

Girth in $GF(q)$ -representable matroids

James Davies¹ | Meike Hatzel² | Kolja Knauer^{3,4} |
 Rose McCarty⁵  | Torsten Ueckerdt⁶

¹Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge, UK

²Institute for Basic Science (IBS), Daejeon, South Korea

³Departament de Matemàtiques i Informàtica, Universitat de Barcelona, Barcelona, Spain

⁴Centre de Recerca Matemàtica, Edifici C, Campus Bellaterra, Bellaterra, Spain

⁵School of Mathematics and School of Computer Science, Georgia Institute of Technology, Atlanta, Georgia, USA

⁶Institute of Theoretical Informatics, Karlsruhe Institute of Technology, Karlsruhe, Germany

Correspondence

Rose McCarty, School of Mathematics and School of Computer Science, Georgia Institute of Technology, 225 North Avenue Northwest, Atlanta, GA 30332, USA.
 Email: rmccarty3@gatech.edu

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Abstract

We prove a conjecture of Geelen, Gerards, and Whittle that for any finite field $GF(q)$ and any integer t , every cosimple $GF(q)$ -representable matroid with sufficiently large girth contains either $M(K_t)$ or $M(K_t)^*$ as a minor.

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1 | INTRODUCTION

The *girth* of a matroid M is the minimum number of elements in a circuit of M , or ∞ if M has no circuits. Examples of cosimple matroids with large girth include the graphic matroid

of a 3-edge-connected graph with large girth and $M(K_t)^*$, the dual of the graphic matroid of the t -vertex clique K_t . Geelen, Gerards, and Whittle [5, Conjecture 5.4] conjectured that every cosimple $GF(q)$ -representable matroid of large girth contains one of these examples as a minor. We prove their conjecture.

Theorem 1. *For any finite field $GF(q)$ and any integer t , there exists an integer $f(t, q)$ such that every cosimple $GF(q)$ -representable matroid with girth at least $f(t, q)$ contains either $M(K_t)$ or $M(K_t)^*$ as a minor.*

Theorem 1 generalizes the theorem of Thomassen [30] that any graph of minimum degree at least three and sufficiently high girth contains K_t as a minor. Thomassen's theorem is celebrated, and there are several strengthenings known for graphs [13, 15]. By considering the cographic case, we can see that Theorem 1 also generalizes the classic lemma due to Mader [14] (and optimized by Thomason [29] and Kostochka [11, 12]) which says that any sufficiently dense graph contains K_t as a minor. Both graphic and cographic matroids must be included as potential outcomes in Theorem 1; this is because not every graph has a small cycle or cut. However, perhaps the condition about representativity could be relaxed. We discuss this possibility in Section 3.

It was believed that a proof of Theorem 1 would require the use of a structure theorem for matroid minors [5]. Yet our proof of Theorem 1 is surprisingly short; it relies on a previously unexplored connection between the Matroid Growth Rate Theorem [7] and Haussler's Shallow Packing lemma [10]. In the context of simple $GF(q)$ -representable matroids with a forbidden graphic minor, the Growth Rate Theorem of Geelen and Whittle [7] bounds the number of elements of the matroid by a linear function of its rank (we remark that there is also a more general Growth Rate Theorem of Geelen, Kung, and Whittle [6]). This theorem directly generalizes Mader's theorem [14].

Our key observation is that for $GF(q)$ -representable matroids, the Growth Rate Theorem (Theorem 2) can be interpreted in terms of the shatter function of an associated set system. This observation allows us to apply powerful tools such as Haussler's Shallow Packing Lemma [10] (Lemma 3). These concepts are fundamental notions in discrete and computational geometry [16, 28, 32], the combinatorics of set systems [25, 26, 31], and first-order logic [1, 4, 22]. However, Theorem 1 is the first application we know of to matroids.[†]

The associated set system we consider in order to apply Haussler's Shallow Packing Lemma is inspired by fundamental graphs. If M can be represented by the columns of a binary matrix $[I \mid A]$ where I is an identity matrix whose columns correspond to a basis B of M , then the *fundamental graph* with respect to B is the bipartite graph whose bipartite adjacency matrix is A . Thus, binary matroids are determined by their fundamental graphs. Fundamental graphs were originally defined for binary matroids [2, 19]. However, they can also be defined and used for general matroids [8, 9]. However, this usually requires more care, since matroids are not generally determined by their fundamental graphs. Instead of taking this approach, we will define an associated set system that stores more information about the matroid than its fundamental graph.

[†] A special case of this connection was implicitly used by the fourth author in order to motivate a conjecture [17, Conjecture 3.5.4] about the neighbourhood complexity of graphs with a forbidden vertex-minor.

2 | THE PROOF

In this section, we prove Theorem 4, which immediately implies Theorem 1. First, we need to introduce some notation, as well as the Growth Rate Theorem and Haussler's Shallow Packing Lemma.

The Growth Rate Theorem for $GF(q)$ -representable matroids of Geelen and Whittle [7] says the following. We remark that Nelson, Norin, and Omana [18] have recently improved the bounds on $\ell(t, q)$ to a singly exponential function.

Theorem 2 [7]. *For any integers t and q , there exists an integer $\ell(t, q)$ such that any simple rank- n $GF(q)$ -representable matroid with no $M(K_t)$ minor has at most $\ell(t, q) \cdot n$ elements.*

To state Haussler's Shallow Packing Lemma, we need to introduce some definitions. Given a finite ground set V , a *set system* \mathcal{F} on V is a subset of 2^V . (We do not allow multisets.) The *shatter function* of \mathcal{F} , denoted by $\pi_{\mathcal{F}}(m)$, is the maximum size of \mathcal{F} when restricted to any m elements in V ; that is, $\pi_{\mathcal{F}}(m)$ is the maximum, over all m -element subsets $W \subseteq V$, of the number of equivalence classes of the relationship \sim_W on \mathcal{F} where two sets $F, F' \in \mathcal{F}$ satisfy $F \sim_W F'$ if $F \cap W = F' \cap W$. For a positive integer δ , we say that two sets $F, F' \in \mathcal{F}$ are δ -separated if their symmetric difference $F \Delta F'$ has size at least δ (that is, there are at least δ elements in V which are in one of F, F' but not the other). We say that \mathcal{F} is δ -separated if any pair of distinct elements in \mathcal{F} are δ -separated.

We use the following version of Haussler's Shallow Packing Lemma [10]. This version is stated as [3, Lemma 2.2], for instance. (Actually [3, Lemma 2.2] is a more general version; we only require the case that $d = 1$.)

Lemma 3 [10]. *For any number $\ell \geq 1$, there exists an integer $c = c(\ell)$ so that for every positive integer δ , if \mathcal{F} is a set system on a finite ground set V so that \mathcal{F} is δ -separated and $\pi_{\mathcal{F}}(m) \leq \ell m$ for every positive integer m , then $|\mathcal{F}| \leq c|V|/\delta$.*

The following theorem immediately implies Theorem 1. We remark that even in the context of graphs (so when M is graphic), this theorem already provides another strengthening of Thomassen's theorem [30].

Theorem 4. *For any finite field $GF(q)$ and any integer t , there exists an integer $k = k(t, q)$ such that if M is a cosimple $GF(q)$ -representable matroid not containing $M(K_t)$ or $M(K_t)^*$ as a minor, then for every basis B of M , there is a circuit C of size at most k with $|C \setminus B| \leq 2$.*

Proof. Let M be a cosimple $GF(q)$ -representable matroid which does not contain $M(K_t)$ or $M(K_t)^*$ as a minor, and let B be a basis of M . By performing row operations and deleting all zero rows, we can obtain a matrix over $GF(q)$ of the form $[I \mid A]$ so that I is a $|B| \times |B|$ identity matrix, and M is represented by the column vectors of $[I \mid A]$ so that the columns of I correspond to the elements in B . Thus we may view A as a $|B| \times |E(M) \setminus B|$ -matrix. Given elements $b \in B$ and $e \in E(M) \setminus B$, we write $A_{b,e}$ for the corresponding entry of A .

Recall that in the binary case, the fundamental graph is the bipartite graph with adjacency matrix A . In this case, the set system would consist of the supports of the columns of A . In the general case, we need to store more information about which element of $GF(q)$ is contained in an

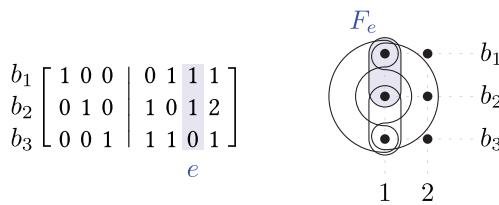


FIGURE 1 A ternary matroid M with a representation $[I \mid A]$ over $GF(3)$ and the corresponding set system \mathcal{F} with $e \in E(M) \setminus B$ and $F_e = \{(b_1, 1), (b_2, 1)\}$ highlighted in blue.

entry $A_{b,e}$ of A . So we now define a set system \mathcal{F} on the ground set $B \times (GF(q) \setminus \{0\})$, which we denote by V for short.

For each element $e \in E(M) \setminus B$, we write F_e for the set of all tuples $(b, \alpha) \in V$ so that $A_{b,e} = \alpha$; see Figure 1 for an example. Then we set $\mathcal{F} = \{F_e : e \in E(M) \setminus B\}$. We may assume that all distinct elements $e, e' \in E(M) \setminus B$ have $F_e \neq F_{e'}$, since otherwise e and e' are represented by the same column vector in A , and we have found the desired circuit.

Now we prove a key claim. Let $\ell = \ell(t, q)$ be the integer from the Growth Rate Theorem (Theorem 2). So any simple rank- n $GF(q)$ -representable matroid with no $M(K_t)$ minor has at most $\ell \cdot n$ elements.

Claim 4.1. For any positive integer m , we have $\pi_{\mathcal{F}}(m) \leq \ell q \cdot m$.

Proof. Let $W \subseteq V$ be an m -element set.

Let B_W be the projection of W onto B . That is, B_W is the set of all $b \in B$ such that there exists $\alpha \in GF(q) \setminus \{0\}$ so that $(b, \alpha) \in W$. Thus $|B_W| \leq m$. Consider taking the matrix $[I \mid A]$ which represents M and deleting from it the rows corresponding to elements in $B \setminus B_W$. The column matroid of this matrix is a minor of M ; it is obtained from M by contracting the elements in $B \setminus B_W$. Thus, by the Growth Rate Theorem (Theorem 2), its simplification (that is, the matroid obtained by removing loops and only keeping one element from each parallel class) has at most $\ell \cdot m$ elements.

Now consider two elements $e, e' \in E(M) \setminus B$ with $F_e \cap W \neq F_{e'} \cap W$. Let $(b, \alpha) \in W$ be an element in one of these sets but not the other. Then one of $A_{b,e}$ and $A_{b,e'}$ is equal to α and the other is not. So in particular, the columns corresponding to e and e' are distinct even when restricted to rows in B_W . Finally, let us consider what happens when we take the simplification of a $GF(q)$ -represented matroid with distinct columns. It has at most one loop for the all zero vector, and each parallel class has at most $q - 1$ elements. It follows that $\pi_{\mathcal{F}}(m) \leq (q - 1)(\ell \cdot m) + 1 \leq \ell q \cdot m$, as desired. \square

Next we apply Haussler's Shallow Packing Lemma (Lemma 3). We write $c = c(\ell, q)$ for the function from Lemma 3, and we set $\delta = \ell qc + 1$. Note that δ is just a function of t and q . By Claim 4.1 and Haussler's lemma, either \mathcal{F} is not δ -separated, or $|\mathcal{F}| \leq c|V|/\delta$.

First, suppose that $|\mathcal{F}| \leq c|V|/\delta$. Recall that all distinct elements in $E(M) \setminus B$ correspond to distinct sets in \mathcal{F} . So $|E(M) \setminus B| = |\mathcal{F}|$. Since the dual of M is a simple $GF(q)$ -representable matroid with no $M(K_t)$ minor, the Growth Rate Theorem (Theorem 2) yields $|E(M^*)| \leq \ell|E(M) \setminus B| = \ell|\mathcal{F}|$. Thus

$$|V| = (q - 1)|B| \leq q|E(M^*)| \leq \ell q|\mathcal{F}| \leq \ell qc|V|/\delta.$$

So $\delta \leq \ell qc$, however we chose $\delta = \ell qc + 1$, a contradiction.

Thus \mathcal{P} is not δ -separated. So there exist distinct elements $e, e' \in E(M) \setminus B$ so that there are fewer than δ elements in the symmetric difference of F_e and $F_{e'}$. Thus there are fewer than δ rows of $[I \mid A]$ where the columns of e and e' differ. Let $B' \subseteq B$ be the basis elements corresponding to those rows. Given an element $e \in E(M)$, we write \vec{e} for the corresponding column vector of $[I \mid A]$. So $\vec{e} - \vec{e}'$ is in the span of $\{\vec{b} : b \in B'\}$. Thus $B' \cup \{e, e'\}$ contains a circuit of M . Since $|B'| \leq \delta - 1$, the theorem holds with $k = \delta + 1$. \square

3 | CONCLUSION

In this section, we discuss possible extensions of Theorem 1 that relax the condition of being $\text{GF}(q)$ -representable.

We write $U_{2,q}$ for the q -element line, and, more generally, $U_{t,q}$ for the uniform matroid with q elements and rank t . That is, $U_{t,q}$ is the q -element matroid where the circuits are the sets of size $t + 1$. The Growth Rate Theorem of Geelen and Whittle [7] also applies to matroids that forbid $U_{2,q+2}$ as a minor, rather than just to $\text{GF}(q)$ -representable matroids. (Recall that $\text{GF}(q)$ -representable matroids do not have $U_{2,q+2}$ minors; see for instance [20, Corollary 6.5.3].)

Theorem 5 [7]. *For any integers t and q , there exists an integer $\ell(t, q)$ such that any simple rank- n matroid with no $U_{2,q+2}$ or $M(K_t)$ minor has at most $\ell(t, q) \cdot n$ elements.*

In light of Theorem 1 and the Growth Rate Theorem (Theorem 5), it is natural to conjecture the following.

Conjecture 6. *For any positive integer t , there exists an integer $p(t)$ such that every cosimple matroid with girth at least $p(t)$ contains either $U_{2,t+2}$, $U_{t,t+2}$, $M(K_t)$, or $M(K_t)^*$ as a minor.*

For large t , both $U_{t,t+2}$ and $M(K_t)^*$ have large girth, while $M(K_t)$ has a cosimple minor of large girth. However, it is less satisfying to forbid the line $U_{2,t+2}$. So it is natural to ask the more general question: What are the unavoidable cosimple matroids of large girth? To frame this problem precisely, let us consider a property \mathcal{P} of classes of matroids. For example, we write $\mathcal{P}_{\text{girth}}$ for the property ‘the class contains cosimple matroids of arbitrarily large girth’. A class of matroids \mathcal{M} is *minor-minimal* with respect to \mathcal{P} if \mathcal{M} is minor-closed, \mathcal{M} has property \mathcal{P} , and no proper minor-closed subclass of \mathcal{M} has property \mathcal{P} .

As an example, the class of all graphic matroids is minor-minimal with respect to $\mathcal{P}_{\text{girth}}$ due to Thomassen’s theorem [30]. Likewise, the class of all cographic matroids is minor-minimal with respect to $\mathcal{P}_{\text{girth}}$ due to Mader’s theorem [15]. It is straightforward to see that the closure of all colines $U_{t,t+2}$ under minors is also minor-minimal with respect to $\mathcal{P}_{\text{girth}}$. We conjecture that classes with property $\mathcal{P}_{\text{girth}}$ have a finite characterization.

Conjecture 7. *There exist a finite number of classes $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k$ which are minor-minimal with respect to property $\mathcal{P}_{\text{girth}}$ such that a class of matroids \mathcal{M} has property $\mathcal{P}_{\text{girth}}$ if and only if it contains at least one of $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_k$.*

Matroids are not well-quasi-ordered under minors, even though graphs famously are, as proven by Robertson and Seymour [24]. Moreover, Conjecture 7 is an example of ‘second-level better-quasi-ordering’, and it is wide open whether graphs are second-level better quasi-ordered under

minors. See [21] for recent progress on this question and [23] for a brief discussion by Robertson and Seymour. Still, we are optimistic about Conjecture 7 since there are some natural matroids to forbid.

In particular, let us write $B(G)$ for the *bicircular* matroid of a graph G . Bicircular matroids were introduced by Simões Pereira [27], and we refer the reader there for definitions. It can be proven using Thomassen's theorem [30] that the class of bicircular matroids is minor-minimal with respect to $\mathcal{P}_{\text{girth}}$. We conjecture the following, which would imply Conjecture 7.

Conjecture 8. *There exists a function g such that for every integer t , every cosimple matroid with girth at least $g(t)$ contains either $U_{t,t+2}$, $M(K_t)$, $M(K_t)^*$, or $B(K_t)$ as a minor.*

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ORCID

Rose McCarty  <https://orcid.org/0000-0002-9884-3406>

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