Districting:
Multi-Period and Stochastic Approaches

Zur Erlangung des akademischen Grades eines

Doktors der Ingenieurwissenschaften

(Dr.-Ing.)

von der KIT-Fakultät für Wirtschaftswissenschaften

des Karlsruher Instituts für Technologie (KIT)

genehmigte

DISSERTATION

von

M.Sc. Anika Johanna Pomes

Tag der mündlichen Prüfung:	05.11.2025
Hauptreferent:	Prof. Dr. Stefan Nickel
Korreferent:	Prof. Dr. Jörg Kalcsics

Acknowledgements

First, I would like to thank Professor Dr. Stefan Nickel for his guidance and support throughout my doctoral studies. His advice and feedback have been essential for completing this work.

I would also like to thank the members of my doctoral examination committee for their time and for the feedback they provided on my dissertation. In particular, I thank Professor Dr. Jörg Kalcsics (University of Edinburgh) as second reviewer, Professor Dr. Emilia Graß (KIT) as examiner, and Professor Dr. Andreas Geyer-Schulz (KIT) for chairing the examination committee.

I am also grateful to the colleagues at the Institute of Operations Research and the Chair of Discrete Optimization and Logistics at KIT. In particular, I would like to thank Fabian Dunke, with whom I had many helpful and motivating discussions about the content of my work. His feedback, questions and encouragement during the final phase supported me greatly.

My deepest thanks go to my family. I am especially grateful to my mother for her continuous support and strength, and to my sister Tabea, who was always there to listen when I needed it. And finally, I want to thank my daughter Marla, who gave me the greatest motivation to bring this work to an end.

Thank you all for your support.

Abstract

Districting involves grouping smaller areas into larger ones or dividing a larger area into multiple subareas. This thesis examines districting with a focus on multi-period dynamics and uncertainty – two aspects that have received limited attention in existing models since the field's beginning in 1965. The consideration of these two factors not only better reflects reality, but – as shown in this thesis – taking them into account can also lead to better decisions.

A deterministic model is first proposed to serve as a basis, ensuring feasible solutions. The first major contribution of the thesis then consists of an in-depth examination of the multi-period setting, where time-dependent reassignments are allowed and a savings parameter is introduced to balance compactness and reassignments. Computational analyses are conducted to identify when the multi-period model provides advantages over static approaches. This is achieved through detailed computational experiments that perform linear correlation analysis at different levels of granularity in order to provide a comprehensive assessment of the relation between demand fluctuations and values of multi-period solutions.

The next part of the thesis then addresses the topic of districting under uncertainty in detail. To this end, we adapt a two-stage stochastic model. Modifications are made to the definition of the average allowed demand in each district, which is now scenario-dependent. The analysis focuses on performance indicators, including the value of the stochastic solution and the expected value of perfect information. The conditions under which stochastic modeling is beneficial, particularly with demand fluctuations between scenarios, are evaluated.

In the final methodological part, both uncertainty and multi-periodicity are combined in a multi-stage stochastic districting model, supported by a relax-and-fix heuristic. The model is tested on various instances, including larger instances, to assess its applicability and

performance with respect to cost, demand variability, and the number of territorial units and periods.

The methodology proposed in this thesis is applied in a comprehensive case study of home healthcare services in Karlsruhe. This realistic case demonstrates the benefits of the developed models in addressing districting challenges, highlighting their potential impact on decision-making. The case study begins with a deterministic single-period setting, followed by an exploration of a multi-period framework. It then transitions to a stochastic setting before combining both time and uncertainty factors. The case study offers valuable insights into the practical application of the proposed districting models in real-world settings, along with important managerial implications.

Contents

Acronyms

DP Districting Problem.

DPSB Districting Problem with Symmetric Balance.

DVSS Dynamic Value of the Stochastic Solution.

EDP Extended Districting Problem.

EEV Expected result of the Expected Value Problem.

EV Expected Value Problem.

EVPI Expected Value of Perfect Information.

HHC Home Health Care.

MPDP Multi-period Districting Problem.

MSSDP Multi-Stage Stochastic Districting Problem.

SC Static Counterpart.

SCM Supply Chain Management.

SDP Stochastic Districting Problem.

SDPAR Stochastic Districting Problem with Auxiliary Recourse.

VMPS Value of the Multi-Period Solution.

VSS Value of the Stochastic Solution.

WS “Wait and See” Solution.

Chapter 1

Introduction

"The worst form of injustice is pretended justice."

— Plato

Plato's quote above appears to be more closely related to the complex challenge of districting than the author may have intended. This can be seen, for example, in the outward appearance of seemingly fair electoral districts, which can mask deep manipulation for partisan gain.

Districting is a relatively young field within operations research that involves grouping smaller areas into larger ones or dividing a larger area into multiple subareas. The concept of districting dates back to Hess et al. (1965) and is closely related to location planning, sharing many similarities. However, certain constraints and goals in districting are beyond the scope of pure location planning. One central aspect of districting is that it does not always involve planning physical locations or facilities. In fact, in many applications, the centers of a district serve as geographic points with no specific function other than being a reference point. This distinguishes districting from the more well-known and extensively researched field of location planning.

Example 1 (Multiple Periods and Multiple Scenarios in Home Health Care Districting) *In Figure 1.1, two different home health care districting solutions in Karlsruhe (Germany) are shown. The focus is on assigning patients to health aides, with each aide being assigned a specific area that covers the assigned patients.*

Case A: Multiple Periods *Two periods are solved independently, which can lead to high reassignment costs between these periods. Since each period is planned separately with different data, the resulting districts may differ significantly. This discrepancy can disrupt continuity of care for patients and health aides, resulting in increased operational challenges.*

Case B: Multiple Scenarios *Two independent scenarios are evaluated, but only one will be realized. Choosing one of these two solutions is not robust, as it fails to prepare for both scenarios adequately. This approach could result in suboptimal or infeasible solutions if the actual scenario differs from the chosen plan. Additionally, it may incur significant costs and longer waiting times for patients if the scenario that was not selected occurs instead.*

The significant differences between these independent solutions highlight the challenges posed by changes between periods and uncertainty in home health care districting. Addressing these challenges requires an integrated approach that considers multiple periods, multiple scenarios, or both. These topics will be discussed in detail in this thesis, with a comprehensive solution for the application of home health care districting presented in Chapter 6.

(a) Distričting solution for the first period or scenario

(b) Districting solution for the second period or scenario

Figure 1.1: Two independently solved districting plans illustrating the impact of data changes and scenario uncertainty in home health care districting.

Districting and location planning are two important concepts of strategic planning, but districting plays a more crucial role in ensuring fairness and equity across larger geographic areas. While location planning focuses on strategically selecting sites for facilities or businesses by optimizing factors such as accessibility and costs, it primarily addresses the optimization of individual sites. In contrast, districting takes a more comprehensive approach by grouping territorial units into clusters. This process is essential for promoting equity, as it creates similar districts which can be based on various characteristics, including population density, socio-economic factors, and environmental considerations. In most cases, this measure is called demand.

While classical facility location planning is part of strategic planning, as it involves considerations for facilities that require a longer time horizon, districting can be relevant to strategic, tactical, and operational planning. If no facility is built in the center of the district, the districts can be planned individually on a daily basis, allowing for territorial units to be reassigned as needed. Consequently, the areas of application for districting are more extensive, as the construction of a facility does not limit them.

Three essential goals of districting are balance, contiguity, and compactness:

Balance refers to the fair distribution of demand. It ensures that no district or district representative has an unfair advantage. Achieving balance is important because it promotes fairness and equal utilization of workload or other resources.

Contiguity refers to the connectedness. It emphasizes the importance of connecting adjacent or neighboring basic territory units. Contiguity ensures that at least one neighbor is in the same district if the district contains more than just one territorial unit.

Compactness refers to the closeness or concentration of territorial units within a given area. It aims to minimize the dispersion of territorial units, resulting in a more efficient and effective districting plan.

As described above – as part of the Supply Chain Management (SCM) – districting is crucial to strategic planning, focusing on long-term goals and objectives. Districts with a facility located in the center may be challenging regarding rearrangement. Districts without a facility can be rearranged more easily and in shorter cycles. However, in applications where facilities are placed in district centers, replacing the facilities may be difficult, but the boundaries of the districts can often be redefined relatively quickly. Districting can also be used in tactical and operational planning. For example, the districts (and routes within these districts) for postal workers can be planned daily based on the volume of letters and parcels, allowing for adjustments to be made as needed. Rearranging, redefinition – or better – reassigning can not only be seen as a reaction but also as a proactive strategy. In many cases, it is already known that data will change, which can be considered while planning the districts for multiple periods. This thesis introduces a comprehensive model for multi-period districting. In this model, the district centers can change in each period or remain consistent throughout the planning horizon. In contrast, reassignments and updates of the district allocations are allowed at the beginning of each period.

In many practical districting applications, the activity measures of territorial units (such as demands) are not always known in advance for the entire planning horizon. This lack of information can be particularly challenging, for example, in the case of school districts, where the actual pupil numbers, which determine the demand for school classes and teachers, can only be estimated with some uncertainty. One common approach is to use birth rates in the respective territorial units to indicate the future number of students. However, it is essential to note that if a new residential area is designated in subsequent years, the number of pupils may differ significantly from the estimated number. Therefore, it is crucial to consider different scenarios for long-term planning to be prepared for such uncertainties. Another example is home health care, where the need for assistance can only be estimated. To address this issue, this thesis proposes a stochastic model that can effectively capture and model the complexities associated with stochastic districting problems. By considering various scenarios and incorporating stochastic elements, this model provides a more comprehensive and robust framework for long-term planning in districting applications.

Hence, this thesis systematically investigates the effects of both stochasticity and multi-periodicity in districting models, considering each factor individually as well as their combined influence. By analyzing these dimensions separately and together, the work synthesizes a comprehensive understanding of their respective and joint impacts on districting outcomes.

Therefore, we introduce a new model and provide extensive computational results to evaluate its benefits in the context of districting. The literature frequently presents computational tests involving randomly generated data for stochastic and multi-period problems. In this thesis, we examine the data generation process and evaluate whether employing a stochastic or multi-period model is advantageous in these contexts. We also examine which of these two factors has a significant influence on the benefits of using a combined multi-period stochastic model.

In the following subsection, various applications of general districting problems are presented. For a deeper understanding of multi-period and uncertainty considerations, additional literature can be referenced in Chapter 3, Chapter 4, and Chapter 5.

1.1 Districting Applications

Districting is referred to by various names in the literature. Besides districting, terms such as territory design and area planning are also commonly used. In addition to districts, the term areas is frequently mentioned, and individual territorial units are referred to as basic units. In this thesis, we use districting, districts, and territorial units.

Kalcsics and Ríos-Mercado (2019) provide a broad overview of districting problems, including typical criteria and constraints found in various applications, as well as ways to measure and model these criteria. They also review the different areas of application for districting problems and the various solution approaches that have been used.

Districting is defined in the literature from two points of view:

- On the one hand, districting can be seen as grouping smaller territorial units into larger districts (Kong et al., 2019).
- Alternatively, districting can be considered a subdivision of a large district into sub-districts consisting of territorial units (Salazar-Aguilar et al., 2011).

In both cases, some constraints must be fulfilled – such as balance, compactness, and continuity – as some of the most popular ones. These requirements can be addressed in the objective or as constraints. The papers in the following subsections address these requirements in various ways.

Another way to categorize districting applications is by determining whether a physical center needs to be established or not. In some cases, facilities should be placed in the center or a representative territorial unit, for example, in school districting, where the school has to be built physically. This case is more connected to the research field of facility location. In other cases, like in political districting, there are no facilities that have to be placed. Regardless, even if there is no facility, the distance to the representative center is minimized. For sales territories, the distance for the traveling salesman is minimized by minimizing the distances or travel routes simultaneously, thus obtaining the most compact districts possible. Some of these applications are considered in more detail below. Many of these applications are not stochastic or multi-period, but they can be extended to include those aspects. Later, the literature that considers these factors is examined.

The classic applications of districting can be categorized as follows:

- In Section 1.1.1, we focus on political districting, which primarily involves the creation of electoral boundaries.
- Section 1.1.2 discusses public districting, with a particular emphasis on school districting and police districting.
- Applications of districting in the healthcare sector are presented in Section 1.1.3.
- Finally, Section 1.1.4 covers commercial districting.

1.1.1 Political Districting

In a political context, the term districting is often associated with gerrymandering. This involves the manipulation of electoral boundaries to influence election outcomes in favor of a particular party or candidate. The United States has seen several instances of gerrymandering in recent history. One of the most popular graphics in the context of political districting is depicted in Figure 1.2. This satirical image was printed in an article written by Elkanah Tisdale in 1812 as a reaction to the new state senate election district of South Essex created by the Massachusetts legislature under Governor Elbridge Gerry. The districts look like a salamander. The word gerrymandering combines the name Gerry and the word salamander.

Figure 1.2: The Gerry-Mander – A political cartoon from the Boston Gazette, highlighting the manipulation of district boundaries for electoral gain (Tisdale, 1812).

Optimal partisan gerrymandering is extensively studied and remains a prominent topic, particularly in the US. For example, Owen and Grofman (1988) examine optimal partisan gerrymandering within a two-party competition system. They investigate two objective functions: maximizing expected seat share and maximizing the probability of a working legislative majority. The findings indicate that the optimal districting schemes generated under these objectives differ, highlighting conflicts between majority party legislators' self-interest and districting that maximizes party advantage. Additionally, they review the 1982 California congressional plan as a risk-minimizing partisan gerrymander.

In contrast to the previous negative use of districting in a political context, Gurnee and Shmos (2021) introduce a two-stage method for optimizing political districting for fairness, called fairmandering. Their method involves a randomized divide-and-conquer column generation heuristic and a master selection problem to choose the districts to include in the final plan. They analyze the range of possible outcomes and implications of fairness in a study of congressional districts.

Ricca et al. (2008) propose a method for political districting on a given territory using a bi-objective partitioning of a graph into connected components. They use heuristics based on discrete weighted Voronoi regions to obtain compact and balanced districts and discuss some formal properties of these algorithms. The performance of these algorithms is tested on randomly generated rectangular grids and real-life benchmarks, and the resulting district maps are compared with the institutional ones adopted in the Italian political elections from 1994 to 2001.

Dugošija et al. (2020) propose another integer linear programming formulation for the political districting problem. They base their model on the graph representation of political territory. Unlike previous formulations, their model considers population equality, compactness, and contiguity as major criteria. They also present two models, one focusing on compactness as an objective function and the other taking into account the interests of the decision-maker. Finally, they provide numerical examples to illustrate the general aspects of the problem and obtain experimental results using the CPLEX solver.

In his thesis, Goderbauer (2020) develops a multi-stage heuristic to divide Germany into electoral districts that comply with legal requirements, such as electoral equality and administrative boundaries. The algorithm successfully allocates districts that are in accordance with the law and fulfill tolerances more closely than the current districting. The problem of dividing a country into electoral districts is defined in this contribution as a multi-criteria graph partition problem. The multi-stage aspect does not involve a connection across multiple periods, as discussed in later chapters of this thesis.

Arredondo et al. (2021) conduct a study on the design of electoral districts in Mexico, focusing on the representation of minority groups, such as the indigenous community, in the Parliament. They formulate mixed integer linear programs to address this issue, considering criteria such as contiguity and population balance. The study employs a two-phase approach, where the first phase aims to form a fixed number of indigenous districts as prescribed by law, and the second phase focuses on forming non-indigenous districts. They test their procedure on the territory of Chiapas in Mexico and on fictitious problem instances represented by a grid graph, while also comparing their district map with the institutional one currently adopted in Chiapas.

Identified research gap: Political districting primarily concerns the definition of electoral districts, a process governed by different legal frameworks in nearly every country. However, a crucial aspect often neglected in both research and practice is the dynamic and uncertain evolution of electoral districts over time. For instance, demographic indicators such as birth rates or the number of school-aged children could be used years in advance to forecast future voter populations in a specific region. These forecasts, however, are uncertain due to unpredictable factors such as migration, economic changes, or policy shifts. Despite this, current approaches rarely incorporate either the temporal dimension or the uncertainty associated with demographic developments. As a result, electoral districts are typically not designed to be robust against future changes and developments.

1.1.2 Public Districting

Districting has numerous applications in the public sector. It includes the division of school and police districts, as well as the categorization of fire brigade and waste management districts. In some countries, healthcare districting is also part of public districting, but it often belongs to service districting. Therefore, this topic is considered separately in Section 1.1.3.

Caro et al. (2004) conducts a study on the school redistricting problem in a city. The authors review existing approaches, establish the desired properties for a good school districting plan, and propose an optimization model integrated with a geographic information system to generate solutions. Specifically, they formulate the school redistricting problem as an integer programming model, where the objective is to minimize overall student travel distance while satisfying constraints such as school capacity and spatial contiguity. The GIS component is used to manage and visualize spatial data, enabling the integration of geographic factors directly into the optimization process. The authors outline a prototype of the system, address challenges related to its implementation, and examine two case studies from Philadelphia. They analyze the trade-offs involved in the solutions and explore questions of feasibility. The findings suggest that complex spatial issues, such as school redistricting, can be effectively addressed by combining both objective, data-driven analysis and subjective judgment.

Bruno et al. (2017a) addresses a debate in Italy about reducing the number of provinces and rearranging their borders. They formulate a mixed integer model to support the decision-making process for more efficient territorial configurations while safeguarding the accessibility of essential services to the population within their boundaries. They compare scenarios provided on four benchmark problems using real data associated with the most representative Italian regions.

Bruno et al. (2017b) develop mathematical models to analyze amalgamation and redistricting policies in Italy, aimed at supporting stakeholders and policymakers in understanding the impact of administrative reforms in local authorities. Specifically, they formulate a territorial reorganization problem as a mixed-integer linear programming model. This approach optimizes the assignment of basic territorial units to provinces while incorporating constraints to ensure service accessibility and administrative efficiency. The methodology is applied to real-world data from several Italian regions, and various redistricting scenarios are compared using computational experiments.

Liberatore et al. (2020) conduct a systematic review of the literature related to the police districting problem, focusing on the efficient and effective design of patrol sectors to improve performance attributes. They highlight the importance of effectiveness in influencing the ability of police agencies to prevent and stop crime, while also emphasizing the need for a fair and homogeneous distribution of workload to ensure the satisfaction of police agents. The review categorizes contributions in terms of attributes and solution methodology adopted, and provides an annotated bibliography presenting the most relevant elements of each research.

Liberatore et al. (2022) propose an equitable police districting model to balance crime-reduction effectiveness with racial fairness in defining patrol districts. He formulates the model as a mixed-integer program using compromise programming and goal programming. The model is validated using a real-world case study in the Central District of Madrid, Spain, and compared to standard patrolling configurations currently used by the police. The results indicate a trade-off between racial fairness and crime control. They show that integrating the proposed racial criterion greatly improves racial fairness with a limited impact on policing effectiveness. The model's solutions outperform the current patrolling configurations used by the police, suggesting its capability to define efficient patrolling configurations that consider both racial and territorial fairness.

Identified research gap: Public districting, particularly in areas such as school and police districting, involves the design and adjustment of administrative boundaries to ensure adequate service provision and resource allocation. While numerous studies address the optimization of district boundaries, the majority of existing approaches rely on static, single-period models that assume fixed population and demand patterns. However, real-world conditions are subject to significant temporal changes and uncertainties, such as demographic shifts, migration, or unexpected events. For example, the demand for schools or police presence may fluctuate due to birth rate trends, urban development, or socio-economic changes. Yet, existing districting models rarely account for these dynamics. Similarly, uncertainties in future population distributions or crime patterns are rarely incorporated into the planning process, despite their critical impact on long-term effectiveness and robustness of districting decisions. Thus, most public districting plans are not designed to adapt to future developments or to hedge against unforeseen changes, limiting their practical relevance. Addressing these gaps would require the integration of multi-period planning and stochastic modeling techniques, enabling decision-makers to create more robust and future-proof districting solutions.

1.1.3 Healthcare

A good overview and literature review for districting in healthcare can be found in Yanık and Bozkaya (2020). The authors review the districting literature in the healthcare domain to provide readers with the most relevant studies and direction for future research. They classify the healthcare districting problems into three main areas: home care services, primary and secondary healthcare services, and emergency healthcare services. They identify the special characteristics of these different areas and present the modeling approaches, assumptions, and solution methods for each of them. They limit their review mostly to studies that include traditional districting models and formulations as well as solution approaches, discuss gaps in the literature, and provide directions for future areas of research.

Blais et al. (2003) show a districting study for the Côte-des-Neiges local community health clinic in Montreal, aiming to partition a territory into six districts while respecting five districting criteria. They use a tabu search technique to solve the problem and implement the solution over two years, after which the clinic management expresses satisfaction with the results.

Enayati et al. (2020) propose a stochastic service district design model to address the service district design problem for ambulances under uncertainty. The model recommends locating ambulances to waiting sites and assigning demand zones to each ambulance at different backup levels. The objective is to maximize the expected number of covered calls while restricting the workload of each ambulance. The proposed model is evaluated through a discrete-event simulation and shows a significant improvement in mean response time and a reduction in the average workload of ambulances.

Darmian et al. (2021) addresse a districting problem based on a real-world case study involving the partitioning of residential districts for a healthcare system operation. They present a mixed-integer programming model incorporating graph theory to enforce contiguity constraints and other practical criteria. Additionally, they extend robust optimization approaches to handle uncertainty and develop an improved genetic algorithm to tackle computational complexity. The authors present extensive computational results from real-world and randomly generated instances to evaluate the models' applicability, robustness measures, and solution approach. They also examine a hierarchical districting approach for decision-makers to obtain districting decisions at various levels of health services and perform sensitivity analyses on key parameters to provide managerial insights for practitioners.

Identified research gap: Healthcare districting involves designing and adjusting the boundaries of service districts to ensure equitable access to medical resources and efficient allocation of healthcare facilities. While recent research has introduced advanced optimization and spatial modeling techniques, the majority of existing approaches still rely on single-period models that assume fixed demand and population patterns. However, real-world healthcare needs are subject to temporal changes, such as demographic shifts, disease outbreaks, and evolving service requirements, as well as uncertainty in demand. For example, the number of patients requiring care in a region can fluctuate due to seasonal epidemics, migration, or policy changes, and such variability is often difficult to predict with certainty. Despite these challenges, most healthcare districting models do not explicitly account for the dynamic evolution of service needs over time or the stochastic nature of healthcare demand distribution. Hence, the district plans may lack adaptability, potentially leading to inefficiencies or inequities in healthcare provision. Addressing these gaps would require the integration of multi-period planning and stochastic modeling methodologies to create better healthcare districts.

1.1.4 Commercial Districting

Salazar-Aguilar et al. (2011) present the first multi-objective approach for the commercial territory design problem with connectivity constraints. One of the main results of their analysis is that the reduction of tolerance in the balancing constraints changes the efficient Pareto fronts, and the increase in the number of districts makes it harder to achieve balance with respect to the number of customers.

Ríos-Mercado and López-Pérez (2013) present a mixed-integer linear program for a commercial districting problem with disjoint assignment requirements where some specified units must be assigned to different territories, and similarity with an existing plan is required. An optimal design minimizes territory dispersion and similarity with the existing design. They propose a solving procedure that is based on an iterative cut generation strategy within a branch-and-bound framework for large-scale instances.

Bender and Kalcsics describe a multi-period service territory design problem and computational experiments in Bender et al. (2016) and Bender and Kalcsics (2020). They introduce a planning problem related to field service workforce in the context of service territory design applications. The problem consists of partitioning customers into service territories and scheduling customer visits throughout a multi-period planning horizon. They focus on the scheduling subproblem and propose a mixed integer programming model along with a location-allocation heuristic. The authors conduct extensive experiments on real-world instances and find that the proposed heuristic yields high-quality solutions.

Bender et al. (2020) propose a novel two-stage districting approach to solve the problem of assigning drivers and vehicles to customers for parcel delivery while taking into account service consistency and daily demand fluctuations. They present three models for the first stage problem and conduct a case study based on a real-world data set to compare the models and analyze the effects of different factors on the delivery tours and driver workload balance. The results show that only a few adaptations of the districts are necessary for the second stage to achieve feasible daily delivery tours. They also analyze the effects of fleet heterogeneity, driver type, depot location, and planning horizon.

Álvarez-Miranda and Pereira (2021) propose a hybrid method for designing delivery zones to improve the quality of express delivery services. The method combines a preprocessing step, a heuristic for generating delivery zone candidates, and a mathematical model to obtain a final territorial design. They test the method using a case study of a Chilean courier company with low service fulfillment in express deliveries. The results show an improvement of 12 percentage points in the ability to meet conditions associated with express deliveries compared to the current situation, highlighting the validity of the method and its potential impact on critical service factors.

Ríos-Mercado et al. (2021) address a districting problem related to the distribution of bottled beverage products in a commercial firm. They propose a heuristic procedure based on a location-allocation scheme to minimize a dispersion function while addressing multiple-activity balancing and contiguity constraints. The proposed method consists of two phases: determining centroids and allocating units to centroids. They also incorporate a local search phase to improve solution quality. Their approach outperforms existing heuristics regarding feasibility and solution quality while requiring less computational effort.

Identified research gap: Commercial districting – such as the design of sales, delivery, or service territories – is a critical task for ensuring market coverage, operational efficiency, and customer satisfaction. While many studies focus on optimizing district boundaries based on current demand, travel costs, or workload balance, most existing approaches rely on static, single-period models that assume fixed market conditions and customer distributions. In practice, however, commercial environments are highly dynamic: customer bases evolve, market trends shift, and demand patterns are subject to uncertainty due to factors like seasonality, economic fluctuations, or competitor actions. Despite these realities, the literature rarely incorporates multi-period planning or explicitly models uncertainty in commercial districting decisions. Hence, districts may quickly become suboptimal or fail to adapt to unforeseen changes. Recent advances in multi-stage stochastic optimization, such as those applied in energy and transportation planning, demonstrate the value of integrating both temporal and stochastic aspects. Tackling this gap would enable businesses to design more robust and adaptable districting solutions that can better resist market volatility and long-term changes.

After outlining the introduction, motivation, and relevant literature, we now turn to the identification of the central research gap and the clarification of the thesis scope. A key gap in the existing literature is that multi-period and uncertainty aspects are typically examined in a highly application-specific way, with most studies focusing exclusively on the unique requirements of their respective domains, such as healthcare, public services, or commercial districting. What is largely missing, however, is a unifying and comparative perspective that systematically analyzes the influence of multi-periodicity and uncertainty across different districting contexts. This thesis addresses this gap by providing a comprehensive cross-domain investigation of these factors, thereby offering broadly applicable insights that extend beyond individual application areas.

1.2 Research Gap and Scope of this Thesis

In the field of districting, mathematical models and solution methods have evolved significantly over the past few years, with an increasing focus on challenges such as multi-period planning and uncertainty. However, most methodologies still address these aspects separately, typically focusing on either multi-period districting or districting under uncertainty. Table 1.1 provides an overview that highlights the fragmented understanding in the literature regarding the joint impact of these two dimensions. Additionally, there has been limited exploration of approaches that systematically integrate both aspects.

Moreover, existing studies usually treat multi-period and uncertainty aspects in a highly application-specific manner, tailoring models and analyses to the unique requirements of individual domains. To date, no comprehensive study has combined these dimensions to analyze their collective influence across districting contexts as illustrated by the individual examinations in the listed application-related references. This reveals a clear need for models and analyses that jointly consider multi-periodicity and uncertainty, as well as for research that provides cross-domain insights into their effects on districting outcomes.

In addition, the evaluation methods and tools used to evaluate the impact of various factors over multiple periods and stochasticity are often inadequately addressed. As shown in Table 1.1, measures of the value of additional information on multiple periods and uncertainty (VMPS, VSS, DVSS, and EVPI) are only examined in two papers, one of which represents a major contribution of this dissertation. Although some frameworks provide insights into the advantages and disadvantages of different assessment instruments, they typically lack an exhaustive comparative analysis that can identify best practices. This gap presents an opportunity to establish a more organized evaluation framework that effectively measures the impact of multiple factors on districting performance over time.

Furthermore, there is a lack of comprehensive computational analyses and case studies that illustrate real-world applications incorporating both multi-period and uncertainty. While some research has delved into these elements individually, the intersection of multi-period approaches with uncertainty remains significantly underrepresented. The computational experimentation in the listed references is highly heterogeneous because the publications span from the 1960s to the 2020s. This thesis seeks to bridge this gap and reveal the practical implications of considering these combined factors within districting, especially when examined consistently in computational analyses focusing on multi-periodicity and uncertainty.

From a methodological perspective, this thesis develops and analyzes both exact and heuristic techniques for solving these problems. It focuses on developing and analyzing mathematical models that address the complexities introduced by the challenges associated with multi-periods and uncertainty. Through this exploration, the thesis aspires to make substantial contributions to the existing literature on districting and set a foundation for future research.

Motivated by the research gaps identified in Section 1.1 and explored above, we formulate four major research questions (RQ):

Source	exact model	heu-ristic	un-certainty	multi-period	application specific	VMPS	EVPI	(D)VSS
Hess et al. (1965)	x	x			x			
Owen and Grofman (1988)	x		x		x			
Blais et al. (2003)	x	x			x			
Caro et al. (2004)	x				x			
Ricca et al. (2008)		x			x			
Salazar-Aguilar et al. (2011)	x	x			x			
Lei et al. (2012)	x	x	x		x			
Ríos-Mercado and López-Pérez (2013)	x	x			x			
Fazlollahi et al. (2014)	x	x		x	x			
Lei et al. (2015)	x	x		x	x			
Bender et al. (2016)	x	x		x	x			
Lei et al. (2016)	x	x	x	x	x			
Bruno et al. (2017a)	x	x			x			
Bruno et al. (2017b)	x				x			
Bender (2017)	x	x		x	x			
Bender et al. (2018)	x			x	x			
Kalcsics and Ríos-Mercado (2019)	x	x	x	x	x			
Yanık et al. (2019)	x			x	x			
Enayati et al. (2020)	x		x		x			
Bender et al. (2020)	x	x		x	x			
Dugošija et al. (2020)	x				x			
Yanık and Bozkaya (2020)	x	x	x		x			
Goderbauer (2020)	x	x			x			
Darmian et al. (2021)	x	x	x		x			
Arredondo et al. (2021)	x	x			x			
Álvarez-Miranda and Pereira (2021)	x	x			x			
Ríos-Mercado et al. (2021)	x	x			x			
Gurnee and Shmos (2021)	x	x			x			
Diglio et al. (2020)	x		x				x	x
Diglio et al. (2021)	x	x	x					
Diglio et al. (2022)	x	x	x					
Lespay and Suchan (2022)	x	x		x	x			
Liberatore et al. (2022)	x				x			
Pomes et al. (2025)	x	x	x	x		x	x	x

Table 1.1: Comprehensive overview of existing literature and identified research gaps.

RQ1 What aspects and additional problem characteristics can be captured by integrating multi-period considerations and uncertainty into districting models, and how does this enrich the modeling of real-world districting problems compared to traditional single-period deterministic approaches?

RQ2 What evaluation tools and methodologies can measure the impact of multi-period and uncertainty factors on districting performance?

RQ3 How does considering multiple periods and stochasticity affect key performance indicators, such as solution quality, robustness, and fairness, of the resulting districting plans?

RQ4 How can a real-world case study be used to demonstrate and analyze the practical effects of integrating multi-period considerations and uncertainty in districting models?

In conclusion, significant research gaps exist regarding the integration of multi-period considerations and uncertainty in districting methodologies. Additionally, enhanced evaluation tools and practical applications in real-world scenarios are needed. This thesis addresses these gaps by providing models, in-depth analyses, and case studies, which help advance our understanding of how these factors interact within the field of districting.

1.3 Computational Environment and Experimental Setup

Extensive computational experiments are examined in sections Chapter 3 – Chapter 6 to address the identified research gaps. All experiments conducted in this thesis were executed on a machine equipped with an Intel(R) Core(TM) i7-7700 processor running at 3.60 GHz, 64 GB of RAM, and the Windows 10 Pro 64-bit operating system. The models and heuristic algorithms were implemented in Python 3.7, utilizing IBM ILOG CPLEX 12.10 as the optimization solver.

The computational experiments are crucial in supporting the answers to the formulated research questions. Through systematic testing and analysis of the developed models and algorithms, we can quantitatively assess the influence of multi-periodicity and uncertainty (RQ1, RQ3), evaluate the effectiveness of different performance metrics and evaluation methodologies (RQ2), and demonstrate the practical relevance of our approaches via real-world case studies (RQ4). The results from these experiments provide the empirical foundation for the insights and conclusions drawn in this thesis.

1.4 Structure of this Thesis

This thesis aims to improve the understanding of districting through a structured approach that addresses a multi-period planning horizon and uncertainties. The various sections explore these topics as shown in Figure 1.3.

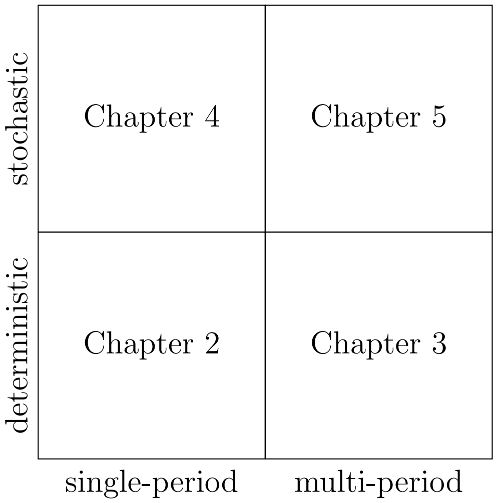


Figure 1.3: Overview of thesis organization along the two dimensions of multi-periodicity and uncertainty.

This thesis is structured as follows:

Chapter 1 In the introductory section, Introduction, we present the motivation behind our focus on districting, highlighting its significance and relevance across different contexts. We also discuss the importance of uncertainty and multi-period perspectives, illustrating these aspects in each context.

Chapter 2 In Deterministic Districting, we start with fundamental definitions and districting criteria relevant to all districting models presented in this thesis. Then, we explore the definitions and characterizations associated with deterministic single-period districting problems.

Chapter 3 In Multi-Period Districting, incorporating a time component enables us to examine multi-period deterministic districting. After reviewing the relevant literature, we present a deterministic multi-period districting model that serves as the basis for the stochastic multi-period model discussed later. This model contains reassignments between periods. Following this, we address the extension for fixed centers over time and introduce a new parameter for adding savings. Afterward, we examine how the dynamics of time affect districting decisions and outcomes by calculating the value of the multi-period solution and its relation to specific input parameters. We provide an analysis to examine the linear correlation between the demand fluctuations and the value of the multi-period solution at different granular levels: first, the number of territorial units with highly fluctuating demand, second, the average demand fluctuations, and finally, the average sum of demand fluctuations in each district.

Chapter 4 In Stochastic Districting, we introduce the concept of uncertainty. First, we provide a brief literature review and discuss an existing two-stage stochastic districting model. Then, we explore potential modifications to the model that could enhance its utility. We present an alternative two-stage stochastic districting model, which also serves as the foundation for the stochastic multi-period model discussed later. In the additional analysis, we focus on single-period stochastic districting and investigate how uncertainty affects the districting process by evaluating the value of the stochastic solution and its relation with the relative demand fluctuations in each district.

Chapter 5 In Multi-Period Stochastic Districting, both key dimensions – time and uncertainty – are integrated through the presentation of a multi-period stochastic districting model. Additionally, we propose a relax-and-fix heuristic to efficiently solve the resulting optimization problems. Extensive computational experiments are conducted to analyze the performance of the model and the heuristics. To assess the need to incorporate both stochasticity and multiple periods, we employ two values: the dynamic value of the stochastic solution and the expected value of perfect information.

Chapter 6 In Districting for Home Health Care: Case Studies, the models introduced in the previous sections are applied to practical case studies within the domain of home healthcare. Detailed numerical examples are presented and thoroughly analyzed to examine the effects of stochasticity and a multi-period planning horizon in a realistic setting. This allows for an in-depth examination of how uncertainty and time influence decision-making in home healthcare districting.

Chapter 7 Finally, the main findings of this work are summarized, and an outlook is provided in Conclusion, Outlook and Further Research, highlighting potential directions and opportunities for future research.

Chapter 2

Deterministic Districting

In this chapter, we present the basic definitions and the foundations of districting, which are relevant to this thesis. We focus on single-period deterministic districting (Figure 2.1) and summarize the first single-period deterministic districting model, discussing its limitations. We will also propose potential solutions to address the issue of infeasibility and suggest extensions to enhance the model.

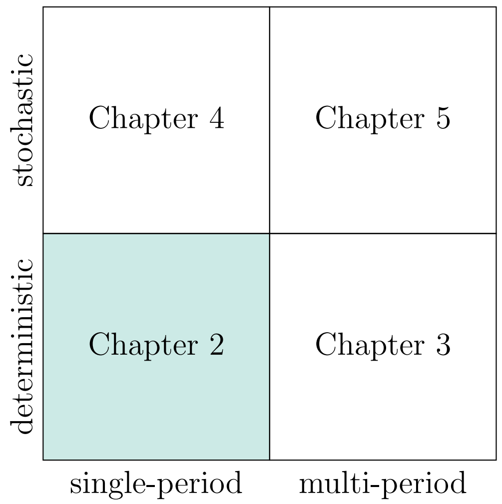


Figure 2.1: Research Focus of Chapter 2: Single-period deterministic districting.

This chapter is organized as follows:

- Section 2.1 presents the formal definitions for the districting context that serve as the foundation for the subsequent chapters.
- In Section 2.2, the first mathematical formulation of a single-period deterministic districting follows. An extension of this first model is also given, which ensures feasibility. This formulation is the basis for most models in the literature.

2.1 Basic Definitions

In the following section, we present the formal definitions that serve as the foundation for the subsequent chapters. An introduction to districting can be found in Laporte et al. (2019) and Ríos-Mercado (2020). The definitions provided here are based on those in Kalcsics and Ríos-Mercado (2019). A tailored version that is sufficient for the scope of this work is outlined in the subsections below:

- The essentials of districting are explained in Section 2.1.1.
- Various districting criteria are discussed in Section 2.1.2. Several examples illustrate the different criteria.

2.1.1 Foundations of Districting

We start with the definition of the territory units in our problem.

Definition 1 (Territory Units)
A Territory Unit (TU) represents a geometric object in the plane. The distance between two TUs $i, j \in I$ is denoted as $c_{ij} = c(i, j)$.

For non-point objects, distances are determined based on representative points, such as the midpoint of a street, the centroid of a polygon, or the distance from one surface to another. Thus, within this thesis, a TU is consistently represented as a geometric point.

Next, we define the activity measures, which are values for each TU that we focus on in the districting plan.

Definition 2 (Activity Measure)
An activity measure of a TU is a quantifiable attribute of the TU that reflects relevant characteristics. This attribute can be either deterministic or stochastic. For each TU $i \in I$, the activity measure is denoted by d_i.

Typical activity measures are demand, service time, estimated sales potential, or number of voters.

In this thesis, we focus only on one activity measure. Later, we incorporate the multi-period and stochastic influences into this measure. However, considering multiple activity measures typically leads to multi-criteria optimization, particularly when the measures conflict with one another.

Definition 3 (Districts)
A district D_k, for $1 \leq k \leq p$, is a subset of TUs. The activity measure of a district is the sum of the activity measures of its TUs. The size of district D_k is defined as $d(D_k) = \sum_{i \in D_k} d_i$.

The total number of districts, p, is typically specified as an input parameter in both the literature and this thesis. However, depending on the context, p can also be treated as a decision variable.

Definition 4 (Districting Problem)
A districting problem is a combinatorial optimization task that involves partitioning a set $I = \{1, \ldots, n\}$ of TUs into districts. The aim is to design districts that satisfy predefined planning criteria in accordance to the activity measure(s).

Various criteria for districting are discussed in Section 2.1.2.

Definition 5 (Solution of a Districting Problem)
A solution to a districting problem is a districting plan that assigns each TU to exactly one of the p districts, such that all constraints are satisfied.

An optimal solution is a feasible solution that achieves the best possible value of the objective function, i.e., it either minimizes or maximizes the objective, depending on the problem formulation.

In districting, the objective function can be formulated in multiple ways. Often, the problem is modeled as a p-median problem where the distances are minimized. However, covering models with a maximization objective function are also possible.

A TU needs to serve as a representative to clearly identify a set of TUs as a district. This facilitates the representation and modeling of the districts.

Definition 6 (Center of a District)
The center of a district is a TU within a district that represents the district. The representative TU of a district is always assigned to itself.

2.1.2 Districting Criteria

This section explains various criteria for districting and offers examples for illustration. The examples in this section utilize a graph-theoretical representation of districting, featuring nodes and edges.

Definition 7 (Complete and Exclusive Assignment)
Let I be the set of all territorial units (TUs), and let $D_1, \ldots, D_p$ denote the districts. A complete and exclusive assignment *requires that:*

$$D_1 \cup D_2 \cup \cdots \cup D_p = I \quad \textit{(completeness)},$$
$$D_l \cap D_k = \varnothing \quad \textit{for all } l \neq k,\ 1 \leq l, k \leq p \quad \textit{(exclusivity)}.$$

According to this definition, each TU is assigned to exactly one district:

- Completeness: Every TU is part of a district (no TU is excluded).
- Exclusivity: No TU is in more than one district (districts are disjoint).

In other words, the collection of districts $\{D_1, \ldots, D_p\}$ forms a partition of the set I of TUs. The requirement of exclusivity is sometimes also called integrity.

For political districting, these criteria are obvious. In sales territory design, unique allocations result in transparent responsibilities for the sales force, avoiding contentions and allowing the establishment of long-term customer relations.

The following Figure 2.2 shows an example of a complete assignment. The TU, which represents the district's center, is shown in bold.

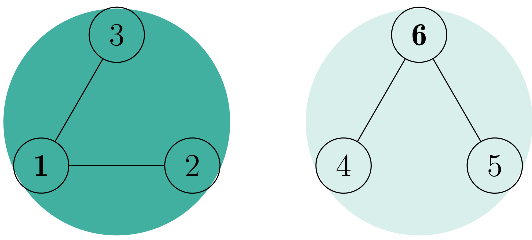


Figure 2.2: Complete and exclusive assignments.

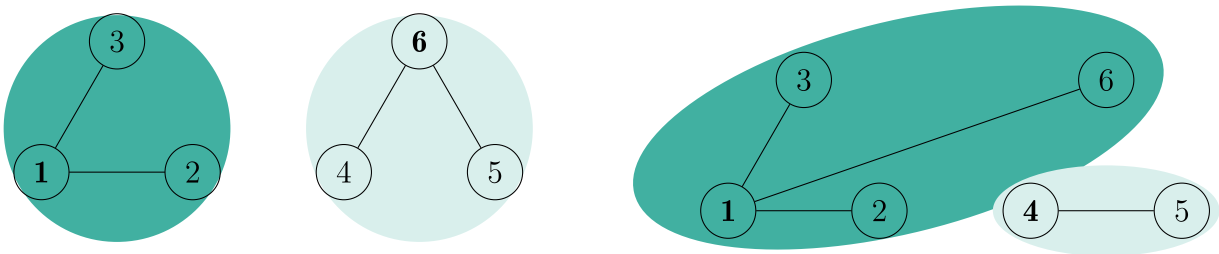
Since districting aims at establishing equal districts, it is important to ensure that all districts have a balanced activity measure. A district's activity measure (or size) is calculated by adding up the activity measures of all the TUs it contains (Definition 2). Unfortunately, due to exclusive assignment (Definition 7) and the discrete nature of the problem, it is generally impossible to achieve perfectly balanced districts. Therefore, the relative percentage deviation of the district sizes from their average size must be calculated to measure balance. The larger this deviation, the more unbalanced the district is (Kalcsics et al., 2005).

Definition 8 (Balance)
Balance is defined as the relative deviation of a district's activity measure from the average activity measure of all districts:

$$bal(D_k) = \frac{|d(D_k) - \mu|}{\mu} \qquad with \qquad \mu = \frac{\sum_{i \in I} d_i}{p}$$

A low value of $bal(D_k)$ reflects a good balance. Conversely, a high value suggests that the district is unbalanced.

Assuming that each node has the same activity measure, Figure 2.3 shows an example of a balanced and an unbalanced solution.



(a) Balanced

(b) Unbalanced

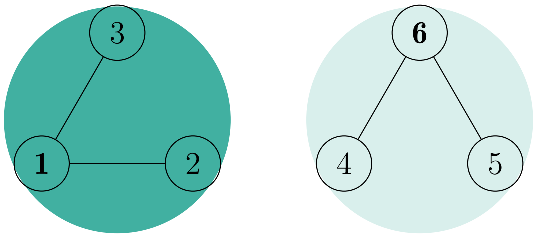
Figure 2.3: Comparison of two assignments: balanced assignment and an unbalanced assignment.

Definition 9 (Compactness)
A district is considered geographically compact if it is somehow round-shaped and undistorted (Kalcsics and Ríos-Mercado, 2019).

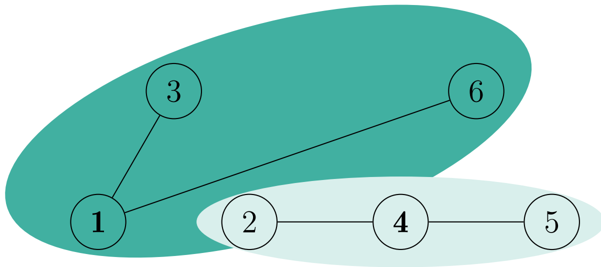
The motivation behind creating compact districts is to prevent gerrymandering and to reduce the travel distances within the districts. Although the idea of compactness may seem intuitive, there is currently no strict definition of compactness, and it strongly depends on the geometric representation of TUs.

Assuming that each node has the same activity measure, Figure 2.4 illustrates an example of both a compact and a non-compact solution. From a visual perspective, it is possible that with different distributions of the activity measure, districts that seem non-compact may actually be compact.

Definition 10 (Contiguity)
Contiguity refers to geographically connected districts, such that any two points within the district can be connected through a path that remains within that district. This ensures that the district is not disconnected and that it is possible to travel between any two TUs within the district without having to leave the district (Kalcsics et al., 2005; Tasnádi, 2011; Kong et al., 2019).



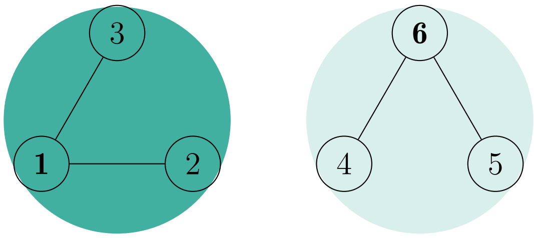
(a) Compact



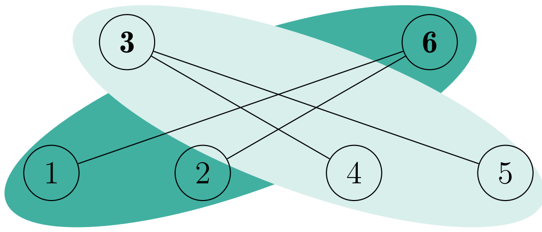
(b) Non-compact

Figure 2.4: Comparison of two assignments: compact assignment and non-compact assignment.

Figure 2.5 shows an example of a contiguous and non-contiguous solution.



(a) Contiguous



(b) Non-contiguous

Figure 2.5: Comparison of two assignments: contiguous assignment and non-contiguous assignment.

When addressing districting criteria such as balance and compactness, these factors can be incorporated into the model in two ways: either directly within the objective function or as constraints. The approach taken will depend on the specific requirements and priorities of the districting problem being addressed.

Butsch (2016) explores different representations of TUs such as polygons, lines, or points, and compares existing measures of compactness for these representations. They also develop new algorithms and approaches to improve the balance and compactness of districts, considering both geometric information and routing distances on road networks. Finally, the authors test their methods on real-world data.

2.2 Deterministic Districting Models

Following the basic definitions of districting introduced in Section 2.1, we delve into the first mathematical formulation in this section. This allows us to discuss the optimization criteria and their mathematical formulation in detail.

Hess et al. (1965) presented the first integer linear program for districting problems in 1965. The problem is modeled as a capacitated p-median facility location problem. In this formulation, capacity is determined by a fraction of the total demand, and each district's demand is allowed to deviate only by a certain proportion from the average district size. Hence, the balance is modeled here as a hard criterion in particular. However, this may lead to infeasible instances. The following section presents a modified version of the original model, which is the basis for all subsequent sections. As noted by Hess et al. (1965), the population serves as the activity measure in this context, forming the basis for evaluating district balance within the model.

The following parameters are used:

I	$= \{1, \dots n\}$ set of TUs
d_i	population of the i–th TU for $i \in I$
c_{ij}	distance between TU i and j for $i, j \in I$
p	number of districts
a	minimum allowable district population, as a percent of the average district population
b	maximum allowable district population, as a percent of the average district population

With the decision variables:

$$x_{ij} \quad = \begin{cases} 1, & \text{if the } i\text{–th population unit is assigned to the district represented by TU } j, \\ 0, & \text{otherwise,} \end{cases} \qquad (i, j \in I)$$

The assumption is that only TUs can be district centers.

Note that $x_{ii} = 1$ for some $i \in I$, indicates that TU i is assigned to itself, which also means that it is selected as the representative TU of its district.

The initial model formulation of the Districting Problem (DP) by Hess et al. is presented as follows:

DP:

$$\min \quad \sum_{i \in I} \sum_{j \in I} c_{ij}^2 d_i x_{ij} \tag{2.1}$$

$$\text{s. t.} \quad \sum_{j \in I} x_{ij} = 1 \qquad \forall i \in I \tag{2.2}$$

$$\sum_{i \in I} x_{ii} = p \tag{2.3}$$

$$\sum_{i \in I} d_i x_{ij} \geq \frac{a}{100} \sum_{i \in I} \frac{d_i}{p} x_{jj} \qquad \forall j \in I \tag{2.4}$$

$$\sum_{i \in I} d_i x_{ij} \leq \frac{b}{100} \sum_{i \in I} \frac{d_i}{p} x_{jj} \qquad \forall j \in I \tag{2.5}$$

$$x_{ij} \in \{0,1\} \qquad \forall i,j \in I \tag{2.6}$$

The objective function 2.1 ensures that the sum of the weighted squared distances is minimized. Constraints 2.2 ensure that each TU must be fully allocated. This, together with Constraints 2.6, ensure complete and exclusive allocation (Definition 7). The number of districts is defined in Constraint 2.3. Constraints 2.4 and 2.5 specify the limits of the set balance, i.e. how much demand or population has to be assigned to each district (Definition 8).

The criteria compactness and contiguity (Definition 9 and Definition 10) defined in Section 2.1.2 are not explicitly formulated as separate hard criteria. In the objective function, the squared distances are minimized to promote compactness. Still, compactness and contiguity cannot be guaranteed.

This formulation is notable due to the presence of balance constraints with lower limit a and upper limit b, which do not need to be symmetrical with the average population in each district $\frac{\sum_{i=1}^{n} d_i}{p}$. Following the publication by Hess et al., the literature has predominantly utilized a symmetric variant. Therefore, two more parameters are required:

μ $= \frac{\sum_{i \in I} d_i}{p}$ average population in each district

α maximum allowed deviation in each district from the average district population μ

The extended model formulation of the Districting Problem with Symmetric Balance (DPSB) is presented as follows:

DPSB:

$$\min \quad \sum_{i,j \in I} c_{ij}^2 d_i x_{ij} \tag{2.7}$$

s. t. $(2.2), (2.3), (2.6)$

$$(1-\alpha)\mu x_{jj} \leq \sum_{i \in I} d_i x_{ij} \qquad \forall j \in I \tag{2.8}$$

$$\sum_{i \in I} d_i x_{ij} \leq (1+\alpha)\mu \qquad \forall j \in I \tag{2.9}$$

$$x_{ij} \leq x_{jj} \qquad \forall i,j \in I \tag{2.10}$$

Constraints 2.8 and 2.9 ensure that the population in each district falls in an acceptable, symmetric range. Additionally, Constraints 2.10 ensure that TUs can only be allocated to centers. We can combine these constraints with 2.9 to get

$$\sum_{i \in I} d_i x_{ij} \leq (1+\alpha)\mu x_{jj} \quad \forall j \in I$$

Finding a feasible solution for problems that can be modeled with Hess' model (DP) is not always possible. The number of districts p cannot be smaller than the number of TUs. Otherwise, Constraint 2.3 can not be fulfilled. This constraint is easy to check, but the feasibility of the balancing Constraints 2.4 and 2.5 are not easy to see in advance.

To ensure at least one feasible solution, the balancing constraint can be adjusted by adding penalty costs to the objective function. This means that if a balanced solution cannot be found, any deviation from the allowed range will incur a penalty in the objective function. To achieve this, two additional variables are introduced to penalize shortages and surpluses in each district:

ψ_j demand surplus in the district represented by TU $j \in I$

φ_j demand shortage in the district represented by TU $j \in I$

Parameters for the surplus and shortage penalties also need to be defined:

g_j unit penalty for surplus at the district represented by TU $j \in I$

h_j unit penalty for shortage at the district represented by TU $j \in I$

The model presented by Hess et al. can be extended to the Extended Districting Problem guaranteeing feasibility :

EDP:

$$\min \quad \sum_{i,j \in I} c_{ij} d_i x_{ij} + \sum_{j \in I} \Big(g_j \psi_j + h_j \varphi_j \Big) \tag{2.11}$$

$$\text{s. t.} \quad (2.2), (2.3), (2.6), (2.10)$$

$$(1-\alpha)\mu x_{jj} \leq \sum_{i \in I} d_i x_{ij} - \psi_j + \varphi_j \qquad \forall j \in I \tag{2.12}$$

$$\sum_{i \in I} d_i x_{ij} - \psi_j + \varphi_j \leq (1+\alpha)\mu \qquad \forall j \in I \tag{2.13}$$

$$\psi_j \geq 0 \qquad \forall j \in I \tag{2.14}$$

$$\varphi_j \geq 0 \qquad \forall j \in I \tag{2.15}$$

The objective function 2.11 minimizes the weighted distance (not squared here). Each unit of length has the same priority, making longer distances less important. Again, we are searching for compact solutions by minimizing distances. Additionally, there is a penalty term due to unbalanced districts from Constraints 2.12 and 2.13. Furthermore, Constraints 2.14 and 2.15 state that ψ and φ must be non-negative. Compactness is addressed in the first part of the objective function, while balance is addressed in the second part.

In the formulation of the model for EDP, two criteria of balance and compactness are included in the objective function. Depending on the available data, a compact solution may already be balanced. However, these two criteria can often be contradictory, requiring consideration of which solution best achieves a balanced outcome based on both criteria. Although two criteria are included, both are derived from a sum of distances and costs, which makes them comparable.

Due to its NP-hardness (Kalcsics and Ríos-Mercado, 2019), larger instances quickly become unsolvable within an acceptable time. When solving the exact model proves to be difficult or time-consuming, it is often more efficient to use heuristics, especially for problems involving a large number of TUs and districts. Effective heuristics can provide satisfactory solutions within a reasonable amount of computational time.

p-median heuristics can be applied to districting problems that can be represented using the extended version. However, this method may lead to unbalanced solutions or incur high penalty costs. For districting, it is crucial to weigh the trade-off between allocation costs and balance or the costs associated with unbalanced districts.

Hess et al. (1965) use a location-allocation heuristic to solve their problem. The location-allocation method involves two phases that are executed alternately. In the first phase, known as the location phase, the centers of the districts are determined. All TUs are assigned to these district centers in the next phase. It is then checked whether it is possible to select better centers. Additionally, new centers are determined if they differ from the previous ones. The stopping criterion is fulfilled when no better centers can be found.

Kalcsics et al. (2005) develop a basic districting model and present two solution approaches. A central idea in their work is introducing a geometric, computational approach to districting, specifically the *Line Partitioning Algorithm*. This method recursively subdivides the set of basic areas (e.g., points representing TUs) using straight lines, thereby generating a partition of the area into districts. The approach is particularly well-suited for large-scale practical problems, as it uses geometry and can be integrated with Geographic Information Systems. The authors also discuss various extensions to the basic model and its applications. The Line Partitioning Algorithm is described in more detail in Kalcsics (2006).

Chapter 3

Multi-Period Districting

This section introduces a new multi-period districting problem, which considers a planning horizon with more than one period (Figure 3.1). The underlying model has already been published in Pomes et al. (2025). In this thesis, we extend the previous work by providing a detailed analysis of the model. Computational analyses are conducted to identify when the multi-period model provides advantages over static approaches. This is achieved through detailed computational experiments that perform linear correlation analysis at different levels of granularity in order to provide a comprehensive assessment of the relation between demand fluctuations and values of multi-period solutions.

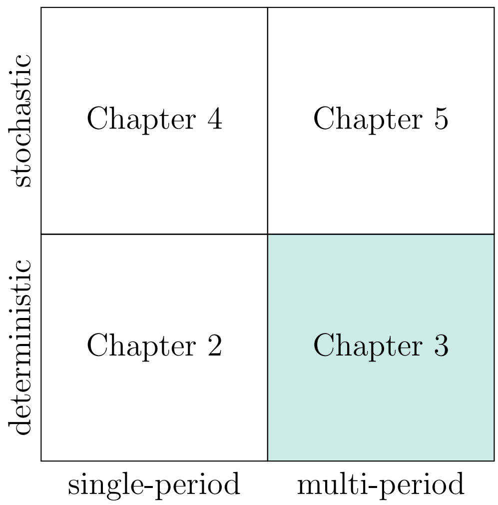


Figure 3.1: Research Focus of Chapter 3: Multi-period deterministic districting.

The main difference between single-period and multi-period districting problems is the possibility of adjustments between periods. In single-period districting, each allocation is fixed, while multi-period districting allows for modifications, called redistricting or reassignments, if necessary.

If the allocation of TUs does not impact the districts themselves, for example, in organizational districts without a specific central institution, the districts can be planned separately for each period. However, in some cases, districts should not be planned independently. For instance, in the healthcare sector, where caregivers should not rotate too frequently, or in the postal sector, where postmen can navigate their district more efficiently if they are already familiar with it, it makes sense to adjust existing districts from period to period rather than planning them from scratch.

This chapter is organized as follows:

- In Section 3.1, a literature review focusing on multi-period deterministic districting problems is provided.
- Section 3.2 presents the multi-period deterministic model.
- Section 3.3 evaluates the benefits of using a multi-period deterministic model for multi-period districting problems.

3.1 Related Literature

Fazlollahi et al. (2014) present a method to reduce the computational load of multi-period optimization models for energy system districts. They use a k-means clustering algorithm to reduce energy demand profiles into typical periods that preserve significant yearly characteristics. The authors divide each period into segments to reduce complexity while respecting peak demands. The method is demonstrated through two case studies, showing that a limited number of typical periods is sufficient to accurately represent an entire equipment's lifetime.

Bender et al. (2016) and Bender (2017) introduce a multi-period service territory design problem that primarily focuses on scheduling. In this problem, customers must be visited multiple times within the planning horizon, and a location-allocation heuristic based on Hess et al. (1965) is utilized. Customers have different needs, and service providers possess various skills. The challenge is to balance customer requirements with stable service districts and working time-related objectives for service providers. Approximately 20% of customer visits may need to be rescheduled in the short term. The approach involves solving the districting problem first and then addressing the scheduling subproblem by designing week and day clusters for each service territory.

Bender et al. (2018) presents a branch-and-price algorithm to solve the problem of scheduling customer visits in multi-period service territory design. They find that the algorithm is effective for up to 55 customers and a four-week planning horizon. They use acceleration techniques and a fast pricing heuristic to reduce the symmetry inherent to the problem.

Yanık et al. (2019) propose a multi-period multi-criteria districting problem in a healthcare context, specifically focusing on patient allocation to general practitioners. The number of patients over the time horizon is predetermined. Additionally, patients can be partially assigned, not solely in a binary manner. A similarity measure is introduced and integrated into the target function to ensure minimal changes between periods.

Lespay and Suchan (2022) suggest a territory design for the multi-period vehicle routing problem with time windows, which is solved using a mixed-integer linear program and a proposed heuristic. The algorithm yields high-quality solutions within moderate running times, and a methodology is proposed in which the territories computed by the proposed heuristic on the past demand of one month are used for the operational routing during the following month. The territories obtained with the methodology lead to better service levels with fewer vehicles.

3.2 Multi-Period Districting Model

Based on the EDP presented in Section 2.2, the model is now extended to a multi-period model with T periods. Although solving each period separately in the multi-period view of the problem is possible, doing so could result in selecting new centers for each period and constant reassignment. However, this approach may not be practical in many applications. For instance, when the center represents a facility, selecting new centers in each period is impossible. Therefore, we allow reassignments in our model.

The solution aims to provide a partitioning plan for the entire time horizon with minimal reassignments, or only those necessary. To achieve this, a penalty factor is implemented in the following multi-period districting problem, which must be paid if reassignments occur between two periods. In some applications, the penalty factor represents real costs incurred for reassigning TUs. For instance, a nurse providing home healthcare may require additional time to become familiar with a new patient in their area, which is not necessary for a patient they already know.

In some cases, a solution that is not possible in a previous period due to balancing issues may become balanced in the next period. However, this solution might not be chosen because it would result in additional costs for reassignments. To address this, a new parameter is introduced called s, representing the *savings* that would be received if a more compact solution is chosen in the current period. This would make the compact solution more attractive. If TU i is assigned to the district represented by TU j in the last period and is now being reassigned, then s_{ij} can be saved by removing TU i from the district represented by TU j. This particular parameter is further explained in Section 3.2.2.

Each district has a representative TU center (Definition 6), which can change from one period to another. Therefore, when a TU is assigned to a district, it is essentially assigned to its representative only for a specific period. It is important to note that the TUs are still assumed to be exclusively assigned to districts (Definition 7). To develop a mathematical formulation for the problem, the following sets are used:

I set of TUs

TP set of time periods, $TP = \{1, \ldots, T\}$

Additionally, the parameters that define the problem are introduced:

p_t number of districts in period $t \in TP$

d_{it} demand of TU $i \in I$ in period $t \in TP$

μ_t reference value for the demand assigned to each district in period $t \in TP$

This value is defined as $\mu_t = \frac{1}{p_t} \sum_{i \in I} d_{it}$ $(t \in TP)$, i.e., the exact demand assigned to each district if the districts are perfectly balanced – which may not be possible due to the single assignment assumption

α allowed deviation for the demand assigned to each district in each period w.r.t. the reference value in that period

c_{ij} initial cost for assigning TU $i \in I$ to TU $j \in I$ in the first period

r_{ijt} cost for reassigning TU $i \in I$ to TU $j \in I$ in (the beginning of) period $t \in TP \setminus \{1\}$

s_{ijt} saving for removing TU $i \in I$ from the district represented by TU $j \in I$ in (the beginning of) period $t \in TP \setminus \{1\}$

g_{jt} unit penalty for surplus at the district represented by TU $j \in I$ in period $t \in TP$ w.r.t. the maximum deviation stated by α

h_{jt} unit penalty for shortage at the district represented by TU $j \in I$ in period $t \in TP$ w.r.t. the maximum deviation stated by α

The multi-period districting problem underlying this work can be formulated using the following decision variables:

$$x_{ijt} \quad = \begin{cases} 1, & \text{if TU } i \text{ is assigned to TU } j \text{ in period } t \\ 0, & \text{otherwise} \end{cases}$$

$$(i, j \in I,\, t \in TP)$$

$$v_{ijt} \quad = \begin{cases} 1, & \text{if TU } i \text{ is reassigned to the district represented by TU } j \text{ in period } t \\ 0, & \text{otherwise} \end{cases}$$

$$(i, j \in I,\, t \in TP \setminus \{1\})$$

$$w_{ijt} \quad = \begin{cases} 1, & \text{if TU } i \text{ is removed from the district represented by TU } j \text{ in period } t \\ 0, & \text{otherwise} \end{cases}$$

$$(i, j \in I,\, t \in TP \setminus \{1\})$$

ψ_{jt} demand surplus in the district represented by TU j in period t, $j \in I$, $t \in TP$

φ_{jt} demand shortage in the district represented by TU j in period t, $j \in I$, $t \in TP$

Note that $x_{iit} = 1$ for some $i \in I$ and $t \in TP$, indicates that TU i is assigned to itself in period t, which also means that it is selected as the representative TU of its district in that period.

Definition 11 (Multi-Period Solution)
A multi-period solution refers to the outcome of a multi-period model, encompassing all decision variables associated with the solution. We denote the objective value of this solution as MP.

Considering the above parameters and decision variables, an optimization model can now be formulated for the Multi-period Districting Problem (MPDP):

MPDP:

$$\begin{aligned} \min \quad & \sum_{i \in I} \sum_{j \in I} c_{ij} d_{i1} x_{ij1} \\ & + \sum_{t \in TP \setminus \{1\}} \sum_{i \in I} \sum_{j \in I} (r_{ijt} d_{it} v_{ijt} - s_{ijt} d_{it} w_{ijt}) \\ & + \sum_{t \in TP} \sum_{j \in I} \Big(g_{jt} \psi_{jt} + h_{jt} \varphi_{jt} \Big) && (3.1) \\ \text{s. t.} \quad & \sum_{j \in I} x_{ijt} = 1 && \forall i \in I,\, t \in TP \quad (3.2) \\ & \sum_{i \in I} x_{iit} = p_t && \forall t \in TP \quad (3.3) \\ & x_{ijt} \le x_{jjt} && \forall i, j \in I,\, t \in TP \quad (3.4) \\ & v_{ijt} \ge x_{ijt} - x_{ij,t-1} && \forall i, j \in I\; t \in TP \setminus \{1\} \quad (3.5) \end{aligned}$$

$$w_{ijt} \leq x_{ij,t-1} \qquad \forall i,j \in I,\ t \in TP \setminus \{1\} \qquad (3.6)$$
$$w_{ijt} + x_{ijt} \leq 1 \qquad \forall i,j \in I,\ t \in TP \setminus \{1\} \qquad (3.7)$$
$$(1-\alpha)\mu_t x_{jjt} \leq \sum_{i \in I} d_{it} x_{ijt} - \psi_{jt} + \varphi_{jt} \qquad \forall j \in I,\ t \in TP \qquad (3.8)$$
$$\sum_{i \in I} d_{it} x_{ijt} - \psi_{jt} + \varphi_{jt} \leq (1+\alpha)\mu_t x_{jjt} \qquad \forall j \in I,\ t \in TP \qquad (3.9)$$
$$x_{ijt} \in \{0,1\} \qquad \forall i,j \in I,\ t \in TP \qquad (3.10)$$
$$v_{ijt} \in \{0,1\} \qquad \forall i,j \in I,\ t \in TP \setminus \{1\} \qquad (3.11)$$
$$w_{ijt} \in \{0,1\} \qquad \forall i,j \in I,\ t \in TP \setminus \{1\} \qquad (3.12)$$
$$\psi_{jt} \geq 0 \qquad \forall j \in I,\ t \in TP \qquad (3.13)$$
$$\varphi_{jt} \geq 0 \qquad \forall j \in I,\ t \in TP \qquad (3.14)$$

In the model above, the objective function 3.1 represents the total cost for the entire planning horizon, given by the sum of three terms: (i) the total assignment cost incurred for building districts at the beginning of the planning horizon, (ii) the total cost for redesigning districts, i.e., reassigning TUs over the planning horizon minus the corresponding savings (accounted for since period 2), (iii) costs for shortage or surplus. As often done in districting, demands are used as weights when computing the assignment costs (Kalcsics and Ríos-Mercado, 2019). Constraints 3.2 ensure that every TU is assigned to exactly one district in every period. Constraints 3.5-3.7 quantify the reassignments/removals throughout the planning horizon so that the corresponding costs are paid. In particular, Constraints 3.5 enforces that an actual reassignment is counted whenever a given TU i is assigned in time period t to a district j to which it had not been assigned in the period before (i.e., $x_{ijt} = 1$, and $x_{ij,t-1} = 0$). Constraints 3.6 ensure that the removal of some TU i from district j in time period t cannot be accounted for if that TU is not assigned to that district ($x_{ij,t-1} = 0$). Also, Constraints 3.7 ensure that a TU i can be assigned or reassigned to district j in each time period. Constraints 3.8 and 3.9 are the balancing constraints which guarantee that the demand served by each district in every time period is within the maximum prescribed deviation α from the reference value. Note that surplus and shortages are also considered. Finally, Constraints 3.10-3.14 define the domain of the decision variables.

Remark 1
In the model described above, contiguity is not explicitly considered. However, optimizing for compactness tends to favor contiguity. In other words, a highly compact solution is likely to be contiguous. This conclusion is supported by exact solution approaches for districting Salazar-Aguilar et al. (2011). Nonetheless, many authors do not explicitly address contiguity in their models, particularly when working with point-like basic TUs Kalcsics and Ríos-Mercado (2019). This issue is also present in the work by Diglio et al. (2020) and has resulted in Pomes et al. (2025), which serves as a foundation for this thesis.

3.2.1 Model Extension: Fixed Centers

The multi-period model described earlier does not guarantee that the representatives or centers remain constant over time. However, limiting reassignments as much as possible helps to avoid unnecessary changes. To ensure consistency across all periods, the following constraints can be added:

$$x_{iit} = x_{ii1} \qquad \forall i \in I, t \in TP \setminus \{1\}$$

Of course, the number of districts must remain the same for all periods. Otherwise, no feasible solution can be found. As described earlier, the center must be fixed if, for example, a facility represents it and cannot be changed.

3.2.2 The Influence of the Savings Parameter

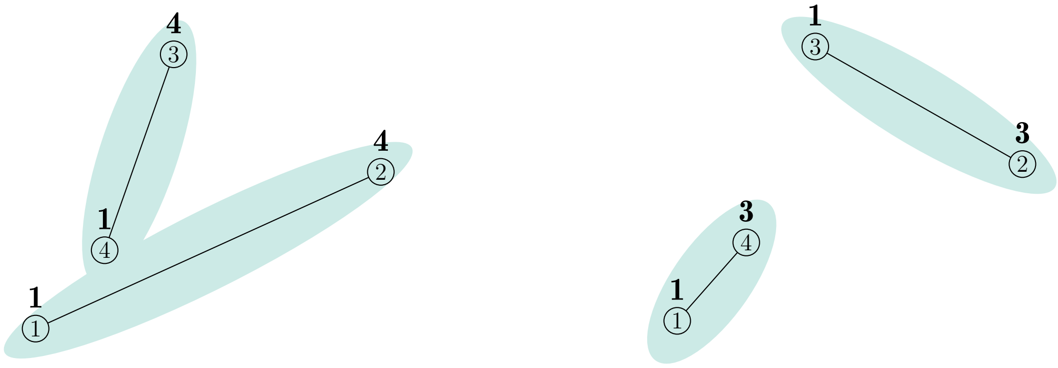
The additional parameter s_{ijt} for all $t \in TP\setminus\{1\}$ has already been introduced above. Computational results for the analysis of savings can be found in Section 5.5.2.2.

By introducing savings, it is possible to trigger reassignments, even when it is not necessary to maintain the balance. Savings allow for more compact solutions, representing the benefits of removing a TU from a district. This assumption is realistic because, once a TU is reassigned, it no longer incurs costs in the district from which it was reassigned. While there are costs associated with the reassignment for a new assignment, the previous district does not have any further expenses related to that TU, which is reflected in the savings. If $s_{ijt} = r_{ijt}$, the planning periods can be planned separately. Example 2 illustrates the benefit of savings.

Example 2 (Impact on the Solution with Positive Savings)
Figure 3.2 shows a two-period example of how savings can trigger a more compact solution without being less balanced. Within the nodes, the index of the TU is indicated, and above it is the weight (demand) of the TU. In period 1 (Figure 3.2a), balance can be achieved with the allocation of TU 3 and 4 as well as 1 and 2 within one district, even at the expense of compactness. In period 2 (Figure 3.2b), the demands change, and a balance can be established by grouping TU 1 and 4, as well as 2 and 3, in the same district. However, this change requires reassignments and, as a result, reassignment costs. Although the districts could be allocated as in period 1 to maintain the same solution, it would result in a less compact solution than the optimal solution for this period independently.

To summarize the benefits of the MPDP:

- It allows for the consideration of different time periods, enabling the adjustment of demand changes over these periods through redistricting.



(a) Period 1 **(b)** Period 2

Figure 3.2: Illustration of a solution to a two-period optimization problem involving four trade units and two districts.

- Changes in the number of districts can be implemented.
- The district representatives can be fixed or change over time.
- The ratio of penalty costs to redistricting costs can be indirectly weighted by the new parameter savings.
- By accounting for both shortages and surpluses, a feasible solution can always be found if the number of districts does not exceed the number of TUs.

In the next section, we evaluate the previously presented MPDP. In particular, we analyze when it is beneficial to utilize the multi-period districting model for multi-period problems.

3.3 Performance Analysis: Model Evaluation and Correlation Comparisons

In the following, the value of the multi-period solution – a frequently used measure to evaluate multi-period models – is explained and defined in Section 3.3.1. In Section 3.3.2, the model and its solution are analyzed to examine the input data and the benefits of using the MPDP.

3.3.1 The Value of the Multi-Period Solution

To evaluate the advantage of the multi-period model over the single-period model, Alumur et al. (2012) first introduced the concept of the value of the multi-period model, which is later defined as the value of the multi-period solution in Laporte et al. (2019). Further analyses of this value for capacitated location problems can be found in Bakker and Nickel (2024).

There exist several ways to define the static counterpart for a multi-period problem. In this thesis, we use the following definition:

Definition 12 (Solution of the Static Counterpart for Multi-Period Districting Problems)
A solution of the Static Counterpart (SC) for the $MPDP$ (SSC_{MP}) is a solution of a simplified single-period problem (SC_{MP}). All parameters are set to their average values calculated over the entire time horizon. This single-period problem is then solved using a single-period model, and the optimal districting decisions are stored. Next, we apply these districting decisions to the multi-period problem. The solution value generated with this procedure is called the SSC_{MP}.

Definition 13 (The Value of the Multi-Period Solution)
The Value of the Multi-Period Solution (VMPS) compares the optimal value of the multi-period problem (MP) and the SSC_{MP}:

$$VMPS = \frac{SSC_{MP} - MP}{MP} \tag{3.15}$$

Frequently, the percentage notation is utilized:

$$\%VMPS = \frac{SSC_{MP} - MP}{MP} \cdot 100 \tag{3.16}$$

3.3.2 Data and Solution Analysis

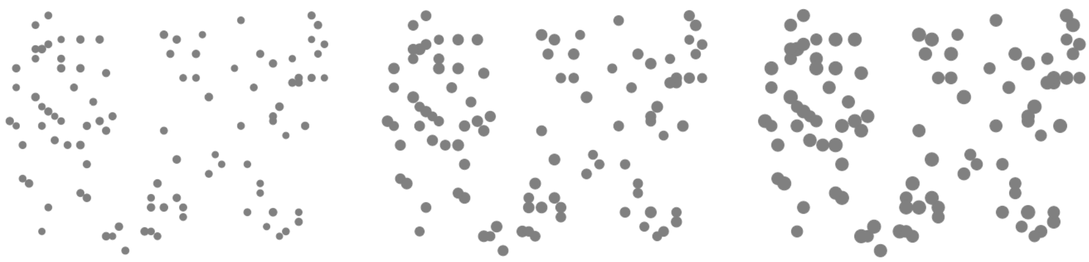
The analysis below focuses on the following questions:

- Is there a relationship between the $\%VMPS$ and the demand of an instance?
- Can we determine from the given data if using the MPDP is more beneficial than its SC_{MP}?

A study on similar questions for Capacitated Facility Location Problems (CFLPs) has been conducted and can be found in Bakker and Nickel (2024). This paper reveals that the fixed cost is the most critical factor in determining the potential value of a multi-period model. The study also indicates that the value of a multi-period approach depends on how effectively relevant cost components can be reduced, rather than the extent to which problem parameters change over time.

The CFLP and districting problems share many similarities. However, there are also some differences that must be considered when analyzing these two types of models. One significant difference is that, unlike the CFLP, classical districting problems do not have fixed costs. This means that the most critical factor for the CFLP, as noted by Bakker and Nickel, has no influence on districting problems and cannot be analyzed. Furthermore, differences in the problem definition impact the $\%VMPS$ analysis. Specifically, districting problems have no capacity, as considered in this thesis. Although the balance criterion has some similarities, it results from the demands and is therefore not given separately from externally. Additionally, there is no unit profit in the presented problem. Hence, the most interesting and primary factor analyzed in the following is demand.

Example 3 (Sufficiency of Static Models for Constantly Increasing Demand)
Consider a scenario where a multi-period model is not necessary, even though changes occur over time. This situation occurs, for example, if demand increases, as shown in Figure 3.3. Suppose we have an optimal solution for the first period, which remains optimal for the subsequent periods, i.e., periods 2 and 3. This is because the areas remain balanced, similar to the first period. Therefore, in this case, there is no need to use a multi-period model. The static counterpart, either with the expected value or the values of one of the three periods, can be used instead to find the optimal solution for the multi-period problem.



(a) Period 1 **(b)** Period 2 **(c)** Period 3

Figure 3.3: A multi-period instance with constantly increasing demand over three periods. The size of the nodes represents the demand.

In the context of multi-period districting, the $\%VMPS = 0$ in cases where the demand either constantly decreases or alternates between increases and decreases. Therefore, if the relative distribution of demand remains identical in all time units, it can be inferred that using a multi-period model is unnecessary.

Next, the properties of various instances are analyzed to determine the benefits of using the MPDP in cases with no constant increase or decrease in demand.

As previously stated, there are cases where $\%VMPS = 0$ despite demand changes. Thus, it is important to investigate the relative demand (w.r.t. to the overall demand) of each TU and its changes. The upcoming section analyzes scenarios where fluctuations impact $\%VMPS$ and where the correlation between the two is significant.

Definition 14 (Relative Demand)
The relative demand of a TU $i \in I$ is defined as

$$d_i^{rel} = \left(\frac{d_{it}}{\sum_{j \in I} d_{jt}} \right)_{t \in TP}$$

Definition 15 (Maximum Fluctuation of a TU)
The maximum fluctuation of a TU $i \in I$ is defined as the difference between the maximum relative demand for this TU and its minimum value:

$$\Delta_i^{abs} = \max(d_i^{rel}) - \min(d_i^{rel}) \qquad \forall i \in I$$

It is important to note that $\Delta_i^{abs} = 0$ for all TUs $i \in I$ in the example shown in Figure 3.3 since $\max(d_i^{rel}) - \min(d_i^{rel}) = 0$ for all TUs.

Definition 16 (Relative Maximum Fluctuation of a TU)
The relative maximum fluctuation for a TU i is defined as the relative difference between the highest demand value over time and the lowest:

$$\Delta_i^{rel} = \frac{\max(d_i^{rel}) - \min(d_i^{rel})}{\max(d_i^{rel})} \qquad \forall i \in I$$

Example 4 (Relative Maximum Demand Fluctuation Calculation)
If a particular TU i contains 10% of the total demand in the initial period and 15% in the subsequent period, its relative share and influence on total demand demonstrate a fluctuation of 33.3% (i.e., $\Delta_i^{rel} = 0.33$). The overall demand for this specific TU may remain constant across both periods. As previously stated, our analysis focuses exclusively on relative demands, which may vary for a TU even in the presence of stable absolute demands for this TU.

Definition 17 (Relative Maximum Fluctuation of an Instance)
The relative maximum fluctuation of an instance is defined as the average value of the demands of all TUs $i \in I$ of this instance:

$$\Delta^{rel} = \frac{\sum_{i \in I} \Delta_i^{rel}}{|I|}$$

For the following evaluations, we use the sample correlation coefficient:

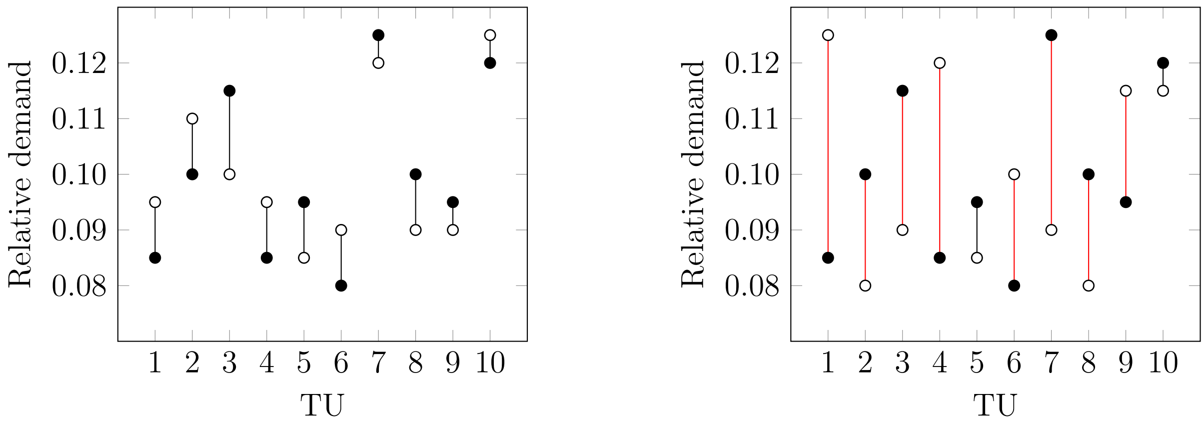
Definition 18 (Sample Correlation Coefficient)
As part of the analytical methodology, the use of sample correlation coefficients is employed – a statistical tool that measures the strength and direction of the linear relationship between two variables:

$$Cor(x,y) = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

where n is the sample size, x_i, y_i are the individual sample points indexed with i, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ (the sample mean) and analogously for $\bar{y}$ (Devore and Berk, 2012).

The relationship between the relative demand and the fluctuations of TUs in a given instance is explained in more detail in the following example.

Example 5 (Relative Demand Profiles: Low vs. High Fluctuations)
Figure 3.4 illustrates two different relative demand profiles. The analysis includes two instances over two periods. In the figures, the black dots represent the relative demand for the TU in the first period, while the white dots indicate the relative demand for the same TU in the second period. The difference or fluctuation between the relative demands in both periods is shown by a line connecting the points. The lines represent the Δ_i^{abs}. A high Δ_i^{abs} is marked with a red line. In Figure 3.4a, there are no high relative fluctuations, while in Figure 3.4b, there are numerous high relative fluctuations.



(a) Instance 1

(b) Instance 2

Figure 3.4: Two instances with two periods and different relative demand profiles.

In Figure 3.4, we can also see that the instance on the left has a low number of high-fluctuating TUs and the instance on the right has a high number of high-fluctuating TUs. The exact count of high-fluctuating TUs depends on the definition or, more specifically, the level of the threshold value used to determine what qualifies as a *high fluctuating* TU. This threshold is denoted in the following as τ.

Using this threshold value τ, we can define the number of highly fluctuating TUs of an instance:

Definition 19 (Number of Highly Fluctuating TUs of an Instance)
The number of TUs with a higher fluctuation of the relative demand than the threshold value τ is defined as:

$$r = \left|\left\{i \in I \mid \max(d_i^{rel}) - \min(d_i^{rel}) \geq \tau\right\}\right|$$

With Definition 15 this can also be written as:

$$r = \left|\left\{i \in I \mid \Delta_i^{abs} \geq \tau\right\}\right|$$

The correlation between the number of TUs with a higher fluctuation of the relative demand than the threshold value τ and the $\%VMPS$ is denoted as $Cor\,(r, \%VMPS)$.

It is assumed that a high number of high fluctuating TUs r is connected to a high $\%VMPS$ or at least that these two values are positively correlated. Also, the relative maximum fluctuation of an instance Δ^{rel} can impact the $\%VMPS$ and is analyzed.

We subsequently provide a detailed analysis of the following:

- In Section 3.3.2.1, the correlation between r and the $\%VMPS$ is examined.
- In Section 3.3.2.2, the focus is on the correlation between the average relative demand fluctuation over a two-period planning horizon and the $\%VMPS$.
- In Section 3.3.2.3, a combined approach is adopted that utilizes the SSC_{MP} (Definition 12).

3.3.2.1 Correlation Analysis: High Fluctuating TUs and the VMPS

In this section, we analyze whether and how the number of highly fluctuating TUs of an instance r correlates with the $\%VMPS$ (Definition 19).

We analyze six different combinations of number of TUs ($|I|$) and number of districts (p), consisting of two periods, all generated using the same distribution function. For each of the six combinations, 500 instances are generated and 80 different values for the threshold parameter τ.

The results are presented in Figure 3.5 and Table 3.1. The correlation between the number of TUs with a fluctuation higher than τ and the $\%VMPS$ varies between zero and 0.36, depending on the level of τ. The correlation is mostly positive, indicating a positive linear relationship between the number of high-fluctuating TUs and the $\%VMPS$. Notably, the peak correlation occurs at a relative deviation, represented by τ, of approximately 0.14. As we can also see in the figure, if the threshold value τ is set too high, no TUs are identified as having high fluctuations. Conversely, if it is set too low, all ten TUs are classified as high-fluctuating.

In conclusion, while we cannot predict the $\%VMPS$ solely by counting high-fluctuating TUs in the data, a positive linear relationship exists between them. Additionally, the likelihood of an instance having a high percentage of the $\%VMPS$ increases with a larger number of fluctuating TUs.

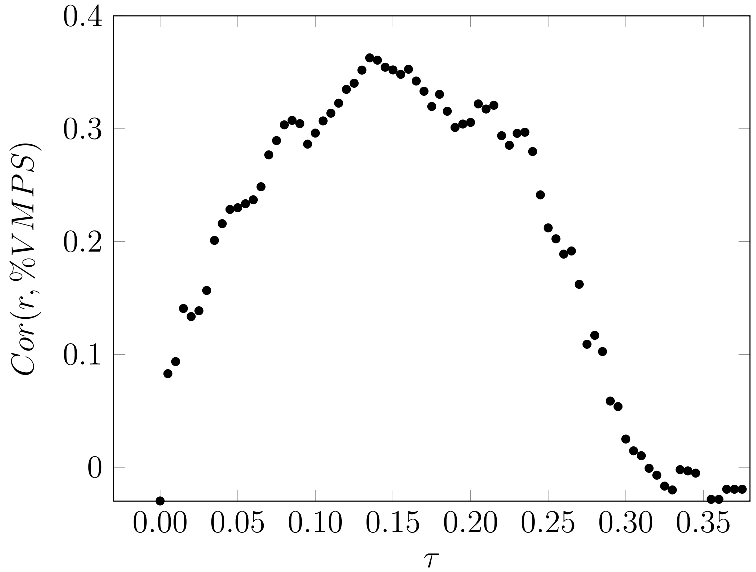


Figure 3.5: Linear correlation between r and the $\%VMPS$ for $|I| = 10$, $p = 5$, and two periods depending on τ.

The results for different numbers of TUs and districts are presented in Table 3.1, where 3000 instances in total are analyzed. This demonstrates that the average $\%VMPS$ is higher when the ratio of TUs to districts is smaller. In these cases, a single TU has a greater impact on the balance of a district, which, in practice, significantly influences the objective value and the advantages of using the multi-period model that considers all relevant information.

As we can also see, both the average CPU runtimes for solving the multi-period model ($\overline{CPU_{MP}}$) and the static counterpart for the multi-period model ($\overline{CPU_{SC_{MP}}}$) increase with the size of the instance, but the multi-period model does so at a greater rate.

Next, we analyze another detail level and focus on the correlation between the average relative maximum fluctuations and the $\%VMPS$, and compare the correlations with the results above.

3.3.2.2 Correlation Analysis: Average Fluctuation and the VMPS

In this section, the correlation between the average value of the relative maximum fluctuation of the relative demand Δ^{rel} in the planning horizon of two periods and the $\%VMPS$ is analyzed.

$\|I\|$	p	$Cor(r, \%VMPS)$	τ	$\overline{\%VMPS}$	$\overline{CPU_{MP}}$	$\overline{CPU_{SC_{MP}}}$
10	2	0.19	0.24	5.0058	0.14	0.07
10	5	0.36	0.14	8.7515	0.15	0.06
20	2	0.27	0.24	6.2648	0.77	0.19
20	5	0.37	0.24	11.4022	1.77	0.20
20	10	0.39	0.17	14.5296	0.67	0.21
100	2	0.15	0.04	4.6143	26.91	4.51

Table 3.1: Computational results for various values of τ, multiple numbers of TUs, multiple numbers of districts, and two periods.

A computational analysis, using the same instances as in Section 3.3.2.1 is conducted with the outcomes presented in Table 3.2. The following points should be highlighted:

- In all test cases, the average $\%VMPS > 0$. This means that every instance variant includes at least one instance with a positive $\%VMPS$, making them suitable for our analysis. If the randomly generated instances all had a $\%VMPS = 0$, no meaningful correlation could be determined.
- The correlation between Δ^{rel} and the $\%VMPS$ is smaller if the number of TUs per district is higher. In other words, a higher average number of TUs per district tends to decrease the correlation.
- The correlation between Δ^{rel} and the $\%VMPS$ is about as high as the correlation between the $\%VMPS$ and the number of TUs with a high Δ^{rel} value (Table 3.1).

$\|I\|$	p	$Cor(\Delta^{rel}, \%VMPS)$	$\overline{\%VMPS}$	$\overline{CPU_{MP}}$	$\overline{CPU_{SC_{MP}}}$
10	2	0.17	5.0058	0.14	0.07
10	5	0.38	8.7515	0.15	0.06
20	2	0.22	6.2648	0.77	0.19
20	5	0.31	11.4022	1.77	0.2
20	10	0.38	14.5296	0.67	0.21
100	2	0.09	4.6143	26.91	4.51

Table 3.2: Computational results for multiple numbers of TUs, multiple numbers of districts, and two periods.

The results depicted in Figure 3.6 illustrate the relation between Δ^{rel} and the $\%VMPS$ for the instances with 10 TUs, 5 districts, and 2 periods. While many $\%VMPS$ values are observed to be close to zero, a few instances have a $\%VMPS$ of up to 50. However, this does not necessarily imply that higher Δ^{rel} values always lead to a higher $\%VMPS$. In fact, the correlation between Δ^{rel} and $\%VMPS$ across 500 instances, which involve 10 TUs, 5 districts, and 2 periods, is $Cor(\Delta^{rel}, \%VMPS) = 0.38$.

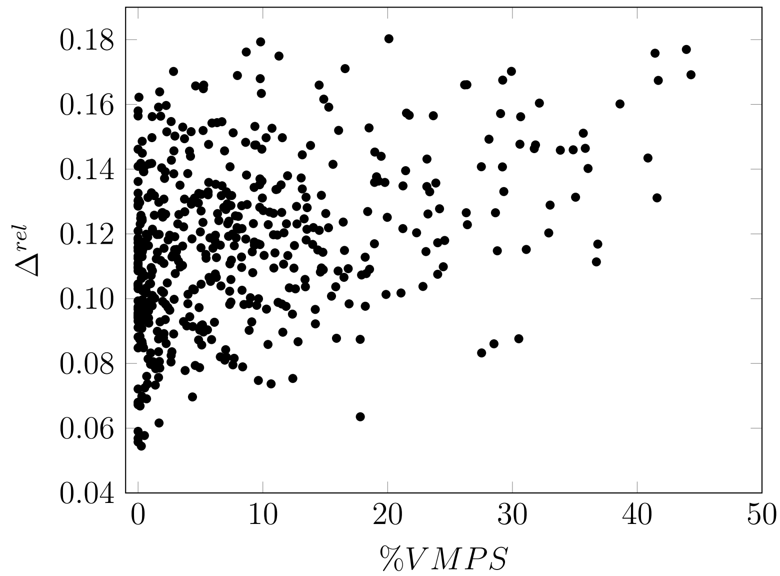


Figure 3.6: Relation between Δ^{rel} and $\%VMPS$ for $|I| = 10$, $p = 5$, and two periods.

The observations above are based on instances with two periods. However, in many cases, a multi-period model is typically used for problems that involve more than two periods. We now analyze the effects on correlation when additional periods are added to these instances.

Table 3.3 shows the results for instances with different numbers of periods. The average $\%VMPS$ increases with the number of periods. This is because the number of periods in the multi-period model allows for redistricting, whereas the solution generated with the mean value cannot be adjusted between periods. Hence, it becomes increasingly important to use a multi-period model in such instances.

At the same time, the correlations are decreasing. The values for $Cor(\Delta^{rel}, \%VMPS)$ decrease from 0.17 to 0.04, and for $Cor(r, \%VMPS)$, from 0.19 to 0.12. This indicates a very weak relationship between these variables. This decreasing correlation may be attributable to the method used to calculate Δ^{rel} and to the fact that, as the number of periods increases, the average value becomes less sensitive to individual fluctuations.

As the correlations between the values mentioned above decrease with an increasing number of periods, we explore a new variant in the next section. This variant aims to enhance or

$\|I\|$	p	T	$Cor(\Delta^{rel}, \%VMPS)$	$Cor(r, \%VMPS)$	τ	$\overline{\%VMPS}$	$\overline{CPU_{MP}}$	$\overline{CPU_{SC_{MP}}}$
10	2	2	0.17	0.19	0.24	5.0058	0.14	0.07
10	2	3	0.16	0.17	0.15	7.7396	0.31	0.1
10	2	4	0.09	0.13	0.39	12.1375	0.61	0.12
10	2	5	0.13	0.15	0.24	14.8764	1.14	0.16
10	2	6	0.02	0.1	0.35	15.889	2.00	0.19
10	2	7	0.04	0.12	0.38	18.3192	3.52	0.22

Table 3.3: Computational results and correlations comparison for multiple numbers of periods and $p = 2$.

maintain the stability of the relationship between demands and $\%VMPS$, particularly when analyzed over more than two periods.

3.3.2.3 Correlation Analysis: District Fluctuations and the VMPS

The previous subsections have shown that the fluctuations of relative demand correlate with the $\%VMPS$. However, it is reasonable to assume that other indicators besides the average fluctuations or the absolute number of TUs with high relative fluctuations within an instance have an influence on the $\%VMPS$ or allow conclusions or forecasts. As can be seen in the examples already mentioned, $\%VMPS = 0$ if there is no fluctuation in the relative demand. This is also possible under other circumstances, for example, if the fluctuations are balanced by the TUs in the surroundings, as long as these TUs are in the same district. In the following, we therefore examine how strong the correlations are between the fluctuations within predefined neighborhoods and the $\%VMPS$. We use the districts created by solving the SC_{MP} as the neighborhood groups. Other groupings are also possible.

The following analysis is at a different detail level than the previous two analyses. While we have previously focused on TUs within an instance, this time we shift our attention to the district level. First, we will define the number of districts that exhibit high fluctuations. This definition will once again depend on the threshold value denoted as τ.

Definition 20 (Number of High Fluctuation Districts)
Let D_t be the number of districts in period t for which the relative deviation from the average demand exceeds a threshold value τ. Formally, we define:

$$D_t = \left| \left\{ j \in I \mid \left(\frac{\psi_{jt}}{\mu_t} + \frac{\varphi_{jt}}{\mu_t} \right) x_{jjt} \geq \tau \right\} \right|$$

where ψ_{jt} and φ_{jt} represent the surplus and shortage for district j in period t, μ_t is the average demand in period t and x_{jjt} is is the decision variable which is 1 if j is the representative TU of an district in period t.

The sum over all periods is defined as:

$$\mathcal{D} = \sum_{t \in TP} D_t$$

Note that $\psi_{jt} \cdot \varphi_{jt} = 0$ for all TUs j and periods t since a district can only have a surplus, shortage, or neither.

The following approach is analyzed:

1. Firstly, the SC_{MP} problem is solved.
2. Secondly, $\mathcal{D}$ for the resulting districts are calculated.
3. After that, the MP is solved and the $\%VMPS$ is calculated.
4. Finally, the correlation between $\mathcal{D}$ and the $\%VMPS$ are calculated.

The recommendation is then to calculate the MPDP completely if there is a high value for $\mathcal{D}$ for an instance.

An analysis of 500 instances is conducted, each consisting of 12 TUs and 3 districts. The number of periods varied, but the demand generation remained identical in each instance and period, as in Section 3.3.2.1 and Section 3.3.2.2. The threshold value, or tolerance level τ, is $\frac{0.2}{12}$ for 12 TUs, while the TUs had a randomly generated uniformly distributed demand $d \in [50 \cdot (1 - 0.2), 50 \cdot (1 + 0.2)]$ in each period.

The results are presented in Figure 3.7. They show a positive linear correlation between the number of periods and districts with a high value of $\mathcal{D}$ and the percentage of $\%VMPS$. In Figure 3.7a, there are three districts and two periods, allowing for a maximum of six fluctuations (where $\mathcal{D} \in \{0, \ldots, 6\}$). Instances with a high value of $\mathcal{D}$ tend to have a higher $\%VMPS$. However, there are also cases where $\mathcal{D}$ is high, but $\%VMPS$ remains low.

As shown in Table 3.4, the correlation $Cor(\mathcal{D}, \%VMPS)$ is significantly stronger than the correlation $Cor(\Delta^{rel}, \%VMPS)$. Specifically, the correlation between $\mathcal{D}$ and $\%VMPS$ ranges from $[0.47, 0.54]$, while the correlation between Δ^{rel} and $\%VMPS$ ranges from $[0.11, 0.22]$. This indicates that $\mathcal{D}$ is much more strongly associated with the percentage of $\%VMPS$ compared to Δ^{rel}. In other words, changes in $\mathcal{D}$ have a clearer impact on the $\%VMPS$ values, while Δ^{rel} shows only a weak relationship with $\%VMPS$.

The results also indicate a linear relationship between $\mathcal{D}$ and $\%VMPS$, with an approximate slope of 0.5 across all tested numbers of periods. As illustrated in Figure 3.7, there are no high $\%VMPS$ values associated with low $\mathcal{D}$ values. However, as $\mathcal{D}$ increases, the likelihood of observing a higher $VMPS$ also increases.

It is advisable to run the MPDP only if the solution generated by solving the SC_{MP} has a high value of $\mathcal{D}$. Otherwise, there is a high probability of a lower $\%VMPS$. However, the MPDP can always be used to make an exact statement. Nevertheless, by making a forecast in advance, a lot of computing time can be saved.

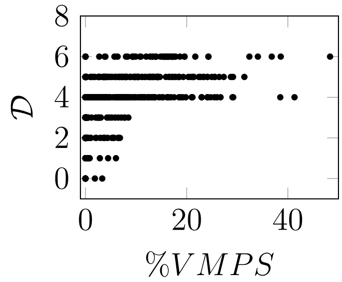
$\lvert I\rvert$	p	T	$Cor(\mathcal{D}, \%VMPS)$	$Cor(\Delta^{rel}, \%VMPS)$	$\overline{\%VMPS}$	$\overline{CPU_{MP}}$	$\overline{CPU_{SC_{MP}}}$
12	3	2	0.47	0.20	7.172	0.27	0.08
12	3	3	0.47	0.21	12.2958	0.67	0.13
12	3	4	0.51	0.22	15.7617	1.50	0.18
12	3	5	0.49	0.11	17.8877	3.57	0.22
12	3	6	0.49	0.07	19.1954	9.70	0.26
12	3	7	0.54	0.19	22.0554	27.04	0.30

Table 3.4: Computational results and correlations comparison for multiple numbers of periods and $p = 3$.

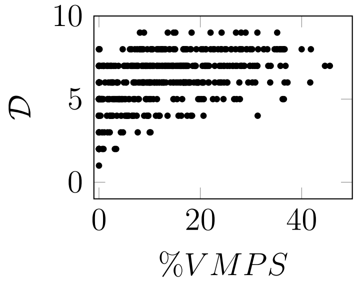
3.4 Conclusion

This chapter first reviews the relevant literature on multi-period districting problems and identifies existing gaps. We introduce a new multi-period model (Pomes et al., 2025) that incorporates reassignment variables, allowing for reassignments between periods. Additionally, the model features the option to integrate savings, helping to manage the motivations for reassigning TUs. Surplus and shortage variables are included to prevent violations of the balancing constraints from rendering the problem unsolvable.

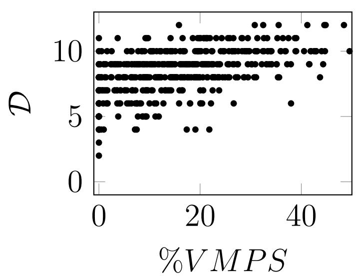
We conduct a detailed performance analysis of the $VMPS$. This analysis examines the circumstances under which applying the MPDP is beneficial in multi-period contexts and when it may be less advantageous. The analysis proceeds through multiple stages, varying in detail. We analyze demand fluctuations and the value of the multi-period solution at different granular levels: first, the number of territorial units with highly fluctuating



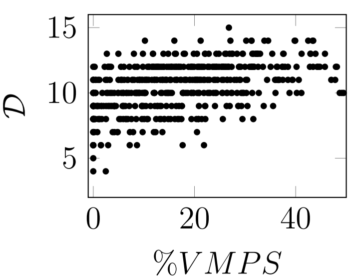
(a) 2 Periods



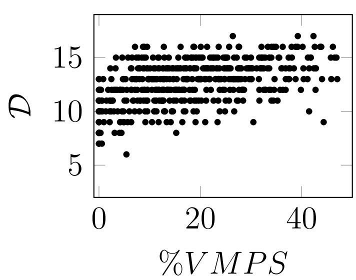
(b) 3 Periods



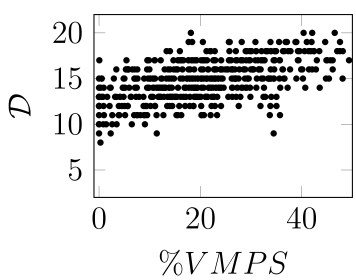
(c) 4 Periods



(d) 5 Periods



(e) 6 Periods



(f) 7 Periods

Figure 3.7: Linear correlations between $\mathcal{D}$ and $\%VMPS$ for $|I| = 12$, $p = 3$, and multiple numbers of periods.

demand, second, the average demand fluctuations, and finally, the average sum of demand fluctuations in each district.

The findings reveal a positive linear correlation between the number of highly fluctuating TUs and the $VMPS$. Additionally, there is a positive linear correlation between average fluctuations and the $VMPS$. The strongest linear correlation is observed at the district level, and this relationship additionally remains stable over several periods. This may be due to the number of periods already being included in the definition of $\mathcal{D}$, whereas the other key figures are only averaged.

Chapter 4

Stochastic Districting

In this chapter, we address districting under uncertainty, where future parameters are not fully known at the time decisions must be made. Unlike the previous chapter, which considered a multi-period setting, we focus here on a stochastic single-period districting problem, as illustrated in Figure 4.1. The foundational model for this chapter is based on the work of Diglio et al. (2020). The novel contribution of this thesis lies in adapting the existing model and providing a comprehensive performance analysis. Specifically, we evaluate the value of the stochastic solution and the expected value of perfect information. Through these analyses, we offer new insights into the benefits of incorporating stochastic elements into single-period districting. The topic of multi-period stochastic districting, which introduces additional complexity and research opportunities, is addressed separately in Chapter 5.

The term *Stochastic Districting* is not clearly defined. The *uncertainty* can refer to various parameters. Distances can be stochastic, for example, in a road network where it is unclear exactly when and where a traffic jam will form, and journey times are only realized during the journey. Costs can be stochastic. Transport prices, for example, can depend on fuel prices and, therefore, cannot always be planned correctly in advance. The number of districts required can also be stochastic. For a parcel delivery company, for example, the delivery areas may vary daily depending on the number of employees available on that day, taking into account holidays, illness, and other factors. In the following, we use the term *Stochastic Districting* to mean districting with uncertain demand and only use this term in this context. Stochasticity in demand may occur, for example, in school districting, where the number of pupils is unknown, or in ambulance districting when the number of patients is uncertain.

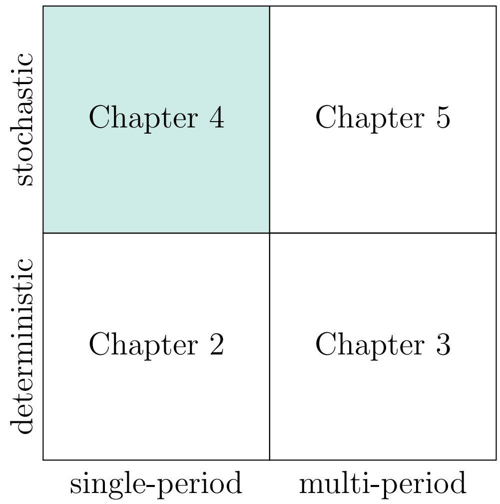


Figure 4.1: Research Focus of Chapter 4: Single-period stochastic districting.

Deterministic models can be used when all information is known. These models are practical if complete information is available, and decisions must be made based on that information. On the other hand, deterministic models can also be used when some information is not yet known. In such cases, for example, the expected values can serve as input parameters, providing some guidance for decision-making. Alternatively, a stochastic model may be utilized to incorporate the stochastic parameter directly into the decision-making process. This enables decision-makers to evaluate different scenarios that may occur, rather than relying solely on one deterministic input.

This chapter is organized as follows:

- In Section 4.1, a literature review focusing on stochastic districting problems is provided.
- Section 4.2 presents a two-stage stochastic model.
- Section 4.3 evaluates the benefits of using a stochastic model for stochastic districting problems.

4.1 Related Literature

A general introduction to stochastic programming is found in Birge and Louveaux (2011). The following literature directly addresses the topic of districting.

Enayati et al. (2020) propose a two-stage stochastic mixed-integer programming model called stochastic service district design to address the service district design problem for ambulance services, aiming to maximize the expected number of emergency calls that are responded to on time while limiting the workload of ambulances. The proposed model recommends how to locate ambulances at the waiting sites in the service area and how to assign a set of demand zones to each ambulance at different backup levels.

Diglio et al. (2020) investigates a stochastic districting problem where a random vector with a given joint probability distribution function represents demand. They propose a two-stage mixed-integer stochastic programming model in which the initial territory design is decided in the first stage, and balancing requirements are met in the second stage through outsourcing and reassignment of TUs based on the known demand. The objective function accounts for the total expected cost and includes the cost for the first-stage assignments plus the expected cost incurred at the second stage. The model extends in different ways to account for practical aspects such as maximum desirable dispersion, reallocation constraints, or similarity of the second-stage solution to the first-stage one. The new modeling framework is tested computationally using instances built with real geographical data.

The same authors investigate a new districting problem with stochastic demands Diglio et al. (2021), aiming to divide a geographic area into contiguous districts balanced with respect to given thresholds. They use a p-median problem with contiguity constraints and chance-constrained balancing requirements. A two-phase heuristic is developed to derive a deterministic equivalent for the problem. Different families of probability distributions for the demands are investigated, and a simulation procedure is proposed to estimate the probability of a given solution to yield a balanced districting. The results of computational tests performed on a set of testbed instances are discussed.

Later, the same authors Diglio et al. (2022) investigate a districting problem with stochastic demand and contiguity constraints. They use chance constraints to model the balancing requirements and propose an approximate deterministic counterpart as a solution algorithm. The algorithm is based on a location-allocation scheme, and they develop two variants of a new heuristic. The authors use an attractiveness function to find a good trade-off between the solutions obtained for single-scenario problems. The results of extensive computational tests are reported.

The results from the papers mentioned above and others can be found in the book *Facility Location Under Uncertainty* (Saldanha-da-Gama and Wang, 2024). In addition to the chapters focused on districting under uncertainty, the book also presents various modeling paradigms and solution techniques for managing uncertainty in general, and specifically in the context of facility location problems.

Lei et al. (2012) introduce a new problem called the vehicle routing and districting problem with stochastic customers and solve it using a two-stage stochastic program. In the first stage, the districting decisions are made, and in the second stage, the expected routing cost of each district is approximated. The authors also consider district compactness as part of the objective function. They develop a large neighborhood search heuristic for their problem, which they test on modified Solomon instances and on modified Gehring and Homberger instances. They provide extensive computational results that confirm the effectiveness of the proposed heuristic.

Lei et al. (2015) introduce a new problem called the Multiple Traveling Salesmen and Districting Problem with Multi-periods and Multi-depots. They consider several factors such as the compactness of subdistricts, the dissimilarity measure of districts, and an equity measure of salesmen's profit as part of the objective function. They estimate the travel costs for salesmen in each subdistrict using an approximation formula and develop an adaptive large neighborhood search metaheuristic for the problem. The proposed metaheuristic is tested on modified Solomon and Gehring and Homberger instances, and the computational results confirm its effectiveness.

4.2 Two-Stage Stochastic Districting Model

Diglio et al. (2020) describe the problem of stochastic districting with uncertain demand. The problem is modeled as a two-stage mixed-integer stochastic program. The first stage defines the decision on the initial districting, and in the second stage, where the demand is known, penalties can occur. They want to minimize the total costs, but not only the sum of costs for the first stage, where they decide the districting plan, but also the expected total costs for the penalties in the second stage.

The following sets are used:

I	set of TUs
S	set of Scenarios

The following parameters are used:

p	number of districts
ξ	$= (d_1, \ldots, d_{\lvert I \rvert})$ random vector with the demand for each TU $i \in I$
$\hat{\mu}$	reference value for the demand assigned to each district
α	allowed deviation for the demand assigned to each district
c_{ij}	cost for assigning TU $i \in I$ to TU $j \in I$
g_j	unit penalty for surplus at the district represented by TU $j \in I$
h_j	unit penalty for shortage at the district represented by TU $j \in I$

As described in Birge and Louveaux (2011), a number of decisions have to be taken before the experiment, these are the first-stage decision variables that represent the assignment decisions:

$$x_{ij} = \begin{cases} 1, & \text{if TU } i \text{ is assigned to the district represented by TU } j \in I \\ 0, & \text{otherwise} \end{cases}$$

A number of decisions can be taken after the experiment, the second-stage decisions:

$\varphi_j(\xi)$ demand shortage in the district represented by TU $j \in I$ under ξ

$\psi_j(\xi)$ demand surplus in the district represented by TU $j \in I$ under ξ

Definition 21 (Stochastic Solution)
A stochastic solution refers to the outcome of a stochastic model, containing all decision variables associated with the solution. We denote the objective value of this solution as SP*.*

Due to the auxiliary variable character of the decision variables in the second stage, Diglio et al. (2020) call the following problem the Stochastic Districting Problem with Auxiliary Recourse (SDPAR):

SDPAR:

$$\min \quad \sum_{i\in I}\sum_{j\in I} c_{ij}x_{ij} + \mathcal{Q}(x) \tag{4.1}$$

$$\text{s. t.} \quad \sum_{j\in I} x_{ij} = 1 \qquad \forall i \in I \tag{4.2}$$

$$\sum_{i\in I} x_{ii} = p \tag{4.3}$$

$$x_{ij} \leq x_{jj} \qquad \forall i, j \in I \tag{4.4}$$

$$x_{ij} \in \{0,1\} \qquad \forall i, j \in I \tag{4.5}$$

with the expected second stage value function $\mathcal{Q} = E_\xi[Q(x,\xi)]$.

$$Q(x,\xi) = \min \sum_{j\in I} g_j\psi_j(\xi) + \sum_{j\in I} h_j\varphi_j(\xi) \tag{4.6}$$

$$\text{s. t.} \quad (1-\alpha)\mu x_{jj} \leq \sum_{i\in I} d_i x_{ij} + \varphi_j(\xi) - \psi_j(\xi) \leq (1+\alpha)\mu x_{jj} \qquad \forall j \in I \tag{4.7}$$

$$\varphi_j(\xi) \geq 0 \qquad \forall j \in I \tag{4.8}$$

$$\psi_j(\xi) \geq 0 \qquad \forall j \in I \tag{4.9}$$

The assignment costs $c_{ij} = l_{ij}E_\xi[d_i]$ are the product of the expected demand and the distance l_{ij} between TU i and j.

The first-stage objective function 4.1 minimizes the assignment costs and the expected second-stage value. Constraints 4.2 ensure complete assignment for each TU (Definition 7) while 4.3 defines the number of districts. Constraints 4.4 ensure that TUs can only be assigned to district representatives, and Constraints 4.5 define the domain of the first-stage decision variables. The second-stage problem is formulated in 4.6 – 4.9. In the objective function 4.6, the expected second-stage value function gets minimized. Constraints 4.7 ensure the balancing by adjusting the surplus or shortage variables. Constraints 4.8 and 4.9 define the domain of the second-stage decision variables.

As there is for each first-stage solution a feasible completion in the second stage, the stochastic program has *relatively complete recourse* (Birge and Louveaux (2011); Diglio et al. (2020). And because the coefficient matrix of the second-stage decision variables is deterministic and can be written as $[I|-I]$, the model has also *simple recourse*.

Diglio et al. (2020) assume that the support Ξ for the random vector ξ is finite. Hence, the problem can be formulated as a deterministic equivalent, and it is possible to index the different scenarios in a finite set $S = \{1, ..., |\Xi|\}$. Moreover, in S, the stochastic demands, as well as the assignment costs and the second-stage decision variables, can be indexed. Hence, $\pi_s \geq 0$ is the probability associated with scenario $s \in S$, and d_{is} is the demand of TU $i \in I$ in scenario $s \in S$. Clearly, we impose that $\sum_{s \in S} \pi_s = 1$.

Also, the second-stage decision variables can now be indexed by the scenario:

- φ_{js} demand shortage in the district represented by TU $j \in I$ in scenario $s \in S$
- ψ_{js} demand surplus in the district represented by TU $j \in I$ in scenario $s \in S$

The detailed representation of the deterministic equivalent of the SDPAR can now be formulated as follows:

SDPAR-DE:

$$\min \quad \sum_{i \in I} \sum_{j \in I} c_{ij} x_{ij} + \sum_{s \in S} \pi_s \Big(\sum_{j \in I} g_j \psi_{js} + \sum_{j \in I} h_j \varphi_{js} \Big) \tag{4.10}$$

$$\text{s. t.} \quad \sum_{j \in I} x_{ij} = 1 \qquad \forall i \in I \tag{4.11}$$

$$\sum_{i \in I} x_{ii} = p \tag{4.12}$$

$$(1-\alpha)\hat{\mu} x_{jj} \leq \sum_{i \in I} d_{is} x_{ij} - \psi_{js} + \varphi_{js} \leq (1+\alpha)\hat{\mu} x_{jj} \qquad \forall j \in I, s \in S \tag{4.13}$$

$$\varphi_{js} \geq 0 \qquad \forall j \in I, s \in S \tag{4.14}$$

$$\psi_{js} \geq 0 \qquad \forall j \in I, s \in S \tag{4.15}$$

$$x_{ij} \leq x_{jj} \qquad \forall i, j \in I \tag{4.16}$$

$$x_{ij} \in \{0, 1\} \qquad \forall i, j \in I \tag{4.17}$$

The model includes $|I|$ assignment constraints, one district count constraint, $2 \cdot |I| \cdot |S|$ balancing constraints (counting upper and lower bounds), $2 \cdot |I| \cdot |S|$ non-negativity constraints for slack variables, and $|I|^2$ constraints linking assignments to representative TUs, with $|I|^2$ binary variables for all assignments.

In the objective function 4.10, the assignment costs and the expected penalty costs are minimized.

The assignment costs c_{ij} for the first-stage decision (which is not scenario dependent) are chosen as the distance l_{ij} multiplied by the expected demand over all scenarios $c_{ij} = l_{ij} \cdot \sum_{s \in S} \pi_s d_{is}$.

The stochastic two-stage model described in Diglio et al. (2020) serves as the basis for extending a multi-period model in the following. However, the definitions of $\hat{\mu}$, g_j and h_j are changed.

In Diglio et al. (2020), the unit penalty g_j and h_j for TU j are set to:

$$g_j = h_j = \max_{i \in I}\{l_{ij} \sum_{s \in S} \pi_s d_{is}\} \qquad \forall j \in I$$

However, in the objective function, the unit penalties are multiplied by the surplus and shortage, which are already demand-dependent. We recommend selecting penalty costs independently of demand because, otherwise, they are considered twice.

The costs

$$g_j = h_j \geq \max_{i \in I}\{l_{ij}\} \qquad \forall j \in I$$

should be chosen to ensure that it is never more advantageous to incur the penalty costs than to assign a TU to a distant TU.

In Diglio et al. (2020) $\hat{\mu}$ is a mean value that runs across all scenarios.

$$\hat{\mu} = \frac{1}{p} \sum_{s \in S} (\pi_s \sum_{i \in I} d_{is}) \tag{4.18}$$

However, the Constraints 4.13 are defined individually for all scenarios:

$$(1 - \alpha)\hat{\mu} x_{jj} \leq \sum_{i \in I} d_{is} x_{ij} - \psi_{js} + \varphi_{js} \leq (1 + \alpha)\hat{\mu} x_{jj} \qquad \forall j \in I, s \in S$$

Therefore, the definition in (4.18) is changed to a scenario-based definition in the following:

$$\mu_s = \frac{1}{p} \sum_{i \in I} d_{is} \qquad \forall s \in S \tag{4.19}$$

The reason for this change is that, in many scenarios, it may not be possible to achieve an average demand of $\hat{\mu}$ in each district, especially when the total demand is already lower or higher than this value. Since Constraints 4.13 are scenario-based, the mean value to be achieved can also be defined as scenario-based.

This change is illustrated in the following example.

Example 6 (Alternative Definition for the Reference Value of the Average Demand) *Let 10 TUs be divided into 2 districts. In the first scenario, all TUs have a demand of 1 ($\sum_{i\in I} d_{i1} = 10$), in the second scenario, a demand of 5 ($\sum_{i\in I} d_{i2} = 50$), both scenarios have the same probability ($\pi_1 = \pi_2 = 0.5$). As defined by Diglio et al. (2020) in equation* (4.18)*, the following result is obtained:*

$$\hat{\mu} = \frac{1}{2}\sum_{s\in S}(0.5\sum_{i\in I} d_{is}) = \frac{1}{2}(0.5\cdot 10 + 0.5\cdot 50) = \frac{1}{2}(5+25) = 15$$

However, this mean value should not be aimed for in either of the two scenarios, as $\sum_{i\in I} d_{i1} = 10$ and $\sum_{i\in I} d_{i2} = 50$ and $p = 2$. In the first scenario, $\hat{\mu}$ would exceed the total demand, and in the second scenario, both districts would fall short of the target demand $\hat{\mu}$ even with a perfect split (50:50). Thus, costs for surplus or shortage would be incurred in each of the two scenarios and the split would only depend on which of these costs is higher or lower.

According to the new definition in (4.19) *and the use of μ_s then applies:*

$$\mu_1 = \frac{1}{2}\cdot\sum_{i\in I} d_{i1} = 5$$
$$\mu_2 = \frac{1}{2}\cdot\sum_{i\in I} d_{i2} = 25$$

Costs in the individual scenarios are only incurred if there are deviations from the respective mean value within the scenarios.

The model on which the further work is based is the following extensive form of the deterministic equivalent called the Stochastic Districting Problem (SDP):

SDP:

$$\begin{aligned}
\min \quad & \sum_{i\in I}\sum_{j\in I} c_{ij}x_{ij} + \sum_{s\in S}\pi_s\Big(\sum_{j\in I} g_j\psi_{js} + \sum_{j\in I} h_j\varphi_{js}\Big) & \\
\text{s. t.} \quad & \sum_{j\in I} x_{ij} = 1 & \forall i\in I \\
& \sum_{i\in I} x_{ii} = p & \\
& \sum_{i\in I} d_{is}x_{ij} - \psi_{js} + \varphi_{js} \le (1+\alpha)\mu_s x_{jj} & \forall j\in I, s\in S \\
& (1-\alpha)\mu_s x_{jj} \le \sum_{i\in I} d_{is}x_{ij} - \psi_{js} + \varphi_{js} & \forall j\in I, s\in S \\
& \varphi_{js} \ge 0 & \forall j\in I, s\in S \\
& \psi_{js} \ge 0 & \forall j\in I, s\in S \\
& x_{ij} \le x_{jj} & \forall i,j\in I \\
& x_{ij} \in \{0,1\} & \forall i,j\in I
\end{aligned}$$

Apart from the modified definition of μ_s described above, the SDP model has the same properties as the deterministic equivalent of SDPAR and, in particular, the recourse properties are preserved.

4.3 Performance Analysis: Model Evaluation and Correlation Comparisons

In Section 4.3.1, we define the most common values used to measure the benefits of stochastic models. In Section 4.3.2, we analyze the SDP and its solution to explore the relation between the input data and the benefits of using the model.

4.3.1 The Value of the Stochastic Solution and the Expected Value of Perfect Information

To evaluate the advantages of using a stochastic model over a deterministic model, various measures have been developed and are commonly used to assess the benefits of using these models. An introduction can be found in Birge and Louveaux (2011). Stochastic problems typically require more time to solve than deterministic models. This is because they involve multiple scenarios and additional variables and constraints to represent uncertainty, significantly increasing the model size and computational tractability. Hence, their benefits must be weighed against their additional burdens, such as extra CPU time.

The following measures are defined for minimization problems, analogous measures can be formulated for maximization problems.

Definition 22 (Expected Value Problem)
The Expected Value Problem (EV) is defined as the deterministic problem obtained by replacing all random variables with their expected values. The Expected result of the Expected Value Problem (EEV) is the expected objective value that results when the solution obtained from solving the EV problem is implemented in the stochastic setting.

Building upon the concept of the Expected Value problem, we can further analyze the performance of solutions in a stochastic setting.

Definition 23 (Value of the Stochastic Solution)
The Value of the Stochastic Solution (VSS) is defined as the difference between the EEV and the solution of the Stochastic Problem (SP):

$$VSS = EEV - SP$$

Frequently, the percentage notation is utilized:

$$\%VSS = \frac{EEV - SP}{SP} \cdot 100$$

To further explore decision-making under uncertainty, we can consider the approach of determining optimal decisions across all scenarios.

Definition 24 ("Wait and See" Solution)
The "Wait and See" Solution (WS) is calculated by first determining the optimal decision for each scenario and then taking the expected value of these optimal values across all scenarios.

Now, we can evaluate the benefits of obtaining perfect information in our decision-making process.

Definition 25 (Expected Value of Perfect Information)
The Expected Value of Perfect Information ($EVPI$) is defined as the difference between the WS and the "here-and-now" (SP) solution value:

$$EVPI = SP - WS$$

Frequently, the percentage notation is utilized:

$$\%EVPI = \frac{SP - WS}{SP} \cdot 100$$

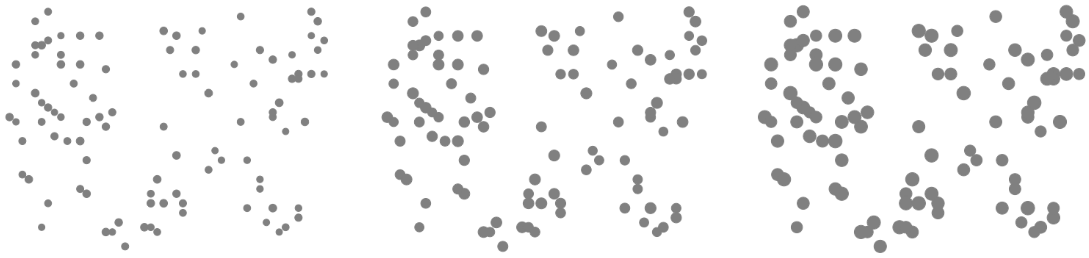
4.3.2 Data and Solution Analysis

In this section, the exploration focuses on the following questions:

- Is there a positive linear relation between the $\%VSS$ or $\%EVPI$ with the stochastic demand of a given instance?
- Can we determine from the given data whether it is beneficial to use a stochastic districting model instead of its static counterpart?

Since the SDP exhibits simple recourse, using the EEV always yields feasible districting decisions for the stochastic problem, although this solution is unlikely to be optimal.

Example 7 (Sufficiency of Static Models for Scenarios with Equal Relative Demands)
An example with three scenarios is shown in Figure 4.2, where each node represents a TU, and the size of the nodes indicates the demand for each TU. Based on the same reasoning as in Section 3.3, it can be said that if a feasible or optimal districting plan is provided for the scenario shown in Figure 4.2a, it is also feasible or optimal for the demand distributions of the scenarios in Figure 4.2b and Figure 4.2c. The total demands differ, but the relative demands are the same in each scenario, thus maintaining the balance.



(a) Scenario 1 **(b)** Scenario 2 **(c)** Scenario 3

Figure 4.2: A stochastic instance with different demands and three scenarios. The size of the nodes represents the demand.

Although Example 7 seems similar to the example in Section 3.3, there are clear differences between the stochastic and the multi-period districting problems. In the stochastic case, a single solution must be determined that is feasible for all scenarios, even though only one scenario will actually occur, and no reassignments are possible. In contrast, in the multi-period case, the demands for each period are known with certainty, and reassigning is allowed.

We now investigate whether a correlation exists with $\%EVPI$ or $\%VSS$, based on the districts found by the solution of the EV and the analysis of this solution.

To evaluate our approach, we define the number of districts with high fluctuations based on a threshold value τ, which indicates whether a district is highly fluctuating or not. This number is defined similarly to the number of highly fluctuating districts in a multi-period setting (Definition 20).

Definition 26 (Number of High Fluctuation Districts)
Let D_s denote the number of districts in scenario s for which the relative deviation from the average demand exceeds a threshold value τ. Formally, we define:

$$D_s = \left| \left\{ j \in I \mid \left(\frac{\psi_{js}}{\mu_s} + \frac{\varphi_{js}}{\mu_s} \right) x_{jjs} \geq \tau \right\} \right|$$

where ψ_{js} and φ_{js} represent the surplus and shortage for district j in scenario s, μ_s is the average demand in scenario s and x_{jjt} is is the decision variable which is 1 if j is the representative TU of an district in scenario s.

The sum over all scenarios is defined as:

$$\mathcal{D} = \sum_{s \in S} D_s$$

Note that $\psi_{js} \cdot \varphi_{js} = 0$ for all TUs j and scenarios s since a district can only have a surplus, shortage, or neither.

The following three-stage approach is analyzed, and the correlations between $\mathcal{D}$ and the $\%VSS$ and $\%EVPI$ are examined.

1. Firstly, the EV problem is solved.
2. Secondly, $\mathcal{D}$ for the resulting districts are calculated.
3. After that, the SP is solved and the $\%VSS$ and $\%EVPI$ are calculated.
4. Finally, the correlation between $\mathcal{D}$ and the $\%VSS$ and $\mathcal{D}$ and the $\%EVPI$ are calculated.

The $\%VSS$ is analyzed in Section 4.3.2.1, while the $\%EVPI$ is examined in Section 4.3.2.2. In both sections, we utilized the same randomly generated 500 instances, each containing 12 TUs. Additionally, 3 districts are used for each number of scenarios, ranging from 2 to 7.

4.3.2.1 Correlation Analysis: District Fluctuations and the VSS

The results of the analysis for the $\%VSS$ are displayed in Figure 4.3 and Table 4.1.

Using the same data generation as in the multi-period model in Section 3.3, it is evident that the $\%VSS$ values are significantly lower than the $\%VMPS$ values. This difference may be attributed to the fact that not all scenarios occur when analyzing scenarios, whereas, in a multi-period model, all future period data occur and must be considered (the probability for each period is 1). It is also interesting that the $\%VSS$ decreases as the number of considered scenarios increases. Consequently, the importance of using a stochastic model decreases as more scenarios are considered. In the multi-period model in Section 3.3, the $\%VMPS$ increases with the number of periods considered. Table 4.1 also shows that the correlation between the number of fluctuations within the groups and the $\%VSS$ is lower than in Section 3.3, but also remains relatively constant (averaging 0.26).

4.3.2.2 Correlation Analysis: District Fluctuations and the EVPI

Figure 4.4 and Table 4.2 show the results of the analysis with the $EVPI$. As can be seen, there is no clear trend indicating that the correlation between the fluctuations in the districts and the $\%EVPI$ consistently increases or decreases as the number of scenarios considered grows, although a slight positive correlation can be observed. In other words, while there appears to be a weak tendency for the correlation to rise with more scenarios, the relation is not strong across all cases examined. This means that simply increasing the number of scenarios does not guarantee a stronger association between district fluctuations and the $\%EVPI$.

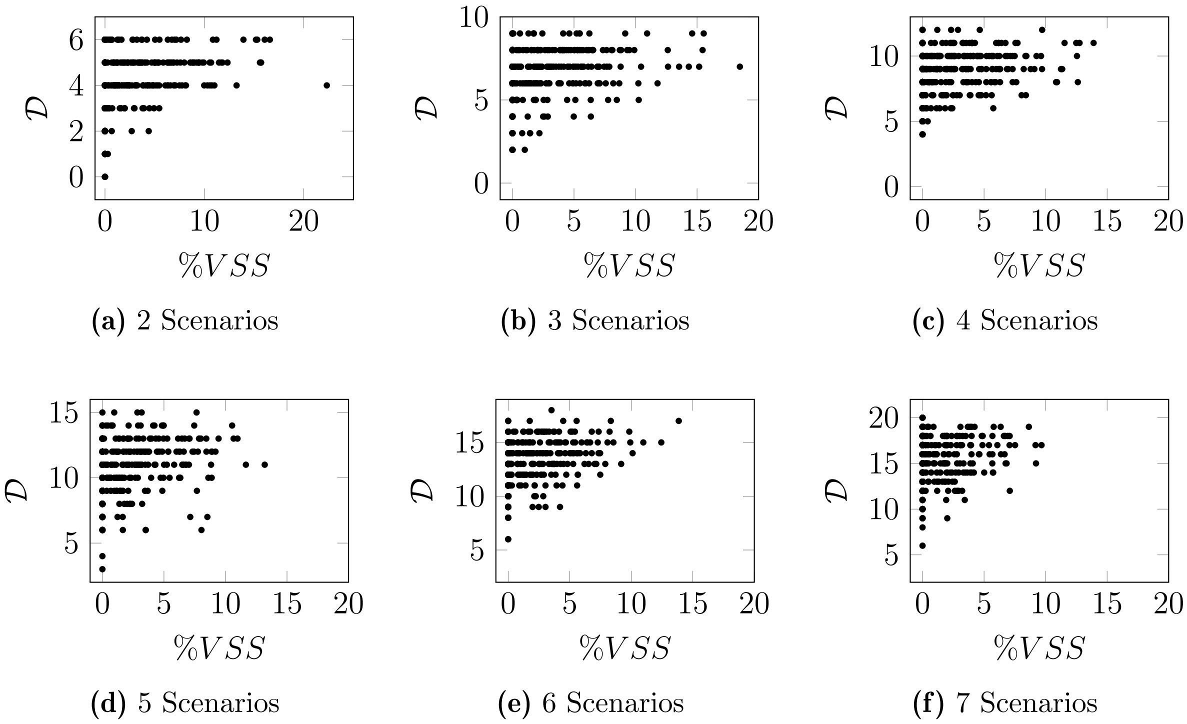


Figure 4.3: Linear correlations between $\mathcal{D}$ and $\%VSS$ for $|I| = 12$, $p = 3$, and multiple numbers of scenarios.

$\|I\|$	p	$\|S\|$	$Cor(\mathcal{D}, \%VSS)$	$\overline{\%VSS}$	$\overline{CPU_{SP}}$	$\overline{CPU_{EV}}$
12	3	2	0.30	2.0005	0.05	0.07
12	3	3	0.28	1.8095	0.05	0.07
12	3	4	0.26	1.5810	0.05	0.07
12	3	5	0.22	1.4872	0.06	0.08
12	3	6	0.31	1.2894	0.06	0.08
12	3	7	0.18	0.8861	0.07	0.08

Table 4.1: Computational results for $|I| = 12$, $p = 3$, and multiple numbers of scenarios.

However, it is worth noting that the average $\%EVPI$ itself tends to increase as the number of analyzed scenarios grows. This may be due to the fact that a larger set of scenarios captures a wider range of possible outcomes and uncertainties, thereby increasing the potential value of having perfect information. As more scenarios are included, the decision-making process can account for a broader spectrum of variability, which in turn may highlight the benefits of reducing uncertainty.

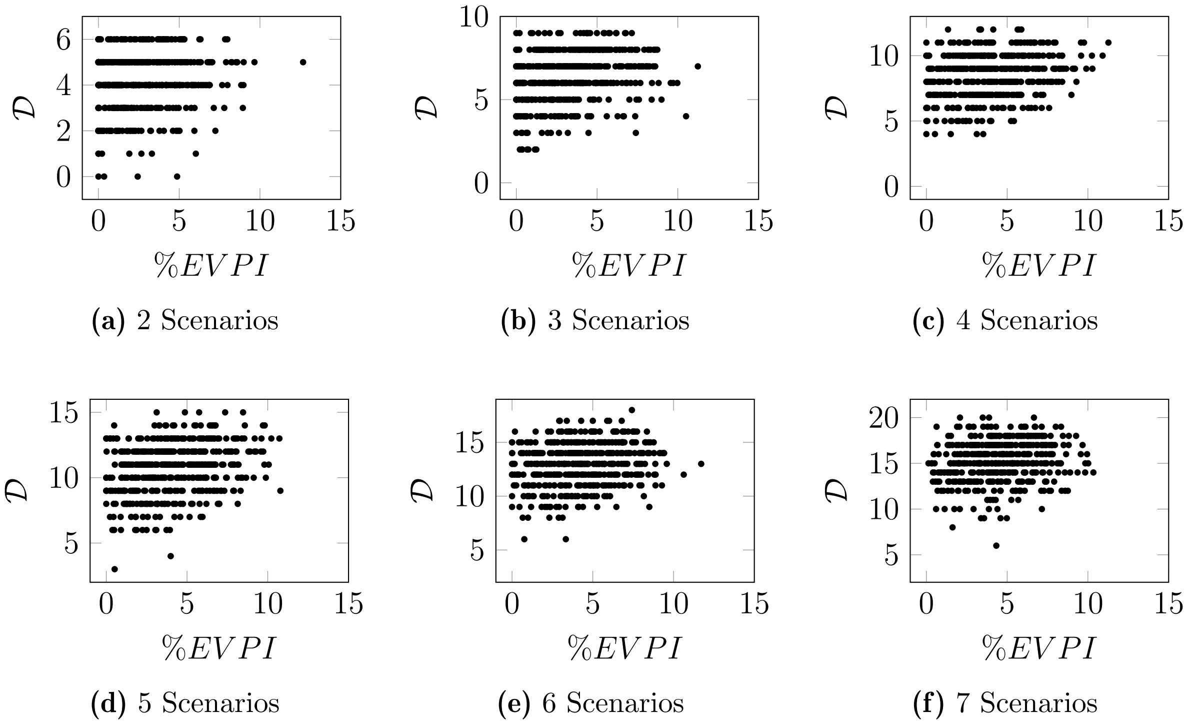


Figure 4.4: Linear correlations between $\mathcal{D}$ and $\%EVPI$ for $|I| = 12$, $p = 3$, and multiple numbers of scenarios.

4.4 Conclusion

This chapter first reviews the relevant literature on stochastic districting problems and identifies existing gaps. We adjust an existing two-stage stochastic model and modify the definition of the reference value μ_s. To ensure feasibility, we use surplus and shortage variables that prevent violations of the balancing constraints from making the problem unsolvable.

A detailed performance analysis of the VSS and $EVPI$ is conducted, examining the conditions under which applying the SDP is beneficial in stochastic contexts and when it may be less advantageous.

$\|I\|$	p	$\|S\|$	$Cor(\mathcal{D}, \%EVPI)$	$\overline{\%EVPI}$	$\overline{CPU_{SP}}$	$\overline{CPU_{EV}}$
12	3	2	0.16	2.4714	0.05	0.07
12	3	3	0.25	3.2667	0.05	0.07
12	3	4	0.23	3.9584	0.05	0.07
12	3	5	0.25	4.1307	0.06	0.08
12	3	6	0.21	4.3616	0.06	0.08
12	3	7	0.14	4.5104	0.07	0.08

Table 4.2: Computational results for $|I| = 12$, $p = 3$, and multiple numbers of scenarios.

The findings reveal a positive linear correlation between the number of highly fluctuating districts and both the VSS and the $EVPI$. To facilitate comparability between VSS and $VMPS$, the demand fluctuations in the scenarios in our tests corresponded to the demand fluctuations of the periods from Section 3.3.2. When we compare these results with the analysis of $VMPS$, it becomes clear that the impact of time outweighs that of uncertainty in the tested instances. This observation raises important questions about the relative effects of these two aspects. This effect may arise because, in a multi-period context, disadvantageous decisions can be avoided in each period by explicitly considering demand developments over time. When these expensive decisions accumulate over time, the overall outcome becomes worse than in a stochastic setting, where a disadvantageous decision occurs at most once. In Chapter 5, both aspects are analyzed together.

Chapter 5

Multi-Period Stochastic Districting

This chapter combines both dimensions that we examined separately: the aspect of multi-periods (Chapter 3) and stochasticity (Chapter 4), as illustrated in Figure 5.1. We examine the multi-stage stochastic model discussed in Pomes et al. (2025) and provide a comprehensive analysis of a multi-period stochastic districting problem, which represents the most complex setting in this thesis.

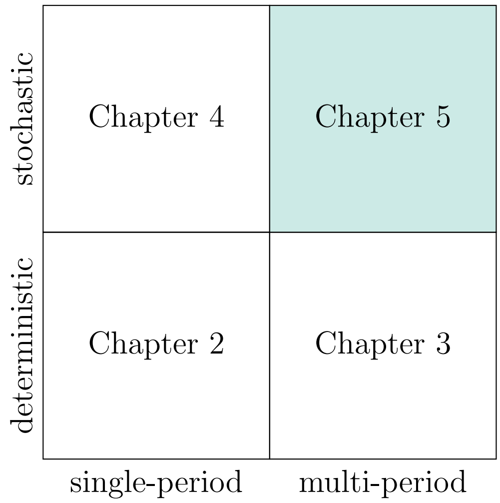


Figure 5.1: Research Focus of Chapter 5: Multi-period stochastic districting.

The benefit of using a multi-period stochastic model lies in its ability to account for uncertainties in demand over multiple time periods.

As discussed in the previous section, demand is not always known in advance. In many real-world problems, demand is uncertain and must be managed or anticipated accordingly. Additionally, it can be important to consider a longer planning horizon during which the underlying parameters may change. These two factors contribute to a more realistic and comprehensive representation of practical districting problems.

Chapter 3 and Chapter 4 already provide a thorough description of the stochastic and multi-period aspects of the problem. These sections serve as the foundation for the subsequent model and analysis that integrates both components. To address uncertain demands in future periods, districts can be reassigned after each period – either for the entire district plan or for specific districts. This structure characterizes it as a multi-stage problem.

This chapter is organized as follows:

- In Section 5.1, a literature review focusing on multi-stage problems is presented.
- Section 5.2 discusses a multi-period stochastic districting problem.
- In Section 5.3, a heuristic approach for the multi-period stochastic districting problem is outlined.
- Section 5.4 covers the dynamic value of the stochastic solution and the expected value of perfect information for multi-stage stochastic districting problems.
- Section 5.5 presents and discusses the computational experiments.
- Finally, Section 5.6 concludes this chapter.

5.1 Related Literature

An introduction to the foundational concepts applied in the following section is provided in the book *Facility Location Under Uncertainty* (Saldanha-da-Gama and Wang, 2024). As mentioned in the previous chapter, this book not only addresses districting under uncertainty but also systematically presents a range of modeling paradigms and solution techniques for managing uncertainty in facility location problems. The text introduces readers to basic and advanced concepts in facility location problems, such as stochastic programming, robust optimization, and chance-constrained programming.

Relevant literature focusing specifically on multi-period districting can be found in Section 3.1, while stochastic districting literature can be viewed in Section 4.1.

Bakker et al. (2020) review methods for optimizing multi-stage problems under uncertainty, noting the growing emphasis on the interaction between uncertainty and time. The authors highlight that existing methods vary in their representation of uncertainty and evaluation of performance, leading to a fragmented understanding. Their review aims to integrate these methods into a cohesive framework to improve sequential decision-making. They emphasize the importance of differentiating between uncertainty models and solution methods, as well as establishing standardized performance metrics for better optimization strategies in uncertain environments.

Lei et al. (2016) propose a solution to a multi-objective dynamic stochastic districting and routing problem. In this problem, the customers of a territory evolve stochastically over a planning horizon, and the authors consider several objectives, including the number of service vehicles, the compactness of the districts, the dissimilarity measure of the districts, and an equity measure of the vehicles' profits. The authors model and solve the problem as a two-stage stochastic program, where districting decisions are made in the first stage, and the expected routing cost of each district is approximated using an approximation formula in the second stage. They develop an enhanced multi-objective evolutionary algorithm, called the preference-inspired co-evolutionary algorithm, which utilizes mating restriction to solve the problem, and compare it with two state-of-the-art multi-objective evolutionary algorithms. Finally, the authors describe a procedure for selecting a preferred design for the proposed problem.

Despite the related literature above, there is a notable lack of studies specifically addressing multi-period stochastic districting. To the best of the author's knowledge, no additional references exist that comprehensively investigate this topic, highlighting a significant gap in the literature. The following chapter is therefore motivated by this gap and aims to advance the understanding of multi-period stochastic districting. For context and background, relevant literature on districting in the context of multi-period models and stochastic optimization is discussed in Section 3.1 and Section 4.1, respectively.

5.2 Multi-Period Stochastic Districting Model

As mentioned in Chapter 4, uncertainty is assumed to be represented by a finite set of previously identified scenarios. In a multi-period setting, this leads to the construction of a scenario tree. Each node in this tree – apart from the root – corresponds to the realization of all the (uncertain) parameters up to that node. The root node represents the initial setting (status quo), directly calling for a here-and-now decision to be made.

Starting from stage 2, at each stage, a new realization of demand occurs. This means that, from stage 2 onward and up to the final stage, the actual demand for the respective period is revealed at each node. At every stage except the last, it is possible to react to these realizations by planning and implementing reassignment decisions, as well as by considering the potential costs of shortages and surpluses. Thus, in stages 2 through the following stages, both reassignment and shortage/surplus considerations are possible and must be evaluated based on the newly revealed demand.

Thus, the nodes in stage 2 are represented by child nodes of the root. The tree proceeds by enumerating all the possible moments in which the costs can be accounted for and the districting can be changed. In the final stage, the only task remaining is to account for the costs of shortages and surpluses based on the observed demand, as no further reassignments are possible.

Figure 5.2 illustrates a multi-stage scenario tree with three stages. In the second stage, the uncertainty associated with that stage is revealed, and similarly, in the third stage, the uncertainty related to that stage is disclosed.

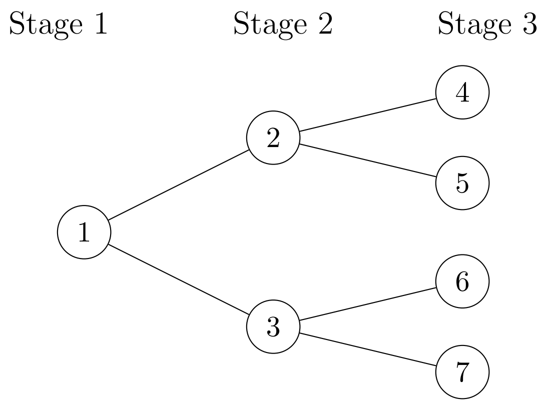


Figure 5.2: A multi-stage scenario tree – three stages in the planning horizon and four scenarios.

Remark 2
It is worth noting that the scenario tree in Figure 5.2, along with the described problem setting, can also be interpreted in time-related terms. Stage 1 marks the beginning of the planning horizon. Stage 2 represents the end of period one and the start of period two. Stage 3 marks the end of period 2 and the planning horizon. In other words, the example above depicts a scenario tree with three stages and two time periods. The demands are realized during a time period, i.e., between two consecutive stages. However, since the problem is not cast within a multi-horizon setting (Escudero and Monge, 2018; Kaut et al., 2014), there is no reason to make a difference between periods and stages. Hence, only the term stage is used hereafter.

In the given terminology, one scenario is a full sequence of events from the first stage to the last one. In other words, one scenario fully determines all the information for the entire planning horizon and thus induces a deterministic multi-period districting problem as defined in Section 3.2. In Figure 5.2, four possible sequences can be observed, i.e., four scenarios. Each scenario is associated with one and only one leaf node in the scenario tree. In the example, there are four scenarios. For instance, the scenario culminating in node 5 consists of the sequence of events leading from the status quo (stage 1) to the possible future observed in stage 2, which is represented by node 2, and, finally, to the possible future in stage 3, which is represented by node 5.

Based on the representation used for uncertainty, a multi-stage stochastic programming model can be developed for the problem. To develop such a model, some additional notation is introduced as follows:

$\mathcal{N}$ set of nodes in the scenario tree

M number of stages in the decision-making process

$\mathcal{N}_m$ set of nodes in stage $m \in M$

Ω set of scenarios in the scenario tree

Since there is only one path from the root node to every leaf in the tree, a scenario is fully identified by the corresponding leaf node.

$\gamma(n)$ immediate predecessor of node $n \in \mathcal{N} \setminus \{1\}$

$m(n)$ stage to which node $n \in \mathcal{N}$ belongs to

π^n probability associated with node $n \in \mathcal{N} \setminus \{1\}$

It is unnecessary to consider the probability associated with node 1 since it is equal to 1 (the node corresponds to the current state of nature). Note also that in each stage, one node occurs for sure. Therefore, the probabilities of the nodes are such that, for every stage $m \in M \setminus \{1\}$, the sum of π^n over $n \in \mathcal{N}_m$ is equal to 1.

In the case of the scenario tree depicted in Figure 5.2, there is $\mathcal{N} = \{1, \ldots, 7\}$ and $\Omega = \{4, 5, 6, 7\}$. For instance, $\mathcal{N}_2 = \{2, 3\}$, $\gamma(5) = 2$, $m(6) = 3$.

To formulate the multi-stage stochastic programming problem, we adopt the notation previously presented, specifically c_{ij} for $i, j \in I$, and α, maintaining their meanings as indicated before. It is important to note that a node $n \in \mathcal{N} \setminus \Omega$ in the scenario tree represents stage $m(n)$. For that stage, it is known that the number of districts should be $p_{m(n)}$. To simplify the notation, this number is represented by p^n, and it is associated with node n. In other words, the value p^n $(n \in \mathcal{N} \setminus \Omega)$ is the same for all nodes in the same stage.

To ensure that a general setting is investigated, it is assumed that all the other parameters are stochastic. This calls for them to be redefined as follows:

d_i^n demand of TU $i \in I$ in stage $m(n)$ if node $n \in \mathcal{N} \setminus \{1\}$ in the scenario tree occurs

Note that the demand of each TU in each stage is unknown beforehand. In the scenario tree, node n belongs to stage $m(n)$, so it represents a possible realization that may be observed. Thus, for some node $n \in \mathcal{N} \setminus \{1\}$, d_i^n represents one possible observation of the demand of TU i in stage $m(n)$.

μ^n $= \frac{1}{p_{m(n)-1}} \sum_{i \in I} d_i^n$ $(n \in \mathcal{N} \setminus \{1\})$ average demand for each district

Reference value for the demand that should be assigned to each district in stage $m(n)$, the stage of node n. As the number of districts p^n is only defined for $n \in \mathcal{N} \setminus \Omega$, we have to shift the stage index by -1.

g_j^n unit penalty cost for surplus at the district represented by TU j in node n $(j \in I,\, n \in \mathcal{N} \setminus \{1\})$

Due to uncertainty, the penalty for surplus can be assessed over time. Hence, in the stochastic model, these costs are associated with nodes in stages $2, \ldots, M$.

h_j^n unit penalty cost for shortage at the district represented by TU j in node n $(j \in I,\, n \in \mathcal{N} \setminus \{1\})$

As for the surplus costs, in the stochastic model, these costs are associated with nodes in stages $2, \ldots, M$.

r_{ij}^n unit cost for reassigning TU i to the district represented by TU j in node n $(i, j \in I,\, n \in \mathcal{N} \setminus (\Omega \cup \{1\}))$

Note that reassignments are made neither at stage 1 (node 1) nor at stage M (nodes in Ω).

s_{ij}^n unit saving for removing TU i from the district represented by TU j in node n $(i, j \in I,\, n \in \mathcal{N} \setminus (\Omega \cup \{1\}))$

As for the reassignment costs, only stages $2, \ldots, M-1$ are considered (from stage 2 to stage $M-1$).

Concerning the decision variables, they are organized into two sets and can be formally defined as follows:

(i) Variables corresponding to planning for a stage. These correspond to (re-)districting decisions.

$$x_{ij}^n = \begin{cases} 1 & \text{if TU } i \text{ is assigned to the district represented by TU } j \text{ in node } n \\ 0 & \text{otherwise.} \end{cases}$$
$$(i, j \in I,\ n \in \mathcal{N} \setminus \Omega)$$

$$v_{ij}^n = \begin{cases} 1 & \text{if TU } i \text{ is reassigned to the district represented by TU } j \text{ in node } n \\ 0 & \text{otherwise.} \end{cases}$$
$$(i, j \in I,\ n \in \mathcal{N} \setminus (\Omega \cup \{1\}))$$

$$w_{ij}^n = \begin{cases} 1 & \text{if TU } i \text{ is removed from the district represented by TU } j \text{ in node } n \\ 0 & \text{otherwise.} \end{cases}$$
$$(i, j \in I,\ n \in \mathcal{N} \setminus (\Omega \cup \{1\}))$$

(ii) Variables accounting for the shortage and surplus in each stage. Note that such values can only be assessed after the demand in the stage is disclosed.

ψ_j^n demand surplus in the district represented by TU j in node n ($j \in I$, $n \in \mathcal{N} \setminus \{1\}$).

φ_j^n demand shortage in the district represented by TU j in node n ($j \in I$, $n \in \mathcal{N} \setminus \{1\}$).

Similar to the stochastic districting problem outlined in Section 4.2, which has simple recourse, we only evaluate costs in the last stage, where penalty costs for surplus or shortage are calculated. In all other stages, (re-)assignment decisions are necessary or allowed.

Using the node-indexed decision variables presented, we can formulate a multi-stage stochastic programming model for the multi-period stochastic districting problem being investigated.

The Multi-Stage Stochastic Districting Problem (MSSDP) can be finally formulated as the following model:

MSSDP:
$$\begin{aligned} \min \quad & \sum_{i \in I} \sum_{j \in I} c_{ij} x_{ij}^1 \left(\sum_{n \in \mathcal{N} | \gamma(n) = 1} \pi^n d_i^n \right) \\ & + \sum_{n \in \mathcal{N} \setminus (\mathcal{N}_1 \cup \mathcal{N}_2)} \sum_{i \in I} \pi^n d_i^n \left(\sum_{j \in I} (r_{ij}^{\gamma(n)} v_{ij}^{\gamma(n)} - s_{ij}^{\gamma(n)} w_{ij}^{\gamma(n)}) \right) \\ & + \sum_{n \in \mathcal{N} \setminus \{1\}} \pi^n \sum_{j \in I} \left(g_j^n \psi_j^n + h_j^n \varphi_j^n \right) \end{aligned} \tag{5.1}$$

$$\text{s. t.} \quad \sum_{j \in I} x_{ij}^n = 1 \qquad \forall i \in I,\ n \in \mathcal{N} \setminus \Omega \tag{5.2}$$

$$\sum_{i \in I} x_{ii}^n = p^n \qquad \forall n \in \mathcal{N} \setminus \Omega \tag{5.3}$$

$$x_{ij}^n \leq x_{jj}^n \qquad \forall i,j \in I,\ n \in \mathcal{N} \setminus \Omega \tag{5.4}$$

$$v_{ij}^n \geq x_{ij}^n - x_{ij}^{\gamma(n)} \qquad \forall i,j \in I,\ n \in \mathcal{N} \setminus (\Omega \cup \{1\}) \tag{5.5}$$

$$w_{ij}^n \leq x_{ij}^{\gamma(n)} \qquad \forall i,j \in I,\ n \in \mathcal{N} \setminus (\Omega \cup \{1\}) \tag{5.6}$$

$$w_{ij}^n + x_{ij}^n \leq 1 \qquad \forall i,j \in I,\ n \in \mathcal{N} \setminus (\Omega \cup \{1\}) \tag{5.7}$$

$$\sum_{i \in I} d_i^n x_{ij}^{\gamma(n)} - \psi_j^n + \varphi_j^n \leq (1+\alpha)\mu^n x_{jj}^{\gamma(n)} \qquad \forall j \in I,\ n \in \mathcal{N} \setminus \{1\} \tag{5.8}$$

$$(1-\alpha)\mu^n x_{jj}^{\gamma(n)} \leq \sum_{i \in I} d_i^n x_{ij}^{\gamma(n)} - \psi_j^n + \varphi_j^n \qquad \forall j \in I,\ n \in \mathcal{N} \setminus \{1\} \tag{5.9}$$

$$x_{ij}^n \in \{0,1\} \qquad \forall i,j \in I,\ n \in \mathcal{N} \setminus \Omega \tag{5.10}$$

$$v_{ij}^n \in \{0,1\} \qquad \forall i,j \in I,\ n \in \mathcal{N} \setminus (\Omega \cup \{1\}) \tag{5.11}$$

$$w_{ij}^n \in \{0,1\} \qquad \forall i,j \in I,\ n \in \mathcal{N} \setminus (\Omega \cup \{1\}) \tag{5.12}$$

$$\psi_j^n \geq 0 \qquad \forall j \in I,\ n \in \mathcal{N} \setminus \{1\} \tag{5.13}$$

$$\varphi_j^n \geq 0 \qquad \forall j \in I,\ n \in \mathcal{N} \setminus \{1\} \tag{5.14}$$

The objective function 5.1 represents the total expected cost (initial districting, plus territory redesign, shortages, and surplus).

Constraints 5.3 ensure that the required number of districts for each node is met. Constraints 5.5,5.6, and 5.7 ensure that both reassignments and removals are accurately accounted for, allowing for the correct calculation of reassignment costs and savings in the objective function. The constraints that are less straightforward are Constraints 5.8 and 5.9. Consider a node $n \in \mathcal{N} \setminus \{1\}$. This means that in stage $m(\gamma(n)) + 1 \equiv m(n)$, the demand is observed as part of the realization leading to node n, i.e., d_i^n, $i \in I$. The shortage and surplus at the district represented by TU j are represented by ψ_j^n and φ_j^n, respectively. However, the shortage and surplus depend on the (re-)districting in stage $m(\gamma(n))+1 \equiv m(n)$, which is represented by variables $x_{ij}^{\gamma(n)}$. On the other hand, the reference value for this stage depends on the observed demand, which explains the use of value μ^n in Constraints 5.8 and 5.9. This explanation is illustrated in Figure 5.3. Constraints 5.10–5.13 specify the variable domain constraints.

In the objective function, different cost factors are considered: penalty costs and (re-) assignment costs/savings. Due to the non-negativity of parameters g_j^n and h_j^n, surplus and shortages can be calculated as follows:

$$\psi_j^n = \max\{0, \sum_{i \in I} d_i^n x_{ij}^{\gamma(n)} - (1+\alpha)\mu^n x_{jj}^{\gamma(n)}\} \qquad j \in I, n \in \mathcal{N} \setminus \{1\}$$

$$\varphi_j^n = \max\{0, (1-\alpha)\mu^n x_{jj}^{\gamma(n)} - \sum_{i \in I} d_i^n x_{ij}^{\gamma(n)}\} \qquad j \in I, n \in \mathcal{N} \setminus \{1\}$$

In practice, they account for the deviation from the upper and lower thresholds in the balancing constraints. These quantities turn out to be crucial for the trade-off between

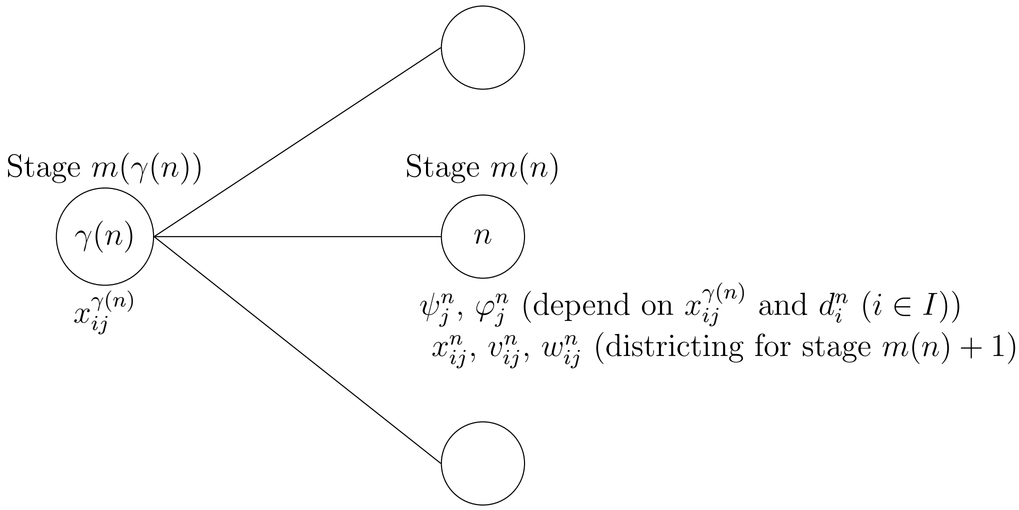


Figure 5.3: Illustration of the building blocks for Constraints 5.8 and 5.9.

two different courses of action that the model is seeking throughout the planning horizon: Reassignment of TUs and performing extra actions to compensate for shortages/surplus. Hence, the problem has a bi-objective character, although it is not explicitly handled in a multi-criteria setting.

5.3 Relax-and-Fix Heuristic

The mathematical model introduced for the MSSDP in the previous section becomes computationally intractable as the number of scenarios in the scenario tree may grow exponentially with the number of stages. Therefore, this section proposes a heuristic algorithm designed to efficiently identify high-quality feasible solutions to the problem.

The procedure relies on resolving the problem by considering a restricted model in which the set of representative TUs that can be selected in each node $n \in N \setminus \Omega$ of the scenario tree is limited to a restricted set of candidates, say C_n. Recall that in the last stage, M, only reassignments, surplus, and shortages based on districting decisions made in stage $M-1$ are accounted for, which explains why no restricted sets are defined in stage M. Thus, the restricted model, which is called (MSSDP-R), results from (MSSDP) with the following additional constraints:

$$\sum_{j \in C_n} x_{ij}^n = 1, \qquad i \in I,\ n \in \mathcal{N} \setminus \Omega, \tag{5.15}$$

$$x_{ij}^n = 0, \qquad i \in I,\ j \in I \setminus C_n,\ n \in \mathcal{N} \setminus \Omega. \tag{5.16}$$

Constraints 5.15 allow the assignment of TU i in node n only to a potential representative, while Constraints 5.16 forbid assignments to non-potential representatives. Note that it is possible to avoid the introduction of variables $x_{ij}^n, i \in I,\ j \in I \setminus C_n,\ n \in \mathcal{N} \setminus \Omega$ although they are kept for a clearer exposition.

The heuristic is formalized in Algorithm 1. Lines 1–8 seek to iteratively define the restricted sets C_n, $n \in N \setminus \Omega$. Initially, no restriction is imposed on the candidate sets C_n (line 1). Thus, all sets coincide with I, and models (MSSDP) and (MSSDP-R) coincide (all the TUs are initially regarded as potential representatives).

A loop starts in stage 1 and ends in stage $M-1$ to define all the restricted sets to consider in each stage. In particular, in iteration m, the sets C_n for nodes n in stage m become fixed. This is accomplished as follows. First, the linear relaxation of (MSSDP-R) is solved. Note that, in the current iteration m, the restricted sets C_n for all stages before m have been fixed, which is already reflected in (MSSDP-R) and thus in its linear relaxation, via Constraints 5.15 and 5.16, that should be updated each time new restricted sets are found. Now, the solution of the linear relaxation is examined, and the values of the self-assignment variables x_{jj}^n for n in stage m are retrieved. Those greater than zero provide a candidate for being a TU representative. Accordingly, every set C_n in stage m is built from scratch using only such TUs (lines 5–8). When the model (MSSDP-R) is solved next time, these restricted sets are updated accordingly.

When the sets C_n are fully determined, the model (MSSDP-R) (that now has embedded all the restricted sets) is solved. This is done in line 9 of Algorithm 1. The obtained solution is the approximation proposed for the optimal solution of MSSDP.

Algorithm 1

1: $C_n \leftarrow I, n \in N \setminus \Omega$ // set of candidate representatives for each node
2: **for** $m \in \{1, .., M-1\}$ **do** // for each stage, except the last one
3: Solve MSSDP-R. Denote the corresponding solution by $\bar{x}$
4: **for** $n \in N_m$ **do** // examine the solution
5: $C_n \leftarrow \emptyset$
6: **for** $j \in I$ **do**
7: **if** $\bar{x}_{jj}^n > 0$ **then**
8: $C_n \leftarrow C_n \cup \{j\}$ // store TU j as a potential representative for node n
9: Solve MSSDP-R
10: **return** $x^\star$ // a feasible solution for MSSDP

Example 8 (Applying the Fix-and-Relax Heuristic for an Instance)
Consider revisiting the scenario tree depicted in Figure 5.2 (three stages, two time periods). When fixing the restricted sets, the first iteration ($m = 1$) refers to the first stage. Here, only node 1 requires analysis. From the solution of the linear relaxation of (MSSDP-R) ($C_n = I$, $n \in N \setminus \Omega$), set C_1 is identified. Then, the second iteration begins. At this point, when solving the model (MSSDP-R), set C_1 is fixed according to the first iteration, and thus, the assignments in node 1 can only be made to potential representatives in set C_1. Since this is the second iteration ($m = 2$), the focus shifts to stage 2 and, consequently,

to nodes 2 and 3. The solution to the linear relaxation of (MSSDP-R) is analyzed, and sets C_2 and C_3 are determined. No further restricted sets need to be identified since the penultimate stage has been reached. Therefore, the process exits the main loop and solves the model (MSSDP-R) with sets C_1, C_2, and C_3 fixed as described.

5.4 Quantifying Uncertainty: Stochastic Solutions and Perfect Information

Evaluating a multi-stage stochastic model involves assessing the advantages of using it over a simpler model. In two-stage stochastic programming, this is often assessed by the value of the stochastic solution (Definition 23). This value can be extended to the multi-stage case in different ways (Escudero et al., 2007; Ziegler, 2012). The underlying idea is to compute an expected value solution and compare it with the multi-stage solution (Saldanha-da-Gama and Wang (2024); Pomes et al. (2025).

Another important value for evaluating multi-period stochastic problems is the $EVPI$, as explained in Definition 25. The subsequent subsection provides a detailed explanation of the calculation of both values.

5.4.1 The Dynamic Value of the Stochastic Solution and the Expected Value of Perfect Information

As described in Section 4.3.1, in stochastic programming, the VSS measures the advantages of incorporating uncertainty into a stochastic problem. Similarly, the dynamic version of the VSS should address uncertainty at each stage of a multi-stage problem, rather than depending only on deterministic expected values.

To calculate the dynamic version of the VSS, we follow this process:

1. Compute the expected values for all random variables and determine the deterministic solution.
2. Fix the first-stage decision.
3. For each node in the subsequent stage of the scenario tree, compute the conditional expected values for all random variables in the subtree.
4. Solve the deterministic problem that is induced by each node.
5. Use these solutions to fix the decisions for the current stage.
6. Proceed to the next stage and repeat the process until the leaf nodes of the scenario tree are reached.

According to this procedure, a model of the EV problem, here EV^n is solved in every node of the scenario tree (Definition 22). We denote its optimal value by Z_{EV^n}. This is done sequentially for stages $1, 2, \ldots, M-1$. Every model EV^n "aggregates" the costs from stage $m(n)$ to the end of the planning horizon.

Following Escudero et al. (2007), the expected result in stage m of using the dynamic solution of the average scenario is defined and denoted as $EDEV_m$ ($m = 1, \ldots, M-1$). This represents the expected value of the optimal results from the problems EV^n, with $n \in \mathcal{N}$, such that $m(n) = m$. In other words, $EDEV_m$ is defined as:

$$EDEV_m = \sum_{n \in \mathcal{N}_m} \pi^n Z_{EV^n}, \quad m = 1, \ldots, M-1.$$

Definition 27 (Dynamic Value of the Stochastic Solution)
The Dynamic Value of the Stochastic Solution ($DVSS$) is defined as:

$$DVSS = EDEV_{M-1} - SP,$$

or

$$\%DVSS = \frac{EDEV_{M-1} - SP}{SP} \cdot 100,$$

where SP is the optimal value of the multi-stage stochastic model.

The $DVSS$ provides valuable insight into how well the expected value solution approximates the optimal solution of the multi-stage stochastic model. Another value to measure the relevance of capturing uncertainty in the problem is the Expected Value of Perfect Information ($EVPI$), which is already defined in Section 4.3.2.

The $EVPI$ measures the maximum price a decision maker would be willing to pay for access to complete information about future demand realization(s).

In contrast to Section 4.3.2, which deals with single-period stochastic problems, we need to calculate the optimal values for each corresponding multi-period deterministic (single-scenario) problem. We denote these values as Z_ω.

According to Definition 24, the value of the WS solution is then defined as:

$$WS = \sum_{\omega \in \Omega} \pi^\omega Z_\omega,$$

and, finally as in Definition 25:

$$EVPI = SP - WS,$$

or

$$\%EVPI = \frac{SP - WS}{WS} \cdot 100.$$

5.5 Computational Experiments

This section reports on the extensive computational experiments performed to validate the proposed model for the MSSDP and the heuristic proposed for approximating its optimal solution.

The results of the computational experiments are structured as follows:

- Section 5.5.1: **Test Data and Implementation Details**
  This section describes the test data used in the computational experiments.
- Section 5.5.2: **Performance Analysis: Model Evaluation and Solution Analysis**
  This section provides an overview of the extensive results obtained from the analysis.
  - Section 5.5.2.1: **In-Depth Analysis of Selected Instances**
    This part offers a detailed examination of the results for a specific instance and extends the analysis further.
  - Section 5.5.2.2: **Cost Breakdown, Demand Variability, and the Role of Savings**
    This section presents additional insights, including:

    (i) Cost Breakdown for Multiple Values of α
    A breakdown of costs through analysis.

    (ii) Varying Demand Scenarios
    An assessment of different demand scenarios.

    (iii) The Role of Savings
    An analysis of parameter savings.
- Section 5.5.3: **Performance Analysis: Heuristic Evaluation and Solution Analysis**
  This section provides results and an analysis of the effectiveness of the proposed heuristic approach.
  - Section 5.5.3.1: **Solution Comparison: Exact Model and Heuristic**
    This part shows the comparison between the solutions derived from the model and the results obtained from the proposed heuristic.
  - Section 5.5.3.2: **In-Depth Exploration of Larger-Sized Instances**
    This section presents additional results for larger-sized instances, focusing on the number of TUs and stages.

    (i) The Impact of the Number of TUs
    Analyzes the impact of the number of TUs on both the solution quality and runtime.

    (ii) The Impact of the Number of Stages
    Examines the impact of the number of stages on both the solution quality and runtime.

5.5.1 Test Data and Implementation Details

For the computational experiments, the set of instances from Diglio et al. (2021) is used as a starting point. These instances use real geographical data corresponding to the province of Novara in Italy. Specifically, there are 88 point-like basic TUs ($|I| = 88$), which are determined as the centroids of the municipalities in the province (Figure 5.5).

To investigate the relevance of considering a multi-stage decision-making process under uncertainty, the focus is on a small case with three stages, as depicted in Figure 5.4. Hence, for illustrative purposes, four scenarios ($|\Omega| = 4$) and seven nodes ($|N| = 7$) are considered. The probabilities of reaching each node $n \in N$ from the immediate predecessor are the same for each node.

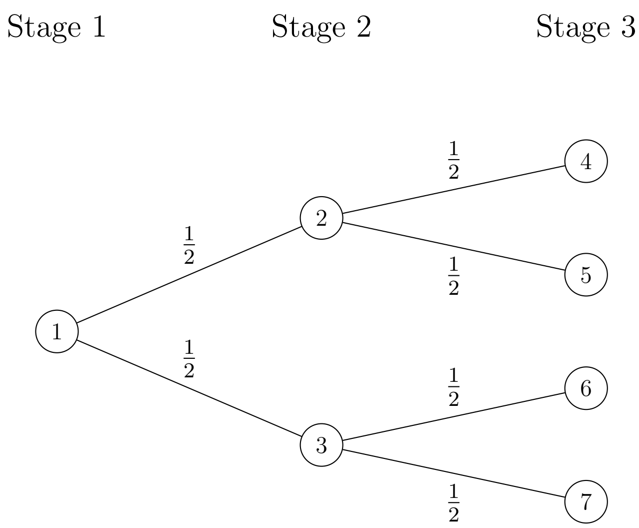


Figure 5.4: A scenario tree with three stages and probabilities of reaching each node starting from its predecessor.

The following introduces a setting where the only stochastic parameter is represented by demands. The corresponding data is obtained as follows. The demands in node 1 are generated assuming they are represented by random variables following a uniform distribution with a fixed expected value equal to 50, and relative standard deviation (RSD) equal to 0.1 [1]. It is assumed that the demands in the lower branch of the scenario tree remain unchanged. This means that for $i \in I$, $d_i^3 = d_i^6 = d_i^7 = d_i^1$.

However, some variability is introduced in the upper branch of the scenario tree, specifically in nodes 2, 4, and 5. For node 2, it is assumed that we have a 50% probability that TUs

[1] For a random variable d, its RSD is given by the ratio between the square root of its variance ($\mathbb{V}ar[d]$) and its expected value ($\mathbb{E}[d]$), namely: $\frac{\sqrt{\mathbb{V}ar[d]}}{\mathbb{E}[d]}$. When considering such values, for a uniform distribution in the range $[a, b]$, it is possible to calculate a as $\mathbb{E}[d](1 - \sqrt{3} \cdot RSD)$, and b as $\mathbb{E}[d](1 + \sqrt{3} \cdot RSD)$.

will experience a 25% reduction in demand compared to that in node 1. To achieve this, a random number λ_i^2 is generated following a Bernoulli distribution with a parameter of 0.50. If the generated random number equals one, the demand in node 2 is computed as 75% of the corresponding demand in node 1. Otherwise, the demand in node 2 will remain the same as that in node 1. The same mechanism applies to nodes 4 and 5, with a 50% probability of a 50% and 75% reduction in demand, respectively.

The generation process can replicate situations where demand originating from TUs may undergo unexpected severe reductions or increases in the future. In summary, there is:

$$d_i^n = \begin{cases} (1-\theta^n)\cdot d_i^{\gamma(n)}, & \text{if } \lambda_i^n = 1, \\ d_i^{\gamma(n)}, & \text{if } \lambda_i^n = 0. \end{cases} \qquad i \in I, n \in \{2,4,5\}$$

with $\theta^2 = 0.25, \theta^4 = 0.50$, and $\theta^5 = 0.75$.

A certain procedure is followed to create ten instances, referred to as *Instances_1*. Additionally, a second set of instances, called *Instances_2*, is created, in which the same demand-generation procedure is applied only to the southern TUs. These TUs are 50% of the centroids with the lowest y-coordinates. Doing so enforced a local trend in demand variation. This approach is expected to make balancing constraints more challenging to meet in the following stages, which may result in higher reassignment/penalty costs. In Figure 5.5, the southern TUs are displayed as empty dots.

Concerning the other parameters underlying the given instances, they are set as follows:

- Assignment costs: $c_{ij} = \ell_{ij}$, $i, j \in I$, where ℓ_{ij} is the Euclidean distance between TUs i and j;
- The probabilities of the nodes $n \in \mathcal{N} \setminus \{1\}$ in the scenario tree are set equal to $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{4}$, and $\frac{1}{4}$ for nodes 2 to 7, respectively.
- The reassignment costs are defined as $r_{ij}^n = \ell_{ij}$, $i, j \in I$, $n \in \mathcal{N} \setminus \{\Omega \cup \{1\}\}$;
- The savings from removing a TU from a district are determined as $s_{ij}^n = \zeta \cdot l_{ij}$, $i, j \in I$, $n \in \mathcal{N} \setminus (\Omega \cup \{1\})$, with $\zeta \in [0, 1]$. Hence, the savings are defined as a fraction of the assignment costs. $\zeta = 0$ is set to zero, which means no savings are considered.
- Finally, for the penalty costs, the maximum distance between any pair of TUs is considered as $g_j^n = h_j^n = \max_{i,j \in I}\{\ell_{ij}\}$, $j \in I$, $n \in \mathcal{N} \setminus \{1\}$. In other words, it is set as the maximum distance between any pair of TUs.

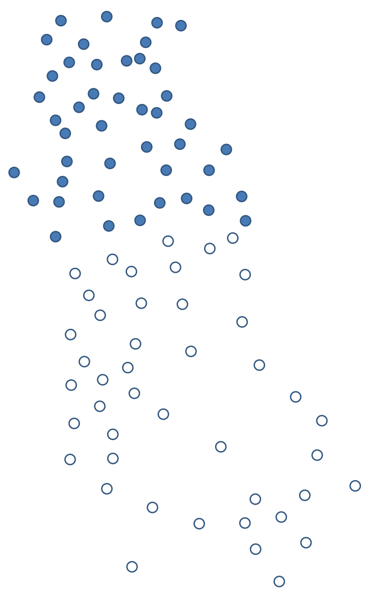


Figure 5.5: Test instance – basic TUs corresponding to the centroids of the province of Novara. Northern centroids are marked in blue, while southern centroids are indicated with empty dots.

Remark 3
The settings discussed above lead to an interesting interpretation of the unit penalty costs. To clarify, we focus on a specific TU, denoted as i. It is supposed that by reassigning this TU in node n to a new district k, a surplus in the district it currently belongs to, say j, can be avoided. The corresponding penalty and reassignment costs can be computed respectively, as $g_j \cdot d_i^n$, and $\ell_{ik} \cdot d_i^n$. Therefore, a reassignment is performed only if $\ell_{ik} < g_j$. The same logic can be applied to shortages. In practice, penalty costs can be interpreted as the maximum distance within which reassignments are worth accepting. In this specific case, a TU is reassigned as long as its distance from the new representative is less than the maximum distance among the TUs. Otherwise, penalties are preferred. This observation is in line with the already mentioned bi-objective character of the problem (Section 5.2).

Once the above parameters are fixed, various experiments are realized by varying the value of the tolerance $\alpha \in \{0.05, 0.10, 0.15, 0.20, 0.25\}$, and the number of districts $p \in \{4, 6\}$. In total, 200 experiments are performed, resulting from five values of α, two values of p, and ten different instances for both *Instances_1* and *Instances_2*.

All experiments are performed with the computational environment described in Section 1.3.

A time limit of three hours is set for the solution of the MSSDP. The procedures for calculating the $DVSS$ and the $EVPI$ are only run when the corresponding MSSDP is solved up to proven optimality within the prescribed time limit. For both procedures, an overall time limit of three hours – 10800 seconds – is considered.

5.5.2 Performance Analysis: Model Evaluation and Solution Analysis

The following section discusses the results obtained by solving the model MSSDP for all the instances generated as described in Section 5.5.1. The results are summarized in Table 5.2 and Table 5.1. For every instance, the presented information includes:

- The computational performance of the model, by showing: (i) the number of optimal solutions – out of ten – obtained within the imposed time limit; (ii) the minimum, maximum, and average optimality gap of the obtained solutions (the gap equals zero when solutions are optimal); (iii) the minimum, maximum, and average computing times. The table displays "t.l." under the maximum computing time column if at least one instance is not optimally solved within the time limit. Of course, if no optimal solutions are obtained, "t.l." would also occur for the minimum and average cases.
- The importance of accounting for uncertainty in the problem can be evaluated using the $\%EVPI$, which represents the value of having perfect information – it measures how much better the solution could be if all uncertainty were resolved. Meanwhile, the $\%DVSS$ can be used to assess how closely the expected value solution approximates the optimal solution to the stochastic problem. These indicators are only calculated when the corresponding model's solutions are optimal. Accordingly, "N/A" is displayed if none of the ten instances are solved to proven optimality.

Starting with a focus on *Instances_1*, it can be observed that the model can optimally solve all the tested instances for $p = 4$ within acceptable computing times, averaging 117 seconds. In fact, the number of optimal solutions is the same for all considered values of the tolerance α, equaling 10. It is also worth highlighting that computing times tend to increase as α decreases. This result is not unexpected, as lower values of α tighten the balancing constraints, thus making them harder to meet. This emphasizes the trade-off between the (re-)assignments and penalty costs when seeking the optimal solution to the problem. Interestingly, this finding is reflected by the distributions of the $\%DVSS$ and $\%EVPI$. In particular, as α increases, the $\%DVSS$ reduces to about 8% for $\alpha = 0.20$ and $\alpha = 0.25$. This suggests that as instances become relatively easier to solve, deterministic problems with expected values can produce better approximate solutions to the MSSDP. Nevertheless, the above values are not negligible. Furthermore, note that the $\%DVSS$ equals 15.50% on average, with a peak of more than 30% for $\alpha = 0.10$. Therefore, it can be argued that explicitly considering uncertainty in the model is crucial. The relevance of hedging against uncertainty is also supported by the obtained values for the $\%EVPI$.

A similar behavior is observed for $p = 6$. However, these instances proved harder to solve, especially for $\alpha = 0.05$. Indeed, under such settings, the model can not obtain the optimal solutions for two instances, exhibiting an optimality gap equal to 1.71% in the worst case. Recall that an increase in the value of p reduces the lower and upper thresholds used in the balancing constraints. This effect is amplified if it jointly occurs with a decrease of α. Finally, no significant differences are found in terms of the $\%DVSS$ and $\%EVPI$.

The computational performance of the model drastically reduces when tackling *Instances_2*. In particular, optimal solutions are not obtained for $p = 6$ and $\alpha = 0.05$. Remarkably, the corresponding average optimality gap is about 7% in this case. Also, the running times increase significantly, being higher than 3000 and 5000 seconds for $p = 4$ and $p = 6$, respectively. These results highlight that resorting to heuristic procedures is necessary to reduce the computational effort required to obtain feasible (and, hopefully, higher-quality) solutions to the problem, even for small-sized instances, as shown in Section 5.5.3.

The key difference between *Instances_1* and *Instances_2* lies in the procedure for generating demand. In *Instances_2*, the same demand-generation process used for *Instances_1* is applied exclusively to the southern TUs. This induces a local trend in demand variation, concentrating changes in the southern TUs. Therefore, the poorer performance observed with *Instances_2* can be attributed to this geographically localized shift in demand.

Additionally, it is observed that the uncertainty considerations turn out to be more relevant for these instances, as the higher values of $\%DVSS$ and $\%EVPI$ reveal, which suggests that capturing uncertainty becomes more relevant when it only affects a subset of locally-distributed TUs. More detailed results can be found in Appendix A, Table A1 – Table A4. Moreover, a deeper look into these indicators is given in Section 5.5.2.1.

5.5.2.1 In-Depth Analysis of Selected Instances

This subsection provides information about two specific instances to illustrate the relevance of capturing uncertainty in the problem, in addition to what is already shown in Section 5.5.2.

Recall that the instances tested are of two types, namely *Instances_1* and *Instances_2*, which differ depending on whether the demand generation mechanism described in the previous section applies to all the TUs or only the southern ones. The instances analyzed are those defined by $p = 4$ and $\alpha = 0.10$: one from *Instances_1*, and one from *Instances_2*. They are denoted by *Instance_a* and *Instance_b*, respectively.

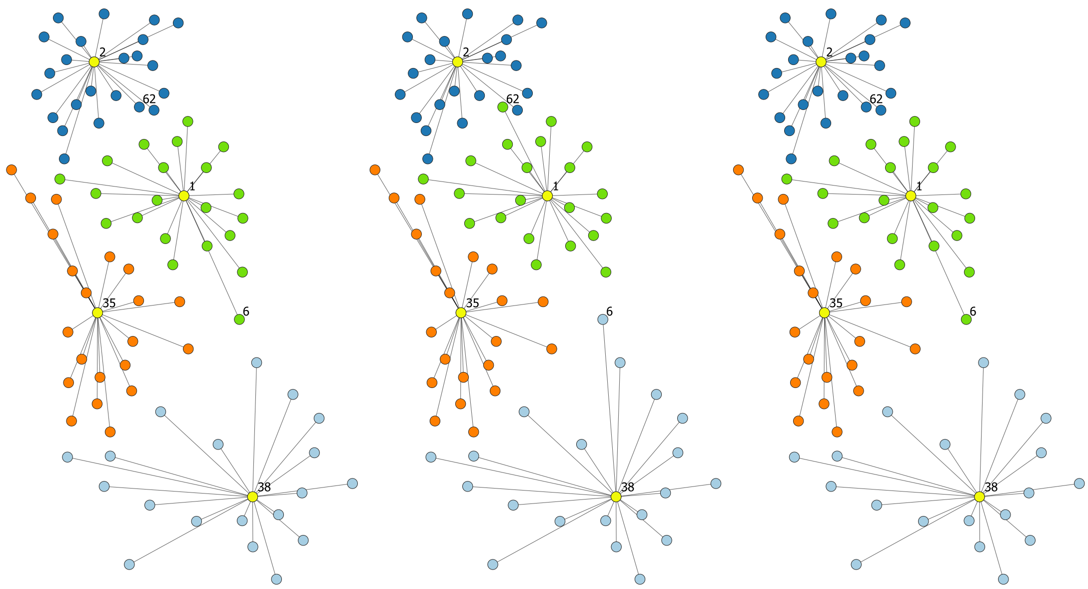
First, the focus is on *Instance_a*. Figure 5.6 depicts the maps of the first- and second-stage solutions (i.e., nodes 1, 2, and 3). In the figure, the representatives of the four districts are highlighted in yellow. As reallocation decisions only apply in stage 2, the districting plans obtained for nodes 2 and 3 hold for their immediate successors (in this case, the leaves of the scenario tree). Therefore, the corresponding third-stage solutions are not shown.

Instance	p	α	# Optimal (out of 10)	Optimality Gap (%)			Computing time (sec.)			$\%DVSS$			$\%EVPI$		
				Min	Max	Avg	Min	Max	Avg	Min	Max	Avg	Min	Max	Avg
Instances_1	4	0.05	10	0	0	0	26	3101	390	2.61	28.01	11.25	0.4	2.5	1.39
		0.1	10	0	0	0	19	165	59	6.77	49.65	30.45	0.1	1.76	0.86
		0.15	10	0	0	0	23	386	101	11.35	38.57	19.57	0.39	2.06	1.15
		0.2	10	0	0	0	13	29	19	0.96	28.33	8.25	0.26	1.53	0.95
		0.25	10	0	0	0	12	21	14	0.1	23.46	7.96	0.29	1.39	0.61
		Total	50	0	0	0	12	3101	117	0.1	49.65	15.50	0.1	2.5	0.99
	6	0.05	8	0	1.71	0.22	163	t.l.	4738	6	37.12	21.55	1.45	4.4	2.73
		0.1	10	0	0	0	13	296	88	0.42	32.14	15.65	0.1	2.17	1.14
		0.15	10	0	0	0	14	70	29	5.61	43.71	17.25	0.32	1.59	0.86
		0.2	10	0	0	0	11	27	17	2.05	22.08	10.69	0.12	1.45	0.59
		0.25	10	0	0	0	10	21	14	0	16.3	7.72	0	1.24	0.38
		Total	48	0	1.71	0.04	10	t.l.	977	0	43.71	14.57	0	4.4	1.14

Table 5.1: Summary of the results obtained by solving the MSSDP with *Instances_1*.

Instance	p	α	# Optimal (out of 10)	Optimality Gap (%)			Computing time (sec.)			%$DVSS$			%$EVPI$		
				Min	Max	Avg	Min	Max	Avg	Min	Max	Avg	Min	Max	Avg
Instances_2	4	0.05	2	0	1.95	0.76	1307	t.l.	9775	44.58	50.23	47.4	5.43	7.05	6.24
		0.1	8	0	0.88	0.13	247	t.l.	3928	43.56	55.11	48.58	2.96	8.52	6.42
		0.15	10	0	0	0	46	8362	1525	42.44	65.79	55.69	2.62	8.79	5.77
		0.2	10	0	0	0	37	1594	468	41.48	59.19	50.5	3.13	8.2	5.72
		0.25	10	0	0	0	71	588	297	24.34	60.68	43.91	2.51	7.26	5.09
		Total	40	0	1.95	0.18	37	t.l.	3199	24.34	65.79	49.22	2.51	8.79	5.85
	6	0.05	0	2.7	11.56	7.13	t.l.	t.l.	t.l.	N/A	N/A	N/A	N/A	N/A	N/A
		0.1	1	0	2.5	1.28	8417	t.l.	10568	32.27	32.27	32.27	2.7	2.7	2.7
		0.15	10	0	0	0	140	10385	3640	42.21	52.57	48.35	1.86	6.53	4.15
		0.2	10	0	0	0	30	453	127	33.63	54.15	47.33	1.2	4.92	2.57
		0.25	10	0	0	0	17	238	48	18.15	56.01	37.73	1.07	3.19	1.73
		Total	31	0	11.56	1.68	17	t.l.	5037	18.15	56.01	41.42	1.07	6.53	2.79

Table 5.2: Summary of the results obtained by solving the MSSDP with *Instances_2*.



(a) First stage – node 1 **(b)** Second stage – node 2 **(c)** Second stage – node 3

Figure 5.6: Solutions for *Instance_a* (*Instances_1*, $p = 4, \alpha = 0.10$).

Firstly, it is observed that the model suggests a first-stage solution that remains unchanged in node 3 (Figure 5.6a and Figure 5.6c). Nonetheless, some variations are observed in node 2 (Figure 5.6b). In particular, TUs 6 and 62 are reallocated, respectively, from the green and dark blue districts to the cyan and green ones. Such behavior is consistent with the characteristics of the instances. Indeed, demands do not vary across the lower branch of the tree (Nodes 3, 6, and 7). Additionally, it indicates that the model aims for a relatively "robust" first-stage solution that remains stable over time, adapting when necessary to changing demands to fulfill balancing requirements.

The need for such reassignments can be explained as follows: if the first-stage solution changes, it necessarily means that such a districting plan fails to be balanced in at least one of the following nodes. To confirm this fact, the demand vector associated with each node $n \in \{2, \ldots, N\}$ needs to be considered and checked whether the first-stage solution is balanced for that demand occurrence. In this case, some violations of the balancing requirements must be seen to justify the observed reallocations as a viable (and "cheaper") action to avoid or reduce penalty costs.

Table 5.3 shows the computation of demand associated with each first-stage district represented by TU j in node n as $\sum_{i \in I} d_i^n \overline{x}_{ij}^1$, where $\overline{x}_{ij}^1$ denotes the values of the first-stage decision variables. Additionally, the table reports the total demand in each node (i.e., $\sum_{i \in I} d_{ij}^n$) and the corresponding lower and upper bounds in the balancing requirements (LB and UB, respectively, with $LB = (1/p)(1-\alpha) \sum_{i \in I} d_{ij}^n$ and $UB = (1/p)(1 + \alpha) \sum_{i \in I} d_{ij}^n$). By using this information, districts violating these thresholds can be easily identified.

As highlighted in bold in the table, such a circumstance occurs in two cases, both in node 4: a surplus in district 2 and a shortage in district 38. Accordingly, it is suggested to modify the first-stage solution in node 2 by:

(i) "unloading" district 2 through the removal of TU 62 and its reassignment to district 1;

(ii) "loading" district 38 with TU 6 (reassigned from district 1). Due to such moves, balancing constraints are never violated in the first-stage solution. Therefore, no penalty costs are paid.

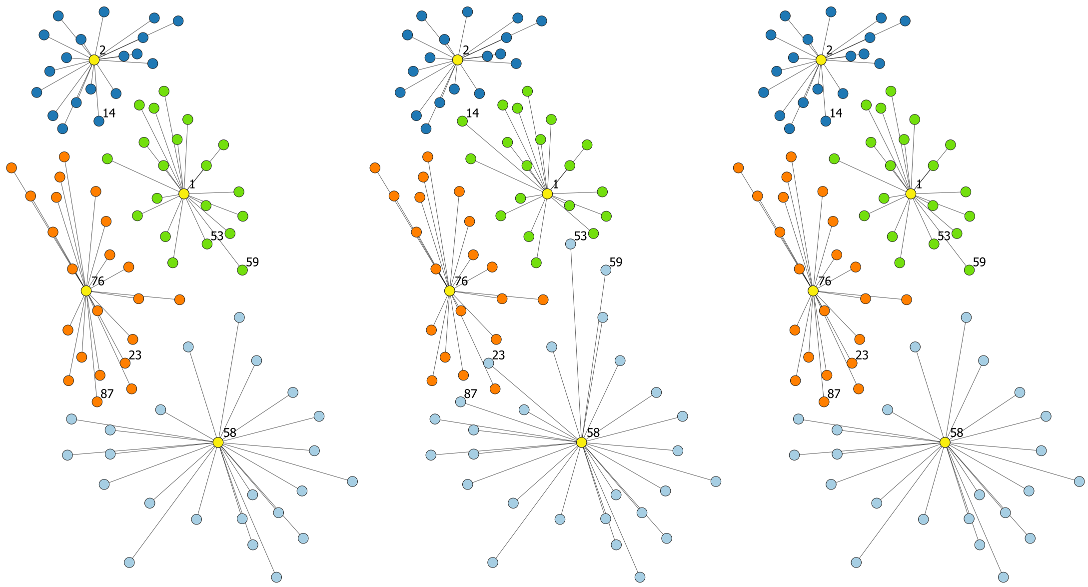
District	node 1	node 2	node 3	node 4	node 5	node 6	node 7
1	1106	982.5	1106	660.75	611.25	1106	1106
2	1195	1039.25	1195	**792.25**	684.69	1195	1195
35	1103	975.75	1103	780.25	563.63	1103	1103
38	993	907	993	**634**	637	993	993
Total demand	4397	3904.5	4397	2867.25	2496.56	4397	4397
LB	989.33	878.51	989.33	645.13	561.73	989.33	989.33
UB	1209.18	1073.74	1209.18	788.49	686.55	1209.18	1209.18

Table 5.3: *Instance_a* (*Instances_1*, $p = 4, \alpha = 0.10$): Analysis of the balance in the first-stage solution.

Similar considerations can be drawn for *Instance_b*, depicted in Figure 5.7. However, it should be noted that five reassignments are performed in node 2: TU 14 from the dark blue to the green district; TUs 23, 53, 59, and 87 from the green and orange districts to the cyan one. This outcome reflects that demand variations only interest southern TUs; hence, adjustments are needed to avoid penalty costs, especially in the cyan district.

For the above solutions, the $\%DVSS$ and the $\%EVPI$ are also computed. The $\%DVSS$ equals 41.76% and 55.11% for *Instance_a* and *Instance_b*, respectively. These values reveal that a sequence of deterministic problems, obtained by replacing the uncertain demands with their expectations, produces a poor approximation to the MSSDP.

Furthermore, this implies that the two approaches produce distinct solutions to the problem. Some differences can be observed between the first-stage solutions yielded by the MSSDP and the EV problem rooted in node 1 (EV^1, see Section 5.4), as shown in Figure 5.8 and Figure 5.9. In particular, it is worth underlining that the latter is characterized



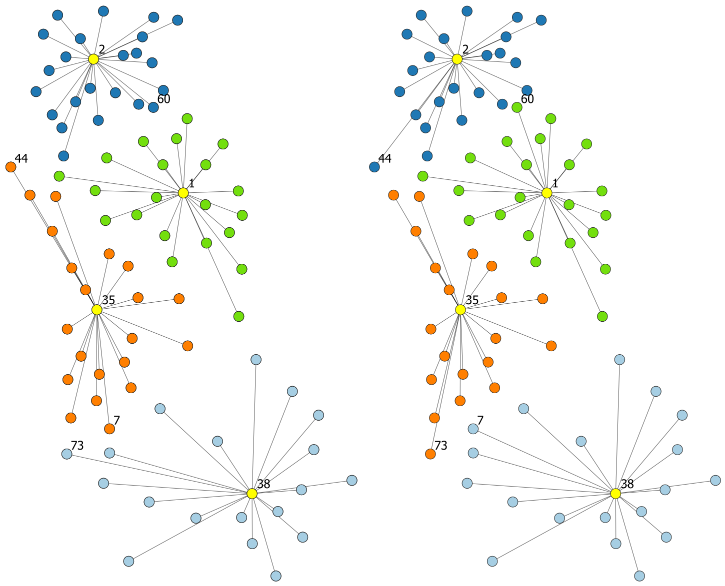
(a) First stage – node 1 **(b)** Second stage – node 2 **(c)** Second stage – node 3

Figure 5.7: Solutions for *Instance_b* (*Instances_2*, $p = 4, \alpha = 0.10$).

by better compactness values (i.e., lower first-stage allocation costs). Nevertheless, it turns out to be weak when employed as a first-stage solution to the stochastic problem. The model's behavior can be observed: paying more for the initial districting to obtain significant savings in the expected reallocation/penalty costs.

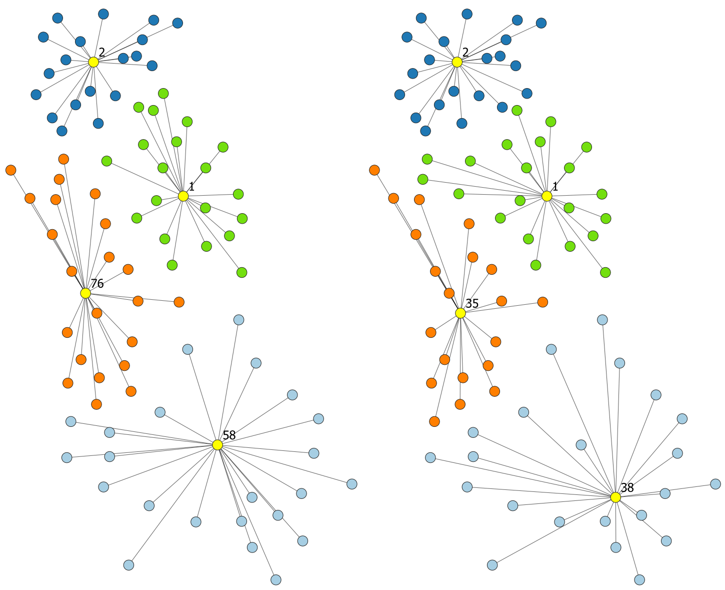
The relevance of capturing uncertainty in these two particular instances is also shown by the value of the $\%EVPI$, which equals 0.94% for *Instance_a* and 7.25% for *Instance_b*. Both the $\%DVSS$ and $\%EVPI$ are higher in the case of *Instance_b*. As shown Section 5.5.2 – Section 5.5.3, such a finding holds in more general terms for the whole set of *Instances_2*. This is an interesting result emerging from the given analysis, as it proves that deterministic approximations can be less effective and also that accessing perfect information about the future becomes more relevant when uncertainty only affects a subset of locally distributed TUs.

To better understand the above values of the $\%DVSS$ and $\%EVPI$, the following Table 5.4 reports for both *Instance_a* and *Instance_b*: (i) the objective function (OF) and its three components, i.e., (ii) the initial assignment costs (AssCosts), (iii) the reassignment costs (ReassCosts), and (iv) the penalty costs (PenCosts) for both the MSSDP model, which has its first-stage solution illustrated in Figure 5.8a, and the expected value approximation of the solution, referred to as $EDEV_{M-1}$, for which the first-stage solution is represented as EV^1 in Figure 5.8b. Hence, the assignment costs in the table are, in fact, the assignment costs associated with the above displayed solutions. The same values are also shown for WS.



(a) MSSDP **(b)** EV^1

Figure 5.8: *Instance_a* (*Instances_1*, $p = 4, \alpha = 0.10$) – First-stage solutions yielded by the MSSDP and the Expected Value solution.



(a) MSSDP **(b)** EV^1

Figure 5.9: *Instance_b* (*Instances_2*, $p = 4, \alpha = 0.10$) – First-stage solutions yielded by the MSSDP and the Expected Value solution.

Instance		OF	AssCosts	ReassCosts	PenCosts
Instance_a	MSSDP	$2.95 \cdot 10^7$	$2.93 \cdot 10^7$	$2.15 \cdot 10^5$	0.00
	$EDEV_{M-1}$	$4.18 \cdot 10^7$	$2.91 \cdot 10^7$	$1.10 \cdot 10^7$	$1.66 \cdot 10^6$
	WS	$2.92 \cdot 10^7$	$2.92 \cdot 10^7$	0.00	0.00
Instance_b	MSSDP	$3.35 \cdot 10^7$	$3.19 \cdot 10^7$	$1.60 \cdot 10^6$	0.00
	$EDEV_{M-1}$	$5.19 \cdot 10^7$	$3.11 \cdot 10^7$	$1.89 \cdot 10^7$	$2.01 \cdot 10^6$
	WS	$3.12 \cdot 10^7$	$3.12 \cdot 10^7$	$5.78 \cdot 10^4$	0.00

Table 5.4: Cost breakdown for *Instance_a* and *Instance_b*.

As the table reports, the reassignment costs in the $EDEV_{M-1}$ are much higher than in the MSSDP. Also, penalty costs are paid for the $EDEV_{M-1}$. This explains the difference between the overall objective function values and, hence, the high $\%DVSS$. When looking at the WS solution, neither reassignment nor penalty costs are identified. Additionally, the initial assignment costs are slightly lower than the MSSDP, which explains the low value found for the $\%EVPI$.

The following Section 5.5.2.2 offers additional insights and results from the MSSDP solution, including a breakdown of costs, scenarios with increased demand, and an examination of how savings influence the outcomes.

5.5.2.2 Cost Breakdown, Demand Variability, and the Role of Savings

This section explores three further aspects of relevance to the problem:

(i) a cost breakdown analysis, to assess the values of the $\%DVSS$ and $\%EVPI$ discussed above for multiple values of α

(ii) the effect of considering both scenarios with decreasing and increasing demand

(iii) the effect of savings.

(i) Cost Breakdown for Multiple Values of α

Hereafter, a cost breakdown analysis of the obtained solutions is presented to explain the values of the $\%DVSS$ and $\%EVPI$ that emerged from the performed experiments. To this end, a subset of solutions is considered – specifically, those obtained for *Instances_1*, $p = 4$, and $\alpha = 0.10, 0.15, 0.20$ (i.e., rows 2, 3, and 4, respectively, in Table 5.1).

For these 30 solutions, Table 5.5 reports: (i) the objective function (OF) and its three components, i.e., (ii) the initial assignment costs (AssCosts), (iii) the reassignment costs (ReassCosts), and (iv) the penalty costs (PenCosts) for the MSSDP, its expected value approximation, i.e., the $EDEV_{M-1}$, and the wait-and-see (WS) solution. For brevity, the above indicators are expressed as the average of the ten solutions obtained for each value of α.

α		OF	AssCosts	ReassCosts	PenCosts
0.10	MSSDP	$2.93 \cdot 10^7$	$2.92 \cdot 10^7$	$1.10 \cdot 10^5$	$4.60 \cdot 10^4$
	$EDEV_{M-1}$	$3.83 \cdot 10^7$	$2.90 \cdot 10^7$	$8.29 \cdot 10^6$	$9.15 \cdot 10^5$
	WS	$2.91 \cdot 10^7$	$2.91 \cdot 10^7$	0.00	$4.66 \cdot 10^3$
0.15	MSSDP	$2.88 \cdot 10^7$	$2.86 \cdot 10^7$	$8.51 \cdot 10^4$	$5.39 \cdot 10^4$
	$EDEV_{M-1}$	$3.44 \cdot 10^7$	$2.84 \cdot 10^7$	$4.87 \cdot 10^6$	$1.12 \cdot 10^6$
	WS	$2.85 \cdot 10^7$	$2.85 \cdot 10^7$	$3.93 \cdot 10^3$	$3.14 \cdot 10^3$
0.20	MSSDP	$2.82 \cdot 10^7$	$2.80 \cdot 10^7$	$1.64 \cdot 10^5$	$2.40 \cdot 10^4$
	$EDEV_{M-1}$	$3.05 \cdot 10^7$	$2.79 \cdot 10^7$	$1.78 \cdot 10^6$	$8.42 \cdot 10^5$
	WS	$2.79 \cdot 10^7$	$2.79 \cdot 10^7$	$5.45 \cdot 10^3$	$1.27 \cdot 10^4$

Table 5.5: Cost breakdown for a subset of obtained optimal solutions (88-TUs, $p = 4$, *Instances_1*).

The comparison between the MSSDP and $EDEV_{M-1}$ explains the high observed $\%DVSS$. As the table shows, the objective function of the $EDEV_{M-1}$ is higher than the MSSDP regardless of the value of α. Indeed, approximating the stochastic program by its expected value leads to a more compact first-stage solution (i.e., the Expected Value solution rooted in node 1, i.e., the EV^1). The latter, in fact, has lower initial assignment costs than the first-stage solution of the MSSDP (see the "AssCosts"-column). However, this turns out to be ineffective when hedging against uncertainty, leading to more expensive reassignments and penalties for unmet balancing.

The comparison between the WS and MSSDP reveals that, regardless of α, accessing perfect information about uncertainty can foster the identification of more compact first-stage solutions (the initial assignment costs – "AssCosts" – are lower than the WS). Also, it helps hedge against uncertainty more effectively, as the lower reassignment costs and penalties reveal. However, the differences are somewhat limited when one looks at the values involved, which clarifies the low observed $\%EVPI$ values.

(ii) Varying Demand Scenarios

So far, instances with decreasing demand scenarios have been considered. In this section, additional tests are performed where changes in the demand can occur also in the lower branch of the tree, i.e., in nodes 3, 6, and 7. Specifically, it is assumed that demands can increase following the procedure for their generation described in Section 5.5.1 (and using the same values of θ therein defined). In practice, demands in node 3 can increase by 25% compared to node 1, while in nodes 6 and 7 they can increase by 50% and 75% w.r.t. node 3, respectively. These increases are represented by the random variable λ_i, which takes the value 1 with probability 0.5 and 0 with probability 0.5. Mathematically, the following holds:

$$d_i^n = \begin{cases} (1-\theta^n)\cdot d_i^{\gamma(n)}, & n \in \{2,4,5\}, \qquad \text{if } \lambda_i^n = 1, \\ (1+\theta^n)\cdot d_i^{\gamma(n)}, & n \in \{3,6,7\}, \qquad \text{if } \lambda_i^n = 1, \\ d_i^{\gamma(n)}, & n \in \mathcal{N} \setminus \{1\}, \qquad \text{if } \lambda_i^n = 0. \end{cases}$$

with $\theta^2 = \theta^3 = 0.25, \theta^4 = \theta^6 = 0.50$, and $\theta^5 = \theta^7 = 0.75$.

We test the 88-TUs of the type of *Instances_1* are considered and solved for $p = 4$ and $\alpha \in \{0.10, 0.15, 0.20\}$, thus resulting in 30 new experiments (10 instances for each combination of p and α). Table 5.6 summarizes the main findings, by showing, for each value of α the average (i) $\%DVSS$, (ii) $\%\ EVPI$, and (iii) CPU Time (in sec.). Specifically, results are given for both cases, i.e., without and with increasing demands (denoted by "w/o" and "w", respectively). Note that results for the "with"-cases are the same as in Table 5.1 and Table 5.2.

	$\%DVSS$		$\%EVPI$		CPU Times	
α	w/o	w	w/o	w	w/o	w
0.10	30.45	31.85	0.87	1.05	59	154
0.15	19.57	33.18	1.15	1.13	101	82
0.20	8.25	18.03	0.95	1.28	19	95

Table 5.6: Results for the 88-TUs instances without (w/o) and with (w) increasing demand scenarios.

The table reveals several key findings. Firstly, the values of %DVSS are consistently higher for the instances including increasing demand scenarios. This indicates that deterministic approximations are less accurate when considering mixed-demand scenarios. Secondly, slightly higher values of the %EVPI are observed, meaning that accessing perfect information has higher relevance in such a setting. Thirdly, the instances appear to be relatively more challenging to solve — except for $\alpha = 0.15$ — as the corresponding running times underscore. Overall, these initial findings suggest that this line of investigation holds promise. However, further and more extensive analysis is needed to confirm their generalizability.

(iii) The Role of Savings

This section aims to demonstrate how savings can influence the solutions generated by the proposed model. To achieve this, additional computational experiments are conducted. For illustrative purposes, we consider a specific instance selected from the ten generated for *Instances_2* and test it with parameters $p = 4$ and $\alpha = 0.25$, by varying the savings s^n_{ij}. Recall that savings obtained by removing a TU from a district are defined as follows: $s^n_{ij} = \zeta \cdot l_{ij}$, $i, j \in I$, $n \in \mathcal{N} \setminus (\Omega \cup \{1\})$, with $\zeta \in [0, 1]$. Hence, the savings are a fraction of the assignment costs. In these new experiments, ζ varied from 0 (no savings considered) to 1 (where the savings equal the initial assignment cost) with a rate of 0.2. The obtained results are summarized in Table 5.7, which reports, for each tested value of ζ: (i) the objective function (OF) and its three components, i.e., (ii) the initial assignment costs (AssCosts), (iii) the reassignment costs (ReassCosts, net of the savings), (iv) the penalty costs (PenCosts), and (v) the CPU Times needed to solve these instances up to optimality (in seconds).

ζ	OF	AssCosts	ReassCosts	PenCosts	CPU Time
0	$3.16 \cdot 10^7$	$3.00 \cdot 10^7$	$1.57 \cdot 10^6$	$3.01 \cdot 10^4$	588
0.2	$3.15 \cdot 10^7$	$3.00 \cdot 10^7$	$1.45 \cdot 10^6$	$3.01 \cdot 10^4$	3105
0.4	$3.14 \cdot 10^7$	$3.00 \cdot 10^7$	$1.34 \cdot 10^6$	$3.01 \cdot 10^4$	4677
0.6	$3.13 \cdot 10^7$	$3.00 \cdot 10^7$	$1.22 \cdot 10^6$	$3.01 \cdot 10^4$	21 941
0.8	$3.05 \cdot 10^7$	$2.88 \cdot 10^7$	$1.74 \cdot 10^6$	0.00	17 289
1	$2.81 \cdot 10^7$	$9.83 \cdot 10^7$	$-7.02 \cdot 10^7$	0.00	5272

Table 5.7: Assessing the impact of savings (88-TUs instances, $p = 4$, $\alpha = 0.25$, *Instances_2*).

As the table shows, the model produces the same optimal solutions for ζ values up to 0.6. Indeed, the initial assignment costs and the penalty costs are the same, while the reassignment costs are obviously different because of the different savings being considered.

When looking at the solution obtained for $\zeta = 0.8$, it can be observed that the first-stage solution has lower initial assignment costs but higher reassignment costs. However, such reassignments allow the model to identify a fully balanced solution, i.e., with penalty costs equal to 0. This fact underscores the impact of savings. Indeed, if they are in play, the model can seek more compact first-stage solutions and has an incentive for numerous (cheaper) reassignments in the second stage. In turn, these reassignments can help avoid penalty costs.

For lower values of ζ, instead, penalty costs have to be paid. Indeed, to avoid them and maintain balance, numerous (highly expensive) reassignments would be needed. This explains why, to reduce reassignments and penalties, a less compact first-stage solution is observed. It is worth noticing that these findings further underline the inherent multi-objective character of the model.

For $\zeta = 1$, the initial assignment costs are high, suggesting a poorly compact first-stage solution. However, the value of reassignment costs is negative, indicating that the savings outweigh the reassignment costs. This means that numerous and less expensive reassignments are made to evolve the initial solution into a more compact and fully balanced districting plan (penalty costs are again zero).

Finally, it is observed that the impact of savings is significant, with computational times at least five times greater than in the case whith $\zeta = 0$.

5.5.3 Performance Analysis: Heuristic Evaluation and Solution Analysis

This step of analysis consists of assessing the performance of the heuristic introduced in Section 5.3. To this end, Algorithm 1 is applied to the whole set of generated instances and benchmarked its results against those obtained from the implementation of the model (MSSDP).

5.5.3.1 Solution Comparison: Exact Model and Heuristic

A brief comparative assessment is reported in Table 5.8, which displays, for each instance: (i) the minimum, maximum, and average gap between the objective function values (Δ_{OF}); (ii) the number of optimal solutions yielded by the model ("Model") and the heuristic ("Heur"); (iii) the minimum, maximum, and average computing times required by the model and the heuristic. Note that the above-mentioned gap is computed as $\Delta_{OF} = 100 \cdot (Z_{Heur} - Z_{Model})(Z_{Model})$, with Z_{Model} and Z_{Heur} denoting the objective function values for the solutions obtained by the model and heuristic, respectively. Thus, a negative deviation indicates cases in which the heuristic solution outperforms the best feasible solution found by the model. Clearly, such a circumstance may occur only when the model does not achieve an optimal solution within the imposed time limit.

As the table shows, the heuristic produces good solutions, particularly for *Instances_1* and $p = 4$, where it achieves the optimal solution in 42 out of 50 cases. In the remaining ones, the corresponding gaps are small, being equal – at most – to 0.83% (for $\alpha = 0.05$) and 0.03% on average. The number of optimal solutions achieved by the heuristic increases to 46 for $p = 6$. Again, the few non-zero gaps between the objective function values are limited and equal to 0.36% in the worst case, i.e., for $\alpha = 0.10$. These results highlight an interesting outcome. Specifically, if the focus is on $\alpha = 0.05$, it can be found that the minimum and maximum gaps equal -0.12% and 0%, respectively. Therefore, it can be concluded that the heuristic either attains the optimal solution to the problem or outperforms the model when its solution reaches the time limit prescribed by producing higher-quality solutions. These (near-)optimal solutions are obtained at an acceptable computational effort. It should be observed that while computing times are comparable for higher values of α, savings become significant as α decreases. On average, it is noticed that the heuristic runs for 28 and 61 seconds for $p = 4$ and $p = 6$, respectively. In particular, in the latter case, it is worth underlining that the algorithm produces almost the same number of optimal solutions by lowering the computing times from 977 to 61 seconds.

These findings are confirmed when focusing on the more challenging *Instances_2*. The number of optimal solutions reached by the heuristic reduces (19 out of 31). However, the gaps are relatively small and equal to 0.85% in the worst case ($p = 6, \alpha = 0.15$). It must be highlighted that the heuristic can produce significantly better solutions whenever the model fails to achieve the optimal solutions to the problem within the imposed time limit. In fact, the average gap equals -1.66% for $p = 6, \alpha = 0.05$, with an improvement of 3.55% in the best case. Although increased w.r.t. to the first set of instances, running times remain acceptable and, above all, they are one order of magnitude lower than the corresponding solution times of the model (on average, 111 vs. 3199 seconds for $p = 4$; 478 vs. 5307 seconds for $p = 6$). More detailed results are given in Appendix A, Table A1 – Table A4. Overall, these findings validate the proposed heuristic and classify it as effective for the investigated problem.

5.5.3.2 In-Depth Exploration of Larger-Sized Instances

The last step of the empirical analysis consists of additional computational tests to assess the capability of the model and the heuristic to solve larger-sized instances for the investigated problem in terms of (i) an increased number of TUs and (ii) a higher number of stages (and scenarios).

(i) The Impact of the Number of TUs

For these computations, the 120-TUs instances (i.e., $|I| = 120$) used by Diglio et al. (2020) are considered, and the same experimental setting as above is replicated. Thus, 200 additional experiments are performed, summarized in Table 5.9. For the sake of brevity, only the

			Δ_{OF} (%)			# Optimal (out of 10)		Computing time (sec.)					
								Model			Heur		
Instance	p	α	Min	Max	Avg	Model	Heur	Min	Max	Avg	Min	Max	Avg
Instances_1	4	0.05	0	0.83	0.08	10	9	26	3101	390	26	40	30
		0.1	0	0.21	0.03	10	8	19	165	59	27	28	28
		0.15	0	0.22	0.06	10	5	23	386	101	27	31	28
		0.2	0	0	0	10	10	13	29	19	26	27	27
		0.25	0	0	0	10	10	12	21	14	25	27	26
		Total	0	0.83	0.03	50	42	12	3101	117	25	40	28
	6	0.05	-0.12	0	-0.02	8	8	162	t.l.	4738	29	754	192
		0.1	0	0.36	0.04	10	9	13	296	88	26	39	30
		0.15	0	0	0	10	10	14	70	29	26	42	30
		0.2	0	0.02	0	10	9	11	27	16	26	30	27
		0.25	0	0	0	10	10	10	21	14	25	28	26
		Total	-0.12	0.36	0	48	46	10	t.l.	977	25	754	61

			Δ_{OF} (%)			# Optimal (out of 10)		Computing time (sec.)					
								Model			Heur		
Instance	p	α	Min	Max	Avg	Model	Heur	Min	Max	Avg	Min	Max	Avg
Instances_2	4	0.05	-0.1	0.6	0.26	2	2	1307	t.l.	9775	56	1388	396
		0.1	-0.14	0.59	0.14	8	4	246	t.l.	3928	31	147	64
		0.15	0	0.36	0.07	10	8	45	8362	1525	29	58	36
		0.2	0	0.31	0.05	10	7	37	1594	468	27	35	31
		0.25	0	0.67	0.19	10	5	71	588	297	27	44	30
		Total	-0.14	0.67	0.14	40	26	37	t.l.	3199	27	1388	111
	6	0.05	-3.55	0.1	-1.66	0	0	t.l.	t.l.	t.l.	1019	3502	1588
		0.1	-0.26	0.32	-0.03	1	1	8417	t.l.	10567	67	1693	657
		0.15	0	0.85	0.14	10	5	140	10385	3640	36	161	84
		0.2	0	0.31	0.03	10	8	29	453	126	29	37	32
		0.25	0	0.18	0.06	10	5	17	238	48	27	32	29
		Total	-3.55	0.85	-0.29	31	19	17	t.l.	5037	27	3502	478

Table 5.8: Assessing the performance of the proposed heuristic.

average values of some relevant indicators are reported in the latter, while a more extensive overview is provided in Appendix A, Table A5 – Table A8.

Three main elements seem to emerge from these extended experiments. First, capturing uncertainty in the problem remains crucial. In particular, the average values of $\%DVSS$ are often higher than those obtained for the 88-TUs instances (Table 5.1 and Table 5.2), thus revealing that the size of the problem seems to affect this aspect. Second, and as expected, the latter has a clear effect on the computational performance of the model. Indeed, although most of the tested instances are optimally solved within the imposed time limit, the average running times are significantly increased w.r.t. to the former tests. Finally, the heuristic confirms its effectiveness, being capable of attaining (near-)optimal or even improved solutions at an acceptable computational effort.

(ii) The Impact of the Number of Stages

For these experiments, the 88-TUs instances are considered. To this end, demands are generated by using the method described in Section 5.5.1. Recall that the following values of θ are assumed for three stages: 0.25 for node 2, 0.5 for node 4, and 0.75 for node 5. Recall also that, for a given node n, θ^n expresses the percentage by which the demand for a generic TU i may reduce w.r.t. to its predecessor.

The scenario tree shown in Figure 5.2 is used as a reference and extended to include four (Figure 5.10) and five stages (Figure 5.11), resulting in eight and 16 scenarios, respectively.

In the case of four stages, nodes 8 and 9 would be successors of node 4, with associated values of $\theta = 0.5$ and $\theta = 0.75$. The same applied to nodes 10 and 11, successors of node 5. No variations in the demand occurred for nodes 3, 6, 7, and their successors. The same reasoning is assumed for a five-stage scenario tree. It is worth underlining that in this generation process, no differentiation is made between the northern and southern TUs, which means that demand changes can apply to all the TUs (as for *Instances_1*). Finally, the equiprobability of the scenarios is still assumed, i.e., with each scenario having a probability equal to $\frac{1}{|\Omega|}$ to occur.

The initial focus is on four stages. Various tests are performed by setting $p = 4$ and varying $\alpha \in \{0.10, 0.15, 0.20\}$. Again, for each combination of p and α, ten instances are generated by varying the demands. This resulted in 30 additional experiments, whose results are summarized in Table 5.10.

It is important to underline that no time limit is imposed for these runs. Hence, all the tested instances are solved up to proven optimality, and the corresponding $\%DVSS$ and $\%EVPI$ are calculated. Interestingly, it can be observed that both indicators are higher (even significantly) when compared to those reported in Table 5.1 and Table 5.2. This indicates that capturing uncertainty becomes more and more relevant as the number of stages (and scenarios) increases in the investigated problem.

Instance	p	α	Avg $\%DVSS$	Avg $\%EVPI$	# Optimal (out of 10) Model	Heur	Avg Δ_{OF} (%)	Avg Computing Times (sec.) Model	Heur
Instances_1	4	0.05	42.43	1.17	10	9	0.02	738	91
		0.1	12.1	1.26	10	9	0.01	1030	59
		0.15	33.9	1.23	10	9	0.01	224	58
		0.2	23.05	1.24	10	10	0	325	57
		0.25	27.04	1	10	10	0	125	54
		Total	27.7	1.18	50	47	0.01	488	64
	6	0.05	30.23	2.14	8	7	0.03	5531	228
		0.1	15.61	1.11	10	7	0.02	373	58
		0.15	19.04	0.54	10	10	0	63	55
		0.2	17.2	0.34	10	10	0	39	54
		0.25	14.72	0.3	10	10	0	29	53
		Total	19.36	0.88	48	44	0.01	1207	89

Instance	p	α	Avg $\%DVSS$	Avg $\%EVPI$	# Optimal (out of 10) Model	Heur	Avg Δ_{OF} (%)	Avg Computing Times (sec.) Model	Heur
Instances_2	4	0.05	N/A	N/A	0	0	-0.33	t.l.	4011
		0.1	67.62	4.33	6	3	0.07	6468	669
		0.15	58.4	3.96	10	1	0.17	1232	114
		0.2	56.07	4.12	10	8	0.05	2035	186
		0.25	49.77	3.63	10	9	0	1247	77
		Total	57.96	4.01	36	21	-0.01	4358	1012
	6	0.05	N/A	N/A	0	0	-0.56	t.l.	2748
		0.1	26.01	3.25	1	1	-0.07	9747	355
		0.15	41.67	2.38	10	9	0.01	1856	66
		0.2	35.65	1.33	10	9	0	88	56
		0.25	44.4	0.72	10	10	0	42	54
		Total	36.93	1.92	31	29	-0.12	4509	656

Table 5.9: Results for the 120-TUs instances.

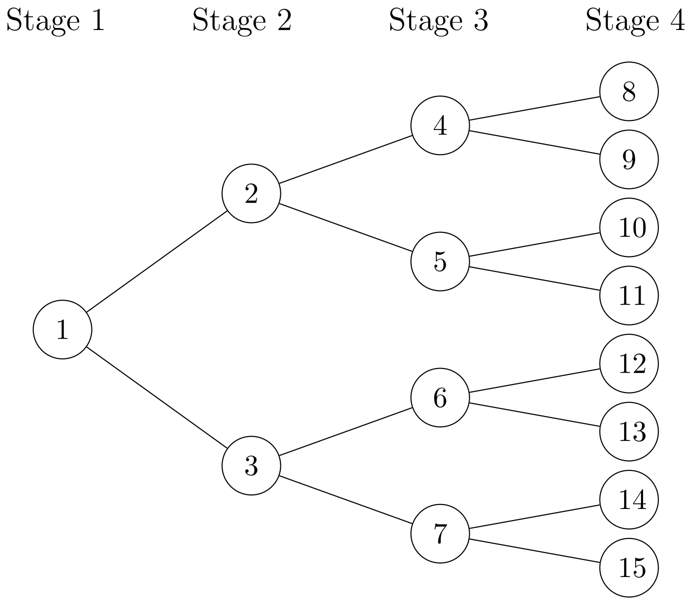


Figure 5.10: A multi-stage scenario tree – four stages in the planning horizon and eight scenarios.

	Avg	Avg	# Optimal (out of 10)		Avg Δ_{OF}	Avg Computing Times (sec.)	
α	$\%DVSS$	$\%EVPI$	Model	Heur	(%)	Model	Heur
0.10	34.34	2.01	10	9	0.02	13735	231
0.15	27.15	2.06	10	9	0.00	10690	167
0.20	18.97	1.93	10	8	0.03	8484	136
Total	26.82	2.00	30	27	0.02	10970	178

Table 5.10: Results for the 88-TUs instances with four stages.

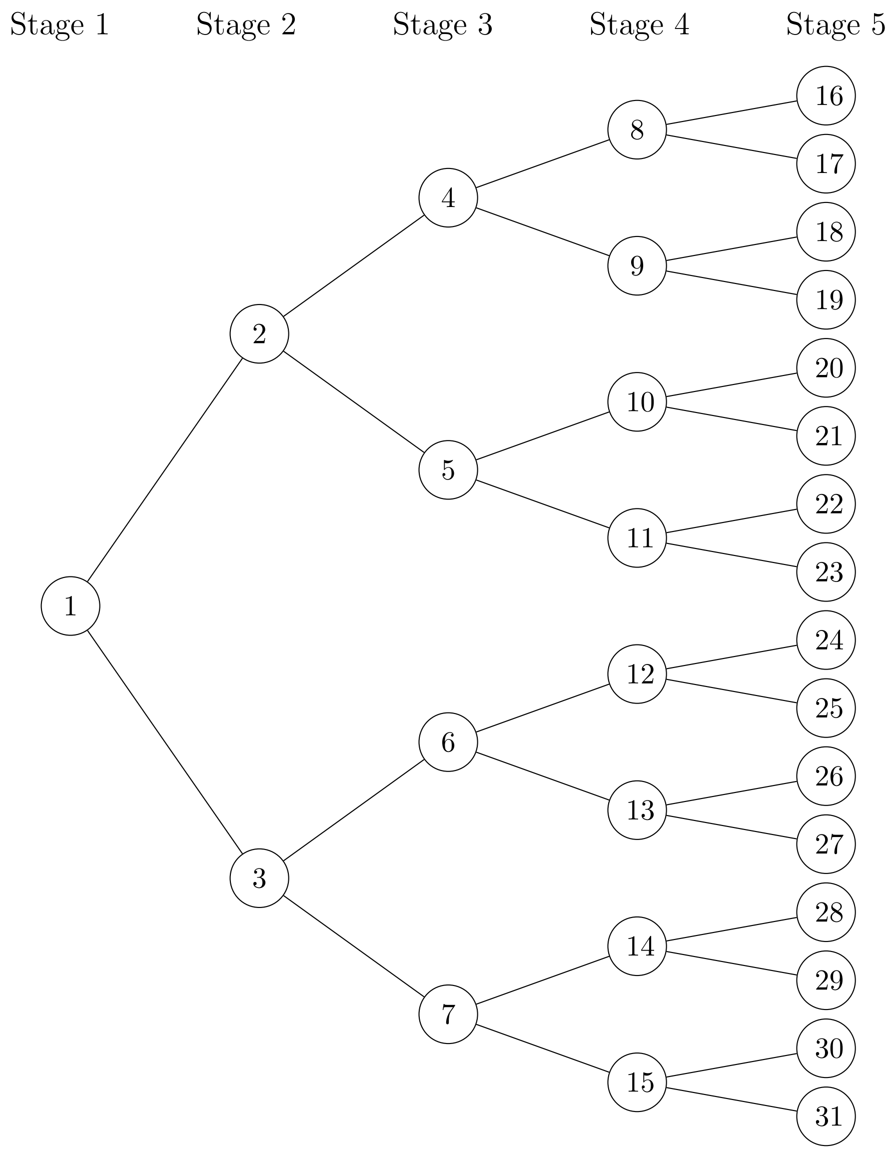


Figure 5.11: A multi-stage scenario tree – five stages in the planning horizon and 16 scenarios.

Moreover, it can be noticed that the heuristic still classifies as effective, being able to achieve 27 optimal solutions out of 30, with negligible optimality gaps in the remanding cases (on average equal to 0.02% – see the "Avg Δ_{OF}"-column). Finally, the computing times remain reasonable, with the output of the algorithm converging on average in 178 seconds (231 seconds in the worst case, for $\alpha = 0.10$), that is, about 60 times faster than the solver. Further details on the performed experiments can be found in Appendix A, Table A9.

Ten additional experiments are performed for five stages, corresponding to ten different demand generations for the 88-TUs instances, with $p = 4$ and $\alpha = 0.20$. In this case, a time limit of 3 hours is again imposed. The detailed results for these runs are reported in Appendix A, Table A10, and some summary statistics are reported in Table 5.11. In the latter table, the following indicators are also displayed in addition to already presented information for previous tests:

- The average optimality gap – across the ten experiments – of the solutions obtained by CPLEX (Avg GAP_{opt});
- The average relative gap – across the ten experiments – between the objective function of the solutions yielded by the heuristic and the lower bound provided by CPLEX (Avg GAP_{HLB}). It is measured as $GAP_{HLB} = \frac{Z_H - Z_{LB}}{Z_{LB}}$, with Z_H and Z_{LB} denoting the objective function values of the heuristic solution and the lower bound, respectively.

# Optimal (out of 10)		Avg Δ_{OF}	Avg GAP_{opt}	Avg GAP_{HLB}	Avg Computing Times (sec.)	
Model	Heur	(%)	(%)	(%)	Model	Heur
2	0	-4.76	5.58	0.89	10185	2803

Table 5.11: Results for the 88-TUs instances with five stages.

As the table shows, the solver attains only two optimal solutions within the given time limit, with an average running time of more than 10000 seconds. The optimality gap reported by CPLEX is not negligible for the remaining cases, averaging 5.58% (as shown in the "Avg GAP_{opt}"-column) and reaching 22.87% in the worst case (Table 5.11). In contrast, the heuristic runs approximately four times faster, in just over 2800 seconds. More importantly, it outperforms the solver significantly in terms of solution quality. The proposed algorithm improves the best integer solution returned by the solver by an average of about 4.76% (as shown in the "Avg Δ_{OF}"-column). Remarkably, the relative gap of the heuristic from the lower bound provided by CPLEX is limited and equal to 0.89% on average (see the "Avg GAP_{HLB}"-column).

The results once again confirm the effectiveness of the heuristic, demonstrating that it can represent a valuable tool for solving medium-sized instances involving up to five stages and

16 scenarios. However, results for six stages, which are not presented for brevity, show that the running times of the heuristic increase significantly. This indicates the need for more refined solution methods to handle instances involving higher-cardinality scenario sets.

5.6 Conclusion

In this chapter, the two aspects of multi-period and uncertainty from Chapter 3 and Chapter 4 are combined and developed into a new model for multi-period stochastic districting problems.

The proposed multi-stage stochastic programming approach modeled uncertainty using a finite set of scenarios, represented as a scenario tree. At the root node of the scenario tree (the start of the planning horizon), an initial districting plan is created.

Given the time-dependent and uncertain nature of demands, districts can be adjusted at each stage by reassigning some TUs to maintain balance. Additionally, the model incorporated special actions to address shortages or surpluses in each district at every node.

The objective of the proposed model is to minimize a cost function composed of three components: i) initial districting costs – costs associated with creating the initial districting plan, ii) reassignment costs – costs incurred from reassigning TUs during subsequent stages, iii) penalty costs – costs for shortage and surplus.

A heuristic algorithm is developed to find approximate solutions to this complex problem efficiently. The heuristic solves a restricted version of the model, focusing on a subset of candidate district representatives. These candidates are identified using insights gained from the linear relaxation of the problem, which helps guide the search for optimal or near-optimal solutions.

Extensive computational tests conducted with generated instances based on the literature and on real geographical data demonstrate the models' ability to tackle computational challenges and, most importantly, highlight the relevance of capturing uncertainty in the problem being examined. To this end, appropriate measures such as the $DVSS$ and the $EVPI$ are computed. Besides, the proposed heuristic proved effective, producing near-optimal solutions with reduced computational effort.

Chapter 6

Districting for Home Health Care: Case Studies

In this chapter, we analyze various case studies on districting in Home Health Care (HHC). The models developed in this thesis are tested using this realistic case. By dividing a geographical area into smaller districts based on demographic characteristics, HHC providers can serve a particular district more efficiently. This helps to optimize the use of resources, such as health aides and equipment, while reducing travel time and costs. Districting also allows health aides to become more familiar with their communities, enabling them to understand their patients' unique healthcare needs better. By tailoring their care to the specific needs of patients, healthcare providers can significantly improve the quality of care and efficiency. Moreover, districting helps improve communication between health aides and patients.

Overall, districting can play a critical role in HHC by improving the quality of care while reducing costs. Healthcare organizations can carefully design and implement districting strategies to ensure that patients receive the best possible care in the comfort of their own homes.

The importance of districting increases significantly when we consider uncertain demand, which is frequently a factor in HHC. Many people may not be aware that they will need assistance until a crisis arises. This uncertainty complicates the planning and assignment of health aides within the planning area.

Moreover, the need for assistance is not static, it can fluctuate over time. New patients may require care, while existing patients might no longer need health services as their conditions improve or change. This dynamic nature of demand underscores the necessity for flexible and responsive districting strategies that can accommodate these shifts in patient needs.

Key roles in HHC, such as HHC providers, patients, and health aides, can be interpreted in the context of districting and applied as follows:

HHC Provider The HHC Provider refers to an individual or organization interested in creating a balanced and efficient districting plan. The HHC provider divides his health aides into different districts and serves as the decision-maker.

Patient In the context of HHC, each TU is associated with a specific number of patients who require service. This patient count is used as the activity measure (Definition 2) for the respective TU.

Health Aide Health aides are individuals who represent a district. Each health aide is responsible for a specific district and provides assistance to all patients within that district. The number of health aides corresponds to the districts that the HHC provider must assign their patients to.

In this chapter, we examine the significance of districting in HHC and its impact on both patients and health aides under different conditions.

- In Section 6.1, we use the extended model (EDP) introduced in Section 2.2 to examine fundamental districting aspects. Specifically, we analyze how varying the number of districts, denoted as p, influences both the geographical shape of these districts and the costs incurred by the HHC provider as p increases. Additionally, we conduct a cost analysis to clarify how much total cost can be reduced by increasing the value of α. This analysis investigates the relation between compactness and balance, specifically assessing the selected centers within the city area of Karlsruhe and analyzing the resulting districts for various values of α.
- In Section 6.2, we analyze a specific multi-period instance for the HHC provider in Karlsruhe and solve it with the MPDP, focusing on the geographical shapes of the resulting districts over two different periods. Furthermore, we calculate the $VMPS$ to determine whether it is beneficial for the HHC provider to implement the model from Section 3.2.
- In Section 6.3, we analyze a stochastic setting for the HHC provider, calculating the $EVPI$ and the VSS to evaluate the SDP model presented in Section 4.2. Additionally, we examine the emerging centers based on 100 randomly selected instances, investigating the variation among these centers and analyzing the similarity of the resulting groupings of TUs.
- In Section 6.4, we consider a multi-period stochastic case for the HHC provider. To assess whether using the MSSDP model from Section 5.2 is worthwhile, we calculate the $DVSS$ and $EVPI$ as outlined in that section.

- In Section 6.5, the results of the previous sections are summarized and provide guidance to the HHC provider on how to proceed with its planning.

The TUs used in the following sections are located in the Karlsruhe region of Germany and are depicted in Figure 6.1. The city of Karlsruhe spans approximately from latitude 48.99°N to 49.03°N and longitude 8.36°E to 8.44°E. Initially, it is assumed that there is one patient in each TU. This assumption is extended in the multi-period and stochastic case. The city area of Karlsruhe is divided into districts to ensure that all TUs are clearly assigned to a health aide.

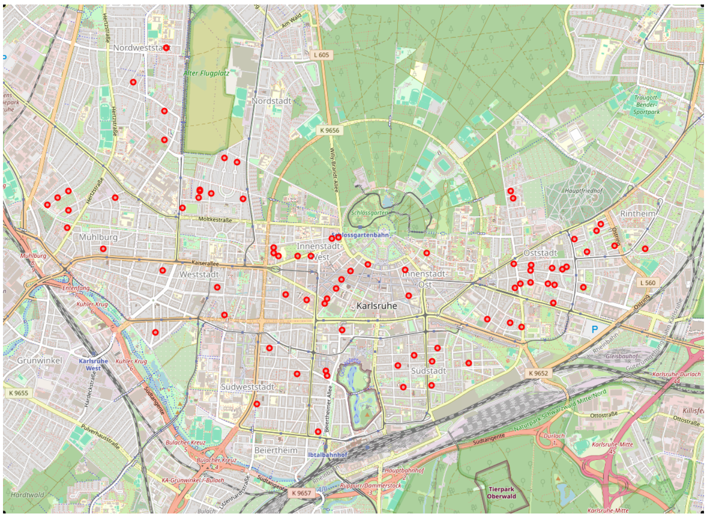


Figure 6.1: Map of Karlsruhe highlighting the marked TUs for the case study on districting within HHC for different settings of uncertainty, with both single-period and multi-period models. Created with `www.openstreetmap.org`.

6.1 Case Study: Single-period Deterministic Case

In this section, we examine a deterministic setting for an HHC provider to divide its patients, the TUs in the center of Karlsruhe depicted in Figure 6.1, into smaller districts for their health aides. There are 80 TUs, each representing a single household that requires service from a health aide. The required number of health aides for each TU is assumed to be one. The aim is to assign the patients to districts such that each health aide has a similar number of patients, while the districts are as compact as possible. The model EDP described in Section 2.2 and the following parameters are used.

The parameters and data described above, including the 80 TUs with demands d_i and the distance matrix c_{ij} based on Euclidean distances, are used consistently throughout the analyses. The parameter μ is computed from the total demand and the number of districts p, while α and p vary depending on the specific study.

Parameter	Description	Value/Formula
I	Set of TUs	$\{1,\ldots,80\}$
d_i	Demand for all TUs $i \in I$	$[1,\ldots,1]$
c_{ij}	Euclidean distances between all TUs $i, j \in I$	
μ	Average demand for each district	$\frac{\sum_{i \in I} d_i}{p}$
α	Allowed deviation from μ for each district	
p	Number of districts	

We analyze the following:

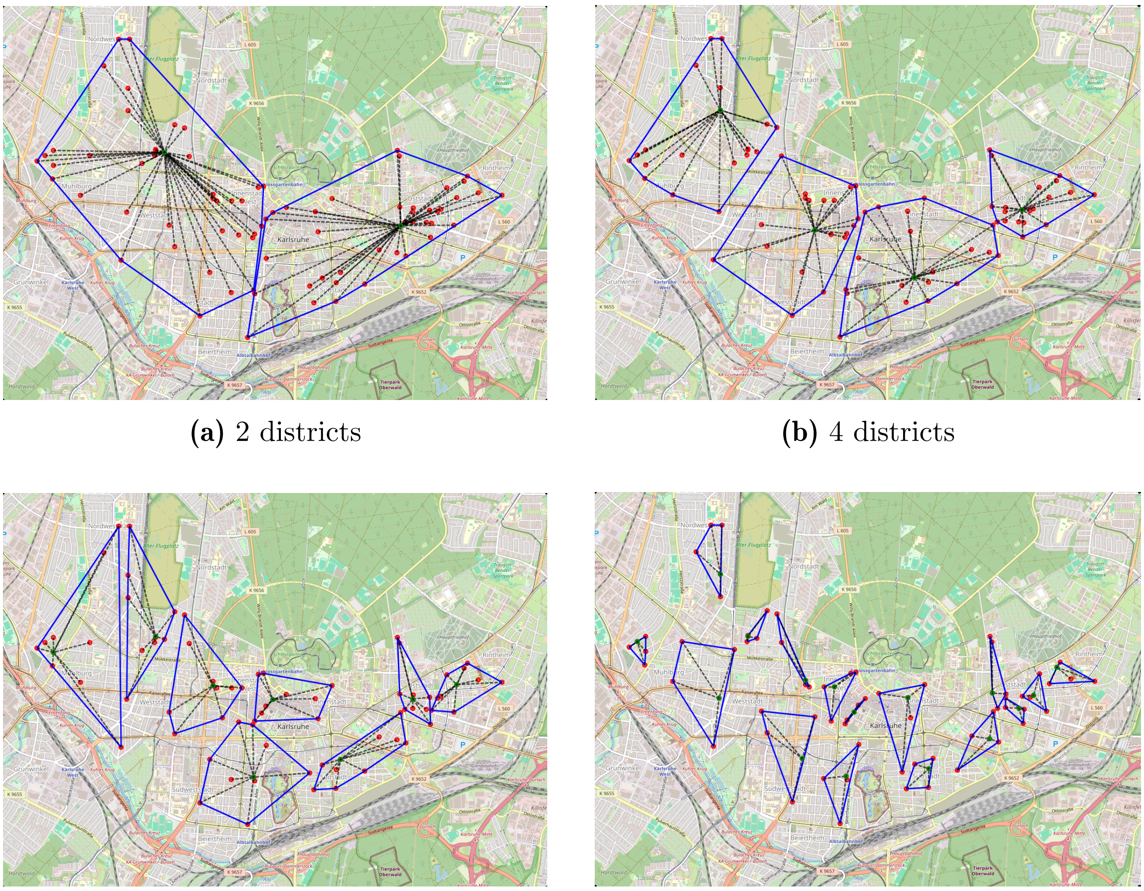
- In Section 6.1.1, we conduct a comprehensive analysis of the resulting districts by systematically varying the number of districts p. This examination not only investigates the geometric configurations of the districts but also assesses the resulting changes in the objective function values, which include assignment costs and penalty costs.
- The influence on the objective value for varying values of α is analyzed and discussed in Section 6.1.2. By systematically varying α, we examine the sensitivity of the objective values against more tolerant restrictions and less balanced solutions.
- An analysis of the resulting centers in the Karlsruhe region for different values of α is conducted in Section 6.1.3. We also examine whether varying the value of α affects only the centers of the emerging districts or if it also alters the groups themselves.

6.1.1 Comparative Analysis of Varying the Number of Districts

This subsection focuses on how district sizes vary as the number of districts increases. The first step examines the district shapes, and the second evaluates cost changes.

In Figure 6.2, the solution for four different numbers of districts is illustrated. Two districts are shown in Figure 6.2a, four districts are generated in Figure 6.2b, eight districts are shown in Figure 6.2c, and 16 districts in Figure 6.2d. The value of $\alpha = 0.001$ indicates that every imbalance is penalized. With 80 TUs, it is possible to perfectly balance the districts for $p = 2, 4, 8, 16$, with $|TU| = 40, 20, 10, 5$ in each district. We can observe that doubling the number of districts does not result in halving the geographical size of each district.

In the next step, we analyze the influence of different numbers of districts p on the objective value.



(a) 2 districts

(b) 4 districts

(c) 8 districts

(d) 16 districts

Figure 6.2: Comparison of district shapes for different numbers of districts p, in a single-period deterministic model for districting in HHC.

The main observation here is that no penalty costs arise if the patients can be perfectly balanced ($\frac{80}{p} \in \mathbb{N}$). For every number of districts, the number of patients in all districts does not differ by more than one patient, which is the best-balanced outcome. For instance, if 80 patients have to be divided into three districts, the distribution is 27 : 27 : 26. This implies that the penalty factors g and h are set high enough to ensure that the solution remains as balanced as possible. If the values are set too low, it may disrupt the balance of the solution. In these instances, a more compact solution is preferable.

In Figure 6.3, it can be seen that the objective value does not consistently decrease as the number of districts increases. This effect is due to the balancing constraints that must be satisfied and the penalty costs incurred if these constraints are not met. The penalty costs are represented by dark gray bars. Contrarily, the overall compactness, or assignment costs, steadily decreases, as indicated by the light gray bars.

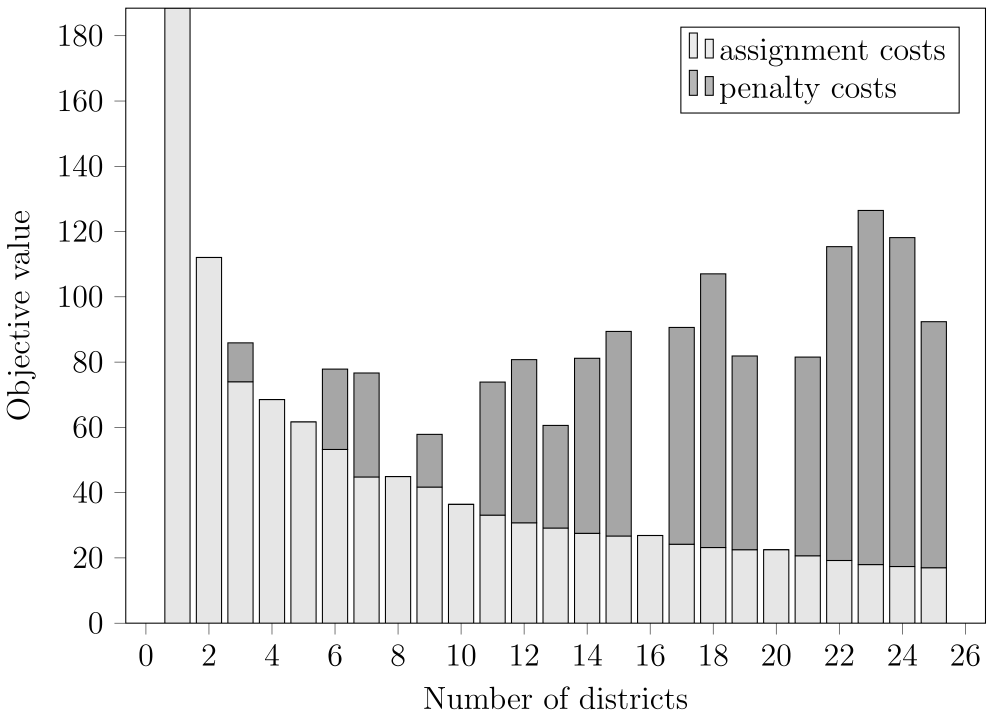


Figure 6.3: Comparison of objective values based on different numbers of districts p in a deterministic, single-period HHC setting, classified into assignment and penalty costs.

It can be summarized that the compactness, defined as the sum of the distances between the TUs and their district centers, is consistently increasing (as indicated by the decreasing assignment costs). However, as the number of districts increases, the differences or improvements in costs become smaller. Therefore, the HHC provider must carefully consider when it is worthwhile to hire additional health aides.

Districts would be most balanced if every health aide managed the same workload –

meaning each health aide would handle the same number of patients—thus eliminating any penalty costs. In practice, however, achieving this balance is not always feasible or necessary. In this analysis, we used a threshold of $\alpha = 0.001$. While this may be too strict for practical applications, it is essential to understand the costs involved.

6.1.2 Comparative Analysis of the Objective Value with Varying Values of the Maximum Allowed Deviation

The following analysis examines the impact on the objective value of varying values for α, which represents the allowed deviation from the average value μ in each district. In this case, the number of districts is $p = 4$.

The value α represents the balancing aspect between different health aides. The HHC provider faces the choice of paying for overtime to create more compact yet unbalanced districts or avoiding overtime to establish completely balanced districts. However, the latter may lead to increased travel time for all health aides.

The compactness is represented by the assignment costs, while overtime leads to additional costs for surplus. Shortage can be interpreted as costs that have to be paid but could have been avoided, which is also penalized in the objective function.

Previously, $\alpha = 0.001$ was chosen, resulting in penalty costs when the number of patients is not divisible by the number of districts. The analysis focuses on whether certain sets of TUs are consistently grouped in a single district and whether some TUs switch for different values of α.

The values for α range from 0.1 to 0.95 in increments of 0.05, so we analyze 18 different settings for the same instance. However, the objective values do not show significant differences as shown in Figure 6.4. Finding a fully balanced solution for all values within the range $[(1-\alpha)\mu; (1+\alpha)\mu]$ for each district means that no penalty costs need to be paid.

The lowest value for $\alpha = 0.1$ leads to the highest sum of travel distances 66.10 km between the representative of a district and the TUs of the district. The lowest sum of travel distances is 61.64 km while the allowed deviation between the workloads is $\alpha = 0.95$, which is a very unbalanced workload for each district. With a higher allowed tolerance for an unbalanced workload, the objective value can only improve by 6.75%.

The interpretation is as follows: the travel time for all health aides does not increase much when a balanced solution is prioritized. Thus, it is more important to focus on equalizing the workload among health aides rather than solely minimizing travel distances by maximizing compactness.

The recommendation for the decision-maker is as follows: It is advisable to accept a small increase in travel time in order to achieve more balanced districts. This approach results in a more equitable distribution of patients and a fairer workload for health aides, all without significantly increasing costs.

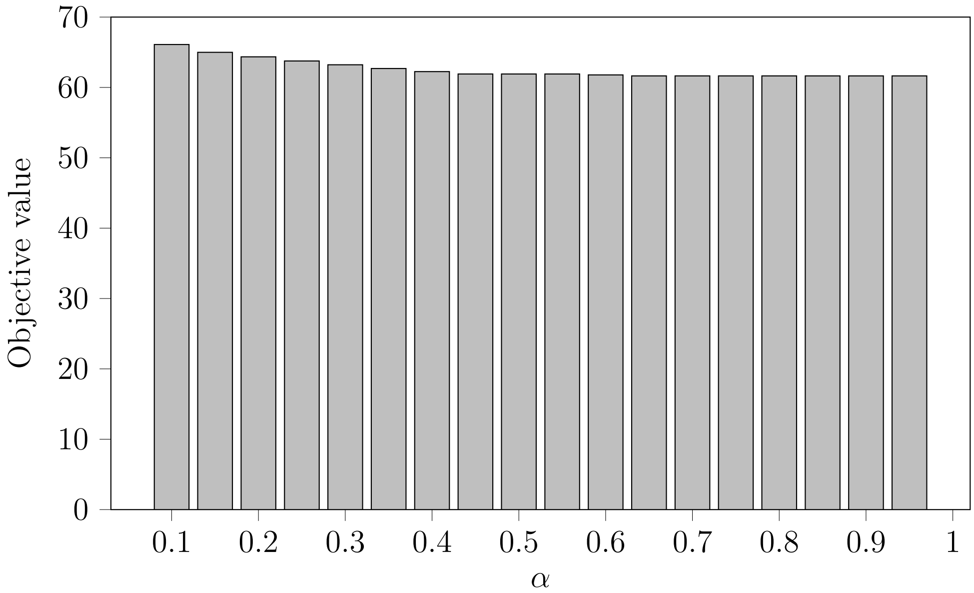


Figure 6.4: Comparison of objective values for different values of α in a deterministic, single-period HHC setting, classified into assignment costs and penalty costs.

6.1.3 Comparative Analysis of Centers with Varying Values of the Maximum Allowed Deviation

In the next step, the focus turns to the chosen district representatives to observe whether they remain consistent or how they vary for the different values of α. The district centers are evaluated for $\alpha = [0.1, 0.15, \ldots, 0.9, 0.95]$ while the number of districts remains $p = 4$. Afterward, we analyze the set of TUs that are clustered together.

The concept of a district center in the context of HHC is not entirely straightforward. It can be interpreted as the ideal starting point for health aides when visiting patients. This location allows for the shortest travel distances to patients within the same area. If health aides do not need to return to the center between patient visits, optimizing their route could resemble a traveling salesperson problem (TSP).

For all 18 instances, 10 different representatives were chosen. Only one center was consistently chosen in all solutions (Figure 6.5, marked in purple). TUs that were never selected as a center are marked with a grey x. The frequency of TUs selected as a center can be seen from the color scale on the right. We can see that the 10 selected centers form four groups, which we examine further below.

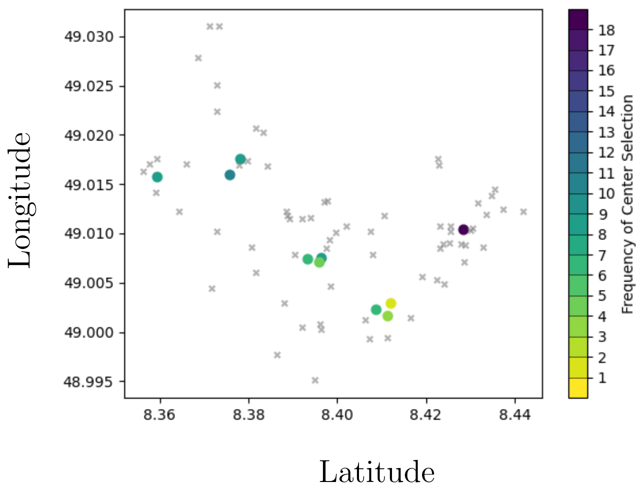


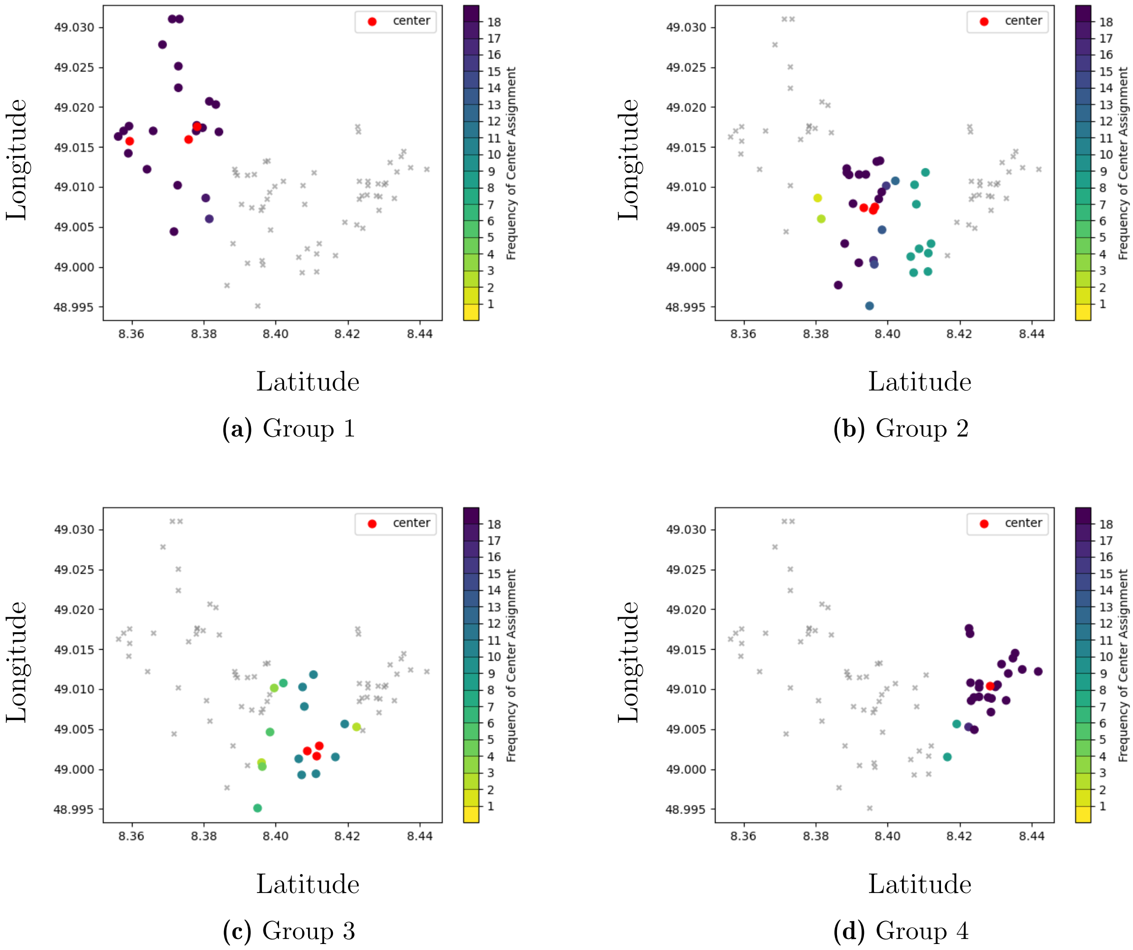
Figure 6.5: Selected centers for 18 instances with varying values of α in a deterministic, single-period setting with $p = 4$.

We now examine the set of TUs that are grouped together within a district, even if they are assigned to different representatives for specific values of α. To this end, we conduct a stability analysis of the assignments to the district centers.

Figure 6.6 displays the results of this analysis. Each subfigure corresponds to one district (group), labeled according to its geographic position as left outer, left central, right central, and right outer. The selected district centers are highlighted in red. TUs marked in color indicate assignments to these centers at least once in all 18 runs, with the frequency shown by the color scale on the right. TUs that are not assigned to this group of centers in any of the 18 runs are represented by grey x.

In Figure 6.6a and Figure 6.6d, we observe that the districts labeled left outer and right outer, which are located at the edges of Karlsruhe city center, are almost identical across all values of α. In contrast, the districts left central and right central shown in Figure 6.6b and Figure 6.6c display considerable instability. Depending on the value of α, these TUs are sometimes included in one district or another.

In summary, based on the data used for this case study, the districts at the edge of the planning area demonstrate greater stability. In contrast, the two groupings in the central part of the city of Karlsruhe show more fluctuations in the assignments.



(a) Group 1

(b) Group 2

(c) Group 3

(d) Group 4

Figure 6.6: Comparison of resulting TU groups and their centers with 18 different values of α in a deterministic single-period setting with $p = 4$.

6.2 Case Study: Multi-period Case

In this scenario, TUs are considered as small neighborhoods rather than individual patients. These neighborhoods can have multiple patients residing within them. The aim of the HHC provider is to distribute the patients among their health aides for several periods. The following parameters and the model (MPDP) described in Section 3.2 is used:

Parameter	Description	Value/Formula
I	Set of TUs	$\{1, \ldots, 80\}$
TP	Set of periods	$\{1, 2\}$
c_{ij}	Euclidean distances between all TUs $i, j \in I$	
α	Allowed deviation from μ for each district	0.001
r_{ij}^t	Unit costs for reassigning TU i to district j in period t $(i, j \in I, t \in TP)$)	c_{ij}
s_{ij}^t	Savings for removing TU i from district j in period t $(i, j \in I, t \in TP)$	0
g_j^t	Unit costs for surplus at district j in period t $(j \in I, t \in TP)$	$\max_{i,j \in I}\{c_{ij}\}$
h_j^t	Unit costs for shortage at district j in period t $(j \in I, t \in TP)$	$\max_{i,j \in I}\{c_{ij}\}$
p	Number of districts	4
d_i^t	Demand for all TUs $i \in I$ for all periods $t \in TP$	
μ^t	Average demand for each district for all periods $t \in TP$	$\frac{\sum_{i \in I} d_i^t}{p}$

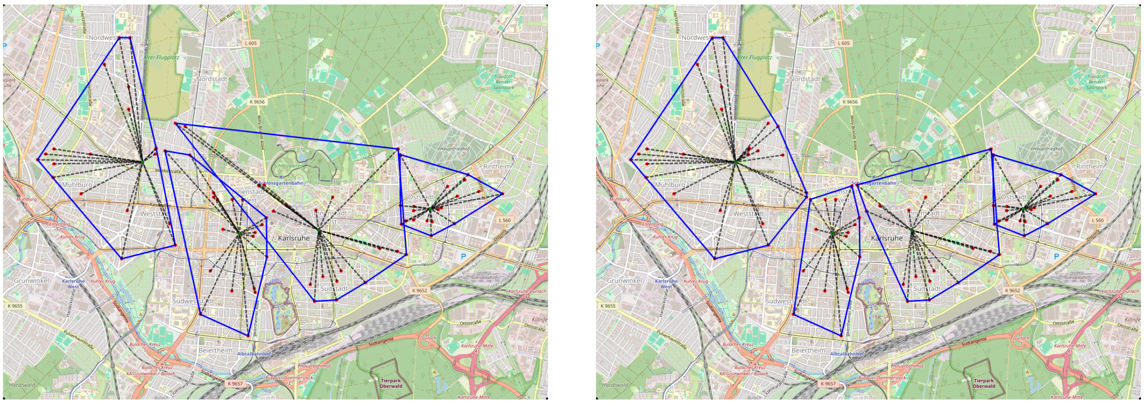
According to demographic evolution, the rising share of elderly people in the neighborhood and the range of services for senior citizens in the southern part of the city are expected to increase significantly over the next few years. As a result, the demand for HHC is predicted to be twice as high in the second period. The HHC provider would like to define its districts today to ensure that patients are informed early about their personal health aides.

- In Section 6.2.1, we analyze the district shapes across different periods and compare them with the solutions of the assignments from the single-period solution using the average scenario.

- In Section 6.2.2, we analyze the $VMPS$ for the two-period problem of the HHC provider described above.

6.2.1 Comparative Analysis of District Shapes across Different Periods

In this subsection, we analyze the district shapes in a two-period setting. In Figure 6.7, we can see that the centers remain the same for both periods, but the assignment decisions differ between both periods.



(a) Period 1 **(b)** Period 2

Figure 6.7: Solutions for two periods in a deterministic, multi-period setting with reassignments and $p = 4$.

In the next step, we compare the resulting districts from the two-period problem solved using the multi-period model with the deterministic single-period solution derived from the average scenario problem. The demands are calculated as follows: $d_i^{avg} = \frac{d_i^1 + d_i^2}{2}$ for all $i \in I$. The resulting districts are shown in Figure 6.8. Three out of the four centers remain the same in the SC_{MP} solution. The districts in both the westernmost and easternmost regions of Karlsruhe closely resemble those found in the two-period solution. The main differences arise in the two central districts, where the average-period solution cannot adjust the assignments.

In the experiments mentioned above, 7 reassignments were made between periods 1 and 2, and the centers remained consistent for both periods. The SC_{MP} solution, while tending to lead to an unbalanced outcome, incurs lower assignment costs compared to when the periods are taken into account. The distribution of the TUs in the SC_{MP} solution to the districts is as follows: 18 : 19 : 20 : 23. Consequently, this results in the need to incur penalty costs. The cost comparison is shown in Table 6.1.

After analyzing the shapes and costs of the resulting districts, it seems that the HHC provider would benefit from using the multi-period model to address its multi-period prob-

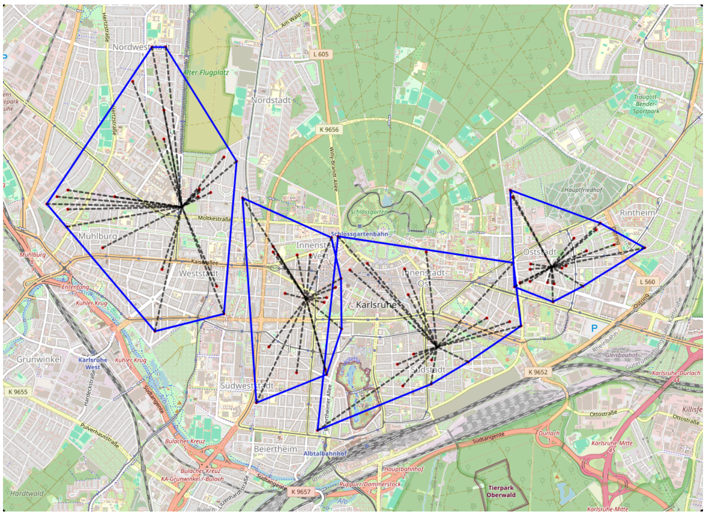


Figure 6.8: Solution for the deterministic single-period model using average demands from two periods of the multi-period setting and $p = 4$.

		Costs		
Model	Objective Value	Assignment	Reassignment	Penalty
MP	0.7281	0.6646	0.0635	-
SC_{MP}	1.6206	0.6061	-	1.0146

Table 6.1: Cost breakdown and comparison of the MP solution and the SSC_{MP}.

lem. This is demonstrated in the following Section 6.2.2, where the $VMPS$ is calculated and interpreted.

6.2.2 Analysis of the VMPS in a Multi-Period Setting

To evaluate the solution, the $VMPS$ is used, as introduced in Section 3.3.1. The solution of the multi-period problem (MP) is compared to the solution of a static counterpart (SSC_{MP}). There exist several ways to define a static counterpart. For this purpose, the single-period problem is formulated with the average demands over the planning horizon for each TU. Alternatively, either the maximum values or the values at the beginning of the planning horizon could be used to formulate the static counterpart.

$$\%VMPS = \frac{SSC_{MP} - MP}{MP} \cdot 100 = \frac{1.6206 - 0.7281}{0.7281} \cdot 100 = 122.5\%$$

In this case, the value can be interpreted as the cost of ignoring the possibility of redistricting patients for the second period or the cost of being unable to redistrict. The HHC provider should be willing to pay this amount to enable reassignments to the districts. The benefit of using the multi-period model to generate a multi-period solution is very beneficial to the HHC provider due to the high penalty costs in the SSC_{MP} (Table 6.1).

To summarize, the districts are changing in response to the changes in demand, leading to some patients being assigned new health aides. While this reassignment may require patients to adjust to a different health aide, it ensures that no individual health aide becomes overloaded and that all health aides share the workload equally. This approach benefits the patients as well since health aides who are not overburdened are less likely to be absent due to stress-related illnesses or dismissal. Without this balance, patients would have to adjust to new health aides regardless, if their assigned health aides become overworked and unavailable.

6.3 Case Study: Stochastic Case

In this section, we examine a stochastic approach for the HHC application. The HHC provider is entering the Karlsruhe HHC market and must divide patients into different districts. Their districting plan is based on various demand scenarios, which depend on anticipated demographic changes and city development, each associated with a specific probability of occurrence. While it is common practice to plan districts using an average scenario and the single-period deterministic model (EDP) Section 2.2, the HHC provider also has the option to utilize the stochastic model (SDP) presented in Section 4.2. The parameters used in the analysis are outlined below, followed by the presentation of the results in the subsequent subsections.

Parameter	Description	Value/Formula
I	Set of TUs	$\{1,\ldots,80\}$
S	Set of scenarios	
c_{ij}	Euclidean distances between all TUs $i,j \in I$	
g_j^s	Unit costs for surplus at district j in scenario s	
	$(j \in I,\ s \in S \setminus \{1\})$	$\max_{i,j\in I}\{c_{ij}\}$
h_j^s	Unit costs for shortage at district j in scenario s	
	$(j \in I,\ s \in S \setminus \{1\})$	$\max_{i,j\in I}\{c_{ij}\}$
α	Allowed deviation from μ for each district	0.001
p	Number of districts	
d_i^s	Demand for all TUs $i \in I$ for all scenarios $s \in S$	
μ^s	Average demand for each district in scenarios $s \in S$	$\frac{\sum_{i\in I} d_i^s}{p}$

- In Section 6.3.1, we analyze the district shapes using a stochastic model in the stochastic setting.
- In Section 6.3.2, we evaluate the stochastic model solution against the average scenario solution and the single-period deterministic model using the VSS.
- In Section 6.3.3, we compare the stochastic model solution to the average scenario solution using the $EVPI$.
- In Section 6.3.4, we analyze the district centers under different scenarios.

6.3.1 Comparative Analysis of District Shapes with Multiple Scenarios

In this subsection, we analyze the district shapes resulting from solving the stochastic model applied to the HHC provider with multiple scenarios. Consider the two periods described in Section 6.2.1 as two distinct single-period scenarios that the HHC provider anticipates for the future, which have the same probability of occurrence. This means that demand in the second scenario doubles in the southern TUs compared to the first scenario, and $p = 4$.

The result from the stochastic model is depicted in Figure 6.9. Since the demands differ from north to south, the districts in the solution show a vertical form to balance the uncertainty. As the TUs are distributed more horizontally in the city area of Karlsruhe, the resulting districts appear very distorted and not compact. This observation is also reflected in the cost breakdown shown in Table 6.2. The assignment costs of the stochastic solution are 29.64% higher than the average assignment costs for each of the two scenarios solved separately.

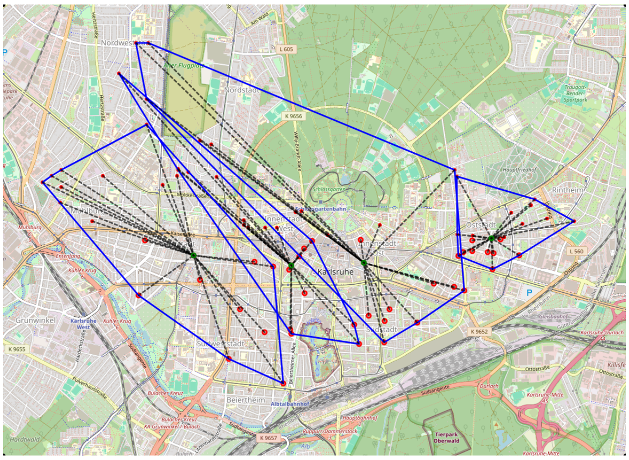


Figure 6.9: Solution for a stochastic, single-period problem with two scenarios and $p = 4$.

In contrast, the districts in the SC_{SP} are the same as in the solution of SC_{MP} shown in Figure 6.8. Hence, it makes no difference in the solution if the scenarios occur in a row in two periods like in Section 6.2.1 or if the scenarios occur with a specific probability in the same period.

In the stochastic case, the decision made by the HHC provider regarding the modeling approach for the planning of their district plays a crucial role in its overall shape and structure. When the HHC provider uses the stochastic model that incorporates multiple scenarios, it allows for a more comprehensive analysis that considers various potential outcomes and uncertainties. This approach can lead to a more complex and nuanced district design that better reflects real-world variability.

Otherwise, if the HHC provider chooses to create its district based only on an average scenario, it misses important factors that could influence its development. This simpler model might also provide a less accurate representation of the potential challenges and opportunities the district could face. Ultimately, the choice of modeling technique has important implications for how the districts function, making it essential for the HHC provider to carefully consider which approach best aligns with its goals.

The comparison above shows that the districts generated by the stochastic model have notably different shapes compared to those produced by the single-period deterministic models with the average scenario. In the following two subsections, we investigate whether these differences are solely geographical or if the solutions also exhibit quantitative differences.

The results of the experiments are presented in a cost comparison found in Table 6.2. It is evident that the assignment costs of the less compact solution, illustrated in Figure 6.9, are more effectively hedged against uncertainty. This is because the less compact solution incurs no penalty costs, unlike the SSC_{SP}.

		Costs	
Model	Objective Value	Assignment	Penalty
SP	1.0275	1.0275	-
SC_{SP}	1.2321	0.7248	0.5073
Single Scenario			
Scenario 1	0.6163	0.6163	-
Scenario 2	0.8297	0.8297	-

Table 6.2: Cost breakdown and comparison of the SP solution and the SSC_{SP}.

6.3.2 Analysis of the VSS in a Stochastic Setting

To quantify the benefit of using the stochastic model, we use the VSS, as outlined in Section 4.3.1.

By contrasting the EEV and SP two values, we can quantify the benefits of incorporating uncertainty into our decision-making process:

$$\%VSS = \frac{EEV - SP}{SP} \cdot 100 = \frac{1.2321 - 1.0275}{1.0275} \cdot 100 = 19.91\%$$

The $\%VSS$ indicates that explicitly considering uncertainty in the model provides a significant advantage over purely deterministic planning. Specifically, a $\%VSS$ of 19.91% means that the expected total cost obtained from the stochastic model is 19.91% lower than the expected cost when using the average scenario in the deterministic model. In other words, this is the expected cost of ignoring uncertainty when making a decision.

If the HHC provider has the chance to utilize the SDP for their districting plan rather than relying on average demand values, it is recommended to use it to save costs.

6.3.3 Analysis of the EVPI in a Stochastic Setting

Another value to measure the value of additional information is the $EVPI$ as described in Section 4.3.1. The $EVPI$ measures the maximum amount a decision-maker would be willing to pay to obtain perfect information about future demands. A high $EVPI$ indicates that this information is highly valuable, suggesting that uncertainty plays a significant role in the decision-making process. Conversely, a low $EVPI$ implies that acquiring further information would not enhance the decision-maker's solution.

By contrasting the WS and SP solutions, we can quantify the advantages of incorporating additional information into the decision-making process:

$$\%EVPI = \frac{SP - WS}{SP} \cdot 100 = 29.93\%$$

This value reflects the amount that the HHC provider should be willing to pay to obtain better information about their patient's demands.

6.3.4 Comparative Analysis of Centers with different Scenarios

In this subsection, we examine the distribution of center locations resulting from 100 randomly generated instances within the Karlsruhe city area. Each instance consists of:

1. Stochastic values $d_i \in \{1, 2\} \quad i \in I$, for all scenarios
2. A constant value of $\alpha = 0.001$ for all instances
3. $p = 2$

Our objective is to analyze the district centers across these randomly generated scenarios, focusing on the similarity and variability of the chosen district representatives.

Figure 6.10 illustrates the distribution of representative TUs across the 100 randomly generated instances. TUs that were selected as centers in at least one instance are highlighted, while those that were never chosen as centers are marked with a grey x. This visualization provides insights into the frequency of center selection for each TU, highlighting districts that were consistently avoided as centers and revealing potential patterns or preferences in center placement.

The visualization in Figure 6.11 shows how frequently TUs are assigned to each of the two groups, which are highlighted in red. The results show a clear pattern: most of the TUs tend to stay within the same district, even when the representative TU for that district varies across different scenarios. This consistency indicates a strong underlying structure in

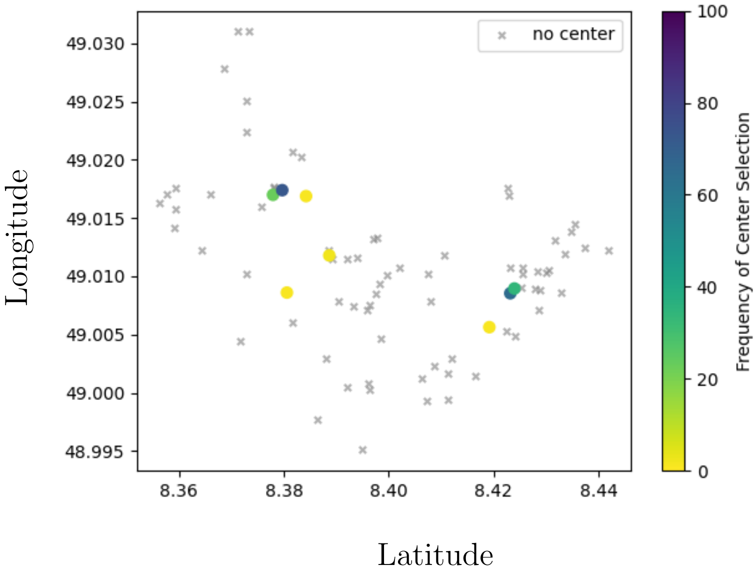


Figure 6.10: Selected centers for 100 randomly generated instances in a stochastic, single-period setting with $p = 2$.

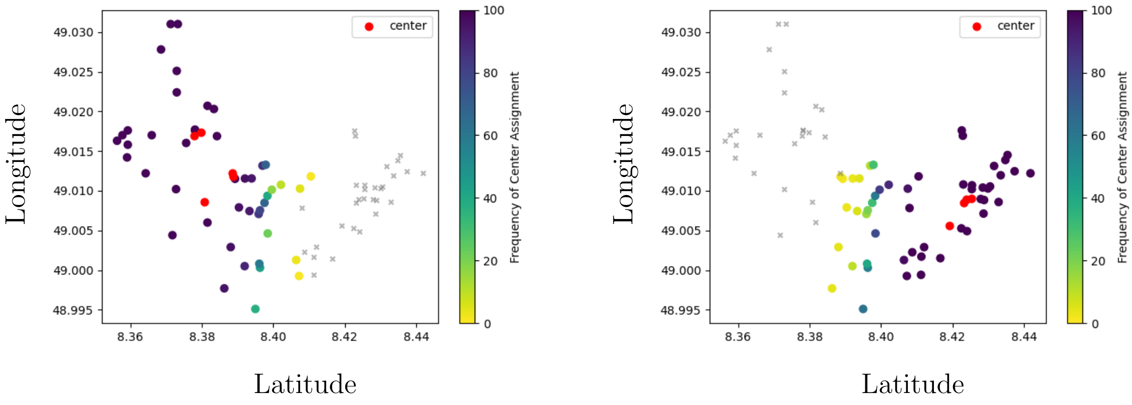
the districting solution. However, the analysis also uncovers a trend at district boundaries. A small set of TUs located along these borders frequently shifts assignments between the two districts.

6.4 Case Study: Multi-period Stochastic Case

In the following section, we analyze a multi-period stochastic setting for the HHC provider. Extensive experiments related to the MSSDP are detailed in Chapter 5.

In this multi-stage setting, we examine a situation incorporating elements from the previous two cases (Section 6.2 and Section 6.3). The setting focuses on demographic changes in the next few years in Karlsruhe and the need for care for individuals who could become patients for the HHC provider. Since the future is uncertain for the HHC provider, different scenarios are used to make a decision.

The objective is to group the patients among the health aides into districts to ensure that current neighborhoods receive adequate care over multiple periods, while also preparing for potential changes in the future. This multi-stage districting challenge requires us to plan for both present and future factors, considering possible variations in demand. We examine different scenarios for the future to get a districting plan for the HHC provider within this



(a) Center Group 1 **(b)** Center Group 2

Figure 6.11: Comparison of resulting districts and their centers with 100 randomly generated instances in a stochastic, single-period setting and $p = 2$.

dynamic and stochastic environment, addressing the uncertainties of future demand and the need for flexibility in response to changing circumstances.

The green nodes (2 and 4) in Figure 6.12 in the first scenario indicate the shifts in demand such as in Section 6.2 and Section 6.3 where the demand in the southern TUs is assumed to be 2. In all other nodes, a demand of 1 is assumed for each TU. The probability assigned to each node is assumed to be 0.5. Therefore, the assumed probability for scenario 1 is 0.25.

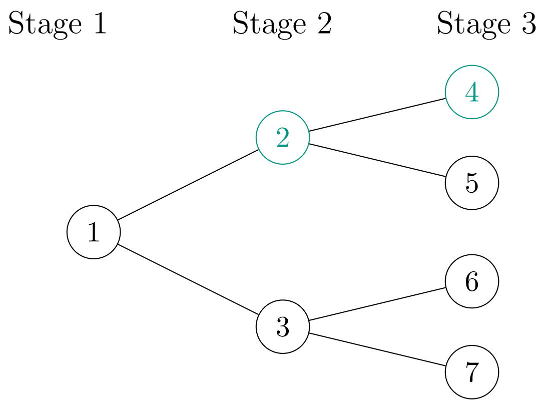


Figure 6.12: A multi-stage scenario tree with 4 scenarios and 3 stages for a stochastic, multi-period districting problem in HHC.

In the following subsections, we analyze the resulting costs and two key performance measures introduced in Chapter 5.

- In Section 6.4.1 we have a closer look into the cost structur of the multi-stage solution.

Parameter	Description	Value/Formula
I	Set of TUs	$\{1, \ldots, 80\}$
$\mathcal{N}$	Set of nodes	
c_{ij}	Euclidean distances between all TUs $i, j \in I$	
r_{ij}^n	Unit costs for reassigning TU i to district j in node n	
	$(i, j \in I,\, n \in \mathcal{N} \setminus (\Omega \cup \{1\}))$	c_{ij}
s_{ij}^n	Savings for removing TU i from district j in node n	
	$(i, j \in I,\, n \in \mathcal{N} \setminus (\Omega \cup \{1\}))$	0
g_j^n	Unit costs for surplus at district j in node n	
	$(j \in I,\, n \in \mathcal{N} \setminus \{1\})$	$\max_{i,j \in I}\{c_{ij}\}$
h_j^n	Unit costs for shortage at district j in node n	
	$(j \in I,\, n \in \mathcal{N} \setminus \{1\})$	$\max_{i,j \in I}\{c_{ij}\}$
α	Allowed deviation from μ for each district	0.001
p	Number of districts	4
d_i^n	Demand for all TUs $i \in I$ for all nodes $n \in \mathcal{N}$	
μ^n	Average demand for each district and node $(i \in I, n \in \mathcal{N})$	$\frac{\sum_{i \in I} d_i^n}{p}$

- In Section 6.4.2, we discuss the $DVSS$ to evaluate the advantages of using the multi-stage model introduced in Section 5.2 over a dynamic version of the expected multi-period model, as described in Section 5.4.1.
- In Section 6.4.3, we compare the solution of the multi-stage model with the WS solution presented in Section 5.4.1 and calculate the $EVPI$.

6.4.1 Cost Breakdown and Solution Analysis

In the following, we analyze the different cost structures of the solutions from the multi-stage model, the solution from the static counterpart (EDEV), and the single scenario cases.

In Table 6.3, we observe that the objective function values from the multi-stage solution differ from those of the static counterpart solution in terms of the cost structure. The multi-stage solution has no reassignment costs or penalty costs, while the static counterpart solution has lower assignment costs (27.78% lower). However, the static solution incurs some reassignment costs and higher penalty costs due to the lack of hedging against uncertainty.

		Costs		
Model	Objective Value	Assignment	Reassignment	Penalty
SP	1.0275	1.0275	-	-
$EDEV$	2.0677	0.7420	0.3935	0.9321
Single Scenario				
Scenario 1	0.8297	0.8297	-	-
Scenario 2	0.9482	0.8529	0.0953	-
Scenario 3	0.6163	0.6163	-	-
Scenario 4	0.6163	0.6163	-	-

Table 6.3: Cost breakdown and comparison of the MSSDP solution, the $EDEV$, and the results from the single scenario cases.

The districts generated by the multi-stage model are the same as those shown in Figure 6.9. In Scenario 1, the outcomes are equivalent to those in Scenario 2 in the stochastic single-period case, while Scenarios 3 and 4 reflect Scenario 1 from the stochastic single-period context (refer to Table 6.2). Scenario 2 is unique due to a change in demand over time. However, this change is not directly comparable to the change observed in the multi-period deterministic case. In this scenario, the initial demand is 1 in the northern TUs and 2 in the southern TUs, but then it returns to a demand of 1 in each TU, which differs from what is illustrated in Table 6.1.

6.4.2 Analysis of the DVSS for a Multi-Stage Setting

To evaluate the result, the $DVSS$ is used, as described in Section 5.4.1. This value compares the solution of the multi-stage model with the dynamic version of the EED, the $EDEV$. It quantifies the benefit of using the multi-stage model for the multi-stage setting instead of using the average of each stage and solving the multi-period model.

$$\%DVSS = \frac{EDEV - SP}{SP} \cdot 100 = 101.23\%$$

In the described setting, the $DVSS$ for the HHC provider is relatively high. This reflects the costs associated with ignoring uncertainty in the decision-making process. Hence, the HHC provider should use the multi-stage model to avoid these costs.

As can be seen in Table 6.3 the high value of the $DVSS$ results from the fact that the more compact solution of the $EDEV$ leads to reassignment and penalty costs in the later stages.

6.4.3 Analysis of the EVPI for a Multi-Stage Setting

Based on Definition 25, we use the $EVPI$ to evaluate the solution of the multi-period stochastic solution (SP) and the wait-and-see solution (WS):

$$\%EVPI = \frac{SP - WS}{SP} \cdot 100 = 27.01\%$$

This value reflects the amount the HHC provider should be willing to pay on average for perfect information about the future of the city development and demographic change.

The relatively low value of $\%EVPI$ (compared to $\%DVSS$) in Table 6.3 can be attributed to the fact that the MSSDP is able to hedge against uncertainty, allowing it to find a solution without incurring reassignments or penalty costs. Consequently, if the average of the single-scenario solutions is not significantly better than this solution, the $\%EVPI$ remains relatively low. Additionally, in Scenario 2, the single-scenario solution incurs reassignment costs, which contributes to an increase in the WS value.

From the results in Section 6.4.1, Section 6.4.2, and Section 6.4.3, it is clear that the HHC provider would benefit from using the MSSDP model to plan its districts in this uncertain situation.

6.5 Conclusions and Outcomes for a HHC Provider in Karlsruhe

The benefits of using the models outlined in this thesis for the HHC provider are summarized in Table 6.4. Each identified benefit is compared to the outcomes derived from using average values and the corresponding static counterpart, which represents the alternative method most commonly used in practice. This traditional approach is often favored when instances are too large or complex to be effectively handled by a more complex model, such as MSSDP. Runtime limitations frequently force practitioners to resort to these simpler methodologies, as they offer a more immediate solution. However, the models proposed in this thesis highlight the potential for more detailed results.

By distinguishing between these approaches, the implications for HHC providers become clearer, emphasizing the importance of selecting the right models to drive better outcomes.

Model	$\%VMPS$	$\%(D)VSS$	$\%EVPI$
MPDP	122.5	-	-
SDP	-	19.91	29.93
MSSDP	-	101.23	27.01

Table 6.4: Comparison of the benefits of different district models.

In summary, the results indicate that the temporal aspect has a more significant influence on the objective function value than the stochastic aspect, based on the data analyzed. When periods are aggregated, the objective function value is noticeably lower compared to when scenarios are aggregated. Although the $\%VMPS$ for the MPDP is very high, the $\%VSS$ for the SDP is relatively low. On the other hand, the $\%DVSS$ for the MSSDP is once again very high. This suggests that the DVSS is more sensitive to the multi-period aspect than to uncertainty in these instances.

From a managerial perspective, this case study highlights the importance of analyzing data fluctuations before making decisions. Decision-makers should assess whether significant variations exist in the data across different scenarios, time periods, or both. If significant fluctuations are identified, it is advisable to use the MSSDP model, taking both aspects into account. However, if computational capacity is limited, valuable insights can still be obtained by running the MPDP and SP models separately, focusing on the most critical aspect.

Chapter 7

Conclusion, Outlook and Further Research

The study of districting is a relatively young research area, dating back to 1965. Since then, it has seen extensive investigation and practical application in various fields. Recently, there has been an increasing focus on two main aspects: multi-period dynamics and uncertainty, which have been tailored separately to specific applications. The increased focus on multi-period dynamics and uncertainty in districting research can be attributed to several aspects. One reason is that demographic shifts, economic changes, and political developments have become increasingly dynamic and unpredictable, which requires models that can adapt to these factors. Another reason for the heightened interest in these areas is the wish to move beyond static analyses of districting problems. Traditional models often treated districting as a single-period or single-scenario decision, overlooking the implications of future changes. The multi-period and stochastic approaches enable a more comprehensive understanding of how districting decisions can impact the future. This shift in perspective facilitates the examination of potential methods that better reflect real-world complexities, where data can change. Moreover, these approaches are particularly worthwhile, as they enable decision-making at all levels, including tactical and strategic, where decisions can have significant impacts. For such decisions, sufficient computational resources should be allocated to adequately address the multi-period and uncertainty aspects in the decision-making process. This work aims to make a meaningful contribution to advancing research in this field.

7.1 Conclusion

The thesis is based on a comprehensive analysis of various districting issues, specifically multi-period and stochastic problems, as well as the combination of both aspects. It starts with a general overview of the districting topic, providing background information on its challenges and complexities. The existing models and studies are often highly problem-specific and do not typically account for factors such as uncertainty and multiple periods. In Chapter 1, these gaps are identified in the current research, which are later addressed. The models presented in this thesis help to close these gaps.

In Chapter 2, we define the fundamental concepts and criteria for districting, providing clear illustrations to enhance understanding. We introduce the first districting model, initially presented by Hess et al. (1965), and examine the symmetries in the balancing constraints. We also address the limitations of this model, which can pose practical challenges. This thesis proposes an adaptive deterministic model as a foundation for the subsequent models discussed in this thesis, as it consistently has a feasible solution. At the end of the chapter, we also present the most common heuristics for solving a deterministic single-period districting problem.

As the first extension, we add time to our problem and look at deterministic multi-period districting problems. In Chapter 3, we start with an overview of the related literature. We present a model for multi-period districting, as published in Pomes et al. (2025). In the proposed model, reassignments between periods are allowed, and an additional parameter for savings is introduced, allowing for the preference of more compact solutions instead of solutions with minimal reassignments. This is particularly useful in practical applications, such as when redistricting incurs only minor costs and focuses on creating compact districts. With this, a transition is created between completely independent periods and dependent periods. We also provide an additional constraint for fixing centers, which can be beneficial in situations where, for example, a facility must be built in the center or when other strategic decisions cannot be altered in the short term. Furthermore, we examine the conditions under which it is beneficial to use the proposed multi-period model and those under which it is not. Computational experiments are conducted at various levels of granularity concerning the TUs and the resulting districts from the static counterpart. This analysis examines the relation between the input data and the key performance indicator $VMPS$.

In Chapter 4, we shift our focus to districting under uncertainty. Instead of examining periods, we now analyze different scenarios. After conducting a literature review, we present an adapted version of an existing two-stage districting model from the literature. We modify the definition of μ, the average allowed demand in each district, which is not scenario-dependent in the previous model. The modifications made to this model are illustrated with examples. We examine the key performance measures VSS and $EVPI$, which serve as indicators of the benefits of using a stochastic model. Similarly to the computational experiments discussed in the previous chapter, we evaluate the types of data that make it advantageous to use a stochastic model, as well as settings where it may not be beneficial. We analyze here the relations between demand fluctuations in the input data and the VSS as well as the $EVPI$.

Furthermore, both aspects – uncertainty and multi-periodicity – are combined and discussed in Chapter 5. We present a multi-stage stochastic districting problem along with a relax-and-fix heuristic that addresses real-world districting challenges. This multi-stage stochastic model is also published in Pomes et al. (2025). Rigorous validation tests are conducted to support this research. To measure the benefit of using the MSSDP, we use a dynamic version of the VSS and the $EVPI$. The validation process involves extensive testing under various scenarios to verify the effectiveness and applicability of the proposed model and heuristic across different instances. The test instances are described in detail before discussing the results, which include cost breakdowns, an analysis of demand variability, and the role of savings. Furthermore, we conduct a comprehensive exploration of larger instances and analyze the impact of the number of TUs and the number of stages.

Finally, all models are analyzed through a comprehensive case study involving home healthcare services in Karlsruhe in Chapter 6. This real-world application demonstrates the viability and usefulness of the developed models in addressing districting challenges, showcasing their potential impact on practical decision-making. We begin with a deterministic single-period setting and then examine a multi-period setting. Next, we examine a stochastic setting before incorporating both time and uncertainty. The case study offers insights into the practical implementation of the proposed districting models in real-world scenarios, providing managerial implications.

Overall, the thesis thoroughly examines districting issues, offers new models for multi-period and stochastic problems, and validates their usefulness through rigorous testing and real-world application. This comprehensive analysis contributes to the existing knowledge on districting and provides practical insights.

The research questions outlined in Chapter 1 can now be answered through the results of this thesis:

RQ1 *What aspects and additional problem characteristics can be captured by integrating multi-period considerations and uncertainty into districting models, and how does this enrich the modeling of real-world districting problems compared to traditional single-period deterministic approaches?*

Multi-period considerations allow for a more dynamic approach to districting, acknowledging that decisions made in one period can affect subsequent periods. In Chapter 3, we present and analyze a multi-period model that incorporates the option of reassignments between periods. Generally, it is recommended to utilize a multi-period model whenever there is a possibility for reassignments, as the $VMPS$ is always greater than or equal to zero. The primary drawback is that finding the exact solution may require more computational time. The centers can be fixed with additional constraints presented in the same section. This is particularly useful when a facility must be constructed at the center or when other restrictions prevent the reassignment of the center. In Chapter 4, we present an adapted stochastic model that allows for multiple scenarios in the decision-making process. Since the value of the VSS is always greater than or equal to zero, using a stochastic model is recommended when there is more than one scenario. The solution obtained through this approach is at least as good as that of the deterministic solution from the static counterpart. One of the key advantages of this model is that it enables recourse possibilities in the event of surplus or shortage, allowing for more flexible decisions based on the realization of each scenario. However, the only drawback is the possibility of increased computational time required. In Chapter 5, we introduce a multi-period stochastic model for districting that allows for reassignments across multiple periods and incorporates multiple scenarios within each time period. This model integrates the additional flexibilities offered by both aspects. Districting and reassignment decisions for each path of the scenario tree are determined before the outcomes of the random variables are known, enabling decision-making under uncertainty.

RQ2 *What evaluation tools and methodologies can measure the impact of multi-period and uncertainty factors on districting performance?*

In this thesis, we analyze various key performance measures that are essential for evaluating the effectiveness of multi-period and stochastic models. We discuss the $VMPS$ for multi-period problems, as well as the VSS and $EVPI$ for stochastic problems. Additionally, we examine the $DVSS$ and $EVPI$ specifically for multi-period stochastic problems. As a further aspect of the analysis, we analyze the relations between these values and the input data. We develop several measures to quantify demand fluctuations in the data. The primary parameter we analyze is the demand in various scenarios, as this value must be balanced in classical districting problems. Our focus is solely on fluctuations in demand. Next, we examine settings with fluctuations in relative demand to determine if there is a linear correlation between these changes and the effectiveness of the models. We utilize various measures, such as fluctuations in demand over different periods or scenarios, to analyze their correlation with $VMPS$, VSS, and $EVPI$. It is demonstrated that fluctuations alone do not justify the use of a multi-period or stochastic model for the problem. When the relative changes in demand remain consistent, there are instances where the same solution can be optimal across multiple periods or scenarios. Our computational results also include an evaluation of computational times, as these may influence the decision to use a static counterpart versus a dynamic or deterministic versus stochastic approach.

RQ3 *How does considering multiple periods and stochasticity affect key performance indicators, such as solution quality, robustness, and fairness, of the resulting districting plans?*

This thesis employs various approaches to check under which conditions neglecting multi-periodicity or stochastic data may result in higher costs. It also explores when using a multi-periodic model for multi-periodic instances and a stochastic model for stochastic data is appropriate. In Chapter 3 and Chapter 4, we observe that multiple periods or stochasticity do not always lead to high values for VSS or $VMPS$. There are instances where the presence of stochasticity and multiple periods does not affect these values, particularly when the relative demand remains consistent across each period or scenario. As a further aspect of the analysis, we examine the relation between fluctuations in relative demand across periods or scenarios and their positive influence on VSS and $VMPS$, while the influence on $EVPI$ is much lower. In the case study presented in Chapter 6, we observe that the temporal aspect has a more significant influence on the objective function value than the stochastic aspect, as indicated by the analyzed data. When periods are aggregated, the objective function value is noticeably lower compared to when scenarios are aggregated. This suggests that the DVSS is more sensitive to the multi-period aspect than to uncertainty in these instances.

RQ4 *How can a real-world case study be used to demonstrate and analyze the practical effects of integrating multi-period considerations and uncertainty in districting models?*

The thesis presents a case study focused on home healthcare services in Karlsruhe in Chapter 6. This real-world application demonstrates how the developed models, which integrate multi-period approaches and stochastic considerations, address districting challenges. Adding these considerations to the HHC decision-making process helps to prevent overworked health aides and reduce waiting times for patients. As in many districting applications, there are two goals to consider: compactness and balance. In the HHC context, compactness refers to minimizing the travel times to the patients. At the same time, balance ensures that the health aides assigned to patients remain as consistent as possible across different periods or scenarios. Integrating both of these factors into the objective function is important. The implications of this case study highlight that the proposed multi-period and stochastic models can significantly enhance practical decision-making for the HHC provider. The presented study also demonstrates that the temporal aspect is more significant than the stochastic aspect. Neglecting both aspects could result in increased costs. However, it is evident that considering the temporal aspect can either lead to higher costs or, when taken into account, result in lower costs.

7.2 Outlook and Further Research

The results of this thesis exhibit several limitations that suggest various directions for future research, including data generation, the runtime of computational experiments, missing data and result analyses, and the need for a real multi-criteria approach or other model extensions.

In our computational experiments, we frequently establish a time limit on runtime to maintain a manageable threshold. Without this limit, the runtimes could become excessively long. Given the complexity of the models presented, we restrict our experiments to smaller sizes, which involve fewer periods or scenarios. This limitation drives the need for heuristics or more efficient algorithms. Furthermore, incorporating machine learning techniques to automate the selection process of representative scenarios could yield improvements in efficiency. Additionally, examining which characteristics in the data lead to specific groups being assigned to the same district under different scenarios or in different periods is crucial for understanding the mechanisms at play. Whether there is a consistent methodology for selecting representatives for various random scenarios or different periods has been raised, which is essential for future investigations. Understanding the characteristics that influence a TU's selection as a representative may offer predictive insights into which representatives might qualify. This exploration could simplify or enhance heuristic approaches.

The proposed relax-and-fix heuristic serves as an initial approach to addressing the MSSDP heuristically. However, there is significant potential for improvement. A critical line of research would involve defining more sophisticated algorithms capable of handling larger-sized instances with a higher number of TUs, more time periods, and more complex scenario trees. This could also include the exploration of heuristics and the application of metaheuristics to optimize performance further. Analyzing computation times across different implementations, including those that employ line partitioning or other algorithms from computational geometry, could provide valuable insights into the efficiency of these methods.

Furthermore, the bi-objective character of the proposed model, with its focus on compactness and balance, highlights the need for clear multi-objective extensions of MSSDPs. To enhance the analysis of districting problems in general, future research should focus on developing more specific approaches to multi-objective optimization, incorporating parameters such as the tolerance level α for the balancing constraints in the objective function or other districting criteria.

Especially in practical applications, the consideration of different periods raises the question of whether the frequency of reassignments should be limited. For instance, no citizen wants to vote in a different district each election, and siblings should ideally be placed in the same school district. This is a specific problem that could be further explored in the future by integrating additional conditions into the multi-period models.

List of Figures

List of Tables

Bibliography

Alumur, S. A., Nickel, S., Saldanha-da-Gama, F., and Verter, V. (2012). Multi-period reverse logistics network design. *European Journal of Operational Research*, 220(1):67–78.

Álvarez-Miranda, E. and Pereira, J. (2021). A Districting Application with a Quality of Service Objective. *Mathematics*, 10(1):13.

Arredondo, V., Martínez-Panero, M., Pena, T., and Ricca, F. (2021). Mathematical political districting taking care of minority groups. *Annals of Operations Research*, 305:375–402.

Bakker, H., Dunke, F., and Nickel, S. (2020). A structuring review on multi-stage optimization under uncertainty: Aligning concepts from theory and practice. *Omega*, 96:102080.

Bakker, H. and Nickel, S. (2024). The Value of the Multi-period Solution revisited: When to model time in capacitated location problems. *Computers & Operations Research*, 161:106428.

Bender, M. (2017). *Recent Mathematical Approaches to Service Territory Design*. PhD thesis, Karlsruhe Institute of Technology, Karlsruhe.

Bender, M. and Kalcsics, J. (2020). Multi-Period Service Territory Design. In Ríos-Mercado, R. Z., editor, *Optimal Districting and Territory Design*, volume 284, pages 129–152. Springer International Publishing, Cham.

Bender, M., Kalcsics, J., and Meyer, A. (2020). Districting for parcel delivery services – A two-Stage solution approach and a real-World case study. *Omega*, 96:102283.

Bender, M., Kalcsics, J., Nickel, S., and Pouls, M. (2018). A branch-and-price algorithm for the scheduling of customer visits in the context of multi-period service territory design. *European Journal of Operational Research*, 269(1):382–396.

Bender, M., Meyer, A., Kalcsics, J., and Nickel, S. (2016). The multi-period service territory design problem - An introduction, a model and a heuristic approach. *Transportation Research Part E: Logistics and Transportation Review*, 96:135157.

Birge, J. R. and Louveaux, F. (2011). *Introduction to Stochastic Programming.* Springer New York, New York, NY.

Blais, M., Lapierre, S. D., and Laporte, G. (2003). Solving a home-care districting problem in an urban setting. *Journal of the Operational Research Society*, 54(11):11411147.

Bruno, G., Diglio, A., Melisi, A., and Piccolo, C. (2017a). A Districting Model to Support the Redesign Process of Italian Provinces. In Sforza, A. and Sterle, C., editors, *Optimization and Decision Science: ODS, Sorrento, Italy, September 4-7 2017*, volume v.217 of *Springer Proceedings in Mathematics and Statistics Ser*, pages 245–256. Springer, Cham.

Bruno, G., Genovese, A., and Piccolo, C. (2017b). Territorial amalgamation decisions in local government: Models and a case study from Italy. *Socio-Economic Planning Sciences*, 57:6172.

Butsch, A. (2016). *Districting Problems: New Geometrically Motivated Approaches.* PhD thesis, Operations Research, Karlsruhe.

Caro, F., Shirabe, T., Guignard, M., and Weintraub, A. (2004). School redistricting: Embedding GIS tools with integer programming. *Journal of the Operational Research Society*, 55(8):836–849.

Darmian, S. M., Fattahi, M., and Keyvanshokooh, E. (2021). An optimization-based approach for the healthcare districting under uncertainty. *Comput. Oper. Res.*

Devore, J. L. and Berk, K. N. (2012). *Modern Mathematical Statistics with Applications.* Springer Texts in Statistics. Springer New York, New York, NY.

Diglio, A., Nickel, S., and Saldanha-da-Gama, F. (2020). Towards a stochastic programming modeling framework for districting. *Annals of Operations Research*, 292(1):249–285.

Diglio, A., Peiró, J., Piccolo, C., and Saldanha-da-Gama, F. (2021). Solutions for districting problems with chance-constrained balancing requirements. *Omega*, 103:102430.

Diglio, A., Peiró, J., Piccolo, C., and Saldanha-da-Gama, F. (2022). Approximation schemes for districting problems with probabilistic constraints. *European Journal of Operational Research.*

Dugošija, D., Savić, A., and Maksimović, Z. (2020). A new integer linear programming formulation for the problem of political districting. *Annals of Operations Research*, 288:247–263.

Enayati, S., Özaltın, O. Y., and Mayorga, M. E. (2020). Designing Ambulance Service Districts Under Uncertainty. *Optimal Districting and Territory Design*, 284:153–170.

Escudero, L. F., Garín, A., Merino, M., and Pérez, G. (2007). The value of the stochastic solution in multistage problems. *TOP*, 15(1):4864.

Escudero, L. F. and Monge, J. F. (2018). On capacity expansion planning under strategic and operational uncertainties based on stochastic dominance risk averse management. *Computational Management Science*, 15(3-4):479–500.

Fazlollahi, S., Bungener, S. L., Mandel, P., Becker, G., and Maréchal, F. (2014). Multi-objectives, multi-period optimization of district energy systems: I. Selection of typical operating periods. *Computers & Chemical Engineering*, 65:54–66.

Goderbauer, S. (2020). *Mathematical Optimization for Optimal Decision-Making in Practice: Energy Systems and Political Districting*. PhD thesis, RWTH Aachen.

Gurnee, W. and Shmos, D. B. (2021). Fairmandering: A column generation heuristic for fairness-optimized political districting. In *SIAM Conference on Applied and Computational Discrete Algorithms*, pages 88–99.

Hess, S. W., Weaver, J. B., Siegfeldt, H. J., Whelan, J. N., and Zitlau, P. A. (1965). Nonpartisan Political Redistricting by Computer. *Operations Research*, 13(6):998–1006.

Kalcsics, J. (2006). *Unified Approaches to Territory Design and Facility Location: Zugl.: Saarbrücken, Univ., Diss., 2006*. Operations Research. Shaker, Aachen.

Kalcsics, J., Nickel, S., and Schröder, M. (2005). Towards a Unified Territorial Design Approach - Applications, Algorithms and GIS Integration. *Sociedad de Estadlstica e Investigación Operativa*.

Kalcsics, J. and Ríos-Mercado, R. Z. (2019). Districting Problems. In Laporte, G., Nickel, S., and Saldanha-da Gama, F., editors, *Location Science*, pages 705–743. Springer, Cham, Switzerland, second edition.

Kaut, M., Midthun, K. T., Werner, A. S., Tomasgard, A., Hellemo, L., and Fodstad, M. (2014). Multi-horizon stochastic programming. *Computational Management Science*, 11(1-2):179–193.

Kong, Y., Zhu, Y., and Wang, Y. (2019). A center-based modeling approach to solve the districting problem. *International Journal of Geographical Information Science*, 33(2):368384.

Laporte, G., Nickel, S., and Saldanha-da Gama, F., editors (2019). *Location Science*. Springer, Cham, Switzerland, second edition.

Lei, H., Laporte, G., and Guo, B. (2012). Districting for routing with stochastic customers. *EURO Journal on Transportation and Logistics*, 1(1-2):6785.

Lei, H., Laporte, G., Liu, Y., and Zhang, T. (2015). Dynamic design of sales territories. *Computers & Operations Research*, 56:8492.

Lei, H., Wang, R., and Laporte, G. (2016). Solving a multi-objective dynamic stochastic districting and routing problem with a co-evolutionary algorithm. *Computers & Operations Research*, 67:1224.

Lespay, H. and Suchan, K. (2022). Territory Design for the Multi-Period Vehicle Routing Problem with Time Windows. *Computers & Operations Research*, 145:105866.

Liberatore, F., Camacho-Collados, M., and Quijano-Sánchez, L. (2022). Equity in the Police Districting Problem: Balancing Territorial and Racial Fairness in Patrolling Operations. *Journal of Quantitative Criminology*.

Liberatore, F., Camacho-Collados, M., and Vitoriano, B. (2020). Police Districting Problem: Literature Review and Annotated Bibliography. *Optimal districting and territory design*, pages 9–29.

Owen, G. and Grofman, B. (1988). Optimal partisan gerrymandering. *Political Geography Quarterly*, 7(1):5–22.

Pomes, A., Diglio, A., Nickel, S., and Saldanha-da-Gama, F. (2025). Multi-stage stochastic districting: Optimization models and solution algorithms. *Annals of Operations Research*.

Ricca, F., Scozzari, A., and Simeone, B. (2008). Weighted Voronoi region algorithms for political districting. *Mathematical and Computer Modelling*, 48(9-10):1468–1477.

Ríos-Mercado, R. Z. (2020). *Optimal Districting and Territory Design*, volume 284. Springer International Publishing, Cham.

Ríos-Mercado, R. Z., Álvarez-Socarrás, A., Castrillón-Escobar, A., and López-Locés, M. C. (2021). A location-allocation-improvement heuristic for districting with multiple-activity balancing constraints and p-median-based dispersion minimization. *Comput. Oper. Res.*, 126:105106.

Ríos-Mercado, R. Z. and López-Pérez, J. F. (2013). Commercial territory design planning with realignment and disjoint assignment requirements. *Omega*, 41(3):525–535.

Salazar-Aguilar, M. A., Rios-Mercado, R. Z., and Cabrera-Ríos, M. (2011). New Models for Commercial Territory Design. *Networks and Spatial Economics*, 11(3):487507.

Saldanha-da-Gama, F. and Wang, S. (2024). *Facility Location under Uncertainty: Models, Algorithms and Applications*. Springer, Cham.

Tasnádi, A. (2011). The Political Districting Problem: A Survey. *Society and Economy. In Central and Eastern Europe - Journal of the Corvinus University of Budapest*, 33(3):543–554.

Tisdale, E. (1812). The Gerry-Mander. *Boston Gazette*.

Yanık, S. and Bozkaya, B. (2020). A Review of Districting Problems in Health Care. *Springer*, Optimal districting and territory design:31–55.

Yanık, S., Kalcsics, J., Nickel, S., and Bozkaya, B. (2019). A multi-period multi-criteria districting problem applied to primary care scheme with gradual assignment. *International Transactions in Operational Research*, 26(5):1676–1697.

Ziegler, H.-P. (2012). *Algorithms for Linear Stochastic Programs and Their Application in Supply Chain Network Design Problems*. PhD thesis, Karlsruhe Institute of Technology, Karlsruhe.

Appendix A: Extensive results

This appendix provides the detailed results for the computations reported in Section 5.5.3. Specifically, Table A1 –Table A4 refer to the 88-TUs instances presented in Section 5.5.2, while Table A5 –Table A8 refer to the 120-TUs instances discussed in Section 5.5.3.2. Besides, Table A9 and Table A10 refer to the 88-TUs instances with four and five stages, respectively, also reported in Section 5.5.3.2.

Each table conveys results for a specific combination of (i) typology of instance (i.e., *Instances_1* or *Instances_2*), and (ii) value of p (4 or 6), and reports the following information:

- Value of α
- Number of performed experiments, "No. Exp" from 1 to 10 (ten different instances were generated per combination of the number of TUs, α, and p)
- Value of the objective function of the best integer solution found by resolving model MSSDP via CPLEX within the imposed time limit of 3 hours (Z_M)
- Value of the objective function returned by the heuristic (Z_H)
- CPU time (in seconds) taken by CPLEX (T_M)
- CPU time (in seconds) taken by the heuristic (T_M)
- Optimality gap (i.e., the MIP relative gap, in %) returned by CPLEX upon termination (if $GAP_{opt} = 0$, the corresponding solution is optimal)
- Value of the objective function of the lower bound provided by CPLEX upon termination (Z_{LB} coincides with Z_M if the solution is optimal)
- Relative gap (in %) between the values of the objective functions of the solutions returned by the heuristic and CPLEX ($GAP_{HM} = \frac{Z_H - Z_M}{Z_M}$)

- Relative gap (in %) between the values of the objective functions of the solutions returned by the heuristic and the lower bound given by CPLEX ($GAP_{HLB} = \frac{Z_H - Z_{LB}}{Z_{LB}}$). Notably, it coincides with the optimality gap (GAP_{opt}) if the optimal solution was attained.

The last two rows of each table inform on the average and maximum values for T_M, T_H, MIP_{gap}, GAP_{HM}, and GAP_{HLB}. Finally, "t.l." denotes occurrences where the imposed time limit was reached.

Table A1: Results for 88-TUs instances, $p = 4$, *Instances_1*.

α	No. Exp	Z_M	Z_H	T_M	T_H	GAP_{opt}	Z_{LB}	GAP_{HM}	GAP_{HLB}
0.05	1	29976058.22	29976058.22	29	26	0.00	29976058.22	0.00	0.00
	2	29827800.81	29827800.81	42	26	0.00	29827800.81	0.00	0.00
	3	29916961.54	29916961.54	33	28	0.00	29916961.54	0.00	0.00
	4	30063459.05	30063459.05	168	31	0.00	30063459.05	0.00	0.00
	5	30267588.98	30267588.98	87	30	0.00	30267588.98	0.00	0.00
	6	29791120.42	29791120.42	3101	40	0.00	29791120.42	0.00	0.00
	7	29739176.12	29739176.12	26	28	0.00	29739176.12	0.00	0.00
	8	30082932.50	30082932.50	35	30	0.00	30082932.5	0.00	0.00
	9	30028424.58	30278385.91	265	28	0.00	30028424.58	0.83	0.83
	10	29954673.67	29954673.67	113	28	0.00	29954673.67	0.00	0.00
0.10	1	29404769.66	29404769.66	26	28	0.00	29404769.66	0.00	0.00
	2	29343876.61	29343876.61	19	28	0.00	29343876.61	0.00	0.00
	3	29469178.93	29469178.93	165	28	0.00	29469178.93	0.00	0.00
	4	29048983.33	29110748.71	35	27	0.00	29048983.33	0.21	0.21
	5	29661694.08	29661694.08	125	28	0.00	29661694.08	0.00	0.00
	6	28937782.84	28937782.84	33	28	0.00	28937782.84	0.00	0.00
	7	29210622.02	29210622.02	23	27	0.00	29210622.02	0.00	0.00
	8	29536484.02	29536484.02	32	27	0.00	29536484.02	0.00	0.00
	9	29202984.73	29202984.73	32	28	0.00	29202984.73	0.00	0.00
	10	29320286.36	29337004.71	103	28	0.00	29320286.36	0.06	0.06
0.15	1	29020287.27	29020287.27	170	28	0.00	29020287.27	0.00	0.00
	2	28926954.26	28926954.26	65	28	0.00	28926954.26	0.00	0.00
	3	28817008.90	28845870.21	79	29	0.00	28817008.9	0.10	0.10
	4	28511466.86	28534662.14	37	27	0.00	28511466.86	0.08	0.08
	5	29063301.07	29126316.97	386	31	0.00	29063301.07	0.22	0.22
	6	28451879.21	28451879.21	94	29	0.00	28451879.21	0.00	0.00
	7	28737235.46	28737235.46	35	27	0.00	28737235.46	0.00	0.00
	8	29055139.46	29055139.46	72	28	0.00	29055139.46	0.00	0.00
	9	28752683.16	28793129.09	44	27	0.00	28752683.16	0.14	0.14
	10	28552566.21	28578390.21	23	27	0.00	28552566.21	0.09	0.09
0.20	1	28234132.89	28234132.89	23	27	0.00	28234132.89	0.00	0.00
	2	28329195.77	28329195.77	25	26	0.00	28329195.77	0.00	0.00
	3	28204615.48	28204615.48	17	26	0.00	28204615.48	0.00	0.00
	4	27944613.61	27944613.61	13	26	0.00	27944613.61	0.00	0.00
	5	28359444.34	28359444.34	29	27	0.00	28359444.34	0.00	0.00
	6	27786036.78	27786036.78	15	26	0.00	27786036.78	0.00	0.00
	7	28144895.89	28144895.89	19	26	0.00	28144895.89	0.00	0.00
	8	28335080.94	28335828.59	16	27	0.00	28335080.94	0.00	0.00
	9	28197083.12	28197083.12	20	27	0.00	28197083.12	0.00	0.00
	10	28004647.80	28004647.80	15	26	0.00	28004647.8	0.00	0.00
0.25	1	27812143.68	27812143.68	15	26	0.00	27812143.68	0.00	0.00
	2	27908249.81	27908249.81	12	25	0.00	27908249.81	0.00	0.00
	3	27882448.11	27882448.11	13	26	0.00	27882448.11	0.00	0.00
	4	27715370.54	27715370.54	12	26	0.00	27715370.54	0.00	0.00
	5	28014064.77	28014064.77	21	26	0.00	28014064.77	0.00	0.00
	6	27460014.04	27460014.04	13	26	0.00	27460014.04	0.00	0.00
	7	27776811.73	27776811.73	13	26	0.00	27776811.73	0.00	0.00
	8	27970538.14	27970538.14	13	27	0.00	27970538.14	0.00	0.00
	9	27790063.84	27790063.84	14	26	0.00	27790063.84	0.00	0.00
	10	27833468.00	27833468.00	12	26	0.00	27833468.00	0.00	0.00
			Average	117	28	0.00		0.03	0.03
			Max	3101	40	0.00		0.83	0.83

Table A2: Results for 88-TUs instances, $p = 6$, *Instances_1*.

α	No. Exp	Z_M	Z_H	T_M	T_H	GAP_{opt}	Z_{LB}	GAP_{HM}	GAP_{HLB}
0.05	1	23393036.57	23393036.57	6172	82	0.00	23393036.57	0.00	0.00
	2	23156401.85	23156401.85	860	66	0.00	23156401.85	0.00	0.00
	3	23475833.31	23475833.31	274	32	0.00	23475833.31	0.00	0.00
	4	23872497.29	23864179.85	t.l.	754	0.48	23757909.3	-0.03	0.45
	5	23605241.94	23605241.94	4569	117	0.00	23605241.94	0.00	0.00
	6	23437144.06	23409920.37	t.l.	386	1.71	23036368.9	-0.12	1.62
	7	22819670.54	22819670.54	162	29	0.00	22819670.54	0.00	0.00
	8	23050501.09	23050501.09	253	30	0.00	23050501.09	0.00	0.00
	9	23536037.89	23536037.89	7774	290	0.00	23536037.89	0.00	0.00
	10	23240706.05	23240706.05	5702	138	0.00	23240706.05	0.00	0.00
0.10	1	22645626.42	22645626.42	23	27	0.00	22645626.42	0.00	0.00
	2	22768758.94	22768758.94	50	33	0.00	22768758.94	0.00	0.00
	3	23040302.10	23040302.10	296	39	0.00	23040302.1	0.00	0.00
	4	22974399.34	22974399.34	256	28	0.00	22974399.34	0.00	0.00
	5	22732183.29	22732183.29	31	33	0.00	22732183.29	0.00	0.00
	6	22425832.73	22425832.73	115	27	0.00	22425832.73	0.00	0.00
	7	22402012.49	22402012.49	15	26	0.00	22402012.49	0.00	0.00
	8	22376240.81	22376240.81	13	27	0.00	22376240.81	0.00	0.00
	9	22611503.57	22693220.64	53	36	0.00	22611503.57	0.36	0.36
	10	22497242.62	22497242.62	28	27	0.00	22497242.62	0.00	0.00
0.15	1	22444594.99	22444594.99	20	28	0.00	22444594.99	0.00	0.00
	2	22559006.68	22559006.68	26	31	0.00	22559006.68	0.00	0.00
	3	22691634.46	22691634.46	34	31	0.00	22691634.46	0.00	0.00
	4	22571620.73	22571620.73	70	28	0.00	22571620.73	0.00	0.00
	5	22605407.36	22605407.36	27	31	0.00	22605407.36	0.00	0.00
	6	22026141.36	22026141.36	20	26	0.00	22026141.36	0.00	0.00
	7	22314782.01	22314782.01	14	27	0.00	22314782.01	0.00	0.00
	8	22341834.03	22341834.03	19	27	0.00	22341834.03	0.00	0.00
	9	22427758.98	22427758.98	43	42	0.00	22427758.98	0.00	0.00
	10	22387022.15	22387022.15	19	27	0.00	22387022.15	0.00	0.00
0.20	1	22276047.98	22276047.98	14	26	0.00	22276047.98	0.00	0.00
	2	22399690.92	22399690.92	14	26	0.00	22399690.92	0.00	0.00
	3	22457788.18	22457788.18	21	27	0.00	22457788.18	0.00	0.00
	4	22375862.52	22375862.52	17	27	0.00	22375862.52	0.00	0.00
	5	22494690.47	22494690.47	27	30	0.00	22494690.47	0.00	0.00
	6	21902828.50	21906546.23	12	27	0.00	21902828.5	0.02	0.02
	7	22255314.49	22255314.49	14	26	0.00	22255314.49	0.00	0.00
	8	22209313.70	22209313.70	11	26	0.00	22209313.7	0.00	0.00
	9	22227724.25	22227724.25	22	27	0.00	22227724.25	0.00	0.00
	10	22252815.11	22252815.11	12	26	0.00	22252815.11	0.00	0.00
0.25	1	22263613.55	22263613.55	17	27	0.00	22263613.55	0.00	0.00
	2	22345046.65	22345046.65	11	26	0.00	22345046.65	0.00	0.00
	3	22359813.37	22359813.37	15	28	0.00	22359813.37	0.00	0.00
	4	22150376.91	22150376.91	11	25	0.00	22150376.91	0.00	0.00
	5	22406287.16	22406287.16	21	28	0.00	22406287.16	0.00	0.00
	6	21878260.03	21878260.03	11	27	0.00	21878260.03	0.00	0.00
	7	22249902.44	22249902.44	17	26	0.00	22249902.44	0.00	0.00
	8	22203946.17	22203946.17	15	26	0.00	22203946.17	0.00	0.00
	9	22107873.34	22107873.34	13	25	0.00	22107873.34	0.00	0.00
	10	22207321.11	22207321.11	10	25	0.00	22207321.11	0.00	0.00
			Average	568	61	0.04		0.00	0.05
			Max	7774	754	1.71		0.36	1.62

Table A3: Results for 88-TUs instances, $p = 4$, *Instances_2*.

α	No. Exp	Z_M	Z_H	T_M	T_H	GAP_{opt}	Z_{LB}	GAP_{HM}	GAP_{HLB}
0.05	1	35290396.75	35501163.99	t.l.	417	0.94	34958667.02	0.60	1.55
	2	34798989.86	34979516.03	t.l.	885	1.45	34294404.51	0.52	2.00
	3	33909262.36	33909262.36	1307	60	0.00	33909262.36	0.00	0.00
	4	34162344.29	34361400.33	t.l.	237	0.42	34018862.44	0.58	1.01
	5	33979032.90	34183871.16	t.l.	56	0.27	33887289.51	0.60	0.88
	6	33871591.34	33871591.34	9998	284	0.00	33871591.34	0.00	0.00
	7	35879613.21	35842071.90	t.l.	1388	1.95	35179960.75	-0.10	1.88
	8	33154982.58	33148578.21	t.l.	77	0.70	32922897.7	-0.02	0.69
	9	33398568.10	33484133.11	t.l.	97	0.56	33211536.12	0.26	0.82
	10	34294466.11	34363467.81	t.l.	457	1.30	33848638.05	0.20	1.52
0.10	1	33790603.32	33989583.63	981	141	0.00	33790603.32	0.59	0.59
	2	33491109.05	33564154.57	1218	34	0.00	33491109.05	0.22	0.22
	3	32519397.36	32519397.36	1821	44	0.00	32519397.36	0.00	0.00
	4	33068239.25	33068239.25	t.l.	54	0.41	32932659.47	0.00	0.41
	5	33036745.27	32989993.92	t.l.	147	0.88	32746021.91	-0.14	0.75
	6	32538442.48	32538442.48	1028	37	0.00	32538442.48	0.00	0.00
	7	33538392.89	33645206.17	558	33	0.00	33538392.89	0.32	0.32
	8	31889815.17	31889815.17	246	31	0.00	31889815.17	0.00	0.00
	9	32188305.03	32309586.33	1636	78	0.00	32188305.03	0.38	0.38
	10	33057193.36	33057193.36	10180	38	0.00	33057193.36	0.00	0.00
0.15	1	32828405.69	32945419.85	413	35	0.00	32828405.69	0.36	0.36
	2	32635421.50	32741155.57	2490	34	0.00	32635421.5	0.32	0.32
	3	31490037.80	31490037.80	395	31	0.00	31490037.8	0.00	0.00
	4	32128068.04	32128068.04	1396	35	0.00	32128068.04	0.00	0.00
	5	31838939.89	31838939.89	365	34	0.00	31838939.89	0.00	0.00
	6	31846610.17	31846610.17	954	36	0.00	31846610.17	0.00	0.00
	7	32774304.93	32774304.93	8362	58	0.00	32774304.93	0.00	0.00
	8	31093709.23	31093709.23	45	29	0.00	31093709.23	0.00	0.00
	9	31298878.16	31298878.16	137	29	0.00	31298878.16	0.00	0.00
	10	31957736.19	31957736.19	693	38	0.00	31957736.19	0.00	0.00
0.20	1	32356582.90	32356582.90	1594	35	0.00	32356582.9	0.00	0.00
	2	32050212.51	32148130.08	601	32	0.00	32050212.51	0.31	0.31
	3	30961390.42	30973196.04	123	28	0.00	30961390.42	0.04	0.04
	4	31611938.76	31611938.76	305	31	0.00	31611938.76	0.00	0.00
	5	30957934.55	30957934.55	37	27	0.00	30957934.55	0.00	0.00
	6	31418070.71	31468500.31	780	32	0.00	31418070.71	0.16	0.16
	7	32008354.48	32008354.48	671	34	0.00	32008354.48	0.00	0.00
	8	30678403.17	30678403.17	73	29	0.00	30678403.17	0.00	0.00
	9	30627812.00	30627812.00	136	28	0.00	30627812	0.00	0.00
	10	31373113.40	31373113.40	364	31	0.00	31373113.4	0.00	0.00
0.25	1	31635192.65	31847424.16	588	30	0.00	31635192.65	0.67	0.67
	2	31499049.06	31623512.88	215	30	0.00	31499049.06	0.40	0.40
	3	30403467.16	30403467.16	71	28	0.00	30403467.16	0.00	0.00
	4	31110202.60	31165957.18	407	28	0.00	31110202.6	0.18	0.18
	5	30487016.46	30487016.46	76	27	0.00	30487016.46	0.00	0.00
	6	31005517.00	31197384.70	526	44	0.00	31005517	0.62	0.62
	7	31487643.61	31487643.61	345	29	0.00	31487643.61	0.00	0.00
	8	30118185.11	30126878.77	119	27	0.00	30118185.11	0.03	0.03
	9	30299027.56	30299027.56	292	27	0.00	30299027.56	0.00	0.00
	10	30718896.10	30718896.10	334	27	0.00	30718896.1	0.00	0.00
			Average	3199	111	0.18		0.14	0.32
			Max	10806	1388	1.95		0.67	2.00

Table A4: Results for 88-TUs instances, $p = 6$, *Instances_2*.

α	No. Exp	Z_M	Z_H	T_M	T_H	GAP_{opt}	Z_{LB}	GAP_{HM}	GAP_{HLB}
0.05	1	29363622.61	28601850.08	t.l.	1072	10.01	26424323.99	-2.59	8.24
	2	27498050.72	27300526.61	t.l.	1347	4.15	26356881.62	-0.72	3.58
	3	27276869.88	27202313.31	t.l.	1444	5.37	25812101.97	-0.27	5.39
	4	27769496.24	27413535.02	t.l.	2346	7.64	25647906.73	-1.28	6.88
	5	27644255.83	26946385.47	t.l.	1075	8.09	25407835.53	-2.52	6.06
	6	27305890.66	26691961.48	t.l.	1204	7.20	25339866.53	-2.25	5.34
	7	30257847.67	29611293.73	t.l.	1299	7.75	27912864.48	-2.14	6.08
	8	25536678.26	25562789.96	t.l.	1019	2.70	24847187.95	0.10	2.88
	9	26562904.86	26204340.72	t.l.	1577	6.85	24743345.88	-1.35	5.90
	10	28653206.40	27637061.46	t.l.	3502	11.56	25340895.74	-3.55	9.06
0.10	1	26140081.36	26071590.24	t.l.	403	0.94	25894364.6	-0.26	0.68
	2	25395556.21	25412068.00	t.l.	173	1.45	25027320.64	0.07	1.54
	3	24797133.68	24797133.68	t.l.	425	0.00	24797133.68	0.00	0.00
	4	25444189.04	25424756.66	t.l.	1106	0.42	25337323.45	-0.08	0.35
	5	25139736.76	25102846.34	t.l.	1054	0.27	25071859.47	-0.15	0.12
	6	24969679.54	25049331.66	t.l.	1693	0.00	24969679.54	0.32	0.32
	7	26026656.06	26043762.91	t.l.	120	1.95	25519136.27	0.07	2.06
	8	24117467.28	24117467.28	8417	67	0.70	23948645.01	0.00	0.70
	9	24480420.53	24432461.37	t.l.	856	0.56	24343330.18	-0.20	0.37
	10	25243557.50	25225584.70	t.l.	670	1.30	24915391.25	-0.07	1.24
0.15	1	24879428.35	24879428.35	10385	113	0.00	24879428.35	0.00	0.00
	2	24459339.46	24459339.46	1650	40	0.00	24459339.46	0.00	0.00
	3	23544834.76	23744699.46	140	53	0.00	23544834.76	0.85	0.85
	4	24359496.27	24375642.53	2166	132	0.41	24259622.34	0.07	0.48
	5	24005205.50	24044978.04	2802	59	0.88	23793959.69	0.17	1.05
	6	23938368.11	23972486.40	764	53	0.00	23938368.11	0.14	0.14
	7	24781811.20	24781811.20	8327	161	0.00	24781811.2	0.00	0.00
	8	23745519.71	23795005.82	206	36	0.00	23745519.71	0.21	0.21
	9	23678914.60	23678914.60	291	37	0.00	23678914.6	0.00	0.00
	10	24173206.16	24173206.16	9672	156	0.00	24173206.16	0.00	0.00
0.20	1	24135927.84	24135927.84	453	37	0.00	24135927.84	0.00	0.00
	2	23846945.95	23846945.95	30	29	0.00	23846945.95	0.00	0.00
	3	23264360.31	23268580.28	129	30	0.00	23264360.31	0.02	0.02
	4	23758302.65	23758302.65	130	36	0.00	23758302.65	0.00	0.00
	5	23391466.01	23391466.01	34	32	0.00	23391466.01	0.00	0.00
	6	23450866.24	23450866.24	89	33	0.00	23450866.24	0.00	0.00
	7	23776845.40	23849593.25	267	32	0.00	23776845.4	0.31	0.31
	8	23498665.10	23498665.10	29	30	0.00	23498665.1	0.00	0.00
	9	23392028.39	23392028.39	68	30	0.00	23392028.39	0.00	0.00
	10	23415612.68	23415612.68	36	31	0.00	23415612.68	0.00	0.00
0.25	1	23630365.79	23630365.79	238	32	0.00	23630365.79	0.00	0.00
	2	23533086.99	23533086.99	18	27	0.00	23533086.99	0.00	0.00
	3	22993223.18	23008391.93	32	27	0.00	22993223.18	0.07	0.07
	4	23478874.08	23478874.08	44	30	0.00	23478874.08	0.00	0.00
	5	23206288.95	23236425.12	37	30	0.00	23206288.95	0.13	0.13
	6	23167689.62	23167689.62	18	28	0.00	23167689.62	0.00	0.00
	7	23366206.90	23366206.90	24	28	0.00	23366206.9	0.00	0.00
	8	23406065.26	23448460.08	23	30	0.00	23406065.26	0.18	0.18
	9	23299997.00	23333619.93	26	28	0.00	23299997	0.14	0.14
	10	23036897.45	23055603.64	17	29	0.00	23036897.45	0.08	0.08
			Average	5037	478	0.00		-0.29	1.41
			Max	10812	3502	0.00		0.85	9.06

Table A5: Results for 120-TUs instances, $p = 4$, *Instances_1*.

α	No. Exp	Z_M	Z_H	T_M	T_H	GAP_{opt}	Z_{LB}	GAP_{HM}	GAP_{HLB}
0.05	1	48828804.78	48946424.05	610	190	0.00	48828804.78	0.24	0.24
	2	48261381.46	48261381.46	361	62	0.00	48261381.46	0.00	0.00
	3	48532701.52	48532701.52	169	109	0.00	48532701.52	0.00	0.00
	4	48460854.11	48460854.11	173	72	0.00	48460854.11	0.00	0.00
	5	48606758.35	48606758.35	4175	144	0.00	48606758.35	0.00	0.00
	6	48608174.19	48608174.19	333	73	0.00	48608174.19	0.00	0.00
	7	48420302.25	48420302.25	180	57	0.00	48420302.25	0.00	0.00
	8	48471146.93	48471146.93	864	82	0.00	48471146.93	0.00	0.00
	9	48333377.90	48333377.90	448	61	0.00	48333377.9	0.00	0.00
	10	48533933.55	48533933.55	65	61	0.00	48533933.55	0.00	0.00
0.10	1	48313054.36	48313054.36	2558	64	0.00	48313054.36	0.00	0.00
	2	47579951.30	47579951.30	106	57	0.00	47579951.3	0.00	0.00
	3	48228643.10	48261438.24	932	66	0.00	48228643.1	0.07	0.07
	4	47808153.18	47808153.18	276	57	0.00	47808153.18	0.00	0.00
	5	47838800.24	47838800.24	284	65	0.00	47838800.24	0.00	0.00
	6	48067536.68	48067536.68	4277	63	0.00	48067536.68	0.00	0.00
	7	48044410.89	48044410.89	107	58	0.00	48044410.89	0.00	0.00
	8	47385597.70	47385597.70	56	55	0.00	47385597.7	0.00	0.00
	9	47719351.12	47719351.12	1634	55	0.00	47719351.12	0.00	0.00
	10	47906340.45	47906340.45	72	56	0.00	47906340.45	0.00	0.00
0.15	1	47686752.69	47686752.69	183	62	0.00	47686752.69	0.00	0.00
	2	46858391.10	46858391.10	100	53	0.00	46858391.1	0.00	0.00
	3	47651291.18	47682481.91	472	63	0.00	47651291.18	0.07	0.07
	4	47226874.92	47226874.92	103	58	0.00	47226874.92	0.00	0.00
	5	47148622.64	47148622.64	73	54	0.00	47148622.64	0.00	0.00
	6	47327318.73	47327318.73	186	63	0.00	47327318.73	0.00	0.00
	7	47632970.14	47632970.14	799	58	0.00	47632970.14	0.00	0.00
	8	46896842.67	46896842.67	67	53	0.00	46896842.67	0.00	0.00
	9	47039694.94	47039694.94	72	60	0.00	47039694.94	0.00	0.00
	10	47303574.46	47303574.46	182	55	0.00	47303574.46	0.00	0.00
0.20	1	47166567.01	47166567.01	1692	66	0.00	47166567.01	0.00	0.00
	2	46387950.33	46387950.33	60	54	0.00	46387950.33	0.00	0.00
	3	46932184.36	46932184.36	465	55	0.00	46932184.36	0.00	0.00
	4	46710683.71	46710683.71	208	64	0.00	46710683.71	0.00	0.00
	5	46736899.17	46736899.17	95	55	0.00	46736899.17	0.00	0.00
	6	46724226.22	46724226.22	362	56	0.00	46724226.22	0.00	0.00
	7	46806838.52	46806838.52	66	57	0.00	46806838.52	0.00	0.00
	8	46275896.84	46275896.84	36	54	0.00	46275896.84	0.00	0.00
	9	46518645.88	46518645.88	177	58	0.00	46518645.88	0.00	0.00
	10	46776544.84	46776544.84	95	56	0.00	46776544.84	0.00	0.00
0.25	1	46520561.22	46520561.22	92	55	0.00	46520561.22	0.00	0.00
	2	45898071.67	45898071.67	61	53	0.00	45898071.67	0.00	0.00
	3	46396966.90	46396966.90	59	55	0.00	46396966.9	0.00	0.00
	4	46199587.48	46199587.48	79	54	0.00	46199587.48	0.00	0.00
	5	46092293.03	46092293.03	62	53	0.00	46092293.03	0.00	0.00
	6	46270081.34	46270081.34	66	53	0.00	46270081.34	0.00	0.00
	7	46129740.73	46129740.73	56	54	0.00	46129740.73	0.00	0.00
	8	46055111.88	46055111.88	58	54	0.00	46055111.88	0.00	0.00
	9	45997866.79	45997866.79	629	56	0.00	45997866.79	0.00	0.00
	10	46280109.55	46280109.55	88	54	0.00	46280109.55	0.00	0.00
			Average	488	64	0.00		0.01	0.01
			Max	4277	190	0.00		0.24	0.24

Table A6: Results for 120-TUs instances, $p = 6$, *Instances_1*.

α	No. Exp	Z_M	Z_H	T_M	T_H	GAP_{opt}	Z_{LB}	GAP_{HM}	GAP_{HLB}
0.05	1	37859434.99	37859434.99	204	64	0.00	37859435	0.00	0.00
	2	37836765.89	37836765.89	10745	140	0.00	37836766	0.00	0.00
	3	37387783.25	37387783.25	134	74	0.00	37387783	0.00	0.00
	4	38261546.33	38317197.18	t.l.	225	0.30	38146762	0.15	0.45
	5	37612487.06	37612487.06	6613	143	0.00	37612487	0.00	0.00
	6	37606796.70	37606796.70	6640	300	0.00	37606797	0.00	0.00
	7	37766402.65	37766402.65	t.l.	633	0.19	37694646	0.00	0.19
	8	37831525.40	37831525.40	2102	106	0.00	37831525	0.00	0.00
	9	37509906.60	37584526.81	4040	422	0.00	37509907	0.20	0.20
	10	38110904.50	38110904.50	3216	168	0.00	38110905	0.00	0.00
0.10	1	37468994.54	37474027.21	2440	62	0.00	37468995	0.01	0.01
	2	37222251.80	37222251.80	138	58	0.00	37222252	0.00	0.00
	3	37004218.83	37004218.83	39	53	0.00	37004219	0.00	0.00
	4	37121877.20	37121877.20	113	56	0.00	37121877	0.00	0.00
	5	36989833.92	36989833.92	491	56	0.00	36989834	0.00	0.00
	6	37012706.30	37012706.30	37	59	0.00	37012706	0.00	0.00
	7	37089325.07	37156853.75	176	63	0.00	37089325	0.18	0.18
	8	37083210.85	37083210.85	115	55	0.00	37083211	0.00	0.00
	9	36634901.31	36650284.62	76	54	0.00	36634901	0.04	0.04
	10	37508597.26	37508597.26	106	59	0.00	37508597	0.00	0.00
0.15	1	36996197.77	36996197.77	79	59	0.00	36996198	0.00	0.00
	2	36869670.17	36869670.17	34	53	0.00	36869670	0.00	0.00
	3	36778118.64	36778118.64	37	54	0.00	36778119	0.00	0.00
	4	36671586.66	36671586.66	41	55	0.00	36671587	0.00	0.00
	5	36675176.03	36675176.03	53	53	0.00	36675176	0.00	0.00
	6	36922016.05	36922016.05	190	58	0.00	36922016	0.00	0.00
	7	36745472.43	36745472.43	45	54	0.00	36745472	0.00	0.00
	8	36637949.80	36637949.80	52	55	0.00	36637950	0.00	0.00
	9	36448852.86	36448852.86	47	54	0.00	36448853	0.00	0.00
	10	37268163.48	37268163.48	47	56	0.00	37268163	0.00	0.00
0.20	1	36796632.23	36796632.23	38	54	0.00	36796632	0.00	0.00
	2	36773741.08	36773741.08	38	52	0.00	36773741	0.00	0.00
	3	36762289.39	36762289.39	42	54	0.00	36762289	0.00	0.00
	4	36520787.12	36520787.12	31	52	0.00	36520787	0.00	0.00
	5	36514810.85	36514810.85	34	53	0.00	36514811	0.00	0.00
	6	36823870.39	36823870.39	70	60	0.00	36823870	0.00	0.00
	7	36562154.80	36562154.80	27	53	0.00	36562155	0.00	0.00
	8	36447736.35	36447736.35	31	53	0.00	36447736	0.00	0.00
	9	36356173.41	36356173.41	26	53	0.00	36356173	0.00	0.00
	10	37060944.88	37060944.88	52	57	0.00	37060945	0.00	0.00
0.25	1	36756854.82	36756854.82	36	53	0.00	36756855	0.00	0.00
	2	36615044.84	36615044.84	26	55	0.00	36615045	0.00	0.00
	3	36700091.21	36700091.21	32	52	0.00	36700091	0.00	0.00
	4	36441208.90	36441208.90	27	54	0.00	36441209	0.00	0.00
	5	36466345.53	36466345.53	23	53	0.00	36466346	0.00	0.00
	6	36686928.31	36686928.31	25	52	0.00	36686928	0.00	0.00
	7	36562154.80	36562154.80	28	51	0.00	36562155	0.00	0.00
	8	36383878.77	36383878.77	27	52	0.00	36383879	0.00	0.00
	9	36356173.41	36356173.41	31	53	0.00	36356173	0.00	0.00
	10	36941821.36	36941821.36	31	54	0.00	36941821	0.00	0.00
			Average	1207	89	0.01		0.01	0.02
			Max	10810	633	0.30		0.20	0.45

Table A7: Results for 120-TUs instances, $p = 4$, *Instances_2*.

α	No. Exp	Z_M	Z_H	T_M	T_H	GAP_{opt}	Z_{LB}	GAP_{HM}	GAP_{HLB}
0.05	1	52815608.80	52800501.77	t.l.	2355	0.31	52651880.41	-0.03	0.28
	2	53522881.26	53312036.26	t.l.	2172	1.15	52907368.13	-0.39	0.76
	3	53110039.41	52839218.61	t.l.	5173	1.46	52334632.83	-0.51	0.96
	4	53600883.48	53565959.36	t.l.	3107	1.01	53059514.56	-0.07	0.95
	5	53098585.23	53322324.60	t.l.	3872	0.38	52896810.61	0.42	0.80
	6	53080918.92	53079204.29	t.l.	5118	0.54	52794281.96	0.00	0.54
	7	52601158.96	52331942.22	t.l.	4036	1.32	51906823.66	-0.51	0.82
	8	54695064.39	53762334.14	t.l.	10177	3.06	53021395.42	-1.71	1.40
	9	52615123.80	52578140.35	t.l.	816	0.94	52120541.64	-0.07	0.88
	10	54101350.40	53868385.40	t.l.	3288	1.79	53132936.23	-0.43	1.38
0.10	1	51201512.77	51233836.41	1769	124	0.00	51201512.77	0.06	0.06
	2	51615712.52	51615712.52	4369	219	0.00	51615712.52	0.00	0.00
	3	50957671.99	50957671.99	2657	175	0.00	50957671.99	0.00	0.00
	4	51902800.13	51976304.67	t.l.	2166	0.39	51700379.21	0.14	0.53
	5	51551039.84	51551039.84	t.l.	1176	0.12	51489178.59	0.00	0.12
	6	51688248.20	51756375.11	t.l.	1997	0.45	51455651.08	0.13	0.58
	7	50578538.01	50688437.22	1121	172	0.00	50578538.01	0.22	0.22
	8	51869910.88	51869910.88	t.l.	235	0.07	51833601.94	0.00	0.07
	9	50865723.60	50963948.47	6196	302	0.00	50865723.6	0.19	0.19
	10	51881181.11	51881181.11	5328	124	0.00	51881181.11	0.00	0.00
0.15	1	50490090.68	50582333.85	630	90	0.00	50490090.68	0.18	0.18
	2	50695690.39	50695690.39	1240	106	0.00	50695690.39	0.00	0.00
	3	50197212.06	50326362.15	452	90	0.00	50197212.06	0.26	0.26
	4	50931423.47	51072792.13	539	113	0.00	50931423.47	0.28	0.28
	5	50522638.07	50565925.74	535	103	0.00	50522638.07	0.09	0.09
	6	50763122.06	50892228.22	2775	116	0.00	50763122.06	0.25	0.25
	7	50095000.16	50192442.18	341	99	0.00	50095000.16	0.19	0.19
	8	50860963.12	50961006.56	2387	164	0.00	50860963.12	0.20	0.20
	9	50370390.99	50449991.93	1046	80	0.00	50370390.99	0.16	0.16
	10	50852346.81	50921732.59	2375	176	0.00	50852346.81	0.14	0.14
0.20	1	49646311.73	49646311.73	2091	93	0.00	49646311.73	0.00	0.00
	2	50063071.25	50063071.25	1321	101	0.00	50063071.25	0.00	0.00
	3	50074427.04	50158490.28	1696	978	0.00	50074427.04	0.17	0.17
	4	50566084.85	50566084.85	1282	73	0.00	50566084.85	0.00	0.00
	5	50167420.83	50167420.83	1280	100	0.00	50167420.83	0.00	0.00
	6	50119743.90	50119743.90	1046	60	0.00	50119743.9	0.00	0.00
	7	49939383.05	49939383.05	1640	105	0.00	49939383.05	0.00	0.00
	8	50064894.64	50209321.83	133	74	0.00	50064894.64	0.29	0.29
	9	50244521.10	50244521.10	1178	108	0.00	50244521.1	0.00	0.00
	10	50325122.15	50325122.15	8680	169	0.00	50325122.15	0.00	0.00
0.25	1	48852237.43	48852237.43	106	60	0.00	48852237.43	0.00	0.00
	2	49286533.56	49286533.56	1554	64	0.00	49286533.56	0.00	0.00
	3	49330255.00	49344406.13	3475	93	0.00	49330255	0.03	0.03
	4	49780867.06	49780867.06	214	65	0.00	49780867.06	0.00	0.00
	5	49201311.50	49201311.50	881	56	0.00	49201311.5	0.00	0.00
	6	49337331.41	49337331.41	2310	67	0.00	49337331.41	0.00	0.00
	7	49341004.99	49341004.99	2602	153	0.00	49341004.99	0.00	0.00
	8	49935505.60	49935505.60	682	62	0.00	49935505.6	0.00	0.00
	9	49548292.52	49548292.52	107	65	0.00	49548292.52	0.00	0.00
	10	49371452.68	49371452.68	537	89	0.00	49371452.68	0.00	0.00
			Average	4358	1012	0.26		-0.01	0.26
			Max	10811	10177	3.06		0.42	1.40

Table A8: Results for 120-TUs instances, $p = 6$, *Instances_2*.

α	No. Exp	Z_M	Z_H	T_M	T_H	GAP_{opt}	Z_{LB}	GAP_{HM}	GAP_{HLB}
0.05	1	40911468.04	40911447.68	t.l.	3886	1.26	40395983.54	0.00	1.28
	2	42127398.86	41698392.38	t.l.	1995	3.52	40644514.42	-1.02	2.59
	3	42282276.70	41572717.85	t.l.	4770	4.83	40240042.74	-1.68	3.31
	4	41860521.34	41585480.84	t.l.	1540	2.26	40914473.56	-0.66	1.64
	5	41335990.97	41248783.94	t.l.	4524	1.92	40542339.94	-0.21	1.74
	6	41643710.39	41295515.77	t.l.	3673	2.72	40511001.47	-0.84	1.94
	7	41266644.61	41023400.52	t.l.	2836	3.07	39999758.62	-0.59	2.56
	8	42282094.57	42131546.47	t.l.	1466	1.61	41601352.85	-0.36	1.27
	9	41331166.96	41223422.80	t.l.	1451	1.37	40764929.97	-0.26	1.12
	10	42006038.73	41999415.93	t.l.	1344	1.85	41228927.01	-0.02	1.87
0.10	1	39062930.30	39062930.30	181	65	0.00	39062930.3	0.00	0.00
	2	39737276.04	39737276.04	t.l.	328	0.31	39614090.48	0.00	0.31
	3	39766108.41	39586523.31	t.l.	595	1.66	39105991.01	-0.45	1.23
	4	39974099.40	39946262.67	t.l.	416	0.94	39598342.87	-0.07	0.88
	5	39427095.20	39427095.20	t.l.	243	0.19	39352183.72	0.00	0.19
	6	39661153.95	39604904.24	t.l.	638	0.45	39482678.76	-0.14	0.31
	7	39176330.04	39161122.88	t.l.	430	0.82	38855084.13	-0.04	0.79
	8	40329851.67	40323047.24	t.l.	288	0.54	40112070.47	-0.02	0.53
	9	39606017.29	39606017.29	t.l.	191	0.43	39435711.42	0.00	0.43
	10	39886013.01	39886013.01	t.l.	352	0.64	39630742.53	0.00	0.64
0.15	1	38306539.83	38306539.83	70	58	0.00	38306539.83	0.00	0.00
	2	38695150.55	38695150.55	412	68	0.00	38695150.55	0.00	0.00
	3	38275946.71	38275946.71	360	71	0.00	38275946.71	0.00	0.00
	4	38824378.07	38824378.07	44	56	0.00	38824378.07	0.00	0.00
	5	38272304.63	38272304.63	383	60	0.00	38272304.63	0.00	0.00
	6	38486066.89	38486066.89	7524	107	0.00	38486066.89	0.00	0.00
	7	37981949.76	37981949.76	45	55	0.00	37981949.76	0.00	0.00
	8	38943156.18	38943156.18	85	60	0.00	38943156.18	0.00	0.00
	9	38667780.44	38670551.37	55	58	0.00	38667780.44	0.01	0.01
	10	38672028.88	38706902.60	9586	72	0.00	38672028.88	0.09	0.09
0.20	1	37868391.42	37868391.42	58	54	0.00	37868391.42	0.00	0.00
	2	37979154.82	37989936.21	55	55	0.00	37979154.82	0.03	0.03
	3	37769794.59	37769794.59	70	57	0.00	37769794.59	0.00	0.00
	4	38420468.01	38420468.01	36	55	0.00	38420468.01	0.00	0.00
	5	37759104.25	37759104.25	61	55	0.00	37759104.25	0.00	0.00
	6	37775115.13	37775115.13	178	57	0.00	37775115.13	0.00	0.00
	7	37572137.48	37572137.48	34	55	0.00	37572137.48	0.00	0.00
	8	38343088.45	38343088.45	63	55	0.00	38343088.45	0.00	0.00
	9	38283978.80	38283978.80	31	55	0.00	38283978.8	0.00	0.00
	10	37880184.44	37880184.44	295	63	0.00	37880184.44	0.00	0.00
0.25	1	37528021.25	37528021.25	34	54	0.00	37528021.25	0.00	0.00
	2	37640623.33	37640623.33	46	55	0.00	37640623.33	0.00	0.00
	3	37480587.61	37480587.61	51	54	0.00	37480587.61	0.00	0.00
	4	38155626.27	38155626.27	40	53	0.00	38155626.27	0.00	0.00
	5	37464246.79	37464246.79	40	53	0.00	37464246.79	0.00	0.00
	6	37441497.23	37441497.23	37	55	0.00	37441497.23	0.00	0.00
	7	37320504.28	37320504.28	38	54	0.00	37320504.28	0.00	0.00
	8	37917190.28	37917190.28	43	54	0.00	37917190.28	0.00	0.00
	9	38062278.92	38062278.92	33	53	0.00	38062278.92	0.00	0.00
	10	37487135.87	37487845.46	56	53	0.00	37487135.87	0.00	0.00
			Average	4509	656	0.61		-0.12	0.50
			Max	10811	4770	4.83		0.09	3.31

Table A9: Results for 88-TUs instances with four stages ($p = 4$, *Instances_1*).

α	No. Exp	Z_M	Z_H	T_M	T_H	GAP_{opt}	Z_{LB}	GAP_{HM}	GAP_{HLB}
0.10	1	29599522.27	29599522.27	5743	131	0.00	29599522.27	0.00	0.00
	2	30776164.85	30776164.85	5432	169	0.00	30776164.85	0.00	0.00
	3	30690752.53	30690752.53	37250	518	0.00	30690752.53	0.00	0.00
	4	30179190.42	30179190.42	6048	236	0.00	30179190.42	0.00	0.00
	5	30255143.93	30255143.93	23172	142	0.00	30255143.93	0.00	0.00
	6	30372990.20	30372990.20	5454	134	0.00	30372990.2	0.00	0.00
	7	29312039.26	29312039.26	21612	166	0.00	29312039.26	0.00	0.00
	8	29139737.85	29139737.85	2809	112	0.00	29139737.85	0.00	0.00
	9	29010838.37	29082589.16	4612	117	0.00	29010838.37	0.25	0.25
	10	30730320.93	30730320.93	25218	582	0.00	30730320.93	0.00	0.00
0.15	1	29098115.73	29098115.73	1576	118	0.00	29098115.73	0.00	0.00
	2	30292679.01	30292679.01	8453	149	0.00	30292679.01	0.00	0.00
	3	29971690.82	29971690.82	2649	111	0.00	29971690.82	0.00	0.00
	4	29639701.95	29639701.95	25937	138	0.00	29639701.95	0.00	0.00
	5	29605293.30	29605293.30	30118	200	0.00	29605293.3	0.00	0.00
	6	29983897.14	29983897.14	12283	126	0.00	29983897.14	0.00	0.00
	7	28487613.62	28487613.62	10340	148	0.00	28487613.62	0.00	0.00
	8	28514815.90	28514815.90	842	128	0.00	28514815.9	0.00	0.00
	9	28137386.11	28137838.22	593	108	0.00	28137386.11	0.00	0.00
	10	29913855.49	29913855.49	14104	449	0.00	29913855.49	0.00	0.00
0.20	1	28443990.86	28443990.86	410	120	0.00	28443990.86	0.00	0.00
	2	29812518.38	29891550.16	19353	240	0.00	29812518.38	0.27	0.27
	3	29486391.20	29501363.10	10170	109	0.00	29486391.2	0.05	0.05
	4	28989098.46	28989098.46	4503	123	0.00	28989098.46	0.00	0.00
	5	29051994.92	29051994.92	19105	157	0.00	29051994.92	0.00	0.00
	6	29452350.32	29452350.32	15893	150	0.00	29452350.32	0.00	0.00
	7	27764416.69	27764416.69	408	113	0.00	27764416.69	0.00	0.00
	8	27990925.99	27990925.99	456	115	0.00	27990925.99	0.00	0.00
	9	27851441.78	27851441.78	356	109	0.00	27851441.78	0.00	0.00
	10	29248631.51	29248631.51	14190	119	0.00	29248631.51	0.00	0.00
			Average	10970	178	0.00		0.02	0.02
			Max	37250	582	0.00		0.27	0.27

Table A10: Results for 88-TUs instances with five stages ($p = 4$, *Instances_1*).

α	No. Exp	Z_M	Z_H	T_M	T_H	GAP_{opt}	Z_{LB}	GAP_{HM}	GAP_{HLB}
0.20	1	28651162.80	28650635.22	8046.00	1154.00	0.00	28651163	0.00	0.00
	2	31885682.36	29993536.39	t.l.	10432.00	7.77	29408165	-5.93	1.99
	3	29988545.00	29708213.20	t.l.	4588.00	1.94	29406767	-0.93	1.03
	4	37320255.03	29186888.70	t.l.	1416.00	22.87	28785113	-21.79	1.40
	5	29754053.60	29183113.44	t.l.	3033.00	2.87	28900112	-1.92	0.98
	6	30021672.48	29680565.49	t.l.	3193.00	2.67	29220094	-1.14	1.58
	7	27896820.37	27898341.04	7227.00	849.00	0.00	27896820	0.01	0.01
	8	30958514.44	28166577.14	t.l.	1313.00	9.59	27989593	-9.02	0.63
	9	29868434.27	28060066.99	t.l.	957.00	6.60	27897118	-6.05	0.58
	10	29582271.26	29350227.70	t.l.	1097.00	1.49	29141495	-0.78	0.72
			Average	10185	2803	5.58		-4.76	0.89
			Max	10823	10432	22.87		0.01	1.99