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Asymptotic independence in more than two dimensions and its implications on risk management

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Abstract In extreme value theory, the presence of *asymptotic independence* signifies that joint extreme events across multiple variables are unlikely. Although well understood in a bivariate context, the concept remains relatively unexplored when addressing the nuances of simultaneous occurrence of extremes in higher dimensions. In this article, we propose a notion of *mutual asymptotic independence* to capture the behaviour of joint extremes in dimensions larger than two and contrast it with the classical notion of *(pairwise) asymptotic independence*. Additionally, we define *k-wise asymptotic independence*, which captures the tail dependence in between pairwise and mutual asymptotic independence. The concepts are compared using examples of Archimedean, Gaussian, and Marshall–Olkin copulas, among others. Finally, we discuss the implications of these new notions of asymptotic independence on assessing the risk in complex systems under distributional ambiguity.

Résumé Dans le cadre de la théorie des valeurs extrêmes, la notion d'*indépendance asymptotique* traduit le fait que la survenue simultanée d'événements extrêmes sur plusieurs variables est peu probable. Bien que ce concept soit bien maitrisé en dimension bivariée, son étude demeure limitée lorsqu'il s'agit d'appréhender la cooccurrence d'événements extrêmes dans des dimensions supérieures. Cet article introduit la notion d'*indépendance asymptotique mutuelle* pour caractériser le comportement conjoint des valeurs extrêmes en dimension supérieure à deux, par opposition à la notion classique d'*indépendance asymptotique par paires*. Les auteurs introduisent également la notion d'*indépendance asymptotique entre k-uplets*, qui permet de mettre en évidence des formes intermédiaires de dépendance extrême, situées entre l'indépendance asymptotique par paires et l'indépendance asymptotique mutuelle. Ces différentes notions sont illustrées et comparées à travers plusieurs exemples, notamment les copules archimédiennes, gaussiennes et de Marshall-Olkin. Enfin, les implications de ces nouvelles notions d'indépendance asymptotique sont examinées dans le contexte de l'évaluation du risque pour des systèmes complexes affectés par une incertitude sur la loi de probabilité sous-jacente.

Keywords Asymptotic independence; copula models; Gaussian copula; multivariate extremes; risk contagion.
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1 Introduction

In many multivariate models, we observe that the likelihood of a joint occurrence of extreme values in two or more variables is negligible in comparison to the occurrence of an extreme value in one variable. In this context, the notion of *asymptotic independence* looms large in the study of joint extreme values in probability distributions, although mostly restricted to the bivariate set-up. A random vector $(Z_1, Z_2) \in \mathbb{R}^2$ with identically distributed marginals is *asymptotically (right-tail/upper-tail) independent* if

$$\mathrm{P}(Z_1 > t, Z_2 > t) = o(\mathrm{P}(Z_1 > t)), \quad t \to \infty, \tag{1.1}$$

or, equivalently, $\mathrm{P}(Z_1 > t | Z_2 > t) \to 0$ as $t \to \infty$. For the rest of this article, we focus only on extremes in the non-negative quadrant and drop the terms *right-/upper-tail* for convenience.

Often called *Sibuya's condition*, (1.1) was shown by Sibuya (1960) for bivariate normal random vectors with any correlation $\rho < 1$. Such a limit behaviour has also been found to hold for bivariate distributions with an arbitrary choice of marginals possessing a variety of dependence structures, including the Frank copula, Ali–Mikhail–Haq copula, Gaussian copula, and Farlie–Gumbel–Morgenstern copula; see Ledford and Tawn (1996, 1998), Coles et al. (1999), and Heffernan (2000). It is widely believed that the presence of asymptotic independence hinders the computation of joint tail probabilities, and this has led to a variety of techniques for modelling and estimating rare tail probabilities when such a property is present; see Ledford and Tawn (1996), Resnick (2002), Ramos and Ledford (2009), Lehtomaa and Resnick (2020), Das et al. (2013), and Das et al. (2022). Nevertheless, for random vectors in dimensions higher than two, limited expositions are available, and multivariate asymptotic independence is often understood to be (1.1) holding for all pairs of variables, which we call *pairwise asymptotic independence*. Such a notion of multivariate asymptotic independence may have its origins in the study of extremes. For instance, in Resnick (2008, Chapter 5.5), a multivariate distribution is called *multivariate asymptotically independent* if it is in the maximum domain of attraction of a multivariate extreme value distribution with independent marginals. Additionally, it is shown that such a characterization of "multivariate asymptotic independence" is equivalent to having "pairwise asymptotic independence" (assuming identical marginals in the maximum domain of attraction of a univariate extreme value distribution); see Resnick (2008, Proposition 5.27) and Galambos (1978, Corollary 5.3.1). In this article, we show that asymptotic tail independence often has a much subtler form that goes beyond pairwise asymptotic independence.

Asymptotic independence for bivariate joint tails is also popularly understood using the coefficient of tail dependence η defined in Ledford and Tawn (1996). If Z_1 and Z_2 are identically unit Fréchet-distributed with distribution function $F(z) = \mathrm{e}^{-1/z}$, $z > 0$, and

$$\mathrm{P}(Z_1 > t, Z_2 > t) = t^{-1/\eta}\ell(t), \qquad t \to \infty,$$

where $1/2 \leq \eta < 1$ and ℓ is slowly varying at infinity (i.e., $\ell(tz)/\ell(z) \to 1$ as $t \to \infty, \forall\, z > 0$), then η represents this *coefficient of tail dependence*. According to Ledford and Tawn (1996), (i) $\eta = 1/2$ and $\ell(t) \geq 1$ signifies *near independence*, (ii) $\eta = 1$ and $\ell(t) \nrightarrow 0$ as $t \to \infty$ signifies *upper tail dependence*, and, finally, (iii) either $1/2 < \eta < 1$, or $\eta = 1$ and $\ell(t) \to 0$ as $t \to \infty$ signifies *positive association*.

The coefficient of tail dependence is a two-dimensional concept and has been extended to d dimensions as *upper tail order* by Hua and Joe (2011) through the survival copula. Prior to further discussions, we review the notions of copula and survival copula.

A copula $C : [0,1]^d \to [0,1]$ is a multivariate distribution function with identical uniform $[0,1]$ marginals. From Sklar's Theorem (Sklar, 1959; Nelsen, 2006; Durante and Sempi, 2016), we know that for any d-dimensional random vector $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ with distribution function F and marginal distributions $F_1, \dots, F_d$, there exists a copula $C : [0,1]^d \to [0,1]$ such that

$$F(z_1, \dots, z_d) = C(F_1(z_1), \dots, F_d(z_d)),$$

for $(z_1, \dots, z_d) \in \mathbb{R}^d$, and if the marginals are continuous, the copula is uniquely given by

$$C(u_1, \dots, u_d) = F(F_1^{\leftarrow}(u_1), \dots, F_d^{\leftarrow}(u_d))$$

for $0 < u_1, \dots, u_d < 1$, where

$$F_j^{\leftarrow}(u_j) := \inf\{z \in \mathbb{R} : F_j(z) \geq u_j\}$$

is the generalized inverse of F_j for $j \in \{1, \dots, d\}$. In this article, we are particularly concerned with the probability of joint extremes where the survival copula $\widehat{C} : [0,1]^d \to [0,1]$, which is also a copula, plays an important

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role; see Durante and Sempi (2016, Chapter 1), McNeil et al. (2015, Section 5.1.5), and Nelsen (2006, Section 2.6). The survival copula $\widehat{C}$ satisfies

$$\mathrm{P}(Z_1 > z_1, \dots, Z_d > z_d) = \widehat{C}(\bar{F}_1(z_1), \dots, \bar{F}_d(z_d))$$

for $(z_1, \dots, z_d) \in \mathbb{R}^d$, where $\bar{F}_j = 1 - F_j$ is the tail function of F_j for $j \in \{1, \dots, d\}$. Of course, the survival copula and the copula are directly related through

$$\widehat{C}(u_1, \dots, u_d) = 1 + \sum_{\substack{S \subseteq \{1,\dots,d\} \\ S \neq \emptyset}} (-1)^{|S|} C_S(1 - u_j \, : \, j \in S)$$

for $0 \leq u_1, \dots, u_d \leq 1$, where $|S|$ is the cardinality of the set S and C_S is the appropriate $|S|$-dimensional marginal copula of C. In dimension $d = 2$, this reduces to

$$\widehat{C}(u_1, u_2) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2)$$

for $0 \leq u_1, u_2 \leq 1$.

Returning to the notion of *tail dependence*, if a d-dimensional survival copula $\widehat{C}$ satisfies

$$\widehat{C}(u, \dots, u) = u^{\kappa} \ell(u), \quad 0 \leq u \leq 1, \tag{1.2}$$

for some slowly varying function ℓ at 0 and some constant $\kappa > 0$, then κ is called the *upper tail order*. Following Hua and Joe (2011), (i) the case $\kappa = d$ signifies *near (asymptotic) independence* (for $d = 2$, we have $\kappa = 1/\eta$), (ii) the case $\kappa = 1$ and $\ell(u) \nrightarrow 0$ (as $u \downarrow 0$) signifies *(asymptotic) upper tail dependence*, and, (iii) the case where $1 < \kappa < d$ is called *upper intermediate tail dependence*. From the definition of *tail order*, we can see that for $d = 2$, the survival copulas in both the cases of "near independence" and "upper intermediate tail dependence" exhibit asymptotic independence in the sense of (1.1); in this article, we gain a better understanding of these ideas for cases where $d > 2$.

Note that for *independence* of multiple random variables, it is well known that "pairwise independence" for all pairs of random variables is not equivalent to their "mutual independence" (cf. Hogg et al. (2013, Chapter 2)). In a similar vein, we propose here the concepts of *pairwise asymptotic independence* in Section 2 and *mutual asymptotic independence* in Section 3. With the new notion of mutual asymptotic independence, we explore the ideas of "near independence" and "intermediate upper tail dependence" through all subsequent dimensions $2, 3, \dots, d$, going beyond just the d-dimensional characterization as given in (1.2). For models that lie in between pairwise and mutually asymptotically independent models, we introduce the concept of *k-wise asymptotic independence* for $k \in \{2, \dots, d\}$ in Section 4. In particular, we investigate and compare the various notions of *asymptotic independence* and illustrate them using popular copula models. Moreover, we obtain the following three key results for the popular Gaussian copula, which have broader theoretical and practical implications:

(i) a formulation of precise necessary and sufficient conditions for mutual asymptotic independence to hold;

(ii) a derivation of the correct tail orders; and

(iii) the existence of Gaussian copula models exhibiting k-wise asymptotic independence but not $(k + 1)$-wise asymptotic independence.

Besides the Gaussian copula, we also provide examples to show the breadth of asymptotic (in)dependence behaviour using the Archimedean copula family. We apply the new notions of asymptotic independence in Section 5 to show its implications on assessing the risk in complex systems under distributional ambiguity. The different notions of asymptotic independence influence risk contagion in financial systems differently and hence may lead to an underestimation or overestimation of risk if applied improperly. In particular, we exhibit this phenomenon using two pertinent conditional risk measures, namely conditional tail probabilities and Contagion Value-at-Risk or Conditional Value-at-Risk (CoVaR) in dimensions $d > 2$. Finally, in Section 6, we conclude with some broader implications of interpreting asymptotic independence in this new light. All proofs for the results presented in this article are provided in the Appendix.

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1.1 Notations

We denote by $\mathbb{I}_d = \{1, \dots, d\}$ an index set with $d \geq 1$ elements, and the cardinality of a set $S \subseteq \mathbb{I}_d$ by $|S|$. For a random vector $\boldsymbol{Z} = (Z_1, \dots, Z_d)$, we write $\boldsymbol{Z} \sim F$ if $\boldsymbol{Z}$ has distribution function F; moreover, we understand that marginally $Z_j \sim F_j$ for $j \in \mathbb{I}_d$. For any non-empty sets $S \subseteq \mathbb{I}_d$, the copula and survival copula of the corresponding $|S|$-dimensional marginal are denoted by C_S and $\widehat{C}_S$, respectively. Moreover, if $d = 1$, we have $C_S(u) = \widehat{C}_S(u) = u$ for $0 \leq u \leq 1$. For a given vector $\boldsymbol{z} \in \mathbb{R}^d$ and $S \subseteq \mathbb{I}_d$, we denote by $\boldsymbol{z}^\top$ the transpose of $\boldsymbol{z}$ and by $\boldsymbol{z}_S \in \mathbb{R}^{|S|}$ the vector obtained by deleting the components of $\boldsymbol{z}$ in $\mathbb{I}_d \setminus S$. Similarly, for non-empty $S \subseteq \mathbb{I}_d$, Σ_S denotes the appropriate submatrix of a given matrix $\Sigma \in \mathbb{R}^{d \times d}$ after removing all rows and columns with indices in $\mathbb{I}_d \setminus S$. Furthermore, $\mathbf{0}_d = (0, \dots, 0)^\top$ and $\mathbf{1}_d = (1, \dots, 1)^\top$ are vectors in $\mathbb{R}^d$, and $\boldsymbol{I}_d$ is the identity matrix in $\mathbb{R}^{d \times d}$; subscripts are dropped when evident from the context. Vector operations are understood component-wise: for example, for vectors $\boldsymbol{z} = (z_1, \dots, z_d)$ and $\boldsymbol{y} = (y_1, \dots, y_d)$, $\boldsymbol{z} \leq \boldsymbol{y}$ means $z_j \leq y_j, \forall j \in \mathbb{I}_d$. For functions $f, g : (0, \infty) \to (0, \infty)$, we write $f(u) \sim g(u)$ as $u \downarrow 0$ if $\lim_{u \downarrow 0} f(u)/g(u) = 1$. Moreover, a function $\ell : (0, \infty) \to (0, \infty)$ is called *slowly varying at 0* if $\lim_{u \downarrow 0} \ell(uz)/\ell(u) = 1, \forall z > 0$.

2 Pairwise asymptotic independence

Note that the definition in (1.1) can be easily generalized to distributions with potentially unequal marginals; any random vector $(Z_1, Z_2) \in \mathbb{R}^2$ with continuous marginals $Z_j \sim F_j, j \in \{1, 2\}$ is asymptotically independent if

$$\widehat{C}(u, u) = \mathrm{P}(F_1(Z_1) > 1 - u, F_2(Z_2) > 1 - u) = o(u), \quad u \downarrow 0, \tag{2.1}$$

where $\widehat{C}$ is the survival copula of F. Note that the limit properties in (1.1) and (2.1) remain equivalent when the marginals of (Z_1, Z_2) are *completely tail-equivalent*, that is, $\mathrm{P}(Z_1 > t)/\mathrm{P}(Z_2 > t) \to 1$ as $t \to \infty$. Although not all extreme sets are of this form, this definition has been a key concept in the modelling of joint extremes.

An interesting feature of this definition of asymptotic independence is that it is based on tail sets tethered along the main diagonal (t, t) (in (1.1)) or $(1 - u, 1 - u)$ (in (2.1)). It is easy to check that (2.1) is equivalent to

$$\widehat{C}(au, bu) = o(u), \quad u \downarrow 0$$

for some $a, b > 0$ (Balkema and Nolde, 2010, Theorem 2). Curiously, an equivalent result for the distribution function of a bivariate random vector does not hold: even if (1.1) holds, it does not necessarily hold for diagonals of the form (at, bt) for arbitrary $a, b > 0$; see Das and Fasen-Hartmann (2024, Proposition 3.9) for an example with normally distributed marginals (Z_1, Z_2) with $\mathrm{Cor}(Z_1, Z_2) = \rho > 0$, where $\mathrm{P}(Z_1 > at, Z_2 > t) = O(\mathrm{P}(Z_2 > t))$, as $t \to \infty$, for $0 < a \leq \rho$.

Although (1.1) and (2.1) are widely applied for bivariate random vectors, a proper multivariate characterization of *asymptotic independence* has been relatively scarce. A definition often used and based on all pairwise comparisons following (2.1) is given next.

Definition 1 (Pairwise asymptotic independence)**.** An $\mathbb{R}^d$-valued random vector $\boldsymbol{Z} \sim F$ with continuous marginal distributions, copula C, and associated survival copula $\widehat{C}$ is *pairwise asymptotically independent* if $\forall\, j, \ell \in \mathbb{I}_d, j \neq \ell$,

$$\widehat{C}_{\{j,\ell\}}(u, u) = o(u), \qquad u \downarrow 0. \tag{2.2}$$

We interchangeably say $\boldsymbol{Z}, F$, or C exhibits pairwise asymptotic independence.

Remark 1.

(a) More generally, for any multivariate distribution with non-continuous marginal distributions, its copula is no longer unique; hence, we cannot extend Definition 1 straightforwardly in such a case. For convenience, since we are primarily concerned with the asymptotic behaviour of the survival copula, we will assume for the rest of the article that all distributions will have continuous marginal distributions, ensuring that both their copula and associated survival copula are unique. Note that, since we are only concerned with tail probabilities in this article, it actually suffices to have the marginal distributions to be continuous above a fixed threshold (that means they are eventually continuous).

(b) In contrast to asymptotic independence, a d-dimensional copula C with $d \geq 2$ exhibits asymptotic upper tail dependence if its survival copula $\widehat{C}$ satisfies

$$\lim_{u \downarrow 0} \frac{\widehat{C}(u, \dots, u)}{u} = \lambda \in (0, 1). \tag{2.3}$$

Obviously, (2.3) implies that (2.2) cannot hold. If (2.3) holds, then it is equivalent to having $\kappa = 1$ and $\ell(u) \nrightarrow 0$ as $u \downarrow 0$ for the upper tail order defined in (1.2).

2.1 Examples

Pairwise asymptotic independence exists in many multivariate distributions. We note a few examples here.

Example 1 (Independence). If all components of a random vector $\boldsymbol{Z} \in \mathbb{R}^d$ are independent, then

$$\mathrm{P}(Z_1 > z_1, \dots, Z_d > z_d) = \prod_{j=1}^{d} \bar{F}_j(z_j) = \widehat{C}^{\mathrm{ind}}(\bar{F}_1(z_1), \dots, \bar{F}(z_d))$$

for $(z_1, \dots, z_d) \in \mathbb{R}^d$, where $C^{\mathrm{ind}} : [0,1]^d \to [0,1]^d$ is the *independence copula* given by

$$C^{\mathrm{ind}}(u_1, \dots, u_d) = \prod_{j=1}^{d} u_j \tag{2.4}$$

for $0 \leq u_1, \dots, u_d \leq 1$ with survival copula $\widehat{C}^{\mathrm{ind}}(u_1, \dots, u_d) = C^{\mathrm{ind}}(u_1, \dots, u_d)$. For any distinct $j, \ell \in \mathbb{I}_d$, the (j, l) marginal survival copula is

$$\widehat{C}^{\mathrm{ind}}_{\{j,\ell\}}(u_1, u_2) = u_1 u_2, \quad 0 \leq u_1, u_2 \leq 1.$$

Thus, clearly (2.2) holds, and hence, the independence copula exhibits pairwise asymptotic independence.

Example 2 (Marshall–Olkin dependence). The Marshall–Olkin (MO) distribution is used in reliability theory to capture the failure of subsystems in a networked system. Here we consider a particular MO dependence; see Lin and Li (2014) and Das and Fasen-Hartmann (2023). Assume that for every non-empty set $S \subseteq \mathbb{I}_d$, there exists a parameter $\lambda_S > 0$ and $\Lambda := \{\lambda_S : \emptyset \neq S \subseteq \mathbb{I}_d\}$. Then, the generalized MO copula $C^{\mathrm{MO}(\Lambda)}$ can be defined by its associated survival copula $\widehat{C}^{\mathrm{MO}(\Lambda)}$, with rate parameter Λ given by

$$\widehat{C}^{\mathrm{MO}(\Lambda)}(u_1, \dots, u_d) = \prod_{i=1}^{d} \prod_{|S|=i} \bigwedge_{j \in S} u_j^{\eta_j^S} \tag{2.5}$$

for $0 \leq u_1, \dots, u_d \leq 1$, where

$$\eta_j^S = \lambda_S \Bigg/ \left(\sum_{J \supseteq \{j\}} \lambda_J \right), \quad j \in S \subseteq \mathbb{I}_d. \tag{2.6}$$

For any distinct $j, \ell \in \mathbb{I}_d$, we can compute that

$$\widehat{C}^{\mathrm{MO}(\Lambda)}_{\{j,\ell\}}(u, u) = u^{\eta^*_{j\ell}}$$

with

$$\eta^*_{j\ell} = \sum_{\substack{S \subseteq \mathbb{I}_d \\ j \in S, \ell \notin S}} \eta_j^S + \sum_{\substack{S \subseteq \mathbb{I}_d \\ j \notin S, \ell \in S}} \eta_\ell^S + \sum_{\substack{S \subseteq \mathbb{I}_d \\ j, \ell \in S}} \max\{\eta_j^S, \eta_\ell^S\} > 1. \tag{2.7}$$

Clearly, since $\eta^*_{j\ell} > 1, \forall j \neq \ell$, $C^{\mathrm{MO}(\Lambda)}$ possesses pairwise asymptotic independence for any choice of Λ; see also Lin and Li (2014, Proposition 3). An even larger class of MO copulas has been introduced in Lin and Li (2014) which are also pairwise asymptotically independent.

Although λ_S is allowed to take any positive value for the non-empty $S \subset \mathbb{I}_d$, we discuss below two particularly interesting choices of the parameters; see also Das and Fasen-Hartmann (2023, Example 2.14).

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(a) Equal parameter values for all sets: Here, $\lambda_S = \lambda$ for all non-empty $S \subseteq \mathbb{I}_d$ where $\lambda > 0$, and we denote the survival copula by $\widehat{C}^{\mathrm{MO}^=}$. We can check from (2.6) that the value of λ is irrelevant here. Clearly in this case, $\eta_j^S = 1/2^{d-1}$, for all $j \in S$ and non-empty $S \subset \mathbb{I}_d$. Hence, we can compute the value of $\eta^*_{j\ell}$ defined in (2.7) as

$$\eta^*_{j\ell} = \eta^*_{12} = \sum_{\substack{S\subseteq \mathbb{I}_d\\ 1\in S}} \frac{1}{2^{d-1}} + \sum_{\substack{S\subseteq \mathbb{I}_d\\ 1\notin S, 2\in S}} \frac{1}{2^{d-1}} = \frac{2^{d-1}+2^{d-2}}{2^{d-1}} = \frac{3}{2}.$$

Therefore, for all $j, \ell \in S$ with $j \neq \ell$,

$$\widehat{C}^{\mathrm{MO}^=}_{\{j,\ell\}}(u,u) = u^{3/2}, \quad 0 \le u \le 1.$$

(b) Parameters proportional to the cardinality of the sets: Here, $\lambda_S = |S|\lambda$ for all non-empty $S \subseteq \mathbb{I}_d$ where $\lambda > 0$ and we denote the survival copula by $\widehat{C}^{\mathrm{MO}^\infty}$. Also, the value of λ is irrelevant, and for all $j \in S$ and non-empty subset $S \subset \mathbb{I}_d$, we have

$$\eta_j^S = \frac{|S|}{(d+1)2^{d-2}}.$$

We compute again the value of $\eta^*_{j\ell}$ defined in (2.7) as

$$\eta^*_{j\ell} = \eta^*_{12} = \sum_{\substack{S\subseteq \mathbb{I}_d\\ 1\in S}} \frac{|S|}{(d+1)2^{d-2}} + \sum_{\substack{S\subseteq \mathbb{I}_d\\ 1\notin S, 2\in S}} \frac{|S|}{(d+1)2^{d-2}} = \frac{(d+1)2^{d-2}+d2^{d-3}}{(d+1)2^{d-2}} = 1 + \frac{d}{2(d+1)}.$$

Therefore, for all $j, \ell \in S$ with $j \neq \ell$,

$$\widehat{C}^{\mathrm{MO}^\infty}_{\{j,\ell\}}(u,u) = u^{1+d/(2(d+1))}, \quad 0 \le u \le 1.$$

The generalized MO copulas with these particular choices of parameters, as in (a) and (b), are also known as Caudras–Augé copulas (Cuadras and Augé, 1981) and have been used in Lévy frailty models for survival analysis. Moreover, if the marginals are identically distributed, then the associated random vector turns out to be exchangeable (Durrett, 1991).

Example 3 (Archimedean copula)**.** Archimedean copulas constitute a widely utilized family of copula models for constructing multivariate distributions (Joe, 2015; Charpentier and Segers, 2009). A d-dimensional copula C^ϕ is Archimedean if

$$C^\phi(u_1,\dots,u_d) := \phi^{\leftarrow}(\phi(u_1)+\cdots+\phi(u_d)) \tag{2.8}$$

for $0 \le u_1,\dots,u_d \le 1$, where the generator function $\phi : [0,1] \to [0,\infty]$ is convex and decreasing, with $\phi(1) = 0$ and $\phi^{\leftarrow}(y) = \inf\{u \in [0,1] : \phi(u) \le y\}$ for $y \in (0,\infty)$. Necessary and sufficient conditions on the function ϕ such that C^ϕ in (2.8) is a copula are given in McNeil and Nešlehová (2009). Note that the survival copula $\widehat{C}^\phi$ of an Archimedean copula C^ϕ is, in general, not Archimedean. A popular choice of ϕ is the Laplace transform of any positive random variable.

Tail dependence in such copulas has been studied in Charpentier and Segers (2009), Larsson and Nešlehová (2011), and Hua and Joe (2011), and sufficient conditions to obtain pairwise asymptotic independence exist. Suppose the random vector $\boldsymbol{Z}$ having an Archimedean copula C^ϕ has a generator ϕ satisfying

$$\lim_{u\downarrow 0} \frac{u\phi'(1-u)}{\phi(1-u)} = 1.$$

Then we may conclude from Charpentier and Segers (2009, Theorem 4.1 and equation (4.4)) that $\boldsymbol{Z}$ is pairwise asymptotically independent. In contrast, if the limit

$$\theta_1 := \lim_{u\downarrow 0} \frac{u\phi'(1-u)}{\phi(1-u)} \in (1,\infty)$$

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exists and is larger than 1, then $\boldsymbol{Z}$ has asymptotic upper tail dependence, that is, its survival copula $\widehat{C}^{\phi}$ satisfies (2.3). We observe from Charpentier and Segers (2009, Table 1) that for many Archimedean copulas, we have $\theta_1 = 1$ and thus they are pairwise asymptotically independent; this includes the Frank copula, Clayton copula, and Ali–Mikhail–Haq copula; see also Nelsen (2006, Table 4.1) for further details.

Example 4 (Gaussian copula)**.** The Gaussian dependence structure is perhaps the most popular one used in practice. Let Φ_Σ denote the distribution function of a d-variate normal distribution with all marginal means 0, variances 1, and a positive-definite correlation matrix $\Sigma \in \mathbb{R}^{d\times d}$, and let Φ denote the standard normal distribution function. Then for $0 < u_1, \ldots, u_d < 1$,

$$C^{\Sigma}(u_1, \ldots, u_d) = \Phi_\Sigma(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_d))$$

denotes the Gaussian copula with correlation matrix Σ. Note that by radial symmetry, any Gaussian copula is equal to its own survival copula, that is, $\widehat{C}^{\Sigma} = C^{\Sigma}$. Pairwise asymptotic independence has been well established for the bivariate normal distribution, as well as the bivariate Gaussian copula if the correlation is less than 1 (Sibuya, 1960; Ledford and Tawn, 1996). Hence, we may immediately conclude that for $d \geq 2$, a Gaussian copula C^{Σ} exhibits pairwise asymptotic independence if Σ is positive definite. In fact, it is possible to find the exact tail order for the Gaussian survival copula for any $S \subseteq \mathbb{I}_d$ with $|S| \geq 2$, where the precise result is given in Section 3.2.2.

3 Mutual asymptotic independence

Pairwise asymptotic independence has often either been used as a natural extension of asymptotic independence (Balkema and Nolde, 2010; Guillou et al., 2018), or taken as a consequence from other relevant properties (de Haan and Ferreira, 2006, Remark 6.2.5), or implicitly assumed (Lalancette et al., 2021) in a variety of works. Next, we define a notion that captures the global joint concurrent tail behaviour of random vectors portrayed by many popular multivariate dependence structures, including, for example, the Gaussian, MO, or various Archimedean copulas, but not restricted to the mere replication of pairwise comparisons of tails.

Definition 2 (Mutual asymptotic independence)**.** An $\mathbb{R}^d$-valued random vector $\boldsymbol{Z} \sim F$ with continuous marginal distributions, copula C, and associated survival copula $\widehat{C}$ is mutually asymptotically independent if for all $S \subseteq \mathbb{I}_d$ with $|S| \geq 2$, we have

$$\lim_{u\downarrow 0} \frac{\widehat{C}_S(u, \ldots, u)}{\widehat{C}_{S\setminus\{\ell\}}(u, \ldots, u)} = 0, \quad \forall\, \ell \in S, \tag{3.1}$$

where we define $0/0 := 0$. We interchangeably say $\boldsymbol{Z}, F$, or C possesses mutual asymptotic independence.

Remark 2. Some explanation is due here in order to distinguish between the traditional notion of multivariate asymptotic independence (sometimes called "mutual" asymptotic independence) in dimensions $d > 2$ (Resnick (2008, Chapter 5.5), Galambos (1978, Chapter 5.2), and McNeil et al. (2015, Chapter 7.6)), and the notion defined in Definition 2. Owing to Resnick (2008, Proposition 5.27) under the constraint that the marginals are continuous, identically distributed, and in the maximum domain of attraction of a univariate extreme value distribution, pairwise asymptotic independence is equivalent to a distribution having "multivariate asymptotic independence", meaning that the distribution lies in the maximum domain of attraction of an extreme value distribution with independent marginals (the limit distribution is a product measure). Our notion of mutual asymptotic independence is not equivalent to multivariate asymptotic independence as we demonstrate in this article, and in particular, in Example 5.

When $d = 2$, both (2.2) and (3.1) reduce to (2.1) and hence are equivalent. Assuming $d \geq 3$ and mutual asymptotic independence, if we take all choices of $S \subseteq \mathbb{I}_d$ with $|S| = 2$, then (3.1) is just a restatement of (2.2), implying pairwise asymptotic independence. We summarize this in the next proposition.

Proposition 1. *If an $\mathbb{R}^d$-valued random vector $\boldsymbol{Z}$ with continuous marginal distributions is mutually asymptotically independent, then it is also pairwise asymptotically independent.*

The reverse implication of Proposition 1 is not necessarily true, as we see in the following example, which mimics the consequences for the analogous notions of classical "mutual" and "pairwise independence" (Hogg et al., 2013).

Example 5. The difference between pairwise and mutual independence can be shown using an $\mathbb{R}^3$-valued random vector with Bernoulli marginals (Hogg et al., 2013, Chapter 2). We adopt a similar approach, but using uniform marginals. Consider i.i.d. uniform [0,1] random variables U, V and W. Then $\boldsymbol{Z}^* = (U, V, W)$ is mutually asymptotically independent (cf. Example 6) and hence pairwise asymptotically independent as well. Now consider $\boldsymbol{Z} = (Z_1, Z_2, Z_3) \sim F$ such that

$$\boldsymbol{Z} = \begin{cases} (U, V, \min(U, V)), & \text{with prob. } 1/3, \\ (U, \min(U, V), V), & \text{with prob. } 1/3, \\ (\min(U, V), U, V), & \text{with prob. } 1/3. \end{cases}$$

First note that for $0 < z < 1$, marginally,

$$F_j(z) = \mathrm{P}(Z_j \le z) = 2z/3 + 1/3[1 - (1-z)^2] = 4z/3 - z^2/3, \quad j \in \{1, 2, 3\},$$

and hence the Z_j's are identically distributed.

(i) If $\widehat{C}$ denotes the survival copula of $\boldsymbol{Z}$, then we can check that for any $\{j, \ell\} \subset \{1, 2, 3\}$,

$$\begin{aligned} \widehat{C}_{\{j,\ell\}}(u, u) &= \mathrm{P}(Z_j > 2 - \sqrt{1+3u}, Z_\ell > 2 - \sqrt{1+3u}) \\ &= \mathrm{P}(U > 2 - \sqrt{1+3u}, V > 2 - \sqrt{1+3u}) \\ &= (\sqrt{1+3u} - 1)^2 = 9u^2/4 + o(u^2), \quad u \downarrow 0. \end{aligned} \tag{3.2}$$

Hence, $\boldsymbol{Z}$ exhibits pairwise asymptotic independence.

(ii) But

$$\begin{aligned} \widehat{C}(u, u, u) &= \mathrm{P}(U > 2 - \sqrt{1+3u}, V > 2 - \sqrt{1+3u}) \\ &= (\sqrt{1+3u} - 1)^2 = 9u^2/4 + o(u^2), \quad u \downarrow 0, \end{aligned}$$

implying that $\boldsymbol{Z}$ does not have mutual asymptotic independence since (3.1) fails for $S = \{1, 2, 3\}$.

(iii) We can compute that if $\boldsymbol{Z}^{(1)}, \dots, \boldsymbol{Z}^{(n)}$ are i.i.d. F and if $\boldsymbol{M}_n$ is the random vector of component-wise maxima given by

$$\boldsymbol{M}_n = \left(\bigvee_{i=1}^n Z_1^{(i)}, \bigvee_{i=1}^n Z_2^{(i)}, \bigvee_{i=1}^n Z_3^{(i)} \right),$$

then with $a_n = 3/2n$ and $b_n = 1$, we have for $\boldsymbol{x} \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathrm{P}(\boldsymbol{M}_n \le a_n \boldsymbol{x} + b_n) = \lim_{n\to\infty} F^n(a_n \boldsymbol{x} + b_n) = G(\boldsymbol{x}) = \prod_{i=1}^3 \Psi_1(x_i), \tag{3.3}$$

where $\Psi_1(\cdot)$ is a univariate extreme value distribution given by $\Psi_1(x) = \min(e^x, 1), x \in \mathbb{R}$. Thus $F \in \mathrm{MDA}(G)$, where G is indeed a product measure according to (3.3), implying multivariate asymptotic independence although F does not have mutual asymptotic independence as shown in (ii).

For illustration, we showed the multivariate asymptotic independence by hand, but the pairwise asymptotic independence in (i) and Resnick (2008, Proposition 5.27) already imply multivariate asymptotic independence.

3.1 Examples: Part I

It is instructive to note examples of mutual asymptotic independence in various distributions.

Example 6 (Independence). Suppose C^{ind} is the independence copula as given in (2.4), then the survival copula for any non-empty subset $S \subset \mathbb{I}_d$ satisfies

$$\widehat{C}_S^{\text{ind}}(u, \dots, u) = u^{|S|}, \qquad 0 \le u \le 1.$$

Thus, (3.1) holds for all such S with $|S| \ge 2$, and hence C^{ind} exhibits mutual asymptotic independence.

Example 7 (Marshall–Olkin dependence). In Example 2, we stated that any random vector $\boldsymbol{Z}$ with dependence given by the generalized MO survival copula $\widehat{C}^{\text{MO}(\Lambda)}$ as defined in (2.5) is pairwise asymptotically independent. In fact, by Lin and Li (2014, Proposition 3) we can conclude that Z is indeed mutually asymptotically independent as well.

3.2 Examples: Part II

In this section, we discuss examples that are pairwise asymptotically independent but may not be mutually asymptotically independent. This will include a large class of examples from the Archimedean copula and the Gaussian copula family.

3.2.1 Archimedean copulas

Recall the Archimedean copula C^{ϕ} defined in Example 3. The following result provides sufficient conditions on the generator ϕ for a random vector with Archimedean copula C^{ϕ} to possess both pairwise and mutual asymptotic independence.

Theorem 1 (Archimedean copula with mutual asymptotic independence). *Let the dependence structure of an $\mathbb{R}^d$-valued random vector $\boldsymbol{Z}$ with continuous marginal distributions be given by an Archimedean copula C^{ϕ} with generator ϕ as in (2.8). Suppose $\phi^{\leftarrow}$ is d-times continuously differentiable and $(-D)^j \phi^{\leftarrow}(0) < \infty \ \forall \ j \in \mathbb{I}_d$. Then $\boldsymbol{Z}$ possesses both pairwise and mutual asymptotic independence.*

The proof follows directly from Charpentier and Segers (2009, Theorem 4.3). The Archimedean copulas of Theorem 1 have the property that for any subset $S \subset \mathbb{I}_d$ with $|S| \ge 2$, the survival copula $\widehat{C}_S^{\phi}$ of the $|S|$-dimensional marginal behaves like the independence copula near the tails, that is,

$$\widehat{C}_S^{\phi}(u, \dots, u) \sim u^{\kappa_S}, \quad u \downarrow 0,$$

where the upper tail order of C_S^{ϕ} is $\kappa_S = |S|$ (also follows from Charpentier and Segers (2009, Theorem 4.3)). In particular, the upper tail order for C^{ϕ} is $\kappa = \kappa_{\mathbb{I}_d} = d$, and hence these copulas are also "nearly independent" (see paragraph below (1.2)); several popular Archimedean copula models such as the Frank copula, Clayton copula, and Ali–Mikhail–Haq copula (Charpentier and Segers, 2009, Table 1) fall into this class exhibiting both pairwise and mutual asymptotic independence. In contrast, there are also Archimedean copulas exhibiting pairwise asymptotic independence but not mutual asymptotic independence. The following result provides sufficient conditions on the generator ϕ to obtain such Archimedean copulas.

Theorem 2 (Archimedean copula with only pairwise asymptotic independence). *Let the dependence structure of a random vector $\boldsymbol{Z} \in \mathbb{R}^d$ with continuous marginal distributions be given by an Archimedean copula C^{ϕ} with generator ϕ as in (2.8). Suppose $\phi'(1) = 0$ and*

$$L(u) := -\phi'(1-u) - u^{-1}\phi(1-u)$$

is a positive function, which is slowly varying at 0. Then, $\boldsymbol{Z}$ possesses pairwise asymptotic independence but does not possess mutual asymptotic independence.

The proof follows directly from Charpentier and Segers (2009, Theorem 4.6 and Corollary 4.7). Now, the Archimedean copulas of Theorem 2 have a different characteristic in the sense that for any subset $S \subset \mathbb{I}_d$ with $|S| \ge 2$, the survival copula $\widehat{C}_S^{\phi}$ on the $|S|$-dimensional marginal behaves as

$$\widehat{C}_S^{\phi}(u, \dots, u) \sim u\ell(u), \quad u \downarrow 0,$$

where ℓ is a slowly varying function at 0 (this follows from Charpentier and Segers (2009, Corollary 4.7)). Hence, the upper tail order of C_S^{ϕ} is $\kappa_S = 1$ for all $S \subset \mathbb{I}_d$ with $|S| \geq 2$. To obtain an example of such a copula, fix some parameter $\theta \in (0, \infty)$ and define the generator

$$\phi_\theta(u) = \frac{1-u}{(-\log(1-u))^\theta}, \quad 0 \leq u \leq 1.$$

Then C^{ϕ_θ} satisfies the assumptions of Theorem 2, resulting in an Archimedean copula with pairwise but not mutual asymptotic independence; we refer to Charpentier and Segers (2009, Table 1).

3.2.2 Gaussian copula

In Example 4, we observed that any random vector with a Gaussian copula having a positive-definite correlation matrix has pairwise asymptotic independence. Interestingly, not all such models will have mutual asymptotic independence. The following theorem provides the exact condition for this.

Theorem 3. *Let the dependence structure of an $\mathbb{R}^d$-valued random vector $\boldsymbol{Z}$ with continuous marginal distributions be given by a Gaussian copula C^Σ with positive-definite correlation matrix Σ. Then, $\boldsymbol{Z}$ exhibits mutual asymptotic independence if and only if $\Sigma_S^{-1}\mathbf{1}_{|S|} > \mathbf{0}_{|S|}$ for all non-empty sets $S \subseteq \mathbb{I}_d$.*

The proof of the theorem is quite involved, requiring a few auxiliary results based on the recently derived knowledge on the asymptotic behaviour of tail probabilities of a multivariate distribution with identically distributed Pareto marginals and Gaussian copula C^Σ in Das and Fasen-Hartmann (2024). Hence the proof has been relegated to Appendix A. Curiously, the ingredients of the proof allow us to find the tail asymptotics of the survival copula of any $|S|$-dimensional marginal in terms of its tail order.

Proposition 2. *Let C^Σ be a Gaussian copula with positive-definite correlation matrix Σ. Then for any subset $S \subset \mathbb{I}_d$ with $|S| \geq 2$, we have as $u \downarrow 0$,*

$$\widehat{C}_S^{\Sigma}(u, \dots, u) \sim u^{\kappa_S} \ell_S(u), \tag{3.4}$$

where ℓ_S is slowly varying at 0 and

$$\kappa_S = \min_{\{\boldsymbol{z} \in \mathbb{R}^{|S|} : \boldsymbol{z} \geq \mathbf{1}_S\}} \boldsymbol{z}^\top \Sigma_S^{-1} \boldsymbol{z}.$$

A proof of this result is given in Appendix A as well.

Remark 3. A few interesting features of Proposition 2 and related results are to be noted here.

(a) Although Proposition 2 only gives the tail order of $\widehat{C}_S^{\Sigma}$, in fact, the exact tail asymptotics for $\widehat{C}_S^{\Sigma}(u\boldsymbol{v}_S)$ as $u \downarrow 0$ for $\boldsymbol{v}_S = (v_s)_{s \in S}$, $v_s \in (0, 1)$, including the slowly varying function are available in Proposition 4 in Appendix A.

(b) The upper tail order κ_S in (3.4) is obtained as a solution to a quadratic programming problem; the exact solution is given in Lemma 2 in Appendix A.

(c) With respect to (3.4), for subsets $S, T \subset \mathbb{I}_d$ with $S \subsetneq T$ and $|S| \geq 2$, it is possible to have (i) $\kappa_S < \kappa_T$, (ii) $\kappa_S = \kappa_T$, with $\ell_S(u) \sim c\, \ell_T(u), u \downarrow 0$ for $c > 0$, and (iii) $\kappa_S = \kappa_T$, with $\ell_S(u) = o(\ell_T(u)), u \downarrow 0$. In Example 8, we can observe both (i) and (ii) holding under different assumptions; an example for (iii) with Pareto marginals and Gaussian copulas is available in Das and Fasen-Hartmann (2024, Remark 5).

(d) In Hua and Joe (2011, Example 1), the authors already state that the tail order κ of a Gaussian copula with positive-definite correlation matrix Σ is $\kappa = \mathbf{1}^\top \Sigma^{-1} \mathbf{1}$ (cf. Joe (2015, Section 4.3.2)). However, to the best of our knowledge, the aforementioned paper does not specify that $\Sigma^{-1}\mathbf{1} > \mathbf{0}$ is indeed a necessary condition for the result, since otherwise the statement is not valid; in fact, if $\Sigma^{-1}\mathbf{1} \not> \mathbf{0}$, then $\kappa < \mathbf{1}^\top \Sigma^{-1} \mathbf{1}$ is possible (cf. Lemma 2).

Example 8. For the purpose of illustration, we provide a positive-definite correlation matrix (with $d = 3$) for a Gaussian copula parameterized by a single parameter ρ which exhibits mutual asymptotic independence for

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only certain values of ρ and only pairwise asymptotic independence for other feasible values; see Das and Fasen-Hartmann (2024, Example 1(b)) for further details. Throughout, we denote by $\ell_j(u)$, $j \in \mathbb{N}$, a slowly varying function at 0. Consider the Gaussian copula C^{Σ} with correlation matrix

$$\Sigma = \begin{pmatrix} 1 & \rho & \sqrt{2}\rho \\ \rho & 1 & \sqrt{2}\rho \\ \sqrt{2}\rho & \sqrt{2}\rho & 1 \end{pmatrix},$$

where $\rho \in \left((1-\sqrt{17})/8, (1+\sqrt{17})/8\right) \approx (-0.39, 0.64)$, which ensures the positive definiteness of Σ. Clearly, pairwise asymptotic independence holds for all such ρ values.

(i) Suppose $\rho < 1/(2\sqrt{2}-1) \approx 0.55$. Then one can check that $\Sigma^{-1}\mathbf{1} > \mathbf{0}$, and hence mutual asymptotic independence holds as well. In fact, we can find the behaviour of the survival copula (using Proposition 4 or Das and Fasen-Hartmann (2024, Example 1(b))): As $u \downarrow 0$,

$$\widehat{C}^{\Sigma}(u,u,u) \sim u^{\frac{3-(4\sqrt{2}-1)\rho}{1+\rho-4\rho^2}} \ell_1(u). \tag{3.5}$$

We also find that as $u \downarrow 0$,

$$\begin{aligned} \widehat{C}^{\Sigma}_{\{13\}}(u,u) &= \widehat{C}^{\Sigma}_{\{23\}}(u,u) \sim u^{\frac{2}{1+\sqrt{2}\rho}} \ell_2(u), \quad \text{and} \\ \widehat{C}^{\Sigma}_{\{12\}}(u,u) &\sim u^{\frac{2}{1+\rho}} \ell_3(u). \end{aligned} \tag{3.6}$$

(ii) On the other hand, if $\rho \geq 1/(2\sqrt{2}-1) \approx 0.55$, then $\Sigma^{-1}\mathbf{1} \not> \mathbf{0}$ and the copula does not have mutual asymptotic independence. In this case, the behaviour of the two-dimensional marginal survival copulas will still be given by (3.6), but the tail behaviour as seen in (3.5) does not hold anymore. Now, as $u \downarrow 0$, we have

$$\widehat{C}^{\Sigma}(u,u,u) \sim u^{\frac{2}{1+\rho}} \ell_4(u).$$

In fact, we can check that $\ell_4(u) \sim \beta\, \ell_3(u)$ as $u \downarrow 0$ for some constant $\beta > 0$ (Das and Fasen-Hartmann, 2024, Example 1(b)), and hence

$$\widehat{C}^{\Sigma}(u,u,u) \sim \beta\, \widehat{C}^{\Sigma}_{\{12\}}(u,u), \quad u \downarrow 0,$$

also verifying that mutual asymptotic independence does indeed not hold here.

4 k-wise asymptotic independence

The fact that some multivariate models exhibit pairwise asymptotic independence yet not mutual asymptotic independence naturally prompts an inquiry into the existence of models that lie in between these two notions. The following definition provides an answer.

Definition 3 (k-wise asymptotic independence). An $\mathbb{R}^d$-valued random vector $\boldsymbol{Z} \sim F$ with continuous marginal distributions, copula C, and associated survival copula $\widehat{C}$ is *k-wise asymptotically independent* for a fixed $k \in \{2, \ldots, d\}$ if for all $S \subseteq \mathbb{I}_d$ with $2 \leq |S| \leq k$, we have

$$\lim_{u \downarrow 0} \frac{\widehat{C}_S(u,\ldots,u)}{\widehat{C}_{S\setminus\{\ell\}}(u,\ldots,u)} = 0, \quad \forall\ \ell \in S,$$

where we define $0/0 := 0$. We interchangeably say $\boldsymbol{Z}, F$, or C possesses k-wise asymptotic independence.

Remark 4. The concept of k-wise asymptotic independence allows us to assess dependence for multidimensional extremes in finer detail. If a random vector $\boldsymbol{Z}$ exhibits k-wise asymptotic independence but not $(k+1)$-wise asymptotic independence, then there exists a combination of exactly k components in $\boldsymbol{Z}$ such that when these are large, another component is large as well; moreover, fewer than k large components cannot produce a large value in another component. Consequently, lower values of k reflect a stronger dependence in the extremes.

Note that for any d-dimensional copula, d-wise asymptotic independence is the same as mutual asymptotic independence (and of course 2-wise is the same as pairwise). Again, following Proposition 1, we may check that mutual asymptotic independence indeed implies k-wise asymptotic independence for all $k \in \{2, \dots, d\}$. The converse of the previous implication is, of course, not true; the examples in the following section also show this.

Obviously, an equivalent characterization of k-wise asymptotic independence is the following.

Proposition 3. *An $\mathbb{R}^d$-valued random vector $\boldsymbol{Z}$ is k-wise asymptotically independent if and only if for all $S \subseteq \mathbb{I}_d$ with $|S| = k$, the $\mathbb{R}^k$-valued random vector $\boldsymbol{Z}_S$ is mutually asymptotically independent.*

4.1 Examples

Indeed, within the class of Archimedean copulas as well as the class of Gaussian copulas with dimensions $d > 2$, we find examples of models that exhibit k-wise asymptotic independence, but not $(k+1)$-wise asymptotic independence given any $k \in \{2, \dots, d-1\}$. Consequently, these models are also not mutually asymptotically independent. Let us begin with an investigation of a particular Archimedean copula.

4.1.1 ACIG copula

This Archimedean copula based on the Laplace transform (LT) of an inverse-gamma distribution, called the ACIG copula in short, was introduced in Hua and Joe (2011), and operates like this: if $Y = X^{-1}$ and $X \sim \text{Gamma}(\alpha, 1)$ for $\alpha > 0$, then the generator of this Archimedean copula is given by the LT of Y. The expression of the generator includes the Bessel function of the second kind. Closed-form expressions of the copula C^{ϕ} and survival copula $\widehat{C}^{\phi}$ are not easy to write down; nevertheless, from computations in Hua and Joe (2011, Example 4) and Hua et al. (2014, Example 4.4), we can conclude that for any $d \geq 2$, the survival copula of the ACIG copula with parameter $\alpha > 0$ has the following asymptotic behaviour:

$$\widehat{C}^{\phi}(u, \dots, u) \sim \beta_d u^{\kappa_d}, \qquad u \downarrow 0, \tag{4.1}$$

where $\kappa_d = \max\{1, \min\{\alpha, d\}\}$ and $\beta_d > 0$ is a positive constant. Here, κ_d is the tail order of the copula. Therefore, if $\alpha \leq 1$ then $\kappa_d = 1$ for all $d \geq 2$, and if $\alpha > 1$, then $\kappa_d = \min(\alpha, d)$. Note that by the exchangeability property of Archimedean copulas and (4.1), we know that for any $S \subset \mathbb{I}_d$ with $|S| \geq 2$,

$$\widehat{C}^{\phi}_S(u, \dots, u) \sim \beta_{|S|} u^{\kappa_{|S|}}, \qquad u \downarrow 0.$$

Thus, we may conclude that for an ACIG copula with parameter $\alpha > 0$, the following holds:

(i) If $0 < \alpha \leq 1$, the ACIG copula exhibits asymptotic upper tail dependence.

(ii) If $1 < \alpha \leq d-1$, the ACIG copula exhibits pairwise asymptotic independence but not mutual asymptotic independence. If additionally, $k-1 < \alpha \leq k$ for $k \in \{2, \dots, d-1\}$, then the ACIG copula still exhibits i-wise asymptotic independence for all $i \in \{2, \dots, k\}$, but not $(k+1)$-wise asymptotic independence.

(iii) If $\alpha > d-1$, the ACIG copula exhibits k-wise asymptotic independence for all $k \in \{2, \dots, d\}$, and hence exhibits mutual asymptotic independence as well.

4.1.2 Gaussian copula

The Gaussian copula has been popular in modelling dependence in a wide variety of applications. It turns out that a class of Gaussian copula models is also able to capture the presence of k-wise asymptotic independence and not $(k+1)$-wise asymptotic independence. This is demonstrated in the following result, whose proof is given in Appendix B.

Theorem 4. *Suppose $k \in \{2, \dots, d-1\}$ and $S_1 \subseteq S_2 \subseteq \{1, \dots, d\}$ with $|S_1| = k$ and $|S_2| = k+1$. Then there exists a Gaussian copula C^{Σ} and a positive-definite correlation matrix Σ, such that C^{Σ} exhibits k-wise asymptotic independence but not $(k+1)$-wise asymptotic independence and for any $x > 0$,*

$$\lim_{u \downarrow 0} \frac{\widehat{C}^{\Sigma}_{S_2}(u, \dots, u, xu, u, \dots, u)}{\widehat{C}^{\Sigma}_{S_1}(u, \dots, u)} = 1, \tag{4.2}$$

where xu is placed at the unique element in $S_2 \setminus S_1$.

This theorem not only provides the existence of a k-wise asymptotically independent Gaussian copula but also gives the striking feature of the copula behaviour in (4.2), where, surprisingly, the value of x has no influence.

This means that for a random vector $\boldsymbol{Z}$ with Gaussian copula C^{Σ}, as given in Theorem 4, and identically distributed marginals, large values in all the S_1 components result in an extremely large value in the single component of $S_2 \setminus S_1$; hence there is a strong dependence between the extremes of the components of S_1 and that of $S_2 \setminus S_1$. All components in S_1 must be large at the same time; only a few components that are large do not result in a large value in the $S_2 \setminus S_1$ component. This is demonstrated quite nicely in the following example, which was used in the proof of Theorem 4.

Example 9. Let the correlation matrix of the Gaussian copula be given as

$$\Sigma = \begin{bmatrix} \boldsymbol{I}_{d-1} & \rho \mathbf{1}_{d-1} \\ \rho \mathbf{1}_{d-1}^{\top} & 1 \end{bmatrix},$$

for some $\rho \in (0, 1)$. Then, for $\rho \in (1/(k-1), 1/(k-2))$, the Gaussian copula C^{Σ} is $(k-1)$-wise asymptotically independent but not k-wise asymptotically independent (see proof of Theorem 4); therefore, if the first $(k-1)$ components are jointly large, then the last component is also large. But if we consider fewer components than $(k-1)$ components to be large, they have no effect on the size of the last component. Note that here $k - 1 = \lceil \rho^{-1} \rceil = \inf\{m \in \mathbb{N} : \rho^{-1} \le m\}$. Thus, for a high value of ρ, fewer components, namely only the first $\lceil \rho^{-1} \rceil$ components, result in a large value in the last component. Although ρ provides a measure of linear dependence, in this example it also measures the dependence in the extremes, which is reflected in the degree $\lceil \rho^{-1} \rceil$ of asymptotic independence.

For the derivation of worst case measures for risk contagion under distributional ambiguity in the next section, Theorem 4 turns out to be indispensable.

Remark 5. It is important to mention here that although most popular copula families are bivariate in nature, Joe (2015, Chapter 4) lists multiple extensions of bivariate copulas to general high dimensions; many such copulas can be explored for creating models with particular levels of asymptotic independence as necessitated by the context.

5 Implication on risk management under distributional ambiguity

In financial risk management, a variety of risk measures are used to assess the risk contagion between different financial products, including stocks, bonds, and equities. Such contagion or systemic risk measures are often based on conditional probabilities and range from computing regular conditional tail probabilities to CoVaR, marginal expected shortfall (MES), marginal mean excess (MME), and more; see Das and Fasen-Hartmann (2018) and Adrian and Brunnermeier (2016) for details.

Here, we focus on two such measures of risk contagion based on specific conditional tail probabilities and conditional tail quantiles. First, recall that for a random variable Z, the *Value-at-Risk* or VaR at level $1 - \gamma \in (0, 1)$ is defined as

$$\mathrm{VaR}_{\gamma}(Z) := \inf\{y \in \mathbb{R} : \mathrm{P}(Z > y) \le \gamma\} = \inf\{y \in \mathbb{R} : \mathrm{P}(Z \le y) \ge 1 - \gamma\},$$

the $(1-\gamma)$-quantile of Z where $\inf \emptyset := \infty$ (cf. Embrechts et al. (2024)). If $Z \sim F$ and F is continuous and strictly increasing, then F is invertible with inverse F^{-1}, and for $0 < \gamma < 1$, we have $\mathrm{VaR}_{\gamma}(Z) = F^{-1}(1-\gamma)$.

Consider the returns from a portfolio of $d > 1$ stocks being given by the random vector $\boldsymbol{Z} = (Z_1, \dots, Z_d) \sim F$. Suppose we are interested in measuring the risk of Z_1 having an extremely large value, given that all variables

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in some non-empty subset $J \subset \mathbb{I}_d \setminus \{1\}$ with $|J| = \ell$ are at extremely high levels. This can be captured via the following *conditional tail probability* (CTP)

$$\mathrm{P}\big(Z_1 > t | Z_j > t,\ \forall j\ \in J\big), \tag{5.1}$$

as $t \to \infty$. Alternatively, for a level $\gamma \in (0,1)$, we are interested in the risk measure

$$\mathrm{CTP}_\gamma(\boldsymbol{Z}_{1|J}) := \mathrm{P}\big(Z_1 > \mathrm{VaR}_\gamma(Z_1) | Z_j > \mathrm{VaR}_\gamma(Z_j),\ \forall j\ \in J\big), \tag{5.2}$$

as $\gamma \to 0$. Note that (5.1) and (5.2) are equivalent if all the marginal random variables $Z_1, \ldots, Z_d$ are identically distributed. For convenience, we will focus on the measure CTP_γ as defined in (5.2).

A second measure of risk contagion we are interested in is a generalization of the VaR to the multivariate setting given by the *Contagion Value-at-Risk or CoVaR* at confidence level (γ_1, γ_2) for $\gamma_1, \gamma_2 \in (0,1)$, defined as

$$\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J}) := \inf\{z \in \mathbb{R}_+ : \mathrm{P}(Z_1 > z | Z_j > \mathrm{VaR}_{\gamma_2}(Z_j),\ \forall j \in J) \le \gamma_1\}. \tag{5.3}$$

The risk measure CoVaR was introduced in the bivariate setting for $J = 2$ to capture risk contagion, as well as systemic risk by Adrian and Brunnermeier (2016) where they set the conditioning event to be $Z_2 = \mathrm{VaR}_{\gamma_2}(Z_2)$; this was later modified by Girardi and Ergün (2013) to $Z_2 > \mathrm{VaR}_{\gamma_2}(Z_2)$ with the restriction that $\gamma_1 = \gamma_2$; this latter definition has been widely used in dependence modelling (Nolde et al., 2022; Mainik and Schaanning, 2014; Härdle et al., 2016; Das and Fasen-Hartmann, 2025) and is generalized in our definition given in (5.3).

In risk management applications, computing quantities like CTP and CoVaR requires knowledge of the joint distribution of the risk vector $\boldsymbol{Z}$. Even if the univariate distributions of all the marginal variables can be estimated, the joint distribution often remains unknown and relatively more involved for estimation purposes. An approach often used is to provide a worst case value for such risk measures under certain constraints on the joint distribution of the variables. Naturally, for such tail risk measures, constraints can be provided in terms of their joint asymptotic tail behaviour, including pairwise, mutual, or k-wise asymptotic independence. It turns out that under different constraints, we may obtain a different tail behaviour for the worst case measures. To further this discussion, let us define $\mathcal{P}$ to be the class of all probability distributions in $\mathbb{R}^d$ with continuous marginal distributions.

For $k \in \{2, \ldots, d\}$, define the classes of distributions

$$\mathcal{P}_k := \{F \in \mathcal{P} : F \text{ possesses } k\text{-wise asymptotic independence}\},$$

and similarly, the restriction of $\mathcal{P}_k$ to distributions with Gaussian copulas,

$$\mathcal{N}_k := \mathcal{P}_k \cap \{F \in \mathcal{P} : F \text{ has a Gaussian copula } C^\Sigma \text{ with } \Sigma \text{ positive definite}\}.$$

Note that $\mathcal{P}_2$ models the class of pairwise asymptotically independent random vectors, whereas $\mathcal{P}_d$ models the class of mutually asymptotically independent random vectors. By Definition 3, it is easy to check that

$$\mathcal{P} \supseteq \mathcal{P}_2 \supseteq \mathcal{P}_3 \supseteq \cdots \supseteq \mathcal{P}_d \quad \text{and} \quad \mathcal{P} \supseteq \mathcal{N}_2 \supseteq \mathcal{N}_3 \supseteq \cdots \supseteq \mathcal{N}_d. \tag{5.4}$$

Furthermore, these classes are non-empty, since $\mathcal{N}_k \neq \emptyset$ by Theorem 4 and $\mathcal{N}_k \subseteq \mathcal{P}_k$.

Since the joint distributions are unknown, we may want to find the worst case CTP or CoVaR in such cases where $F \in \mathcal{P}_k \subset \mathcal{P}$ or $F \in \mathcal{N}_k$, $k \in \{2, \ldots, d\}$. First, we present the result for the CTP. The proofs of this theorem and all subsequent results in this section are given in Appendix C.

Theorem 5. *Suppose an $\mathbb{R}^d$-valued random vector $\boldsymbol{Z} = (Z_1, \ldots, Z_d) \sim F$, $d \ge 2$, has continuous marginal distributions. Furthermore, suppose $J \subset \mathbb{I}_d \setminus \{1\}$ with $|J| = \ell$.*

(*a*) *If $k \in \{\ell + 1, \ldots, d\}$, then*

$$\sup_{F \in \mathcal{N}_k} \lim_{\gamma \downarrow 0} \mathrm{CTP}_\gamma(\boldsymbol{Z}_{1|J}) = \sup_{F \in \mathcal{P}_k} \lim_{\gamma \downarrow 0} \mathrm{CTP}_\gamma(\boldsymbol{Z}_{1|J}) = 0.$$

(*b*) *If $k \in \{2, \ldots, \ell\}$, then*

$$\sup_{F \in \mathcal{N}_k} \lim_{\gamma \downarrow 0} \mathrm{CTP}_\gamma(\boldsymbol{Z}_{1|J}) = \sup_{F \in \mathcal{P}_k} \lim_{\gamma \downarrow 0} \mathrm{CTP}_\gamma(\boldsymbol{Z}_{1|J}) = 1.$$

The results indicate a qualitatively different behaviour of the worst case CTP depending on whether the tail dependence exhibits k-wise asymptotic independence with $k > |J|$ vis-a-vis $k \leq |J|$. When $k > |J|$, $\mathrm{CTP}_{\gamma}(\boldsymbol{Z}_{1|J})$ converges to 0 as $\gamma \downarrow 0$, suggesting that extremely large losses of Z_j for all $j \in J$ have a negligible influence on extremely large losses of Z_1. In contrast, when $k \leq |J|$, there exists a k-wise asymptotically independent distribution function F, which is also pairwise asymptotically independent, such that extremely large losses of Z_j for all $j \in J$ result, with a probability converging to 1, in an extremely large loss of Z_1. In particular, for the Gaussian copula, that is an astonishing result because it is in contrast to the belief that there are no joint extremes. This shows that for measuring risk contagion, it is important to distinguish between these different concepts of tail independence and that assuming an improper notion of asymptotic independence for our risk portfolio may lead to either underestimation or overestimation of the risk contagion.

In the following, we investigate the asymptotic behaviour of the measure CoVaR. For technical reasons, we restrict the class $\mathcal{P}_k$ slightly; in particular, we will assume that F_1, the distribution of Z_1, is Pareto-distributed, that is, $F_1(z) = 1 - z^{-\alpha}$, $z \geq 1$, for some $\alpha > 0$. Suppose $\mathcal{P}^* := \{F \in \mathcal{P} : F_1 \text{ is Pareto-distributed}\}$. For $k \in \{2, \dots, d\}$, define the classes

$$\mathcal{P}_k^* := \mathcal{P}_k \cap \left\{ F \in \mathcal{P}^* : \sup_{\gamma \in (0, x^{-1}]} \frac{\widehat{C}_S(x\gamma, \gamma, \dots, \gamma)}{\widehat{C}_S(\gamma, \gamma, \dots, \gamma)} < \infty, \ \forall\, S \subseteq \mathbb{I}_d, \forall\, x \geq 1 \right\} \subseteq \mathcal{P}_k,$$

and

$$\mathcal{N}_k^* := \mathcal{N}_k \cap \mathcal{P}^* \subseteq \mathcal{N}_k.$$

Remark 6. Instead of assuming that F_1 follows a Pareto distribution, it is possible to consider a broader class, allowing F_1 to have a regularly varying tail. However, this approach makes the proofs more technical without providing any further valuable insights. Hence, we have shown our results for the smaller class $\mathcal{P}_k^*$ for the purpose of exposition.

Although we reduce the class $\mathcal{P}_k$ to $\mathcal{P}_k^*$, it still remains quite large and contains, in particular, k-wise asymptotically independent Gaussian copulas (with F_1 being Pareto-distributed).

Lemma 1. $\mathcal{N}_k^* \subseteq \mathcal{P}_k^*$ *for* $k \in \{2, \dots, d\}$.

By restricting our consideration to the sets $\mathcal{P}_k^*$ and $\mathcal{N}_k^*$, we derive the subsequent result concerning the asymptotic behaviour of the CoVaR.

Theorem 6. *Suppose an* $\mathbb{R}^d$*-valued random vector* $\boldsymbol{Z} = (Z_1, \dots, Z_d) \sim F$, $d \geq 2$, *has continuous marginal distributions and the marginal* F_1 *is a Pareto distribution. Furthermore, let* $J \subset \mathbb{I}_d \setminus \{1\}$ *with* $|J| = \ell$.

(*a*) *If* $k \in \{\ell + 1, \dots, d\}$, *then for any* $\gamma_1 \in (0, 1)$,

$$\sup_{F \in \mathcal{N}_k^*} \lim_{\gamma_2 \downarrow 0} \frac{\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J})}{\mathrm{VaR}_{\gamma_2}(Z_1)} = \sup_{F \in \mathcal{P}_k^*} \lim_{\gamma_2 \downarrow 0} \frac{\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J})}{\mathrm{VaR}_{\gamma_2}(Z_1)} = 0.$$

(*b*) *If* $k \in \{2, \dots, \ell\}$, *then for any* $\gamma_1 \in (0, 1)$,

$$\sup_{F \in \mathcal{N}_k^*} \lim_{\gamma_2 \downarrow 0} \frac{\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J})}{\mathrm{VaR}_{\gamma_2}(Z_1)} = \sup_{F \in \mathcal{P}_k^*} \lim_{\gamma_2 \downarrow 0} \frac{\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J})}{\mathrm{VaR}_{\gamma_2}(Z_1)} = \infty.$$

Similar to the case of finding the worst case CTP, we observe that the worst case CoVaR also has a qualitatively different behaviour depending on whether the tail dependence exhibits k-wise asymptotic independence with $k > |J|$, or with $k \leq |J|$. When $k > |J|$, the ratio $\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J})/\mathrm{VaR}_{\gamma_2}(Z_1)$ converges to 0, reflecting that $\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J})$ increases at a negligible rate in comparison to $\mathrm{VaR}_{\gamma_2}(Z_1)$ as $\gamma_2 \downarrow 0$ and that $\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J})$ is relatively small; in other words, the required risk reserve capital is low. But if $k \leq |J|$, there exists a $F \in \mathcal{N}_k^* \subseteq \mathcal{P}_k^*$ where $\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J})/\mathrm{VaR}_{\gamma_2}(Z_1)$ converges to ∞, so that $\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J})$ may increase much faster to ∞ than $\mathrm{VaR}_{\gamma_2}(Z_1)$ as $\gamma_2 \downarrow 0$, giving a relatively high $\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J})$ and a higher reserve risk capital requirement.

Remark 7. Computations analogous to the ones carried out in this section can also be done for other measures of risk contagion, such as the marginal expected shortfall (MES), or the marginal mean excess (MME) (Cai et al., 2015; Das and Fasen-Hartmann, 2018). Note that, as when computing CoVaR, we need to restrict $\mathcal{P}_k$ to smaller classes satisfying various technical conditions. We leave these pursuits for interested readers to explore.

6 Conclusion

In this article, we provided a notion of multivariate asymptotic independence, which is useful in comparing extreme events in different dimensions beyond mere pairwise comparisons (which have traditionally been used in the literature). This parallels the dichotomy of mutual independence vis-a-vis pairwise independence for multivariate random vectors. We believe this new notion also provides an alternate pathway for characterizing extremal dependence for high-dimensional problems relating to tail events. We have illustrated this idea using examples of particular copula models, including a few from the Archimedean family, along with the Gaussian and MO copulas. The copulas considered often exhibit at least pairwise asymptotic independence if not mutual asymptotic independence. For both Archimedean and Gaussian copulas, we presented examples exhibiting not only mutual asymptotic independence but also pairwise asymptotic independence without mutual asymptotic independence. In particular, for the Gaussian copula, this result is quite striking since it is in contrast to the common belief that the Gaussian copula does not allow joint extremes. We have also introduced the concept of k-wise asymptotic independence, which generalizes these two notions (pairwise and mutual) and brings them under the same umbrella. Here we have shown that for any $k \in \{2, \dots, d\}$, there exists a k-wise asymptotically independent Gaussian copula (which is not $(k+1)$-wise asymptotically independent if $k < d$). Moreover, we have shown that these assumptions of different notions of asymptotic tail independence significantly impact measures of risk contagion within a financial system, such as CTP or CoVaR, depending on the specific context. Overlooking these concepts and assuming merely pairwise asymptotic independence for models may often lead to a significant underestimation of risks.

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A Proofs of Section 3

First, we present some auxiliary results required for the proof of Theorem 3. The following lemma is from Hashorva and Hüsler (2002, Proposition 2.5 and Corollary 2.7).

Lemma 2. *Let $\Sigma \in \mathbb{R}^{d\times d}$ be a positive-definite correlation matrix. Then for any $S \subset \mathbb{I}_d$ with $|S| \geq 2$, the quadratic programming problem*

$$\mathcal{P}_{\Sigma_S^{-1}} : \min_{\{\boldsymbol{z}\in\mathbb{R}^{|S|}:\boldsymbol{z}\geq\boldsymbol{1}_S\}} \boldsymbol{z}^\top \Sigma_S^{-1} \boldsymbol{z}$$

has a unique solution $\boldsymbol{e}^S \in \mathbb{R}^d$ such that

$$\kappa_S := \min_{\{\boldsymbol{z}\in\mathbb{R}^{|S|}:\boldsymbol{z}\geq\boldsymbol{1}_S\}} \boldsymbol{z}^\top \Sigma_S^{-1} \boldsymbol{z} = \boldsymbol{e}^{S\,\top} \Sigma_S^{-1} \boldsymbol{e}^S > 1.$$

Moreover, there exists a unique non-empty index set $I_S \subseteq S$ *with* $J_S := S \setminus I_S$ *such that the unique solution* $\boldsymbol{e}^S$ *is given by*

$$\boldsymbol{e}^S_{I_S} = \mathbf{1}_{I_S},$$
$$\boldsymbol{e}^S_{J_S} = -[\Sigma_S^{-1}]^{-1}_{J_SJ_S}[\Sigma_S^{-1}]_{J_SI_S}\mathbf{1}_{I_S} \geq \mathbf{1}_{J_S},$$

and $\mathbf{1}_{I_S}\Sigma^{-1}_{I_S}\mathbf{1}_{I_S} = \boldsymbol{e}^{S\,\top}\Sigma_S^{-1}\boldsymbol{e}^S = \kappa_S > 1$ *as well as* $\boldsymbol{z}^\top\Sigma_S^{-1}\boldsymbol{e}^S = \boldsymbol{z}^\top_{I_S}\Sigma^{-1}_{I_S}\mathbf{1}_{I_S}\ \forall\, \boldsymbol{z} \in \mathbb{R}^{|S|}$. *Also, defining* $h_i^S := e_i^\top\Sigma^{-1}_{I_S}\mathbf{1}_{I_S}$ *for* $i \in I_S$, *where* e_i *has only one nonzero entry 1 at the ith co-ordinate, we have* $h_i^S > 0\ \forall i \in I_S$.

Lemma 3. *Let* $\Sigma \in \mathbb{R}^{d\times d}$ *be a positive-definite correlation matrix and* $I := I_{\mathbb{I}_d}$ *be defined as in Lemma 2.*

(*a*) *Suppose* $\Sigma^{-1}\mathbf{1} > \mathbf{0}$. *Then for any* $S \subseteq \mathbb{I}_d$ *with* $S \neq \mathbb{I}_d$, *the inequality* $\kappa_{\mathbb{I}_d} > \kappa_S$ *holds.*

(*b*) *Suppose* $\Sigma^{-1}\mathbf{1} \not> \mathbf{0}$. *Then* $I \neq \mathbb{I}_d$ *and for any set* $S \neq \mathbb{I}_d$ *with* $I \subseteq S \subseteq \mathbb{I}_d$ *the equality* $\kappa_{\mathbb{I}_d} = \kappa_S$ *holds. For* $S \subseteq \mathbb{I}_d$ *with* $S^c \cap I \neq \emptyset$ *we have* $I = I_S$ *and the inequality* $\kappa_{\mathbb{I}_d} > \kappa_S$ *holds.*

Proof. We start with some preliminary calculations. Suppose $S \subseteq \mathbb{I}_d$ with $S^c \cap I \neq \emptyset$. Let $\boldsymbol{e}^* := \boldsymbol{e}^{\mathbb{I}_d}$ be the unique solution of the quadratic programming problem $\mathcal{P}_{\Sigma^{-1}}$ such that $\kappa_{\mathbb{I}_d} = \boldsymbol{e}^{*\top}\Sigma^{-1}\boldsymbol{e}^*$, $\boldsymbol{e}^* \geq \mathbf{1}$ and $[\Sigma^{-1}\boldsymbol{e}^*]_{S^c} \neq \mathbf{0}_{S^c}$ since $[\Sigma^{-1}\boldsymbol{e}^*]_I > \mathbf{0}_I$ and $S^c \cap I \neq \emptyset$ (cf. Lemma 2). First, define $\widetilde{\boldsymbol{e}}_S := \boldsymbol{e}^*_{S^c} + [\Sigma^{-1}]^{-1}_{S^c}[\Sigma^{-1}]_{S^cS}\boldsymbol{e}^*_S$ and note that

$$\begin{aligned}\widetilde{\boldsymbol{e}}_S &= \boldsymbol{e}^*_{S^c} + [\Sigma^{-1}]^{-1}_{S^c}[\Sigma^{-1}]_{S^cS}\boldsymbol{e}^*_S \\ &= [\Sigma^{-1}]^{-1}_{S^c}\Big([\Sigma^{-1}]_{S^c}\boldsymbol{e}^*_{S^c} + [\Sigma^{-1}]_{S^cS}e^*_S\Big) \\ &= [\Sigma^{-1}]^{-1}_{S^c}\Big[\Sigma^{-1}\boldsymbol{e}^*\Big]_{S^c} \neq \mathbf{0}_{S^c}. \end{aligned} \tag{A1}$$

Finally, the Schur decomposition (Lauritzen 2004, eq. (B2))

$$[\Sigma^{-1}]_S = \Sigma_S^{-1} + [\Sigma^{-1}]_{SS^c}[\Sigma^{-1}]^{-1}_{S^c}[\Sigma^{-1}]_{S^cS}$$

along with (A1) implies that

$$\begin{aligned}\kappa_{\mathbb{I}_d} &= \boldsymbol{e}^{*\top}\Sigma^{-1}\boldsymbol{e}^* \\ &= \boldsymbol{e}^{*\top}_S\Sigma_S^{-1}\boldsymbol{e}^*_S + \widetilde{\boldsymbol{e}}_S^\top[\Sigma^{-1}]_{S^c}\widetilde{\boldsymbol{e}}_S && \text{(A2)} \\ &> \boldsymbol{e}^{*\top}_S\Sigma^{-1}\boldsymbol{e}^*_S \geq \min_{\boldsymbol{z}_S \geq \mathbf{1}_S} \boldsymbol{z}_S^\top\Sigma_S^{-1}\boldsymbol{z}_S = \kappa_S. && \text{(A3)}\end{aligned}$$

(a) If $\Sigma^{-1}\mathbf{1} > \mathbf{0}$, then $I = \mathbb{I}_d$ and $\boldsymbol{e}^* = \mathbf{1}$; see Hashorva and Hüsler (2002, Proposition 2.5). Thus, any $S \subseteq \mathbb{I}_d$ with $S \neq \mathbb{I}_d$ satisfies $S^c \cap I \neq \emptyset$ and the result follows from (A3).

(b) If $\Sigma^{-1}\mathbf{1} \not> \mathbf{0}$, then $I \subseteq \mathbb{I}_d$ and $I \neq \mathbb{I}_d$; see Hashorva and Hüsler (2002, Proposition 2.5). Hence, Lemma 2 and $\Sigma_I^{-1}\mathbf{1}_I > \mathbf{0}_I$ imply that

$$\kappa_{\mathbb{I}_d} = \mathbf{1}_I^\top\Sigma_I^{-1}\mathbf{1}_I^\top = \kappa_I.$$

Further, we already know from the Schur decomposition (A2), which is valid independent of the choice of the set S, that $\kappa_{\mathbb{I}_d} \geq \kappa_S \geq \kappa_I$. Hence the only possibility is that $\kappa_{\mathbb{I}_d} = \kappa_S = \kappa_I$. The second statement was already proven in (A3). □

The next proposition provides the tail asymptotics for the Gaussian survival copula using Das and Fasen-Hartmann (2024, Theorem 1).

Proposition 4. *Let* C^Σ *be a Gaussian copula with positive-definite correlation matrix* Σ *and* $S \subset \mathbb{I}_d$ *with* $|S| \geq 2$. *Let* κ_S, I_S, *and* h_s^S, *where* $s \in I_S$, *be defined as in Lemma 2. Now, with* $\boldsymbol{v}_S = (v_s)_{s\in S}$ *where* $v_s \in (0,1), \forall s \in S$, *we have as* $u \downarrow 0$,

$$\widehat{C}^\Sigma_S(u\boldsymbol{v}_S) = (1+o(1))\Upsilon_S(2\pi)^{\frac{\kappa_S}{2}}u^{\kappa_S}(-2\log u)^{\frac{\kappa_S - |I_S|}{2}}\prod_{s\in I_S} v_s^{h_s^S}, \tag{A4}$$

where $\Upsilon_S > 0$ *is a constant.*

Proof. Since (A4) is independent of the marginals of the distribution, consider a random vector $\boldsymbol{Z} \sim G$ in $\mathbb{R}^d$ with standard Pareto marginals, that is, $G_j(z) = \mathrm{P}(Z_j \leq z) = 1 - z^{-1}, z \geq 1, \forall j \in \mathbb{I}_d$, and dependence given by the Gaussian copula C^Σ. Using Das and Fasen-Hartmann (2024, Theorem 1) we have that for $\boldsymbol{z}_S = (z_s)_{s\in S}$ with $z_s > 0\ \forall s \in S$, as $t \to \infty$,

$$\mathrm{P}(Z_s > tz_s,\ \forall\, s \in S) = (1 + o(1))\Upsilon_S(2\pi)^{\frac{\kappa_S}{2}} t^{-\kappa_S}(2\log(t))^{\frac{\kappa_S - |I_S|}{2}} \prod_{s\in I_S} z_s^{-h_s^S}, \tag{A5}$$

where $\Upsilon_S > 0$ is a constant. Then

$$\widehat{C}_S^\Sigma(u\boldsymbol{v}_S) = \mathrm{P}(G_s(Z_s) > 1 - uv_s,\ \forall\, s \in S) = \mathrm{P}(Z_s > u^{-1}v_s^{-1},\ \forall\, s \in S),$$

and the result follows immediately from (A5). □

Lemma 4. *Let C^Σ be a Gaussian copula with positive-definite correlation matrix Σ. Then there exists a $\ell \in \mathbb{I}_d$ such that*

$$\lim_{u\downarrow 0} \frac{\widehat{C}^\Sigma(u, \dots, u)}{\widehat{C}^\Sigma_{\mathbb{I}_d\setminus\{\ell\}}(u, \dots, u)} = c \in (0, 1] \tag{A6}$$

if and only if $\Sigma^{-1}\mathbf{1} \ngeq \mathbf{0}$.

Proof. $\Leftarrow$: Suppose $\Sigma^{-1}\mathbf{1} \ngeq \mathbf{0}$. From Lemma 3(b) we already know that $I \neq \mathbb{I}_d$. Now let $\ell \in \mathbb{I}_d \setminus I$. For $S = \mathbb{I}_d \setminus \{\ell\}$, we have $I \subseteq S \subseteq \mathbb{I}_d$, with $I = I_S$ and $\kappa_{\mathbb{I}_d} = \kappa_S$ (cf. proof of Lemma 3). Now, using (A4) we have

$$\lim_{u\downarrow 0} \frac{\widehat{C}^\Sigma(u, \dots, u)}{\widehat{C}^\Sigma_{\mathbb{I}_d\setminus\{\ell\}}(u, \dots, u)} = \frac{\Upsilon_{\mathbb{I}_d}}{\Upsilon_{\mathbb{I}_d\setminus\{\ell\}}} > 0.$$

$\Rightarrow$: Suppose there exists $\ell \in \mathbb{I}_d$ such that (A6) holds. We prove the statement by contradiction. By way of contradiction, assume $\Sigma^{-1}\mathbf{1} > \mathbf{0}$ holds. Lemma 3 says that for any set $S \subseteq \mathbb{I}_d$ with $S \neq \mathbb{I}_d$, the inequality $\kappa_{\mathbb{I}_d} > \kappa_S$ holds. Again, using (A4) we have with $\kappa^* := \kappa_{\mathbb{I}_d} - \kappa_{\mathbb{I}_d\setminus\{\ell\}}$ and $d^* := d - |I_{\mathbb{I}_d\setminus\{\ell\}}|$,

$$\lim_{u\downarrow 0} \frac{\widehat{C}^\Sigma(u, \dots, u)}{\widehat{C}^\Sigma_{\mathbb{I}_d\setminus\{\ell\}}(u, \dots, u)} = \lim_{u\downarrow 0} \frac{\Upsilon_{\mathbb{I}_d}}{\Upsilon_{\mathbb{I}_d\setminus\{\ell\}}} (\sqrt{2\pi}u)^{\kappa^*}(-2\log u)^{\frac{\kappa^* - d^*}{2}} = 0,$$

which is a contradiction to (A6). □

Proof of Theorem 3. The proof follows now from Lemma 4 by using an analogous argument as given in the proof of Proposition 4. □

Proof of Proposition 2. The proof directly follows from Proposition 4 where a representation for ℓ_S is also provided. □

B Proofs of Section 4

Proof of Theorem 4. First, we define for some $\rho \in (-\frac{1}{\sqrt{k}}, \frac{1}{\sqrt{k}})$ the $\mathbb{R}^{(k+1)\times(k+1)}$-valued positive-definite matrix

$$\Gamma_\rho := \begin{bmatrix} \boldsymbol{I}_k & \rho\mathbf{1}_k \\ \rho\mathbf{1}_k^\top & 1 \end{bmatrix},$$

with inverse

$$\Gamma_\rho^{-1} = \begin{bmatrix} \boldsymbol{I}_k + \frac{\rho^2}{1-k\rho^2}\mathbf{1}_k\mathbf{1}_k^\top & \frac{-\rho}{1-k\rho^2}\mathbf{1}_k \\ \frac{-\rho}{1-k\rho^2}\mathbf{1}_k^\top & \frac{1}{1-k\rho^2} \end{bmatrix}. \tag{B1}$$

Note that

$$\Gamma_\rho^{-1}\mathbf{1}_{k+1} = \left[\frac{1-\rho}{1-k\rho^2}, \dots, \frac{1-\rho}{1-k\rho^2}, \frac{1-k\rho}{1-k\rho^2}\right]^\top.$$

If we restrict $\rho \in [\frac{1}{k}, \frac{1}{\sqrt{k}})$, then the first k components of $\Gamma_\rho^{-1}\mathbf{1}_{k+1}$ are positive and the last component is negative, resulting in $\Gamma_\rho^{-1}\mathbf{1}_{k+1} \not> \mathbf{0}_{k+1}$, and hence due to Theorem 3, a Gaussian copula C^{Γ_ρ} with correlation matrix Γ_ρ is not mutually asymptotically independent, and thus not $(k+1)$-wise asymptotically independent.

Now suppose that $\boldsymbol{X} \in \mathbb{R}^{(k+1)\times(k+1)}$ is a random vector with Gaussian copula C^{Γ_ρ}, where ρ is further restricted to $\rho \in \left(\frac{1}{k}, \min\left(\frac{1}{k-1}, \frac{1}{\sqrt{k}}\right)\right)$. Consider a subset $S \subset \{1, \dots, k+1\}$ with $|S| = j$ such that $j \in \{2, \dots, k\}$.

- If $k+1 \in S$, considering $k+1$ to be the final element of S, we have

$$[\Gamma_\rho]_S^{-1}\mathbf{1}_j = \left[\frac{1-\rho}{1-(j-1)\rho^2}, \dots, \frac{1-\rho}{1-(j-1)\rho^2}, \frac{1-(j-1)\rho}{1-(j-1)\rho^2}\right]^\top > \mathbf{0}_j.$$

- If $k+1 \notin S$, then $[\Gamma_\rho]_S = \boldsymbol{I}_j$ and hence

$$[\Gamma_\rho]_S^{-1}\mathbf{1}_j = \mathbf{1}_j > \mathbf{0}_j.$$

Thus Theorem 3 implies then that $\boldsymbol{X}_J$, for any $J \subseteq \{1, \dots, k+1\}$ with $|J| \le k$, is a mutually asymptotically independent random vector in $\mathbb{R}^J$. Finally, a conclusion of Proposition 3 is that $\boldsymbol{X}$ is k-wise asymptotically independent in $\mathbb{R}^{(k+1)\times(k+1)}$, although it is not $(k+1)$-wise asymptotically independent. From Lemma 2, we know that $I_{\{1,\dots,k+1\}} = \{1, \dots, k\} = I_{\{1,\dots,k\}}$, $\kappa_{\{1,\dots,k+1\}} = \kappa_{\{1,\dots,k\}} = k$, $h_i^{\{1,\dots,k+1\}} = h_i^{\{1,\dots,k\}} = 1$ for $i \in \{1, \dots, k\}$, and, finally, from Proposition 4 that

$$\lim_{u\downarrow 0}\frac{\widehat{C}^{\Gamma_\rho}_{\{1,\dots,k+1\}}(u, \dots, xu)}{\widehat{C}^{\Gamma_\rho}_{\{1,\dots,k\}}(u, \dots, u)} = 1.$$

Note that the constant Υ_S in Proposition 4 is not specified in this article but is given in Das and Fasen-Hartmann (2024, Theorem 1), from which we obtain $\Upsilon_{\{1,\dots,k+1\}} = \Upsilon_{\{1,\dots,k\}}$.

After all, define the $(d\times d)$-dimensional correlation Σ_ρ as a block diagonal matrix having in the first $(d-(k+1))\times(d-(k+1))$ block the identity matrix, zeros in the two off-diagonal blocks, and, in the last $(k+1)\times(k+1)$ block Γ_ρ with $\rho \in \left(\frac{1}{k}, \min\left(\frac{1}{k-1}, \frac{1}{\sqrt{k}}\right)\right)$, that is, the random vector $\boldsymbol{Z}^* = (Z_1^*, \dots, Z_d^*)$ with Gaussian copula C^{Σ_ρ} has the property that $Z_1^*, \dots, Z^*_{d-(k+1)}$ are an independent sequence which is also independent of the random vector $\boldsymbol{X} = (Z^*_{d-k}, \dots, Z^*_d)$ in $\mathbb{R}^{(k+1)\times(k+1)}$ with Gaussian copula C^{Γ_ρ}. Then by analogous arguments as above, $\boldsymbol{Z}^*$ is a k-wise asymptotically independent random vector in $\mathbb{R}^d$ although it is not $(k+1)$-wise asymptotically independent and

$$\lim_{u\downarrow 0}\frac{\widehat{C}^{\Sigma_\rho}_{\{d-k,\dots,d\}}(u, \dots, xu)}{\widehat{C}^{\Sigma_\rho}_{\{d-k,\dots,d-1\}}(u, \dots, u)} = 1.$$

The d-dimensional random vector $\boldsymbol{Z}$ is finally a permutation of $\boldsymbol{Z}^*$ with $\boldsymbol{Z}_{S_2} = \boldsymbol{Z}^*_{\{d-k,\dots,d\}}$, $\boldsymbol{Z}_{S_1} = \boldsymbol{Z}^*_{\{d-k,\dots,d-1\}}$ and $\boldsymbol{Z}_{\mathbb{I}_d\setminus S_2} = \boldsymbol{Z}^*_{\{1,\dots,d-k-1\}}$ and satisfies the requirements of the theorem. □

C Proofs of Section 5

Proof of Theorem 5. For ease of notation, we define $J^* := J \cup \{1\}$. By definition,

$$\mathrm{CTP}_\gamma(\boldsymbol{Z}_{1|J}) = \mathrm{P}\big(Z_1 > \mathrm{VaR}_\gamma(Z_1) | Z_j > \mathrm{VaR}_\gamma(Z_j),\ \forall j \in J\big)$$

$$= \mathrm{P}(Z_1 > F_1^{-1}(1-\gamma) | Z_j > F_j^{-1}(1-\gamma),\ \forall j \in J)$$

$$= \frac{\widehat{C}_{J^*}(\gamma, \dots, \gamma)}{\widehat{C}_J(\gamma, \dots, \gamma)}, \tag{C1}$$

which does not depend on the marginal distributions.

(a) Since $\mathcal{N}_k \subseteq \mathcal{P}_k$, and probabilities are non-negative, it is sufficient to show the statement for $\mathcal{P}_k$. But for any $F \in \mathcal{P}_k$, by the definition of k-wise asymptotic independence and because $|J^*| = \ell + 1 \leq k$, we have $\lim_{\gamma \downarrow 0} \mathrm{CTP}_\gamma(\boldsymbol{Z}_{1|J}) = 0$, and thus (a) holds.

(b) If $d = 2$, there is nothing else to prove. Hence, now assume $d \geq 3$. Since $0 \leq \mathrm{CTP}_\gamma(\boldsymbol{Z}_{1|J}) \leq 1$, to show (b), it is sufficient to provide an example of $F \in \mathcal{N}_\ell \subseteq \mathcal{N}_k \subseteq \mathcal{P}_k$ for $k \in \{2, \dots, \ell\}$, such that for $\boldsymbol{Z} \sim F$, we have $\lim_{\gamma \downarrow 0} \mathrm{CTP}_\gamma(\boldsymbol{Z}_{1|J}) = 1$. To this end, we will choose F with a Gaussian copula C^Σ and positive-definite correlation matrix Σ as identified in Theorem 4, such that F exhibits ℓ-wise asymptotic independence but not $(\ell + 1)$-wise asymptotic independence, and for any $x > 0$

$$\lim_{\gamma \downarrow 0} \frac{\widehat{C}^\Sigma_{J^*}(x\gamma, \gamma, \dots, \gamma)}{\widehat{C}^\Sigma_J(\gamma, \dots, \gamma)} = 1.$$

Hence, $F \in \mathcal{N}_\ell$, and by (C1) we have also

$$\lim_{\gamma \downarrow 0} \mathrm{CTP}_\gamma(\boldsymbol{Z}_{1|J}) = \lim_{\gamma \downarrow 0} \frac{\widehat{C}^\Sigma_{J^*}(\gamma, \dots, \gamma)}{\widehat{C}^\Sigma_J(\gamma, \dots, \gamma)} = 1,$$

which is what we wanted to show. □

Proof of Lemma 1. By definition, we have the relation $\mathcal{N}^*_k \subseteq \mathcal{N}_k \subseteq \mathcal{P}_k$. Since distributions in $\mathcal{N}^*_k$ have a Pareto-distributed marginal in the first component, it remains to show that for any Gaussian copula C^Σ, where Σ is a positive-definite correlation matrix,

$$\sup_{\gamma \in (0, x^{-1}]} \frac{\widehat{C}^\Sigma_S(x\gamma, \gamma, \dots, \gamma)}{\widehat{C}^\Sigma_S(\gamma, \gamma, \dots, \gamma)} < \infty, \tag{C2}$$

for all $S \subseteq \mathbb{I}_d$, and for all $x \geq 1$. However, a conclusion from Proposition 4 is that for any $S \subseteq \mathbb{I}_d$, there exists a constant $h^S_1 \geq 0$ (where $h^S_1 = 0$ if $1 \notin I_S$) so that for any $x > 0$,

$$\lim_{\gamma \downarrow 0} \frac{\widehat{C}^\Sigma_S(x\gamma, \gamma, \dots, \gamma)}{\widehat{C}^\Sigma_S(\gamma, \gamma, \dots, \gamma)} = x^{h^S_1},$$

implying (C2). □

Proof of Theorem 6. First, note that

$$\begin{aligned}\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J}) &= \inf\{z \in \mathbb{R}_+ : \mathrm{P}(Z_1 > z | Z_j > \mathrm{VaR}_{\gamma_2}(Z_j),\ \forall j \in J) \leq \gamma_1\} \\ &= \mathrm{VaR}_{\gamma_2}(Z_1) \inf\{z \in \mathbb{R}_+ : \mathrm{P}(Z_1 > z\mathrm{VaR}_{\gamma_2}(Z_1) | Z_j > \mathrm{VaR}_{\gamma_2}(Z_j),\ \forall j \in J) \leq \gamma_1\}.\end{aligned}$$

Suppose Z_1 is Pareto(α)-distributed, $\alpha > 0$. Then the previous equation reduces to

$$\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J}) = \mathrm{VaR}_{\gamma_2}(Z_1) \inf\left\{z \in \mathbb{R}_+ : \frac{\widehat{C}_{J^*}(z^{-\frac{1}{\alpha}}\gamma_2, \gamma_2, \dots, \gamma_2)}{\widehat{C}_J(\gamma_2, \dots, \gamma_2)} \leq \gamma_1\right\}, \tag{C3}$$

where $J^* = J \cup \{1\}$.

(a) Suppose $F \in \mathcal{P}_k^*$ and $k \in \{\ell+1, \dots, d\}$. Let $\epsilon \in (0, \gamma_1)$ and

$$K := \sup_{\gamma \in (0, \epsilon^{1/\alpha}]} \frac{\widehat{C}_{J^*}(\epsilon^{-1/\alpha}\gamma, \gamma, \dots, \gamma)}{\widehat{C}_{J^*}(\gamma, \gamma, \dots, \gamma)}, \tag{C4}$$

which is finite for $F \in \mathcal{P}_k^*$ by the definition of $\mathcal{P}_k^*$. Furthermore, $F \in \mathcal{P}_k^* \subseteq \mathcal{P}_k$ implies that there exists a $\gamma_0(\epsilon) \in (0, \gamma_1)$ such that

$$\frac{\widehat{C}_{J^*}(\gamma, \gamma, \dots, \gamma)}{\widehat{C}_{J}(\gamma, \gamma, \dots, \gamma)} \le \frac{\epsilon}{K}, \quad \forall\, \gamma \in (0, \gamma_0(\epsilon)). \tag{C5}$$

Therefore, from (C4) and (C5), for all $0 < \gamma_2 < \min(\epsilon^{1/\alpha}, \gamma_0(\epsilon))$ we have

$$\frac{\widehat{C}_{J^*}(\epsilon^{-\frac{1}{\alpha}}\gamma_2, \gamma_2, \dots, \gamma_2)}{\widehat{C}_{J}(\gamma_2, \dots, \gamma_2)} = \frac{\widehat{C}_{J^*}(\epsilon^{-\frac{1}{\alpha}}\gamma_2, \gamma_2, \dots, \gamma_2)}{\widehat{C}_{J^*}(\gamma_2, \dots, \gamma_2)} \frac{\widehat{C}_{J^*}(\gamma_2, \dots, \gamma_2)}{\widehat{C}_{J}(\gamma_2, \dots, \gamma_2)} \le K \cdot \frac{\epsilon}{K} < \gamma_1,$$

and finally, using (C3), we get

$$\frac{\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J})}{\mathrm{VaR}_{\gamma_2}(Z_1)} \le \epsilon.$$

Since $\epsilon \in (0, \gamma_1)$ is arbitrary, this results in

$$\lim_{\gamma_2 \downarrow 0} \frac{\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J})}{\mathrm{VaR}_{\gamma_2}(Z_1)} = 0.$$

Finally, from Lemma 1 we already know that $\mathcal{N}_k^* \subseteq \mathcal{P}_k^*$, thus the result is true for $\mathcal{N}_k^*$ as well.

(b) We will construct a $\boldsymbol{Z} \sim F \in \mathcal{N}_\ell^*$, so that

$$\lim_{\gamma_2 \downarrow 0} \frac{\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J})}{\mathrm{VaR}_{\gamma_2}(Z_1)} = \infty,$$

which shows the statement. To this end, we will choose $\boldsymbol{Z} \sim F$, which has a Gaussian copula C^Σ with positive-definite correlation matrix Σ as in Theorem 4, such that F exhibits ℓ-wise asymptotic independence but not $(\ell+1)$-wise asymptotic independence and for any $x > 0$,

$$\lim_{u \downarrow 0} \frac{\widehat{C}^\Sigma_{J^*}(xu, u, \dots, u)}{\widehat{C}^\Sigma_{J}(u, \dots, u)} = 1. \tag{C6}$$

Additionally, suppose that the marginal F_1 is Pareto(α)-distributed. Then, $F \in \mathcal{N}_\ell^* \subseteq \mathcal{N}_k^* \subseteq \mathcal{P}_k^*$ for $k \in \{2, \dots, \ell\}$. Because of (C6), for any $M > 0$, there exists a $\gamma_0(M) \in (0,1)$ such that

$$\frac{\widehat{C}^\Sigma_{J^*}(M^{-\frac{1}{\alpha}}\gamma_2, \gamma_2, \dots, \gamma_2)}{\widehat{C}^\Sigma_{J}(\gamma_2, \dots, \gamma_2)} > \frac{\gamma_1 + 1}{2}, \quad \forall\, \gamma_2 \in (0, \gamma_0(M)).$$

From this, we get that $\forall\, \gamma_2 \in (0, \gamma_0(M))$,

$$\frac{\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J})}{\mathrm{VaR}_{\gamma_2}(Z_1)} = \inf\left\{ z \in \mathbb{R}_+ : \frac{\widehat{C}_{J^*}(z^{-\frac{1}{\alpha}}\gamma_2, \gamma_2, \dots, \gamma_2)}{\widehat{C}_{J}(\gamma_2, \dots, \gamma_2)} \le \gamma_1 \right\} \ge M,$$

implying

$$\liminf_{\gamma_2 \downarrow 0} \frac{\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J})}{\mathrm{VaR}_{\gamma_2}(Z_1)} \ge M.$$

Since $M > 0$ is arbitrary, we have

$$\lim_{\gamma_2 \downarrow 0} \frac{\mathrm{CoVaR}_{\gamma_1,\gamma_2}(\boldsymbol{Z}_{1|J})}{\mathrm{VaR}_{\gamma_2}(Z_1)} = \infty,$$

exhibiting the desired property for our chosen F and, hence proving the result. □

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