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Absence of the confinement–induced Efimov effect: a direct proof in a specific geometry

Marvin R. Schulz[1] · Sylvain Zalczer[1]

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Abstract
We consider a system of three particles with identical mass interacting via short-range potentials, such that two of the particles are on parallel lines in a plane and the third one is on a line perpendicular to this plane. In this geometry, we prove that the corresponding Schrödinger operator only has a finite number of eigenvalues under physically reasonable assumptions on the decay of the interaction potentials. Our result disproves a recent prediction made in physics literature.

Keywords Resonances · Virtual levels · Efimov effect · Schrödinger operators

Mathematics Subject Classification 81Q10 · 81V45

1 Introduction

The Efimov effect can be described as follows: The three-body Schrödinger operator of a system of three three-dimensional particles that interact via short-range potentials has an infinite number of negative eigenvalues, if the Hamiltonians of the two-body subsystems have no negative eigenvalues and at least two of them have a zero-energy resonance. Moreover, the eigenvalues form a geometric sequence whose common ratio is independent from the nature of the potentials. Such a curious phenomenon was first predicted by the physicist Vitaly Efimov in 1970 [4]. In 1974, Yafaev gave the first rigorous mathematical proof of it in [31]. The Efimov effect was considered by physicists as a purely theoretical curiosity, until it was observed experimentally in the early 2000s in an ultracold gas of caesium atoms [12]. Efimov effect has since then be studied both by the physics and mathematics community, see for example the review [14], the PhD thesis [2] or the lecture notes [3] for further references.

One particularly interesting question was whether a similar effect could occur in configurations different from the classical situation of three particles in dimension

✉ Marvin R. Schulz
marvin.schulz@kit.edu

1 Karlsruhe Institute of Technology, Karlsruhe, Germany

three. It was proven in [28] that the Efimov effect does not exists for a system of three one- or two-dimensional bosons. Advances in experiments with ultra-cold Fermi-Fermi mixtures such as in [22] make it possible to study situations where different species of particles are confined to distinct subspaces of $\mathbb{R}^3$, see for example [13]. Nishida and Tan discussed the possible existence of a so-called *confinement-induced Efimov effect* in [15] and [16]. In [14, p. 44, Table 1] the existence of the confinement-induced Efimov effect was predicted in various situations.

We consider the case where two particles can move along two parallel lines in a plane and the third particle moves in a line perpendicular to this plane. In [14, p. 44, Table 1], the existence of the confinement-induced Efimov effect was predicted for this configuration.

We briefly recall the picture suggested in the physics literature that may underlie the prediction. In [16], Nishida and Tan argued that scale-invariant behavior, characteristic of the Efimov effect, can emerge in systems with interactions in three-dimensional space—regardless of the specific interpretation of these spatial dimensions. When considering two-dimensional interactions, they suggested that the Efimov effect may still arise in systems composed of anyons. In [14, Sections 8 and 9], several geometrical configurations with mixed dimensions that support this phenomenon were identified. One such example, depicted in Fig. 1, corresponds to a setup listed in [14, p. 44, Table 1], where the relative distances are vectors in $\mathbb{R}^3$.

Although the subsystems of interacting particles are governed by forces in $\mathbb{R}^3$, the effective dimension of two of the underlying two-particle subsystems is two-dimensional and one is one-dimensional. Since the lines in the depicted configuration do not intersect, the particle statistics are irrelevant—whether the particles are bosons, fermions, or anyons. According to [14, Subsection 3.3.4] and [17], the biwire structure of the configuration is conducive to the emergence of the Efimov effect in ultra-cold gases.

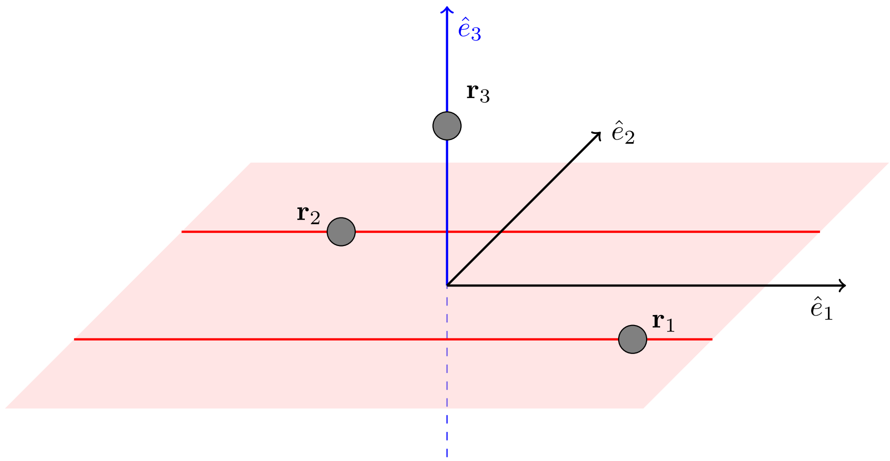

Fig. 1 Geometrically constrained particles moving on separated lines in $\mathbb{R}^3$

At large interparticle distances, the presence of so-called virtual levels in the two–particle subsystems may indeed mediate an effective long-range attraction in specific directions, due to a "conspiracy of potential wells" as described in [11] and [20, Chapter 1.5.2] which for three unconstrained paticles in dimension $d = 3$ leads to the Efimov effect, see [23]. However, given the geometric constraints and the effective dimensionality, this mechanism might not be strong enough to generate an infinite discrete spectrum of the corresponding Schrödinger operator. Arguing solely on the basis of scaling relations as done in [16, 17] seems to miss this point. In fact, we rigorously rule out the existence of any Efimov type effect for this configuration of particles.

Our analysis is based on [33], where Zhislin formulated a useful condition on the finiteness of the discrete spectrum of many particle Schrödinger operators. His approach was further developed and applied to show the absence of Efimov effect in various situations for example in [25, 26, 29] and [30]. In [1] Barth, Bitter and Vugalter showed the absence of Efimov effect in various systems of one- and two-dimensional particles. Since the subsystems in the particle configuration shown in Fig. 1 above are either one- or two-dimensional, we can apply techniques similar to those used in [1] to establish the finiteness of the discrete spectrum.

Our paper is organized as follows. In Sect. 2, we describe the model and state our theorem. In Sect. 3, we give the proof of it in the case where the two-particle operators have no bound states. In Appendix A, we recall some of the results of previous works that we use and in Appendix B, we study the case where at least one of the two-particle systems has a bound state, which is mainly a direct adaptation of [33].

Remark 1.1 It is worth noting that the simple dimensionality-based argument used in [14] does not hold in all cases. For instance, it predicts the existence of the Efimov effect in a system of five purely one-dimensional particles. In this case, the four-particle subsystems—when reduced to center-of-mass coordinates—are described by three spatial parameters. However, as shown in [1], the Efimov effect is known to be absent in purely one-dimensional systems, regardless of the number of particles.

2 Configuration of particles

Inspired by recent predictions from physics for the *confinement-induced Efimov effect* (see, [14, p. 43 f.]) we consider a system of three particles with identical masses interacting via short-range two-body potentials in $\mathbb{R}^3$. Two particles, called 2 and 3, are confined to parallel lines and the third particle called 1 is confined to a line perpendicular to the plane spanned by the two lines on which particles 2 and 3 are moving. Furthermore, the line on which particle 1 moves is assumed to not intersect with the two lines for particles 2 and 3. Compare this to Fig. 1.

For simplicity we assume that the parallel lines have distance 1. The configuration of such a system is determined by three real numbers $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. The particles have positions

$$\mathbf{r}_1 = \begin{pmatrix} x_1 \\ -1/2 \\ 0 \end{pmatrix}, \ \mathbf{r}_2 = \begin{pmatrix} x_2 \\ 1/2 \\ 0 \end{pmatrix}, \ \mathbf{r}_3 = \begin{pmatrix} 0 \\ 0 \\ x_3 \end{pmatrix}. \tag{2.1}$$

The particles interact pairwise. For $i, j \in \{1, 2, 3\}$ with $i < j$, we denote by V_{ij} the potential describing the interaction between particles at $\mathbf{r}_i$ and $\mathbf{r}_j$.

From physics, we know that it is reasonable to assume that the potential V_{ij} solely depends on the distance between the particles i and j. For three unconfined particles in dimension $d = 3$, the Efimov effect can occur only if at least two subsystems possess a so-called virtual level, see [10, 30]. This is a strong restriction on the admissible potentials. If, in addition, all interaction potentials are short-range, the effect may appear [31]. For purely attractive, long-range potentials, the subsystems cannot have virtual levels, and consequently, the effect is absent. We also exclude certain artificial long-range potentials that are attractive at short distances, repulsive at large distances, and in addition produce virtual levels, see [7, Appendix A] for such an example. We therefore assume that all potentials are short-range, *i.e.,* that there exist $R_0 > 0$ and $C, \nu > 0$ such that for all $i, j \in \{1, 2, 3\}$ with $i < j$,

$$\left|V_{ij}(r)\right| \leq \frac{C}{r^{2+\nu}}, \text{ when } r \geq R_0. \tag{2.2}$$

For one–particle Schrödinger operators the Hardy Potential is the borderline case which distinguishes between finitely, respectively, infinitely many negative eigenvalues, see [19, Theorem XIII.6]. Consequently, the assumption in (2.2) is the reasonable assumption for short–range potentials.

Note that $\left|\mathbf{r}_i - \mathbf{r}_j\right| \geq 1/2$ for $i, j \in \{1, 2, 3\}$ with $i < j$. That is, all three particles have a minimal distance from each other. Thus, it is (physically) reasonable to assume that all potentials are bounded.

Remark 2.1 We consider the case where all particles have the same mass and the two parallel lines are equidistant from a third, perpendicular line, with this distance fixed to one. The in-plane distance between the parallel lines does not affect our proof, but our approach does not cover the absence of the Efimov effect when particle masses are arbitrary or when some line is tilted. In such cases, Lemma (A.2) is generally not applicable and the loss of symmetries significantly complicates the analysis.
The identical-particle case in this simple geometry is a natural starting point, and we prove here the finiteness of the discrete spectrum by combining results for purely one-dimensional and purely two-dimensional systems (see [1]). After separating the center of mass in the subsystems, the problem reduces to a three-particle system consisting of two two-dimensional and one one-dimensional subsystem.
A new result for more general configurations has since been obtained, see [8]. However, the result in [8] requires considerably stronger decay assumptions on the interaction potentials than the short-range condition (2.2) used here. Consequently, it does not cover the case treated in this work, which applies under the natural short–range condition. Moreover, [8] requires an extensive analysis of decay properties of zero-energy solutions of critical Schrödinger operators that goes far beyond the purpose of this research.

The corresponding operator for the system described above is

$$H = -\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + \sum_{1\le i<j\le 3} V_{ij}(\left|\mathbf{r}_i - \mathbf{r}_j\right|). \tag{2.3}$$

Here, $\left|\mathbf{r}_i - \mathbf{r}_j\right|$ is to be understood as a function of the variables x_i, x_j using the definition in (2.1). The operator H acts on $L^2(\mathbb{R}^3)$. Under these assumptions H is self-adjoint on the Sobolev space $H^2(\mathbb{R}^3)$ and its form domain is $H^1(\mathbb{R}^3)$. Such a system of three particles can be decomposed into three subsystems of two particles. For the pair of particles $(3, j)$ with $j \in \{1, 2\}$ we consider the operator

$$h_{j3} := -\frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_j^2} + V_{j3}(\left|\mathbf{r}_1 - \mathbf{r}_j\right|) \tag{2.4}$$

which acts on $L^2(\mathbb{R}^2)$. While the full three-particle system is not invariant under any translation in $\mathbb{R}^3$ due to the geometric constraints, the subsystem consisting of particles 1 and 2 is invariant under translations in the $\hat{e}_1$-direction. Thus for the subsystem of particles 1 and 2 we need to study it in the center of mass frame by introducing relative coordinates in $\mathbb{R}^2$. These are given by $q, \xi \in \mathbb{R}$ with

$$q := \frac{1}{\sqrt{2}}(x_2 - x_1), \quad \xi := \frac{1}{\sqrt{2}}(x_1 + x_2). \tag{2.5}$$

Up to a factor of $\sqrt{2}$ the coordinate ξ describes the position of the center of mass of the two particles 2 and 3, while the coordinate q describes the relative motion of these two particles in the e_1-direction.
For any $x \in \mathbb{R}^3$ we denote by $|x|$ the Euclidean norm of the vector. Note that $|\mathbf{r}_1 - \mathbf{r}_2| = \sqrt{2q^2 + 1}$. The operator

$$\left[-\partial_q^2 + V_{23}\left(\sqrt{2q^2+1}\right)\right] \otimes \mathbb{1} + \mathbb{1} \otimes (-\partial_\xi^2) \tag{2.6}$$

corresponds to the pair $(1, 2)$. In the center of mass frame, we have

$$h_{12} := -\partial_q^2 + V_{12}\left(\sqrt{2q^2+1}\right). \tag{2.7}$$

which acts on $L^2(\mathbb{R})$. Let

$$\Sigma := \min_{i<j\in\{1,2,3\}} \inf \sigma(h_{ij}). \tag{2.8}$$

Similar to the HVZ-Theorem [9, 24, 32] (see also [18][Thm. XIII.17]), one sees that $\Sigma \le 0$. It follows from the same theorem that $\sigma_{ess}(H) = [\Sigma, \infty)$. The goal of this paper is to study the discrete spectrum of H. Our main result is

Theorem 2.2 *The operator H has at most a finite number of discrete eigenvalues below Σ.*

Remark 2.3 The theorem above allows any of the two-particle operators to have a virtual level. In [14] it was predicted that the system described by the operator H in Equation (2.3) shows a confinement-induced Efimov effect. Our Theorem 2.2 disproves this claim, which was based on heuristic arguments from physics.

3 Proof of Theorem 2.2

According to the min-max principle, the spectrum of H below Σ is finite if there exists a finite dimensional space $M \subset L^2(\mathbb{R}^3)$ such that, for any $\psi \in L^2(\mathbb{R}^3)$ orthogonal to M,

$$\langle \psi, H\psi \rangle \geq \Sigma \, \|\psi\|^2 . \tag{3.1}$$

Due to [33] (see also Appendix A.1), such a space M exists whenever there exist $b, \beta > 0$ such that for any $\psi \in H^1(\mathbb{R}^3)$ with $\operatorname{supp} \psi \subset \{x \in \mathbb{R}^3 | \, |x| > b\}$

$$L[\psi] := \int_{\mathbb{R}^3} \left(\sum_{i=1}^{3} \left|\partial_{x_i} \psi\right|^2 + \sum_{i<j} V_{ij} \, |\psi|^2 \right) dx - \int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|^\beta} dx \geq \Sigma \, \|\psi\|^2 . \tag{3.2}$$

In the following, we set $\beta = 2 + \nu$, where $\nu > 0$ is the exponent corresponding to the short-range property of the potentials as stated in Equation (2.2). We first prove the theorem in the case $\Sigma = 0$. With a small modification, the case $\Sigma < 0$ is analogous to the one considered in [33]. For the convenience of the reader we give the proof for $\Sigma < 0$ in Appendix B.

Heuristics from physics predicts that the three-particle system breaks up if (at least) one particle is far away from the others. So following ideas of [28], we want to define, for all $i, j \in \{1, 2, 3\}$ with $i < j$, the set of geometric configurations where the particles i and j are close to each other and the third particle is far away. Observe that

$$|\mathbf{r}_1 - \mathbf{r}_2|^2 = 2q^2 + 1, \quad \left|\mathbf{r}_3 - \mathbf{r}_j\right|^2 = x_1^2 + x_3^2 + 1/4, \quad j \in \{1, 2\}. \tag{3.3}$$

Given parameters $b > 0$ and $\gamma \in (0, 1)$ which will be fixed later we define the regions

$$\begin{aligned} K_{12}^b(\gamma) &:= \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : |q| \leq \gamma \sqrt{\xi^2 + x_3^2}, |x| > b \text{ for } q, \xi \text{ defined in (2.5)} \right\}, \\ K_{j3}^b(\gamma) &:= \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \left|(x_j, x_3)\right| \leq \gamma \, |x_k| \, , \text{for } k \in \{1, 2\}, k \neq j, \ |x| > b \right\}. \end{aligned} \tag{3.4}$$

Here, $|(x_j, x_3)|$ is the Euclidean length of the vector (x_j, x_3) in the $(j$-3)-plane. Note that outside of the ball $B_b(0)$ the above sets describe conical regions in $\mathbb{R}^3$. For example, outside of $B_b(0)$ the region $K_{13}^b(\gamma)$ is a conical region around the x_2-direction, where

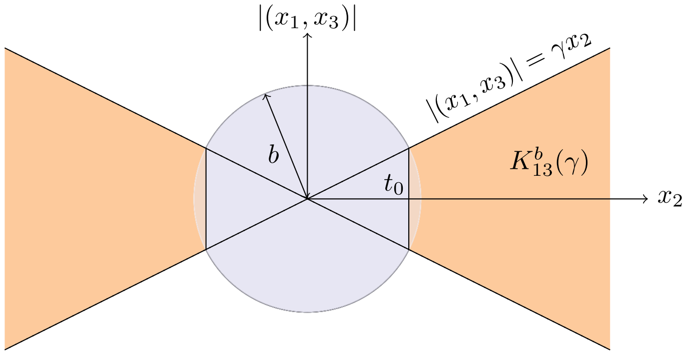

Fig. 2 A sketch of the set $K^b_{13}(\gamma)$ defined in Equation (3.4). The other sets look similar

the particles 1 and 3 are close to each other and particle 2 is far away. Compare this to Fig. 2.

We also define the set Ω_0 of configurations where all three particles are far apart:

$$\Omega_0 := \{x \in \mathbb{R}^3 \,|\, |x| > b\} \setminus \left(\bigcup_{1 \le i < j \le 3} K^b_{ij}(\gamma) \right). \tag{3.5}$$

For a measurable set $\Omega \subset \mathbb{R}^3$, we define the local energies

$$L[\psi, \Omega] := \int_\Omega \left(\sum_{i=1}^3 \left| \partial_{x_i} \psi \right|^2 + \sum_{i<j} V_{ij} \, |\psi|^2 \right) dx - \int_\Omega \frac{|\psi(x)|^2}{|x|^{2+\nu}} dx. \tag{3.6}$$

Then, for any $\psi \in H^1(\mathbb{R}^3)$

$$L[\psi] = \sum_{i<j} L[\psi, K_{ij}] + L[\psi, \Omega_0]. \tag{3.7}$$

We prove the bound (3.2), by estimating $L[\psi, K^b_{ij}(\gamma)]$ and $L[\psi, \Omega_0]$ from below. Notice that for γ small enough the sets $K^b_{ij}(\gamma)$ and Ω_0 are disjoint. In the following we shall assume that $\gamma < 1/4$ is small enough.

Remark 3.1 The prove of of Equation (3.2) for $\Sigma = 0$ proceeds in several steps. We first prove that the local energies in (3.6) can be estimated in terms of integrals over the surface $\partial K^b_{ij}(\gamma)$ only. This is done in Lemma 3.2 and 3.4 below. We then show in Lemma 3.5 how this surface Integrals can be controlled by a small portion of the kinetic on the set Ω_0. At the end of this section, we prove how these Lemmas conclude the main statement.

We start with $L[\psi, K^b_{j3}(\gamma)]$ for $j \in \{1, 2\}$.

Lemma 3.2 *For $j \in \{1, 2\}$, there exist $C, b_0 > 0$ such that for all $b \geq b_0$ and for any $\psi \in H^1(\mathbb{R}^3)$ with $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^3 | \, |x| > b\}$*

$$L[\psi, K^b_{j3}(\gamma)] \geq -C \int_{\partial K^b_{j3}(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma. \tag{3.8}$$

where $\nu > 0$ is defined in equation (2.2).

Proof We prove the estimate for $j = 1$. The proof for $j = 2$ is similar. For any $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ let

$$\zeta := (x_1, x_3) \in \mathbb{R}^2. \tag{3.9}$$

Then by definition of K^b_{13} in (3.4) and (3.3)

$$K^b_{13}(\gamma) = \{(\zeta, x_2) \in \mathbb{R}^3 : |\zeta| \leq \gamma\, |x_2|\, , |(\zeta, x_2)| \geq b\}\,. \tag{3.10}$$

Given $b > 0$ and $\psi \in H^1(\mathbb{R}^3)$ with $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^3 | \, |x| > b\}$, we decompose

$$L[\psi, K^b_{13}(\gamma)] = L_3[\psi, K^b_{13}(\gamma)] + L_4[\psi, K^b_{13}(\gamma)] \tag{3.11}$$

where

$$\begin{aligned} L_3[\psi, K^b_{13}(\gamma)] &:= \int_{K^b_{13}(\gamma)} \left(\left|\nabla_\zeta \psi\right|^2 + V_{13}\left(\sqrt{|\zeta|^2 + \frac{1}{4}}\right) |\psi|^2 \right) d(\zeta, x_2), \\ L_4[\psi, K^b_{13}(\gamma)] &:= \int_{K^b_{13}(\gamma)} \left(\left|\partial_{x_2}\psi\right|^2 + (V_{23} + V_{12})\, |\psi|^2 \right) d(\zeta, x_2) - \int_{K^b_{13}(\gamma)} \frac{|\psi|^2}{|x|^{2+\nu}}\, dx. \end{aligned} \tag{3.12}$$

Let $t_0 := b/\sqrt{1+\gamma^2}$, as it appears in Fig. 2. Then, for all $(x_2, \zeta) \in K^b_{13}(\gamma)$, $|x_2| \geq t_0$ and, since ψ vanishes whenever $|\zeta|^2 + x_2^2 \leq b^2$,

$$L_3[\psi, K^b_{13}(\gamma)] = \int_{|x_2| \geq t_0} \iint_{|\zeta| \leq \gamma |x_2|} \left(\left|\nabla_\zeta \psi\right|^2 + V_{13}\left(\sqrt{|\zeta|^2 + \frac{1}{4}}\right) |\psi|^2 \right) d\zeta\, dx_2\,. \tag{3.13}$$

We remark that V_{13} solely depends on $|\zeta|$ and is short-range and bounded. By assumption $-\Delta + V_{13} = h_{13} \geq 0$. Thus, by Lemma A.2 which is a restatement of [2, Lemma 6.6], there exists some $C_0 > 0$ such that, when b is large enough so that $\gamma t_0 \geq R_0$,

$$L_3[\psi, K^b_{13}(\gamma)] \geq -C_0 \int_{|x_2| \geq t_0} \frac{\int_0^{2\pi} |\psi(x_2, \gamma\, |x_2|\, , \theta)|^2\, d\theta}{(\gamma\, |x_2|)^\nu}\, dx_2\,. \tag{3.14}$$

Remark 3.3 The assumptions of equal masses and the chosen geometry are essential at this point. Without them, the potential V_{13} and the kinetic energy are not simultaneously invariant under rotations, which prevents the application of Lemma A.2. The proof of Lemma A.2 relies heavily on this symmetry. Consequently, one cannot directly adapt that proof to obtain a result for potentials with different symmetries or configurations with arbitrary mass.

By the usual abuse of notation, we use the same symbol ψ for the function in all coordinate systems. To avoid any confusion, we will write down its argument when necessary. We remark that by passing to polar coordinates in the (x_1, x_3) plane introducing $|\zeta| \in \mathbb{R}_+$,$\theta \in [0, 2\pi)$ and fixing $|\zeta| = \gamma\, |x_2|$ the mapping

$$(-\infty, -t_0] \cup [t_0, \infty) \times [0, 2\pi) \to \mathbb{R} \times \mathbb{R}_+ \times [0, 2\pi) \quad \text{with} \quad (x_2, \theta) \mapsto (x_2, \gamma\, |x_2|\, , \theta) \tag{3.15}$$

is a parametrization of the two components of the surface $\partial K^b_{13}(\gamma)$ outside of $B_b(0)$. Since in addition ψ vanishes on $B_b(0)$ and $|x| = \sqrt{1+\gamma^2}\, |x_2|$ on that surface, we can rewrite Equation (3.14) in

$$L_3[\psi, K^b_{13}(\gamma)] \geq -C_1 \int_{\partial K^b_{13}(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma, \tag{3.16}$$

for some constant $C_1 > 0$. Here $d\sigma = |x|\; d\,|x|\, d\theta$ is the surface measure on the set $\partial K^b_{13}(\gamma)$, which explains the additional factor of $|x|$ in the denominator of equation (3.16).
Let us now bound $L_4[\psi, K^b_{13}(\gamma)]$. Note that for any $x \in K^b_{13}(\gamma)$

$$\begin{aligned} |\mathbf{r}_1 - \mathbf{r}_2| &\geq |(x_1, x_2)| \geq |x_2| \geq t_0, \\ |\mathbf{r}_2 - \mathbf{r}_3| &\geq |x_2 - x_3| \geq (1-\gamma)\, |x_2| \geq (1-\gamma) t_0\, . \end{aligned} \tag{3.17}$$

Since $t_0 = b/\sqrt{1+\gamma^2}$, we can use the short-range property of the potentials V_{23}, V_{12} together with (3.17) to find that there exists $C_2 > 0$ such that for $b \geq R_0\sqrt{1+\gamma^2}$, we have for any $x \in K^b_{13}(\gamma)$

$$|(V_{23} + V_{12})(x)| \leq \frac{C_2}{|x_2|^{2+\nu}}. \tag{3.18}$$

Thus

$$L_4[\psi, K^b_{13}(\gamma)] \geq \iint_{\mathbb{R}^2} \int_{|x_2| \geq \max\{t_0, |\zeta|/\gamma\}} \left(\left|\partial_{x_2}\psi\right|^2 - \frac{C_2+1}{|x_2|^{2+\nu}}\, |\psi|^2 \right) dx_2 d\zeta. \tag{3.19}$$

For any fixed $\zeta \in \mathbb{R}^2$, we define $a(\zeta) := \max\{t_0, |\zeta|\, /\gamma\}$. Note that by direct computations $a(\zeta) := \max\{t_0, |\zeta|\, /\gamma\} \geq R_0$ for b large enough. We apply [1, Lemma 6.3] (see Appendix A.5) to the innermost integral. Thus, there exists $C_3 > 0$ such that

$$L_4[\psi, K^b_{13}(\gamma)] \geq -C_3 \iint_{\mathbb{R}^2} \frac{1}{a(\zeta)^{1+\nu}} \Big[|\psi\,(a(\zeta), \zeta)\,|^2 + |\psi\,(-a(\zeta), \zeta)\,|^2 \Big]\, d\zeta\,. \tag{3.20}$$

By construction $\psi(\pm a(\zeta), \zeta)$ vanishes whenever $|\zeta| \leq \gamma t_0$. Then

$$L_4[\psi, K^b_{13}(\gamma)] \geq -C_3\gamma^{1+\nu} \iint_{|\zeta|\geq\gamma t_0} \frac{1}{|\zeta|^{1+\nu}} \Big[|\psi\,(|\zeta|/\gamma, \zeta)\,|^2 + |\psi\,(-|\zeta|/\gamma, \zeta)\,|^2 \Big]\, d\zeta\,. \tag{3.21}$$

Note once more that $\zeta \mapsto (\zeta_1, \pm|\zeta|/\gamma, \zeta_2)$ for $\zeta \in \mathbb{R}^2$ with $|\zeta| \geq \gamma t_0$ is a parametrization of the two components of the surface $\partial K^b_{13}(\gamma)$ outside of $B_b(0)$. Similarly to what we did in order to get from Equation (3.14) to Equation (3.16), we can rewrite Equation (3.21) in

$$L_4[\psi, K^b_{13}(\gamma)] \geq -C_4 \int_{\partial K^b_{13}(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma, \tag{3.22}$$

for some $C_4 > 0$. Inserting the bounds (3.16) and (3.22) into (3.11), we find

$$L[\psi, K^b_{13}(\gamma)] \geq -(C_4 + C_1) \int_{\partial K^b_{13}(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}}\, d\sigma\,. \tag{3.23}$$

This proves the Lemma 3.2. □

Next we provide a lower bound for $L[\psi, K^b_{12}(\gamma)]$. The techniques used are similar to the ones in the proof of Lemma 3.2 but the proof is slightly different due to the different geometry of $K^b_{12}(\gamma)$. We show

Lemma 3.4 *There exist $C, b_0 > 0$ such that, for all $b \geq b_0$ and for any $\psi \in H^1(\mathbb{R}^3)$ with* $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^3 \,|\, |x| > b\}$

$$L[\psi, K^b_{12}(\gamma)] \geq -C \int_{\partial K^b_{12}(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma. \tag{3.24}$$

Proof Recall the definition of the relative coordinates (q, ξ) in Equation (2.5). The set $K^b_{12}(\gamma)$ is invariant under rotations in the (ξ, x_3) plane. Let $\eta := (\xi, x_3) \in \mathbb{R}^2$. Then the set $K^b_{12}(\gamma)$ is

$$K^b_{12}(\gamma) = \left\{ (\eta, q) \in \mathbb{R}^2 \times \mathbb{R} : |q| \leq \gamma\,|\eta|\,, \sqrt{|\eta|^2 + q^2} \geq b \right\}. \tag{3.25}$$

Similar to Lemma 3.2, for some $b > 0$, let $\psi \in H^1(\mathbb{R}^3)$ with $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^3 \,|\, |x| > b\}$ and define

$$L_1[\psi, K_{12}^b(\gamma)] := \int_{K_{12}^b(\gamma)} \left(\left|\partial_q \psi\right|^2 + V_{12}\left(\sqrt{2q^2+1}\right) |\psi|^2 \right) d(q,\eta),$$
$$L_2[\psi, K_{12}^b(\gamma)] := \int_{K_{12}^b(\gamma)} \left(\left|\nabla_\eta \psi\right|^2 + \sum_{j=1}^{2} V_{j3}\,|\psi|^2 \right) d(q,\eta) - \int_{K_{12}^b(\gamma)} \frac{|\psi|^2}{|x|^{2+\nu}}\, dx. \tag{3.26}$$

We decompose

$$L[\psi, K_{12}^b(\gamma)] = L_1[\psi, K_{12}^b(\gamma)] + L_2[\psi, K_{12}^b(\gamma)]. \tag{3.27}$$

Recall that $t_0 = b/\sqrt{1+\gamma^2}$, then

$$L_1[\psi, K_{12}^b(\gamma)] = \iint_{|\eta|\ge t_0} \int_{-\gamma|\eta|}^{\gamma|\eta|} \left|\partial_q \psi\right|^2 + V_{12}\,|\psi|^2 \, dq\, d\eta \tag{3.28}$$

since ψ vanishes whenever $|\eta|^2 + q^2 \le b^2$. Recall that $h_{12} \ge 0$ and that V_{12} is short-range. Then we can apply [1, Lemma 6.2] (see Appendix A.4) to conclude that there exists $D_0 > 0$ such that for all b with $\gamma t_0 = \gamma b/\sqrt{1+\gamma^2} \ge R_0$

$$L_1[\psi, K_{12}^b(\gamma)] \ge -D_0 \iint_{|\eta|\ge t_0} \frac{|\psi(\gamma\,|\eta|\,,\eta)|^2 + |\psi(-\gamma\,|\eta|\,,\eta)|^2}{(\gamma\,|\eta|)^{1+\nu}} d\eta\,. \tag{3.29}$$

Similarly as in the proof of Lemma 3.2, we can rewrite this in

$$L_1[\psi, K_{12}^b(\gamma)] \ge -D_1 \int_{\partial K_{12}^b(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma \tag{3.30}$$

for some $D_1 > 0$.

Let us now bound $L_2[\psi, K_{12}^b(\gamma)]$. On the set $K_{12}^b(\gamma)$

$$\left|\mathbf{r}_3 - \mathbf{r}_j\right| \ge \sqrt{1-4\gamma}\,|\eta| \ge \sqrt{1-4\gamma}\,t_0 \tag{3.31}$$

for $j \in \{1,2\}$ by the construction of the set $K_{12}^b(\gamma)$. Recall that $\gamma < 1/4$ by assumption. Thus, we can use that the potentials V_{13}, V_{23} are short-range. Then, there exists $D_2 > 0$ such that, for b large enough with $t_0 \ge R_0$,

$$L_2[\psi, K_{12}^b(\gamma)] \ge \int_{\mathbb{R}} \iint_{|\eta|\ge \max\{|q|/\gamma, t_0\}} \left|\nabla_\eta \psi\right|^2 - \frac{D_2}{|\eta|^{2+\nu}}\,|\psi|^2\, d\eta dq\,. \tag{3.32}$$

Going into spherical coordinates and applying [1, Lemma 6.7] (see Appendix A.3) to the innermost integral above one can bound the above integral by a surface integral: there exists $D_3 > 0$ such that

$$L_2[\psi, K_{12}^b(\gamma)] \geq -D_3 \int_{|q|\geq \gamma t_0} \frac{\int_0^{2\pi} |\psi(q, |q|/\gamma, \theta)|^2 \, d\theta}{|q|^\nu} dq \,. \tag{3.33}$$

Using that ψ vanishes inside the ball $B_b(0)$ we conclude

$$\begin{aligned} L_2[\psi, K_{12}^b(\gamma)] &\geq -D_3 \int_{|q|\geq \gamma t_0} \frac{\int_0^{2\pi} |\psi(q, |q|/\gamma, \theta)|^2 \, d\theta}{|q|^\nu} dq \\ &\geq -D_4 \int_{\partial K_{12}^b(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma \end{aligned} \tag{3.34}$$

for some $D_4 > 0$, where in the last bound we used that $|q| = \sqrt{1+\gamma^{-2}}|x|$ on $\partial K_{12}^b(\gamma)$. Recall that $d\sigma$ is the surface measure on $\partial K_{12}^b(\gamma)$ similar to Equation (3.16).

For the first to second line of (3.34) we used the fact that the integral is taken over the part of the surface of $\partial K_{12}^b(\gamma)$ where ψ does not vanish. Compare this to the previous case in the proof of Lemma 3.2. Inserting the inequalities in Equation (3.30), (3.34) into Equation (3.27) concludes the proof of the lemma. □

Combining Equation (3.7) and Lemmas 3.2 and 3.4 shows there exists $\hat{C} > 0$ and $b_0 > 0$ such that for $b \geq b_0$

$$L[\psi] \geq L[\psi, \Omega_0] - \hat{C} \sum_{1\leq i<j\leq 3} \int_{\partial K_{ij}^b(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma \,. \tag{3.35}$$

In the next step, we prove that we can compensate the integrals over the surface of the sets $K_{ij}^b(\gamma)$ in the Equation (3.35) by a small portion of the kinetic energy on Ω_0. We do so with the help of the trace theorem and Hardy's inequality on the half line.

Lemma 3.5 *For $1 \leq i < j \leq 3$ and $\gamma' \in (\gamma, 1)$, we define $\Omega_{ij}^b(\gamma, \gamma') := K_{ij}^b(\gamma') \backslash K_{ij}^b(\gamma)$. For all $\varepsilon > 0$, there exists $b_0 > 0$ such that, for all $b \geq b_0$ and for any $\psi \in H^1(\mathbb{R}^3)$ with $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^3 | \, |x| > b\}$*

$$\int_{\partial K_{ij}^b(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma \leq \varepsilon \int_{\Omega_{ij}^b(\gamma,\gamma')} |\nabla \psi|^2 \, dx. \tag{3.36}$$

Proof Let us first prove the lemma for $(i, j) = (1, 3)$. We introduce spherical coordinates $(r, \theta, \varphi) \in \mathbb{R}^+ \times [-\pi/2, \pi/2] \times [0, 2\pi)$:

$$(x_1, x_2, x_3) = (r \cos\theta \cos\varphi, r \sin\theta, r \cos\theta \sin\varphi) \,. \tag{3.37}$$

Defining the opening angle of the set $K_{13}^b(\gamma)$ as $\alpha_0 := \arctan(\gamma)$, we see that the set $K_{13}^b(\gamma)$ takes in those coordinates a very simple form:

$$\left\{(r,\theta,\varphi) : \varphi \in [0,2\pi), |\theta| \geq \frac{\pi}{2} - \alpha_0, r > b\right\} \tag{3.38}$$

and for the measure $d\sigma = \sin(\pi/2 - \alpha_0)d\varphi dr = \cos(\alpha_0)d\varphi dr$. Hence,

$$\int_{\partial K_{13}^b(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma = \int_b^\infty \frac{\int_0^{2\pi} \left|\psi(r,\frac{\pi}{2}-\alpha_0,\varphi)\right|^2 + \left|\psi(r,-\frac{\pi}{2}+\alpha_0,\varphi)\right|^2 d\varphi}{r^\nu} \cos(\alpha_0) dr, \tag{3.39}$$

where we did not write the part of the integral which is in $B_b(0)$ since ψ vanishes there. Next we want to consider a slightly enlarged conical set. For $\gamma' \in (\gamma, 1)$, we define $\alpha_1 := \arctan(\gamma') > \alpha_0$. Then $K_{13}^b(\gamma') \supset K_{13}^b(\gamma)$. We will use the well known trace theorem. For each $r \in [b,\infty)$, we apply [5, Theorem 1, p. 272] to the function

$$\theta \mapsto \int_0^{2\pi} |\psi(r,\theta,\varphi)|^2 d\varphi \tag{3.40}$$

on the interval $\theta \in (\pi/2 - \alpha_1, \pi/2 - \alpha_0) =: I(\alpha_0,\alpha_1)$. We find that there exists a constant $C_4(\gamma,\gamma')$ such that

$$\int_0^{2\pi} \left|\psi(r,\frac{\pi}{2}-\alpha_0,\varphi)\right|^2 d\varphi \leq C_4(\gamma,\gamma') \int_{\pi/2-\alpha_1}^{\pi/2-\alpha_0} \int_0^{2\pi} \left(|\psi|^2 + |\partial_\theta\psi|^2\right) d\varphi d\theta. \tag{3.41}$$

A similar inequality holds in the interval $(-\pi/2+\alpha_0, -\pi/2+\alpha_1)$. Inserting (3.41) into (3.39), we find

$$\begin{aligned}\int_{\partial K_{13}^b(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma &\leq C_4(\gamma,\gamma')\cos(\alpha_0) \int_b^\infty \int_{|\theta|\in I(\alpha_0,\alpha_1)} \int_0^{2\pi} \frac{|\psi|^2 + |\partial_\theta\psi|^2}{r^\nu} d\varphi d\theta dr \\ &\leq \frac{C_4(\gamma,\gamma')\cos(\alpha_0)}{b^\nu} \int_b^\infty \int_{|\theta|\in I(\alpha_0,\alpha_1)} \int_0^{2\pi} \left(|\psi|^2 + |\partial_\theta\psi|^2\right) d\varphi d\theta dr.\end{aligned} \tag{3.42}$$

We remark that the domain of integration in the Equation (3.42) is exactly $\Omega_{13}^b(\gamma,\gamma')$ as defined in the statement of Lemma 3.5. In order to conclude the proof, we transform the right-hand side of Equation (3.36) to spherical coordinates:

$$\int_{\Omega_{13}^b(\gamma,\gamma')} |\nabla\psi|^2 dx \geq \int_b^\infty \int_{|\theta|\in I(\alpha_0,\alpha_1)} \int_0^{2\pi} \left(|\partial_r\psi|^2 + r^{-2}|\partial_\theta\psi|^2\right) r^2 \cos(\theta) d(r,\theta,\varphi). \tag{3.43}$$

For fixed (θ,φ) we want to apply Hardy's inequality on the half-line to the function $\psi(\cdot,\theta,\varphi)$. Note that $\liminf_{r\to\infty} |\psi(r,\theta,\varphi)| = 0$ since $\psi \in H^1(\mathbb{R}^3)$. Thus, we can apply

[6, Theorem 2.65] to find

$$\int_{\Omega_{13}^b(\gamma,\gamma')} |\nabla\psi|^2\, dx \geq \frac{\cos(\alpha_0)}{4} \int_b^\infty \int_{|\theta|\in I(\alpha_0,\alpha_1)} \int_0^{2\pi} \left(|\psi|^2 + |\partial_\theta \psi|^2\right) d(r,\theta,\varphi). \tag{3.44}$$

Combining the inequalities in Equations (3.42) and (3.44), we see that, for all $\varepsilon > 0$, Equation (3.36) holds for $(i, j) = (1, 3)$ for all b large enough. The proof in the case $(i, j) = (2, 3)$ is similar, the only difference being that we exchange x_1 and x_2 in the definition of the spherical coordinates.
Concerning the case $(i, j) = (1, 2)$, we recall the coordinates q and ξ introduced in Equation (2.5). We define spherical coordinates by

$$(q, \xi, x_3) = (r\sin\theta, r\cos\theta\sin\varphi, r\cos\theta\cos\varphi). \tag{3.45}$$

In this set of coordinates the set $K_{12}^b(\gamma)$ reads

$$\{(r,\theta,\varphi) : \varphi \in [0, 2\pi), |\theta| \leq \alpha_0, r > b\} \tag{3.46}$$

where, $\alpha_0 = \arctan(\gamma)$ is again the opening angle of the corresponding conical set, and

$$\int_{\partial K_{23}^b(\gamma)} \frac{|\psi|^2}{|x|^{1+\nu}} d\sigma = \int_b^\infty \frac{\int_0^{2\pi} |\psi(r,\alpha_0,\varphi)|^2 + |\psi(r,-\alpha_0,\varphi)|^2\, d\varphi}{r^\nu} \cos\alpha_0 dr. \tag{3.47}$$

From this point, one can follow the proof in the case $(i, j) = (1, 3)$. In this case we apply the trace theorem to the function

$$\theta \mapsto \int_0^{2\pi} |\psi(r,\theta,\varphi)|^2\, d\varphi \tag{3.48}$$

on the intervals (α_0, α_1) and $(-\alpha_1, -\alpha_0)$ to conclude the statement of this lemma. □

We now complete the proof of Theorem 2.2 by proving that Inequality (3.2) holds. We fix γ' in $(1, \gamma)$ such that the sets $\Omega_{ij}^b(\gamma, \gamma')$ for $1 \leq i < j \leq 3$ do not intersect. Combining Lemma 3.5 together with (3.7) and the results of Lemmas 3.2, 3.4 (for $\varepsilon = 1/2$), we find that there exists a $b_0 > 0$ such that, for any $b > b_0$ and $\psi \in H^1(\mathbb{R}^3)$ with $\operatorname{supp}\psi \subset \{x \in \mathbb{R}^3 |\, |x| > b\}$,

$$L[\psi] \geq L[\psi, \Omega_0] - \frac{1}{2} \sum_{1\leq i<j\leq 3} \int_{\Omega_{ij}^b(\gamma,\gamma')} |\nabla\psi|^2\, dx. \tag{3.49}$$

Let $b \geq b_0$ and remember the definition of $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ in terms of $(x_1, x_2, x_3) \in \mathbb{R}^3$ in (2.1). By construction of the set Ω_0, we have that for all $x \in \Omega_0$ and all $1 \leq i < j \leq 3$

$$\left|\mathbf{r}_i - \mathbf{r}_j\right| \geq \frac{\gamma}{\sqrt{1+\gamma^2}} |x| \geq \frac{\gamma}{\sqrt{1+\gamma^2}} b. \tag{3.50}$$

Hence, we can use that the short-range property of the potentials V_{ij} in Ω_0 and write for some $C > 0$ that

$$L[\psi] \geq \int_{\Omega_0} |\nabla\psi|^2 - \frac{C\,|\psi|^2}{|x|^{2+\nu}} dx - \frac{1}{2}\sum_{i<j} \|\nabla\psi\|^2_{L^2(\Omega^b_{ij}(\gamma,\gamma'))}$$
$$= \int_{\Omega_0} \frac{1}{2}\,|\nabla\psi|^2 - \frac{C\,|\psi|^2}{|x|^{2+\nu}} dx + \frac{1}{2}\,\|\nabla\psi\|^2_{L^2(\Omega_0)} - \frac{1}{2}\sum_{i<j} \|\nabla\psi\|^2_{L^2(\Omega^b_{ij}(\gamma,\gamma'))}\,. \tag{3.51}$$

Remember that we constructed the $\Omega^b_{ij}(\gamma,\gamma')$ such that they are disjoint subsets of Ω_0. As a consequence,

$$\|\nabla\psi\|^2_{L^2(\Omega_0)} - \sum_{i<j} \|\nabla\psi\|^2_{L^2(\Omega^b_{ij}(\gamma,\gamma'))} \geq 0. \tag{3.52}$$

By construction of Ω_0, there exists a set $M \subset S^2$ such that $\Omega_0 = (b,\infty) \times M$. We now finally fix $b \geq b_0$ large enough such that by Hardy's inequality on the half-line (see [6, Theorem 2.65])

$$\int_{\Omega_0} \frac{1}{2}\,|\nabla\psi|^2 - \frac{C}{|x|^{2+\nu}} dx \geq \int_M \int_b^\infty \left(\frac{1}{2}\,|\partial_r\psi|^2 - \frac{C}{r^{2+\nu}}\right) r^2 dr d\omega \geq 0\,. \tag{3.53}$$

Inserting Equation (3.53) and Equation (3.52) into Equation (3.51) gives $L[\psi] \geq 0$ which proves Equation (3.2) for $\Sigma = 0$ and therefore concludes the statement of Theorem 2.2.

Appendix A. Some Lemmas

For the convenience of the reader we repeat here some lemmas from other publications without proofs.

In [33] Zhislin gave the following criterion for the finiteness of the discrete spectrum of a Schrödinger operator. The following Lemma is a straight forward adaption of [1, Lemma C.1]

Lemma A.1 *Let $H = -\Delta + V$ in $L^2(\mathbb{R}^3)$ with V bounded. Let $\Sigma \leq 0$ and assume there exist $\beta, b, \varepsilon > 0$ such, that*

$$\int_{\mathbb{R}^3} \left(|\nabla\psi|^2 + V\,|\psi|^2\right) dx - \varepsilon \int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|^\beta} \geq \Sigma\,\|\psi\|^2 \tag{A.1}$$

for any $\psi \in H^1(\mathbb{R}^3)$ with supp $\psi \subset \{x \in \mathbb{R}^3 |\, |x| > b\}$. *Then the operator H has at most a finite number of eigenvalues below Σ.*

In the proof of the main theorem we make use of some lemmas from [1]. For convenience we repeat these lemmas here. The following is a restatement of [2, Lemma 6.6]

Lemma A.2 *Consider $h = -\Delta + V$ in $L^2(\mathbb{R}^2)$ with V bounded and V short-range (see Equation (2.2)). Assume $h \geq 0$ and V radial symmetric then there exists a $c_0 > 0$ such, that for any $b_0 > R_0$*

$$\int_{|x|\geq b_0} \left(|\nabla\psi|^2 + V\,|\psi|^2 \right) dx \geq -c_0 b_0^{-\nu} \int_0^{2\pi} |\psi(b_0,\theta)|^2\, d\theta \tag{A.2}$$

where R_0, ν are the constants from the short-range property of V.

The next lemma is a restatement of [1, Lemma 6.7]

Lemma A.3 *Let $c_0 > 0$. Then for any sufficiently large $b > 0$ and for any $\psi \in H^1(\mathbb{R}^2)$*

$$\int_{|x|\geq b} \left(|\nabla\psi(x)|^2 - c_0\,|x|^{-2-\nu}\,|\psi(x)|^2 \right) dx \geq -\frac{c_0 b^{-\nu}}{\pi} \int_0^{2\pi} |\psi(b,\theta)|^2\, d\theta\,. \tag{A.3}$$

The next lemma is a restatement of [1, Lemma 6.2]

Lemma A.4 *Consider $h = -\Delta + V$ in $L^2(\mathbb{R})$ with V bounded and V short-range (see Equation (2.2)). Assume $h \geq 0$ then there exists $c > 0$, such that for any $b_0 \geq R_0$ and $\psi \in H^1(\mathbb{R})$*

$$\int_{b_0}^{b_0} (\left|\psi'(t)\right|^2 + V(t)\,|\psi(t)|^2) dt \geq -c b_0^{-1-\nu} (|\psi(b_0)|^2 + |\psi(-b_0)|^2) \tag{A.4}$$

where R_0, ν are the constants from the short-range property of V.

The next lemma is a restatement of [1, Lemma 6.3]

Lemma A.5 *Let $c_0 \geq 0$. Then for any sufficiently large $b > 0$ and for any $\psi \in H^1(\mathbb{R})$*

$$\int_b^\infty \left(\left|\psi'(t)\right|^2 - c_0 t^{-2-\nu}\,|\psi(t)|^2 \right) dt \geq -2c_0 b^{-1-\nu}\,|\psi(b)|^2\,. \tag{A.5}$$

We use several times Hardy's inequality on the half-line. Let us copy here the version of [6, Theorem 2.65]

Theorem A.6 *Let $\rho \in \mathbb{R}\backslash\{1\}$. Let u be weakly differentiable on $\mathbb{R}_+$ with $u' \in L^2(\mathbb{R}_+, r^\rho dr)$ and assume that*

$$\liminf_{r\to 0} |u(r)| = 0 \text{ if } \rho < 1, \quad \liminf_{r\to\infty} |u(r)| = 0 \text{ if } \rho > 1, \tag{A.6}$$

Then

$$\int_0^\infty |u(r)|^2\, r^{-2+\rho}\, dr \leq \left(\frac{2}{\rho - 1} \right)^2 \int_0^\infty \left|u'(r)\right|^2 r^\rho dr. \tag{A.7}$$

The constant on the right side is optimal.

The following lemma is a straight forward adaptation of [27, Lemma 5.1]. Recall the definitions of $K_{ij}^b(\gamma)$ in Equation (3.4), then

Lemma A.7 *Given* $\varepsilon, \gamma > 0$ *for all* $i, j \in \{1,2,3\}$, $i < j$ *there exist* $\tilde{\gamma} \in (0, \gamma)$ *and continuous functions* $u_{ij}, v_{ij} : \mathbb{R}^3 \to \mathbb{R}$ *with piecewiese continuous derivatives such that*

$$u_{ij}^2 + v_{ij}^2 = 1, \quad u_{ij}(x) = \begin{cases} 1 & x \in K_{ij}^0(\tilde{\gamma}) \\ 0 & x \notin K_{ij}^0(\gamma) \end{cases}, \quad \left|\nabla u_{ij}\right|^2 + \left|\nabla v_{ij}\right|^2 \le \varepsilon \left(\frac{v_{ij}^2}{|x|^2} + \frac{u_{ij}^2}{\left|q_{ij}\right|^2} \right) \tag{A.8}$$

where $q_{12} = q = \frac{1}{\sqrt{2}}(x_1 - x_3) \in \mathbb{R}$ *and* $q_{j3} = (x_3, x_j) \in \mathbb{R}^2$.

Appendix B. Proof of Theorem 2.2 when Σ < 0

We now cover the case when Σ defined in Equation (2.8) is negative. As in the previous case we will prove that there exists a $b > 0$ such that

$$L[\psi] := \int_{\mathbb{R}^3} \left(\sum_{i=1}^{3} \left|\partial_{x_i} \psi\right|^2 + \sum_{i<j} V_{ij} \, |\psi|^2 \right) dx - \int_{\mathbb{R}^3} \frac{|\psi(x)|^2}{|x|^{2+\nu}} dx \ge \Sigma \, \|\psi\| \tag{B.1}$$

for any $\psi \in H^1(\mathbb{R}^3)$ with supp $\psi \subset \{x \in \mathbb{R}^3 | \, |x| > b\}$. As in the case $\Sigma = 0$, we consider the sets $K_{ij}^b(\gamma)$ and Ω_0 defined in (3.4) and (3.5). Contrary to the previous case, we cannot apply the lemmas stated in Appendix A, since we would need to integrate a constant function on a infinite-measure domain. For this reason, we localize the functional L with smooth cut-off functions. For fixed $\varepsilon < 1/8$ we consider the family of cut-off functions u_{ij} defined in Lemma A.7 together with $\mathcal{V} := \sqrt{1 - \sum_{i<j \in \{1,2,3\}} u_{ij}^2}$. The family $\{\mathcal{V}, u_{ij}$ for $i, j \in \{1,2,3\}$ with $i < j\}$ forms a partition of unity by construction. According to the IMS localization formula,

$$L[\psi] = \sum_{1 \le i < j \le 3} \left(L[u_{ij}\psi] - \langle \psi, \left|\nabla u_{ij}\right|^2 \psi \rangle \right) + L[\mathcal{V}\psi] - \langle \psi, |\nabla \mathcal{V}|^2 \psi \rangle. \tag{B.2}$$

Recall the definition of $\Omega_{ij}^b(\tilde{\gamma}, \gamma)$ in the statement of Lemma 3.5. Note that, on each $\Omega_{ij}^b(\tilde{\gamma}, \gamma)$, $\mathcal{V} = v_{ij} = \sqrt{1 - u_{ij}^2}$. Moreover, $\nabla \mathcal{V}$ vanishes outside of

$$\bigcup_{1 \le i < j \le 3} \Omega_{ij}^b(\tilde{\gamma}, \gamma)$$

by construction. Similarly, ∇u_{ij} vanishes outside of $\Omega_{ij}^b(\tilde{\gamma}, \gamma)$.

Consequently by Lemma A.7,

$$\sum_{1\le i<j\le 3}\langle\psi,\left|\nabla u_{ij}\right|^2\psi\rangle+\langle\psi,|\nabla\mathcal{V}|^2\psi\rangle=\sum_{1\le i<j\le 3}\int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)}\left(\left|\nabla u_{ij}\right|^2+\left|\nabla v_{ij}\right|^2\right)|\psi|^2\,dx$$
$$\le\varepsilon\sum_{1\le i<j\le 3}\int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)}\left(\frac{v_{ij}^2}{|x|^2}+\frac{u_{ij}^2}{\left|q_{ij}\right|^2}\right)|\psi|^2\,dx. \tag{B.3}$$

Therefore,

$$L[\psi]\ge\sum_{1\le i<j\le 3}\left(L[u_{ij}\psi]-\varepsilon\int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)}\frac{u_{ij}^2}{\left|q_{ij}\right|^2}|\psi|^2\,dx\right)$$
$$+L[\mathcal{V}\psi]-\varepsilon\sum_{1\le i<j\le 3}\int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)}\frac{\mathcal{V}^2}{|x|^2}|\psi|^2\,dx. \tag{B.4}$$

We show the following

Lemma B.1 *There is a $b_0>0$ such that for all $b\ge b_0$ and for any $\psi\in H^1(\mathbb{R}^3)$ with* $\operatorname{supp}\psi\subseteq\{x\in\mathbb{R}^3\,|\,|x|>b\}$ *with $i,j\in\{1,2,3\}$, $i\ne j$,*

$$L[u_{ij}\psi]-\varepsilon\int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)}\frac{u_{ij}^2}{\left|q_{ij}\right|^2}|\psi|^2\,dx\ge\Sigma\left\|u_{ij}\psi\right\|^2 \tag{B.5}$$

where q_{ij} was defined in Lemma A.7.

Proof Let

$$\xi_{ij}:=\begin{cases}(\xi,x_3)\in\mathbb{R}^2, & (ij)=(12)\\ x_2\in\mathbb{R}, & (ij)=(13)\\ x_1\in\mathbb{R}, & (ij)=(23)\end{cases},\quad\tilde{\psi}:=\psi u_{ij}. \tag{B.6}$$

Let $W_{ij}:=\left[\sum_{1\le l<m\le 3}V_{lm}\right]-V_{ij}$ and define

$$\tilde{L}_1[\tilde{\psi}]:=\langle\tilde{\psi},(-\Delta_{q_{ij}}+V_{ij})\tilde{\psi}\rangle,$$
$$\tilde{L}_2[\tilde{\psi}]:=\langle\tilde{\psi},(-\Delta_{\xi_{ij}}+W_{ij})\tilde{\psi}\rangle-\int_{\mathbb{R}^3}\frac{\left|\tilde{\psi}(x)\right|^2}{|x|^{2+\nu}}dx. \tag{B.7}$$

Then,

$$L[\tilde{\psi}]=\tilde{L}_1[\tilde{\psi}]+\tilde{L}_2[\tilde{\psi}]. \tag{B.8}$$

We distinguish between the cases $\inf\sigma(h_{ij})>\Sigma$ and $\inf\sigma(h_{ij})=\Sigma$. We start with the case $\inf\sigma(h_{ij})>\Sigma$. Let $\kappa:=\inf\sigma(h_{ij})-\Sigma>0$. Then

$$\tilde{L}_1[\tilde{\psi}]\ge(\Sigma+\kappa)\left\|\tilde{\psi}\right\|^2. \tag{B.9}$$

Using the short-range property of W_{ij} and the fact that, on $\Omega^b_{ij}(\tilde{\gamma},\gamma)$, $|q_{ij}| \geq \tilde{\gamma}\,|\xi_{ij}|$, we find that there exists $C > 0$ and $b_0 > 0$ such that for any $b \geq b_0$

$$\tilde{L}_2[\tilde{\psi}] - \varepsilon \int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{u_{1j}^2}{|q_{ij}|^2} |\psi|^2\, dx \geq -\langle \tilde{\psi}, \Delta_{\xi_{ij}} \tilde{\psi}\rangle - C \int_{\mathbb{R}^3} \frac{|\tilde{\psi}(x)|^2}{|\xi_{ij}|^2} dx\,. \tag{B.10}$$

Since $-\Delta_{\xi_{ij}} \geq 0$, and on $K^b_{ij}(\gamma)$, $|\xi_{ij}| \geq |x|/\sqrt{1+\gamma^2} \geq b/\sqrt{1+\gamma^2}$, we find

$$\tilde{L}_2[\tilde{\psi}] - \varepsilon \int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{u_{1j}^2}{|q_{ij}|^2} |\psi|^2\, dx \geq -C \int_{\mathbb{R}^3} \frac{|\tilde{\psi}(x)|^2}{|\xi_{ij}|^2} dx \geq -\frac{C_1}{b^2} \left\|\tilde{\psi}\right\|^2 \tag{B.11}$$

for some $C_1 > 0$ independent of b. Choose b large enough such that $\kappa \geq C_1/b^2$. Then, combining Equation (B.9) and Equation (B.11) concludes the statement in this case.

Next we consider the case where $\inf \sigma(h_{ij}) = \Sigma$ such that Σ is an eigenvalue of h_{ij} and thus there exists a $\varphi_0 \in L^2(dq_{ij})$ with $\|\varphi_0\| = 1$ such that $\langle \varphi_0, h_{ij}\varphi_0\rangle = \Sigma$. We define

$$f(\xi_{ij}) := \langle \varphi_0, \tilde{\psi}\rangle_{L^2(dq_{ij})}, \quad g(q_{ij},\xi_{ij}) := \tilde{\psi}(q_{ij},\xi_{ij}) - f(\xi_{ij})\varphi_0(q_{ij}). \tag{B.12}$$

Note that φ_0 and g are orthogonal in the usual $L^2(dq_{ij})$-sense by construction. Since Σ is an isolated nondegenerate eigenvalue of h_{ij}, there exists some $\kappa' > 0$ such that

$$\tilde{L}_1[\tilde{\psi}] = \langle \tilde{\psi}, h_{ij}\tilde{\psi}\rangle \geq \Sigma \left\|\tilde{\psi}\right\|^2 + \kappa' \|g\|^2\,. \tag{B.13}$$

One can choose κ' as the distance of Σ and the remaining spectrum of h_{ij}. Using the short-range property of W_{ij} and Lemma A.7, there exists $C > 0$ such that for $b > 0$ large enough

$$\begin{aligned}\tilde{L}_2[\tilde{\psi}] - \varepsilon \int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{u_{ij}^2}{|q_{ij}|^2} |\psi|^2\, dx &\geq \frac{1}{2}\langle \tilde{\psi}, -\Delta_{\xi_{ij}}\tilde{\psi}\rangle - C \int \frac{|\tilde{\psi}|^2}{|\xi_{ij}|^{2+\delta}} d(q_{ij},\xi_{ij}) \\ &\quad + \frac{1}{2}\langle \tilde{\psi}, -\Delta_{\xi_{ij}}\tilde{\psi}\rangle - \varepsilon \int_{\Omega^b_{ij}(\gamma,\tilde{\gamma})} \frac{|\tilde{\psi}|^2}{|q_{ij}|^2} d(q_{ij},\xi_{ij})\,.\end{aligned} \tag{B.14}$$

Note that the variable ξ_{ij} is either one- or two-dimensional. Depending on its dimension we use Hardy's inequality on the half line or the two-dimensional Hardy's inequality (see [21]) to estimate the first line on the right hand side in equation (B.14). The inequality is applicable since $\tilde{\psi}$ vanishes whenever $|\xi_{ij}|$ is small enough by construction. Thus

$$\frac{1}{2}\langle\tilde{\psi}, -\Delta_{\xi_{ij}}\tilde{\psi}\rangle - C\int \frac{|\tilde{\psi}|^2}{|\xi_{ij}|^{2+\delta}} d(q_{ij}, \xi_{ij}) \geq 0. \tag{B.15}$$

It remains to prove the non-negativity of the right-hand side of (B.14). Recall that φ_0 and g are orthogonal in the $L^2(dq_{ij})$-sense and thus

$$\langle\tilde{\psi}, -\Delta_{\xi_{ij}}\tilde{\psi}\rangle = \langle|\varphi_0|^2 f, -\Delta_{\xi_{ij}} f\rangle + \langle g, -\Delta_{\xi_{ij}} g\rangle \geq \langle f, -\Delta_{\xi_{ij}} f\rangle_{L^2(d\xi_{ij})}. \tag{B.16}$$

We use $(a+b)^2 \leq 2a^2 + 2b^2$ to find that there exists $C' > 0$ such that for b large enough

$$\begin{aligned}\int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{|\tilde{\psi}|^2}{|q_{ij}|^2} d(q_{ij}, \xi_{ij}) &\leq 2\int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{|\varphi_0|^2 |f|^2}{|q_{ij}|^2} d(q_{ij}, \xi_{ij}) \\ &\quad + 2\int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{|g|^2}{|q_{ij}|^2} d(q_{ij}, \xi_{ij}) \\ &\leq 2\int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{|q_{ij}|^2 |\varphi_0|^2 |f|^2}{|q_{ij}|^4} d(q_{ij}, \xi_{ij}) + \frac{C'}{b^2}\|g\|^2,\end{aligned} \tag{B.17}$$

where in the last line, we used that, on $\Omega^b_{ij}(\tilde{\gamma},\gamma)$,

$$|q_{ij}|^2 \geq \frac{\tilde{\gamma}^2}{1+\tilde{\gamma}^2}|x|^2 \geq \frac{\tilde{\gamma}^2}{1+\tilde{\gamma}^2} b^2. \tag{B.18}$$

Since φ_0 is an eigenfunction of h_{ij} associated with a discrete eigenvalue, it decays exponentially at infinity (*cf.* for example [18], Theorem XIII.39). Hence for any $\delta_1 > 0$ there exists $q_0 > 0$ such that $|q_{ij}|^2 |\varphi_0(q_{ij})|^2 \leq \delta_1$ for any $|q_{ij}| \geq q_0$. Thus for b large enough

$$\int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{|q_{ij}|^2 |\varphi_0|^2 |f|^2}{|q_{ij}|^4} d(q_{ij}, \xi_{ij}) \leq \frac{\delta_1}{\tilde{\gamma}^4}\int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{|f|^2}{|\xi_{ij}|^4} d(q_{ij}, \xi_{ij}). \tag{B.19}$$

From equations (B.19) and (B.17), we find

$$\begin{aligned}&\frac{1}{2}\langle\tilde{\psi}, -\Delta_{\xi_{ij}}\tilde{\psi}\rangle - \varepsilon\int_{\Omega^b_{ij}(\gamma,\tilde{\gamma})} \frac{|\tilde{\psi}|^2}{|q_{ij}|^2} d(q_{ij}, \xi_{ij}) \\ &\geq \frac{1}{2}\langle f, -\Delta_{\xi_{ij}} f\rangle_{L^2(d\xi_{ij})} - \varepsilon\frac{\delta_1}{\tilde{\gamma}^4}\int_{\Omega^b_{ij}(\gamma,\tilde{\gamma})} \frac{|f|^2}{|\xi_{ij}|^4} d(q_{ij}, \xi_{ij}) - \varepsilon\frac{C'}{b^2}\|g\|^2.\end{aligned} \tag{B.20}$$

As in the proof of Equation (B.15), we can again use Hardy's inequality depending on the dimension of the variable ξ_{ij} to find

$$\frac{1}{2}\langle f, -\Delta_{\xi_{ij}} f\rangle_{L^2(d\xi_{ij})} - \varepsilon \frac{\delta_1}{\tilde{\gamma}^4} \int_{\Omega^b_{ij}(\gamma,\tilde{\gamma})} \frac{|f|^2}{|\xi_{ij}|^4} d(q_{ij}, \xi_{ij}) \geq 0 \tag{B.21}$$

for $b > 0$ large enough. Combining equations (B.14), (B.15) (B.20) and (B.21), we find

$$\tilde{L}[\tilde{\psi}] - \varepsilon \int_{\Omega^b_{ij}(\gamma,\tilde{\gamma})} \frac{u_{ij}^2}{|q_{ij}|^2} |\psi|^2 \, d(q_{ij}, \xi_{ij}) \geq \Sigma \left\|\tilde{\psi}\right\|^2 + \left(\kappa' - \varepsilon \frac{C'}{b^2}\right) \|g\|^2 . \tag{B.22}$$

The statement in equation (B.5) follows by taking b large enough. □

We have thus proved that

$$L[\psi] \geq \sum_{1\leq i<j\leq 3} \left\|u_{ij}\psi\right\|^2 \Sigma + L[\mathcal{V}\psi] - \varepsilon \sum_{1\leq i<j\leq 3} \int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{\mathcal{V}^2}{|x|^2} |\psi|^2 \, dx. \tag{B.23}$$

By construction $\mathcal{V}$ is supported in

$$\Omega_0(\tilde{\gamma}) := \mathbb{R}^3 \setminus \bigcup_{1\leq i<j\leq 3} K^b_{ij}(\tilde{\gamma}) . \tag{B.24}$$

Therefore, by applying the short-range property to any of the potentials we can write

$$L[\mathcal{V}\psi] \geq \int_{\Omega_0(\tilde{\gamma})} \left(|\nabla \mathcal{V}\psi|^2 - \frac{C}{|x|^{2+\nu}} |\mathcal{V}\psi|^2\right) dx. \tag{B.25}$$

Hence, we can estimate the last terms in Equation (B.23). Combining the previous inequalities shows that for $b > 0$ large enough

$$L[\mathcal{V}\psi] - \varepsilon \sum_{1\leq i<j\leq 3} \int_{\Omega^b_{ij}(\tilde{\gamma},\gamma)} \frac{|\mathcal{V}\psi|^2}{|x|^2} dx \geq \int_{\mathbb{R}^3} \left(|\nabla \mathcal{V}\psi|^2 - \frac{2\varepsilon}{|x|^2} |\mathcal{V}\psi|^2\right) dx \geq 0 \tag{B.26}$$

by Hardy's inequality in dimension three. Thus we have shown

$$L[\psi] \geq \sum_{1\leq i<j\leq 3} \left\|u_{ij}\psi\right\|^2 \Sigma \geq \|\psi\|^2 \Sigma \tag{B.27}$$

since $\Sigma < 0$. This is precisely the statement in Equation (B.1).

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no Conflict of interest.

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