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Dual variational methods for time-harmonic nonlinear Maxwell's equations

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Abstract. We prove the existence of infinitely many nontrivial solutions for time-harmonic nonlinear Maxwell's equations on bounded domains and on $\mathbb{R}^3$ using dual variational methods. In the dual setting we apply a new version of the Symmetric Mountain Pass Theorem that does not require the Palais-Smale condition.

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1. Introduction

Nonlinear boundary value problems of the form

$$\begin{aligned} &\nabla \times \left(\mu(x)^{-1}\nabla \times E\right) - \omega^2\varepsilon(x)E = f(x,E) \quad \text{in } \Omega, \\ &E \times \nu = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

originate from Maxwell's equations for time-harmonic electric fields $E(x)e^{i\omega t}$ propagating in an optically nonlinear medium. Here, $\omega \in \mathbb{R}$ is the frequency of the wave, μ denotes the permeability matrix, ε is the permittivity matrix of the propagation medium and ν denotes the outer unit normal field of the domain $\Omega \subset \mathbb{R}^3$. The nonlinearity $f(x,E) \in \mathbb{R}^3$ represents the superlinear part of the electric displacement field within the propagation medium, see [21, pp. 825-826]. Several existence results for nontrivial solutions of nonlinear Maxwell boundary value problems like (1) have been proved [6–8,28] under various assumptions on the data, notably for bounded C^2-domains $\Omega \subset \mathbb{R}^3$ and the model nonlinearity $f(x,E) = |E|^{p-2}E$ for $2 < p < 6$ with uniformly positive definite matrices ε, μ such that $\varepsilon \in W^{1,\infty}(\Omega;\mathbb{R}^{3\times 3})$, see Theorem 2.2 and Proposition 3.1 in [7]. In this paper we set up an alternative approach by implementing the dual variational method for

$$\nabla \times \left(\mu(x)^{-1}\nabla \times E\right) - \omega^2\varepsilon(x)E = f(x,E) \quad \text{in } \Omega, \quad \mu(x)^{-1}(\nabla \times E) \times \nu = 0 \quad \text{on } \partial\Omega. \tag{2}$$

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By analogy with classical elliptic boundary value problems we will call (1) a Dirichlet problem and (2) a Neumann problem. We refer to Remark 32 for a justification of this nomenclature. Given that the Dirichlet problem has already been studied to some extent, we focus on the Neumann problem in the following.

In our first main result we show that (2) has a ground state and infinitely many bound state solutions under the following assumptions on the data:

(A1) Ω is a bounded C^1-domain satisfying an exterior ball condition.

(A2) $\varepsilon, \mu \in L^\infty(\Omega; \mathbb{R}^{3\times 3})$ are uniformly positive definite with $\varepsilon \in W^{1,3}(\Omega; \mathbb{R}^{3\times 3})$.

(A3) $f : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ is measurable with $f(x, E) = f_0(x, |E|)|E|^{-1}E$ where, for almost all $x \in \Omega$,

$$s \mapsto f_0(x, s) \text{ is positive, differentiable and increasing on } (0, \infty),$$
$$s \mapsto s^{-1} f_0(x, s) \text{ is increasing on } (0, \infty)$$

and there are $c_1, c_2 > 0$ and $2 < p < 6$ such that

$$\frac{1}{2} f_0(x, s)s - \int_0^s f_0(x, t)\, dt \geq c_1 s^p \geq c_2 f_0(x, s)s \quad \text{for all } s \geq 0.$$

As we explain further below, the $W^{1,3}$-regularity for ε is needed to ensure Sobolev-type embeddings of the function spaces we are working in. To find solutions of (2) under the given assumptions, it is reasonable to perform a Helmholtz decomposition where a given vector field E is splitted according to $E = E_1 + E_2$ such that εE_1 is divergence-free and E_2 is curl-free in a suitable sense. To make this rigourous we introduce the Hilbert space $\mathcal{H} := H(\mathrm{curl}; \Omega)$ as the completion of $C^\infty(\Omega; \mathbb{R}^3)$ with respect to the inner product

$$\langle E, F\rangle := \int_\Omega \mu(x)^{-1}(\nabla \times E) \cdot (\nabla \times F) + \varepsilon(x)E \cdot F\, dx. \tag{3}$$

The appropriate function space for (2) turns out to be $\mathcal{V} \oplus \mathcal{W}$ where

$$\mathcal{V} := \Big\{E_1 \in \mathcal{H} : \int_\Omega \varepsilon(x)E_1 \cdot \nabla\Phi\, dx = 0 \text{ for all } \Phi \in C^1(\overline{\Omega})\Big\},$$
$$\mathcal{W} := \big\{\nabla u : u \in W^{1,p}(\Omega)\big\}.$$

Note that $\mathcal{V}, \mathcal{W}$ are formally orthogonal to each other and that the curl operator vanishes identically on $\mathcal{W}$. The Sobolev-type embeddings of $\mathcal{V}$ that we shall prove later ensure that the associated Euler functional

$$\begin{aligned} I(E) := \frac{1}{2} \int_\Omega \mu(x)^{-1}(\nabla \times E_1) \cdot (\nabla \times E_1)\, dx \\ -\frac{\omega^2}{2} \int_\Omega \varepsilon(x)E \cdot E\, dx - \int_\Omega F(x, E)\, dx \end{aligned} \tag{4}$$

is continuously differentiable where $E = E_1 + E_2$ with $E_1 \in \mathcal{V}, E_2 \in \mathcal{W}$. Here, $F(x, \cdot)$ denotes the primitive of $f(x, \cdot)$ with $F(x, 0) = 0$. A weak solution $E \in \mathcal{V} \oplus \mathcal{W}$ of (2) is then defined as a solution of the Euler-Lagrange equation $I'(E) = 0$. A nontrivial weak solution having least energy among all nontrivial weak solutions is called a ground state. Our first result reads as follows.

Theorem 1. *Assume (A1),(A2),(A3) and $\omega^2 \geq 0$. Then (2) has a ground state and infinitely many bound states in $\mathcal{V} \oplus \mathcal{W}$.*

Almost the same proof gives the corresponding result for the Dirichlet problem (1) and we shall comment on the necessary modifications in Appendix 6. In particular, the proof of Theorem 1 indicates an alternative method to prove [7, Theorem 2.2] under slightly different assumptions on the data. It is noteworthy that our proof, which relies on the dual variational method, avoids saddle-point reductions to the Nehari-Pankov manifold and the rather involved critical point theory for strongly indefinite functionals.

We demonstrate that the dual variational approach is applicable on $\mathbb{R}^3$ as well. This problem is substantially different given that it resembles a nonlinear Helmholtz equation rather than an elliptic boundary value problem. We have to make our assumptions on the permittivity ε and permeability μ more restrictive by requiring both to be constant and scalar: $(\varepsilon, \mu) \equiv (\varepsilon_0, \mu_0)$ where $\varepsilon_0\mu_0 \in (0, \infty)$. This leads to the problem

$$\nabla \times \nabla \times E - \omega^2\varepsilon_0\mu_0 E = f(x, E) \qquad \text{in } \mathbb{R}^3. \tag{5}$$

Our aim is to prove the existence of infinitely many L^p-solutions for this problem. The main difference compared to the case of a bounded domain is that the curl-curl operator on $\mathbb{R}^3$ does not come with discrete point spectrum in $(0, \infty)$ but continuous spectrum just like the Helmholtz operator. In particular, a resolvent at the spectral parameter $\omega^2\varepsilon_0\mu_0 > 0$ does not exist, but some sort of right inverse can be constructed by means of the Limiting Absorption Principle. This linear operator $\mathcal{R}$, defined in (27) below, enjoys boundedness and compactness properties as an operator from the divergence-free functions in $L^{p'}(\mathbb{R}^3; \mathbb{R}^3)$ to the divergence-free functions in $L^p(\mathbb{R}^3; \mathbb{R}^3)$ provided that $4 < p < 6$. Evequoz and Weth [14] showed how to exploit these properties in the context of dual variational methods for nonlinear Helmholtz equations. In order to adapt this to the Maxwell setting on $\mathbb{R}^3$ we sharpen our assumptions on the nonlinearity.

(A4) f satisfies (A3) with $4 < p < 6$ where $f_0(\cdot, s)$ is $\mathbb{Z}^3$-periodic for all $s \in \mathbb{R}$ and there are $c, C > 0$ with

$$cs^{p-2} \le \partial_s f_0(x, s) \le Cs^{p-2} \qquad \text{for almost all } x \in \mathbb{R}^3 \text{ and all } s \in \mathbb{R}.$$

Our main result about (5) reads as follows.

Theorem 2. *Assume (A4) and $\omega^2\varepsilon_0\mu_0 \in (0, \infty)$. Then the equation (5) admits a dual ground state and infinitely many geometrically distinct solutions in $L^p(\mathbb{R}^3; \mathbb{R}^3)$.*

Here, the notion of a dual ground state is the same as in [14], i.e., $P := f(x, E)$ is a ground state for some associated dual functional J involving the operator $\mathcal{R}$, see Section 5 and in particular (28) further below. As a new feature compared to the Nonlinear Helmholtz Equation [13] our analysis makes use of the so-called Div-Curl-Lemma.

In view of the result on bounded domains, one may aim for an extension of Theorem 2 to more general permittivities ε and permeabilites μ. Here the main challenge is the construction of a bounded linear operator $\mathcal{R}$ with analogous properties as well as a corresponding Helmholtz Decomposition Theorem.

We emphasize that, up to our knowledge, this is the first variational existence result for (5) in the case $\omega^2\varepsilon_0\mu_0 > 0$ and it is much stronger than Theorem 3(ii) in [19] which has been obtained by a fixed point argument. Note that much research has been devoted to the complementary case $\omega^2\varepsilon_0\mu_0 \le 0$ within the framework of cylindrically symmetric and thus divergence-free solutions [3,5,9,12,18] where the variational analysis is somewhat parallel to the well-studied case of stationary nonlinear Schrödinger equations on $\mathbb{R}^3$. This follows from the identity $\nabla \times \nabla \times E = -\Delta E$ for divergence-free vector fields E.

Our strategy is to prove the above-mentioned results using dual variational methods based on a partially new variant of the Symmetric Mountain Pass Theorem (SMPT). Note that the classical SMPT [1, Corollary 2.9] is not sufficient to prove Theorem 2 given that the Palais-Smale condition does not hold. For this reason we first provide a critical point theorem (Theorem 5) that allows to prove both our main theorems simultaneously. In this result, the C^1-functional is only required to be "PS-attracting" in the sense of Definition 4 below. Having proved this result in Section 2, we apply it in our proofs of Theorem 1 in Section 4 and Theorem 2 in Section 5. Section 3 provides the underlying Linear Theory.

2. A Symmetric Mountain Pass Theorem without PS-condition

In this section we prove a variant of the Symmetric Mountain Pass Theorem with the distinguishing feature that it does not require the Palais-Smale condition. The idea for this abstract result and its proof is due to Szulkin-Weth [26] and an earlier paper by Bartsch-Ding [4] where the existence of infinitely many geometrically distinct solutions has been proved for some periodic nonlinear Schrödinger equation of the form

$$-\Delta u + V(x)u = f(x,u) \qquad \text{in } \mathbb{R}^n$$

with an odd nonlinearity $f(x,\cdot)$. Several refinements and applications can be found in the literature. Here we want to highlight the contributions by Squassina and Szulkin [24] and Evequoz [13] given their relevance for our paper. Our first goal is to provide an abstract critical point theorem for functionals $J : Z \to \mathbb{R}$ on a general Banach space with some sort of Mountain Pass Geometry. In general, the PS-condition for J may fail, but some useful but weaker compactness property, a "PS-attracting" property, is required to hold. This property is essentially extracted from the paper by Szulkin and Weth [26] and functionals satisfying the PS-condition are easily seen to be PS-attracting, see Remark 6(a). As a consequence, our variant of the SMPT generalizes the original one and yields the existence of infinitely many solutions in various applications. In the presence of discrete translational equivariance it provides the existence of infinitely many geometrically distinct solutions, see Remark 6(b).

For a given functional $J \in C^1(Z)$ we define the sets of critical points

$$\mathcal{K} := \left\{u \in Z : J'(u) = 0\right\}, \qquad \mathcal{K}_d := \{u \in \mathcal{K} : J(u) = d\}.$$

We first introduce the notion of a $\mathcal{K}$-decomposition where $\mathcal{K}$ is subdivided into suitable bounded and symmetric pieces that are discrete. Here, a set $A \subset Z$ is called symmetric provided that it is even, i.e., we have $x \in A$ if and only if $-x \in A$.

Definition 3. Let $J \in C^1(Z)$ be even. We say that J admits a $\mathcal{K}$-decomposition $\mathcal{K} = \bigcup_{i\in I} \mathcal{K}^i$ for some set I if the following holds for all $i, j \in I$:

(i) $\mathcal{K}^i$ is bounded, symmetric and non-empty,
(ii) $\inf\{\|u - v\| : u, v \in \mathcal{K}^i \cup \mathcal{K}^j, u \neq v\} > 0$,
(iii) $\{J(u) : u \in \mathcal{K}^i\}$ is a finite set.

Such a $\mathcal{K}$-decomposition is called finite if $\#I < \infty$ and infinite if $\#I = \infty$.

We stress that (ii) needs to be checked for $i = j$ as well and that the lower bound in (ii) need not be uniform with respect to i, j. As to (iii), note that the $\mathcal{K}^i$ need not be finite. In fact, we have $\#\mathcal{K}^i = \infty$ in relevant applications, see Remark 6(b). To formulate our result we introduce the notion of a PS-attracting functional.

Definition 4. We say that J is PS-attracting if for any given Palais-Smale sequences $(v_n)_n, (w_n)_n$ of J we have $\|v_n - w_n\| \to 0$ as $n \to \infty$ or

$$\limsup_{n\to\infty} \|v_n - w_n\| \geq \kappa \quad \text{where } \kappa := \inf\left\{\|u - v\| : u, v \in \mathcal{K}, u \neq v\right\}.$$

We stress that there is no need to prove $\kappa > 0$. The importance of this observation is discussed in Remark 6(c). In fact, even in situations where the Palais-Smale condition holds, κ can be 0. For instance, consider any functional $J \in C^1(\mathbb{R})$ satisfying $J(x) = 1$ for $1 \leq |x| \leq 2$ as well as $|J'(x)| \geq c > 0$ outside some compact set. Obviously, J satisfies the PS-condition, but the set of critical points is not discrete and hence κ must be zero. The definition of κ involves the set $\mathcal{K}$ that is, of course, unknown a priori. Nevertheless, we shall see below that this compactness property can be verified in applications by proving that PS-sequences with a positive distance converge, after applying suitable isometries, to distinct critical points. The assumptions on the energy functional are the following:

(G_1) Z is a Banach space Z with $Z = Z^+ \oplus Z^-$ and $\dim(Z^-) < \infty, \dim(Z^+) = \infty$.
(G_2) $J \in C^1(Z)$ is even with $J(0) = 0$ and, for some $\rho > 0$,

$$\inf_{S_\rho^+} J > 0 \qquad \text{where} \qquad S_\rho^+ = \{v \in Z^+ : \|v\| = \rho\}.$$

(G_3) For any given $m \in \mathbb{N}$ there is a finite-dimensional subspace $Z_m \subset Z^+$ such that $J(u) \to -\infty$ uniformly for $u \in Z^- \oplus Z_m$ as $\|u\| \to \infty$ and $\dim(Z_m) \nearrow +\infty$ as $m \to \infty$.

(G_4) J is PS-attracting.

Note that $Z^- = \{0\}$ is a valid choice in (G_1) and the uniformity in (G_3) need only hold for a fixed $m \in \mathbb{N}$. Our variant of the Symmetric Mountain Pass Theorem without Palais-Smale condition reads as follows.

Theorem 5. *Assume* $(G_1), (G_2), (G_3), (G_4)$. *Then every* $\mathcal{K}$*-decomposition of* J *is infinite. In particular,* J *has infinitely many pairs of critical points.*

Remark 6.

(a) If $J \in C^1(Z)$ satisfies the PS-condition, then J is PS-attracting. Indeed, for two PS-sequences $(v_n), (w_n)$ with $\limsup_{n\to\infty} \|v_n - w_n\| \geq \mu > 0$ we may subsequently pass to subsequences such that $(v_n), (w_n)$ strongly converge to v, w, respectively. The continuity of J' implies that v, w are critical points of J with $\|v - w\| \geq \mu > 0$. So $v \neq w$ and
$$\limsup_{n\to\infty} \|v_n - w_n\| \geq \|v - w\| \geq \kappa.$$
So J is PS-attracting. This implication shows that Theorem 5 contains the original Symmetric Mountain Pass Theorem [1, Corollary 2.9] and even the more general version [25, Theorem 6.5] as special cases.

(b) In the context $\mathbb{Z}^d$-equivariant stationary nonlinear Schrödinger equations this theorem applies to $\mathcal{K}$-decompositions given by sets of the form
$$\mathcal{K}^v = \{v(\cdot + n) : n \in \mathbb{Z}^d\} \cup \{-v(\cdot + n) : n \in \mathbb{Z}^d\}.$$
Indeed, in the papers [13,24] the authors implicitly check the properties required in Definition 3, notably that different orbits have positive distance to each other, see [13, Lemma 3.1]. The PS-attracting property of the energy functional is verified using the Nonvanishing Property [14, Theorem 2.3] of $\mathcal{R}$ as well as the Rellich-Kondrachov theorem. Our reasoning in the proof of Theorem 2 follows the same lines.

(c) Theorem 4.2 in [4] is more general regarding the regularity assumptions on the functional and it even allows for infinite-dimensional Z^- up to modifications of the topology on Z^-. On the other hand, in assumption (Φ_5) the authors assume $\kappa > 0$. This is responsible for the fact that this Theorem cannot be applied directly in typical situations, but must be used in some indirect reasoning just as in the proof of Theorem 1.2 in [4]. In particular, [4, Theorem 4.2] does not generalize the SMPT. Theorem 5 removes this inconvenience.

The strategy to prove Theorem 5 is as follows: We assume for contradiction that the claim is false, i.e.,
$$\text{There is a finite } \mathcal{K} - \text{decomposition for } J. \tag{6}$$
With this assumption and Definition 3(ii) we know that κ from Definition 4 is positive. This enables us to perform a deformation argument involving the minmax values
$$d_k := \inf_{A \in \Sigma, \iota^*(A) \geq k} \sup_{u \in A} J(u).$$
where $\Sigma := \{A \subset Z : A \text{ is symmetric and compact}\}$ and ι^* is defined as in [13,24], namely
$$\iota^*(A) := \min\left\{\gamma(h(A) \cap S_\rho^+) : \ h \in \mathcal{H}\right\} \qquad \text{where}$$
$$\mathcal{H} := \big\{h : Z \to h(Z) \text{ is an odd homeomorphism with } J(h(u)) \leq J(u)$$
$$\text{for all } u \in Z\big\}$$
with ρ as in (G_2) and the Krasnoselski genus γ. Exploiting (6) and hence κ from Definition 4 is positive we show that the d_k are critical values of J and form an increasing sequence. In particular, infinitely many different critical values exist. This, however, contradicts the assumption (6) given that Definition 3(iii) implies that finite $\mathcal{K}$-decomposition allow for at most finitely many critical values. This contradiction finishes the proof.

Proposition 7. *Given the assumptions of Theorem 5 we have* $0 < d_k \leq d_{k+1} < \infty$ *for all* $k \in \mathbb{N}$.

Proof. The inequality $d_k \leq d_{k+1}$ holds by definition. These values are not $+\infty$ given that, for any given $k \in \mathbb{N}$, the combination of $(G_1), (G_2), (G_3)$ and the argument from [24, Lemma 2.16 (iv)] or [4, Lemma 4.5] implies the existence of a set $A \in \Sigma$ with $\iota^*(A) \geq k$. Moreover, any $A \in \Sigma$ with $\iota^*(A) \geq 1$ implies $\gamma(h(A) \cap S_\rho^+) \geq 1$ for some $h \in \mathcal{H}$, in particular $h(A) \cap S_\rho^+ \neq \emptyset$. So

$$d_k \geq d_1 = \inf_{A \in \Sigma, \iota^*(A) \geq 1} \sup_{u \in A} J(u) \geq \inf_{h(A) \cap S_\rho^+ \neq \emptyset} \sup_{u \in A} J(h(u)) \geq \inf_{S_\rho^+} J > 0.$$

□

In the following, we write $U_\delta(\mathcal{K}_d) := \{z \in Z : \operatorname{dist}(z, \mathcal{K}_d) < \delta\}$, so $U_\delta(\emptyset) = \emptyset$.

Proposition 8. *In addition to the assumptions of Theorem 5 suppose (6). Let κ be given as in Definition 4.*

(a) For $0 < \delta < \kappa$ and $d \in \mathbb{R}$ such that for all non-empty $K \subset \mathcal{K}_d$ we have $\tau > 0$ where

$$\tau := \inf \left\{ \|J'(v)\| : v \in U_\delta(K) \setminus U_{\frac{\delta}{2}}(K) \right\}.$$

(b) If $d \in \mathbb{R} \setminus \{0\}$ then $\gamma(\mathcal{K}_d) \leq 1$.

(c) For all $d \in \mathbb{R}$ there is $\varepsilon_0 > 0$ such that $\mathcal{K}_{\tilde{d}} = \emptyset$ for $0 < |d - \tilde{d}| < \varepsilon_0$.

Proof. To prove (a) assume for contradiction that there is a sequence $(v_n)_n \subset U_\delta(K) \setminus U_{\delta/2}(K)$ such that $J'(v_n) \to 0$. Then we can find a sequence $(w_n) \subset K \subset \mathcal{K}_d$ with

$$0 < \frac{\delta}{2} \leq \|v_n - w_n\| \leq \delta < \kappa \quad \text{for all } n \in \mathbb{N}.$$

Since $(v_n), (w_n)$ are PS-sequences and J is PS-attracting, this contradicts our choice of κ. So we must have $\tau > 0$ and (a) is proved. Since J is PS-attracting and $d \neq 0$, $\mathcal{K}_d$ is a discrete set that does not contain 0. Hence, $\mathcal{K}_d = \bigcup_{i \in I} \{v_i, -v_i\}$ is a disjoint union for some set I, then $h : \mathcal{K}_d \to \mathbb{R} \setminus \{0\}, \pm v_i \mapsto \pm 1$ defines an odd and continuous map, so $\gamma(\mathcal{K}_d) \leq 1$. This proves (b).

Finally, (6) and Definition 4(iii) imply that the set of critical values $\{J(u) : u \in \mathcal{K}\}$ is finite and hence discrete. This gives (c). □

To prove the theorem we consider a pseudo-gradient field on $Z \setminus \mathcal{K}$, i.e., a locally Lipschitz continuous map $H : Z \setminus \mathcal{K} \to Z$ with

$$\|H(w)\| \leq 2\|J'(w)\|, \qquad J'(w)[H(w)] \geq \|J'(w)\|^2. \tag{7}$$

We define the associated flow η as the unique maximal solution of the initial value problem

$$\frac{d}{dt}\eta(t, w) = -H(\eta(t, w)), \qquad \eta(0, w) = w \qquad \text{for } w \in Z \setminus \mathcal{K}. \tag{8}$$

The maximal existence time in positive time direction is denoted by $T^+(w)$. The flow η is continuous at all points (t, w) with $w \in Z \setminus \mathcal{K}$ and $0 \leq t < T^+(w)$ by unique local solvability of (8). The most important property is that $t \mapsto J(\eta(t, w))$ is decreasing on $[0, T^+(w))$ due to

$$\frac{d}{dt}\Big(J(\eta(t, w))\Big) = -J'(\eta(t, w))[H(\eta(t, w))] \leq -\|J'(\eta(t, w))\|^2 \text{for } 0 < t < T^+(w).$$

We use this in the following deformation argument involving suitable sublevel sets $J^c := \{u \in Z : J(u) \leq c\}$.

Lemma 9. *In addition to the assumptions of Theorem 5 suppose (6), let $d := d_k$ for some $k \in \mathbb{N}$ and $0 < \delta < \kappa$. Then there exists $\varepsilon > 0$ such that*

$$\lim_{t \to T^+(w)} J(\eta(t, w)) < d - \varepsilon \quad \text{for all } w \in J^{d+\varepsilon} \setminus U_\delta(\mathcal{K}_d). \tag{9}$$

Moreover, the entrance time map

$$e : J^{d+\varepsilon} \setminus U_\delta(\mathcal{K}_d) \to [0, \infty), \quad w \mapsto \min\left\{t \geq 0 : J(\eta(t, w)) \leq d - \varepsilon\right\}$$

is well-defined, even and continuous.

Proof. In the following we verify the claim for

$$0 < \varepsilon < \min\left\{\frac{\delta\tau}{\sqrt{32}}, \frac{\kappa}{2}, \varepsilon_0\right\}$$

where $\tau, \kappa, \varepsilon_0 > 0$ are as in Proposition 8. Assume for contradiction that (9) does not hold for such ε. Then we find some $w \in J^{d+\varepsilon} \setminus \overline{U_\delta(\mathcal{K}_d)}$ such that

$$d - \varepsilon \leq J(\eta(t,w)) \leq d + \varepsilon \quad \text{for all } t \in [0, T^+(w)). \tag{10}$$

Using (10) we first show that the flow converges to some critical point at energy level d, i.e.,

$$\lim_{t \to T^+(w)} \eta(t,w) = w^* \qquad \text{where } w^* \in \mathcal{K}_d. \tag{11}$$

We first prove this in the case $T^+(w) < \infty$. For $0 < \tilde{t} < t < T^+(w)$ we have

$$\begin{aligned} \|\eta(t,w) - \eta(\tilde{t},w)\| &\leq \int_{\tilde{t}}^t \|H(\eta(s,w))\|\, ds \overset{(7)}{\leq} 2\int_{\tilde{t}}^t \|J'(\eta(s,w))\|\, ds \\ &\overset{(7)}{\leq} 2\int_{\tilde{t}}^t \sqrt{J'(\eta(s,w))[H(\eta(s,w))]}\, ds \\ &\leq 2\sqrt{t-\tilde{t}}\left(\int_{\tilde{t}}^t J'(\eta(s,w))[H(\eta(s,w))]\, ds\right)^{\frac{1}{2}} \\ &\overset{(8)}{=} 2\sqrt{t-\tilde{t}}\Big(J(\eta(\tilde{t},w)) - J(\eta(t,w))\Big)^{\frac{1}{2}} \\ &\overset{(10)}{\leq} 2\sqrt{t-\tilde{t}}\,(2\varepsilon)^{\frac{1}{2}}. \end{aligned} \tag{12}$$

Since $T^+(w)$ is finite, this estimate implies that $\eta(t,w)$ is Cauchy and hence converges as $t \to T^+(w)$. The limit w^* must be a critical point of J because otherwise the trajectory $t \mapsto \eta(t,w)$ could be continued beyond $T^+(w)$ thanks to the local solvability of the initial value problem (8) with initial data in $Z \setminus \mathcal{K}$. So we have $w^* \in \mathcal{K}$ and (10) gives $d - \varepsilon \leq J(w^*) \leq d + \varepsilon$. From $0 < \varepsilon < \varepsilon_0$ and Proposition 8(c) we conclude $w^* \in \mathcal{K}_d$, so (11) is proved.

Next we show (11) in the case $T^+(w) = \infty$. Since the map $t \mapsto J(\eta(t,w))$ is decreasing and bounded from below in view of (10), we find that $J(\eta(t,w))$ converges as $t \to +\infty$. We now deduce that $\eta(t,w)$ converges as well. Assume for contradiction that this is not the case. Then there exists a sequence $(t_n)_n \subset [0,\infty)$ with $t_n \to \infty$ and $\|\eta(t_n,w) - \eta(t_{n+1},w)\| = \varepsilon$ for every n. Choose the smallest $t_n^1 \in (t_n, t_{n+1})$ such that $\|\eta(t_n,w) - \eta(t_n^1,w)\| = \frac{\varepsilon}{3}$ and let $\mu_n := \min_{s\in[t_n,t_n^1]} \|J'(\eta(s,w))\|$. Then

$$\begin{aligned} \frac{\varepsilon}{3} &= \left\|\eta(t_n^1,w) - \eta(t_n,w)\right\| \leq \int_{t_n}^{t_n^1} \|H(\eta(s,w))\| ds \overset{(7)}{\leq} 2\int_{t_n}^{t_n^1} \|J'(\eta(s,w))\| ds \\ &\leq \frac{2}{\mu_n}\int_{t_n}^{t_n^1} \|J'(\eta(s,w))\|^2 ds \overset{(7)}{\leq} \frac{2}{\mu_n}\int_{t_n}^{t_n^1} J'(\eta(s,w))[H(\eta(s,w))] ds \\ &\overset{(8)}{=} \frac{2}{\mu_n}\Big(J(\eta(t_n,w)) - J(\eta(t_n^1,w))\Big). \end{aligned}$$

Since $J(\eta(t,w))$ converges as $t \to \infty$, we have $J(\eta(t_n,w)) - J(\eta(t_n^1,w)) \to 0$ and thus $\mu_n \to 0$ as $n \to \infty$. So there exist $s_n^1 \in [t_n, t_n^1]$ such that $J'(v_n) \to 0$, where $v_n := \eta(s_n^1, w)$. Similarly, we find a largest $t_n^2 \in (t_n^1, t_{n+1})$ for which $\|\eta(t_{n+1},w) - \eta(t_n^2,w)\| = \frac{\varepsilon}{3}$ and then $w_n := \eta(s_n^2, w)$ satisfies $J'(w_n) \to 0$. As $\|v_n - \eta(t_n,w)\| \leq \frac{\varepsilon}{3}$ and $\|w_n - \eta(t_{n+1},w)\| \leq \frac{\varepsilon}{3}$,

$$(v_n)_n, (w_n)_n \quad \text{are Palais-Smale sequences with} \quad \frac{\varepsilon}{3} \leq \|v_n - w_n\| \leq 2\varepsilon < \kappa.$$

This, however, contradicts our choice of κ, so the assumption was false. Hence, $\eta(t,w)$ converges as $t \to \infty$. The limit must be a critical point of J because otherwise $\frac{d}{dt}\big(J(\eta(t,w))\big)$

would be uniformly negative for large t, which violates (10). So (11) also holds in the case $T^+(w) = +\infty$.

From (11) we infer that the flow $t \mapsto \eta(t,w)$ eventually enters the region $U_\delta(\{w^*\})$ at some time $t_1 \in (0, T^+(w))$ and remains outside of the region $U_{\delta/2}(\{w^*\})$ until some time $t_2 \in (t_1, T^+(w))$. Formally,

$$t_1 := \max\big\{t \in \big[0, T^+(w)\big) : \eta(t,w) \notin U_\delta(\{w^*\})\big\},$$
$$t_2 := \inf\big\{t \in \big(t_1, T^+(w)\big) : \eta(t,w) \in U_{\delta/2}(\{w^*\})\big\}.$$

As in (12) we get the inequality

$$\frac{\delta}{2} \le \|\eta(t_2,w) - \eta(t_1,w)\| \le 2\sqrt{t_2 - t_1}\,(2\varepsilon)^{1/2}, \quad \text{and thus } t_2 - t_1 \ge \frac{\delta^2}{32\varepsilon}.$$

On the other hand, Proposition 8(a) gives $\|J'(\eta(s,w)\| \ge \tau > 0$ for all $s \in [t_1, t_2]$ and thus

$$\begin{aligned}
d &\overset{(11)}{=} \lim_{t \to T^+(w)} J\left(\eta\left(t,w\right)\right) \;\le\; J\left(\eta\left(t_2,w\right)\right) \\
&\overset{(10)}{\le} d + \varepsilon + J\left(\eta\left(t_2,w\right)\right) - J\left(\eta\left(t_1,w\right)\right) \\
&\overset{(8)}{=} d + \varepsilon - \int_{t_1}^{t_2} J'(\eta(s,w))[H(\eta(s,w))]\,ds \\
&\overset{(7)}{\le} d + \varepsilon - \int_{t_1}^{t_2} \|J'(\eta(s,w))\|^2\,ds \\
&\le d + \varepsilon - \tau^2\left(t_2 - t_1\right) \;\le\; d + \varepsilon - \frac{\tau^2\delta^2}{32\varepsilon} \\
&< d,
\end{aligned}$$

which yields a contradiction. So (10) is false and (9) is proved.

Finally, thanks to (9) we know that $e(w) \in [0, T(w))$ is a well-defined finite nonnegative number characterized by $J(\eta(e(w),w)) = \min\{J(w), d - \varepsilon\}$. Since J and η are continuous and $t \mapsto J(\eta(t,w))$ is decreasing near each $w \in J^{d+\varepsilon} \setminus U_\delta(\mathcal{K}_d)$, we obtain that the entrance time map is continuous. □

Proof of Theorem 5. Assume for contradiction that (6) is true. We then show

$$\mathcal{K}_{d_k} \neq \emptyset \qquad \text{and} \qquad d_{k+1} > d_k \quad \text{for all } k \in \mathbb{N}.$$

Indeed, by the continuity property of the genus and Proposition 8(b), there exists $\delta > 0$ such that $\gamma(\bar{U}) = \gamma(\mathcal{K}_{d_k}) \le 1$ where $U := U_\delta(\mathcal{K}_{d_k})$ and $0 < \delta < \kappa$. We may then choose $\varepsilon > 0$ as in Lemma 9. By definition of ι^* we can find $A \in \Sigma$ with $\iota^*(A) \ge k$ and $\sup_A J \le d_k + \varepsilon$. Then we have $A \setminus U \in \Sigma$ and, by (9),

$$\sup_{u \in A \setminus U} J(\eta(e(u),u)) \le d_k - \varepsilon. \tag{13}$$

The map $\tilde{h}(u) := \eta(e(u),u)$ is well-defined, odd and continuous on $J^{d_k+\varepsilon} \setminus U$ and can be extended to some function $h \in \mathcal{H}$ that is defined on $J^{d_k+\varepsilon}$. For instance, define for $u \in J^{d_k+\varepsilon}$

$$\begin{aligned}
&h(u) := \eta(e(u),u) && \text{if } u \in \operatorname{dist}(u, \mathcal{K}_{d_k}) \ge \delta, \\
&h(u) := \eta\Big((2\lambda\delta^{-1} - 1)e(u), u\Big) && \text{if } \operatorname{dist}(u, \mathcal{K}_{d_k}) = \lambda \in \big(\tfrac{\delta}{2}, \delta\big), \\
&h(u) := u && \text{if } \operatorname{dist}(u, \mathcal{K}_{d_k}) \le \tfrac{\delta}{2}.
\end{aligned}$$

From the definition of d_l and (13) we get $\iota^*(h(A\setminus U)) = \iota^*(\tilde{h}(A\setminus U)) \le k-1$. So the mapping properties of ι^* from [24, Lemma 2.16] and Proposition 8(b) give[1]

$$k \le \iota^*(A) \le \iota^*(A\setminus U) + \gamma(\overline{U}) \le \iota^*(h(A\setminus U)) + \gamma(\mathcal{K}_{d_k}) \le k-1+\gamma(\mathcal{K}_{d_k}).$$

So $\gamma(\mathcal{K}_{d_k}) \le 1$ implies $\mathcal{K}_{d_k} \neq \emptyset$. Moreover, $d_k = d_{k+1}$ is impossible. Indeed, under this assumption we may even choose $A \in \Sigma$ with $\iota^*(A) \ge k+1$ with the properties given above, so

$$k+1 \le \iota^*(A) \le k-1+\gamma(\mathcal{K}_{d_k}) \le k,$$

a contradiction. Given that the critical levels are nondecreasing, we obtain $d_{k+1} > d_k$ for all $k \in \mathbb{N}$. So there are infinitely many critical levels and every $\mathcal{K}$-decomposition of J is infinite. So the assumption (6) is false, which is all we had to prove. □

3. The linear Neumann problem on bounded domains

We are first interested in a solution theory for the linear boundary value problem

$$\begin{aligned} &\nabla \times \big(\mu(x)^{-1}\nabla \times E\big) - \lambda\varepsilon(x)E = \varepsilon(x)g \quad \text{in } \Omega, \\ &(\mu(x)^{-1}\nabla \times E) \times \nu = 0 \text{on } \partial\Omega \end{aligned} \tag{14}$$

where $\lambda \in \mathbb{R}$ and $E \in \mathcal{V}$. A weak solution $E \in \mathcal{V}$ of (14) is characterized by

$$\int_\Omega \mu(x)^{-1}(\nabla\times E)\cdot(\nabla\times\Phi)\,dx - \lambda\int_\Omega \varepsilon(x)E\cdot\Phi\,dx = \int_\Omega \varepsilon(x)g\cdot\Phi\,dx \text{ for all } \Phi\in\mathcal{V}. \tag{15}$$

In order to identify the right function space for g we use a Helmholtz Decomposition. Define

$$\begin{aligned} X^{p'} &:= \Big\{E \in L^{p'}(\Omega;\mathbb{R}^3) : \int_\Omega \varepsilon(x)E\cdot\nabla\phi\,dx = 0 \text{ for all } \phi\in W^{1,p}(\Omega)\Big\}, \\ Y^{p'} &:= \big\{\nabla u : u \in W^{1,p'}(\Omega)\big\}. \end{aligned}$$

By the Gauss-Green formula elements of $X^{p'}$ are not only divergence-free in Ω, but also satisfy $\varepsilon(x)E\cdot\nu = 0$ on $\partial\Omega$ in a distributional sense. We also need

$$\begin{aligned} (Y^{p'})^{\perp_\varepsilon} &:= \Big\{f \in L^p(\Omega;\mathbb{R}^3) : \int_\Omega \varepsilon(x)f\cdot g\,dx = 0 \text{ for all } g \in Y^{p'}\Big\}, \\ (X^{p'})^{\perp_\varepsilon} &:= \Big\{g \in L^p(\Omega;\mathbb{R}^3) : \int_\Omega \varepsilon(x)f\cdot g\,dx = 0 \text{ for all } f \in X^{p'}\Big\}. \end{aligned}$$

From [2, Theorem 1] and $W^{1,3}(\Omega) \subset \mathrm{VMO}(\Omega)$ [?, Theorem 3.3(ii)] we get the following.

Proposition 10. *(Auscher, Qafsaoui) Assume (A1),(A2) and* $1 < p < \infty$. *Then for any given* $f \in L^{p'}(\Omega;\mathbb{R}^3)$ *the boundary value problem*

$$\nabla\cdot\big(\varepsilon(x)(f+\nabla u)\big) = 0 \quad \text{in } \Omega, \qquad \varepsilon(x)(f+\nabla u)\cdot\nu = 0 \quad \text{on } \partial\Omega \tag{16}$$

has a weak solution in $u \in W^{1,p'}(\Omega)$ *that is unique up to constants and satisfies* $\|\nabla u\|_{p'} \lesssim \|f\|_{p'}$.

This preliminary result provides a Helmholtz Decomposition that allows to set up a Fredholm theory for (14). To see this define

$$\mathcal{L}E := \varepsilon(x)^{-1}\nabla\times\big(\mu(x)^{-1}\nabla\times E\big), \qquad \langle E,F\rangle_\varepsilon := \int_\Omega \varepsilon(x)E\cdot F\,dx.$$

Then the problem reads $(\mathcal{L}-\lambda)E = g$ where $\mathcal{L}$ is a selfadjoint operator in the Hilbert space $(L^2(\Omega;\mathbb{R}^3), \langle\cdot,\cdot\rangle_\varepsilon)$. In the following we write $E \perp_\varepsilon F$ if $\langle E,F\rangle_\varepsilon = 0$ and analogously for subspaces of $L^2(\Omega;\mathbb{R}^3)$.

[1] Lemma 2.16 in [24] is stated and proved only for $Z^- = \{0\}$, but the same proof yields the result for any finite-dimensional Z^-.

Proposition 11. *Assume (A1),(A2) and* $1 < p < \infty$. *Then we have*

$$L^{p'}(\Omega;\mathbb{R}^3) = X^{p'} \oplus Y^{p'} \qquad \text{with} \quad (Y^p)^{\perp_\varepsilon} = X^{p'}, Y^p = (X^{p'})^{\perp_\varepsilon}.$$

Proof. First of all, $X^{p'}, Y^{p'}$ are closed subspaces of $L^{p'}(\Omega;\mathbb{R}^3)$ and the intersection of these subspaces is $\{0\}$. Indeed, for $E \in X^{p'} \cap Y^{p'}$ we have $E = \nabla u$ for some $u \in W^{1,p'}(\Omega)$ with

$$\int_\Omega \varepsilon(x)\nabla u \cdot \nabla\phi \, dx = 0 \quad \text{for all } \phi \in W^{1,p}(\Omega).$$

Proposition 10 for $f = 0$ then implies $\nabla u = 0$, i.e., $E = 0$. We thus conclude that the sum is direct and $X^{p'} \oplus Y^{p'} \subset L^{p'}(\Omega;\mathbb{R}^3)$. To prove equality we define, for any given $f \in L^{p'}(\Omega;\mathbb{R}^3)$, $\Pi f := f + \nabla u$ where u is the solution from Proposition 10. Then Π is a bounded linear operator on $L^{p'}(\Omega;\mathbb{R}^3)$ with

$$\int_\Omega \varepsilon(x)(\Pi f)\cdot\nabla\phi\, dx = \int_\Omega \varepsilon(x)(f+\nabla u)\cdot\nabla\phi\, dx = 0 \qquad \text{for all } \phi \in W^{1,p}(\Omega)$$

in view of (16). We conclude $\Pi : L^{p'}(\Omega;\mathbb{R}^3) \to X^{p'}$, so $\nabla u \in Y^{p'}$ implies $f = \Pi f - \nabla u \in X^{p'} \oplus Y^{p'}$. We thus obtain

$$L^{p'}(\Omega;\mathbb{R}^3) = X^{p'} \oplus Y^{p'}.$$

The equality $(Y^p)^{\perp_\varepsilon} = X^{p'}$ holds by definition. To show $Y^p = (X^{p'})^{\perp_\varepsilon}$ first note that the inclusion $\subset$ is trivial. So let us assume $g \in (X^{p'})^{\perp_\varepsilon}$, in particular $g \in L^p(\Omega;\mathbb{R}^3)$. Since $f := \varepsilon(x)^{-1}(\nabla\times\Phi) \in X^{p'}$ whenever $\Phi \in C_0^\infty(\Omega;\mathbb{R}^3)$, we get

$$\int_\Omega (\nabla\times\Phi)\cdot g\, dx = \int_\Omega \varepsilon(x) f \cdot g(x)\, dx = 0 \qquad \text{for all } \Phi \in C_0^\infty(\Omega;\mathbb{R}^3). \tag{17}$$

This and Lemma 33 imply that g must be a gradient, which proves the claim. □

Remark 12. In the case of homogeneous and isotropic permittivities, i.e., $\varepsilon(x) \equiv \varepsilon_0$ with $\varepsilon_0 \in (0,\infty)$, the above Helmholtz Decomposition Theorem was proved by Fujiwara-Morimoto in [16] for smooth domains and Simader and Sohr [23, Theorem 1.4] extended this result to C^1-domains. We refer to [15, Theorem 11.1],[17, Theorem 1.3] for alternative proofs and variants of this result in Lipschitz domains.

In order to use the Fredholm theory of symmetric compact operators we first show the following.

Proposition 13. *Assume (A1),(A2). Then* $\mathcal{V}$ *is a closed subspace of* $H^1(\Omega;\mathbb{R}^3)$.

Proof. The space $\mathcal{V}$ is a subset of $F(\Omega,\varepsilon,\nu)$ defined in [22] given that all vector fields $E \in \mathcal{V}$ satisfy

$$E \in L^2(\Omega;\mathbb{R}^3), \quad \nabla\times E \in L^2(\Omega;\mathbb{R}^3), \quad \nabla\cdot(\varepsilon E) = 0 \text{ in } \Omega, \quad (\varepsilon E)\cdot\nu = 0 \text{ on } \partial\Omega.$$

By [22, Theorem 1.1] and (A1),(A2) the norms $\|\cdot\|_{H^1(\Omega;\mathbb{R}^3)}$ and $\|\cdot\|$ from (3) are equivalent on $F(\Omega,\varepsilon,\nu)$ and in particular on $\mathcal{V}$, which proves the result. □

Note that [22, Theorem 1.1] requires the exterior ball condition for Ω, which is why we included it in (A1). As a consequence of the previous proposition, $\mathcal{V}$ inherits all embeddings of $H^1(\Omega;\mathbb{R}^3)$. The Rellich-Kondrachov Theorem as well as the definitions of $\mathcal{V}, X^p$ imply

$$\mathcal{V} \hookrightarrow X^p \quad \text{boundedly for } 1 \le p \le 6 \text{ and compactly for } 1 \le p < 6 \tag{18}$$

given that the Sobolev-critical exponent for the embeddings of $H^1(\Omega;\mathbb{R}^3)$ into $L^p(\Omega;\mathbb{R}^3)$ is $p = 6$ by our assumption on the domain regularity (A1). The Riesz Representation Theorem in $\mathcal{V}$ implies that $\mathcal{L}+1$ has a bounded resolvent from $\mathcal{V}'$ into $\mathcal{V}$, which, by (18), is compact as an operator from $X^{p'}$ to X^p for $1 \le p < 6$ and in particular from the Hilbert space X^2 into itself. We write

$$\sigma(\mathcal{L}) := \{\lambda \in \mathbb{R} : \mathcal{L}(E) = \lambda E \text{ for some } E \in \mathcal{V}\setminus\{0\}\}$$

for the spectrum of $\mathcal{L}$ and denote by $\mathrm{Eig}_\lambda \subset \mathcal{V}$ the (possibly trivial) eigenspace of $\mathcal{L}$ associated with $\lambda \in \mathbb{R}$. In view of (18) we have $\mathrm{Eig}_\lambda \hookrightarrow X^p$ and, for notational convenience, we write Eig_λ^p whenever this set is considered as a closed subspace of X^p and hence of $L^p(\Omega;\mathbb{R}^3)$. Fredholm theory for selfadjoint operators in Hilbert spaces gives the following.

Proposition 14. *Assume (A1),(A2),* $1 \le p \le 6$ *and* $\lambda \in \mathbb{R}$.

(i) The selfadjoint operator $\mathcal{L}$ *in the Hilbert space* $(X^2, \langle\cdot,\cdot\rangle_\varepsilon)$ *has countably many eigenvalues such that the corresponding eigenfunctions form an orthonormal basis of* $(X^2, \langle\cdot,\cdot\rangle_\varepsilon)$ *and of* $(\mathcal{V}, \langle\cdot,\cdot\rangle)$*. All of these eigenvalues are positive and they tend to* $+\infty$.

(ii) The linear problem (14) with $g \in X^{p'}$ *has a weak solution if and only if* $g \perp_\varepsilon \mathrm{Eig}_\lambda^p$*. The solution is unique up to elements of* Eig_λ.

(iii) For all $\lambda \in \mathbb{R} \setminus \sigma(\mathcal{L})$ *the resolvent* $(\mathcal{L}-\lambda)^{-1} : X^{p'} \to \mathcal{V}$ *is bounded for all* $p \le 6$ *and it is compact provided that* $p < 6$*. The analogous statement is true for* $\lambda \in \sigma(\mathcal{L})$ *if* $X^{p'}, \mathcal{V}$ *are replaced by the subspaces* $\{g \in X^{p'} : g \perp_\varepsilon \mathrm{Eig}_\lambda^p\}$ *and* $\{E \in \mathcal{V} : E \perp \mathrm{Eig}_\lambda\}$.

Proof. The claims are standard except for the positivity and the unboundedness of the eigenvalues. The eigenvalues are unbounded from above given that the corresponding Rayleigh quotients are unbounded from above over $\mathcal{V}$. To see this one may choose nontrivial divergence-free test functions with shrinking support, e.g., $E_\tau(x) := \chi(\tau|x-y|^2)(y_2 - x_2, x_1 - y_1, 0)$ for suitable $\chi \in C_0^\infty(\mathbb{R}), y \in \Omega$ and $\tau \nearrow \infty$. Moreover, every eigenpair $(E,\lambda) \in \mathcal{V} \times \mathbb{R}$ of $\mathcal{L}$ satisfies, in view of (A2),

$$\lambda\|E\|_\varepsilon^2 = \lambda\int_\Omega \varepsilon(x)E\cdot E\,dx = \int_\Omega \mu(x)^{-1}(\nabla\times E)\cdot(\nabla\times E)\,dx \gtrsim \int_\Omega |\nabla\times E|^2\,dx.$$

As a consequence, eigenvalues λ satisfy $\lambda \ge 0$ and $\lambda = 0$ if and only if the associated eigenfunction satisfies $\nabla \times E = 0$. Given that $E \in \mathcal{V}$, the latter is equivalent to $E = 0$ because Lemma 33 implies $E = \nabla u$ for some $u \in H^1(\Omega)$ and thus, by definition of $\mathcal{V}$, $\|\nabla u\|_\varepsilon^2 = \langle E, \nabla u\rangle_\varepsilon = 0$. As a consequence, λ must be positive. □

Now we extend the considerations to the linear problem (14) for right hand sides $g \in X^{p'} \oplus Y^q$ for suitable $q \in [1,\infty]$. The right ansatz for the solution space is $\mathcal{V} \oplus Y^q$. A weak solution $E \in \mathcal{V} \oplus Y^q$ of (14), with $E = E_1 + E_2, E_1 \in \mathcal{V}, E_2 \in Y^q$, is supposed to satisfy

$$\int_\Omega \mu(x)^{-1}(\nabla\times E_1)\cdot(\nabla\times\Phi_1)\,dx - \lambda\int_\Omega \varepsilon(x)E\cdot\Phi\,dx = \int_\Omega \varepsilon(x)g\cdot\Phi\,dx \quad \text{for all } \Phi \in \mathcal{V}\oplus Y^q.$$

This extends the notion of a weak solution in $\mathcal{V}$ given by (15). To have well-defined integrals we assume[2] $1 \le p \le 6$ as before as well as $q \ge \max\{2, p'\}$. By orthogonality, this is equivalent to

$$\begin{aligned}\int_\Omega \mu(x)^{-1}(\nabla\times E_1)\cdot(\nabla\times\Phi_1)\,dx - \lambda\int_\Omega \varepsilon(x)E_1\cdot\Phi_1\,dx &= \int_\Omega \varepsilon(x)g_1\cdot\Phi_1\,dx \qquad \text{for all } \Phi_1 \in \mathcal{V},\\ -\lambda\int_\Omega \varepsilon(x)E_2\cdot\Phi_2\,dx &= \int_\Omega \varepsilon(x)g_2\cdot\Phi_2\,dx \qquad \text{for all } \Phi_2 \in Y^q.\end{aligned}$$

[2] Defining the integral

$$\int_\Omega \varepsilon(x)g\cdot\Phi\,dx := \int_\Omega \varepsilon(x)g_1\cdot\Phi_1\,dx + \int_\Omega \varepsilon(x)g_2\cdot\Phi_2\,dx$$

by formal orthogonality with respect to $\langle\cdot,\cdot\rangle_\varepsilon$ we can even relax the assumption $q \ge \max\{2,p'\}$ to $q \ge 2$. Indeed, $g_1 \in X^{p'}, \Phi_1 \in \mathcal{V} \subset X^p, 1 \le p \le 6$ justifies the first integral and $g_2, \Phi_2 \in Y^q, q \ge 2$ justifies the second one. However, given that our applications only concern the range $2 < p < 6$, this extension is not needed in this paper.

From Proposition 14 we get the solution theory for the first equation and the solution of the second equation is trivial by our choice of the solution space. We obtain the following.

Theorem 15. *Assume (A1),(A2) as well as* $g = g_1 + g_2$ *where* $g_1 \in X^{p'}, g_2 \in Y^q$ *for exponents* $1 \le p \le 6, \max\{2, p'\} \le q \le \infty$ *and* $\lambda \in \mathbb{R}$.

(i) If $\lambda \in \mathbb{R} \setminus (\sigma(\mathcal{L}) \cup \{0\})$ *then the unique weak solution* $E \in \mathcal{V} \oplus Y^q$ *of* (14) *is given by*

$$E = (\mathcal{L} - \lambda)^{-1} g_1 + \lambda^{-1} g_2.$$

(ii) If $\lambda \in \sigma(\mathcal{L})$ *then* (14) *admits weak solutions if and only if* $g_1 \perp_\varepsilon \mathrm{Eig}_\lambda^p$. *In this case all weak solutions* $E \in \mathcal{V} \oplus Y^q$ *are given by*

$$E \in (\mathcal{L} - \lambda)^{-1} g_1 + \lambda^{-1} g_2 + \mathrm{Eig}_\lambda.$$

(iii) If $\lambda = 0$ *then* (14) *admits weak solutions if and only if* $g_2 = 0$. *In this case all weak solutions* $E \in \mathcal{V} \oplus Y^q$ *are given by*

$$E \in \mathcal{L}^{-1} g_1 + Y^q.$$

Later, in the discussion of the nonlinear Neumann problem for $\lambda = \omega^2 \in \sigma(\mathcal{L})$, it will turn out convenient to choose g_1 as in (ii). To this end, we introduce for $1 \le p \le 6$ and $\lambda \in \mathbb{R}$

$$X_\lambda^{p'} := \Big\{ f \in X^{p'} : \int_\Omega \varepsilon(x) f \cdot \phi \, dx = 0 \text{ for all } \phi \in \mathrm{Eig}_\lambda^p \Big\}.$$

This definition makes sense for $1 \le p \le 6$ and $\lambda \in \mathbb{R}$. We then have $g_1 \in X^{p'}, g_1 \perp_\varepsilon \mathrm{Eig}_\lambda^p$ if and only if $g_1 \in X_\lambda^{p'}$. This is why the function space $X_\lambda^{p'}$ and its properties will be needed later on.

Proposition 16. *Assume* $\frac{6}{5} \le p \le 6$ *and* $\lambda \in \mathbb{R}$. *Then* $X^{p'} = X_\lambda^{p'} \oplus \mathrm{Eig}_\lambda^{p'}$ *with*

$$(X_\lambda^{p'})^{\perp_\varepsilon} = \mathrm{Eig}_\lambda^p \oplus Y^p, \qquad (\mathrm{Eig}_\lambda^{p'} \oplus Y^{p'})^{\perp_\varepsilon} = X_\lambda^p.$$

Proof. We only prove $(X_\lambda^{p'})^{\perp_\varepsilon} \subset \mathrm{Eig}_\lambda^p \oplus Y^p$, so let $f \in (X_\lambda^{p'})^{\perp_\varepsilon} \subset L^p(\Omega; \mathbb{R}^3)$ and define

$$\Pi(f) := f - \sum_{i=1}^n \langle f, \phi_i \rangle_\varepsilon \phi_i \in L^p(\Omega; \mathbb{R}^3) \text{ where } \{\phi_1, \ldots, \phi_n\} \text{ is an ONB of } (\mathrm{Eig}_\lambda^p, \langle \cdot, \cdot \rangle_\varepsilon).$$

With this definition we find $\Pi(f) \in (X^{p'})^{\perp_\varepsilon} = Y^p$ by Proposition 11. This implies $f \in \mathrm{Eig}_\lambda^p \oplus Y^p$. □

4. Proof of Theorem 1

The nonlinear Neumann problem (2) reads

$$(\mathcal{L} - \omega^2) E = P \qquad \text{where } P := \varepsilon(x)^{-1} f(x, E) \text{ and } E \in \mathcal{V} \oplus \mathcal{W}.$$

Aiming for a dual formulation of this problem, we solve the linear problem and treat the resulting equation as a problem for the vector field P. The inversion of the map $E \mapsto P$ is possible thanks to our assumption on the nonlinearity f. In the Appendix (Proposition 34) we show that (A3) implies that an inverse $\psi(x, \cdot) := f(x, \cdot)^{-1}$ exists almost everywhere with the following properties:

(A3') $\psi : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ is measurable with $\psi(x, P) = \psi_0(x, |P|)|P|^{-1} P$ where, for almost all $x \in \Omega$,

$$z \mapsto \psi_0(x, z) \text{ is positive, increasing and differentiable on } (0, \infty),$$
$$z \mapsto z^{-1} \psi_0(x, z) \text{ is decreasing on } (0, \infty).$$

Moreover, there are $c_1, c_2 > 0$ and $2 < p < 6$ such that for almost all $x \in \Omega$ and $z > 0$ we have

$$\int_0^z \psi_0(x,s)\,ds - \frac{1}{2}\psi_0(x,z)z \geq c_1|z|^{p'} \geq c_2\psi_0(x,z)z \tag{19}$$

We anticipate Proposition 34 for the sake of the presentation.

Proposition 17. *Assume that $f : \Omega\times\mathbb{R}^3 \to \mathbb{R}^3$ satisfies (A3). Then $\psi(x,\cdot) := f(x,\cdot)^{-1}$ exists for almost all $x \in \Omega$ and satisfies (A3').*

As a consequence, solving the nonlinear Neumann problem amounts to solving the quasilinear problem

$$(\mathcal{L} - \omega^2)\big(\psi(x, \varepsilon(x)P)\big) = P. \tag{20}$$

The case distinction (i),(ii),(iii) in Theorem 15 leads to a separate discussion of (20) according to the following cases:

$$\text{(I)} \quad \omega^2 \in (0,\infty)\setminus\sigma(\mathcal{L}) \qquad \text{or} \qquad \text{(II)} \quad \omega^2 \in \sigma(\mathcal{L}) \qquad \text{or} \qquad \text{(III)} \quad \omega^2 = 0.$$

Case (I) will be treated in full detail whereas our presentation of the cases (II),(III) focusses on the modifications with respect to (I).

4.1. Case (I)

Here we assume $\omega^2 \in (0,\infty)\setminus\sigma(\mathcal{L})$.

The first step is to prove that the original problem is equivalent to finding ground and bound states of the functional

$$\begin{aligned} J(P) := &\int_\Omega \Psi(x,\varepsilon(x)P)\,dx + \frac{1}{2\omega^2}\int_\Omega P_2\cdot\varepsilon(x)P_2\,dx \\ &- \frac{1}{2}\int_\Omega (\mathcal{L}-\omega^2)^{-1}P_1\cdot\varepsilon(x)P_1\,dx \end{aligned} \tag{21}$$

for $P \in Z := X^{p'} \oplus Y^2$. It is straightforward to check $J \in C^1(Z)$ with Fréchet derivative

$$\begin{aligned} J'(P)[h] = &\int_\Omega \psi(x,\varepsilon(x)P)\cdot\varepsilon(x)h\,dx + \omega^{-2}\int_\Omega P_2\cdot\varepsilon(x)h_2\,dx \\ &- \int_\Omega (\mathcal{L}-\omega^2)^{-1}P_1\cdot\varepsilon(x)h_1\,dx \end{aligned}$$

for all $h \in Z$. Here one uses that $(\mathcal{L}-\omega^2)^{-1}$ is symmetric with respect to $\langle\cdot,\cdot\rangle_\varepsilon$. Exploiting the formulas for $(X^{p'})^{\perp_\varepsilon}$ and $(Y^2)^{\perp_\varepsilon}$ from Proposition 11 we find that the Euler-Lagrange equation of J reads

$$\psi(x,\varepsilon(x)P) = (\mathcal{L}-\omega^2)^{-1}P_1 - \omega^{-2}P_2 \qquad \text{for } P_1 \in X^{p'},\ P_2 \in Y^2. \tag{22}$$

Lemma 18. *Assume (A1),(A2),(A3) and $\omega^2 \in (0,\infty)\setminus\sigma(\mathcal{L})$. Then $I'(E) = 0, E \in \mathcal{V}\oplus\mathcal{W}$ if and only if $J'(P) = 0, P \in Z$, where P, E are related to each other via $P = \varepsilon(x)^{-1}f(x,E)$, $E = \psi(x,\varepsilon(x)P)$ with ψ as in Proposition 17.*

Proof. Assume $I'(E) = 0$ where $E = E_1+E_2$ for $E_1 \in \mathcal{V}, E_2 \in \mathcal{W}$, define $P := \varepsilon(x)^{-1}f(x,E)$. From $E \in \mathcal{V}\oplus\mathcal{W} \subset L^p(\Omega;\mathbb{R}^3)$ and the growth properties of f from (A3) we infer $P \in L^{p'}(\Omega;\mathbb{R}^3)$. We write $P = P_1 + P_2$ according to the Helmholtz Decomposition from Proposition 11. According to the formula of I from (4) we get for $\Phi_1 \in \mathcal{V}, \Phi_2 = 0$

$$\begin{aligned} 0 &= \int_\Omega \mu(x)^{-1}(\nabla\times E_1)\cdot(\nabla\times\Phi_1)\,dx - \omega^2\int_\Omega\varepsilon(x)E\cdot\Phi_1\,dx - \int_\Omega f(x,E)\cdot\Phi_1\,dx \\ &= \int_\Omega\mu(x)^{-1}(\nabla\times E_1)\cdot(\nabla\times\Phi_1)\,dx - \omega^2\int_\Omega\varepsilon(x)E_1\cdot\Phi_1\,dx - \int_\Omega\varepsilon(x)P_1\cdot\Phi_1\,dx. \end{aligned}$$

Choosing instead $\Phi_1 = 0$ and $\Phi_2 \in \mathcal{W}$ we get

$$0 = -\omega^2\int_\Omega \varepsilon(x)E_2\cdot\Phi_2\,dx - \int_\Omega \varepsilon(x)P_2\cdot\Phi_2\,dx$$

for all $\Phi_2 \in \mathcal{W}$. This implies

$$E_1 = (\mathcal{L} - \omega^2)^{-1} P_1, \qquad E_2 = -\omega^{-2} P_2.$$

In particular, $P_2 = -\omega^2 E_2 \in \mathcal{W} = Y^p \subset Y^2$, so

$$P = P_1 + P_2 \in X^{p'} \oplus Y^2 = Z \quad \text{and}$$
$$\psi(x, \varepsilon(x)P) = E = E_1 + E_2 = (\mathcal{L} - \omega^2)^{-1} P_1 - \omega^{-2} P_2.$$

So (22) holds and we conclude $J'(P) = 0$. To prove the reverse implication assume $J'(P) = 0$ for $P \in X^{p'} \oplus Y^2$ so that (22) holds. This implies $P \in L^{p'}(\Omega; \mathbb{R}^3)$ and hence $E := \psi(x, \varepsilon(x)P) \in L^p(\Omega; \mathbb{R}^3)$ in view of (A3'). Recall that (A3') is equivalent to (A3) thanks to Proposition 17. From (22) we even get $E_1 = (\mathcal{L} - \omega^2)^{-1} P_1 \in \mathcal{V}$ by Proposition 14(iii), so $E \in \mathcal{V} \oplus \mathcal{W}$. So (22) and Theorem 15(i) imply $I'(E) = 0$. □

Knowing that the dual problem (22) and the original one are equivalent, we may now focus on proving the existence of critical points of J with the aid of Theorem 5.

Proposition 19. *Assume (A1),(A2),(A3'). Then J from (21) satisfies Palais-Smale condition.*

Proof. Let (P_n) be a Palais-Smale sequence for J, so $J(P_n) \to c \in \mathbb{R}$ and $J'(P_n) \to 0$. From (A3') we get

$$\begin{aligned} c + o(1)\|P_n\|_{p'} &= J(P_n) - \frac{1}{2} J'(P_n)[P_n] \\ &= \int_\Omega \Psi(x, P_n) - \frac{1}{2}\psi(x, P_n) \cdot P_n \, dx \gtrsim \int_\Omega |P_n|^{p'} \, dx = \|P_n\|_{p'}^{p'}, \end{aligned}$$

so (P_n) is bounded in $L^{p'}(\Omega; \mathbb{R}^3)$. This and $J(P_n) \to c$ imply that (P_n^2) is bounded in $L^2(\Omega; \mathbb{R}^3)$ as well, so we may assume $P_n^1 \rightharpoonup P^1$ in $X^{p'}$ and $P_n^2 \rightharpoonup P^2$ in Y^2. From $J'(P_n) \to 0, P_n \rightharpoonup P$ and the compactness of $(\mathcal{L} - \omega^2)^{-1} : X^{p'} \to X^p$, see Proposition 14(iii), we get as $n \to \infty$.

$$\begin{aligned} o(1) &= J'(P_n)[P_n - P] - J'(P)[P_n - P] \\ &= \int_\Omega \big(\psi(x, \varepsilon(x)P_n) - \psi(x, \varepsilon(x)P)\big) \cdot \varepsilon(x)(P_n - P) \\ &\quad + \omega^{-2}(P_n^2 - P^2) \cdot \varepsilon(x)(P_n^2 - P^2) \, dx \\ &\quad - \int_\Omega (\mathcal{L} - \omega^2)^{-1}(P_n^1 - P^1) \cdot \varepsilon(x)(P_n^1 - P^1) \, dx \\ &= \int_\Omega \big(\psi(x, \varepsilon(x)P_n) - \psi(x, \varepsilon(x)P)\big) \cdot \varepsilon(x)(P_n - P) \\ &\quad + \omega^{-2}(P_n^2 - P^2) \cdot \varepsilon(x)(P_n^2 - P^2) \, dx + o(1). \end{aligned}$$

The integrand is nonnegative and hence converges to zero in $L^1(\Omega)$. By a Corollary of the Riesz-Fischer Theorem, a subsequence, still denoted by (P_n), is pointwise almost everywhere bounded by some function $H \in L^1(\Omega)$ and converges to zero pointwise almost everywhere. Since $\psi(x, \cdot)$ is strictly monotone for almost all $x \in \Omega$ and $\varepsilon(x)$ is uniformly positive definite, we deduce $P_n \to P$ and $P_n^2 \to P^2$ pointwise almost everywhere. Furthermore, combining

$$\big(\psi(x, \varepsilon(x)P_n) - \psi(x, \varepsilon(x)P)\big) \cdot \varepsilon(x)(P_n - P) + \omega^{-2}(P_n^2 - P^2) \cdot \varepsilon(x)(P_n^2 - P^2) \leq H \quad \text{on } \Omega$$

with the estimates $\psi(x, \varepsilon(x)P_n) \cdot \varepsilon(x) P_n \gtrsim |P_n|^{p'}$ and $|\psi(x, \varepsilon(x)P_n)| \lesssim |P_n|^{p'-1}$ from (A3') gives

$$|P_n|^{p'} + |P_n^2|^2 \leq \hat{H} \qquad \text{on } \Omega$$

where $\hat{H} \in L^1(\Omega)$ is defined in terms of $H, P, \psi(x, \varepsilon(x)P)$. So the Dominated Convergence Theorem implies $P_n \to P$ in $L^{p'}(\Omega)$ and $P_n^2 \to P^2$ in $L^2(\Omega)$, hence $\|P_n - P\| \to 0$. In particular, $J'(P) = 0$ and the claim is proved. □

We finally prove the existence of critical points. As usual, a ground state is a nontrivial critical point with least energy among all nontrivial critical points. We will see that ground states for J give rise to ground states for I and hence minimal energy solutions for the original problem.

Theorem 20. *Assume (A1),(A2),(A3') and $\omega^2 \in (0,\infty)\setminus\sigma(\mathcal{L})$. Then J admits a ground state and infinitely many bound states.*

Proof. The Banach space $Z = X^{p'} \oplus Y^2$ satisfies (G_1) with $Z^+ = Z, Z^- = \{0\}$. We check that the functional $J \in C^1(Z)$ has the Mountain Pass Geometry $(G_2),(G_3)$. Indeed, (G_2) is straightforward. To prove (G_3) let $m \in \mathbb{N}$ be arbitrary. Then define Z_m as the span of m linearly independent eigenfunctions associated with eigenvalues $\lambda_1,\ldots,\lambda_m \in (\omega^2,\infty)\cap\sigma(\mathcal{L})$. With this choice, we get for a linear combination of eigenfunctions $P := \sum_{i=1}^m c_i\phi_i \in Z_m$

$$\begin{aligned}\int_\Omega (\mathcal{L}-\omega^2)^{-1}P\cdot\varepsilon(x)P\,dx &= \sum_{i,j=1}^m c_ic_j\int_\Omega(\mathcal{L}-\omega^2)^{-1}\phi_i\cdot\varepsilon(x)\phi_j\,dx\\ &= \sum_{i,j=1}^m \frac{c_ic_j}{\lambda_i-\omega^2}\int_\Omega \phi_i\cdot\varepsilon(x)\phi_j\,dx\\ &= \sum_{i=1}^m \frac{c_i^2}{\lambda_i-\omega^2}\|\phi_i\|_\varepsilon^2\\ &\ge \min\{(\lambda_i-\omega^2)^{-1} : i=1,\ldots,m\}\cdot\|P\|_\varepsilon^2\\ &\ge c_m\|P\|^2\end{aligned}$$

for some $c_m > 0$. Hence, $P = P_1$ and $P_2 = 0$ gives for some $C > 0$

$$J(P) = \int_\Omega \Psi(x,\varepsilon(x)P)\,dx - \frac{1}{2}\int_\Omega(\mathcal{L}-\omega^2)^{-1}P\cdot\varepsilon(x)P\,dx \le C\|P\|_{p'}^{p'} - \frac{c_m}{2}\|P\|^2,$$

which implies $J(P)\to-\infty$ as $P\in Z_m, \|P\|\to\infty$. Finally, Proposition 19 shows that J satisfies the Palais-Smale condition, so (G_4) holds by Remark 6(a). Theorem 5 then implies the existence of infinitely many finite energy solutions. Finally, a critical point exists at the mountain pass level [27, Theorem 1.15] and this solution is in fact a ground state by [27, Theorem 4.2]. □

Proof of Theorem 1 for $\omega^2 \in (0,\infty)\setminus\sigma(\mathcal{L})$: Since f satisfies (A3), the function $\psi(x,\cdot) := f(x,\cdot)^{-1}$ satisfies (A3') by Proposition 17. So Theorem 20 implies that the functional J has a ground state $P^\star \in Z\setminus\{0\}$. By Lemma 18, $E^\star(x) := \psi(x,P^\star(x))$ satisfies $E^\star \in \mathcal{V}\oplus\mathcal{W}\setminus\{0\}$ as well as $I'(E^\star) = 0$. In fact, $E^\star$ is even a ground state, which is proved[3] as in [20]. Moreover, Theorem 20 provides infinitely many other nontrivial critical points of J and hence of I, which finishes the proof of Theorem 1. □

[3] One may argue as in the proof of [20, Theorem 15]. In the notation of that paper,

$$X := \mathcal{V}\oplus\mathcal{W},\qquad Y := Z = X^{p'}\oplus Y^2,\qquad G := J,\qquad \varphi(h) := \int_\Omega h\,dx$$

$$Q_1(E,\tilde{E}) := \int_\Omega \mu(x)^{-1}(\nabla\times E)\cdot(\nabla\times\tilde{E}) - \omega^2\varepsilon(x)E\cdot\tilde{E}\,dx,$$

$$Q_2(E,\tilde{E}) := \int_\Omega(\mathcal{L}-\omega^2)^{-1}E\cdot\tilde{E}\,dx.$$

The theorem requires the equivalence of the original and the dual problem, see eq. (10) in [20], which we checked in Lemma 18. Even though F,G are in general not twice continuous differentiable, the same argument as in [20] gives the claim.

4.2. Case (II)

We proceed as in the previous section and start by defining the energy functional

$$\begin{aligned} J(P) := &\int_\Omega \Psi(x,\varepsilon(x)P)\,dx + \frac{1}{2\omega^2}\int_\Omega P_2\cdot\varepsilon(x)P_2\,dx \\ &- \frac{1}{2}\int_\Omega (\mathcal{L}-\omega^2)^{-1}P_1\cdot\varepsilon(x)P_1\,dx, \end{aligned} \tag{23}$$

on the Banach space $Z_{\omega^2} := X^{p'}_{\omega^2}\oplus Y^2$. Note that by our choice of Z_{ω^2} the resolvent is well-defined as a linear operator acting on $P_1\in X^{p'}_{\omega^2}$, see Theorem 15(i). We have $J\in C^1(Z_{\omega^2})$ with

$$\begin{aligned} J'(P)[h] = &\int_\Omega \Psi(x,\varepsilon(x)P)\cdot\varepsilon(x)h\,dx + \omega^{-2}\int_\Omega P_2\cdot\varepsilon(x)h_2\,dx \\ &- \int_\Omega (\mathcal{L}-\omega^2)^{-1}P_1\cdot\varepsilon(x)h_1\,dx \end{aligned}$$

for all $h\in Z_{\omega^2}$ and the Euler-Lagrange equation $J'(P)=0$ actually reads, in view of Proposition 16,

$$\psi(x,\varepsilon(x)P)\in(\mathcal{L}-\omega^2)^{-1}P_1 - \omega^{-2}P_2 + \mathrm{Eig}^p_{\omega^2}\,. \tag{24}$$

Lemma 21. *Assume (A1),(A2),(A3) and $\omega^2\in\sigma(\mathcal{L})$. Then $I'(E)=0, E\in\mathcal{V}\oplus\mathcal{W}$ if and only if $J'(P)=0, P\in X^{p'}_{\omega^2}\oplus Y^2$, where P,E are related to each other via $P=\varepsilon(x)^{-1}f(x,E)$, $E=\psi(x,\varepsilon(x)P)$ with ψ as in Proposition 17.*

Proof. From $P\in Z_{\omega^2}\subset L^{p'}(\Omega;\mathbb{R}^3)$ and from the fact that ψ satisfies $(A3')$ we get $E:=\psi(x,\varepsilon(x)P)\in L^p(\Omega;\mathbb{R}^3)$, in particular $E_2\in Y^p=\mathcal{W}$. Now if $J'(P)=0$, then (24) implies

$$E=E_1+E_2,\qquad E_1\in(\mathcal{L}-\omega^2)^{-1}P_1+\mathrm{Eig}^p_{\omega^2},\quad E_2=-\omega^{-2}P_2$$

From $P_1\in X^{p'}$ we infer $(\mathcal{L}-\omega^2)E_1=P_1$ in the weak sense, in particular $E_1\in\mathcal{V}$. Then $I'(E)=0$ follows and the claim is proved. □

The verification of the Palais-Smale condition and the existence proof for critical points is the same as above as it suffices to replace $X^{p'}$ by $X^{p'}_{\omega^2}$ in the proof.

Theorem 22. *Assume (A1),(A2),(A3') and $\omega^2\in\sigma(\mathcal{L})$. Then J from (23) satisfies the Palais-Smale condition and admits ground states as well as infinitely many bound states.*

Proof of Theorem 1 for $\omega^2\in\sigma(\mathcal{L})$: Same reasoning as in the case $\omega^2\in(0,\infty)\setminus\sigma(\mathcal{L})$ up to replacing Theorem 20 by Theorem 22. □

4.3. Case (III)

We finally deal with the static case $\omega^2=0$ where the boundary value problem does not involve the permittivity matrix ε any more, so may without loss of generality assume $\varepsilon(x):=I_{3\times3}$ and ignore (A2). The energy functional is given by $J:X^{p'}\to\mathbb{R}$ with

$$J(P):=\int_\Omega \Psi(x,P)\,dx - \frac{1}{2}\int_\Omega \mathcal{L}^{-1}P\cdot P\,dx.$$

Once more, this functional belongs to $C^1(X^{p'})$ and the Euler-Lagrange equation reads

$$\psi(x,P)\in\mathcal{L}^{-1}P+Y^p$$

For the definition of $\mathcal{L}^{-1}:X^{p'}\to\mathcal{V}\hookrightarrow X^p$ and the Euler-Lagrange equation see Theorem 15(iii). Proceeding as above we get the following.

Lemma 23. *Assume (A1),(A3) and $\omega^2=0$. Then $I'(E)=0, E\in\mathcal{V}\oplus\mathcal{W}$ if and only if $J'(P)=0, P\in X^{p'}$ where P,E are related to each other via $P=f(x,E)$, $E=\psi(x,P)$ and ψ is given by Proposition 17.*

Theorem 24. *Assume (A1),(A3') and $\omega^2 = 0$. Then $J \in C^1(X^{p'})$ satisfies the Palais-Smale condition and admits ground states as well as infinitely many bound states.*

***Proof of Theorem* 1 *for* $\omega^2 = 0$:** Same reasoning as in the case $\omega^2 \in (0,\infty) \setminus \sigma(\mathcal{L})$ up to replacing Theorem 20 by Theorem 24. ☐

5. Proof of Theorem 2

We now use the dual variational method to prove the existence of infinitely many L^p-solutions to the Nonlinear time-harmonic Maxwell's equation (5) under the assumption

$$\varepsilon(x) = \varepsilon_0 \in (0,\infty), \mu(x) = \mu_0 \in (0,\infty), \quad \omega^2 > 0, \quad f(\cdot, E) \text{ satisfies (A4).} \tag{25}$$

So the equation to solve is

$$\nabla \times \nabla \times E - \lambda E = f(x,E) \quad \text{in } \mathbb{R}^3 \qquad \text{where } \lambda := \omega^2 \varepsilon_0 \mu_0.$$

To this end we adapt the strategy that Evequoz and Weth [14] used to prove the existence of dual ground states for the Nonlinear Helmholtz Equation. In order to implement the dual variational method in the Maxwell setting, we use the Helmholtz Decomposition on $\mathbb{R}^3$. We recall that $\dot{W}^{1,p}(\mathbb{R}^3;\mathbb{R}^3)$ is a homogeneous Sobolev space, i.e., the closure of test functions with respect to $u \mapsto \|\nabla u\|_p$. We define

$$X^{p'} := \left\{ E \in L^{p'}(\mathbb{R}^3;\mathbb{R}^3) : \int_\Omega E \cdot \nabla \Phi \, dx = 0 \text{ for all } \Phi \in \dot{W}^{1,p}(\mathbb{R}^3;\mathbb{R}^3) \right\},$$
$$Y^{p'} := \left\{ \nabla u : u \in \dot{W}^{1,p'}(\mathbb{R}^3;\mathbb{R}^3) \right\}.$$

Proposition 25. *Assume $1 < p < \infty$. Then $L^{p'}(\mathbb{R}^3;\mathbb{R}^3) = X^{p'} \oplus Y^{p'}$ with $(X^{p'})^\perp = Y^p, (Y^{p'})^\perp = X^p$. We have, in the distributional sense,*

$$\nabla \times \nabla \times E_1 = -\Delta E_1 \quad \text{for } E_1 \in X^{p'}, \qquad \nabla \times \nabla \times E_2 = 0 \quad \text{for } E_2 \in Y^{p'}. \tag{26}$$

Proof. For any given $E \in L^{p'}(\mathbb{R}^3;\mathbb{R}^3)$ we define

$$\widehat{E_1}(\xi) := \hat{E}(\xi) - |\xi|^{-2}(\xi \cdot \hat{E}(\xi))\xi, \qquad \widehat{E_2}(\xi) := |\xi|^{-2}(\xi \cdot \hat{E}(\xi))\xi$$

Mikhlin's Multiplier Theorem implies that E_1, E_2 indeed belong to $L^{p'}(\mathbb{R}^3;\mathbb{R}^3)$. It is straightforward to check $(X^{p'})^\perp = Y^p, (Y^{p'})^\perp = X^p$. Then (26) follows from the identity $\nabla \times \nabla \times \Phi = -\Delta\Phi + \nabla(\nabla \cdot \Phi)$ for $\Phi \in C_0^\infty(\mathbb{R}^3;\mathbb{R}^3)$ and from the fact that the curl operator annihilates gradients. ☐

As a consequence, it suffices to find solutions for

$$(-\Delta - \lambda)E_1 - \lambda E_2 = f(x, E_1 + E_2) \quad \text{in } \mathbb{R}^3.$$

To prove the existence of nontrivial solutions to this problem we study a dual problem that, in contrast to the situation on bounded domains studied earlier, is not equivalent to the original problem. This is due to the fact that $-\Delta - \lambda$ is not invertible on $X^{p'}$, but admits some sort of a right inverse $\mathcal{R}$ defined via the Limiting Absorption Principle for the Helmholtz equation. We define $\mathcal{R} := \Re[(-\Delta - \lambda - i0)^{-1}]$, i.e.,

$$\mathcal{R} : X^{p'} \to X^p, \quad g \mapsto \lim_{\varepsilon \to 0^+} \Re\left(\mathcal{F}^{-1}\left(\frac{\hat{g}}{|\cdot|^2 - \lambda - i\varepsilon}\right)\right). \tag{27}$$

It is known [14, Theorem 2.1] that this is a well-defined bounded linear operator for $4 \le p \le 6$. Note that $\frac{2(n+1)}{n-1} = 4$ corresponds to the Stein-Tomas exponent and $\frac{2n}{n-2} = 6$ to the Sobolev-critical exponent for $n = 3$. Introducting $P := f(x, E_1 + E_2)$ we find that it is sufficient to find a solution to

$$E_1 = \mathcal{R}(P^1), \qquad E_2 = -\lambda^{-1} P^2.$$

This leads to the study of the energy functional

$$J(P) = \int_{\mathbb{R}^3} \Psi(x,P)\,dx + \frac{1}{2\lambda}\int_{\mathbb{R}^3} |P^2|^2\,dx - \frac{1}{2}\int_{\mathbb{R}^3} P^1\cdot\mathcal{R}(P^1)\,dx \tag{28}$$

that is well-defined on the Banach space

$$Z := X^{p'} \oplus (Y^2 \cap Y^{p'}) \quad \text{with norm } \|P\| := \|P\|_{p'} + \|P^2\|_2.$$

Note that the assumption $P_2 \in Y^2$ is not sufficient. The functional J is continuously differentiable and the Euler-Lagrange equation reads

$$\psi(x,P) = \mathcal{R}(P_1) - \lambda^{-2}P_2. \tag{29}$$

It turns out that solutions P of this dual problem lead to solutions of the function space $\mathcal{V}\oplus\mathcal{W}$ where

$$\mathcal{V} := \mathcal{R}(X^{p'}), \quad \mathcal{W} := Y^p \cap Y^{p'}, \qquad \text{so } \mathcal{V}\oplus\mathcal{W} \subset L^p(\mathbb{R}^3;\mathbb{R}^3).$$

Moreover, local elliptic regularity theory implies $\mathcal{V} \subset W^{2,p'}_{\text{loc}}(\mathbb{R}^3;\mathbb{R}^3)$. A function $E = E_1 + E_2 \in \mathcal{V}\oplus\mathcal{W}$ is called a weak solution of (5) if

$$\int_{\mathbb{R}^3} (\nabla\times E_1)\cdot(\nabla\times\Phi_1) - \lambda E\cdot\Phi\,dx = \int_{\mathbb{R}^3} f(x,E)\cdot\Phi\,dx$$
$$\text{for all } \Phi \in (\mathcal{V}\oplus\mathcal{W})\cap C_0^\infty(\Omega;\mathbb{R}^3)$$

and all integrals are well-defined by the choice of our space of test functions.

Lemma 26. *Assume* (25). *Then* $J'(P) = 0, P \in Z$ *implies* $I'(E) = 0, E \in \mathcal{V}\oplus\mathcal{W} \subset L^p(\mathbb{R}^3;\mathbb{R}^3)$ *where* $E = \psi(x,P)$ *and* ψ *is given by Proposition 17.*

Proof. Let $P \in Z$ satisfy $J'(P) = 0$, so (29) holds. From $Z \subset L^{p'}(\mathbb{R}^3;\mathbb{R}^3)$ and the growth conditions of ψ from (A3') we deduce $E := \psi(x,P) \in L^p(\mathbb{R}^3;\mathbb{R}^3)$. The Euler-Lagrange equation gives

$$E = E_1 + E_2, \qquad E_1 = \mathcal{R}(P_1), \quad E_2 = -\lambda^{-1}P_2$$

and thus $E_1 \in \mathcal{V}, E_2 \in \mathcal{W}$ and $I'(E) = 0$ in the sense of (30). ☐

Now we construct infinitely many nontrivial critical points for J with the aid of Theorem 5. We shall need the following local compactness property that generalizes [13, Lemma 2.2]. As a new feature, it uses Div-Curl-Lemma.

Proposition 27. *Assume* (25). *Then for any PS-sequence* (P_n) *of* J *there is a subsequence* (P_{n_j}) *and a critical point* P *of* J *such that*

$$P_{n_j} \rightharpoonup P \text{ in } Z, \quad P_{n_j} \to P \text{ in } L^{p'}_{loc}(\mathbb{R}^3;\mathbb{R}^3), \quad P^2_{n_j} \to P^2 \text{ in } L^2_{loc}(\mathbb{R}^3;\mathbb{R}^3).$$

Proof. As in the case of a bounded domain, one finds that (P_n) is bounded and w.l.o.g. weakly convergent to some $P \in Z$. From $J'(P_n) \to 0$ and $P_n \rightharpoonup P$ we infer for all bounded balls $B \subset \mathbb{R}^3$

$$\begin{aligned}
o(1) &= (J'(P_n) - J'(P))[(P_n - P)\mathbb{1}_B] \\
&= \int_B \big(\psi(x,P_n) - \psi(x,P)\big)\cdot(P_n - P) + \lambda^{-1}(P_n^2 - P^2)\cdot(P_n - P)\,dx \\
&- \int_B (P_n - P)\cdot\mathcal{R}(P_n^1 - P^1)\,dx \\
&= \int_B \big(\psi(x,P_n) - \psi(x,P)\big)\cdot(P_n - P) + \lambda^{-1}|P_n^2 - P^2|^2\,dx \\
&- \int_B (P_n - P)\cdot\mathcal{R}(P_n^1 - P^1)\,dx \\
&+ \lambda^{-1}\int_B (P_n^2 - P^2)\cdot(P_n^1 - P^1)\,dx \quad \text{as } n\to\infty.
\end{aligned}$$

We emphasize that for a fixed $n \in \mathbb{N}$ the last term does not necessarily vanish given that the integral is over B and not $\mathbb{R}^3$. The compactness of $\mathbb{1}_B\mathcal{R}$ established in [14, Lemma 4.1 (i)] implies

$$\lim_{n\to\infty} \int_B (P_n - P) \cdot \mathcal{R}(P_n^1 - P^1)\, dx = 0.$$

The Div-Curl-Lemma [11, Theorem 4.5.16] and $\nabla \cdot (P_n^1 - P^1) = 0$, $\nabla \times (P_n^2 - P^2) = 0$ give

$$\lim_{n\to\infty} \int_B (P_n^2 - P^2) \cdot (P_n^1 - P^1)\, dx = 0.$$

So we conclude

$$\lim_{n\to\infty} \int_B \big(\psi(x, P_n) - \psi(x, P)\big) \cdot (P_n - P) + \lambda^{-1}|P_n^2 - P^2|^2\, dx = 0.$$

As in the proof of Proposition 19 we obtain, up to the choice of a suitable subsequence, $P_n \to P$ in $L^{p'}(B)$ and $P_n^2 \to P^2$ in $L^2(B)$ as well as $J'(P) = 0$. □

Proposition 28. *Assume $P, Q \in L^{p'}(\mathbb{R}^3; \mathbb{R}^3)$ and (A4). Then*

$$\|P - Q\|_{p'} \lesssim \left(\int_{\mathbb{R}^3} \big(\psi(x, P) - \psi(x, Q)\big) \cdot (P - Q)\, dx\right)^{\frac{1}{2}} \cdot (\|P\|_{p'} + \|Q\|_{p'})^{\frac{2-p'}{2}}.$$

Proof. We content ourselves with proving

$$\big(\psi(x, P) - \psi(x, Q)\big) \cdot (P - Q) \gtrsim (|P| + |Q|)^{p'-2}|P - Q|^2$$

given that the remaining argument based on Hölder's inequality is straightforward. Define $f(\tau) := \psi(x, Q + \tau(P - Q)) \cdot (P - Q)$. Then

$$\big(\psi(x, P) - \psi(x, Q)\big) \cdot (P - Q) = f(1) - f(0) = f'(\tau) \quad \text{for some } \tau \in (0, 1).$$

So it remains to estimate $f'(\tau)$ from below. To do this we recall $\psi(x, \xi) = \psi_0^*(x, |\xi|)\xi$ where $\psi_0^*(x, z) := z^{-1}\psi_0(x, z)$ for almost all $x \in \mathbb{R}^3$, $z > 0$. Since $s \mapsto s^{-1}f_0(x, s)$ is increasing, we have $\partial_z\psi_0^*(x, z) \le 0$ for all $z > 0$. Hence, for $\xi_\tau := Q + \tau(P - Q)$,

$$\begin{aligned}
f'(\tau) &= D\psi(x, \xi_\tau)[P - Q] \cdot (P - Q) \\
&= \psi_0^*(x, |\xi_\tau|)|P - Q|^2 + |\xi_\tau|\partial_z\psi_0^*(x, |\xi_\tau|)\big(|\xi_\tau|^{-1}\xi_\tau \cdot (P - Q)\big)^2 \\
&\ge \big(\psi_0^*(x, |\xi_\tau|) + |\xi_\tau|\partial_z\psi_0^*(x, |\xi_\tau|)\big)|P - Q|^2 \\
&= \partial_z\psi_0(x, z)|_{z=|\xi_\tau|}|P - Q|^2 \\
&\overset{(A4)}{\gtrsim} |\xi_\tau|^{p'-2}|P - Q|^2 \\
&\gtrsim (|P| + |Q|)^{p'-2}|P - Q|^2.
\end{aligned}$$

Here we used $p', q' < 2$. □

Theorem 29. *Assume (25). Then J has a ground state and infinitely many geometrically distinct critical points.*

Proof. We apply Theorem 5 to the infinite-dimensional Banach space $Z = X^{p'} \oplus (Y^2 \cap Y^{p'})$. The assumptions $(G_1), (G_2)$ of that theorem are straightforward to check for $Z^- := \{0\}$. As to (G_3), for any given $m \in \mathbb{N}$ choose $Z_m := \operatorname{span}\{\phi_1, \ldots, \phi_m\} \subset Z$ where

$$\hat{\phi}_j(\xi) := \chi_j(|\xi|^2) \begin{pmatrix} -\xi_1 \\ \xi_2 \\ 0 \end{pmatrix}$$

where $\chi_j \in C_0^\infty(\mathbb{R})$ and $\emptyset \subsetneq \operatorname{supp}(\chi_i) \subset (\omega^2\varepsilon_0\mu_0 + j, \omega^2\varepsilon_0\mu_0 + j + 1)$.

Then $\{\phi_1, \ldots, \phi_m\}$ is a linearly independent set of divergence-free Schwartz functions. Moreover, all elements of Z_m are divergence-free, so $Z_m \subset X^{p'}$. Exploiting that all norms are equivalent on finite-dimensional spaces, one finds a $c_m > 0$ such that

$$\int_{\mathbb{R}^3} \mathcal{R}(P^1) \cdot P^1 \, dx = \int_{\mathbb{R}^3} \mathcal{R}(P) \cdot P \, dx = \int_{\mathbb{R}^3} \frac{|\hat{P}(\xi)|^2}{|\xi|^2 - \omega^2 \varepsilon_0 \mu_0} \, d\xi \geq c_m \|P\|^2 \quad \text{for all } P \in Z_m.$$

Furthermore,

$$\frac{1}{2\lambda} \int_{\mathbb{R}^3} |P^2|^2 \, dx = 0 \qquad \text{for all } P \in Z_m.$$

These facts imply $J(P) \to -\infty$ uniformly as $P \in Z_m, \|P\| \to \infty$ and (G_3) is proved. Finally, we verify (G_4) following [13, Lemma 3.2]. Assume we have two PS-sequences $(P_n), (Q_n)$ for J. In the case

$$\int_{\mathbb{R}^3} (P_n^1 - Q_n^1) \cdot \mathcal{R}(P_n^1 - Q_n^1) \, dx \to 0$$

we get

$$\begin{aligned} o(1) &= (J'(P_n) - J'(Q_n))[P_n - Q_n] \\ &= \int_{\mathbb{R}^3} \big(\psi(x, P_n) - \psi(x, Q_n)\big) \cdot (P_n - Q_n) + \lambda^{-1} |P_n^2 - Q_n^2|^2 \, dx \\ &- \int_{\mathbb{R}^3} (P_n^1 - Q_n^1) \cdot \mathcal{R}(P_n^1 - Q_n^1) \, dx \\ &= \int_{\mathbb{R}^3} \big(\psi(x, P_n) - \psi(x, Q_n)\big) \cdot (P_n - Q_n) \\ &+ \lambda^{-1} |P_n^2 - Q_n^2|^2 \, dx + o(1) \qquad \text{as } n \to \infty, \end{aligned}$$

which implies $\|P_n - Q_n\| \to 0$ in view of Proposition 28. In the complementary case the nonvanishing property of $\mathcal{R}$ from [14, Theorem 2.3] gives, for a suitable ball $B \subset \mathbb{R}^3$, a positive number ζ and $x_n \in \mathbb{Z}^3$, that $\tilde{P}_n = P_n(\cdot - x_n), \tilde{Q}_n := Q_n(\cdot - x_n)$ are still PS-sequences of J with

$$\int_B |\tilde{P}_n^1 - \tilde{Q}_n^1|^{p'} \, dx \geq \zeta > 0 \qquad \text{for all } n \in \mathbb{N}.$$

Here, the $\mathbb{Z}^3$-periodicity of f is used. By Proposition 27 we can select a subsequence, still denoted by $\tilde{P}_n, \tilde{Q}_n$ with weak limits $P, Q \in \mathcal{K}$ satisfying

$$\int_B |P^1 - Q^1|^{p'} \, dx \geq \zeta > 0.$$

Hence, $P, Q \in \mathcal{K}, P \neq Q$ and thus, by weak convergence,

$$\limsup_{n \to \infty} \|P_n - Q_n\| \geq \|P - Q\| \geq \kappa.$$

So J is PS-attracting and (G_4) is proved. So all assumptions of Theorem 5 hold, hence any $\mathcal{K}$-decomposition of J must be infinite. In particular, by Remark 6(b), there are infinitely many periodic orbits in $\mathcal{K}$, which proves the claim. ☐

Proof of Theorem 2. It suffices to combine the existence of infinitely many periodic orbits in $\mathcal{K}$ from Theorem 29 with the fact that critical points of J yields critical points of I by Lemma 26. ☐

6. Comments on the Dirichlet problem

Only few adjustments are necessary to treat the Dirichlet problem in the same way as the Neumann problem. Essentially it suffices to modify the function spaces. The linear Dirichlet boundary value problem reads

$$\nabla \times \big(\mu(x)^{-1} \nabla \times E\big) - \lambda \varepsilon(x) E = \varepsilon(x) g \quad \text{in } \Omega, \qquad E \times \nu = 0 \quad \text{on } \partial\Omega. \tag{30}$$

In contrast to the Neumann problem, the right function spaces for E now is

$$\mathcal{V}_0 := \Big\{ E_1 \in \mathcal{H}_0 : \int_\Omega \varepsilon(x) E_1 \cdot \nabla\Phi \, dx = 0 \text{ for all } \Phi \in C_0^1(\Omega) \Big\}$$

where $\mathcal{H}_0$ is defined as the closure of test functions with respect to the inner product $\langle\cdot,\cdot\rangle$ defined in (3). In particular, $\mathcal{V}_0 \subset \mathcal{V}$ is a closed subspace, but it is also a closed subspace of $H_0^1(\Omega;\mathbb{R}^3)$. This follows just as in the Neumann case with the aid of [22, Theorem 1.1]. A weak solution of (30) satisfies

$$\int_\Omega \mu(x)^{-1}(\nabla\times E)\cdot(\nabla\times\Phi)\,dx - \lambda\int_\Omega \varepsilon(x)E\cdot\Phi\,dx = \int{}_\Omega \varepsilon(x)g\cdot\Phi\,dx \text{for all } \Phi\in\mathcal{V}_0.$$

We introduce

$$X_0^{p'} := \Big\{ E \in L^{p'}(\Omega;\mathbb{R}^3) : \int_\Omega \varepsilon(x)E\cdot\nabla\phi\,dx = 0 \text{ for all } \phi\in W_0^{1,p}(\Omega)\Big\},$$
$$Y_0^{p'} := \big\{\nabla u : u \in W_0^{1,p'}(\Omega)\big\}.$$

Notice that $X_0^{p'} \supsetneq X^{p'}$ and $Y_0^{p'} \subsetneq Y^{p'}$ given that elements of $X_0^{p'}$ do not carry any information about the boundary behaviour in contrast to elements of $X^{p'}$. For instance, in the case of a ball $\Omega = \{x\in\mathbb{R}^3 : |x|<1\}$ we have $E(x) := \varepsilon(x)^{-1}(x_2,x_1,0) \in X_0^{p'} \setminus X^{p'}$. Indeed, the vector field $\varepsilon(x)E$ is divergence-free in Ω with $\varepsilon(x)E(x)\cdot\nu(x) = \varepsilon(x)E(x)\cdot x = 2x_1x_2 \not\equiv 0$ on $\partial\Omega$. As before we define

$$(Y_0^{p'})^{\perp_\varepsilon} := \Big\{ f\in L^p(\Omega;\mathbb{R}^3) : \int_\Omega \varepsilon(x) f\cdot g\,dx = 0 \text{ for all } g\in Y_0^{p'}\Big\},$$
$$(X_0^{p'})^{\perp_\varepsilon} := \Big\{ g\in L^p(\Omega;\mathbb{R}^3) : \int_\Omega \varepsilon(x) f\cdot g\,dx = 0 \text{ for all } f\in X_0^{p'}\Big\}.$$

The Helmholtz Decomposition for the Dirichlet problem relies on the following result [2, Theorem 1].

Proposition 30. *[Auscher, Qafsaoui] Assume (A1),(A2) and $1<p<\infty$. Then, for any given $f\in L^{p'}(\Omega;\mathbb{R}^3)$ the boundary value problem*

$$\nabla\cdot\big(\varepsilon(x)(f+\nabla u)\big) = 0 \quad \text{in } \Omega$$

has a unique weak solution in $u\in W_0^{1,p'}(\Omega)$. It satisfies $\|\nabla u\|_{p'} \lesssim \|f\|_{p'}$.

Proposition 31. *Assume (A1),(A2) and $1<p<\infty$. Then we have*

$$L^{p'}(\Omega;\mathbb{R}^3) = X_0^{p'} \oplus Y_0^{p'} \qquad \text{with} \quad (Y_0^p)^{\perp_\varepsilon} = X_0^{p'}, Y_0^p = (X_0^{p'})^{\perp_\varepsilon}.$$

Proof. The proof of $L^{p'}(\Omega;\mathbb{R}^3) = X_0^{p'}\oplus Y_0^{p'}$ is the same as the one of Proposition 11 up to replacing Proposition 10 by Proposition 30. We only show $(X_0^{p'})^{\perp_\varepsilon} \subset Y_0^p$. For $f\in(X_0^{p'})^{\perp_\varepsilon}$ we have $f\in(X^{p'})^{\perp_\varepsilon}$ thanks to $X_0^{p'} \supset X^{p'}$. So Proposition 11 gives $f=\nabla F$ for some $F\in W^{1,p}(\Omega)$. We want to show that we even have $F\in W_0^{1,p}(\Omega)$ after substracting a suitable constant. To see this let $\phi\in C^1(\partial\Omega)$ be arbitrary with zero average and let $u\in W^{1,p'}(\Omega)$ denote the unique weak solution of

$$-\Delta u = 0 \quad \text{in } \Omega, \qquad \nabla u\cdot\nu = \phi \quad \text{on } \partial\Omega.$$

This is possible by [15, Theorem 9.2]. Then u is harmonic with $\varepsilon(x)^{-1}\nabla u\in X_0^{p'}$. Hence,

$$0 = \int_\Omega \varepsilon(x)\nabla F\cdot\varepsilon(x)^{-1}\nabla u\,dx = \int_\Omega \nabla F\cdot\nabla u\,dx = \int_{\partial\Omega} F\phi\,d\sigma.$$

As a consequence,

$$0 = \int_{\partial\Omega} F\phi\,d\sigma \qquad \text{whenever} \quad \int_{\partial\Omega}\phi\,d\sigma = 0.$$

So F is constant on $\partial\Omega$ and after substracting the constant, we find $f = \nabla F$ for some $F \in W_0^{1,p}(\Omega)$, so $f \in Y_0^p$. Here we used that $F \in W_0^{1,p}(\Omega)$ holds if and only if $F \in W^{1,p}(\Omega)$ has zero trace [10, p.315]. This finishes the proof. □

As in the Neumann setting $\mathcal{V}_0$ inherits the embeddings from $H_0^1(\Omega;\mathbb{R}^3)$, which allows to set up Fredholm theory for the linear Dirichlet problem (30) using

$$\mathcal{V}_0 \hookrightarrow X_0^p \quad \text{boundedly for } 1 \le p \le 6 \text{ and compactly for } 1 \le p < 6.$$

In this way one finds that Theorem 15 admits a counterpart for the Dirichlet problem with $\mathcal{V}, \mathcal{W}, X^{p'}, Y^q$ replaced by $\mathcal{V}_0, \mathcal{W}_0, X_0^{p'}, Y_0^q$, respectively. Replacing the function spaces in the discussion of the nonlinear problems the leads to existence results for infinitely many solutions of the nonlinear Dirichlet problem under the same assumptions (A1),(A2),(A3). We close this section by a remark on the interpretation of the boundary condition $E \times \nu = 0$ on $\partial\Omega$.

Remark 32. In the context of (1) the metallic boundary condition $E \times \nu = 0$ on $\partial\Omega$ holds in the sense

$$\int_\Omega \Big((\nabla \times \Phi) \cdot E - \Phi \cdot (\nabla \times E) \Big)\, dx = 0 \qquad \text{for all } \Phi \in C^1(\overline{\Omega};\mathbb{R}^3).$$

In fact, this identity holds for all $E \in C_0^\infty(\Omega;\mathbb{R}^3)$ and hence, by density with respect to $\|\cdot\|$, for all $E \in \mathcal{H}_0$. It encodes the boundary condition given that, under suitable regularity assumptions, the integral equals, by the Divergence Theorem,

$$\int_\Omega \nabla \cdot (\Phi \times E)\, dx = \int_{\partial\Omega} (\Phi \times E) \cdot \nu \, d\sigma = \int_{\partial\Omega} (E \times \nu) \cdot \Phi \, d\sigma$$

for the outer unit normal field $\nu : \partial\Omega \to \mathbb{R}^3$. This is analogous to the classical Dirichlet problem for the Laplacian where the zero trace boundary condition comes with the space $H_0^1(\Omega)$. This motivates the name "Dirichlet problem" for (1). In the context of (2) the boundary condition $(\mu(x)^{-1}\nabla \times E) \times \nu = 0$ on $\partial\Omega$ is encoded in the Euler-Lagrange equation for the functional I over $\mathcal{H}$. So it shows up as a free boundary condition, which is analogous to the Neumann problem for the Laplacian.

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Appendix A. Some technical results

Lemma 33. *Assume (A1),(A2) and $1 \le p < \infty$. If $f \in L^p(\Omega;\mathbb{R}^3)$ satisfies $\nabla \times f = 0$ in the distributional sense, then $f = \nabla u$ for some $u \in W^{1,p}(\Omega)$.*

Proof. By assumption we have

$$\int_\Omega f \cdot (\nabla \times \Phi)\, dx = 0 \qquad \text{for all } \Phi \in C_0^\infty(\Omega;\mathbb{R}^3).$$

Take mollifiers $(\eta_\tau)_{\tau>0}$ that form a smooth approximation of the identity with $\text{supp}(\eta_\tau) \subset B_\tau(0)$ and define

$$f_\tau(x) := (\eta_\tau * f)(x) := \int_\Omega \eta_\tau(x-y) f(y)\, dy.$$

For any given $\Phi \in C_0^\infty(\Omega;\mathbb{R}^3)$ we have $\Phi_\tau \in C_0^\infty(\Omega)$ for all positive $\tau < \text{dist}(\text{supp}(\Phi), \partial\Omega)$. This implies, using integration by parts, $(\nabla \times \Phi)_\tau = \nabla \times \Phi_\tau$ as well as

$$\int_\Omega (\nabla \times f_\tau) \cdot \Phi\, dx = \int_\Omega f_\tau \cdot (\nabla \times \Phi)\, dx = \int_\Omega f \cdot (\nabla \times \Phi)_\tau\, dx = \int_\Omega f \cdot (\nabla \times \Phi_\tau)\, dx \overset{(17)}{=} 0.$$

Given that the test function Φ is arbitrary as long as $\tau < \text{dist}(\text{supp}(\Phi), \partial\Omega)$, we conclude that for any given strictly contained ball $B \subset\subset \Omega$ we have $\nabla \times f_\tau = 0$ in B provided that $0 < \tau < \text{dist}(B, \partial\Omega)$. Since f_τ is smooth and any such ball is simply connected, we have

$$f_\tau|_B = \nabla u_{B,\tau} \quad \text{for a unique } u_{B,\tau} \in C^\infty(B) \text{ with}$$
$$\int_B u_{B,\tau}\, dx = 0 \qquad \text{where } 0 < \tau < \text{dist}(B, \partial\Omega).$$

By Poincaré's Inequality there is a constant $C = C(B)$ depending on B such that

$$\|u_{B,\tau} - u_{B,\delta}\|_{W^{1,p}(B)} \le C\|\nabla u_{B,\tau} - \nabla u_{B,\delta}\|_{L^p(B)} = \|f_\tau - f_\delta\|_p \to 0 \quad \text{as } \tau, \delta \to 0^+.$$

We conclude $u_{B,\tau} \to u_B$ for some $u \in W^{1,p}(B)$ and it is standard to verify

$$f = \nabla u_B \quad \text{on } B.$$

If $(\chi_i)_{i\in\mathbb{N}}$ is a partition of unity subordinate to some open cover $\Omega = \bigcup_{i\in\mathbb{N}} B_i$ with balls $B_i \subset \Omega$, it is straightforward to check

$$f = \nabla u \quad \text{on } \Omega \qquad \text{where } u := \sum_{i\in\mathbb{N}} \chi_i F_{B_i}.$$

Since $f \in L^p(\Omega)$, we have $u \in W^{1,p}(\Omega)$, which is all we had to show. □

Proposition 34. *Assume that $f : \Omega \times \mathbb{R}^3 \to \mathbb{R}^3$ satisfies (A3). Then $\psi(x,\cdot) := f(x,\cdot)^{-1}$ exists for almost all $x \in \Omega$ and satisfies (A3').*

Proof. By assumption (A3), for almost all $x \in \Omega$ the function $z \mapsto f_0(x,z)$ is positive, differentiable and increasing on $(0,\infty)$ with $f_0(x,z) \to 0$ as $z \to 0$ and $f_0(x,z) \to +\infty$ as $z \to \infty$. In particular, $f_0(x,\cdot) : [0,\infty) \to [0,\infty)$ admits a positive, differentiable and increasing inverse $\psi_0(x,\cdot) := f_0(x,\cdot)^{-1}$ for such $x \in \Omega$. Moreover, $z \mapsto z^{-1}\psi_0(x,z)$ is decreasing on $(0,\infty)$ because $s \mapsto s^{-1} f_0(x,s)$ is increasing on $(0,\infty)$. This implies $f(x,\cdot)^{-1} = \psi(x,\cdot)$ where $\psi(x,P) := \psi_0(x,|P|)|P|^{-1}P$ and the claimed properties of ψ_0 except (19).

To prove (19) note that (A3) implies $z := f_0(x,s) \sim s^{p-1}$ and thus $\psi_0(x,z) = s \sim z^{p'-1}$. This implies the second inequality in (19).

Furthermore, by differentiation we find the identity

$$\int_0^{f_0(x,s)} \psi_0(x,t)\, dt + \int_0^s f_0(x,t)\, dt = s f_0(x,s) \quad \text{for all } s \ge 0$$

and conclude with the aid of assumption (A3)

$$\int_0^z \psi_0(x,t)\,dt - \frac{1}{2}\psi_0(x,z)z = \frac{1}{2}f_0(x,s)s - \int_0^s f_0(x,t)\,dt \gtrsim s^p \sim z^{p'}.$$

This provides the first inequality in (19) and the claim is proved. □

References

[1] Ambrosetti, A., Rabinowitz, P.H.: Dual variational methods in critical point theory and applications. J. Funct. Anal. **14**(4), 349–381 (1973)

[2] Auscher, P., Qafsaoui, M.: Observations on $W^{1,p}$ estimates for divergence elliptic equations with VMO coefficients. Bollettino dell'Unione Matematica Italiana **8**5–B(2), 487–509 (2002)

[3] Azzollini, A., Benci, V., D'Aprile, T., Fortunato, D.: Existence of static solutions of the semilinear Maxwell equations. Ric. Mat. **55**(2), 283–297 (2006)

[4] Bartsch, T., Ding, Y.: On a nonlinear Schrödinger equation with periodic potential. Math. Ann. **313**, 15–37 (1999)

[5] Bartsch, T., Dohnal, T., Plum, M., Reichel W.: Ground states of a nonlinear curl-curl problem in cylindrically symmetric media. NoDEA Nonlinear Differential Equations Appl., **23**(5):Art. 52, 34, (2016)

[6] Bartsch, T., Mederski, J.: Ground and Bound State Solutions of Semilinear Time-Harmonic Maxwell Equations in a Bounded Domain. Arch. Ration. Mech. Anal. **215**, 283–306 (2015)

[7] Bartsch, T., Mederski, J.: Nonlinear time-harmonic Maxwell equations in an anisotropic bounded medium. J. Funct. Anal. **272**(10), 4304–4333 (2017)

[8] Bartsch, T., Mederski, J.: Nonlinear time-harmonic Maxwell equations in domains. J. fixed point theory appl. **19**, 959–986 (2017)

[9] Bieganowski, B.: Solutions to a nonlinear Maxwell equation with two competing nonlinearities in $\mathbb{R}^3$. Bulletin of the Polish Academy of Sciences Mathematics, (2020)

[10] Brezis, H.: Functional analysis. Sobolev spaces and partial differential equations. Universitext. Springer, New York (2011)

[11] Chang, K.-C.: Methods in nonlinear analysis. Springer Monographs in Mathematics. Springer-Verlag, Berlin (2005)

[12] D'Aprile, T., Siciliano, G.: Magnetostatic solutions for a semilinear perturbation of the Maxwell equations. Adv. Differential Equations **16**(5–6), 435–466 (2011)

[13] Evequoz, G.: On the periodic and asymptotically periodic nonlinear Helmholtz equation. Nonlinear Anal. Theory Methods Appl. **152**, 88–101 (2017)

[14] Evequoz, G., Weth, T.: Dual variational methods and nonvanishing for the nonlinear Helmholtz equation. Adv. Math. **280**, 690–728 (2015)

[15] Fabes, E., Mendez, O., Mitrea, M.: Boundary Layers on Sobolev-Besov Spaces and Poisson's Equation for the Laplacian in Lipschitz Domains. J. Funct. Anal. **159**(2), 323–368 (1998)

[16] Fujiwara, D., Morimoto, H.: An L^r-theorem of the Helmholtz decomposition of vector fields. Journal of the Faculty of Science, the University of Tokyo. Sect. 1 A, Mathematics **24**, 685–700 (1977)

[17] Geng, J., Shen, Z.: The Neumann problem and Helmholtz decomposition in convex domains. J. Funct. Anal. **259**(8), 2147–2164 (2010)

[18] Hirsch, A., Reichel, W.: Existence of cylindrically symmetric ground states to a nonlinear curl-curl equation with non-constant coefficients. Z. Anal. Anwend. **36**(4), 419–435 (2017)

[19] Mandel, R.: Uncountably many solutions for nonlinear Helmholtz and curl-curl equations. Adv. Nonlinear Stud. **19**(3), 569–593 (2019)

[20] Mandel, R.: Ground states for Maxwell's equations in nonlocal nonlinear media. Partial Differ. Equ. Appl., **3**, (2022)

[21] Mederski, J.: Ground states of time-harmonic semilinear Maxwell equations in $\mathbb{R}^3$ with vanishing permittivity. Arch. Ration. Mech. Anal. **218**(2), 825–861 (2015)

[22] Prokhorov, A., Filonov, N.: Regularity of electromagnetic fields in convex domains. J. Math. Sci. **210**, 793–813 (2015)

[23] Simader, C.G., Sohr, H.: A new approach to the Helmholtz Decomposition and the Neumann problem in L^q-spaces for Bounded and Exterior Domains, pages 1–35. (1992)

[24] Squassina, M., Szulkin, A.: Multiple solutions to logarithmic Schrödinger equations with periodic potential. Calc. Var. Partial Differential Equations, pages 585– 597, (2015)

[25] Struwe, M.: Variational methods, volume 34 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics. Springer-Verlag, Berlin, fourth edition, 2008. Applications to nonlinear partial differential equations and Hamiltonian systems

[26] Szulkin, A., Weth, T.: The method of Nehari manifold. In Handbook of nonconvex analysis and applications, pages 597–632. Int. Press, Somerville, MA, (2010)

[27] Willem, M.: Minimax theorems. Progress in Nonlinear Differential Equations and their Applications, vol. 24. Birkhäuser Boston Inc, Boston, MA (1996)

[28] Yang, M., Ye, W., Zhang, S.: On a critical time-harmonic Maxwell equation in nonlocal media. Proceedings of the Royal Society of Edinburgh: Section A Mathematics, pages 1–45, (2024)

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