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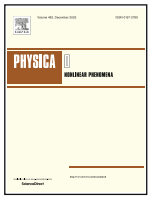

Nonlinear dynamics of periodic Lugiato-Lefever waves against sums of co-periodic and localized perturbations

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ABSTRACT

In recent years, essential progress has been made in the nonlinear stability analysis of periodic Lugiato-Lefever waves against co-periodic and localized perturbations. Inspired by considerations from fiber optics, we introduce a novel iteration scheme which allows to perturb against sums of co-periodic and localized functions. This unifies previous stability theories in a natural manner.

1. Introduction

We study the Lugiato-Lefever equation on the extended real line

$$\partial_t u = -\beta i \partial_x^2 u - (1 + i\alpha)u + i|u|^2 u + F, \quad \beta \in \{-1, 1\}, \quad \alpha \in \mathbb{R}, \quad F > 0, \tag{1.1}$$

for $u : \mathbb{R} \times [0, \infty) \to \mathbb{C}$, which is a model from nonlinear optics [1]. An important observation is that the principal part $-\beta i \partial_x^2 u$ is dispersive while the damping term $-u$ causes energy dissipation. The forcing term F again adds energy to the physical system and allows for pattern formation as predicted by Lugiato and Lefever in [2]. The most fundamental patterns such as pulses, small-amplitude or dnoidal periodic waves are found in [3–6] and in [7] and [8], respectively. Recently, in [9], the authors have obtained large-amplitude periodic waves.

In this paper, we prove nonlinear L^∞-stability of T-periodic standing waves against initial perturbations from the space $L^2_{\mathrm{per}}(0,T) \oplus L^2(\mathbb{R})$[1] under diffusive spectral stability assumptions. These spectral assumptions are only established for the waves in [9] and [7].

We emphasize that sums of periodic and localized perturbations are not necessarily localized or periodic and thus our result is a nontrivial unification of the theories [10] (co-periodic) and [11] (localized). For the precise formulation of our main result, we refer to Section 2.5.

In view of fully nonlocalized perturbations, the recently developed nonlinear stability theory for dissipative semilinear systems [12] is not immediately applicable to all pattern-forming semilinear systems such as the Lugiato-Lefever equation. From this perspective, the present paper is the first to accommodate nonlocalized perturbations and to combine $H^l_{\mathrm{per}}(0,T)$- and H^k-theory. On the other hand, considerations from fiber optics, see [13] and Remark 2.1, where combinations of localized and co-periodic effects naturally occur, motivate the investigation of these types of perturbations. Interpretating the Lugiato-Lefever equation as a variant of the cubic nonlinear Schrödinger equation, a related inspiration for this paper comes from the so-called *tooth problem* asking whether solutions of the nonlinear Schrödinger equation with not necessarily small initial data from $L^2_{\mathrm{per}}(\mathbb{R}) \oplus L^2(\mathbb{R})$ exist globally; we refer to [14] and [15] for answers to the tooth problem and background information.

Our central challenge is to develop the right modulational ansatz to make a nonlinear iteration argument close. Inspired by [16], we introduce both a temporal and a localized spatio-temporal phase modulation to capture the critical dynamics of the perturbation induced by

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[1] The term $\oplus$ denotes the direct sum, that is for every $u \in L^2_{\mathrm{per}}(0,T) \oplus L^2(\mathbb{R})$, we find precisely one $w \in L^2_{\mathrm{per}}(0,T)$ and $v \in L^2(\mathbb{R})$ such that $u = w + v$.

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translational invariance of (1.1). Moreover, we employ novel $L^2 - L^\infty$-estimates in the nonlinear iteration argument to control the interaction between the periodic and localized components of the perturbation. For more details on the strategy of the proof, we refer to Section 2.6. The outlook section Section 6 is devoted to the robustness of our approach as well as its possible extension to periodic wave trains in viscous conservation laws where the handling of fully nonlocalized perturbations is similarly challenging as for the Lugiato-Lefever equation but for different reasons. In case of the Lugiato-Lefever equation, a crucial difficulty towards extending to a fully nonlocalized stability result is to choose a suitable class of perturbations which contains all C^∞-functions. The generic space of perturbations is given by

$$C^m_{\mathrm{ub}}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} : f \text{ is } m\text{-times differentiable with uniformly continuous and bounded derivatives}\},$$

$m \in \mathbb{N}$, whenever studying reaction-diffusion systems [12,17,18]. The solutions of (1.1) with initial data in $H^1_{\mathrm{per}}(\mathbb{R}) \oplus H^1(\mathbb{R})$ naturally lie in $C_{\mathrm{ub}}(\mathbb{R})$ due to Sobolev embedding. However, it is shown in [19] that $t \mapsto ||e^{i\partial_x^2 t}u_0||_{L^\infty}$ blows up in finite times for certain $u_0 \in C_{\mathrm{ub}}(\mathbb{R})$ and therefore it is convenient to study other spaces than $C_{\mathrm{ub}}(\mathbb{R})$ to approach a fully nonlocalized stability result for the Lugiato-Lefever equation (1.1). More suitable variants are given by the so-called modulation spaces $M^m_{\infty,1}(\mathbb{R})$, $m \in \mathbb{N}_0$, which are introduced in [20]. We discuss such an extension in Section 6.4.

2. Preparation and main result

We reformulate the Lugiato-Lefever equation as a semilinear system with a $\mathbb{C}$-linear part by splitting into real- and imaginary variables. Then, we construct perturbed solutions with initial data in $L^2_{\mathrm{per}}(0,T) \oplus L^2(\mathbb{R})$ and derive the associated perturbation equations. At the end of this section, we impose spectral properties and formulate our main result.

2.1. Reformulation as real system

As $|u|^2u$ is not differentiable with respect to $u \in \mathbb{C}$, we introduce $\mathbf{u} := (\mathbf{u}_r, \mathbf{u}_i)^T := (\mathrm{Re}(u), \mathrm{Im}(u))^T : \mathbb{R} \to \mathbb{R}^2$ which transforms (1.1) into the real system

$$\mathbf{u}_t = \mathcal{J}\left(\begin{pmatrix} -\beta & 0 \\ 0 & -\beta \end{pmatrix}\mathbf{u}_{xx} + \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}\mathbf{u}\right) - \mathbf{u} + \mathcal{N}(\mathbf{u}) + \begin{pmatrix} F \\ 0 \end{pmatrix}, \tag{2.1}$$

where

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{N}(\mathbf{u}) = |\mathbf{u}|^2 J\mathbf{u} = \begin{pmatrix} -\mathbf{u}_i^3 - \mathbf{u}_r^2\mathbf{u}_i \\ \mathbf{u}_r\mathbf{u}_i^2 + \mathbf{u}_r^3 \end{pmatrix}.$$

2.2. The perturbed solution in $L^2_{per}(0,T) \oplus L^2(\mathbb{R})$

We assume the existence of a periodic standing wave.

(H1) There exists a smooth, nonconstant and T-periodic stationary solution $\phi_0 : \mathbb{R} \to \mathbb{C}$ of (1.1).

We set $\phi := (\phi_r, \phi_i)^T := (\mathrm{Re}(\phi_0), \mathrm{Im}(\phi_0))^T : \mathbb{R} \to \mathbb{R}^2$ and construct a solution of (2.1) with initial datum $\phi + \mathbf{w}_0 + \mathbf{v}_0$ in $L^2_{\mathrm{per}}(0,T) \oplus L^2(\mathbb{R})$ with $\mathbf{w}_0 \in L^2_{\mathrm{per}}(0,T)$ and $\mathbf{v}_0 \in L^2(\mathbb{R})$ by following the strategy of [14]. We first solve

$$\begin{aligned} \mathbf{w}_t &= \mathcal{J}\left(\begin{pmatrix} -\beta & 0 \\ 0 & -\beta \end{pmatrix}\mathbf{w}_{xx} + \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}\mathbf{w}\right) - \mathbf{w} + \mathcal{N}(\mathbf{w}) + \begin{pmatrix} F \\ 0 \end{pmatrix} \\ \mathbf{w}(0) &= \phi + \mathbf{w}_0 \end{aligned} \tag{2.2}$$

in $L^2_{\mathrm{per}}(0,T)$. Then, we transfer from the solution $\mathbf{w}$ of (2.1) to a solution $\mathbf{u}$ of (2.1) with initial datum $\phi + \mathbf{w}_0 + \mathbf{v}_0$ by solving the perturbed problem

$$\begin{aligned} \mathbf{v}_t &= \mathcal{J}\left(\begin{pmatrix} -\beta & 0 \\ 0 & -\beta \end{pmatrix}\mathbf{v}_{xx} - \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}\mathbf{v}\right) - \mathbf{v} + \mathcal{N}(\mathbf{v}+\mathbf{w}) - \mathcal{N}(\mathbf{w}) \\ \mathbf{v}(0) &= \mathbf{v}_0. \end{aligned} \tag{2.3}$$

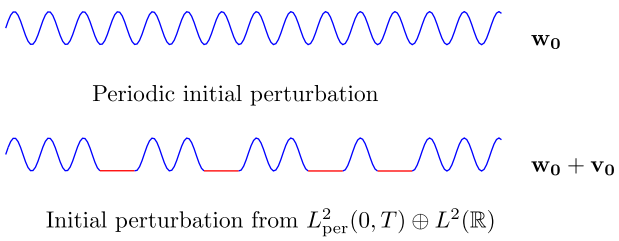

Fig. 1. For the sake of illustration, we reduce to the real part of an initial perturbation $\mathbf{w}_0 + \mathbf{v}_0$ with $\mathbf{w}_0 \in L^2_{\mathrm{per}}(0,T)$ and $\mathbf{v}_0 \in L^2(\mathbb{R})$. This figure demonstrates that $\mathbf{v}_0$ can in particular be chosen such that $\mathbf{v}_0 + \mathbf{w}_0$ coincides with $\mathbf{w}_0$ except for finitely many periods for which the signal vanishes, which motivates the name "tooth space" for $L^2_{\mathrm{per}}(0,T) \oplus L^2(\mathbb{R})$ ("$L^2(\mathbb{R})$ knocks out teeth of $L^2_{\mathrm{per}}(0,T)$").

In summary, if we solve (2.2) and (2.3), then $\mathbf{u} = \mathbf{w} + \mathbf{v}$ is a solution of (2.1) with $\mathbf{u}(0) = \phi + \mathbf{w}_0 + \mathbf{v}_0$.

Remark 2.1 (Interpretation from fiber optics). *The cubic nonlinear Schrödinger equation on $L^2_{per}(0,T) \oplus L^2(\mathbb{R})$ can be understood as model from nonlinear fiber optics by considering $t \geq 0$ as point on a fiber and $x \in \mathbb{R}$ as time variable, cf. [13]. The stationary periodic solution ϕ is then the signal at any point on the fiber if it is chosen as input signal. Prescribing a signal at the initial point of the fiber by $\phi + L^2_{per}(0,T)$, we ask how the signal as function depending on the time $x \in \mathbb{R}$ looks at the place $t \geq 0$ on the fiber. Adding an $L^2(\mathbb{R})$-perturbation corresponds to temporally limited changes of the input signal. In particular, one may switch off the periodic signal for finitely many times as illustrated in Fig. 1. Global existence of the perturbed solutions then translates to the observation that the fiber has infinite length while stability of ϕ is interpreted as that the signal stays close to ϕ at any point $t \geq 0$ on the fiber.*

2.3. Unmodulated perturbation equations

Given a solution $\mathbf{u}(t) = \mathbf{w}(t) + \mathbf{v}(t)$ of (2.1), we derive the unmodulated perturbation equations by splitting the perturbation as

$\tilde{\mathbf{u}}(t) = \mathbf{u}(t) - \phi = (\mathbf{w}(t) - \phi) + \mathbf{v}(t)$ and setting $\tilde{\mathbf{w}}(t) = \mathbf{w}(t) - \phi$.

This gives the coupled perturbation system

$$\begin{aligned} \tilde{\mathbf{w}}_t &= \mathcal{L}_0(\phi)\tilde{\mathbf{w}} + \mathcal{R}_1(\phi)(\tilde{\mathbf{w}})\mathbf{v}_t = \mathcal{L}_0(\phi)\mathbf{v} + \mathcal{R}_2(\phi)(\tilde{\mathbf{w}}, \mathbf{v}) \\ \tilde{\mathbf{w}}(0) &= \mathbf{w}_0, \qquad\qquad \mathbf{v}(0) = \mathbf{v}_0, \end{aligned} \tag{2.4}$$

where $\mathcal{L}_0(\phi)$ is the linearization of (2.1) about ϕ, given by

$$\mathcal{L}_0(\phi) = \mathcal{J}\begin{pmatrix} -\beta\partial_x^2 - \alpha + 3\phi_r^2 + \phi_i^2 & 2\phi_r\phi_i \\ 2\phi_r\phi_i & -\beta\partial_x^2 - \alpha + \phi_r^2 + 3\phi_i^2 \end{pmatrix} - \mathcal{I}, \tag{2.5}$$

first residual nonlinearity is given by

$$\mathcal{R}_1(\phi)(\tilde{\mathbf{w}}) = \mathcal{N}(\tilde{\mathbf{w}} + \phi) - \mathcal{N}(\phi) - \mathcal{N}'(\phi)\tilde{\mathbf{w}}, \tag{2.6}$$

and the second residual nonlinearity is defined by

$$\mathcal{R}_2(\phi)(\tilde{\mathbf{w}}, \mathbf{v}) = \mathcal{R}_1(\phi)(\mathbf{v} + \tilde{\mathbf{w}}) - \mathcal{R}_1(\phi)(\tilde{\mathbf{w}}) = \mathcal{R}_{2,1}(\phi)(\tilde{\mathbf{w}}, \mathbf{v}) + \mathcal{R}_{2,2}(\phi)(\tilde{\mathbf{w}}, \mathbf{v}), \tag{2.7}$$

with

$$\begin{aligned} \mathcal{R}_{2,1}(\phi)(\tilde{\mathbf{w}}, \mathbf{v}) &= \mathcal{N}(\mathbf{v} + \tilde{\mathbf{w}} + \phi) - \mathcal{N}(\tilde{\mathbf{w}} + \phi) - \mathcal{N}'(\tilde{\mathbf{w}} + \phi)\mathbf{v}, \\ \mathcal{R}_{2,2}(\phi)(\tilde{\mathbf{w}}, \mathbf{v}) &= \mathcal{N}'(\tilde{\mathbf{w}} + \phi)\mathbf{v} - \mathcal{N}'(\phi)\mathbf{v}. \end{aligned}$$

Fix some constant $K > 0$. Then, there exists a constant $C > 0$ such that for $\mathbf{v}, \tilde{\mathbf{w}} \in \mathbb{C}$ with $|\mathbf{v}|, |\tilde{\mathbf{w}}| \leq K$ we have the nonlinear bounds

$$|\mathcal{R}_1(\phi)(\tilde{\mathbf{w}})| \leq C|\tilde{\mathbf{w}}|^2, \quad |\mathcal{R}_{2,1}(\phi)(\tilde{\mathbf{w}}, \mathbf{v})| \leq C|\mathbf{v}|^2, \quad |\mathcal{R}_{2,2}(\phi)(\tilde{\mathbf{w}}, \mathbf{v})| \leq C|\mathbf{v}||\tilde{\mathbf{w}}|. \tag{2.8}$$

For the local wellposedness of (2.4), we refer to Section 4.1.

We also abbreviate $\mathcal{L}_0 = \mathcal{L}_0(\phi)$, $\mathcal{R}_1 = \mathcal{R}_1(\phi)$, $\mathcal{R}_2 = \mathcal{R}_2(\phi)$, $\mathcal{R}_{2,1} = \mathcal{R}_{2,1}(\phi)$ and $\mathcal{R}_{2,2} = \mathcal{R}_{2,2}(\phi)$ whenever ϕ is the original periodic wave profile.

2.4. Spectral assumptions on ϕ

Consider the Bloch operators $\mathcal{L}(\xi) = e^{-i\xi\cdot}\mathcal{L}_0 e^{i\xi\cdot}$, $\xi \in [-\frac{\pi}{T}, \frac{\pi}{T})$, posed on $L^2_{\text{per}}(0,T)$ with domain $D(\mathcal{L}(\xi)) = H^2_{\text{per}}(0,T)$. Since $\mathcal{L}(\xi)$ has compact resolvent, its spectrum consists of isolated eigenvalues of finite algebraic multiplicity only.

We introduce the standard *diffusive spectral stability* assumptions, cf. [10,11,16,21].

(D1) We have $\sigma_{L^2}(\mathcal{L}_0) \subset \{\lambda \in \mathbb{C} : \text{Re}(\lambda) < 0\} \cup \{0\}$;
(D2) There exists a constant $\theta > 0$ such that for any $\xi \in [-\frac{\pi}{T}, \frac{\pi}{T})$ we have $\text{Re}\,\sigma_{L^2_{\text{per}}(0,T)}(\mathcal{L}(\xi)) \le -\theta\xi^2$;
(D3) 0 is a simple eigenvalue of $\mathcal{L}(0)$.

The spectrum of $\mathcal{L}_0$ on $L^2(\mathbb{R})$ is the union of the spectra of the Bloch operators, i.e.,

$$\sigma_{L^2}(\mathcal{L}_0) = \bigcup_{\xi\in[-\frac{\pi}{T},\frac{\pi}{T})} \sigma_{L^2_{\text{per}}(0,T)}(\mathcal{L}(\xi)).$$

The spectrum $\sigma_{L^2}(\mathcal{L}_0)$ is purely essential, see e.g. [22,23]. We emphasize that the periodic solutions of (1.1) established in [7] and [9] satisfy (H1) and (D1)-(D3). First, Assumption (D3) together with the translational invariance of (1.1) imply that the kernel of $\mathcal{L}(0)$ is spanned by ϕ'. Therefore, 0 is also a simple eigenvalue of the adjoint operator $\mathcal{L}(0)^*$. By $\widetilde{\Phi}_0 \in H^2_{\text{per}}(0,T)$, we denote the corresponding eigenfunction satisfying

$$\langle \widetilde{\Phi}_0, \phi' \rangle_{L^2(0,T)} = 1.$$

For some sufficiently small $\xi_0 \in (0, \frac{\pi}{T})$, we summarize the following properties and refer to [11, Lemma 2.1] for their justifications and further properties of $\sigma_{L^2}(\mathcal{L}_0)$ and $\sigma_{L^2_{\text{per}}(0,T)}(\mathcal{L}(\xi))$.

(P1) There exist T-periodic and smooth eigenfunctions Φ_ξ with $\Phi_0 = \phi'$ associated to the simple eigenvalue $\lambda_c(\xi)$ of $\mathcal{L}(\xi)$, i.e., $\mathcal{L}(\xi)\Phi_\xi = \lambda_c(\xi)\Phi_\xi$ for $\xi \in (-\xi_0, \xi_0)$.
(P2) There exist T-periodic and smooth eigenfunctions $\tilde{\Phi}_\xi$ associated to the simple eigenvalue $\overline{\lambda_c(\xi)}$ of the adjoint operator $\mathcal{L}(\xi)^*$, i.e., $\mathcal{L}(\xi)^*\tilde{\Phi}_\xi = \overline{\lambda_c(\xi)}\tilde{\Phi}_\xi$ for $\xi \in (-\xi_0, \xi_0)$.
(P3) For any $m \in \mathbb{N}_0$, the expansions

$$\left|\lambda_c(\xi) - ia\xi + d\xi^2\right| \lesssim |\xi|^3, \qquad \left\|\Phi_\xi - \phi_0'\right\|_{H^m(0,T)} \lesssim |\xi|,$$

hold for $\xi \in (-\xi_0, \xi_0)$ with some coefficients $a \in \mathbb{R}$ and $d > 0$.

We define the spectral projections onto $\text{span}\{\Phi_\xi\}$ as

$$\Pi(\xi)g = \Phi_\xi \langle \widetilde{\Phi}_\xi, \mathbf{g}\rangle_{L^2(0,T)}, \quad \mathbf{g} \in L^2_{\text{per}}(0,T).$$

2.5. Formulation of main result

We are now in the position to state our main result.

Theorem 2.1. *Assume (H1) and (D1)-(D3). There exist constants $C, \varepsilon > 0$ such that for initial data $\mathbf{w}_0 \in H^6_{\text{per}}(0,T)$ and $\mathbf{v}_0 \in H^3(\mathbb{R})$ with*

$$E_0 := ||\mathbf{w}_0 + \mathbf{v}_0||_{H^6_{\text{per}}(0,T)\oplus H^3(\mathbb{R})} < \varepsilon$$

there exist a solution

$$\mathbf{u}(t) \in C([0,\infty); C^2_{\text{ub}}(\mathbb{R})) \cap C^1([0,\infty); C_{\text{ub}}(\mathbb{R})) \tag{2.9}$$

of (2.1) with initial condition $\mathbf{u}(0) = \phi + \mathbf{w}_0 + \mathbf{v}_0$, some smooth function $\gamma \in C([0,\infty), H^5(\mathbb{R}))$ and a constant $\sigma_* \in \mathbb{R}$ with the properties

$$||\mathbf{u}(t) - \phi||_{W^{2,\infty}} \le CE_0 \tag{2.10}$$

and

$$||\mathbf{u}(\cdot,t) - \phi_0(\cdot + \sigma_* + \gamma(\cdot,t))||_{L^\infty} \le C(1+t)^{-\frac{3}{4}}E_0, \tag{2.11}$$

for all $t \ge 0$.

We briefly discuss the regularity assumptions in Theorem 2.1. The assumption $\mathbf{v}_0 \in H^3(\mathbb{R})$ is justified by the fact that the regularity control on $\mathbf{v}$ mainly proceeds along the lines of a standard $L^1 \cap H^k$-nonlinear stability analysis [11,24] using the nonlinear damping estimate established in [25]. The reason for the regularity assumption on $\mathbf{w}_0$ is that $\mathbf{v}$ is considered as a perturbation of the periodic solution $\mathbf{w}$, which yields expressions in the modulated perturbation equations for $\mathbf{v}$ where $\mathbf{w}$ appears with two spatial derivatives. As we need to control $\mathbf{v}$ in $H^3(\mathbb{R})$, this leads to three more derivatives on $\mathbf{w}$. Therefore, we need to bound the fifth derivative in L^∞ which is covered by assuming $\mathbf{w}_0 \in H^6_{\text{per}}(0,T)$. Since we estimate the H^1-norm of the residual in Section 5.2 in order to find an L^∞-estimate (in the spirit of the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$), it suffices to demand $\mathbf{w}_0 \in H^5_{\text{per}}(0,T)$ and $\mathbf{v}_0 \in H^2(\mathbb{R})$ when one only aims to establish pure L^2-estimates on $\mathbf{v}$ and henceforth obtain (2.11) in $L^\infty(\mathbb{R})$ with lower decay rate $(1+t)^{-\frac{1}{2}}$.

Comparing (2.11) to the associated estimate in [11], we loose an algebraic decay factor of $\frac{1}{4}$ and one might ask whether we can compensate this lack of decay by taking the assumption $\mathbf{v}_0 \in L^1(\mathbb{R}) \cap H^3(\mathbb{R})$. However, this assumption does not improve the decay rates due to the coupling terms $\mathcal{R}_2(\tilde{\mathbf{w}}, \mathbf{v})$ in (2.4) which cannot be controlled in $L^1(\mathbb{R})$ but only in $L^2(\mathbb{R})$, cf. Remark 4.1.

Another interesting contrast to the result in [11] is the constant phase shift arising in the modulational estimate (2.11). The reason for this constant phase shift precisely originates from the fact the we not only enclose localized- but also co-periodic perturbation. This is reflected in designing the modulational approach to exploit the orbital stability result from [10].

Remark 2.2 (Uniqueness of solutions). *The solution $u(t)$ in Theorem 2.1 is unique from the sum space*

$$C([0,\infty); H^6_{per}(0,T) \oplus H^3(\mathbb{R})) \cap C^1([0,\infty); H^4_{per}(0,T) \oplus H^1(\mathbb{R})).$$

The reason is as follows. Taking a solution $u(t)$ of (2.1) from the sum space, then it fulfills (2.1) pointwisely. Since the sum $L^2_{per}(0,T) \oplus L^2(\mathbb{R})$ is direct, we can split $u(t)$ in a periodic part which satisfies (2.2) and a localized part which necessarily satisfies (2.3). These both parts are uniquely determined through Propositions 4.1 and 4.2. This implies that $u(t)$ is the unique solution of (2.1) from the sum space.

Concerning fully nonlocalized solutions of (2.1), local well-posedness of (2.1) with initial data from the modulation space $M_{\infty,1}(\mathbb{R})$ follows by standard semigroup theory, invoke e.g. [26, Theorem 1.2 in Section 6.1], using the observations that the principal linear part of the Eq. (2.1) generates a C_0-semigroup on $M_{\infty,1}(\mathbb{R})$ and that $M_{\infty,1}(\mathbb{R})$ admits the algebra property. For the latter, we refer to [27, Lemma 5.10]. As an immediate consequence, there exists a unique (mild) solution of (2.1) as element from $C([0,t), M_{\infty,1}(\mathbb{R}))$, $t > 0$, whenever $\mathbf{u}_0 \in C^2_{ub}(\mathbb{R}) \hookrightarrow M_{\infty,1}(\mathbb{R})$. In particular, this shows that the solution $\mathbf{u}(t)$ of (2.1) with $\mathbf{u}(0) = \mathbf{w}_0 + \mathbf{v}_0$ in Theorem 2.1 satisfying (2.9) is unique, see also [26, Corollary 1.3 in Section 6.1].

2.6. Strategy of proof

The main task in the proof of Theorem 2.1 is to find a suitable way to modulate the perturbations allowing to close a nonlinear argument through iterative estimates on their Duhamel formulae. The construction of the perturbations $\hat{\mathbf{w}}(t)$ and $\mathbf{v}(t)$ shows that $\hat{\mathbf{w}}(t)$ is independent of $\mathbf{v}(t)$. Therefore, we first modulate $\hat{\mathbf{w}}(t)$ by introducing a temporal modulation function $\sigma(t)$. Precisely, we choose $\hat{\mathbf{w}}(x,t) = \mathbf{w}(x - \sigma(t), t) - \phi(x)$. Then, we do not only interpret $\mathbf{v}(t)$ as perturbation of $\mathbf{w}(t)$ but do the same for their modulated variants. This leads to the following inverse-modulated perturbation

$$\hat{\mathbf{v}}(x,t) = \hat{\mathbf{u}}(x,t) - \hat{\mathbf{w}}(x,t) := \mathbf{u}(x - \sigma(t) - \gamma(t), t) - \hat{\mathbf{w}}(x,t) - \phi(x).$$

We arrive at a coupled system which allows to make an a-posteriori choice of the spatio-temporal phase modulation $\gamma : [0,\infty) \times \mathbb{R} \to \mathbb{R}^2$. We are then in the position to exploit the exponential decay of $|\sigma_t(t)|$ and

$||\hat{\mathbf{w}}(t)||_{L^\infty}$. We also introduce the forward-modulated perturbation

$$\begin{aligned}\mathring{\mathbf{v}}(x,t) &= \mathring{\mathbf{u}}(x,t) - \hat{\mathbf{w}}(x+\sigma(t)+\gamma(x,t),t)\\ &:= \mathbf{u}(x,t) - \phi(x+\sigma(t)+\gamma(x,t),t) - \hat{\mathbf{w}}(x+\sigma(t)+\gamma(x,t),t)\end{aligned}$$

which obeys a semilinear equation and can thus be used to control regularity in the nonlinear iteration. We control regularity by deriving a nonlinear damping estimate on $\mathring{\mathbf{v}}(t)$ and subsequently relating $\mathring{\mathbf{v}}(t)$ to $\hat{\mathbf{v}}(t)$. Finally, we show the following stability estimates: for suitable constants $C, \delta > 0$ and small initial conditions

$$E_0 = E_p + E_l \text{ with } E_p = ||\mathbf{w}_0||^6_{H^6_{\text{per}}} \text{ and } E_l = ||\mathbf{v}_0||_{H^3},$$

we find

$$|\sigma_t(t)|, ||\hat{\mathbf{w}}(t)||_{H^6_{\text{per}}(0,T)} \leq Ce^{-\delta t}E_p, \quad |\sigma| \leq CE_p, \quad t \geq 0, \tag{2.12}$$

and

$$||\gamma_x(t)||_{H^4}, ||\gamma_t(t)||_{H^3}, ||\mathring{\mathbf{v}}(t)||_{H^3} \leq C(1+t)^{-\frac{1}{2}}E_l, \quad ||\gamma(t)||_{H^5} \leq CE_l, \quad t \geq 0. \tag{2.13}$$

Using L^∞-estimates on the propagators, we arrive at the refined L^∞-estimate

$$||\mathring{\mathbf{v}}(t)||_{L^\infty} \leq C(1+t)^{-\frac{3}{4}}E_l, \quad t \geq 0. \tag{2.14}$$

Applying (2.12) and (2.14), Theorem 2.1 follows from the observations

$$||\mathring{\mathbf{u}}(t)||_{L^\infty} \leq ||\hat{\mathbf{w}}(\cdot+\sigma(t)+\gamma(\cdot,t),t)||_{L^\infty} + ||\mathring{\mathbf{v}}(t)||_{L^\infty}, \quad t \geq 0,$$

and

$$||\mathbf{u}(t)-\phi||_{W^{2,\infty}} \leq C\big(||\mathring{\mathbf{u}}(t)||_{W^{2,\infty}} + |\sigma(t)| + ||\gamma(t)||_{W^{2,\infty}}\big), \quad t \geq 0,$$

and Sobolev embedding. The estimate (2.12) can be derived along the lines of the co-periodic stability analysis in [10] and therefore the remaining task of this paper is to prove (2.13) and (2.14).

Notation.. Let S be a set, and let $A, B\colon S \to \mathbb{R}$. Throughout the paper, the expression "$A(x) \lesssim B(x)$ for $x \in S$", means that there exists a constant $C > 0$, independent of x, such that $A(x) \leq CB(x)$ holds for all $x \in S$.

3. Linear estimates

We collect well-known facts about the semigroup generated by the linearization $\mathcal{L}_0$ on $L^2_{\text{per}}(0,T)$ and on $L^2(\mathbb{R})$. Let $\chi\colon [0,\infty) \to \mathbb{R}$ be a smooth temporal cut-off function satisfying $\chi(t) = 0$ for $t \in [0,1]$ and $\chi(t) = 1$ for $t \in [2,\infty)$.

3.1. Semigroup decomposition and estimates on $L^2_{per}(0,T)$

We write

$$\tilde{S}_1(t) = (e^{\mathcal{L}_0 t} - \chi(t)\Pi(0))\mathbf{g}$$

and have the following linear estimate.

Proposition 3.1 ([10,28]). *Assume (H1) and (D1)-(D3). There exist constants $\delta_0, C > 0$ such that the estimate*

$$||\tilde{S}_1(t)\mathbf{g}||_{H^6_{per}(0,T)} \leq Ce^{-\delta_0 t}||\mathbf{g}||_{H^6_{per}(0,T)}$$

is valid for all $\mathbf{g} \in H^6_{per}(0,T)$.

3.2. Semigroup decomposition and estimates on $L^2(\mathbb{R})$

Assume (H1) and (D1)-(D3). Let $0 \leq \rho \leq 1$ be a smooth cut-off function with support in $(-\xi_0, \xi_0)$ and $\rho \equiv 1$ on $(-\frac{\xi_0}{2}, \frac{\xi_0}{2})$. Using Floquet-Bloch theory and recalling (P1)-(P3), we decompose

$$e^{\mathcal{L}_0 t}\mathbf{g} = \tilde{S}_2(t)\mathbf{g} + \phi' s_p(t)\mathbf{g},$$

with

$$s_p(t)\mathbf{g}(x) = \frac{\chi(t)}{2\pi}\int_{\mathbb{R}}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \rho(\xi)e^{i\xi(x-y)+\lambda_c(\xi)t}\tilde{\Phi}^*_\xi(y)\,d\xi\mathbf{g}(y)\,dy,$$

$$S_r(t)\mathbf{g}(x) = \frac{\chi(t)}{2\pi}\int_{\mathbb{R}}\int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} \rho(\xi)e^{i\xi(x-y)+\lambda_c(\xi)t}(\Phi_\xi(x) - \phi'(x))\tilde{\Phi}^*_\xi(y)\,d\xi\mathbf{g}(y)\,dy,$$

$$S_e(t)\mathbf{g} = e^{\mathcal{L}_0 t}\mathbf{g} - \phi' s_p(t)\mathbf{g} - S_r(t)\mathbf{g},$$

and

$$\tilde{S}_2(t)\mathbf{g} = S_e(t)\mathbf{g} + S_r(t)\mathbf{g}, \quad \mathbf{g} \in L^2(\mathbb{R}), (x,t) \in \mathbb{R}\times[0,\infty),$$

as in [11,24,29]. The principal observations are that $S_e(t)$ decays exponentially from $H^k(\mathbb{R})$ to $H^k(\mathbb{R})$, while $s_p(t)$ admits decay analogously to the heat semigroup and $S_r(t)$ decays with the additional rate $(1+t)^{-\frac{1}{2}}$ compared $s_p(t)$ due to the term $\Phi_\xi - \phi'_0$ and (P3). We summarize the following linear estimates.

Proposition 3.2. *Assume (H1) and (D1)-(D3). Let $l, j \in \mathbb{N}_0$ and $k \in \{0,1,2\}$. Then there exists a constant $C_{l,j} > 0$ such that*

$$||\partial_x^l\partial_t^j s_p(t)\partial_x^k\mathbf{g}||_{L^2} \leq C_{l,j}(1+t)^{-\frac{l+j}{2}}||\mathbf{g}||_{L^2}, \quad \mathbf{g} \in L^2(\mathbb{R})$$

$$||\partial_x^l\partial_t^j s_p(t)\mathbf{g}||_{L^2} \leq C_{l,j}(1+t)^{-\frac{1}{4}-\frac{l+j}{2}}||\mathbf{g}||_{L^1}, \quad \mathbf{g} \in L^2(\mathbb{R})\cap L^1(\mathbb{R}),$$

for all $t \geq 0$. Furthermore, there exists a constant $C > 0$ such that

$$||\tilde{S}_2(t)\mathbf{g}||_{L^2} \leq C(1+t)^{-\frac{3}{4}}||\mathbf{g}||_{L^1\cap L^2}, \quad \mathbf{g} \in L^2(\mathbb{R})\cap L^1(\mathbb{R})$$

and

$$||\tilde{S}_2(t)\mathbf{g}||_{L^2} \leq C(1+t)^{-\frac{1}{2}}||\mathbf{g}||_{L^2}, \quad \mathbf{g} \in L^2(\mathbb{R}),$$

for all $t \geq 0$.

Proof. The first two estimates are precisely shown in [11, Lemma 3.1]. The third one is a consequence of [11, Lemma 3.1 & Lemma 3.2]. Adapting the proof of the estimate $\tilde{S}_c(t)$ in [11, Lemma 3.2], one immediately finds the last estimate. □

Proposition 3.3 (L^∞-estimates). *Assume (H1) and (D1)-(D3). There exist constants $C, \delta_1 > 0$ such that*

$$||\tilde{S}_2(t)\mathbf{g}||_{L^\infty} \leq C\big(e^{-\delta_1 t}||\mathbf{g}||_{H^1} + (1+t)^{-1}||\mathbf{g}||_{L^1\cap L^2}\big), \quad \mathbf{g} \in H^1(\mathbb{R})\cap L^1(\mathbb{R}),$$

$$||\tilde{S}_2(t)\mathbf{g}||_{L^\infty} \leq C(1+t)^{-\frac{3}{4}}||\mathbf{g}||_{H^1}, \quad \mathbf{g} \in H^1(\mathbb{R}),$$

and

$$||\partial_x s_p(t)\mathbf{g}||_{L^\infty} \leq C(1+t)^{-\frac{3}{4}}||\mathbf{g}||_{L^2}, \quad \mathbf{g} \in L^2(\mathbb{R}),$$

$$||\partial_x s_p(t)\mathbf{g}||_{L^\infty} \leq C(1+t)^{-1}||\mathbf{g}||_{L^1}, \quad \mathbf{g} \in L^1(\mathbb{R}),$$

for all $t \geq 0$.

Proof. The last two estimates are shown in [11, Lemma 3.2]. The first two estimates are consequences of [11, Lemma 3.1] together with [24, Corollary 3.4] and [29, Proposition 3.1]. □

4. Nonlinear iteration scheme

4.1. Local existence of the solutions

Using standard semigroup theory, see e.g. [30] or [26], we establish

Proposition 4.1. *Let $\mathbf{w}_0 \in H^6_{per}(0,T)$. There exist a maximal time $T_{max} \in (0,\infty]$ and a unique solution $\mathbf{w} \in C([0,T_{max}); H^6_{per}(0,T)) \cap C^1([0,T_{max}); H^4_{per}(0,T))$ of (2.2) with $\mathbf{w}(0) = \phi + \mathbf{w}_0$. If $T_{max} < \infty$, then*

$$\limsup_{t\uparrow T_{max}} ||\mathbf{w}(t)||_{H^4_{per}(0,T)} = \infty. \tag{4.1}$$

Having the solution $\mathbf{w}$ at hand, the local existence of $\mathbf{v}$ follows.

Proposition 4.2. *Let $\mathbf{w}$ and T_{max} be as in Proposition 4.1. Let $\mathbf{v}_0 \in H^3(\mathbb{R})$. There exist a maximal time $\tau_{max} \leq T_{max}$ and a unique solution $\mathbf{v} \in C([0,\tau_{max}); H^3(\mathbb{R})) \cap C^1([0,\tau_{max}); H^1(\mathbb{R}))$ of (2.3) with $\mathbf{v}(0) = \mathbf{v}_0$. If $\tau_{max} < T_{max}$, then*

$$\limsup_{t\uparrow\tau_{max}} ||\mathbf{v}(t)||_{H^1} = \infty. \tag{4.2}$$

Proof. Due to the embedding $H^1_{\text{per}}(0,T) \hookrightarrow L^\infty(\mathbb{R})$, we have that

$$\mathbf{v} \mapsto \mathcal{N}(\mathbf{v}+\mathbf{w}) - \mathcal{N}(\mathbf{w})$$

is locally Lipschitz-continuous as map from $H^1(\mathbb{R})$ to $H^1(\mathbb{R})$. Moreover,

$$\mathcal{J}\left(\begin{pmatrix} -\beta & 0 \\ 0 & -\beta \end{pmatrix}\partial_x^2 - \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}\right) - \mathcal{I}$$

generates a C_0-semigroup on $H^1(\mathbb{R})$ with domain $H^3(\mathbb{R})$. Therefore, Proposition 4.2 follows from [26, Theorem 1.4 of Section 6.1]. □

4.2. Inverse-modulated perturbations

We first modulate $\mathbf{u}(t)$, that is, we consider

$$\mathbf{u}(x-\sigma(t),t) - \phi(x) = (\mathbf{w}(x-\sigma(t),t) - \phi(x)) + \mathbf{v}(x-\sigma(t),t)$$

for some $\sigma : [0,\infty) \to \mathbb{R}$ with $\sigma(0) = 0$ to be defined a-posteriori. Then, we set

$$\hat{\mathbf{w}}(x,t) = \mathbf{w}(x-\sigma(t),t) - \phi(x). \tag{4.3}$$

Motivated by the fact that $\mathbf{v}(t)$ is a perturbation of $\mathbf{w}(t) = \tilde{\mathbf{w}}(t) + \phi$, we subsequently define

$$\hat{\mathbf{v}}(x,t) = \mathbf{u}(x-\sigma(t)-\gamma(x,t),t) - \hat{\mathbf{w}}(x,t) - \phi(x) \tag{4.4}$$

for some $\gamma : \mathbb{R}\times[0,\infty) \to \mathbb{R}$ with $\gamma(\cdot,0) = 0$ to be defined a-posteriori. We find the modulated perturbation equations for $\hat{\mathbf{w}}(t)$,

$$\begin{aligned} &(\partial_t - \mathcal{L}_0)(\hat{\mathbf{w}} + \phi'\sigma) = \mathcal{R}_1(\hat{\mathbf{w}}) - \sigma_t\hat{\mathbf{w}}_x \\ &\hat{\mathbf{w}}(0) = \mathbf{w}_0 - \phi. \end{aligned} \tag{4.5}$$

and the one for $\hat{\mathbf{v}}(t)$,

$$\begin{aligned} &(\partial_t - \mathcal{L}_0)(\hat{\mathbf{v}} + \phi'\gamma - \gamma_x\hat{\mathbf{w}} - \gamma_x\hat{\mathbf{v}}) = \mathcal{R}_3(\hat{\mathbf{w}},\hat{\mathbf{v}},\gamma) - \sigma_t\hat{\mathbf{v}}_x + (1-\gamma_x)\mathcal{R}_{2,2}(\hat{\mathbf{w}},\hat{\mathbf{v}}) \\ &+ \mathcal{T}(\hat{\mathbf{w}},\gamma)\hat{\mathbf{v}}(0) = \mathbf{v}_0, \end{aligned} \tag{4.6}$$

where

$$\mathcal{R}_3(\hat{\mathbf{w}},\hat{\mathbf{v}},\gamma) = \mathcal{Q}(\hat{\mathbf{w}},\hat{\mathbf{v}},\gamma) + \partial_x\mathcal{S}(\hat{\mathbf{v}},\gamma) + \partial_x^2\mathcal{P}(\hat{\mathbf{v}},\gamma)$$

and

$$\mathcal{Q}(\hat{\mathbf{w}},\hat{\mathbf{v}},\gamma) = (1-\gamma_x)\mathcal{R}_{2,1}(\hat{\mathbf{w}},\hat{\mathbf{v}}),$$

$$\mathcal{S}(\hat{\mathbf{v}},\gamma) = -\gamma_t\hat{\mathbf{v}} + \beta\mathcal{J}\left(\frac{\gamma_{xx}}{(1-\gamma_x)^2}\hat{\mathbf{v}} - \frac{\gamma_x^2}{1-\gamma_x}\phi'\right),$$

$$\mathcal{P}(\hat{\mathbf{v}},\gamma) = -\beta\mathcal{J}\left(\gamma_x + \frac{\gamma_x}{1-\gamma_x}\right)\hat{\mathbf{v}},$$

$$\begin{aligned} \mathcal{T}(\hat{\mathbf{w}},\gamma) = &-\gamma_x\mathcal{R}_1(\hat{\mathbf{w}}) - \partial_x\left(\gamma_t\hat{\mathbf{w}} - \beta\mathcal{J}\left(\frac{\gamma_{xx}}{(1-\gamma_x)^2}\hat{\mathbf{w}}\right)\right) \\ &- \partial_x^2\left(\beta\mathcal{J}\left(\gamma_x + \frac{\gamma_x}{1-\gamma_x}\right)\hat{\mathbf{w}}\right). \end{aligned}$$

We delegate the derivation of (4.6) to Appendix A. The main observation is that in the nonlinearities of (4.6) any $\hat{\mathbf{w}}$- and σ_t-term is paired with a γ_x, $\hat{\mathbf{v}}$ or γ_t contribution suggesting that we have sufficient control for an L^2-iteration scheme since we expect exponential decay for $||\hat{\mathbf{w}}(t)||_{L^\infty}$ and $|\sigma_t(t)|$ from [10] while we control $\gamma_x(t)$, $\hat{\mathbf{v}}(t)$ and $\gamma_t(t)$ in $H^k(\mathbb{R})$. To this end, we establish the following straightforward bounds which follow by Hölder's inequality and Sobolev embedding.

Lemma 4.1. *Fix a constant $c > 0$ such that $||f||_{L^\infty} \le \frac{1}{2}$ for all $f \in H^1(\mathbb{R})$ with $||f||_{H^1} \le c$ due to Sobolev embedding. There exists a constant $C > 0$ such that*

$$\begin{aligned} &L^1\text{-bound: } ||\mathcal{R}_3(\hat{\mathbf{w}},\hat{\mathbf{v}},\gamma)||_{L^1} \le C\Big(||\hat{\mathbf{v}}||_{L^2}^2 + ||(\gamma_x,\gamma_t)||_{H^2\times H^1}(||\hat{\mathbf{v}}||_{H^2} + ||\gamma_x||_{L^2})\Big), \\ &L^2\text{-bound: } ||\mathcal{R}_3(\hat{\mathbf{w}},\hat{\mathbf{v}},\gamma)||_{L^2} \le C\Big(||\hat{\mathbf{v}}||_{H^1}^2 + ||(\gamma_x,\gamma_t)||_{H^2\times H^1}(||\hat{\mathbf{v}}||_{H^2} + ||\gamma_x||_{L^2})\Big), \\ &H^1\text{-bound: } ||\mathcal{R}_3(\hat{\mathbf{w}},\hat{\mathbf{v}},\gamma)||_{H^1} \le C\Big(||\hat{\mathbf{v}}||_{H^1}^2 + ||(\gamma_x,\gamma_t)||_{H^3\times H^2}(||\hat{\mathbf{v}}||_{H^3} + ||\gamma_x||_{H^1})\Big), \end{aligned}$$

and

$$\begin{aligned} &L^2\text{-bounds: } ||\sigma_t\hat{\mathbf{v}}_x||_{L^2} \le C|\sigma_t||\hat{\mathbf{v}}||_{H^1}, \quad ||(1-\gamma_x)\mathcal{R}_{2,2}(\hat{\mathbf{w}},\hat{\mathbf{v}})||_{L^2} \\ &\qquad \le C||\hat{\mathbf{v}}||_{L^2}||\hat{\mathbf{w}}||_{H^1_{per}(0,T)}, \\ &\qquad ||\mathcal{T}(\hat{\mathbf{w}},\gamma)||_{L^2} \le C||(\gamma_x,\gamma_t)||_{H^2\times H^1}||\hat{\mathbf{w}}||_{H^3_{per}(0,T)}, \\ &H^1\text{-bounds: } ||\sigma_t\hat{\mathbf{v}}_x||_{H^1} \le C|\sigma_t||\hat{\mathbf{v}}||_{H^2}, \quad ||(1-\gamma_x)\mathcal{R}_{2,2}(\hat{\mathbf{w}},\hat{\mathbf{v}})||_{H^1} \\ &\qquad \le C||\hat{\mathbf{v}}||_{H^1}||\hat{\mathbf{w}}||_{H^2_{per}(0,T)}, \\ &\qquad ||\mathcal{T}(\hat{\mathbf{w}},\gamma)||_{H^1} \le C||(\gamma_x,\gamma_t)||_{H^3\times H^2}||\hat{\mathbf{w}}||_{H^4_{per}(0,T)}, \end{aligned}$$

hold for all $\hat{\mathbf{v}} \in H^3(\mathbb{R})$, $\hat{\mathbf{w}} \in H^4_{per}(0,T)$, $(\gamma_t,\gamma_x) \in H^2(\mathbb{R})\times H^3(\mathbb{R})$ and $\sigma_t \in \mathbb{R}$ provided

$$||\hat{\mathbf{w}}||_{H^4_{per}(0,T)}, ||\hat{\mathbf{v}}||_{H^1}, ||\gamma_x||_{H^3} \le c.$$

4.3. Modulation in the purely co-periodic setting

The forward-modulated perturbation $\mathring{\mathbf{w}}(x,t) = \mathbf{w}(x,t) - \phi(x+\sigma(t))$ fulfills the semilinear system

$$(\partial_t - \mathcal{L}_0)(\mathring{\mathbf{w}}(t) - \phi'\sigma) = \mathcal{R}_4(\mathring{\mathbf{w}}(t),\sigma(t)) + (\phi'(\cdot+\sigma(t)) - \phi')\sigma_t(t)$$

with

$$\mathcal{R}_4(\mathring{\mathbf{w}}(t),\sigma(t)) = \mathcal{R}_1(\phi(\cdot+\sigma(t)))(\mathring{\mathbf{w}}(t)) - (\mathcal{N}'(\phi) - \mathcal{N}'(\phi(\cdot+\sigma(t))))\mathring{\mathbf{w}}(t).$$

Introducing the temporal modulation function

$$\begin{aligned} \sigma(t) = &\chi(t)\Pi(0)\tilde{\mathbf{w}}_0 + \int_0^t \chi(t-s)\Pi(0)\big(\mathcal{R}_4(\mathring{\mathbf{w}}(s),\sigma(s)) \\ &+ (\phi'(\cdot+\sigma(s)) - \phi')\sigma_s(s)\big)\,ds \end{aligned} \tag{4.7}$$

gives rise to the Duhamel formula

$$\mathring{\mathbf{w}}(t) = \tilde{S}_1(t)\tilde{\mathbf{w}}_0 + \int_0^t \tilde{S}_1(t-s)\big(\mathcal{R}_4(\mathring{\mathbf{w}}(s),\sigma(s)) + (\phi'(\cdot+\sigma(s)) - \phi')\sigma_s(s)\big)\,ds. \tag{4.8}$$

By a standard fixed point argument, we have local existence of σ.

Proposition 4.3. *Let $\mathbf{w}$ and T_{max} be as in Proposition 4.1. There exists a maximal time $t_{max,\sigma} \le T_{max}$ such that (4.7) with $\mathring{\mathbf{w}}(x,t) = \mathbf{w}(x,t) - \phi(x+\sigma(t))$ has a unique solution*

$$\sigma \in C^1([0,t_{max,\sigma});\mathbb{R}) \text{ with } \sigma(0) = 0 \text{ and } |(\sigma(t),\sigma_t(t))| < \frac{1}{2}, \quad t \in [0,t_{max,\sigma}).$$

If $t_{max,\sigma} < \tau_{max}$, then $\limsup_{t\uparrow t_{max,\sigma}} |(\sigma(t),\sigma_t(t))| \ge \frac{1}{2}$.

Furthermore, using the mean value theorem, the following bounds hold.

Lemma 4.2. *Let $K > 0$. There exists a constant $C > 0$ such that*

$$\begin{aligned} &||\mathcal{R}_4(\mathring{\mathbf{w}},\sigma)||_{H^6_{per}(0,T)} \le C||\mathring{\mathbf{w}}||_{H^6_{per}(0,T)}\Big(||\mathring{\mathbf{w}}||_{H^6_{per}(0,T)} + |\sigma|\Big), \\ &||(\phi'(\cdot+\sigma) - \phi')\sigma_t||_{H^6_{per}(0,T)} \le C|\sigma||\sigma_t|, \end{aligned}$$

provided

$$|\sigma| + |\sigma_t| + ||\mathring{\mathbf{w}}||_{H^6_{per}(0,T)} \le K.$$

4.4. Choice of the spatio-temporal phase modulation

We have the Duhamel formula for $\hat{\mathbf{v}}(t)$,

$$\begin{aligned} \hat{\mathbf{v}}(t) = &e^{\mathcal{L}_0 t}\mathbf{v}_0 - \phi'\gamma(t) + \int_0^t e^{\mathcal{L}_0(t-s)}\Big(\mathcal{R}_3(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(s),\gamma(s)) \\ &- \sigma_s(s)\hat{\mathbf{v}}_x(s) + (1-\gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{v}}(s),\hat{\mathbf{w}}(s)) + \mathcal{T}(\hat{\mathbf{w}}(s),\gamma(s))\Big)\,ds \\ &+ \gamma_x(t)\hat{\mathbf{w}}(t) + \gamma_x(t)\hat{\mathbf{v}}(t), \end{aligned}$$

under the condition that $\gamma(0) = 0$. We make the implicit choice

$$\begin{aligned} \gamma(t) = &s_p(t)\mathbf{v}_0 + \int_0^t s_p(t-s)\Big(\mathcal{R}_3(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(s),\gamma(s)) - \sigma_s(s)\hat{\mathbf{v}}_x(s) \\ &+ (1-\gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{v}}(s),\hat{\mathbf{w}}(s)) + \mathcal{T}(\hat{\mathbf{w}}(s),\gamma(s))\Big)\,ds \end{aligned} \tag{4.9}$$

which reduces the Duhamel formula for $\hat{\mathbf{v}}(t)$ to

$$\begin{aligned}\hat{\mathbf{v}}(t) =&\tilde{S}_2(t)\mathbf{v}_0 + \gamma_x(t)\hat{\mathbf{w}}(t) + \gamma_x(t)\hat{\mathbf{v}}(t)\\ &+\int_0^t \tilde{S}_2(t-s)\Big(\mathcal{R}_3(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(s),\gamma(s)) - \sigma_s(s)\hat{\mathbf{v}}_x(s)\\ &+(1-\gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{v}}(s),\hat{\mathbf{w}}(s)) + \mathcal{T}(\hat{\mathbf{w}}(s),\gamma(s))\Big)\,ds.\end{aligned} \tag{4.10}$$

Setting $t = 0$ in (4.9) and using that $s_p(0) = 0$, one indeed verifies that $\gamma(0) = 0$.

Proposition 4.4. *Let $\mathbf{w}$ and T_{max} as in Proposition 4.1, $\mathbf{v}$, $\mathbf{v}_0$ and τ_{max} as in Proposition 4.2 and σ and $t_{max,\sigma}$ as in Proposition 4.3. Furthermore, let $0 < c < \frac{1}{2}$ be a constant such that $||f||_{L^\infty} \le \frac{1}{c}||f||_{H^1}$ for all $f \in H^1(\mathbb{R})$. There exists a maximal time $t_{max,\gamma} \le \min\{\tau_{max}, t_{max,\sigma}\}$ such that (4.9) with $\hat{\mathbf{w}}(t) = \mathbf{w}(\cdot - \sigma(t)) - \phi$ and $\hat{\mathbf{v}}(t) = \mathbf{u}(\cdot - \sigma(t) - \gamma(\cdot,t),t) - \hat{\mathbf{w}}(t) - \phi$ has a unique solution*

$$\gamma \in C([0,t_{max,\gamma}); H^5(\mathbb{R})) \cap C^1([0,t_{max,\gamma}); H^3(\mathbb{R})) \text{ with } \gamma(0) = 0$$

satisfying

$$||(\gamma(t),\gamma_t(t))||_{H^5\times H^3} < \frac{c}{2}, \quad t \in [0,t_{max,\gamma}). \tag{4.11}$$

If $t_{max,\gamma} < \min\{\tau_{max}, t_{max,\sigma}\}$, then

$$\limsup_{t\uparrow t_{max,\gamma}} ||\gamma(t),\gamma_t(t)||_{H^5\times H^3} \ge \frac{c}{2}. \tag{4.12}$$

Proof. See Appendix C. □

Corollary 4.1. *Let $\mathbf{w}$ and T_{max} as in Proposition 4.1 and $\mathbf{v}$, $\mathbf{v}_0$ and τ_{max} as in Proposition 4.2. Let γ, σ and $t_{max,\sigma}$, $t_{max,\gamma}$ as in Propositions 4.3 and 4.4. Then, the inverse-modulated perturbation $\hat{\mathbf{v}} \in C([0,t_{max,2}); L^2(\mathbb{R}))$ defined by (4.4) satisfies (4.10) with $\hat{\mathbf{w}}(t) = \mathbf{w}(\cdot - \sigma(t)) - \phi$ and $\hat{\mathbf{v}}(t) \in H^3(\mathbb{R})$ for all $t \in [0,t_{max,\gamma})$.*

Proof. Let $t \in [0,t_{\max,\gamma})$. We observe

$$\begin{aligned}\hat{\mathbf{v}}(x,t) &= \mathbf{v}(x-\sigma(t)-\gamma(x,t),t) + \mathbf{w}(x-\sigma(t)-\gamma(x,t),t) - \hat{\mathbf{w}}(x,t) - \phi(x)\\ &= \mathbf{v}(x-\sigma(t)-\gamma(x,t),t) + \mathbf{w}(x-\sigma(t)-\gamma(x,t),t) - \mathbf{w}(x-\sigma(t),t)\end{aligned}$$

yielding on the one hand, with $\mathbf{w}(t) \in H^5_{\text{per}}(0,T) \hookrightarrow W^{4,\infty}(\mathbb{R})$ and the mean value theorem,

$$||\mathbf{w}(x-\sigma(t)-\gamma(x,t),t) - \mathbf{w}(x-\sigma(t),t)||_{H^3} \lesssim ||\gamma(t)||_{H^3}.$$

On the other hand, since $\gamma(t) \in H^4 \hookrightarrow W^{3,\infty}$, $\sup_{s\in[0,t]}||\gamma_x(s)||_{L^\infty} \le \frac{1}{2}$ and $\mathbf{v}(t) \in H^3(\mathbb{R})$, we conclude

$$\hat{\mathbf{v}}(t) \in H^3(\mathbb{R})$$

with the help of the chain and substitution rule. □

Remark 4.1. *We can now provide some intuition where the decay $(1+t)^{-\frac{1}{2}}$ for $\mathbf{v}$ and γ_x in (2.13) originates from. For this purpose, assume that $|\sigma_t|$ admits exponential decay while $||\hat{\mathbf{v}}||_{L^2} \approx ||\hat{\mathbf{v}}_x||_{L^2}$ decays at rate $(1+t)^{-\kappa}$ with $\kappa \ge \frac{1}{2}$. Considering the term $||\sigma_t\hat{\mathbf{v}}_x||_{L^2} \le ||\hat{\mathbf{v}}_x||_{L^2}|\sigma_t|$ in (4.9) and (4.10), this yields the integral*

$$\int_0^t (1+t-s)^{-\frac{1}{2}}(1+s)^{-\kappa}e^{-s}\,ds \lesssim (1+t)^{-\frac{1}{2}}.$$

By noting that $\sigma_t\hat{\mathbf{v}}_x$ cannot be estimated in any L^p-norm with $1 \le p < 2$ due to the lack of localization of $\hat{\mathbf{v}}$, we can only close an iterative argument with $\kappa = \frac{1}{2}$ at best. This shows that even with the additional assumption $\mathbf{v}_0 \in L^1(\mathbb{R})$, the decay rate on $\hat{\mathbf{v}}$ cannot be improved.

4.5. Forward-modulation of $\mathbf{v}$ and nonlinear damping estimates

We wish to control $||\hat{\mathbf{v}}||_{H^3}$ in terms of $||\hat{\mathbf{v}}||_{L^2}$, γ_t, γ_x, σ_t and $\hat{\mathbf{w}}$ in the nonlinear iteration argument in order to control regularity, cf. Lemma 4.1. For this purpose, we introduce the forward-modulated perturbation $\mathring{\mathbf{v}}$, that is,

$$\begin{aligned}\mathring{\mathbf{v}}(x,t) &= \mathbf{u}(x,t) - \hat{\mathbf{w}}(x+\sigma(t)+\gamma(x,t),t) - \phi(x+\sigma(t)+\gamma(x,t))\\ &= \mathbf{u}(x,t) - \mathbf{w}(x+\gamma(x,t),t)\\ &= \mathbf{v}(x,t) + \mathbf{w}(x,t) - \mathbf{w}(x+\gamma(x,t),t),\end{aligned} \tag{4.13}$$

which satisfies the semilinear system, cf. Appendix B,

$$(\partial_t - \mathcal{L}_0(\mathring{\phi}))\mathring{\mathbf{v}}(t) = \mathcal{R}_2(\mathring{\phi})(\tilde{\mathbf{w}}(\cdot+\gamma(\cdot,t),t),\mathring{\mathbf{v}}(t)) + \mathcal{R}_5(\tilde{\mathbf{w}}(t),\gamma(t),\gamma_t(t)) \tag{4.14}$$

with

$$\begin{aligned}\mathcal{L}_0(\mathring{\phi}) =&\mathcal{J}\begin{pmatrix}-\beta\partial_x^2 - \alpha + 3\mathring{\phi}_1^2 + \mathring{\phi}_2^2 & 2\mathring{\phi}_1\mathring{\phi}_2\\ 2\mathring{\phi}_1\mathring{\phi}_2 & -\beta\partial_x^2 - \alpha + \mathring{\phi}_1^2 + 3\mathring{\phi}_2^2\end{pmatrix}\\ &- \mathcal{I}, \quad \mathring{\phi}(x,t) = \phi(x+\gamma(x,t)),\end{aligned}$$

and

$$\begin{aligned}&\mathcal{R}_5(\tilde{\mathbf{w}}(t),\gamma(t),\gamma_t(t))\\ &= -\tilde{\mathbf{w}}_x(\cdot+\gamma(\cdot,t),t)\gamma_t(t) - \phi'(\cdot+\gamma(\cdot,t))\gamma_t(t)\\ &\quad - \beta\mathcal{J}\Big(\tilde{\mathbf{w}}_x(\cdot+\gamma(\cdot,t),t)\gamma_{xx}(t) + \tilde{\mathbf{w}}_{xx}(\cdot+\gamma(\cdot,t),t)(2\gamma_x(t)+\gamma_x(t)^2)\\ &\qquad + \phi'(\cdot+\gamma(\cdot,t))\gamma_{xx}(t) + \phi''(\cdot+\gamma(\cdot,t))(2\gamma_x(t)+\gamma_x(t)^2)\Big).\end{aligned}$$

Note that $\mathcal{R}_2(\mathring{\phi})$ is as defined in (2.7) and $\tilde{\mathbf{w}}(t) = \mathbf{w}(t) - \phi$ is the umodulated perturbation of $\mathbf{w}(t)$. We setup the local existence for the forward-modulated perturbation $\mathring{\mathbf{v}}(t)$ which is a straightforward consequence of Propositions 4.1, 4.3 and 4.4.

Corollary 4.2. *Let $\mathbf{w}$ and T_{max} as in Proposition 4.1 and $\mathbf{v}$, and τ_{max} as in (4.2). Let γ, σ and $t_{max,\sigma}$, $t_{max,\gamma}$ as in Propositions 4.3 and 4.4. Then, the forward-modulated perturbation $\mathring{\mathbf{v}} \in C([0,t_{max,\gamma}); H^3(\mathbb{R})) \cap C^1([0,t_{max,\gamma}); H^1(\mathbb{R}))$ defined by (4.13) satisfies (4.14) with $\tilde{\mathbf{w}}(t) = \mathbf{w}(t) - \phi$.*

Lemma 4.3. *Let $j = 1,2,3$. Fix $K > 0$. There exists some $C > 0$ such that for $t \in [0,t_{max,\gamma})$, we obtain*

$$\begin{aligned}||\partial_x^j\mathcal{R}_5(\tilde{\mathbf{w}}(t),\gamma(t),\gamma_t(t))||_{L^2} &\le C(||\gamma_x(t)||_{H^{j+1}} + ||\gamma_t(t)||_{H^j}),\\ ||\partial_x^j\mathcal{R}_2(\mathring{\phi})(\tilde{\mathbf{w}}(\cdot+\gamma(\cdot,t),t),\mathring{\mathbf{v}}(t))||_{L^2} &\le C||\mathring{\mathbf{v}}(t)||_{H^j}\end{aligned}$$

provided

$$\sup_{0\le s\le t}\Big(||\tilde{\mathbf{w}}(s)||_{H^6_{per}(0,T)} + ||\mathring{\mathbf{v}}(s)||_{H^3} + ||(\gamma_x(s),\gamma_s(s))||_{H^4\times H^3}\Big) \le K.$$

Proof. We bound

$$\begin{aligned}&||\partial_x^j(\beta\mathcal{J}\big(\tilde{\mathbf{w}}_{xx}(\cdot+\gamma(\cdot,t),t)(2\gamma_x(t)+\gamma_x(t)^2) + \tilde{\mathbf{w}}_x(\cdot+\gamma(\cdot,t),t)\gamma_{xx}(t)\big)\\ &+\tilde{\mathbf{w}}_x(\cdot+\gamma(\cdot,t),t)\gamma_t(t))||_{L^2}\\ &\lesssim ||\tilde{\mathbf{w}}(t)||_{H^{j+3}_{\text{per}}(0,T)}||\gamma_x(t)||_{H^{j+1}} + ||\tilde{\mathbf{w}}(t)||_{H^{j+2}_{\text{per}}(0,T)}||\gamma_t(t)||_{H^j},\end{aligned}$$

$j = 1,2,3$, where we used the Sobolev embedding $H^k_{\text{per}}(0,T) \hookrightarrow W^{k-1,\infty}(\mathbb{R})$. Similarly, one proceeds for terms involving ϕ instead of $\tilde{\mathbf{w}}$. Also it is straightforward to check the bound for $\mathcal{R}_2$. □

Proceeding as in [11,16,25], we are now in the position to derive a nonlinear damping estimate for $\mathring{\mathbf{v}}$.

Proposition 4.5. *Let γ, σ and $t_{max,\sigma}$, $t_{max,\gamma}$ as in Propositions 4.3 and 4.4. Let $\mathbf{w}$ and T_{max} as in Proposition 4.1. Take $\mathbf{v}$, $\mathbf{v}_0$ and τ_{max} as in Proposition 4.2. Define $\mathring{\mathbf{v}}$ through (4.13) and set $\tilde{\mathbf{w}}(t) = \mathbf{w}(t) - \phi$. Fix $K > 0$. There exists a constant $C > 0$ such that*

$$\begin{aligned}||\mathring{\mathbf{v}}(t)||^2_{H^3} \le C\Big(&e^{-t}||\mathbf{v}_0||^2_{H^3} + ||\mathring{\mathbf{v}}(t)||^2_{L^2}\\ &+\int_0^t e^{-(t-s)}\Big(||\mathring{\mathbf{v}}(s)||^2_{L^2} + ||\gamma_x(s)||^2_{H^4} + ||\gamma_t(t)||^2_{H^3}\Big)\,ds\Big)\end{aligned} \tag{4.15}$$

for all $t \in [0,t_{max,\gamma})$ provided

$$\sup_{0\le s\le t}\Big(||\tilde{\mathbf{w}}(s)||_{H^6_{per}(0,T)} + ||\mathring{\mathbf{v}}(s)||_{H^3} + ||(\gamma_x(s),\gamma_s(s))||_{H^3\times H^2}\Big) \le K. \tag{4.16}$$

Proof. First we use that $\mathbf{v}_0 \in H^5(\mathbb{R})$ gives a solution $\mathbf{v} \in C([0,\tau_{\max}); H^5(\mathbb{R})) \cap C^1([0,\tau_{\max}); H^3(\mathbb{R}))$ of (2.4) arguing analogously as in Proposition 4.2. Since $H^5(\mathbb{R})$ is dense in $H^3(\mathbb{R})$ an approximation argument as in [30, Proposition 4.3.7] yields the result for $\mathbf{v}_0 \in H^3(\mathbb{R})$.

Fix $K > 0$. Let $t \in [0, t_{\max,\gamma})$ such that (4.16) holds. The forward-modulated perturbation $\mathring{\mathbf{v}}$ is designed such that the principal part in (4.14) is given by $\partial_t - \mathcal{L}_0(\mathring{\phi})$. This is the reason why we choose the same energies as introduced in [11,16] and [25], that is,

$$E_j(t) = ||\partial_x^j \mathring{\mathbf{v}}(t)||_{L^2}^2 - \frac{1}{2\beta}\langle \mathcal{J} M(\mathring{\phi})\partial_x^{j-1}\mathring{\mathbf{v}}, \partial_x^{j-1}\mathring{\mathbf{v}}\rangle, \quad j = 1,2,3,$$

with

$$M(\mathring{\phi}) = 2\begin{pmatrix} -2\mathring{\phi}_r\mathring{\phi}_i & \mathring{\phi}_r^2 - \mathring{\phi}_i^2 \\ \mathring{\phi}_r^2 - \mathring{\phi}_i^2 & 2\mathring{\phi}_r\mathring{\phi}_i \end{pmatrix}.$$

We compute, see [25] and [16],

$$\partial_t E_j(t) = -2E_j(t) + R_1(t) + R_2(t)$$

with

$$|R_1(t)| \leq \frac{2}{3}E_j(t) + C_1||\mathring{\mathbf{v}}(t)||_{L^2}^2$$

for some t-independent constant $C_1 > 0$ and

$$\begin{aligned} R_2(t) =& 2\mathrm{Re}\,\langle \partial_x^j \mathcal{R}_5(\hat{\mathbf{w}}(t), \gamma(t), \gamma_t(t)), \partial_x^j \mathring{\mathbf{v}}(t)\rangle_{L^2} \\ &- \frac{1}{\beta}\mathrm{Re}\,\langle \mathcal{J} M(\mathring{\phi})\partial_x^{j-1}\mathcal{R}_5(\hat{\mathbf{w}}(t), \gamma(t), \gamma_t(t)), \partial_x^{j-1}\mathring{\mathbf{v}}(t)\rangle_{L^2}. \end{aligned}$$

By Lemma 4.3, using (4.16), interpolation and Young's inequality, there exists a t-independent constant $C_2 > 0$ such that

$$|R_2(t)| \leq \frac{1}{3}E_j(t) + C_2\Big(||\mathring{\mathbf{v}}(t)||_{L^2}^2 + ||\gamma_x(t)||_{H^{j+1}}^2 + ||\gamma_t(t)||_{H^j}^2\Big).$$

We conclude

$$\partial_t E_j(t) \leq -E_j(t) + C_3\Big(||\mathring{\mathbf{v}}(t)||_{L^2}^2 + ||\gamma_x(t)||_{H^{j+1}}^2 + ||\gamma_t(t)||_{H^j}^2\Big)$$

for some t-independent constant $C_3 > 0$ and $j = 1,2,3$. Integrating the latter and using, for some t-independent constant $C_4 > 0$,

$$||\partial_x \mathring{\mathbf{v}}(t)||_{L^2}^2 \leq E_1(t) + C_4||\mathring{\mathbf{v}}(t)||_{L^2}^2$$

and

$$||\partial_x^j \mathring{\mathbf{v}}(t)||_{L^2}^2 \leq 2E_j(t) + C_4\Big(||\mathring{\mathbf{v}}(t)||_{L^2}^2 + E_{j-1}(t)\Big),$$

$j = 2,3$, which follow by interpolation and Young's inequality, we arrive at (4.15). □

Lemma 4.4. *Fix $K > 0$. Let the assumptions be as in the previous corollary. There exists $C > 0$ such that for all $t \in [0, t_{max,\gamma})$ it holds*

$$||\mathring{\mathbf{v}}(t)||_{L^2} \leq C\big(||\hat{\mathbf{v}}(t)||_{L^2} + ||\gamma_x(t)||_{L^2}\big), \tag{4.17}$$

$$||\hat{\mathbf{v}}(t)||_{H^3} \leq C\big(||\mathring{\mathbf{v}}(t)||_{H^3} + ||\gamma_x(t)||_{H^3}\big) \tag{4.18}$$

and

$$||\mathring{\mathbf{v}}(t)||_{L^\infty} \leq C\big(||\hat{\mathbf{v}}(t)||_{L^\infty} + ||\gamma_x(t)||_{L^\infty}\big), \tag{4.19}$$

provided

$$\sup_{0\leq s\leq t}\Big(||\hat{\mathbf{w}}(s)||_{H^5_{per}} + ||\gamma(s)||_{H^4} + |\sigma(s)|\Big) \leq K \text{ and } \sup_{0\leq s\leq t}||\gamma_x(s)||_{L^\infty} \leq \frac{1}{2}.$$

Proof. Let $t \in [0, t_{\max,\gamma})$. We write $A_t(x) = x - \gamma(x,t) - \sigma(t)$ and $B_t(x) = x + \gamma(x,t) + \sigma(t)$. By the inverse function theorem and the fact that $||\gamma_x(t)||_{L^\infty} \leq \frac{1}{2}$, it is easy to check that A_t is invertible with

$$x = A_t(A_t^{-1}(x)) = A_t^{-1}(x) - \gamma(A_t^{-1}(x), t) - \sigma(t)$$

and therefore

$$A_t^{-1}(x) - B_t(x) = \gamma(A_t^{-1}(x), t) - \gamma(x,t).$$

We also compute

$$\begin{aligned} &\partial_x(A_t^{-1}(x)) = \frac{1}{1-\gamma_x(A_t^{-1}(x),t)}, \quad \partial_x^2(A_t^{-1}(x)) = \frac{\gamma_{xx}(A_t^{-1}(x),t)}{(1-\gamma_x(A_t^{-1}(x),t))^3}, \\ &\partial_x^3(A_t^{-1}(x)) = \frac{\gamma_{xxx}(A_t^{-1}(x),t)}{(1-\gamma_x(A_t^{-1}(x),t))^4} + \frac{3\gamma_{xx}(A_t^{-1}(x),t)^2}{(1-\gamma_x(A_t^{-1}(x),t))^5}. \end{aligned} \tag{4.20}$$

Using

$$\gamma(A_t^{-1}(x),t) - \gamma(x,t) = (A_t^{-1}(x) - x)\int_0^1 \gamma_x(x + \theta(A_t^{-1}(x) - x), t)\, d\theta, \tag{4.21}$$

(4.20) and $||\gamma_x(t)||_{L^\infty} \leq \frac{1}{2}$, we can estimate

$$||A_t^{-1} - B_t||_{L^2} \lesssim ||\gamma_x||_{L^2}, \quad ||A_t^{-1} - B_t||_{H^l} \lesssim ||\gamma_x||_{H^l}, \tag{4.22}$$

for $l = 1,2,3$. This is shown in [31, Lemma 2.7] and [25, Corollary 5.3]. It additionally follows

$$||\hat{\mathbf{v}}(A_t^{-1}(\cdot),t) - \hat{\mathbf{v}}(\cdot,t)||_{H^3} \lesssim ||\hat{\mathbf{v}}(t)||_{H^3}, \quad ||\mathring{\mathbf{v}}(A_t(\cdot),t) - \mathring{\mathbf{v}}(\cdot,t)||_{L^2} \lesssim ||\mathring{\mathbf{v}}(t)||_{L^2}$$

as for [16, (3.24) & (3.25)]. We observe

$$\hat{\mathbf{v}}(A_t^{-1}(x),t) - \hat{\mathbf{v}}(x,t) = (\hat{\mathbf{w}}(B_t(x),t) - \hat{\mathbf{w}}(A_t^{-1}(x),t)) + (\phi(A_t^{-1}(x)) - \phi(B_t(x)))$$

and estimate with (4.22) and the mean value theorem,

$$||\hat{\mathbf{v}}(A_t^{-1}(\cdot),t) - \hat{\mathbf{v}}(\cdot,t)||_{H^3} \lesssim ||A_t^{-1} - B_t||_{H^3} \lesssim ||\gamma_x||_{H^3}.$$

On the other hand,

$$\begin{aligned} \hat{\mathbf{v}}(x,t) - \mathring{\mathbf{v}}(A_t(x),t) =& -\hat{\mathbf{w}}(x,t) + \hat{\mathbf{w}}(x + \gamma(A_t(x),t) - \gamma(x,t), t) \\ &+ (-\phi(x) + \phi(x + \gamma(A_t(x),t) - \gamma(x,t))) \end{aligned}$$

and by and the mean value theorem and with

$$||\gamma(A_t(\cdot),t) - \gamma(\cdot,t)||_{L^2} \lesssim ||\gamma_x(t)||_{L^2},$$

which follows by (4.21), we obtain

$$||\hat{\mathbf{v}}(\cdot,t) - \mathring{\mathbf{v}}(A_t(\cdot),t)||_{L^2} \lesssim ||\gamma_x(t)||_{L^2}.$$

Putting everything together, we arrive at (4.17) and (4.18). The relation (4.19) follows analogously. □

5. Nonlinear stability analysis

5.1. The nonlinear iteration

We first establish nonlinear stability of $\mathbf{w}$.

Theorem 5.1. *Let $\mathbf{w}_0 \in H^6_{per}(0,T)$. Let $\mathbf{w}$, T_{max} be as in Proposition 4.1 and let σ and $t_{max,\sigma}$ as in Proposition 4.3. There exist $C, \delta_2, \varepsilon_p > 0$ such that for*

$$E_p := ||\mathbf{w}_0||_{H^6_{per}(0,T)} < \varepsilon_p,$$

the functions $\mathbf{w}(t)$ and $\sigma(t)$ exist globally, i.e. $t_{max,\sigma} = T_{max} = \infty$, and

$$\begin{aligned} &|\sigma(t)|, ||\mathbf{w}(t) - \phi||_{H^6_{per}(0,T)} \leq CE_p, \quad |\sigma_t(t)|, ||\hat{\mathbf{w}}(t)||_{H^6_{per}(0,T)}, ||\hat{\mathbf{w}}(t) \\ &\quad ||_{H^6_{per}(0,T)} \leq Ce^{-\delta_2 t}E_p, \end{aligned}$$

for all $t \geq 0$.

Proof. The estimates on $|\sigma(t)|$, $|\sigma_t(t)|$ and $||\hat{\mathbf{w}}(t)||_{H^6_{per}(0,T)}$ follow by a standard procedure, see [23] and [10], taking Proposition 3.1 and Lemma 4.2 for (4.8) into account. We observe that

$$||\hat{\mathbf{w}}(t)||_{H^6_{per}(0,T)} = ||\hat{\mathbf{w}}(\cdot + \sigma(t), t)||_{H^6_{per}(0,T)} = ||\mathring{\mathbf{w}}(t)||_{H^6_{per}(0,T)}$$

for all $t \geq 0$, to finish the proof. □

Let $\mathbf{w}_0 \in H^6_{per}(0,T)$ and ε_p as in Theorem 5.1. We set $E_l = ||\mathbf{v}_0||_{H^3}$ and choose $0 < c < \frac{1}{2}$ as in Proposition 4.4. Under the assumption $E_p < \varepsilon_p$, we consider the template function $\eta : [0, t_{\max,\gamma}) \to \mathbb{R}$ given by

$$\eta(t) = \sup_{0\leq s\leq t}\Big((1+s)^{\frac{1}{2}}\big(||\hat{\mathbf{v}}(s)||_{H^3(\mathbb{R})} + ||(\gamma_x(s), \gamma_s(s))||_{H^4\times H^3}\big) + ||\gamma(t)||_{L^2}\Big)$$

and show that there exists a constant $C \geq 1$ independent of E_l and E_p such that

$$\eta(t) \leq C(E_l + \eta(t)^2 + \eta(t)E_p) \tag{5.1}$$

for every $t \in [0, t_{\max,\gamma})$ with $\eta(t) < \frac{c}{2}$. By Proposition 4.4, we have the property: if $t_{\max,\gamma} < \infty$, then

$$\limsup_{t \uparrow t_{\max,\gamma}} \eta(t) \geq \frac{c}{2}. \tag{5.2}$$

Furthermore, η is continuous by Proposition 4.4 and Corollaries 4.1 and 4.2 and monotonically increasing.

Iteration argument. Suppose we have proven (5.1). First we take $E_p < \min\{\varepsilon_p, \frac{1}{2C}\}$ which gives

$$\eta(t) \leq 2C(E_l + \eta(t)^2). \tag{5.3}$$

Now, choose $4C^2E_l < \frac{c}{2}$. Assuming there exists some $t \in [0, t_{\max,\gamma})$ such that $\eta(t) \geq 4CE_l$, the continuity of η provides a t_0 with $\eta(t_0) = 4CE_l < \frac{c}{2}$. Therefore, (5.3) and $c \in (0, \frac{1}{2})$ imply

$$\eta(t_0) \leq 2C\big(E_l + (16C^2E_l)E_l\big) < 4CE_l.$$

This is a contradiction and we arrive at

$$\sup_{t \in [0, t_{\max,\gamma})} \eta(t) \leq 4CE_l < \frac{c}{2} \tag{5.4}$$

and hence (5.2) cannot hold. We conclude that (5.4) holds with $t_{\max,\gamma} = \tau_{\max} = \infty$. This proves (2.13) and it suffices to justify (5.1). For this purpose, let $t \in [0, t_{\max,\gamma})$ with $\eta(t) < \frac{c}{2}$.

Bound on $\hat{\mathbf{v}}$. We first bound $||\hat{\mathbf{v}}(s)||_{H^3}$ for which we use (4.18), that is,

$$||\hat{\mathbf{v}}(s)||_{H^3} \lesssim (1+s)^{-\frac{1}{2}}\eta(s), \tag{5.5}$$

for $s \in [0, t]$. Together with the nonlinear bounds, Lemma 4.1, Proposition 3.2 and Theorem 5.1, we arrive at

$$\begin{aligned}
||\hat{\mathbf{v}}(s)||_{L^2} &\lesssim (1+s)^{-\frac{1}{2}}E_l + ||\gamma_x(s)||_{L^2}||\hat{\mathbf{w}}(s)||_{L^\infty} + ||\gamma_x(s)||_{H^1}||\hat{\mathbf{v}}(s)||_{L^2} \\
&\quad + \int_0^s (1+s-\tau)^{-\frac{3}{4}}||\mathcal{R}_3(\hat{\mathbf{w}}(\tau), \hat{\mathbf{v}}(\tau), \gamma(\tau))||_{L^1\cap L^2}\, d\tau \\
&\quad + \int_0^s (1+s-\tau)^{-\frac{1}{2}}\Big(||\sigma_t(\tau)\hat{\mathbf{v}}_x(\tau)||_{L^2} + ||\mathcal{T}(\hat{\mathbf{w}}(\tau), \gamma(\tau))||_{L^2} \\
&\qquad\qquad + ||(1-\gamma_x(\tau))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(\tau), \hat{\mathbf{v}}(\tau))||_{L^2}\Big)\, d\tau \\
&\lesssim (1+s)^{-\frac{1}{2}}E_l + (1+s)^{-\frac{1}{2}}\eta(t)^2 + (1+s)^{-\frac{1}{2}}\eta(t)E_p \\
&\quad + \eta(t)^2\int_0^s (1+s-\tau)^{-\frac{3}{4}}(1+\tau)^{-1}d\,\tau \\
&\quad + \eta(t)E_p\int_0^s (1+s-\tau)^{-\frac{1}{2}}(1+\tau)^{-\frac{1}{2}}e^{-\delta_2\tau}d\,\tau \\
&\lesssim (1+s)^{-\frac{1}{2}}\big(E_l + \eta(t)^2 + \eta(t)E_p\big),
\end{aligned} \tag{5.6}$$

for $s \in [0, t]$, where we used that $\eta(t) \leq \frac{1}{2}$.

Bounds on γ. We estimate, with (5.5), Proposition 3.2, Lemma 4.1 and Theorem 5.1,

$$\begin{aligned}
||\partial_s^k\partial_x^l\gamma(s)||_{L^2} &\leq ||\partial_s^k\partial_x^l s_p(s)||_{L^2\to L^2}||\mathbf{v}_0||_{L^2} \\
&\quad + \int_0^s ||\partial_s^k\partial_x^l s_p(s-\tau)||_{L^1\to L^2}||\mathcal{R}_3(\hat{\mathbf{w}}(\tau), \hat{\mathbf{v}}(\tau), \gamma(\tau))||_{L^1}\, d\tau \\
&\quad + \int_0^s ||\partial_s^k\partial_x^l s_p(s-\tau)||_{L^2\to L^2}\Big(||\sigma_t(\tau)\hat{\mathbf{v}}_x(\tau)||_{L^2} + ||\mathcal{T}(\hat{\mathbf{w}}(\tau), \gamma(\tau))||_{L^2} \\
&\quad + ||(1-\gamma_x(\tau))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(\tau), \hat{\mathbf{v}}(\tau))||_{L^2}\Big)\, d\tau \\
&\lesssim (1+s)^{-\frac{k+l}{2}}E_l + \eta(s)^2\int_0^s (1+s-\tau)^{-\frac{1}{4}-\frac{k+l}{2}}(1+\tau)^{-1}d\,\tau \\
&\quad + \eta(s)E_p\int_0^s (1+s-\tau)^{-\frac{k+l}{2}}(1+\tau)^{-\frac{1}{2}}e^{-\delta_2\tau}d\,\tau \\
&\lesssim (E_l + \eta(t)^2 + \eta(t)E_p)\begin{cases} 1, & \text{if } l+k=0 \\ (1+s)^{-\frac{1}{2}}, & \text{if } l+k=1 \\ (1+s)^{-1}, & \text{otherwise,} \end{cases}
\end{aligned} \tag{5.7}$$

for every $s \in [0, t]$.

Bounds on $\hat{\mathbf{v}}$. Invoking (5.6) and (5.7), (4.17) yields

$$||\hat{\mathbf{v}}(s)||_{L^2} \lesssim (1+s)^{-\frac{1}{2}}\big(E_l + \eta(t)^2 + \eta(t)E_p\big)$$

for all $0 \leq s \leq t$. Finally, with Proposition 4.5 and (5.7), we obtain

$$||\hat{\mathbf{v}}(s)||_{H^3} \lesssim (1+s)^{-\frac{1}{2}}\big(E_l + \eta(t)^2 + \eta(t)E_p\big),$$

for all $0 \leq s \leq t$, using $\eta(t) \leq \frac{1}{2}$.

We have shown the key inequality (5.1).

Remark 5.1. *The proof of (5.1) reveals that the nonlinear iteration argument closes as long as $|\sigma_t(t)|$ and $||\hat{\mathbf{w}}(t)||_{H^6_{per}(0,T)}$ decay of order $(1+t)^{-\kappa}$ for some $\kappa > \frac{1}{2}$.*

5.2. Refined L^∞-estimates

By Proposition 3.3 and the nonlinear bounds in Lemma 4.1 and reinserting the L^2-estimates (2.13), we obtain

$$\begin{aligned}
||\gamma_x(t)||_{L^\infty} &\lesssim ||\partial_x s_p(t)||_{L^2\to L^\infty}||\mathbf{v}_0||_{L^2} \\
&\quad + \int_0^t ||\partial_x s_p(t-s)||_{L^1\to L^\infty}||\mathcal{R}_3(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(s), \gamma(s))||_{L^1}\, ds \\
&\quad + \int_0^t ||\partial_x s_p(t-s)||_{L^2\to L^\infty}\Big(||\sigma_t(s)\hat{\mathbf{v}}_x(s)||_{L^2} + ||\mathcal{T}(\hat{\mathbf{w}}(s), \gamma(s))||_{L^2} \\
&\quad + ||(1-\gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(s))||_{L^2}\Big)\, ds \\
&\lesssim (1+t)^{-\frac{3}{4}}E_l + E_l\Big(\int_0^t (1+t-s)^{-1}(1+s)^{-1}\, ds \\
&\quad + \int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{1}{2}}e^{-\delta_2 s}\, ds\Big) \\
&\lesssim (1+t)^{-\frac{3}{4}}E_l,
\end{aligned}$$

for all $t \geq 0$. Furthermore, we have

$$\begin{aligned}
||\hat{\mathbf{v}}(t)||_{L^\infty} &\lesssim (1+t)^{-\frac{3}{4}}E_l + \int_0^t (1+t-s)^{-1}||\mathcal{R}_3(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(s), \gamma(s))||_{L^1\cap L^2}\, ds \\
&\quad + \int_0^t e^{-\delta_1(t-s)}||\mathcal{R}_3(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(s), \gamma(s))||_{H^1}\, ds \\
&\quad + \int_0^t (1+t-s)^{-\frac{3}{4}}\Big(||\sigma_t(s)\hat{\mathbf{v}}_x(s)||_{H^1} \\
&\quad + ||(1-\gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(s))||_{H^1} + ||\mathcal{T}(\hat{\mathbf{w}}(s), \gamma(s))||_{H^1}\Big)\, ds \\
&\lesssim (1+t)^{-\frac{3}{4}}E_l + E_l\Big(\int_0^t (1+t-s)^{-1}(1+s)^{-1}\, ds \\
&\quad + \int_0^t (1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{1}{2}}e^{-\delta_2 s}\, ds\Big) \\
&\lesssim (1+t)^{-\frac{3}{4}}E_l,
\end{aligned}$$

for all $t \geq 0$. With the help of (4.19), we deduce

$$||\hat{\mathbf{v}}(t)||_{L^\infty} \lesssim (1+t)^{-\frac{3}{4}}E_l,$$

for all $t \geq 0$. Finally, we observe that for $\sigma_* := \int_0^\infty \sigma_s(s)\, ds$, we have

$$|\sigma_* - \sigma(t)| \leq \int_t^\infty |\sigma_s(s)|\, ds \lesssim e^{-\delta_1 t}E_p$$

and therefore

$$||\phi(\cdot + \sigma_* + \gamma(\cdot, t), t) - \phi(\cdot + \sigma(t) + \gamma(\cdot, t), t)||_{L^\infty} \lesssim e^{-\delta_1 t}E_p,$$

for all $t \geq 0$. As described in Section 2.6, Theorem 2.1 follows.

6. Discussion

6.1. Applicability of the scheme to viscous conservation laws

A satisfying L^∞-stability theory considering C_{ub}-perturbations is established in parabolic reaction-diffusion systems [18] which can be extended beyond the parabolic setting as shown in the context of the

FitzHugh-Nagumo system [12]. On the other hand, there are crucial obstacles in establishing pure L^∞-stability results for viscous conservation laws as described in [12,17]. An interesting ingredient of the L^2-analysis in [31] is that the authors introduce a sum of spatio-temporal modulation functions to capture more than one critical mode arising from the presence of the conservation laws. Therefore, a key difficulty lies in the fact that the critical dynamics is governed by a coupled Whitham system for which one cannot immediately apply the Cole-Hopf transform as done in the scalar case for which this Whitham system reduces to the Burgers' equation [12,18]. This begs the question of whether we can apply our $L^2_{\mathrm{per}}(0,T)\oplus L^2(\mathbb{R})$-scheme to this setting, cf. [31] and [32]. In order to generate a result like Theorem 2.1 in this setting, the question reduces to whether such a sum of spatio-temporal modulation functions is compatible with our approach. As the $L^2_{\mathrm{per}}(0,T)$-theory can be settled by a standard procedure, we strongly expect that it is relatively straightforward to allow for $L^2_{\mathrm{per}}(0,T)\oplus L^2(\mathbb{R})$-perturbations by precisely following our nonlinear analysis, in particular using the modified versions of inverse- and forward-modulated perturbations in Sections 4.2 and 4.5, and by respecting the semigroup decomposition and estimates established in [31, pp. 149] and the nonlinear damping estimate in [31, pp. 146].

6.2. Uniformly subharmonic plus localized perturbations

Considering the uniformly subharmonic nonlinear stability result for the Lugiato-Lefever equation in [16], the question arises whether we can show a nonlinear stability result involving perturbations $\mathbf{w}_0+\mathbf{v}_0$ with $\mathbf{w}_0\in H^6_{\mathrm{per}}(0,NT)\cap L^1_{\mathrm{per}}(0,NT)$, $\mathbf{v}_0\in H^3(\mathbb{R})$ and which is uniform in $N\in\mathbb{N}$. We note that as in [16] the additional condition $L^1_{\mathrm{per}}(0,NT)$ is crucial in [16] to guarantee the uniformity in N.

Suppose (H1) and (D1)-(D3). Now, given a solution $\mathbf{u}=\mathbf{w}+\mathbf{v}$ of (1.1) where $\mathbf{w}$ solves (2.2) and $\mathbf{v}$ solves (2.3) with $\mathbf{u}(0)=\mathbf{w}_0+\mathbf{v}_0$, we broadly sketch a possible scheme for the claimed stability estimate

$$||\mathbf{u}(t)||_{L^\infty}\lesssim(1+t)^{-\frac{3}{4}}\Big(||\mathbf{w}_0||_{H^6_{\mathrm{per}}(0,NT)\cap L^1_{\mathrm{per}}(0,NT)}+||\mathbf{v}_0||_{H^3(\mathbb{R})}\Big),\quad t\geq 0, \tag{6.1}$$

uniformly in $N\in\mathbb{N}$. Keeping the main lines of this paper, we might introduce the inverse-modulated perturbations

$$\hat{\mathbf{w}}(x,t)=\mathbf{w}(x-\sigma(t)-\gamma_1(x,t),t)-\phi,$$
$$\hat{\mathbf{v}}(x,t)=\mathbf{u}(x-\sigma(t)-\gamma_1(x,t)-\gamma_2(x,t),t)-\hat{\mathbf{w}}(x,t)-\phi(x)$$

and the forward-modulated perturbations

$$\mathring{\mathbf{w}}(x,t)=\mathbf{w}(x,t)-\phi(x+\sigma(t)+\gamma_1(x,t)),$$
$$\mathring{\mathbf{v}}(x,t)=\mathbf{u}(x,t)-\hat{\mathbf{w}}(x+\sigma(t)+\gamma_1(x,t)+\gamma_2(x,t),t)-\phi(x+\sigma(t)+\gamma_1(x,t)+\gamma_2(x,t)),$$

with modulation functions

$$\sigma:[0,\infty)\to\mathbb{R},\gamma_1:[0,\infty)\to L^2_{\mathrm{per}}(0,NT)\text{ and }\gamma_2:[0,\infty)\to L^2(\mathbb{R}).$$

To derive suitable perturbation equations for $\hat{\mathbf{v}}$ and $\mathring{\mathbf{v}}$, we refer to Appendices A and B. Finally, with the established decay rates, see [16, Theorem 1.4],

$$||\partial_x\gamma_1(t)||_{L^2_{\mathrm{per}}(0,NT)},|\sigma_t(t)|,||\hat{\mathbf{w}}(t)||_{L^2_{\mathrm{per}}(0,NT)}\sim O((1+t)^{-\frac{3}{4}}),$$

uniformly in N, one may invoke Remark 5.1 and [16, Proposition 3.7]). It remains to prove versions of Proposition 4.5 and Lemma 4.4 to deduce (6.1).

6.3. Nonlocalized phase modulations

An interesting feature of the modulational stability estimate (2.11) is that we capture the most critical dynamics of the periodic wave by a phase modulation from $\mathbb{R}+H^3(\mathbb{R})$. This class of functions covers the simplest nontrivial nonlocalized phase perturbations one could think of. Therefore, an interesting open question is whether we can allow for perturbations from $L^2_{\mathrm{per}}(0,T)\oplus L^2(\mathbb{R})$ as well as (L^∞-large) initial modulations $\gamma_0-\sigma_*$ such that $||\gamma_0'||_{H^3}$ is small to conclude an estimate such as (2.11). We refer to [25] for a nonlinear stability result against localized perturbations in this context, allowing for a nonlocalized initial phase modulation.

6.4. Fully nonlocalized perturbations

Local well-posedness results in nonlinear Schrödinger-type equations, possibly with (periodic) potentials, have been established for initial data from the modulation space $M^m_{\infty,1}(\mathbb{R})$, see e.g. [33,34], [35, pages 245–252] and references therein. At the same time, global in time results are widely open. Due to the additional dissipativity in (1.1) compared to the classical cubic nonlinear Schrödinger equation, we expect the possibility to extend our result to perturbations from $M^m_{\infty,1}(\mathbb{R})$ for sufficiently large $m\in\mathbb{N}$. This might be achieved by conceptually following the lines of [12] through the complex inversion formula of the semigroup. Nevertheless, it turns out that high-frequency damping and regularity control are delicate challenges and it seems that essentially new ideas have to be developed.

CRediT authorship contribution statement

Joannis Alexopoulos: Writing – original draft.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

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Appendix A. Derivation of (4.6)

We write $\hat{\mathbf{u}}(x,t)=\mathbf{u}(x-\sigma(t)-\gamma(x,t),t)-\phi(x)$, insert (4.5) and [16, (3.4)] to obtain

$$\begin{aligned}(\partial_t-\mathcal{L}_0)(\hat{\mathbf{v}})&=(\partial_t-\mathcal{L}_0)\hat{\mathbf{u}}-(\partial_t-\mathcal{L}_0)\hat{\mathbf{w}}\\&=-(\partial_t-\mathcal{L}_0)(\phi'\gamma)+(1-\gamma_x)\mathcal{R}_1(\hat{\mathbf{u}})-\mathcal{R}_1(\hat{\mathbf{w}})+\sigma_t\hat{\mathbf{w}}_x\\&\quad+\partial_x S(\hat{\mathbf{u}},\gamma,\gamma_t,\sigma_t)+\partial_x^2\mathcal{P}(\hat{\mathbf{u}},\gamma)+(\partial_t-\mathcal{L}_0)(\gamma_x\hat{\mathbf{u}}),\end{aligned}$$

where, recalling $\hat{\mathbf{u}}=\hat{\mathbf{v}}+\hat{\mathbf{w}}$,

$$\begin{aligned}\widetilde{S}(\hat{\mathbf{u}},\gamma,\gamma_t,\sigma_t)&=-\gamma_t\hat{\mathbf{u}}-\sigma_t\hat{\mathbf{u}}+\beta\mathcal{J}\left(\frac{\gamma_{xx}}{(1-\gamma_x)^2}\hat{\mathbf{u}}-\frac{\gamma_x^2}{1-\gamma_x}\phi'\right)\\&=\widetilde{S}(\hat{\mathbf{v}},\gamma,\gamma_t,\sigma_t)-\gamma_t\hat{\mathbf{w}}-\sigma_t\hat{\mathbf{w}}+\beta\mathcal{J}\left(\frac{\gamma_{xx}}{(1-\gamma_x)^2}\hat{\mathbf{w}}\right)\\\mathcal{P}(\hat{\mathbf{u}},\gamma)&=-\beta\mathcal{J}\left(\gamma_x+\frac{\gamma_x}{1-\gamma_x}\right)\hat{\mathbf{u}}=\mathcal{P}(\hat{\mathbf{v}},\gamma)+\mathcal{P}(\hat{\mathbf{w}},\gamma).\end{aligned}$$

We emphasize that the critical terms $\sigma_t\hat{\mathbf{w}}$ and $(\partial_t - \mathcal{L}_0)(\phi'\sigma)$ cancel out. We arrive at

$$\begin{aligned}(\partial_t - \mathcal{L}_0)(\hat{\mathbf{v}} + \phi'\gamma - \gamma_x\hat{\mathbf{w}} - \gamma_x\hat{\mathbf{v}}) =& \mathcal{Q}(\hat{\mathbf{w}},\hat{\mathbf{v}},\gamma) + \partial_x\mathcal{S}(\hat{\mathbf{v}},\gamma) + \partial_x^2\mathcal{P}(\hat{\mathbf{v}},\gamma)\\ &- \sigma_t\hat{\mathbf{v}}_x + (1-\gamma_x)\mathcal{R}_{2,2}(\hat{\mathbf{w}},\hat{\mathbf{v}}) + \mathcal{T}(\hat{\mathbf{w}},\gamma)\end{aligned}$$

with

$$\begin{aligned}\mathcal{Q}(\hat{\mathbf{w}},\hat{\mathbf{v}},\gamma) &= (1-\gamma_x)(\mathcal{R}_1(\hat{\mathbf{v}}+\hat{\mathbf{w}}) - \mathcal{R}_1(\hat{\mathbf{w}}) - \mathcal{R}_{2,2}(\hat{\mathbf{w}},\hat{\mathbf{v}})) = (1-\gamma_x)\mathcal{R}_{2,1}(\hat{\mathbf{v}},\hat{\mathbf{w}}),\\ \mathcal{S}(\hat{\mathbf{v}},\gamma) &= -\gamma_t\hat{\mathbf{v}} + \beta\mathcal{J}\left(\frac{\gamma_{xx}}{(1-\gamma_x)^2}\hat{\mathbf{v}} - \frac{\gamma_x^2}{1-\gamma_x}\phi'\right),\\ \mathcal{T}(\hat{\mathbf{w}},\gamma) &= -\gamma_x\mathcal{R}_1(\hat{\mathbf{w}}) - \partial_x\left(\gamma_t\hat{\mathbf{w}} - \beta\mathcal{J}\left(\frac{\gamma_{xx}}{(1-\gamma_x)^2}\hat{\mathbf{w}}\right)\right)\\ &\quad - \partial_x^2\left(\beta\mathcal{J}\left(\gamma_x + \frac{\gamma_x}{1-\gamma_x}\right)\hat{\mathbf{w}}\right).\end{aligned}$$

Appendix B. Derivation of (4.14)

We recall that

$$\mathring{\mathbf{v}}(x,t) = \mathbf{u}(x,t) - \mathbf{w}(x+\gamma(x,t),t),$$

set $\mathring{\phi}(x,t) = \phi(x+\gamma(x,t))$ and use the notions (2.5), (2.6) and (2.7). Furthermore, we write

$$\mathcal{D}(\mathbf{u}) = \mathcal{J}\left(\begin{pmatrix}-\beta & 0\\ 0 & -\beta\end{pmatrix}\mathbf{u}_{xx} + \begin{pmatrix}-\alpha & 0\\ 0 & -\alpha\end{pmatrix}\mathbf{u}\right) - \mathbf{u}.$$

Using that $\mathbf{u}$ and $\mathbf{w}$ are solutions of (2.1), we derive

$$\begin{aligned}&(\partial_t - \mathcal{L}_0(\mathring{\phi}))\mathring{\mathbf{v}}(t)\\ &= (\partial_t - \mathcal{D})(\mathbf{u}(t)) - (\partial_t - \mathcal{D})(\mathbf{w})(\cdot+\gamma(\cdot,t),t) - \mathcal{N}'(\mathring{\phi}(t))\mathring{\mathbf{v}}(t)\\ &\quad + \mathcal{R}_5(\tilde{\mathbf{w}}(t),\gamma(t),\gamma_t(t))\\ &= \mathcal{N}(\mathring{\mathbf{v}}(t) + \tilde{\mathbf{w}}(\cdot+\gamma(\cdot,t),t) + \mathring{\phi}(t)) - \mathcal{N}(\tilde{\mathbf{w}}(\cdot+\gamma(\cdot,t),t) + \mathring{\phi}(t))\\ &\quad - \mathcal{N}'(\mathring{\phi})\mathring{\mathbf{v}}(t) + \mathcal{R}_5(\tilde{\mathbf{w}}(t),\gamma(t),\gamma_t(t))\\ &= \mathcal{R}_2(\mathring{\phi})(\tilde{\mathbf{w}}(\cdot+\gamma(\cdot,t),t),\mathring{\mathbf{v}}(t)) + \mathcal{R}_5(\tilde{\mathbf{w}}(t),\gamma(t),\gamma_t(t))\end{aligned}$$

with

$$\begin{aligned}&\mathcal{R}_5(\tilde{\mathbf{w}}(t),\gamma(t),\gamma_t(t))\\ &= -\mathbf{w}_x(\cdot+\gamma(\cdot,t),t)\gamma_t(t) - \beta\mathcal{J}\left(\mathbf{w}_x(\cdot+\gamma(\cdot,t),t)(\gamma_{xx}(t))\right.\\ &\quad\left.+\mathbf{w}_{xx}(\cdot+\gamma(\cdot,t),t)(2\gamma_x(t)+\gamma_x(t)^2)\right)\\ &= -\tilde{\mathbf{w}}_x(\cdot+\gamma(\cdot,t),t)\gamma_t(t) - \phi'(\cdot+\gamma(\cdot,t),t)\gamma_t(t)\\ &\quad - \beta\mathcal{J}\Big(\tilde{\mathbf{w}}_x(\cdot+\gamma(\cdot,t),t)(\gamma_{xx}(t)) + \tilde{\mathbf{w}}_{xx}(\cdot+\gamma(\cdot,t),t)(2\gamma_x(t)+\gamma_x(t)^2)\\ &\quad + \phi'(\cdot+\gamma(\cdot,t),t)(\gamma_{xx}(t)) + \phi''(\cdot+\gamma(\cdot,t),t)(2\gamma_x(t)+\gamma_x(t)^2)\Big).\end{aligned}$$

Recall that $\tilde{\mathbf{w}}(t) = \mathbf{w}(t) - \phi$ denotes the unmodulated perturbation of $\mathbf{w}$.

Appendix C. Proof of Proposition 4.4

We recall the Duhamel formula (4.9) given by

$$\begin{aligned}\gamma(t) =& s_p(t)\mathbf{v}_0 + \int_0^t s_p(t-s)\Big(\mathcal{R}_3(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(s),\gamma(s)) - \sigma_t(s)\hat{\mathbf{v}}_x(s)\\ &+ (1-\gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(s)) + \mathcal{T}(\hat{\mathbf{w}}(s),\gamma(s))\Big)\,ds\end{aligned}$$

whereas

$$\hat{\mathbf{v}}(x,t) = \mathbf{u}(x-\sigma(t)-\gamma(x,t),t) - \hat{\mathbf{w}}(x,t) - \phi(x).$$

To prevent confusion, we write

$$\hat{\mathbf{v}}(t) = \hat{\mathbf{v}}(\gamma(t),t)$$

and for the sake of readability, we introduce

$$\begin{aligned}\tilde{\mathcal{N}}(t,\sigma(s),\gamma(s),s) =& s_p(t-s)\Big(\mathcal{R}_3(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(s),\gamma(s)) - \sigma_t(s)\hat{\mathbf{v}}_x(s)\\ &+ (1-\gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(s)) + \mathcal{T}(\hat{\mathbf{w}}(s),\gamma(s))\Big).\end{aligned}$$

We do a contraction argument. For this purpose, we establish nonlinear bounds.

Lemma C.1. *Fix $0 \le \tau_1 \le \tau_2 < \min\{\tau_{max}, t_{max,\sigma}\}$. Let $j,k \in \mathbb{N}_0$ and fix $K > 0$. Suppose that*

$$\sup_{t\in[0,\tau_2]}\left(||\mathbf{w}(t)||_{W^{1,\infty}} + |\sigma_t(t)| + ||\mathbf{v}(t)||_{W^{1,\infty}}\right) \le K.$$

There exist constants $C > 0$ and $C_{j,k} > 0$ such that we have the bounds

$$\sup_{s\in[\tau_1,\tau_2]}||\hat{\mathbf{v}}(\gamma_1(s),s) - \hat{\mathbf{v}}(\gamma_2(s),s)||_{L^2} \le C\sup_{s\in[\tau_1,\tau_2]}||\gamma_1(s)-\gamma_2(s)||_{L^2}, \tag{C.1}$$

and

$$\begin{aligned}&\sup_{t\in[\tau_1,\tau_2]}\sup_{s\in[\tau_1,t]}||\partial_t^j\partial_x^k(\tilde{\mathcal{N}}(t,\sigma(s),\gamma_2(s),s) - \tilde{\mathcal{N}}(t,\sigma(s),\gamma_2(s),s))||_{L^2}\\ &\qquad\le C_{j,k}\sup_{s\in[\tau_1,\tau_2]}(||\gamma_1(s)-\gamma_2(s)||_{H^2} + ||\partial_t\gamma_1(s) - \partial_t\gamma_2(s)||_{L^2}),\end{aligned} \tag{C.2}$$

for any $\gamma_1,\gamma_2 \in C([\tau_1,\tau_2];H^5(\mathbb{R}))\times C^1([\tau_1,\tau_2];H^3(\mathbb{R}))$ with $\sup_{t\in[\tau_1,\tau_2]}||\partial_x\gamma_1(t)||_{L^\infty}, ||\partial_x\gamma_1(t)||_{L^\infty} \le \frac{1}{2}$.

Proof. Let $t \in [\tau_1,\tau_2]$ and $s \in [\tau_1,t]$. We rewrite

$$\begin{aligned}\hat{\mathbf{v}}(x,\gamma_1(s),s) - \hat{\mathbf{v}}(x,\gamma_2(s),s) =& \mathbf{v}(x-\sigma(s)-\gamma_1(x,s),s)\\ &- \mathbf{v}(x-\sigma(s)-\gamma_2(x,s),s)\\ &+ \mathbf{w}(x-\sigma(s)-\gamma_2(x,s),s)\\ &- \mathbf{w}(x-\sigma(s)-\gamma_1(x,s),s)\end{aligned}$$

and since $\mathbf{w}(x-\sigma(s)-\gamma(x,s),s) - \phi(x) = \hat{\mathbf{w}}(x-\gamma(x,s),s)$, this yields (C.1) by the mean value theorem. Let $j,k \in \mathbb{N}_0$. Recalling the choice of $\mathcal{R}_3 = \mathcal{Q} + \partial_x\mathcal{S} + \partial_x^2\mathcal{P}$ and the estimates on $s_p(t)$ from Proposition 3.2, we obtain

$$\begin{aligned}&||\partial_t^j\partial_x^k s_p(t-s)(\mathcal{R}_3(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(\gamma_1(s),s),\gamma_1(s)) - \mathcal{R}_3(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(\gamma_2(s),s),\gamma_2(s)))||_{L^2}\\ &\quad\le C_{j,k}(||\gamma_1(s)-\gamma_2(s)||_{H^2} + ||\partial_t\gamma_1(s) - \partial_t\gamma_2(s)||_{L^2}),\end{aligned}$$

by taking derivatives on $s_p(t-s)$ and (C.1). Next, with the Cauchy-Schwarz inequality, we find

$$\begin{aligned}&||\partial_t^j\partial_x^k s_p(t-s)\Big((1-\partial_x\gamma_1(s))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(\gamma_1(s),s))\\ &\quad - (1-\partial_x\gamma_2(s))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(\gamma_2(s),s))\Big)||_{L^2} \le C_{j,k}||\gamma_1(s)-\gamma_2(s)||_{H^1}.\end{aligned}$$

Moreover, we have

$$||\partial_t^j\partial_x^k s_p(t-s)\Big(\sigma_t(s)\big(\hat{\mathbf{v}}_x(\gamma_1(s),s) - \hat{\mathbf{v}}_x(\gamma_2(s),s)\big)\Big)||_{L^2} \le C_{j,k}||\gamma_1(s)-\gamma_2(s)||_{L^2}.$$

Together with

$$\begin{aligned}&||\partial_t^j\partial_x^k s_p(t-s)\Big(\mathcal{T}(\hat{\mathbf{w}}(s),\gamma_1(s)) - \mathcal{T}(\hat{\mathbf{w}}(s),\gamma_1(s))\Big)||_{L^2}\\ &\quad\le C_{j,k}(||\gamma_1(s)-\gamma_2(s)||_{H^2} + ||\partial_t\gamma_1(s) - \partial_t\gamma_2(s)||_{L^2}),\end{aligned}$$

again taking derivatives on $s_p(t-s)$, we arrive at (C.2). □

By the choice of s_p, we immediately have that $\gamma(t) = 0$ for $t \in [0,1]$. We need to justify that we can extend the modulation function γ to a maximal time such that the alternative (4.12) holds.

Proof of Proposition 4.4. Let $\tilde{\gamma}$ be a solution of (4.9) on $[0,t_0]$ with some $t_0 > 0$. Lemma C.1 tells us that

$$\begin{aligned}\gamma \mapsto& s_p(t)\mathbf{v}_0 + \int_{t_0}^t s_p(t-s)\Big(\mathcal{R}_3(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(s),\gamma(s)) - \sigma_t(s)\hat{\mathbf{v}}_x(s)\\ &+ (1-\gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(s),\hat{\mathbf{v}}(s)) + \mathcal{T}(\hat{\mathbf{w}}(s),\gamma(s))\Big)\,ds\end{aligned} \tag{C.3}$$

defines a contraction on

$$\begin{aligned}X_{t_0,\tau_0} = \Big\{&\gamma \in C([t_0,t_0+\tau_0];H^5(\mathbb{R}))\cap C^1([t_0,t_0+\tau_0];H^3(\mathbb{R})) :\\ &\sup_{s\in[t_0,t_0+\tau_0]}||(\gamma(s),\gamma_s(s))||_{H^5\times H^3} < \frac{c}{2}\Big\}\end{aligned}$$

for sufficiently small $\tau_0 > 0$. By the Banach fixed point theorem, there exists a unique solution

$$\gamma \in C([t_0,t_0+\tau_0];H^5(\mathbb{R}))\cap C^1([t_0,t_0+\tau_0];H^3(\mathbb{R}))$$

of

$$\begin{aligned}\gamma(t) = s_p(t)\mathbf{v}_0 + \int_0^{t_0} s_p(t-s)\Big(&\mathcal{R}_3(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(\tilde{\gamma}(s), s), \tilde{\gamma}(s)) - \sigma_t(s)\hat{\mathbf{v}}_x(\tilde{\gamma}(s), s)\\ &+ (1-\gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(\tilde{\gamma}(s), s)) + \mathcal{T}(\hat{\mathbf{w}}(s), \tilde{\gamma}(s))\Big)\, ds\\ + \int_{t_0}^{t} s_p(t-s)\Big(&\mathcal{R}_3(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(\gamma(s), s), \gamma(s)) - \sigma_t(s)\hat{\mathbf{v}}_x(\gamma(s), s)\\ &+ (1-\gamma_x(s))\mathcal{R}_{2,2}(\hat{\mathbf{w}}(s), \hat{\mathbf{v}}(\gamma(s), s)) + \mathcal{T}(\hat{\mathbf{w}}(s), \gamma(s))\Big)\, ds\end{aligned} \tag{C.4}$$

for $t \in [t_0, t_0+\tau_0]$. Since (C.3) defines a contraction on X_{t_1,t_2} for every $t_1 \in (0, \min\{\tau_{\max}, t_{\max,\sigma}\})$ and sufficiently small $t_2 > 0$, we conclude that

$$\gamma(t) = \begin{cases} \tilde{\gamma}(t), & t \in [0, t_0] \\ \gamma(t), & t \in [t_0, t_0+\tau_0] \end{cases} \tag{C.5}$$

is the unique solution of (4.9) on $[0, t_0+\tau_0]$ such that $\gamma \in X_{t_0,\tau_0}$. Now, there exists a maximal time $t_{\max,\gamma} \in (0, \min\{\tau_{\max}, t_{\max,\sigma}\}]$ such that (4.11) holds. Assume that $t_{\max,\gamma} < \min\{\tau_{\max}, t_{\max,\sigma}\}$. If (4.12) fails, then there exists a unique solution $\tilde{\gamma}$ of (4.9) with $\sup_{t\in[0,t_{\max,\gamma})} ||(\tilde{\gamma}(t), \tilde{\gamma}_t(t))||_{H^5\times H^3} < \frac{c}{2}$. This again allows to solve (C.4) on $X_{t_{\max,\gamma},\tau_0'}$ for some small τ_0' giving a solution of (4.9) via (C.5) on $[0, t_{\max,\gamma}+\tau_0')$ with $||(\gamma(s), \gamma_s(s))||_{H^5\times H^3} < \frac{c}{2}$ for all $t \in [0, t_{\max,\gamma}+\tau_0')$. This contradicts the maximality of $t_{\max,\gamma}$ and we conclude (4.12). □

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