in $\mathcal{B}$. Hence, we see that each vertex $x \in \mathcal{V}$ is adjacent to at most six vertices in $\mathcal{B} \subseteq \mathcal{F}$ and at most two vertices in $F_{\mathrm{norm}} \subseteq \mathcal{F}$. Thus, the bipartite graph A contains at most $8n$ edges. Note that, in particular, A can be computed in linear time. Indeed, a traversal of the boundary of a face f of $\mathcal{E}$ yields the graph G_f. With a depth-first-search, the blocks B_f can be computed [22]. Since f is a connecting face if and only if $|B_f| \geq 2$, the Generalized Factor instance can be computed in linear time. □

The next two claims establish that A admits a B-factor if and only if G admits a 2-connected 3-augmentation H extending $\mathcal{E}$.

Claim 3.5 *If A admits a B-factor, then G has a 2-connected 3-augmentation H extending $\mathcal{E}$.*

Proof Let A' be a B-factor of A, i.e., a subgraph A' such that $\deg_{A'}(x) \in B(x)$ for every $x \in V(A)$. We shall construct a connected and bridgeless supergraph H' of G. (A connected graph with maximum degree 3 is 2-connected if and only if it is bridgeless.) We construct H' as follows: For each edge $vx \in E(A')$ with $v \in \mathcal{V}$ and $x \in \mathcal{F}$, we add a new half-edge from v into a face f of $\mathcal{E}$. If x represents a face, then $f = x$. Otherwise, let f be the face such that x represents a block in B_f.

Now, for each face f of $\mathcal{E}$, all half-edges ending inside f are connected to a new vertex v_f. Obviously, H' is planar. To see that H' is connected, consider a connecting face f of $\mathcal{E}$. We have $\deg_{A'}(b) \geq 1$ for every $b \in S_f \cup L_f$, so each such b is connected to v_f by at least one edge. Lastly, to prove that H' is bridgeless, we consider three cases:

- A non-bridge (i.e., an edge that is not a bridge) of G is a non-bridge in H' as $H' \supseteq G$.
- A bridge e of G has a unique face f of $\mathcal{E}$ incident to both its sides. Each of the two components of $G - e$ contains at least one leaf block in L_f as a subgraph. As we have $\deg_{A'}(b) \geq 1$ for all $b \in L_f$, there is at least one edge from each leaf block to v_f in H'. Thus, e is a non-bridge in H'.
- No edge incident to a new vertex v_f (for some face f of $\mathcal{E}$) is a bridge, because v_f has at least two edges to every incident component of H': If f is a normal face, then $\deg_{H'}(v_f) \geq 2$ because $\deg_{A'}(f) \neq 1$. Now assume that f is a connecting face, and consider a component C of G_f. If C consists of a single block b (which would be in S_f), then v_f is connected to at least two vertices of b, because we have $\deg_{A'}(b) \geq 2$. Otherwise, if C consists of multiple blocks, then its block-cut-tree has at least two leaves. In this case, v_f is connected to at least one vertex per leaf block $b \in L_f$ because $\deg_{A'}(b) \geq 1$.

Since $\deg_{A'}(v) = 1$ for each $v \in \mathcal{V}$, we see that all vertices in $V(G)$ have degree 3 in H'. We apply a wheel-extension (Observation 2.1) at each new vertex v_f of degree larger than 3, and replace each vertex v_f of degree 2 by the gadget represented in Fig. 5c (this simulates replacing it with a single edge connecting its neighbors, but without the risk of creating a multi-edge). We obtain a 3-regular graph H that is planar, connected, and bridgeless. Finally, H is 2-connected, because it is connected, bridgeless and of maximum degree 3. □

Claim 3.6 *If G has a 2-connected 3-augmentation H extending $\mathcal{E}$, then A has a B-factor.*

Proof Since H extends $\mathcal{E}$, its new vertices and edges must have been added solely into the faces of $\mathcal{E}$.

We have to construct a B-factor A' of A. To this end, we consider the vertices $v \in \mathcal{V}$, i.e., vertices of G with $\deg_G(v) = 2$. Each such v has $\deg_H(v) = 3$, so there is exactly one new edge e incident to v. Let f be the face of $\mathcal{E}$ that e is inside. If f is a normal face, we add the edge vf to A'. Now assume that f is a connecting face. As $\deg_G(v) = 2$, vertex v can be in at most two blocks of G_f. If v is in a singleton block, then this is the only block in B_f containing v. If v lies in exactly two leaf blocks $b_1, b_2 \in B_f$, then both must be bridges, whose other endpoints have degree 1; a contradiction to $\delta(G) \geq 2$. Thus, there is at most one block in $L_f \cup S_f$ containing v. If there exists a block in $L_f \cup S_f$ containing v, let b be this (unique) block. Otherwise, choose an arbitrary $b \in B_f$ containing v. We add the edge vb to A'.

We prove that $\deg_{A'}(x) \in B(x)$ for all $x \in \mathcal{V} \cup \mathcal{F}$. For each vertex $v \in \mathcal{V}$, we added exactly one edge to A', therefore we have $\deg_{A'}(v) = 1$ as required.

For a normal face $f \in F_{\text{norm}}$, it holds that $\deg_{A'}(f)$ is either 0 or at least 2 because H is 2-connected and therefore there are either no or at least two new edges half-edges inside f.

Now, consider a connecting face $f \in F_{\text{conn}}$. Each $b \in S_f$ is a singleton block of G_f. Since H is 2-connected, there are (at least) two paths leaving different vertices $v_1, v_2 \in b$ via new edges through f. Therefore, A' contains the edges $v_1 b$ and $v_2 b$, i.e., $\deg_{A'}(b) \geq 2$ as required. Similarly, each $b \in L_f$ is a leaf-block. Since H is 2-connected, there is (at least) one path leaving $v \in b$ via a new edge through f. Therefore, we have $vb \in E(A')$ and thus $\deg_{A'}(b) \geq 1$ as required. □

It remains to argue that we can compute a B-factor of A efficiently. By inspecting the set $B(x)$ for each $x \in \mathcal{V} \cup \mathcal{F}$, we can see that none of them contains a gap of size 2 or greater. Therefore, we are in a special case of the Generalized Factor problem that can be solved, in $\mathcal{O}(n^4)$ time, by Cornuéjols ' algorithm (see Theorem 2.2).

A closer inspection yields that only for $x \in S_f$ the sets $B(x)$ contain two forbidden degrees. (Note that $\deg_A(v) \leq 2$ for all $v \in \mathcal{V}$: If there is a face f such that v is contained in two blocks of G_f, then both edges incident to v are bridges; thus v is incident to no other face. Otherwise, this follows from $\deg_G(v) \leq 2$, i.e., v being incident to at most two faces.) Therefore, if $S_f = \emptyset$ for every connecting face $f \in F_{\text{conn}}$, then we can even apply the algorithm by Sebő, which takes only $\mathcal{O}(n^2)$ time (see Theorem 2.3). In particular, this is the case if G is connected. □

Remark 3.7 The attentive reader might be tempted to think that we can modify the Generalized Factor instance in the proof above so that it satisfies the conditions of Sebő, even when $S_f \neq \emptyset$. One such attempt resides in splitting each vertex x representing a singleton block into two vertices x_1, x_2 with associated sets $B(x_1), B(x_2)$ which only exclude the value 0. Indeed, the sets $B(x_1), B(x_2)$ fulfill the condition of Sebő, but now some of the vertices in $\mathcal{V}$ might not, see Fig. 8.

In the obtained Generalized Factor instance, all vertices in $\mathcal{V}$ have degree 3, yet $B(x) = \{1\}$ for every $x \in \mathcal{V}$, i.e both 2 and 3 are forbidden values. Thus, the conditions of Sebő (see Theorem 2.3) are not satisfied.

4 Connected 3-Augmentations for a Fixed Embedding

In this section, we find connected 3-augmentations for arbitrary (and possibly disconnected) input graphs G. Refer to the second column of the table in Fig. 1. For the variable embedding setting, the problem can be solved in linear time, see Proposition 5.1.

Here, we present an algorithm for the fixed embedding with runtime $\mathcal{O}(\left|V(G)\right|^{3/2})$. In fact, with a reduction to Maximum Flow, the algorithm can be altered to obtain almost linear runtime, see Remark 4.2.

By Observation 1.3, determining whether a planar embedding admits a connected 3-augmentation is equivalent to deciding whether it can be extended to a subcubic planar connected graph.

Lemma 4.1 *Let G be a planar subcubic graph with an embedding $\mathcal{E}$ and $\Delta(G) \leq 3$. Let n be the number of vertices in G. Then we can compute, in time $\mathcal{O}(n^{3/2})$, a connected subcubic planar supergraph H of G extending $\mathcal{E}$, or conclude that none exists.*

We remark that the proof of Lemma 4.1 is essentially a simpler version of the proof of Lemma 3.3. Again, we reduce the problem to an instance of Generalized Factor. That is, we need to decide whether an auxiliary graph A admits a B-factor. However, here the sets $B(\cdot)$ have no gaps. We obtain an instance of Degree Constrained Sub-

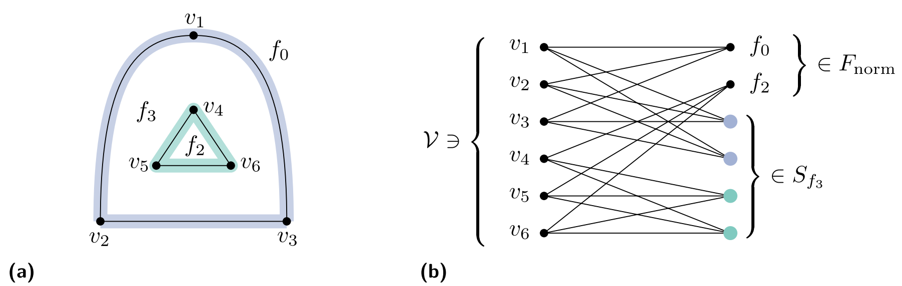


Fig. 8 **a** A planar subcubic graph G. **b** Its modified Generalized Factor instance where each singleton block appears twice

GRAPH that is solvable in time $\mathcal{O}(\left|V(G)\right|^{3/2})$ (cf. Theorem 2.4). A direct application of the algorithm of Sebő would result in a runtime of $\mathcal{O}(\left|V(G)\right|^{2})$ instead.

In contract to Lemma 3.3, we now only need to find a *subcubic* planar connected supergraph. Thus, we do not need to assign exactly $3 - \deg_G(x)$ new edges to a vertex *x* of degree less than 3. In fact, such a vertex can be incident to any number of new edges, ranging from 0 to $3 - \deg_G(x)$. The same replacement rules as in Lemma 3.2 could be applied in order to reduce the problem to an input graph *G* with $\delta(G) \geq 2$. However, this will not be necessary as we can handle in this section vertices of degree 0 and 1 in the same way as vertices of degree 2.

Proof Using the same ideas as in the proof of Lemma 3.3, we reduce the problem of finding a connected supergraph which extends $\mathcal{E}$, to an instance of the GENERALIZED FACTOR problem *A* which fulfills the necessary condition to apply an $\mathcal{O}(n^2)$-time algorithm by Sebő, see Theorem 2.3.

We construct the supergraph *H* of *G* by adding vertices and edges to some faces of $\mathcal{E}$. Thus, the obtained embedding of *H* extends $\mathcal{E}$.

In order to obtain a connected supergraph, faces of $\mathcal{E}$ which are incident to at least two connected components *must* contain new edges (and possibly vertices). We call the set of these the *component-connecting faces* F_{comp}.

For a component-connecting face $f \in F_{\text{comp}}$, we denote by G_f the subgraph of *G* on the vertices and edges incident to *f* (using the same notation as in the proof of Lemma 3.3).

The GENERALIZED FACTOR instance *A* is a bipartite graph with bipartition classes $\mathcal{V}$ and $\mathcal{F}$. Here, $\mathcal{V} := \{v \in V \mid \deg_G(v) \leq 2\}$ contains all vertices of *G* that have degree less than 3 (and hence require new edges). Vertices in $\mathcal{F}$ represent component-connecting faces of $\mathcal{E}$. For each component-connecting face $f \in F_{\text{comp}}$, we add all components of G_f as vertices to $\mathcal{F}$. (If there are two faces *f*, *g* in $\mathcal{E}$ such that G_f and G_g contain two components corresponding to the same subgraph of *G*, then $\mathcal{F}$ contains two such vertices: one corresponding to the component of G_f, and another to the component of G_g.)

Each $c \in \mathcal{F}$ corresponds to a component of G_f for some face of $\mathcal{E}$. In the graph *A*, *c* is incident to all $v \in \mathcal{V}$ which are incident to the component *c* in $\mathcal{E}$. It remains to assign a set $B(x) \subseteq \{0, 1, \ldots, \deg_A(x)\}$ of possible degrees to each vertex $x \in \mathcal{V} \cup \mathcal{F}$:

$$B(x) := \begin{cases} \{k \in \mathbb{N}_0 \mid k \leq \min(3 - \deg_G(x), \deg_A(x))\}, & \text{if } x \in \mathcal{V} \\ \{1, 2, 3, \ldots, \deg_A(x)\}, & \text{if } x \in \mathcal{F}, \end{cases}$$

i.e., each vertex $x \in \mathcal{V}$ can be incident to up to $3 - \deg_G(x)$ edges, and each $c \in \mathcal{F}$ is incident to at least one edge in any *B*-factor.

Observe that the order and size of *A* are linear in *n*: Every vertex $u \in \mathcal{V}$ is incident to at most two faces of $\mathcal{E}$ as $\deg_G(u) \leq 2$. Since *u* belongs to at most one component of G_f for each component-connecting face *f*, we obtain $\left|\mathcal{F}\right| \leq 2n$ and $\deg_A(v) \leq 2$

for every $v \in \mathcal{V}$. In particular, it follows that $|E(A)| \leq 2n$. Note that A can be computed in linear time.

Using a similar argument as in Lemma 3.3, we observe that A admits a B-factor if and only if G has a connected subcubic planar supergraph H extending $\mathcal{E}$. The only difference resides in the fact that a vertex $v \in \mathcal{V}$ might have degree 0 in the B-factor. We then obtain the corresponding supergraph by inserting an augmentation gadget with $i = 3 - \deg_G(v)$ attachments within one of the faces incident to *v* and connecting its subdivision vertices with edges to *v*.

Note that no set *B*(*x*) with $x \in \mathcal{V} \cup \mathcal{F}$ contains a gap. We can therefore apply the algorithm by Gabow (see Theorem 2.4). We now show that the algorithm has runtime $\mathcal{O}(n^{3/2})$. Let $\operatorname{up}(x) = \max(t \in B(x))$ be the maximum value in *B*(*x*). With the handshake lemma, we obtain

$$c := \sum_{x \in \mathcal{V} \cup \mathcal{F}} \operatorname{up}(x) \leq \sum_{x \in \mathcal{V} \cup \mathcal{F}} \deg_A(x) = 2|E(A)|.$$

As $|E(A)| \leq 2n$, Gabow's algorithm computes a B-factor of A in time $\mathcal{O}(n^{3/2})$. □

Remark 4.2 The Generalized Factor instance we constructed in the proof above can be reduced to an instance of Maximum Flow. In theory,[4] this achieves a better, namely almost-linear [6], runtime. The graph of the flow instance is obtained by adding two vertices *s* and *t* (namely the source and sink) and connecting *s* to all vertices in $\mathcal{V}$ and *t* to all vertices in $\mathcal{F}$. Edges incident to *s* are outgoing, edges incident to *t* are incoming. Edges between $\mathcal{V}$ and $\mathcal{F}$ are oriented from $\mathcal{V}$ to $\mathcal{F}$. The edge capacities of edges *sx* with $x \in \mathcal{V}$ encode the sets $B(x) = \{0, 1, 2, \ldots, 3 - \deg_G(x)\}$. This is achieved by setting the edge capacity to $3 - \deg_G(x)$. All other edges have a capacity of 1. It can be easily verified that the obtained graph admits an *s*–*t*-flow of value at least $|\mathcal{F}|$ if and only if A admits a B-factor.

5 Connected and 2-Connected 3-Augmentations for Variable Embeddings

Here we consider the 3-augmentation problem for an arbitrary input graph *G* and a connected or 2-connected output graph *H* in the variable embedding setting, corresponding to the second and third column of the table in Fig. 1 respectively. We present a linear-time algorithm for connected output and a quadratic-time algorithm for 2-connected output, which yields Statement 2 and 4 of Theorem 1.2.

In fact, we are mainly concerned with finding 2-connected 3-augmentations, i.e., Statement 4. Our task boils down to finding a suitable planar embedding of *G* such that for each vertex *v* of *G* and each missing edge at *v*, we can assign an incident face

[4] The algorithm in [6] is randomized and not practical. Our combinatorial, but asymptotically slower, algorithm is preferable here.

at v that should contain the new edge. Let us note that this might only work for some planar embeddings of G. See Fig. 9b for a negative example.

We show Statement 4 of Theorem 1.2 in three steps. First, we show that for G to admit a 2-connected 3-augmentation we may restrain ourselves to all inclusion-maximal 2-connected components, called *blocks*, of G, and checking whether each admits a 2-connected 3-augmentation. As all blocks can be found in linear time [32], we may assume henceforth that the input graph G is 2-connected. Second, we generalize Theorem 3.1 to multi-graphs, i.e., we present a polynomial-time algorithm that given a planar multi-graph G with a fixed planar embedding $\mathcal{E}$ tests whether G admits a 2-connected 3-augmentation $H \supseteq G$ with a planar embedding whose restriction to G equals $\mathcal{E}$. Finally, for a 2-connected graph G with variable embedding, we use an SPQR-tree of G to efficiently go through the possible planar embeddings of G with a dynamic program and to identify one such embedding that allows for a 2-connected 3-augmentation, or conclude that no such exists.

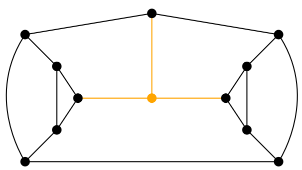
The next proposition settles the 3-augmentation problem for connected output, and yields a reduction to 2-connected input for the decision problem with 2-connected output.

Proposition 5.1 *For a disconnected subcubic graph G with connected components $G_1, \ldots, G_\ell$, with $\ell \geq 2$, we have that*

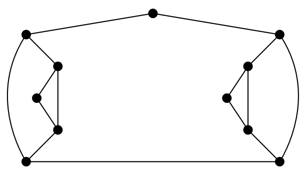
(i) *G has a connected 3-augmentation if and only if no G_i is 3-regular*
(ii) *G has a 2-connected 3-augmentation if and only if each G_i has a 2-connected 3-augmentation and no G_i is 3-regular.*

Proof Let $G_1, \ldots, G_\ell$ denote the components of G.

(i) Suppose there exists a connected 3-augmentation H of G. Note that H is also a connected 3-augmentation of each G_i. As H is connected, each G_i has a vertex incident to at least one edge in $E(H) - E(G)$, i.e., no G_i is 3-regular. Suppose now that no G_i is 3-regular. By Observation 1.3, each G_i admits a connected 3-augmentation H_i. Note that each H_i contains at least one new edge e_i. Consider an embedding of $H_1 \cup \cdots \cup H_\ell$ where each e_i lies on the outer face. Finally, add an augmentation gadget A with 2ℓ attachments into the outer face, delete $e_1, \ldots, e_\ell$, and connect the 2ℓ degree-2 vertices of $H_1 \cup \cdots \cup H_\ell$ with the



(a) A planar graph G_1 with a 2-connected 3-augmentation H_1 (orange).



(b) A different planar embedding of G_1 that does not allow for a 2-connected 3-augmentation.

Fig. 9 Example instances for the 2-connected 3-augmentation problem

2ℓ degree-2 vertices of A by a non-crossing matching. The resulting graph is a connected 3-augmentation of G.

(ii) Again, any 2-connected 3-augmentation of G is in particular a 2-connected 3-augmentation of each G_i. If such an augmentation of G exists, no G_i is 3-regular. On the other hand, if each G_i admits a 2-connected 3-augmentation H_i and no G_i is 3-regular, the same construction as in the proof above yields a connected 3-augmentation H of G. Note that H is bridgeless. As a graph of maximum degree 3 is 2-connected if it is connected and bridgeless, the claim follows. □

The first claim of Proposition 5.1 yields a linear-time algorithm for arbitrary input and connected output in the variable embedding setting. This corresponds to Statement 2 of Theorem 1.2.

The remainder of Sect. 5 only considers 2-connected output. The second claim of above proposition shows that we may assume that the input is connected. By the following proposition, we may even assume that it is 2-connected.

Proposition 5.2 *A connected subcubic graph G admits a 2-connected 3-augmentation if and only if each block of G admits a 2-connected 3-augmentation.*

Proof Suppose G contains no bridge. As G is connected and subcubic, it follows that G is 2-connected, i.e. the only block of G corresponds to the whole graph and the claim follows.

We may therefore assume that G contains a bridge $e = uv$. Let G_1 be the connected component of $G - e$ containing u, and let $G_2 = G - G_1$ denote the remaining graph. It is enough to show that if G_1 and G_2 have 2-connected 3-augmentations H_1 respectively H_2, then G also has a 2-connected 3-augmentation. To this end, consider an edge $e_1 \in E(H_1) - E(G_1)$ incident to u and an edge $e_2 \in E(H_2) - E(G_2)$ incident to v. These edges exist as $\deg_{G_1}(u), \deg_{G_2}(v) \leq \Delta(G) - 1 \leq 2$ but $\deg_{H_1}(u) = \deg_{H_2}(v) = 3$. Choose a planar embedding of $H_1 \cup H_2$ with e_1 and e_2 being outer edges. Denoting by a, b the endpoints of e_1, e_2 different from u, v, we see that $(H_1 - e_1) \cup (H_2 - e_2) \cup \{uv, ab\}$ is a connected 3-augmentation of G with no bridges, i.e., a 2-connected 3-augmentation. □

5.1 The Fixed Embedding Setting for Multigraphs

In order to settle the variable embedding setting for 2-connected output, we use the algorithm for fixed embedding as a crucial subroutine. Yet, the embedded graphs we consider might not be simple. The following lemma yields a linear-time reduction from multi-graphs to simple graphs. Its proof is based on the gadget represented Figure 5c we used in the proof of Lemma 3.2 to avoid parallel edges.

Lemma 5.3 *Let G be an n-vertex planar multi-graph of maximum degree $\Delta(G) \leq 3$ with an embedding $\mathcal{E}$. There is a simple planar graph G' with $\Delta(G') \leq 3$ on at most $13n$ vertices with an embedding $\mathcal{E}'$ such that G has a 2-connected 3-augmentation extending $\mathcal{E}$ if and only if G' has one extending $\mathcal{E}'$.*

Proof Replacing parallel edges of G with the gadget represented in Fig. 5c yields a simple planar graph G' of maximum degree at most 3 whose embedding $\mathcal{E}'$ is closely related to $\mathcal{E}$. As every vertex of G is incident to at most three edges, we have $|E(G)| \leq 3n$. We introduced at most four new vertices for each edge of G, thus $|V(G')| \leq |V(G)| + 4|E(G)| \leq 13n$. Note that a 2-connected 3-augmentation of G extending $\mathcal{E}$ yields one of G' extending $\mathcal{E}'$ and vice versa. □

We can therefore generalize Theorem 3.1 to multi-graphs.

Corollary 5.4 *Let G be an n-vertex planar multi-graph of maximum degree $\Delta(G) \leq 3$ with a fixed planar embedding $\mathcal{E}$. Then we can compute in time $\mathcal{O}(n^2)$ a 2-connected 3-augmentation H of G with a planar embedding $\mathcal{E}_H$ whose restriction to G equals $\mathcal{E}$, or conclude that no such 3-augmentation exists.*

5.2 2-Connected 3-Augmentations for 2-Connected Input and Variable Embeddings

Even an unlabeled 2-connected subcubic planar graph G can have exponentially many different planar embeddings (e.g., the $(2 \times n)$-grid graph). Thus, iterating over all embeddings of G and applying the algorithm from Theorem 3.1 to each of them is not a polynomial-time algorithm and is hence not a feasible approach for us. In this section we describe how to use the SPQR-tree of G to efficiently find a planar embedding $\mathcal{E}$ of G such that there is a 3-augmentation H of G extending $\mathcal{E}$, or conclude that no such embedding exists. The algorithm from Proposition 5.4 will be an important subroutine. We show the following.

Proposition 5.5 *Let G be an n-vertex 2-connected subcubic planar graph. Then we can compute in time $\mathcal{O}(n^2)$ a 2-connected 3-augmentation H of G or conclude that no such H exists.*

Together with Proposition 5.1 and Proposition 5.2, this yields Statement 4 of Theorem 1.2.

Overview. The proof of Proposition 5.5 uses a bottom-up dynamic programming approach on the SPQR-tree T of G rooted at a Q-vertex ρ corresponding to some edge e_ρ in G. Consider a vertex $\mu \neq \rho$ in T. Let uv be the virtual edge in $\operatorname{skel}(\mu)$ that is associated with the parent edge of μ.

Recall that each embedding $\mathcal{E}$ of G with e_ρ on the outer face, when restricted to the pertinent graph $\operatorname{pert}(\mu)$, gives an embedding $\mathcal{E}_\mu$ of $\operatorname{pert}(\mu)$ whose inner faces are also inner faces of $\mathcal{E}$, and with u and v being outer vertices of $\mathcal{E}_\mu$. The outer face of $\mathcal{E}_\mu$ is composed of two (not necessarily edge-disjoint) u-v-paths; the left and right outer path of $\mathcal{E}_\mu$, which are contained in the left and right outer face of $\mathcal{E}_\mu$ inside $\mathcal{E}$, respectively; see Fig. 10 for an example. We seek to partition the (possibly exponentially many) planar embeddings of $\operatorname{pert}(\mu)$ with u, v on its outer face into a constant number of equivalence classes based on how many edges in a 2-connected 3-augmentation of G could possibly "connect" $\operatorname{pert}(\mu)$ with the rest of the graph G

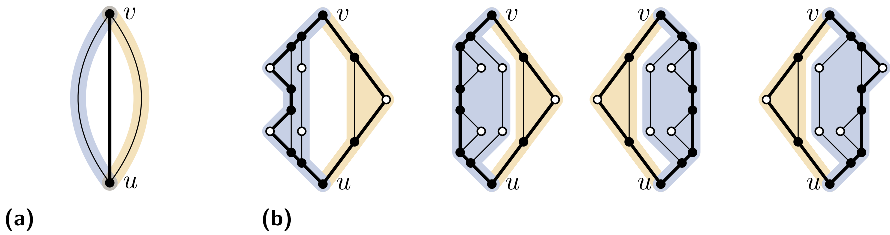


Fig. 10 **a** The skeleton of a P-vertex μ with its virtual edge *uv* (thick). **b** Four of the 32 embeddings of E_μ with its left and right outer *uv*-path (thick)

inside the left or right outer face of $\mathcal{E}_\mu$ inside $\mathcal{E}$. This corresponds[5] to the number of degree-2 vertices on the left and right side in so-called *inner augmentations* of $\mathcal{E}_\mu$. Loosely speaking, it will be enough for us to distinguish three cases for the left side (0, 1, or at least 2 connections), the symmetric three cases for the right side, and to record which of the nine resulting combinations are possible. Note that this grouping of embeddings of $\mathcal{E}_\mu$ into constantly many classes is the key insight that allows an efficient dynamic program.

Whether a particular equivalence class is realizable by some planar embedding $\mathcal{E}_\mu$ of $\mathrm{pert}\,(\mu)$ will depend on the vertex type of μ (S-, P- or R-vertex) and the realizable equivalence classes of its children $\mu_1, \ldots, \mu_k$. In the end, we shall conclude that the whole graph *G* has a 2-connected 3-augmentation if and only if for the unique child μ of the root ρ of *T* the equivalence class of embeddings of $\mathrm{pert}\,(\mu)$ for which neither the left nor the right side has any connections is non-empty.

Most of our arguments are independent of SPQR-trees and we instead consider so-called *uv*-graphs, which are slightly more general than pertinent graphs. We shall introduce inner augmentations of *uv*-graphs, which then give rise to label sets for *uv*-graphs, both in a fixed and variable embedding setting. These label sets encode the aforementioned number of connections between the *uv*-graph as a subgraph of *G* and the rest of *G* in a potential 2-connected 3-augmentation. After showing that we can compute variable label sets by resorting to the fixed embedding case and Proposition 5.4, we present the final dynamic program along the rooted SPQR-tree *T* of *G*.

uv -Graphs and Labels.A *uv-graph* is a connected multigraph G_{uv} with $\Delta(G_{uv}) \leq 3$, two distinguished vertices *u*, *v* of degree at most 2, together with a planar embedding $\mathcal{E}_{uv}$ such that *u* and *v* are outer vertices. A connected multigraph $H_{uv} \supseteq G_{uv}$ with planar embedding $\mathcal{E}_H$ is an *inner augmentation* of G_{uv} if

- $\mathcal{E}_H$ extends $\mathcal{E}_{uv}$ and has *u*, *v* on its outer face,
- each of *u*, *v* has the same degree in H_{uv} as in G_{uv},
- every vertex of H_{uv} except for *u*, *v* has degree 1 or 3,
- every degree-1 vertex of H_{uv} lies in the outer face of $\mathcal{E}_H$ and
- every bridge of H_{uv} that is not a bridge of G_{uv} is incident to a degree-1 vertex.

[5] up to the fact that left and right outer path may share degree-2 vertices, each of which sends however its third edge into only one of the left and right outer face.

See Fig. 11 for an example. Because u, v are outer vertices in $\mathcal{E}_H$, one could add another edge e_{uv} (oriented from u to v) into the outer face of $\mathcal{E}_H$ preserving planarity (this edge is not part of the inner augmentation). Then e_{uv} splits the outer face into two faces f_A, f_B that are to the left and to the right of e_{uv}, respectively, when going from u to v.

Each degree-1 vertex of H_{uv} now lies either inside f_A or f_B.

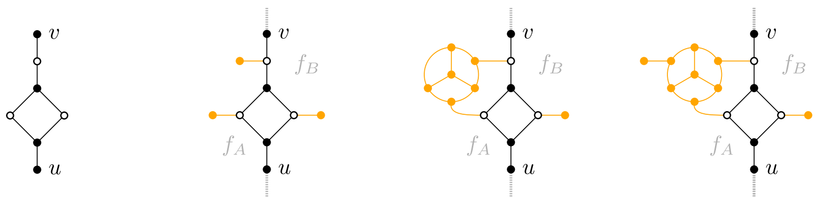
We are interested in the number of degree-1 vertices in each of these faces of $\mathcal{E}_H$ and write $d(H_{uv}, \mathcal{E}_H) = (a, b)$ if an inner augmentation H_{uv} of G_{uv} has exactly a degree-1 vertices inside f_A and exactly b degree-1 vertices inside f_B.

Lemma 5.6 *Let H_{uv} be an inner augmentation of G_{uv} with $d(H_{uv}, \mathcal{E}_H) = (a, b)$. If $a \geq 2$, then G_{uv} has an inner augmentation H^0_{uv} with $d(H^0_{uv}, \mathcal{E}^0_H) = (0, b)$ and an inner augmentation H^1_{uv} with $d(H^1_{uv}, \mathcal{E}^1_H) = (1, b)$. A symmetric statement holds when $b \geq 2$.*

Proof Add the edge uv to the inner augmentation H_{uv} such that there are a degree-1 vertices in f_A. We add an augmentation gadget with a attachments into f_A and identify the a degree-2 vertices of the gadget with the a degree-1 vertices in f_A in a non-crossing way. Ignoring edge uv, the obtained graph is the desired inner augmentation H^0_{uv} with $d(H^0_{uv}, \mathcal{E}^0_H) = (0, b)$; see Figs. 11b, c for an example. We obtain H^1_{uv} by additionally subdividing an edge of the augmentation gadget that is incident to f_A once and by attaching a degree-1 vertex to it into f_A; see Figs. 11b, d. □

Motivated by Lemma 5.6, we focus on inner augmentations H_{uv} with $d(H_{uv}, \mathcal{E}_H) = (a, b)$ where $a, b \in \{0, 1\}$, and assign to H_{uv} in this case the *label* ab with $\mathrm{a}, \mathrm{b} \in \{0, 1\}$.

The *embedded label set* $L_{\mathrm{emb}}(G_{uv}, \mathcal{E}_{uv})$ contains all labels ab such that there is an inner augmentation H_{uv} of G_{uv} with label ab. Allowing other planar embeddings of G_{uv}, we further define the *variable label set* as $L_{\mathrm{var}}(G_{uv}) = \bigcup_{\mathcal{E}} L_{\mathrm{emb}}(G_{uv}, \mathcal{E})$, where $\mathcal{E}$ runs over all planar embeddings of G_{uv} where u and v are outer vertices. As this in particular includes for each embedding $\mathcal{E}$ of G_{uv} also the flipped embedding $\mathcal{E}'$ of G_{uv}, it follows that $\mathrm{ab} \in L_{\mathrm{var}}(G_{uv})$ if and only if $\mathrm{ba} \in L_{\mathrm{var}}(G_{uv})$. Whenever this property holds for a (variable or embedded) label set, we call the label set *symmetric*. Hence, all variable label sets are symmetric, but embedded label sets may or may not be symmetric.



(a) **(b)** $d(H_{uv}, \mathcal{E}_H) = (2, 1)$ **(c)** $d(H_{uv}, \mathcal{E}_H) = (0, 1)$ **(d)** $d(H_{uv}, \mathcal{E}_H) = (1, 1)$

Fig. 11 **a** A uv-graph G_{uv}. **b–d** Inner augmentations H_{uv} (orange) of G_{uv} with an embedding $\mathcal{E}_H$ where $d(H_{uv}, \mathcal{E}_H)$ is equal to (2, 1), (0, 1), (1, 1) respectively

For brevity, let us use $\star$ as a wildcard character, in the sense that if $\{\mathrm{x}0, \mathrm{x}1\}$ is in an embedded or variable label set for some $\mathrm{x} \in \{0, 1\}$, then we shorten the notation and replace them by a label $\mathrm{x}\star$. Symmetrically, we use the notation $\star\mathrm{x}$ and in particular define $\{\star\star\} := \{00, 01, 10, 11\}$. Using this notation, the eight possible symmetric label sets are:

$$\emptyset, \{00\}, \{01, 10\}, \{11\}, \{0\star, \star 0\}, \{00, 11\}, \{1\star, \star 1\}, \{\star\star\} \tag{1}$$

The following lemma reveals the significance of inner augmentations and label sets.

Lemma 5.7 *Let G be a 2-connected graph with $\Delta(G) \leq 3$ and $\mathcal{E}$ be an embedding of G with some outer edge $e = xy$. Further, let G_{uv} be the uv-graph obtained from G by deleting e and adding two new vertices u, v with edges ux and vy into the outer face of $\mathcal{E}$. Then G has a 2-connected 3-augmentation if and only if* $00 \in L_{\mathrm{var}}(G_{uv})$.

Proof First let $H \supseteq G$ be a 2-connected 3-augmentation of G and let $\mathcal{E}_H$ be an embedding of H with $e = xy$ being an outer edge. Then deleting e and adding two new vertices *u*, *v* with edges *ux* and *vy* into the outer face of $\mathcal{E}_H$ results in an inner augmentation H_{uv} of G_{uv} with respect to the embedding of G_{uv} inherited from $\mathcal{E}_H$. As adding an edge e_{uv} from *u* to *v* into H_{uv} gives a graph with no degree-1 vertices, we have $00 \in L_{\mathrm{var}}(G_{uv})$.

Conversely, assume that $00 \in L_{\mathrm{var}}(G_{uv})$. Then there is an embedding $\mathcal{E}_{uv}$ of G_{uv} that allows for some inner augmentation H_{uv} with embedding $\mathcal{E}_H$ for which $H_{uv} + e_{uv}$ has no degree-1 vertices, where $e_{uv} = uv$ denotes a new edge between *u* and *v*. Thus, in H_{uv} the vertices *u* and *v* have degree 1 (as in G_{uv}), every vertex of H_{uv} except *u*, *v* has degree 3, and the only bridges of H_{uv} are the edges *ux* and *vy*. Then we obtain a 2-connected 3-augmentation *H* of *G* by removing *ux*, *vy* from H_{uv} and adding the edge *xy* into the outer face of $\mathcal{E}_H$. In case, H_{uv} already contains the edge *xy*, this is replaced by a copy of $K_4^{(2)}$ with two non-crossing edges between *x*, *y* and the two degree-2 vertices of $K_4^{(2)}$. $\square$

Gadgets. In our algorithm below, we aim to replace certain *uv*-graphs X (with variable embedding) by *uv*-graphs *Y* with fixed embedding $\mathcal{E}_Y$, such that the variable label set $L_{\mathrm{var}}(X)$ equals the embedded label set $L_{\mathrm{emb}}(Y, \mathcal{E}_Y)$. This will allow us to use Proposition 5.4 from the fixed embedding setting as a subroutine.

The following lemma describes seven *uv*-graphs, each with a fixed embedding, corresponding to the seven different non-empty variable label sets as given in (1). For this purpose, each such *gadget* is itself a *uv*-graph *Y* with a fixed embedding $\mathcal{E}_Y$.

Lemma 5.8 *For every uv-graph G_{uv} with $L_{\mathrm{var}}(G_{uv}) \neq \emptyset$ there exists a gadget Y with an embedding $\mathcal{E}_Y$ such that u, v are outer vertices and $L_{\mathrm{emb}}(Y, \mathcal{E}_Y) = L_{\mathrm{var}}(G_{uv})$.*

Proof We distinguish the seven cases of what $L_{\mathrm{var}}(G_{uv})$ is according to (1). In each of our gadgets, which are shown in Fig. 12, vertices *u* and *v* are the only degree 1 vertices. For convenience, let us call the degree-2 vertices in the gadgets *Y* the *white*

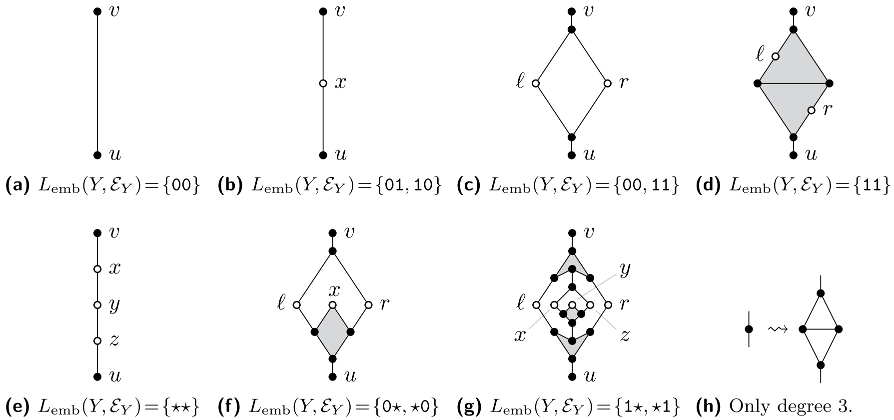


Fig. 12 **a**–**g** The seven gadgets for the seven non-empty variable label sets $L_{\mathrm{var}}(G_{uv})$ of a *uv*-graph G_{uv}. **h** Modification to locally replace a degree-2 vertex by four degree-3-vertices

vertices. Hence, each white vertex *w* (names as in Fig. 12) has exactly one new incident edge in an inner augmentation *H* of *Y*, and we denote this new edge by e_w. Observe that all gadgets, except for the one in Fig. 12e for label set $\{\star\star\}$, have at most one white vertex on either side (left or right) of the outer face of $\mathcal{E}_Y$. Hence whenever such a white vertex *w* has its new edge e_w in the outer face of $\mathcal{E}_Y$, then e_w is a bridge in *H* and its other endpoint has degree 1, thus counting towards $d(H,\mathcal{E}_H)$. Also for convenience, we call an inner face of $\mathcal{E}_Y$ that has 0 or 1 incident white vertex a *gray face*. Observe that gray faces contain no new edges of *H* as inner faces may not contain bridges of *H*. We proceed by going through the seven gadgets one-by-one. Note that in each case it is enough to check which of 00, 01, 10 and 11 are contained in a label set in order to determine it uniquely. For this we always let *H* be any inner augmentation of *Y*.

- Case $L_{\mathrm{var}}(G_{uv})=\{00\}$, i.e., $00\in L_{\mathrm{var}}(G_{uv})$ and $01,10,11\notin L_{\mathrm{var}}(G_{uv})$. Consider the gadget *Y* and its embedding $\mathcal{E}_Y$ represented in Fig. 12a. We have no white vertices and thus only one inner augmentation $H=Y$. Thus, $L_{\mathrm{emb}}(Y,\mathcal{E}_Y)=\{00\}$ follows.
- Case $L_{\mathrm{var}}(G_{uv})=\{01,10\}$, i.e., $01,10\in L_{\mathrm{var}}(G_{uv})$ and $00,11\notin L_{\mathrm{var}}(G_{uv})$. Consider the gadget *Y* and its embedding $\mathcal{E}_Y$ represented in Fig. 12b. The only white vertex *x* has its edge e_x in *H* in the outer face. Clearly there are exactly two possibilities; e_x on the left or on the right. Thus, we have $L_{\mathrm{emb}}(Y,\mathcal{E}_Y)=\{01,10\}$.
- Case $L_{\mathrm{var}}(G_{uv})=\{00,11\}$, i.e., $00,11\in L_{\mathrm{var}}(G_{uv})$ and $01,10\notin L_{\mathrm{var}}(G_{uv})$. Consider the gadget *Y* and its embedding $\mathcal{E}_Y$ represented in Fig. 12c. We have two white vertices ℓ and *r*. If one of e_ℓ, e_r lies in the inner face *f* of $\mathcal{E}_Y$, then the other also lies in *f*, as otherwise there would be a bridge of *H* in *f*. Thus either both edges e_ℓ and e_r are in *f* or none, so $01,10\notin L_{\mathrm{emb}}(Y,\mathcal{E}_Y)$. To obtain $00\in L_{\mathrm{emb}}(Y,\mathcal{E}_Y)$, add an edge $e=e_\ell=e_r$ between ℓ and *r* in *f*. To obtain $11\in L_{\mathrm{emb}}(Y,\mathcal{E}_Y)$, put e_ℓ, e_r into the outer face of $\mathcal{E}_Y$.
- Case $L_{\mathrm{var}}(G_{uv})=\{11\}$, i.e., $11\in L_{\mathrm{var}}(G_{uv})$ and $00,01,10\notin L_{\mathrm{var}}(G_{uv})$. Consider the gadget *Y* and its embedding $\mathcal{E}_Y$ represented in Fig. 12d. We have

two white vertices ℓ, r. Since both inner faces are gray, we have that e_ℓ and e_r lie on separate sides of the outer face and form bridges of *H*. Thus, there exists exactly one inner augmentation of *Y* for this embedding $\mathcal{E}_Y$ and we have $L_{\text{emb}}(Y, \mathcal{E}_Y) = \{11\}$.

- Case $L_{\text{var}}(G_{uv}) = \{\star\star\}$, i.e., $00, 01, 10, 11 \in L_{\text{var}}(G_{uv})$. Consider the gadget *Y* and its embedding $\mathcal{E}_Y$ represented in Fig. 12e. We have three white vertices *x*, *y* and *z*. To obtain $00 \in L_{\text{emb}}(Y, \mathcal{E}_Y)$, put a new vertex into the outer face of $\mathcal{E}_Y$ and connect it to *x*, *y*, *z* by the edges e_x, e_y, e_z, respectively. To obtain $01 \in L_{\text{emb}}(Y, \mathcal{E}_Y)$, add an edge $e = e_x = e_y$ between *x* and *y* in the outer face and put e_z into the outer face on the right. Symmetrically, we obtain $10 \in L_{\text{emb}}(Y, \mathcal{E}_Y)$. To obtain $11 \in L_{\text{emb}}(Y, \mathcal{E}_Y)$, put e_x into the outer face on the left, add a new vertex *w* into the outer face on the right, connect *w* to *y*, *z* by the edges e_y, e_z, respectively, and add a pendant edge at *w* into the outer face of the result.
- Case $L_{\text{var}}(G_{uv}) = \{0\star, \star 0\}$, i.e., $00, 01, 10 \in L_{\text{var}}(G_{uv})$ and $11 \notin L_{\text{var}}(G_{uv})$. Consider the gadget *Y* and its embedding $\mathcal{E}_Y$ represented in Fig. 12f. We have three white vertices ℓ, *x*, *r* and the edge e_x in *H* lies in the face *f* of $\mathcal{E}_Y$ with all three white vertices on its boundary (since e_x may not lie in the gray face). Since e_x is not a bridge, at least one of e_ℓ and e_r lies within *f* as well. Therefore, e_ℓ and e_r cannot both lie in the outer face of *Y*, so $11 \notin L_{\text{emb}}(Y, \mathcal{E}_Y)$ follows. To obtain $00 \in L_{\text{emb}}(Y, \mathcal{E}_Y)$, put a new vertex into *f* and connect it to ℓ, *x*, *r* by the edges e_ℓ, e_x, e_r, respectively. To obtain $10 \in L_{\text{emb}}(Y, \mathcal{E}_Y)$, put e_ℓ into the outer face of $\mathcal{E}_Y$, and add an edge $e = e_x = e_r$ between *x* and *r* in *f*. Symmetrically, we obtain that $01 \in L_{\text{emb}}(Y, \mathcal{E}_Y)$.
- Case $L_{\text{var}}(G_{uv}) = \{1\star, \star 1\}$, i.e., $01, 10, 11 \in L_{\text{var}}(G_{uv})$ and $00 \notin L_{\text{var}}(G_{uv})$. Consider the gadget *Y* and its embedding represented in Fig. 12g. We have five white vertices ℓ, *x*, *y*, *z*, and *r*. Let *f* be the inner face of $\mathcal{E}_Y$ with *x*, *y* and *z* on its boundary. As the other face at *y* is gray, edge e_y lies in *f*. Now suppose that none of e_ℓ, e_r lie in the outer face of $\mathcal{E}_Y$. As then e_ℓ is not a bridge, e_x lies not in *f*. Similarly, as e_r is then not a bridge, e_z lies not in *f*. But then e_y is a bridge in the inner face *f*, which is impossible for an inner augmentation. Hence $00 \notin L_{\text{emb}}(Y, \mathcal{E}_Y)$. To obtain $01 \in L_{\text{emb}}(Y, \mathcal{E}_Y)$, put e_r into the outer face of $\mathcal{E}_Y$, add an edge $e = e_y = e_z$ between *y* and *z* into *f*, and an edge $e' = e_\ell = e_x$ between ℓ and *x* into their common inner face in $\mathcal{E}_Y$. Symmetrically, we obtain $10 \in L_{\text{emb}}(Y, \mathcal{E}_Y)$. Finally, to obtain $11 \in L_{\text{emb}}(Y, \mathcal{E}_Y)$, put both e_ℓ and e_r into the outer face of $\mathcal{E}_Y$, add a new vertex into *f* and connect it to *x*, *y*, *z* by edges e_x, e_y, e_z, respectively. □

We remark that Fig. 12h is not a gadget, and instead a local modification that we use at other places, such as in the proof of Lemma 5.9.

Computing a Label Set. In our algorithm below we want to compute the variable label sets of $\operatorname{pert}(\mu)$ for vertices μ of the rooted SPQR-tree *T* of *G*. As we will see, we can reduce this to a constant number of computations of embedded label sets of certain *uv*-graphs that are specifically crafted to encode all the possible embeddings of $\operatorname{pert}(\mu)$. The following lemma describes how to do this.

Lemma 5.9 *Let G_{uv} be an n-vertex uv-graph and $\mathcal{E}_{uv}$ a planar embedding where u and v are outer vertices. Then we can check each of the following in time $\mathcal{O}(n^2)$:*

- Whether $00 \in L_{\text{emb}}(G_{uv}, \mathcal{E}_{uv})$.
- Whether $01 \in L_{\text{emb}}(G_{uv}, \mathcal{E}_{uv})$ or $10 \in L_{\text{emb}}(G_{uv}, \mathcal{E}_{uv})$.
- Whether $11 \in L_{\text{emb}}(G_{uv}, \mathcal{E}_{uv})$.

In particular, if $L_{\text{emb}}(G_{uv}, \mathcal{E}_{uv})$ is symmetric, then this is sufficient to determine the exact embedded label set $L_{\text{emb}}(G_{uv}, \mathcal{E}_{uv})$.

Proof The eight possible symmetric label sets in (1) correspond bijectively to the eight possible yes-/no-answer combinations of the above three checks. Hence these checks are sufficient to determine $L_{\text{emb}}(G_{uv}, \mathcal{E}_{uv})$, provided it is symmetric.

- To check whether $00 \in L_{\text{emb}}(G_{uv}, \mathcal{E}_{uv})$, add an edge e_{uv} between *u*, *v* into the outer face of $\mathcal{E}_{uv}$. If afterwards *u* and/or *v* has degree 2, then we further replace them with degree-3 vertices using the gadget shown in Fig. 12h. We call the obtained embedded planar multigraph G^+_{uv}. We claim that $00 \in L_{\text{emb}}(G_{uv}, \mathcal{E}_{uv})$ if and only if G^+_{uv} has a 2-connected 3-augmentation extending its planar embedding. Indeed, removing edge e_{uv} from any such 3-augmentation (and possibly undoing the replacements of *u*, *v*) yields an inner augmentation H_{uv} of G_{uv} with label 00. On the other hand, an inner augmentation H_{uv} with label 00 that extends $\mathcal{E}_{uv}$ and an additional edge between *u*, *v* is a 2-connected 3-augmentation, except that *u*, *v* might still have degree 2 (they have degree 1 or 2 in G_{uv} which is increased by one by e_{uv}). In that case, replace them by the gadget shown in Fig. 12h to obtain a 2-connected 3-augmentation of G^+_{uv}.
- Similarly, to check whether 01 or 10 is in $L_{\text{emb}}(G_{uv}, \mathcal{E}_{uv})$, we add a path of length two between *u*, *v* into the outer face of $\mathcal{E}_{uv}$. Let *w* be the middle vertex of this path. If afterwards *u* and/or *v* have degree 2, we replace them with degree-3 vertices using the gadget from Fig. 12h. Call the obtained planar graph G^+_{uv}. We claim that $\{01, 10\} \cap L_{\text{emb}}(G_{uv}, \mathcal{E}_{uv}) \neq \emptyset$ if and only if G^+_{uv} has a 2-connected 3-augmentation extending its planar embedding. If such a 3-augmentation exists, then removing edges *uw* and *vw* (and possibly undoing the replacements of *u*, *v*) yields an inner augmentation H_{uv} of G_{uv} where *w* is the only degree-1 vertex (and in the outer face of H_{uv}). It follows that H_{uv} has label 01 or 10. For the other direction, assume that 01 or 10 is in $L_{\text{emb}}(G_{uv}, \mathcal{E}_{uv})$. Then there is an inner augmentation H_{uv} of G_{uv} with label 01 or 10. In particular there is a degree-1 vertex *w* in the outer face of H_{uv}. Now add edges *uw* and *vw*, and replace each of *u*, *v* that has degree 2 by the gadget from Fig. 12h. This yields a 2-connected 3-augmentation of G^+_{uv}.
- Lastly, to check whether $11 \in L_{\text{emb}}(G_{uv}, \mathcal{E}_{uv})$, we build a graph G^+_{uv} by taking the *uv*-graph from Fig. 12d, embedding it into the outer face of $\mathcal{E}_{uv}$ and identifying the respective *u* and *v* vertices. If afterwards *u*, *v* have degree 2, we replace them by degree-3 vertices using the gadget from Fig. 12h. Let ℓ, r be the two degree-2 vertices as shown in Fig. 12d. Again, we claim that $11 \in L_{\text{emb}}(G_{uv}, \mathcal{E}_{uv})$ if and only if G^+_{uv} has a 2-connected 3-augmentation extending its embedding.

If such a 3-augmentation exists, removing all vertices from the *uv*-graph in Fig. 12d except for u, v, ℓ, r (and possibly undoing the replacements of *u*, *v*) yields an inner augmentation H_{uv} with label 11 (because there are two degree-1 vertices and a *uv*-edge in the outer face of the embedding of H_{uv} would separate them into different faces). To construct a 2-connected 3-augmentation from an inner augmentation H_{uv} of G_{uv} with label 11, consider a hypothetical *uv*-edge separating the two degree-1 vertices of H_{uv} into different faces. Add the graph from Fig. 12d as above in the embedding of H_{uv} where the *uv*-edge would have been. Then identify the two degree-1 vertices with ℓ and *r* in a non-crossing way. This yields a 3-augmentation of G_{uv} after possibly replacing *u* and/or *v* by the gadget from Fig. 12h, as in the previous cases.

In all three cases, the existence of a 2-connected 3-augmentation of G_{uv}^+ can be checked using Proposition 5.4. As only a constant number of vertices and edges were added to G_{uv} this check takes time in $\mathcal{O}(n^2)$. □

- We remark that in the proof of Lemma 5.9 for each label $\mathrm{ab} \in \{00, 01, 11\}$ we actually added the gadget for the embedded label set $\{\mathrm{ab}, \mathrm{ba}\}$ between vertices *u*, *v* (namely the gadgets represented in (a), (b) and (d) of Fig. 12 respectively). Thus, these three gadgets serve a twofold role in our algorithm. For a label $\mathrm{ab} \in \{00, 01, 11\}$they permit to check for a given embedding $\mathcal{E}_X$ of a *uv*-graph X whether $\mathrm{ab} \in L_{\mathrm{emb}}(X, \mathcal{E}_X)$ (cf. Lemma 5.9),
- they provide a *uv*-graph Y with an embedding $\mathcal{E}_Y$ such that $L_{\mathrm{emb}}(Y, \mathcal{E}_Y) = \{\mathrm{ab}, \mathrm{ba}\}$ (cf. Lemma 5.8).**Algorithm for Variable Embedding.** In order to decide whether a given 2-connected planar graph *G* admits some planar embedding which admits a 2-connected 3-augmentation, we use the SPQR-tree T of *G*. Rooting T at some Q-vertex ρ, the pertinent graph $\mathrm{pert}\,(\mu)$ of a vertex μ in *T* is a subgraph of *G*. Moreover, if $u_\mu v_\mu$ is the virtual edge in $\mathrm{skel}\,(\mu)$ associated with the parent edge of μ, then $\mathrm{pert}\,(\mu)$ is a *uv*-graph (with u_μ, v_μ taking the roles of *u*, *v* in the *uv*-graph). Now the variable label set $L_{\mathrm{var}}(\mathrm{pert}\,(\mu))$ is a constant-size representation of all possible labels that any possible embedding of an inner augmentation of $\mathrm{pert}\,(\mu)$ can have (having u_μ and v_μ on its outer face). Determining whether *G* admits a 2-connected 3-augmentation now boils down to computing $L_{\mathrm{var}}(\mathrm{pert}\,(\mu))$ of the unique child μ of the root ρ.

Lemma 5.10 *Let μ be the unique child of the root ρ. Given $L_{\mathrm{var}}(\mathrm{pert}\,(\mu))$, we can decide in constant time whether G admits a 2-connected 3-augmentation.*

Proof Recall that $\mathrm{pert}\,(\rho) = G$. Following the setup of Lemma 5.7, let *x*, *y* be the two unique vertices of $\mathrm{skel}\,(\rho)$ and *xy* be the unique non-virtual edge, i.e., the edge $e_\rho = xy$ of *G*. Let G_{uv} be the *uv*-graph obtained from $G = \mathrm{pert}\,(\rho)$ by deleting $e_\rho = xy$ and adding two new pendant edges *ux*, *vy*. Note that *x* and *y* have the same degree in G_{uv} as in *G*. By Lemma 5.7, *G* has a 2-connected 3-augmentation if and only if $00 \in L_{\mathrm{var}}(G_{uv})$.

To check whether $00 \in L_{\mathrm{var}}(G_{uv})$, note that $\mathrm{pert}(\mu) = G - e_\rho$. In fact, G admits no 2-connected 3-augmentation if one of the variable label sets $L_{\mathrm{var}}(\mathrm{pert}(\mu))$ is empty (cf. Lemma 5.13). That is, we may assume that $\mathrm{pert}(\mu) = G - e_\rho$. Consider the gadget Y with embedding $\mathcal{E}_Y$ from Lemma 5.8 such that $L_{\mathrm{emb}}(Y, \mathcal{E}_Y) = L_{\mathrm{var}}(\mathrm{pert}(\mu))$. Let u' and v' denote the two degree-1 vertices in Y. If both x and y have degree 3 in G (hence also in G_{uv}), then $L_{\mathrm{var}}(G_{uv}) = L_{\mathrm{var}}(\mathrm{pert}(\mu)) = L_{\mathrm{emb}}(Y, \mathcal{E}_Y)$ and we already know whether or not 00 is contained in these label sets.

If x has degree 2 in G (hence also degree 2 in G_{uv}, while degree 1 in $\mathrm{pert}(\mu)$), then x receives a new edge in inner augmentations of G_{uv} but not in inner augmentations of $\mathrm{pert}(\mu)$. For Y to model $L_{\mathrm{var}}(G_{uv})$ instead of $L_{\mathrm{var}}(\mathrm{pert}(\mu))$, we subdivide in Y the edge at u' by a new vertex x'. Similarly, if y has degree 2 in G, we subdivide in Y the edge at v'. For the resulting graph Y' with embedding $\mathcal{E}_{Y'}$ it follows that $L_{\mathrm{var}}(G_{uv}) = L_{\mathrm{emb}}(Y', \mathcal{E}_{Y'})$ and we can check whether 00 is contained in these label sets by calling Lemma 5.9 on Y' with embedding $\mathcal{E}_{Y'}$. This takes constant time, as Y' has constant size. □

The remainder of this section describes how the variable label sets of all vertices in the SPQR-tree can be computed by a bottom-up dynamic program. Here we need to distinguish whether we consider a leaf or an inner vertex of the SPQR-tree.

Lemma 5.11 *The variable label set of each leaf μ of the SPQR-tree is $L_{\mathrm{var}}(\mathrm{pert}(\mu)) = \{00\}$.*

Proof Each leaf of the SPQR-tree is a Q-vertex. That is, the pertinent graph $\mathrm{pert}(\mu)$ is just a single edge and the only inner augmentation of $\mathrm{pert}(\mu)$ is $\mathrm{pert}(\mu)$ itself, and as such has label 00. □

Lemma 5.12 *Let μ be an inner vertex of the SPQR-tree with children $\mu_1, \dots, \mu_k$, such that each variable label set $L_{\mathrm{var}}(\mathrm{pert}(\mu_i))$, $i = 1, \dots, k$, is non-empty and known. Then the variable label set $L_{\mathrm{var}}(\mathrm{pert}(\mu)))$ can be computed in time $\mathcal{O}(\|\mathrm{skel}\,\mu\|^2)$ if μ is an S-, R- or P-vertex.*

Proof Let *uv* be the virtual edge associated with the parent edge of μ. Further, let $u_i v_i$ be the virtual edge associated with the tree edge $\mu\mu_i$, $i = 1, \dots, k$. We remove the edge *uv* in $\mathrm{skel}(\mu)$ to obtain a *uv*-graph G_{uv}.

We first observe that G_{uv} has a unique embedding (up to flipping) with *u* and *v* on the outer face.

- If μ is an R-vertex, its skeleton $\mathrm{skel}(\mu)$ is 3-connected. So by Whitney's Theorem [35] $\mathrm{skel}(\mu)$ has a unique planar embedding up to flipping and the choice of the outer face. Only in two of these embeddings (up to flipping) is the virtual edge *uv* incident to the outer face. Both induce the same planar embedding of G_{uv} with *u* and *v* on the outer face. Due to the one-to-one-correspondence of embeddings of G_{uv} and $\mathrm{skel}(\mu)$, we observe that there is (up to flipping) only one embedding of G_{uv} with *u* and *v* on the outer face.

- If μ is an S-vertex, G_{uv} is a path and its planar embedding is unique.
- If μ is a *P*-vertex, $\operatorname{skel}(\mu)$ consists of two vertices and at least three parallel edges. Since $\Delta(G) \leq 3$, we have that $\operatorname{skel}(\mu)$ contains exactly three parallel edges. Thus, G_{uv} contains at most two parallel edges and has a unique embedding (up to flipping).

Since variable label sets are symmetric, we can ignore the flipped embedding of G_{uv}.

Replace each virtual edge $u_i v_i$ by the gadget *Y* with embedding $\mathcal{E}_Y$ from Lemma 5.8 that realizes the embedded label set $L_{\text{emb}}(Y, \mathcal{E}_Y) = L_{\text{var}}(\operatorname{pert}(\mu_i))$. Call the obtained graph G_μ.

If μ is an S-vertex, we need to handle the vertices in G_μ that belong to $\operatorname{skel}(\mu)$ (i.e., those that have not been introduced by some gadget). Each $w \in \operatorname{skel}(\mu)$ has degree 2 in $\operatorname{skel}(\mu)$. If $\deg_G(w) = 3$ or if *w* is one of *u*, *v*, then we replace *w* by the gadget shown in Fig. 12(h) to replace each such degree-2 vertex by four degree-3 vertices. This is necessary because if $\deg_G(w) = 3$ in *G*, we may not add in an augmentation additional edges to *w* at any time. Additionally, if $w \in \{u, v\}$, then we consider possible new edges at vertex *w* further upwards in the SPQR-tree and not here. By a slight abuse of notation, we still call the obtained graph G_μ and its planar embedding as constructed $\mathcal{E}_\mu$.

Because the embedded label set of a gadget *Y* with fixed embedding $\mathcal{E}_Y$ equals the variable label set of the corresponding subgraph $\operatorname{pert}(\mu_i)$, it follows that $L_{\text{var}}(\operatorname{pert}(\mu)) = L_{\text{emb}}(G_\mu, \mathcal{E}_\mu)$. This equivalent reformulation of the variable label set of $\operatorname{pert}(\mu)$ as the embedded label set of G_μ is the key insight. Each gadget has constant size, so $\|G_\mu\| \in \mathcal{O}(\|\operatorname{skel}(\mu)\|)$. Thus, we can compute $L_{\text{emb}}(G_\mu, \mathcal{E}_\mu)$ in time $\mathcal{O}(\|\operatorname{skel}(\mu)\|^2)$ using Lemma 5.9. If μ is a P-vertex, $\operatorname{skel}(\mu)$ has constant size since $\Delta(G) \leq 3$. That is, $L_{\text{emb}}(G_\mu, \mathcal{E}_\mu)$ can be computed in $\mathcal{O}(\|\operatorname{skel}(\mu)\|^2)$. $\square$

Lemma 5.12 computes the variable label set of an inner vertex of the SPQR-tree, requiring that the variable label sets of its children are non-empty. If this condition is not satisfied, i.e., at least one vertex μ has $L_{\text{var}}(\operatorname{pert}(\mu)) = \emptyset$, then the following lemma applies:

Lemma 5.13 *If $L_{\text{var}}(\operatorname{pert}(\mu)) = \emptyset$ for some vertex μ of the SPQR-tree T of G, then G has no 2-connected 3-augmentation.*

Proof Assuming that *G* has a 2-connected 3-augmentation *H*, we shall show that we have $L_{\text{var}}(\operatorname{pert}(\mu)) \neq \emptyset$ for every vertex μ of *T*. If μ is the root, let *u*, *v* be the two unique vertices in $\operatorname{skel}(\mu)$ (because $\mu = \rho$ is a Q-vertex). If μ is not the root, let *u*, *v* be the endpoints of the virtual edge associated with the parent edge of μ.

By the definition of labels, $L_{\text{var}}(\operatorname{pert}(\mu)) \neq \emptyset$ if there is some inner augmentation of $\operatorname{pert}(\mu)$ for at least one of its planar embeddings with *u*, *v* on its outer face. But the 2-connected 3-augmentation *H* of *G* induces an inner augmentation of $\operatorname{pert}(\mu)$ as follows: Let $\mathcal{E}_H$ be a planar embedding of *H* with outer edge e_ρ and $\mathcal{E}_G$ its restriction to *G*. Recall that then *u*, *v* are outer vertices of $\operatorname{pert}(\mu)$ in $\mathcal{E}_G$. Consider the embedded subgraph of *H* consisting of $\operatorname{pert}(\mu)$ and all vertices and edges of *H* inside inner faces of $\operatorname{pert}(\mu)$ in $\mathcal{E}_G$. For each vertex $w \neq u, v$ on the outer face of $\operatorname{pert}(\mu)$ in $\mathcal{E}_G$

incident to an edge of H in the outer face of $\operatorname{pert}(\mu)$, we add a new pendant edge at w into the outer face of $\operatorname{pert}(\mu)$ in $\mathcal{E}_G$. The resulting graph is an inner augmentation of $\operatorname{pert}(\mu)$ and hence $L_{\text{var}}(\operatorname{pert}(\mu)) \neq \emptyset$. □

Now that we can compute the variable label sets $L_{\text{var}}(\operatorname{pert}(\mu))$ of all vertices of the SPQR-tree, we are finally set up to prove Proposition 5.5. There we claim that we can decide in polynomial time whether a 2-connected planar graph G with $\Delta(G) \leq 3$ has a 2-connected 3-augmentation.

Proof of Proposition 5.5 As mentioned above, we use bottom-up dynamic programming on the SPQR-tree T of G rooted at an arbitrary Q-vertex ρ corresponding to an edge e_ρ in G.

The base cases are the leaves of T, all of which are Q-vertices. By Lemma 5.11, we have $L_{\text{var}}(\operatorname{pert}(\mu)) = \{00\}$.

Now let μ be an inner vertex of T and thus be either an S-, a P- or an R-vertex. All its children $\mu_1, \ldots, \mu_k$ have already been processed and their variable label sets $L_{\text{var}}(\operatorname{pert}(\mu_i))$ are known. Then the variable label set $L_{\text{var}}(\operatorname{pert}(\mu))$ can be computed in time $\mathcal{O}(\|\operatorname{skel}(\mu)\|^2)$ by Lemma 5.12. To apply these lemmas, we need to guarantee that the variable label sets $L_{\text{var}}(\operatorname{pert}(\mu_i))$ of the children are non-empty. If this is not the case, then by Lemma 5.13 graph G has no 2-connected 3-augmentation and we can stop immediately.

Once we computed the label set $L_{\text{var}}(\operatorname{pert}(\mu))$ of the unique child μ of the root ρ, an application of Lemma 5.10 shows that we can now decide in constant time whether G admits a 2-connected 3-augmentation.

The overall runtime is the time needed to construct the SPQR-tree plus the time spent processing each of its vertices. Gutwenger and Mutzel [19] show how to construct the SPQR-tree in time $\mathcal{O}(n)$. The time for the dynamic program traversing the SPQR-tree T is

$$\mathcal{O}\Big(\sum_{\mu \in V(T)} \|\operatorname{skel}(\mu)\|^2\Big) \subseteq \mathcal{O}\Big(\Big(\sum_{\mu \in V(T)} \|\operatorname{skel}(\mu)\|\Big)^2\Big) \subseteq \mathcal{O}(n^2),$$

where the first step uses that for a set of positive integers the sum of their squares is at most the square of their sum, and the second step uses that the SPQR-tree has linear size. □

6 NP-Hardness for 3-Connected 3-Augmentations

In this section, we prove that deciding whether a given planar graph G admits a 3-connected 3-augmentation is NP-complete. In particular, we show that the problem remains NP-complete when restricted to connected graphs G. This implies the NPC-results represented in the fourth column of the table in Fig. 1, corresponding to Statement 5 of Theorem 1.2.

We reduce from the NP-complete problem PLANAR-MONOTONE-3SAT [10]. In intermediate steps of the proof, we obtain graphs with vertices of degree greater than 3. We use wheel-extensions (see Sect. 2) in order to obtain 3-connected 3-augmentations.

Recall that an embedding of any 3-connected 3-augmentation H induces an embedding $\mathcal{E}$ of G. For convenience, we call such a pair $(H, \mathcal{E})$ a *solution* for G. Let us also define a (≤ 2)-*subdivision* of a graph R to be the result of subdividing each edge in R with up to two vertices. Note that, if R is 2-connected, then so is every (≤ 2)-subdivision of R.

Lemma 6.1 *Let G be a graph obtained from a (≤ 2)-subdivision R_2 of a 3-connected planar graph R by attaching a degree-1 vertex to each subdivision vertex. Then G admits a solution $(H, \mathcal{E})$ if and only if no face of $\mathcal{E}$ has exactly one or two incident degree-1 vertices.*

Proof First, assume that $(H, \mathcal{E})$ is a solution for G. Assume for the sake of contradiction that f is a face of $\mathcal{E}$ incident to a set S of exactly one or two degree-1 vertices of G. As H is 3-regular, each vertex in S is incident to two new edges in f. But then, S forms a vertex-cut of cardinality at most 2 in H; a contradiction to H being 3-connected.

For the other direction, let $\mathcal{E}$ be an embedding of G in which no face has exactly one or two incident degree-1 vertices. Our task is to find a solution $(H, \mathcal{E})$ for G, i.e., to insert new vertices and new edges into the faces of $\mathcal{E}$ to obtain a 3-connected 3-regular planar graph H.

To this end, consider any face f of $\mathcal{E}$. If f has no incident degree-1 vertices of G, we insert nothing in f. Otherwise, f has at least $\ell \geq 3$ incident degree-1 vertices, and we identify all these vertices into one vertex v_f of degree ℓ. Let H_1 be the planar graph we obtain by doing this for all faces of $\mathcal{E}$. Clearly, H_1 is planar, $\delta(H_1) \geq 3$, and $R_2 \subset H_1$.

We claim that H_1 is 3-edge-connected, i.e., $\theta(H_1) \geq 3$. First, H_1 is connected, as R_2 is connected. It remains to show that the plane dual H_1^* of H_1 has no loops (i.e., H_1 has no bridges) and no pairs of parallel edges (i.e., H_1 has no 2-edge-cuts). For this, consider any edge e^* of H_1^* and its primal edge e of H_1. If $e \notin E(R_2)$, then e is incident to a vertex v_f of degree $\ell \geq 3$ in a face f of $\mathcal{E}$. In this case e^* is neither a loop nor has a parallel edge in H_1^*.

If $e \in E(R_2)$, then e^* is not a loop, since R_2 is 2-connected. It remains to rule out that two edges $e_1, e_2 \in E(R_2)$ form a 2-edge-cut, i.e., their dual edges e_1^*, e_2^* in H_1^* are parallel. Let f, f' be the two faces of $\mathcal{E}$ incident to e_1 and e_2. As R is 3-connected, e_1 and e_2 both originate from the subdivision(s) of the same edge e_R of R. Consider a subdivision vertex of R_2 between e_1 and e_2. Let v be its new neighbor in H_1; say $v = v_f$ for face f. Then v_f has at least two further neighbors, at least one of which is not a subdivision vertex of e_R, because e_R is subdivided at most twice. But then in the dual H_1^*, the edges e_1^* and e_2^* are incident to different vertices inside f; hence are not parallel.

Finally, we apply a wheel-extension to every vertex v_f, resulting in a planar 3-regular graph H. Further, H contains G as a subgraph and Observation 2.1 yields $\theta(H) \geq \min(\theta(H_1), 3) = 3$. In other words, H is the desired 3-connected 3-augmentation of G. □

By Lemma 6.1, any graph G as described in the lemma admits a 3-connected 3-augmentation if and only if it admits an embedding $\mathcal{E}$ with no face incident to exactly one or two degree-1 vertices. Testing such graphs for such embeddings, however, turns out to be NP-complete.

Theorem 6.2 *Deciding whether a given graph is a subgraph of a 3-regular 3-connected planar graph is NP-complete.*

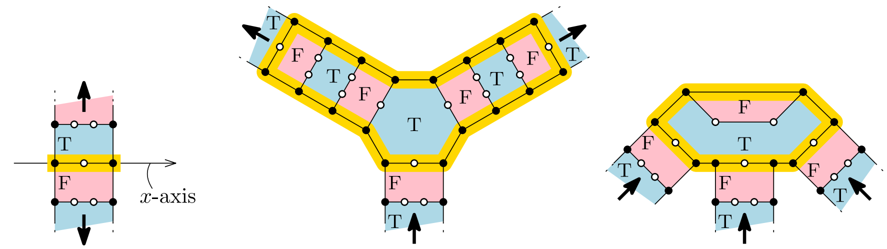
Proof First, we show that the problem is in NP. Let G be a graph that admits a 3-connected 3-augmentation H. We need to show that G also admits a 3-connected 3-augmentation whose size is polynomial in the size of G. To this end, consider the subgraph N of H induced by all new vertices. In H, contract each connected component of N into a single vertex, keeping parallel edges but removing loops. The resulting graph H' is planar and 3-edge-connected, since so was H. Note in particular that the vertices we obtained by contraction have degree at least 3. Next, we apply a wheel-extension to each vertex obtained from the contractions that has degree larger than 3. By Observation 2.1, the resulting graph H'' is 3-regular and 3-edge-connected, hence also 3-connected; in particular, a 3-connected 3-augmentation of G. Moreover, each vertex in H'' has distance at most three to some vertex of G and the maximum degree is bounded by 3, and thus H'' has only $O(|V(G)|)$ many vertices. Thus, our decision problem is in NP.

To show NP-hardness, we reduce from Planar-Monotone-3SAT. An instance of Planar-Monotone-3SAT is a monotone 3SAT-formula Ψ together with its bipartite variable-clause incidence graph I_Ψ and a planar embedding $\mathcal{E}_\Psi$ of I_Ψ. Each clause in Ψ contains either only positive literals (then the clause is called *positive*) or only negative literals (the clause is called *negative*). Moreover, Ψ and the given embedding $\mathcal{E}_\Psi$ satisfy the following:

- Each variable lies on the x-axis and no edge crosses the x-axis.
- Positive clauses lie above the x-axis, negative clauses lie below the x-axis.
- Each clause has at most three literals.

It is known that Planar-Monotone-3SAT is NP-complete [10]. Note that the problem remains NP-hard if we assume that each clause contains exactly three literals and that each variable appears in at least one positive and at least one negative clause. The latter can be achieved by the following reduction rule: Any variable that appears only in positive (negative) form can be set to true (false).

In the remainder of the proof, we construct a planar graph G_Ψ and argue that it admits a 3-connected 3-augmentation if and only if there exists a truth assignment of the variables in Ψ satisfying all clauses. The construction of G_Ψ consists of two steps. We construct a graph R_2 with minimum degree 2 and a unique embedding (up to flipping and the choice of the outer face). Adding pendant edges to all vertices of degree 2, we obtain G_Ψ. In every 3-connected 3-augmentation of $(H, \mathcal{E}_H)$ of G_Ψ, the assignment of the pendant edges to faces corresponds to a truth assignment of the variables in Ψ.



(a) variable gadget **(b)** positive splitter gadget **(c)** positive clause gadget

Fig. 13 Gadgets used in the NP-hardness reduction

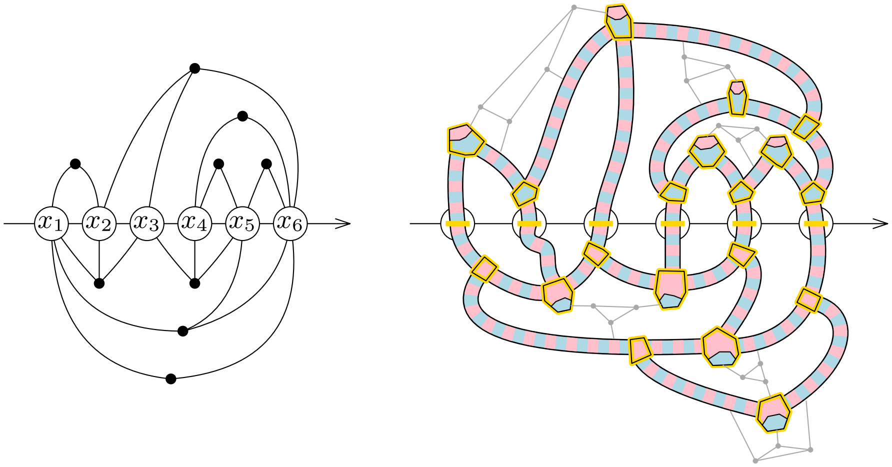


Fig. 14 Illustration of a PLANAR-MONOTONE-3SAT embedding $\mathcal{E}_\Psi$ and a corresponding graph G_Ψ. Extra vertices and edges added for the 3-connectivity of R are shown in gray

Construction of $\mathbf{G}_\Psi$. The graph G_Ψ is obtained from the embedding $\mathcal{E}_\Psi$ of I_Ψ using the gadgets illustrated in Figure 13. In particular, we replace each variable by a copy of the variable gadget in Fig. 13a. As illustrated by the arrows in Fig. 13, these gadgets form the starting point of one *upper* and one *lower corridor* for each variable. Each corridor consists of alternating red (standing for false) and blue (standing for true) faces. The idea is the following: if an augmentation $(H, \mathcal{E}_H)$ of G_Ψ contains for a variable x only new edges within the blue faces of the corridor C_x corresponding to x, the variable x is true; if $\mathcal{E}_H$ only contains new edges within the red faces of C_x, x is false. Above the x-axis, for each variable with k occurrences in positive clauses, we use $k - 1$ positive splitter gadgets, see Fig. 13b, to split its upper corridor into k upper corridors. Then we replace each positive clause by a copy of the positive clause gadget in Fig. 13c, which forms the end of one corridor for each variable appearing in the clause. These corridors are routed without overlap and entirely above the x-axis by following the given embedding $\mathcal{E}_\Psi$ of I_Ψ. We proceed symmetrically below the x-axis, with red and blue swapped, using otherwise isomorphic negative splitter gadgets and negative clause gadgets. See Fig. 14 for a full example.

The resulting graph is a (≤ 2)-subdivision of a 3-regular planar graph R. We refer to the vertices of R as black vertices, and the subdivision vertices as white vertices. Each of the white (subdivision) vertices is incident to one red and one blue face, while each black vertex (of R) is incident to an *uncolored* (neither red nor blue) face. By adding additional vertices into uncolored faces and connecting these to incident edges, we modify R to obtain a 3-connected 3-regular planar graph R' which still has a (≤ 2)-subdivision R_2 including all gadgets and corridors. As R' is 3-connected, the plane embedding of R_2 is unique (up to the choice of the outer face). Finally, we attach a degree-1 vertex to each subdivision vertex, which completes the construction of G_Ψ. See Fig. 15.

Correctness of the reduction. By Lemma 6.1, G_Ψ admits a 3-connected 3-augmentation if and only if G_Ψ admits an embedding $\mathcal{E}$ in which no face has exactly one or two incident degree-1 vertices. Since the embedding of the subgraph R_2 of G_Ψ is fixed, such embedding $\mathcal{E}$ exists if and only if for each subdivision vertex we can choose either the incident red or the incident blue face in such a way that no face is chosen exactly once or twice. Except for the two highlighted faces in the clause gadgets, any pair of neighboring red and blue faces has in total at most five subdivision vertices. Thus, for each variable either all blue faces in all corridors are chosen, corresponding to the variable being set to true, or all red faces in all corridors are chosen, corresponding to the variable being set to false. In each positive clause gadget, the highlighted red face has only two incident subdivision vertices and hence cannot be chosen at all. Thus, the blue face of each positive clause gadget is chosen by two subdivision vertices, and thus must be chosen by at least one further subdivision vertex at the end of a variable corridor. This means that in each positive clause, at least one variable must be set to true. See again Fig. 15 for an illustration. Symmetrically, at least one variable in each negative clause must be set to false, i.e., we have a satisfying truth assignment for Ψ. In the same way, we obtain from a satisfying truth assignment for Ψ a valid choice for each subdivision vertex, i.e., an embedding of G_Ψ as required by Lemma 6.1.

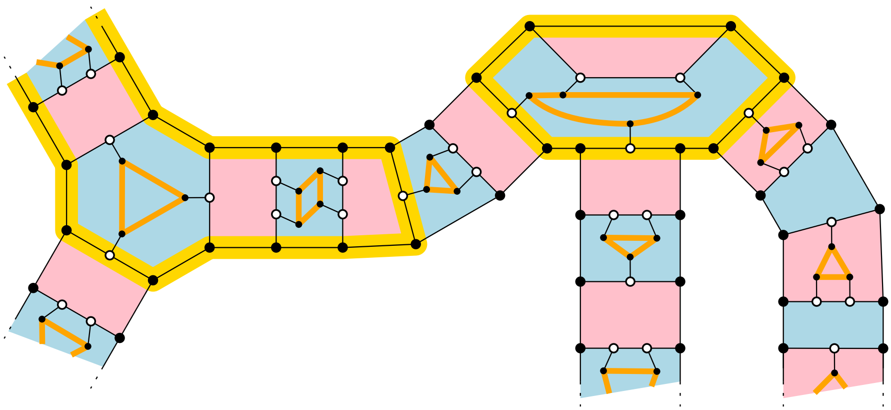


Fig. 15 Part of the graph G_Ψ together with a 3-connected 3-augmentation in orange. This corresponds to a clause with two true variables (left with a splitter gadget, and middle) and one false variable (right)

To summarize, we obtain a planar graph G_Ψ that admits a 3-connected 3-augmentation if and only if the PLANAR-MONOTONE-3SAT-formula Ψ is satisfiable. The size of G_Ψ is polynomial in the size of Ψ. $\square$

Remark 6.3 The (≤ 2)-subdivision R_2 in the above reduction behaves quite similar to G_Ψ. In every 3-connected 3-augmentations of G_Ψ every blue and red face is assigned no or at least three vertices of degree 1. This corresponds to an assignment of the degree-2-vertices (the neighbors of the degree-1-vertices) of R_2 to the red and blue faces. One might assume that the reduction in Theorem 6.2 also works for R_2 instead of G_Ψ. Yet, there exist 3-connected 3-augmentations $(H, \mathcal{E}_H)$ of R_2, where a blue or a red face *f* gets assigned exactly two degree-2-vertices *u*, *v*. Indeed, if $uv \in E(H) \setminus E(R_2)$ is a new edge within *f*, this is the case. In fact, this is the only setting where exactly two degree-2-vertices are assigned to a red or blue face. Note that here the graph R_2 is not an *induced* subgraph of the augmentation *H*. That is, the above reduction also yields NP-completeness of recognizing *induced* subgraphs of 3-connected 3-regular planar graphs, even for 2-connected inputs with a unique embedding.

7 Discussion and Open Problems

Consulting the table in Fig. 1, our results show that for $k \leq 2$ finding *k*-connected 3-augmentations is possible in polynomial time, both in the variable and the fixed embedding setting. On the other hand, Theorem 6.2 shows that finding 3-connected 3-augmentations is NP-complete in the variable embedding setting, even if the input graph is connected. Yet some cases remain open, see Fig. 1 for an overview.

Question 7.1 Is the 3-connected 3-augmentation problem NP-complete for

- for 2-connected input
- and/or fixed embedding?

We suspect these cases for 3-connected 3-augmentations to be NP-complete as well. For one thing, the graphs in our reduction in Sect. 6 are "almost 2-connected" and have "almost a unique embedding", as discussed in Remark 6.3. In particular, it seems difficult to construct splitters for 2-connected input graphs *G*.

Additionally, finding 3-connected 3-augmentations for fixed embedding seems to crucially require a coordination among the new edges, which cannot be modeled as a GENERALIZED FACTOR problem with gaps of length 1. For example, if the input graph *G* is 2-connected (but not already 3-connected) with a fixed embedding, then there is an edge-cut of size 2. See Fig. 16 for an illustration. To establish 3-connectivity in a 3-augmentation $H \supseteq G$, we must connect both sides of the cut through one of the two incident faces f_1, f_2, requiring both sides to coordinate and agree on which of f_1, f_2 to choose.

Our work was motivated by the complexity of the 3-EDGE-COLORABILITY-problem. We showed how to test in polynomial time whether a planar graph *G* is a subgraph

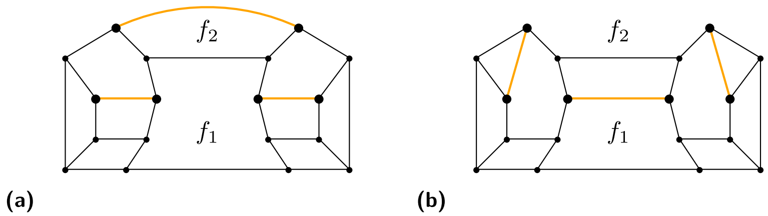


Fig. 16 Two 3-connected 3-augmentations of the same 2-connected graph with fixed embedding

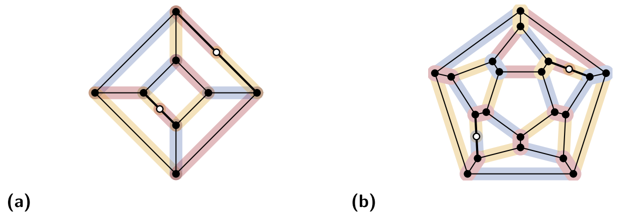


Fig. 17 Two 3-edge-colorable planar graphs with no bridgeless 3-augmentation

of some 2-connected cubic planar graph. This yields a polynomial-time algorithm to recognize subgraphs of bridgeless cubic planar graphs. Recall that such subgraphs admit proper 3-edge-colorings. (This follows from the Four-Color-Theorem [1, 2] and the work of Tait [31].) However, there are 3-edge-colorable planar graphs with no bridgeless 3-augmentation; $K_{2,3}$ is an easy example.

For another class of (potential) examples, consider for instance any 3-connected 3-regular plane graph G (that is, the dual of a plane triangulation) and subdivide (with a new degree-2 vertex each) any set of at least two edges, where no two of these are incident to the same face of G (so their dual edges form a matching in the triangulation), see Fig. 17. The resulting graph G' has only one embedding (up to the choice of the outer face and flipping) and clearly no bridgeless 3-augmentation. On the other hand, Conjecture 7.2 below predicts that G' is 3-edge-colorable.

The computational complexity of the 3-Edge-Colorability-problem for planar graphs remains open, while it is known to be NP-complete already for 3-regular, but not necessarily planar, graphs [21]. One can easily show that a planar subcubic graph is 3-edge-colorable if and only if all of its blocks (inclusion-maximal 2-connected subgraphs) are 3-edge-colorable, i.e., 3-Edge-Colorability reduces to the 2-connected case. A simple counting argument shows that a 2-connected subcubic graph G with exactly one degree-2 vertex is not 3-edge-colorable (independent of whether G is planar or not). The following conjecture, attributed to Grötzsch by Seymour [30], states that in the case of planar graphs, this is the only obstruction.

Conjecture 7.2 *(Grötzsch, cf. [30]) If G is a 2-connected planar graph of maximum degree $\Delta(G) \leq 3$, then G is 3-edge-colorable, unless it has exactly one vertex of degree 2.*

Note that if Conjecture 7.2 is true, 3-Edge-Colorability would be in P, as its condition is easy to check in linear time. Thus a full answer to our initial question, Question 1.1, would most likely also resolve Conjecture 7.2.

Finally, let us also briefly discuss planar graphs of maximum degree larger than 3. In 1965, Vizing conjectured the following, providing a proof only for $\Delta \geq 8$.

Conjecture 7.3 *(Vizing [34]) All planar graphs of maximum degree $\Delta \geq 6$ are Δ-edge-colorable.*

As of today, it is known that all planar graphs of maximum degree $\Delta \geq 7$ are Δ-edge-colorable [18, 28, 36], and optimal edge-colorings can be computed efficiently in these cases. The case $\Delta = 6$ is still open, while for $\Delta = 3, 4, 5$ there are planar graphs of maximum degree Δ that are not Δ-edge-colorable [34]. While the Δ-Edge-Colorability-problem for planar graphs is suspected to be in Pfor $\Delta = 3$ (cf. Conjecture 7.2), it is suspected to be NP-complete for $\Delta = 4, 5$.

Question 7.4 (Chrobak and Nishizeki [7]) Is the Δ-Edge-Colorability-problem for $\Delta = 4, 5$ NP-complete for planar graphs?

Considering the fractional chromatic index $\eta'(G)$ of a graph G, Seymour provided a generalization of Conjecture 7.2.

Conjecture 7.5 *(Seymour's Exact Conjecture [30]) Every planar graph G is $\lceil \eta'(G) \rceil$-edge-colorable.*

It is worth noting that Seymour's Exact Conjecture implies Vizing's Conjecture, as well as the Four-Color-Theorem; see e.g., the survey [5].

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Declarations

Conflict of interest The authors declare no Conflict of interest.

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