Error analysis of exponential integrators for wave-type equations at low regularity

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Contents

Introduction

The analysis of nonlinear dispersive partial differential equations is a large and rich field within mathematics. It has encountered decisive breakthroughs in the past decades, based on methods from harmonic and nonlinear analysis. Important examples of such systems include the nonlinear wave, Schrödinger, and Maxwell equations, but also water-wave models like the Korteweg–de Vries (KdV) equation, cf. [71]. The nonlinear wave equation for instance models a vibrating object with state-dependent force. On the other hand, the nonlinear Schrödinger equation serves as an important amplitude equation in nonlinear optics, see [69].

In a dispersive equation, waves with different frequencies tend to travel at different velocities or in different directions. This makes the solution disperse in space as time evolves. At the same time, in the core examples conserved quantities such as mass or energy are available. This is in contrast to diffusive equations such as the heat equation, which exhibit energy dissipation over time.

The *wellposedness* of nonlinear dispersive equations has been intensively investigated. This means existence and uniqueness of solutions, as well as continuous dependence on the data. To study the long-time behavior as well, it is important to have a good wellposedness theory in a *low-regularity* setting, meaning the case when the initial data come from a function space only requiring a small amount of differentiability, possibly even allowing discontinuities. This is because the natural conservation laws are often associated with such spaces. As opposed to diffusive equations such as the heat equation, no smoothing of the initial data can be expected for fixed times, in general. In the context of dispersive equations, *Strichartz estimates* play an important role. Based on dispersion, they control spatial integrability beyond the estimates following from the energy and Sobolev inequalities. These estimates were first formulated in [68] and later generalized in, e.g., [23, 46]. They are very well suited to treat power-type nonlinearities, and often enable a comprehensive wellposedness theory in situations that would hardly be accessible by other methods, for instance at the level of finite-energy solutions in several spatial dimensions. Bilinear refinements of Strichartz estimates are also of interest, such as *null form estimates* for the wave equation, which exploit cancellations in the nonlinear frequency interactions between waves, cf. [48]. An estimate of that type (with a rather elementary proof) does even exist for the one-dimensional wave equation, which is not a truly dispersive problem.

For nonlinear differential equations, explicit and practically computable solution formulas are not available in most cases. Therefore, one often relies on numerical approximations. If a time-dependent equation separates into a sum of sub-problems that could be solved efficiently individually, *splitting methods* are often attractive, see [30, 51] for an overview. The simplest of those schemes is the *Lie splitting*. Here, we perform one time step of the first sub-problem followed by one step of the second sub-problem to approximate one time step of the full problem. A natural improvement is the *Strang splitting*, where one time step of the full equation is approximated by first computing a half step of the first sub-problem, then a full step of the second sub-problem, and finally again a half step of the first sub-problem. Due to its symmetry, the Strang splitting is formally more accurate than the Lie splitting. Splitting methods often enjoy favorable geometric properties, cf. [27].

In this work we focus on *semilinear* equations, meaning that the equation separates into a linear and a nonlinear part, where the highest-order derivatives appear exclusively in the linear part. A time-stepping scheme which treats the linear part exactly is referred to as an *exponential integrator*, see [28] for an overview on this topic. Such methods are particularly suitable for partial differential equations with periodic boundary conditions in space, since the spatial discretization by means of the Fourier (pseudo)-spectral method enables an efficient computation of the linear propagator. Splitting methods belong to the class of exponential integrators in the case when one of the sub-problems is linear and is treated exactly.

The error analysis of exponential integrators applied to nonlinear dispersive equations is challenging. The seminal paper [50] established second-order convergence of the Strang splitting applied to the cubic Schrödinger equation. The analysis in that paper requires high smoothness, in particular, bounds on four spatial derivatives of the solution. Numerical experiments suggest that this is more than a technical issue, since order reduction is generally observed if the regularity requirements are not fulfilled. Hence, it is interesting to investigate which (possibly fractional) convergence rates can still be proven if the solution is only assumed to belong to a less regular space. Such an analysis was done, e.g., in [18] in the context of splitting methods for the semilinear Schrödinger equation and in [22] for exponential integrators applied to the one-dimensional semilinear wave equation, respectively.

The error analysis in the above works heavily relies on tools such as energy estimates and Sobolev embeddings. However, as it is the case for the wellposedness theory, such "classical" inequalities are often insufficient if the regularity is very low, e.g., at energy level. Therefore, it is natural to exploit variants of the tools used in the wellposedness theory at low regularity also in numerical analysis, such as versions of Strichartz estimates that are discrete in time and/or space. In the literature, this has been investigated mainly for Schrödinger equations so far.

Since the regularizing effect described by Strichartz estimates only exists when time is averaged, straightforward analogues of these inequalities that are discrete in time but continuous in space fail, in general. One way to still obtain such results is to include smoothing operators such as frequency filters. This approach was pursued in the works [36, 53]. The resulting discrete-time Strichartz estimates were then used to obtain error bounds for time discretizations of nonlinear Schrödinger equations on $\mathbb{R}^d$, see also [15, 17]. In [41, 52, 54], the case of periodic boundary conditions was treated by also using discrete Bourgain spaces.

On the other hand, Strichartz estimates which are continuous in time but discrete in space have been studied as well. The papers [38, 39, 40] analyze the impact of a spatial discretization by means of finite differences on the dispersive behavior of the Schrödinger equation. It turns out that the dispersive estimates are not preserved uniformly in the discretization parameter, and several modifications of the discretization procedures are proposed to recover the full dispersion. Fully discrete approximations are also considered in [37]. In more recent works [31, 32, 33, 34], uniform Strichartz estimates with derivative loss are shown for finite difference discretizations of the Schrödinger equations on the full space $\mathbb{R}^d$ as well as on the torus $\mathbb{T}^d$. The corresponding problem for the multi-dimensional wave equation seems to be more difficult and much less is known, cf. [16].

Another related branch of research that recently gained a lot of attention in the literature is the construction and analysis of *low-regularity integrators*. In settings of low regularity, these tailor-made time discretization schemes can outperform more classical exponential integrators thanks to an improved local error structure which requires less regularity. The construction of these methods typically relies on the embedding of nonlinear frequency interactions into the numerical scheme. The first integrators of this type were proposed in [29] and [55] for the KdV and semilinear Schrödinger equations, respectively. For the semilinear wave equation, such a scheme was developed in [49]. It is natural to exploit Strichartz-like estimates for the error analysis of these methods, as it was first done in the articles [53, 54] for the nonlinear Schrödinger equation. See also, e.g., [9, 14, 20, 59] for further important contributions in the context of low-regularity integrators, and [60] for a survey article.

Content of this thesis

The first part of this thesis contains the preprint [62] with minor modifications. However, the analysis of the second-order scheme in Section 2.2 and the associated numerical experiments in Section 2.3 are new.

The second part includes an extended version of the preprint [61]. It has been generalized in several respects, e.g., we treat fractional powers inside the nonlinearity. Moreover, we added the derivation of discrete-time Strichartz estimates on $\mathbb{R}^3$ and the analysis for the scaling-critical quintic equation, both based on our published paper [63]. This concerns Sections 5.1–5.2, 7.6, and parts of Section 6.1 of this thesis. As a new feature, the analysis of the critical defocusing equation on the full space has been extended to a global-in-time result, including scattering of the numerical solution.

We generally concentrate on the error analysis of discrete-time approximations, often working in a semi-discrete setting with continuous space. The second part of this thesis also contains error bounds for full discretizations by means of the Fourier pseudo-spectral method.

Part I. Improved error estimates for low-regularity integrators using space-time bounds

In the first part of this thesis, we analyze three known low-regularity integrators. These are the first- and second-order methods for the semilinear Schrödinger equation from [55] and [9, 56, 59], respectively, as well as the second-order scheme for the semilinear wave equation from [49]. We show that they converge with their full formal orders in certain situations in which previously only reduced convergence orders were known. Here we concentrate on the one-dimensional equations with periodic boundary conditions. The general strategy of proof is the same in all cases. We first derive a suitable representation of the local error. In a second step, the sum of the local error terms is optimally estimated exploiting a known equation-specific space-time inequality for the solution. Here we only use the estimates for continuous time, since discrete-time estimates often involve a loss, see [53, 54, 63]. For the Schrödinger equation, we apply the periodic L^4 Strichartz inequality. In the case of the wave equation, we use a null form estimate, which seems to be a new tool in numerical analysis. The proof of the error bound is then completed in a classical way by a discrete Gronwall argument. Hence, our proof strategy is very flexible, also applicable to higher dimensions, and could possibly be adapted to show error bounds also for other equations and integrators. In this part, we only analyze the temporal semi-discretization, but expect that an extension to a fully discrete setting is possible.

Part II. Error analysis of the Strang splitting for 3D semilinear wave equations with finite-energy solutions

The semilinear wave equation $\partial_t^2 u - \Delta u = \pm |u|^{\alpha-1} u$ is one of the most important model problems for dispersive behavior. Its analytical properties are well understood, see [66, 71]. In view of the energy equality, H^1 (or its homogeneous version $\dot{H}^1$) is the most natural regularity level for solutions $u(t)$ and data. On 3D domains, in the case of powers $\alpha \in (1, 3]$, one can investigate wellposedness by means of the standard tools of evolution equations, whereas the treatment of the case $\alpha \in (3, 5]$ is based on dispersive properties. To our knowledge, in numerical analysis the latter situation has not been studied in this setting previously.

In the second part of this thesis, we treat a variant of the Strang splitting for the time integration of the semilinear wave equation on the full space $\mathbb{R}^3$ as well as the three-dimensional torus $\mathbb{T}^3$ under a finite-energy condition. In the case of a cubic nonlinearity, we show almost second-order convergence in L^2 and almost first-order convergence in the energy space. For the energy-critical quintic nonlinearity, we show first-order convergence in L^2 and convergence without rate in the energy space. To our knowledge, this is the first error analysis performed for a scaling-critical dispersive problem (together with our analysis of the Lie splitting in [63]). Notably, our analysis for the critical defocusing problem on the full space is even global in time. In the case of powers $\alpha \in (3, 5)$, the proven convergence rates are the "interpolated values" of the cases $\alpha = 3$ and $\alpha = 5$. For the torus case, corresponding error bounds for a full discretization using the Fourier pseudo-spectral method in space are also given. Finally, we discuss a numerical example indicating the sharpness of our theoretical results.

We do not treat a low-regularity integrator such as the corrected Lie splitting (which was also considered in our earlier work [63]) in this part of the thesis. The reason for this is that we were not able to show superior convergence behavior of the corrected Lie splitting compared to the Strang splitting in the 3D case with finite-energy data.

Therefore, we stick to the more classical Strang splitting for our analysis.

Compared to the existing literature on time discretizations of semilinear wave equations in low-regularity regimes, we do not impose a global Lipschitz assumption on the nonlinearity as in, e.g., [13, 49]. Moreover, in contrast to the works [10, 22] (and also our Part I), there is no uniform space-time L^∞-bound on the solution u available, since in three dimensions the Sobolev embedding $H^s \hookrightarrow L^\infty$ requires $s > 3/2$, but we only assume H^1 regularity of u.

A main ingredient in our error analysis are continuous- and discrete-time Strichartz estimates. It would be generally favorable to use exclusively the continuous-time Strichartz estimates as in Part I. However, in the case of higher powers $\alpha > 3$, Strichartz estimates in discrete time are needed to ensure the boundedness of the numerical approximations. Therefore, a longer chapter of this part is dedicated to the derivation of the needed estimates in discrete time. We first show them on the full space $\mathbb{R}^3$. The corresponding inequalities on the torus $\mathbb{T}^3$ are then deduced exploiting the finite propagation speed of the wave equation. To show error bounds of order greater than one in the cases $\alpha \in [3, 5)$, we also make use of the integration and summation by parts formulas to exploit cancellations in the error terms. Here we follow ideas from [10], though they have to be carefully adapted to fit to the Strichartz estimates (which were not needed in [10] due to higher regularity assumptions). We treat the subcritical and energy-critical cases separately. The latter one does not require the use of summation by parts since only a first-order error bound is shown. However, already this requires a much more delicate analysis than the first-order error estimates in the subcritical range.

Organization

Chapter 1 gives an overview on the first part of this thesis, which is concerned with low-regularity integrators for the one-dimensional semilinear Schrödinger and wave equations. The error analyses of the first and second-order low-regularity integrators for the Schrödinger equation are carried out in Chapter 2, which also contains some numerical experiments. The next Chapter 3 consists of the error analysis of the corrected Lie splitting applied to the wave equation.

The second part of this thesis is devoted to the three-dimensional semilinear wave equation. Chapter 4 introduces this topic and gives an overview on our results. In Chapter 5, we give a review on Strichartz estimates for the linear wave equation. We provide the proofs of the discrete-time inequalities and state all the Strichartz estimates needed for the forthcoming error analysis. In Chapter 6 we give a brief review on the wellposedness theory of the semilinear wave equation and deduce important estimates for its solution. The error analysis of the Strang splitting is then carried out in Chapter 7. In our last Chapter 8 we discuss a numerical experiment to illustrate our temporal error bounds.

General notations

We use the notation $A \lesssim_{\gamma_1,\ldots,\gamma_m} B$ if there is a constant $c > 0$ (depending on m quantities $\gamma_1, \ldots, \gamma_m$) such that $A \leq cB$, and the symbol $\gtrsim$ is similarly used. If $A \lesssim B$ and $A \gtrsim B$, then we write $A \eqsim B$. We denote by $\mathbb{N}$ the set of non-zero natural numbers and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The torus $\mathbb{T} \coloneqq \mathbb{R}/(2\pi\mathbb{Z})$ is understood as the interval $[-\pi, \pi]$ where one identifies the end-points $-\pi$ and π. Let Ω either be the d-dimensional torus $\mathbb{T}^d$ or a measurable subset of $\mathbb{R}^d$. We denote the standard Lebesgue spaces of p-integrable functions (depending on the context real- or complex-valued) by $L^p(\Omega)$ and the standard L^2-based Sobolev spaces of s-times weakly differentiable functions by $H^s(\Omega)$, where $p \in [1, \infty]$ and $s \in \mathbb{R}$. Occasionally we also make use of the L^p-based Sobolev spaces $W^{k,p}(\Omega)$ for $k \in \mathbb{N}_0$. If it is clear from the context, we often abbreviate $H^s = H^s(\Omega)$ as well as $L^p = L^p(\Omega)$ etc. For $p \in [1, \infty]$, a time interval J, and a Banach space X, we use the Bochner space $L^p(J, X)$ with norm

$$\|F\|_{L^p(J,X)} = \Big(\int_J \|F(t)\|_X^p \,\mathrm{d}t \Big)^{\frac{1}{p}},$$

and the usual modification for $p = \infty$. If a "free" variable t occurs in such a Bochner norm, the time integration is carried out with respect to t. For a step size $\tau > 0$ and a number $n \in \mathbb{N}_0$, the discrete times are usually denoted by $t_n \coloneqq n\tau$. Some additional notations for Part II are defined in Section 4.5.

Part I.

Improved error estimates for low-regularity integrators using space-time bounds

1. Overview

Due to their importance as model problems in mathematical physics, the nonlinear Schrödinger and wave equations have been intensively studied in the past decades, both analytically and numerically. In this part of the thesis we study their numerical time integration in the one-dimensional case with periodic boundary conditions. We treat the semilinear Schrödinger equation

$$\begin{aligned} \mathrm{i}\partial_t u + \partial_x^2 u &= \mu |u|^2 u, \quad (t,x) \in [0,T] \times \mathbb{T}, \\ u(0) &= u_0 \in H^1(\mathbb{T}), \end{aligned} \tag{1.1}$$

where we allow for both signs $\mu \in \{\pm 1\}$. Our second problem is the semilinear wave equation

$$\begin{aligned} \partial_t^2 u - \partial_x^2 u &= g(u), \quad (t,x) \in [0,T] \times \mathbb{T}, \\ u(0) &= u_0 \in H^1(\mathbb{T}), \\ \partial_t u(0) &= v_0 \in L^2(\mathbb{T}), \end{aligned} \tag{1.2}$$

with a general nonlinearity $g \in C^2(\mathbb{R}, \mathbb{R})$. Our regularity assumptions on the initial data are natural in view of the energy conservation laws. In the case of the wave equation, we require u_0 and v_0 to be real-valued.

1.1. The Schrödinger case

In the seminal paper [55], a low-regularity integrator was proposed for the time integration of the nonlinear Schrödinger equation (1.1) (and also its higher-dimensional versions). The scheme computes approximations $u_n \approx u(n\tau)$ via

$$u_{n+1} = \Phi_\tau(u_n) \coloneqq e^{\mathrm{i}\tau\partial_x^2}\Big(u_n - \mathrm{i}\tau\mu(u_n)^2 \varphi_1(-2\mathrm{i}\tau\partial_x^2)\bar{u}_n\Big). \tag{1.3}$$

The operator $\varphi_1(-2\mathrm{i}\tau\partial_x^2)$ can be defined in Fourier space or using the functional calculus for $\varphi_1(z) = (e^z - 1)/z$. For our purposes, the definition via the integral representation

$$\varphi_1(-2\mathrm{i}\tau\partial_x^2) f \coloneqq \frac{1}{\tau}\int_0^\tau e^{-2\mathrm{i}s\partial_x^2} f \,\mathrm{d}s \tag{1.4}$$

for $f \in L^2(\mathbb{T})$ is convenient. The authors in [55] proved a general convergence result which, in the one-dimensional case, reads as follows.

Theorem 1.1 ([55])**.** *Let $r > 1/2$ and $\gamma \in (0,1]$. Assume that the solution u to (1.1) satisfies $u(t) \in H^{r+\gamma}(\mathbb{T})$ for all $t \in [0,T]$. Then there are a constant $C > 0$ and a* *maximum step size* $\tau_0 > 0$ *such that the approximations* u_n *obtained by* (1.3) *satisfy the error bound*

$$\|u(n\tau) - u_n\|_{H^r(\mathbb{T})} \leq C\tau^\gamma$$

for all $\tau \in (0, \tau_0]$ *and* $n \in \mathbb{N}_0$ *with* $n\tau \leq T$. *The numbers* C *and* τ_0 *only depend on* T *and* $\|u\|_{L^\infty([0,T],H^{r+\gamma}(\mathbb{T}))}$.

Note that Theorem 1.1 asserts that we need $u_0 \in H^{r+1}$ in order to obtain first-order convergence in H^r. This is in contrast to more classical schemes such as splitting methods, where first-order convergence in H^r would require the stronger condition $u_0 \in H^{r+2}$, cf. [18, 50]. We will later make use of Theorem 1.1 since it provides an a-priori bound in L^∞ for the numerical solution u_n if τ is small enough. The condition $r > 1/2$ in Theorem 1.1 arises from the use of the algebra property of the Sobolev space $H^r(\mathbb{T})$. The space $L^2(\mathbb{T})$ does not have this property, so it is a natural question if Theorem 1.1 still holds if $r = 0$ and $\gamma = 1$. This was addressed in the follow-up work [53], where the problem (1.1) was considered on the spatial domain $\mathbb{R}^d$ for $d \in \{1, 2, 3\}$. The core difficulty is that the local error of the scheme (1.3) is roughly of the form

$$\tau^2 |\partial_x u|^2 u,$$

cf. p. 731 of [53]. This term can be estimated in L^2 for fixed times provided that $u(t) \in W^{1,4}$, which is not covered by the assumption $u_0 \in H^1$. It is however known that solutions to dispersive equations such as (1.1) enjoy better integrability properties in space if we also involve integration in the time variable. This is formalized using *Strichartz estimates*, which control mixed space-time $L^p L^q$ norms of solutions to linear dispersive equations in terms of the data, cf. Chapter 2.3 of [71]. In [53], the authors proved discrete-time Strichartz estimates and used them to show fractional convergence rates (strictly between 1/2 and 1 depending on the dimension) in L^2 for a frequency-filtered variant of (1.3). In the case $d = 1$, the convergence rate was 5/6. In the subsequent paper [54], the authors analyzed the problem (1.1) on the torus $\mathbb{T}$. They introduced *discrete Bourgain spaces* and used them to prove a convergence rate of almost 7/8 for a significantly refined frequency-filtered variant of (1.3).

In [53, 54], the optimal first-order convergence could not be reached since the discrete-time Strichartz and Bourgain space estimates only hold for frequency localized functions and contain a multiplicative loss depending on $K\tau^{1/2}$, where K is the largest frequency and τ denotes the time step-size. The continuous-time Strichartz estimates however do not suffer from these disadvantages.

In this work we extend Theorem 1.1 to the case $r = 0$ and $\gamma = 1$ with optimal first-order convergence. In contrast to [53, 54], we do not use frequency filtering and discrete-time Strichartz or Bourgain space estimates. Instead, we derive an error representation which allows us to apply the continuous-time periodic Strichartz estimate

$$\|e^{\mathrm{i}t\partial_x^2} f\|_{L^4([0,T]\times\mathbb{T})} \lesssim_T \|f\|_{L^2(\mathbb{T})}. \tag{1.5}$$

A proof of (1.5) can be found in Theorem 1 and the subsequent remark of [74] or Proposition 2.1 of [7]. The idea to use continuous-time Strichartz estimates to control the local error goes back to [36].

We also give error bounds for the second-order variant of (1.3), which was originally proposed in [9, 56, 59]. This scheme is given by

$$u_{n+1} = \tilde{\Phi}_\tau(u_n) \coloneqq e^{\mathrm{i}\tau\partial_x^2}\Big(u_n - \mathrm{i}\tau\mu(u_n)^2[\varphi_1(-2\mathrm{i}\tau\partial_x^2) - \varphi_1'(-2\mathrm{i}\tau\partial_x^2)]\bar{u}_n\Big) \\ - \mathrm{i}\tau\mu(e^{\mathrm{i}\tau\partial_x^2}u_n)^2 e^{\mathrm{i}\tau\partial_x^2}\varphi_1'(-2\mathrm{i}\tau\partial_x^2)\bar{u}_n - \frac{\tau^2}{2}e^{\mathrm{i}\tau\partial_x^2}(|u_n|^4 u_n). \tag{1.6}$$

The operator $\varphi_1'(-2\mathrm{i}\tau\partial_x^2)$ is defined by

$$\varphi_1'(-2\mathrm{i}\tau\partial_x^2)f \coloneqq \frac{1}{\tau^2}\int_0^\tau s e^{-2\mathrm{i}s\partial_x^2} f \,\mathrm{d}s, \tag{1.7}$$

or alternatively using the functional calculus for $\varphi_1'(z) = (ze^z - e^z + 1)/z^2$.

Our convergence result in L^2 for H^1-solutions reads as follows. Its proof is carried out in Section 2.1 for the first-order scheme and in Section 2.2 for the second-order scheme, respectively.

Theorem 1.2. *Assume that the solution u to* (1.1) *satisfies $u(t) \in H^1(\mathbb{T})$ for all $t \in [0, T]$. Then there are a constant $C > 0$ and a maximum step size $\tau_0 > 0$ such that the approximations u_n obtained by* (1.3) *or* (1.6) *satisfy the error bound*

$$\|u(n\tau) - u_n\|_{L^2(\mathbb{T})} \le C\tau$$

for all $\tau \in (0, \tau_0]$ and $n \in \mathbb{N}_0$ with $n\tau \le T$. The numbers C and τ_0 only depend on T and $\|u\|_{L^\infty([0,T],H^1(\mathbb{T}))}$.

To obtain second-order convergence, we need to impose higher regularity assumptions than just $u_0 \in H^1(\mathbb{T})$. In [56], it was shown that for any $r > 1/2$, the scheme (1.6) is second-order convergent in H^r for solutions with $u_0 \in H^{r+2}$. The work [59] established second-order convergence of (1.6) in L^2 under the assumption $u_0 \in H^{9/4}(\mathbb{T})$. In Section 2.2, we prove the following result, which asserts that H^2-regularity is enough to obtain second-order convergence in L^2.

Theorem 1.3. *Assume that the solution u to* (1.1) *satisfies $u(t) \in H^2(\mathbb{T})$ for all $t \in [0, T]$. Then there are a constant $C > 0$ and a maximum step size $\tau_0 > 0$ such that the approximations u_n obtained by* (1.6) *satisfy the error bound*

$$\|u(n\tau) - u_n\|_{L^2(\mathbb{T})} \le C\tau^2$$

for all $\tau \in (0, \tau_0]$ and $n \in \mathbb{N}_0$ with $n\tau \le T$. The numbers C and τ_0 only depend on T and $\|u\|_{L^\infty([0,T],H^2(\mathbb{T}))}$.

Numerically, the first-order convergence of (1.3) for H^1-solutions, as well as the second-order convergence of (1.6) for H^2-solutions, are not clearly visible. On the other hand, the first-order convergence of the second-order scheme for H^1-solution shows up in the numerical experiment as expected from Theorem 1.2. We elaborate on this issue in Section 2.3.

Remark 1.4. We comment on possible extensions of Theorems 1.2 and 1.3 to higher dimensions. The embedding $H^1 \hookrightarrow L^\infty$ as well as the estimate (1.5), which are both crucially exploited in the proof of Theorem 1.2, are then wrong, in general. In two dimensions, they however both only require an arbitrary small amount of extra regularity (see Proposition 3.6 of [7] for the 2D version of (1.5)). Therefore, it is possible to extend Theorem 1.2 to the 2D case under the slightly stronger regularity assumption $u_0 \in H^{1+\varepsilon}$ for some $\varepsilon > 0$. One could also stick to the H^1 assumption if one considers a suitably filtered variant of (1.3) or (1.6) and lowers the convergence rate by ε. Similarly, Theorem 1.3 extends to the two-dimensional case under the assumption $u_0 \in H^{2+\varepsilon}$. The three-dimensional cases seem to be more difficult and we do not know how the optimal result then looks like. The situation becomes easier if the torus $\mathbb{T}^d$ is replaced by the full space $\mathbb{R}^d$, since then a wider range of Strichartz estimates becomes applicable, cf. Chapter 2.3 of [71]. By combing our techniques with those of [53], it seems feasible to show almost first-order convergence in L^2 for a frequency-filtered version of (1.3) under the assumption $u_0 \in H^1(\mathbb{R}^3)$, for instance.

Remark 1.5. It is also possible to extend our analysis to the symmetrized two-step variants of (1.3) and (1.6) that were recently proposed in [20].

1.2. The wave case

For the nonlinear wave equation (1.2), the authors in [49] proposed a low-regularity integrator which was called the *corrected Lie splitting*. It computes approximations $(u_n, v_n) \approx (u(n\tau), \partial_t u(n\tau))$ via

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = e^{\tau A}\left[\begin{pmatrix} u_n \\ v_n \end{pmatrix} + \tau \begin{pmatrix} 0 \\ g(u_n) \end{pmatrix} + \tau^2 \varphi_2(-2\tau A) \begin{pmatrix} -g(u_n) \\ g'(u_n) v_n \end{pmatrix} \right], \tag{1.8}$$

with wave operator $A(u, v) = (v, \partial_x^2 u)$. The operator $\varphi_2(-2\tau A)$ is defined by the integral representation

$$\varphi_2(-2\tau A) w := \frac{1}{\tau^2} \int_0^\tau (\tau - s) e^{-2sA} w \, \mathrm{d}s \tag{1.9}$$

for $w \in H^1 \times L^2$. Similar as in the Schrödinger case, one could equivalently use the functional calculus for $\varphi_2(z) = (e^z - z - 1)/z^2$. In [49], under a Lipschitz condition on the nonlinearity g, it was shown that the scheme (1.8) converges with order 2 in $H^1 \times L^2$ under the regularity assumption $(u_0, v_0) \in H^{1+d/4} \times H^{d/4}$ for spatial dimensions $d \in \{1, 2, 3\}$. This is an improvement compared to classical trigonometric or exponential integrators, where second-order convergence in $H^1 \times L^2$ would require the stronger condition $(u_0, v_0) \in H^2 \times H^1$, cf. [10, 22]. The reason for the particular regularity requirement $(u_0, v_0) \in H^{1+d/4} \times H^{d/4}$ in [49] is that the main part of the local error is roughly of the form

$$\|(\partial_t u)^2 - \nabla u \cdot \nabla u\|_{L^2(\mathbb{T}^d)}, \tag{1.10}$$

cf. equation (2.26) of [49]. This term was then estimated (for fixed times) using the triangle inequality and the Sobolev embedding $H^{d/4}(\mathbb{T}^d) \hookrightarrow L^4(\mathbb{T}^d)$. For the one-dimensional case $d = 1$, the authors in [49] also gave a convergence result under the weaker regularity assumption $(u_0, v_0) \in H^1 \times L^2$. Using an interpolation argument, it was shown that the scheme (1.8) converges almost with order 4/3 in $H^1 \times L^2$. However, the numerical experiments in [49] suggested that the convergence is of order 2 also in this case.

Here, we give a rigorous proof of this second-order convergence. In contrast to the Schrödinger case, the 1D wave equation does not exhibit dispersive behavior. Instead, the idea is to exploit that the expression (1.10) contains a so-called *null form* which allows for improved space-time bounds compared to the above fixed-time approach. Such null form estimates are widely used in the analysis of nonlinear wave equations, cf. [48] or Chapter 6 of [71]. They rely on cancellation of parallel interactions (where waves move together) in the bilinear expression in (1.10). In the one-dimensional case one has the following estimate. If ϕ solves the linear inhomogeneous wave equation $\partial_t^2 \phi - \partial_x^2 \phi = F$ on $[0, T] \times \mathbb{T}$, then one has the inequality

$$\|(\partial_t \phi)^2 - (\partial_x \phi)^2\|_{L^2([0,T]\times\mathbb{T})} \lesssim_T \|\partial_x \phi(0)\|^2_{L^2(\mathbb{T})} + \|\partial_t \phi(0)\|^2_{L^2(\mathbb{T})} + \|F\|^2_{L^1([0,T],L^2(\mathbb{T}))}. \quad (1.11)$$

Note that the right-hand side of (1.11) only contains the L^2 norm of $\partial_{t,x}\phi(0)$ instead of the L^4 norm that would result from the triangle inequality approach. If we replace $\mathbb{T}$ with $\mathbb{R}$, estimate (1.11) can be found in (1.8) of [48] (in a simplified form) or in (6.29) of [71]. For convenience, we give a direct proof of (1.11) on $\mathbb{T}$ based on d'Alembert's formula in Section 3.1.

With the help of estimate (1.11), we are able to show the following improved error bound for the corrected Lie splitting (1.8). The proof is given in Section 3.2. To our knowledge, this is the first time that a null form estimate like (1.11) is used in numerical analysis.

Theorem 1.6. *Assume that the solution u to* (1.2) *satisfies* $(u(t), \partial_t u(t)) \in H^1(\mathbb{T}) \times L^2(\mathbb{T})$ *for all $t \in [0, T]$. Then there are a constant $C > 0$ and a maximum step size $\tau_0 > 0$ such that the approximations (u_n, v_n) obtained by* (1.8) *satisfy the error bound*

$$\|u(n\tau) - u_n\|_{H^1(\mathbb{T})} + \|\partial_t u(n\tau) - v_n\|_{L^2(\mathbb{T})} \le C\tau^2$$

for all $\tau \in (0, \tau_0]$ and $n \in \mathbb{N}_0$ with $n\tau \le T$. The numbers C and τ_0 only depend on g, T, $\|u\|_{L^\infty([0,T],H^1(\mathbb{T}))}$, and $\|\partial_t u\|_{L^\infty([0,T],L^2(\mathbb{T}))}$.

For a numerical example concerning the wave equation (1.2), we refer to Figure 1 of [49]. It shows second-order convergence in $H^1 \times L^2$ of the corrected Lie splitting (1.8) as predicted by our Theorem 1.6. Comparisons with other schemes are also provided.

Remark 1.7. The higher-dimensional versions of the null form estimate (1.11) require more regularity, cf. [48] or inequality (6.29) of [71]. In two dimensions, they could possibly still be used to show an analogue of Theorem 1.6 with a convergence rate greater than one under a suitable growth condition on g. Very recently, convergence rates for a Strang splitting scheme for the 3D semilinear wave equation with power nonlinearity under the assumption $(u_0, v_0) \in H^1 \times L^2$ were obtained in [61], see Part II. We do not know whether in this situation it is possible to show higher rates by using a low-regularity integrator instead.

2. The Schrödinger case

2.1. Error analysis of the first-order low-regularity integrator

In this section we prove Theorem 1.2 in the case of the first-order scheme (1.3). We first state the main hypothesis of this section and then convert the linear Strichartz estimate (1.5) into a bound for the solution u to the nonlinear problem (1.1). In the following we frequently use that for $s > 1/2$, the Sobolev space $H^s(\mathbb{T})$ forms an algebra and embeds into $L^\infty(\mathbb{T})$, cf. Lemma A.2 and Theorem A.1.

Assumption 2.1. There exists a time $T > 0$ and a solution $u \in C([0,T], H^1) \cap C^1([0,T], H^{-1})$ to the nonlinear Schrödinger equation (1.1) with bound

$$M_1 \coloneqq \|u\|_{L^\infty([0,T],H^1)}.$$

Remark 2.2. The equation (1.1) is actually globally wellposed for any initial data $u_0 \in L^2(\mathbb{T})$ and the time T in Assumption 2.1 can be taken arbitrarily large, cf. Theorem 4.45 of [7].

Proposition 2.3. *Let u, T, and M_1 be given by Assumption 2.1. Then we have the estimate*

$$\|\partial_x u\|_{L^4([0,T]\times\mathbb{T})} \lesssim_{M_1,T} 1.$$

Proof. We apply inequality (1.5) to Duhamel's formula

$$u(t) = e^{\mathrm{i}t\partial_x^2} u_0 - \mathrm{i}\mu \int_0^t e^{\mathrm{i}(t-s)\partial_x^2} (|u|^2 u)(s)\,\mathrm{d}s.$$

Using also Minkowski's inequality and Sobolev's embedding $H^1 \hookrightarrow L^\infty$, we get

$$\begin{aligned}\|\partial_x u\|_{L^4([0,T]\times\mathbb{T})} &\lesssim_T \|\partial_x u_0\|_{L^2} + \int_0^T \|\partial_x(|u|^2 u)(s)\|_{L^2}\,\mathrm{d}s \\ &\lesssim_{M_1} 1 + \|u\|^2_{L^2([0,T],L^\infty)} \|\partial_x u\|_{L^\infty([0,T],L^2)} \lesssim_{M_1,T} 1. \qquad \square\end{aligned}$$

We now give a representation of the local error of the low-regularity integrator (1.3). The calculations are inspired by the ones in Section 3 of [59]. But compared to there and also [55], we do not insert the approximation $u(s) \approx e^{\mathrm{i}s\partial_x^2} u_0$ at first. This makes it easier for us to apply Proposition 2.3 in the subsequent Lemma 2.5.

Lemma 2.4. *Let u and T be given by Assumption 2.1. Then, the local error of* (1.3) *is given by*

$$u(\tau) - u_1 = \mu \int_0^\tau \int_0^s e^{\mathrm{i}(\tau-\sigma)\partial_x^2} D(\sigma, s)\,\mathrm{d}\sigma\,\mathrm{d}s,$$

for $\tau \in (0, T]$. *Here we define*

$$D(\sigma, s) = D_1(\sigma, s) + D_2(\sigma, s) + D_3(\sigma, s)$$

with

$$\begin{aligned}
D_1(\sigma, s) &:= \mu u(\sigma)^2 \Big(e^{2\mathrm{i}(\sigma-s)\partial_x^2} (|u|^2 \bar{u})(\sigma) - 2|u(\sigma)|^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2} \bar{u}(\sigma) \Big), \\
D_2(\sigma, s) &:= -2(\partial_x u(\sigma))^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2} \bar{u}(\sigma), \\
D_3(\sigma, s) &:= -4u(\sigma)\partial_x u(\sigma) e^{2\mathrm{i}(\sigma-s)\partial_x^2} \partial_x \bar{u}(\sigma),
\end{aligned}$$

for $0 \le \sigma \le s \le T$.

Proof. By (1.4), we have

$$\tau \varphi_1(-2\mathrm{i}\tau\partial_x^2)\bar{u}_0 = \int_0^\tau e^{-2\mathrm{i}s\partial_x^2} \bar{u}_0 \, \mathrm{d}s.$$

Duhamel's formula, (1.3), and the fundamental theorem of calculus thus imply

$$\begin{aligned}
u(\tau) - u_1 &= -\mathrm{i}\mu e^{\mathrm{i}\tau\partial_x^2} \int_0^\tau \Big(e^{-\mathrm{i}s\partial_x^2} (u^2\bar{u})(s) - u_0^2 e^{-2\mathrm{i}s\partial_x^2} \bar{u}_0 \Big) \, \mathrm{d}s \qquad (2.1) \\
&= \mu e^{\mathrm{i}\tau\partial_x^2} \int_0^\tau (N(s,s) - N(0,s)) \, \mathrm{d}s = \mu e^{\mathrm{i}\tau\partial_x^2} \int_0^\tau \int_0^s \partial_1 N(\sigma, s) \, \mathrm{d}\sigma \, \mathrm{d}s.
\end{aligned}$$

Here, the function $N(\cdot, s) \in C^1([0,\tau], H^{-1}(\mathbb{T}))$ is defined as

$$N(\sigma, s) := -\mathrm{i}e^{-\mathrm{i}\sigma\partial_x^2} \Big(u(\sigma)^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2} \bar{u}(\sigma) \Big). \qquad (2.2)$$

Using the product rule and the differential equation (1.1), we compute the derivative as

$$\begin{aligned}
\partial_1 N(\sigma, s) &= e^{-\mathrm{i}\sigma\partial_x^2} \Big[- \partial_x^2 \Big(u(\sigma)^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2} \bar{u}(\sigma) \Big) - 2\mathrm{i}u(\sigma)\partial_t u(\sigma) e^{2\mathrm{i}(\sigma-s)\partial_x^2} \bar{u}(\sigma) \\
&\quad + 2u(\sigma)^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2} \partial_x^2 \bar{u}(\sigma) - \mathrm{i}u(\sigma)^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2} \partial_t \bar{u}(\sigma) \Big] \\
&= e^{-\mathrm{i}\sigma\partial_x^2} \Big[- 2\partial_x^2 u(\sigma) u(\sigma) e^{2\mathrm{i}(\sigma-s)\partial_x^2} \bar{u}(\sigma) - 2(\partial_x u(\sigma))^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2} \bar{u}(\sigma) \\
&\quad - 4u(\sigma)\partial_x u(\sigma) e^{2\mathrm{i}(\sigma-s)\partial_x^2} \partial_x \bar{u}(\sigma) - u(\sigma)^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2} \partial_x^2 \bar{u}(\sigma) \\
&\quad + 2u(\sigma)\partial_x^2 u(\sigma) e^{2\mathrm{i}(\sigma-s)\partial_x^2} \bar{u}(\sigma) - 2\mu u(\sigma)(|u|^2 u)(\sigma) e^{2\mathrm{i}(\sigma-s)\partial_x^2} \bar{u}(\sigma) \\
&\quad + 2u(\sigma)^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2} \partial_x^2 \bar{u}(\sigma) - u(\sigma)^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2} \partial_x^2 \bar{u}(\sigma) \\
&\quad + \mu u(\sigma)^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2} (|u|^2 \bar{u})(\sigma) \Big] \\
&= e^{-\mathrm{i}\sigma\partial_x^2} \Big[- 2(\partial_x u(\sigma))^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2} \bar{u}(\sigma) - 4u(\sigma)\partial_x u(\sigma) e^{2\mathrm{i}(\sigma-s)\partial_x^2} \partial_x \bar{u}(\sigma) \\
&\quad + \mu u(\sigma)^2 \Big(- 2|u(\sigma)|^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2} \bar{u}(\sigma) + e^{2\mathrm{i}(\sigma-s)\partial_x^2} (|u|^2 \bar{u})(\sigma) \Big) \Big] \\
&= e^{-\mathrm{i}\sigma\partial_x^2} \Big[D_1(\sigma, s) + D_2(\sigma, s) + D_3(\sigma, s) \Big], \qquad (2.3)
\end{aligned}$$

where we exploit the cancellation of all second-order partial derivatives. The derivative is well-defined in $H^{-1}(\mathbb{T})$ since in 1D we can use the embedding $L^1 \hookrightarrow H^{-1}$ and that the multiplication by an H^1 function is a continuous operator on H^{-1}. □

In the next step we bound the sum of the local errors terms, where we will crucially exploit Proposition 2.3 as well as the dual of estimate (1.5).

Lemma 2.5. *Let u, T, and M_1 be given by Assumption 2.1. Then we can bound the sum of local errors of* (1.3) *by*

$$\Big\| \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k-1)\tau\partial_x^2}\Big(u(t_{k+1}) - \Phi_\tau(u(t_k))\Big)\Big\|_{L^2} \lesssim_{M_1,T} \tau,$$

for all $\tau \in (0,T]$ and $n \in \mathbb{N}_0$ with $n\tau \le T$.

Proof. Lemma 2.4 with $u(t_k + \cdot)$ instead of u yields

$$\sum_{k=0}^{n-1} e^{\mathrm{i}(n-k-1)\tau\partial_x^2}\Big(u(t_{k+1}) - \Phi_\tau(u(t_k))\Big) = \mu \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2} \int_0^\tau \int_0^s e^{-\mathrm{i}\sigma\partial_x^2} D(t_k+\sigma, t_k+s)\,\mathrm{d}\sigma\,\mathrm{d}s.$$

We now use the decomposition $D = D_1 + D_2 + D_3$ from Lemma 2.4. For the first term we even obtain

$$\Big\| \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2} \int_0^\tau \int_0^s e^{-\mathrm{i}\sigma\partial_x^2} D_1(t_k+\sigma, t_k+s)\,\mathrm{d}\sigma\,\mathrm{d}s\Big\|_{H^1} \lesssim_{M_1,T} n\tau^2 \lesssim_T \tau,$$

using the algebra property of H^1 in 1D. The second term is controlled via

$$\begin{aligned}
&\Big\| \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2} \int_0^\tau \int_0^s e^{-\mathrm{i}\sigma\partial_x^2} D_2(t_k+\sigma, t_k+s)\,\mathrm{d}\sigma\,\mathrm{d}s\Big\|_{L^2} \\
&\quad \le \sum_{k=0}^{n-1} \int_0^\tau \int_0^s \|D_2(t_k+\sigma, t_k+s)\|_{L^2}\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\quad \lesssim \sum_{k=0}^{n-1} \int_0^\tau \int_0^\tau \|(\partial_x u(t_k+\sigma))^2\|_{L^2} \|e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar{u}(t_k+\sigma)\|_{L^\infty}\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\quad \lesssim \tau \sum_{k=0}^{n-1} \int_0^\tau \|\partial_x u(t_k+\sigma)\|_{L^4}^2 \|u(t_k+\sigma)\|_{H^1}\,\mathrm{d}\sigma \lesssim_{M_1} \tau \|\partial_x u\|_{L^2([0,T],L^4)}^2 \\
&\quad \lesssim_T \tau \|\partial_x u\|_{L^4([0,T]\times\mathbb{T})}^2 \lesssim_{M_1,T} \tau,
\end{aligned}$$

by means of Hölder's inequality, the Sobolev embedding $H^1 \hookrightarrow L^\infty$, and Proposition 2.3. The term involving D_3 is first rewritten as

$$\sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2} \int_0^\tau \int_0^s e^{-\mathrm{i}\sigma\partial_x^2} D_3(t_k+\sigma, t_k+s)\,\mathrm{d}\sigma\,\mathrm{d}s = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^s e^{\mathrm{i}(n\tau-\sigma)\partial_x^2} D_3(\sigma, s)\,\mathrm{d}\sigma\,\mathrm{d}s$$

$$= \int_0^{n\tau} e^{\mathrm{i}(n\tau-\sigma)\partial_x^2} \int_\sigma^{\lceil \frac{\sigma}{\tau} \rceil \tau} D_3(\sigma, s) \, \mathrm{d}s \, \mathrm{d}\sigma,$$

where the application of Fubini's theorem is justified since the double integral converges absolutely in H^{-1}. We next apply the dual of the periodic Strichartz estimate (1.5), which reads

$$\Big\| \int_0^T e^{-\mathrm{i}t\partial_x^2} F(t) \, \mathrm{d}t \Big\|_{L^2} \lesssim_T \|F\|_{L^{\frac{4}{3}}([0,T]\times\mathbb{T})}. \tag{2.4}$$

Using Hölder's inequality with $\frac{3}{4} = \frac{1}{\infty} + \frac{1}{4} + \frac{1}{2}$, as above we infer that

$$\begin{aligned}
&\Big\| \int_0^{n\tau} e^{\mathrm{i}(n\tau-\sigma)\partial_x^2} \int_\sigma^{\lceil \frac{\sigma}{\tau} \rceil \tau} D_3(\sigma, s) \, \mathrm{d}s \, \mathrm{d}\sigma \Big\|_{L^2} \\
&\quad \lesssim_T \Big\| \sigma \mapsto \int_\sigma^{\lceil \frac{\sigma}{\tau} \rceil \tau} D_3(\sigma, s) \, \mathrm{d}s \Big\|_{L^{\frac{4}{3}}([0,T]\times\mathbb{T})} \\
&\quad \lesssim \Big\| \sigma \mapsto \|u(\sigma)\|_{L^\infty} \|\partial_x u(\sigma)\|_{L^4} \int_\sigma^{\lceil \frac{\sigma}{\tau} \rceil \tau} \|e^{2\mathrm{i}(\sigma-s)\partial_x^2} \partial_x \bar{u}(\sigma)\|_{L^2} \, \mathrm{d}s \Big\|_{L^{\frac{4}{3}}([0,T])} \\
&\quad \lesssim_{M_1} \tau \|\partial_x u\|_{L^{\frac{4}{3}}([0,T],L^4)} \lesssim_T \tau \|\partial_x u\|_{L^4([0,T]\times\mathbb{T})} \lesssim_{M_1,T} \tau.
\end{aligned}$$

□

We can now finish the proof of the global error bound in a classical way with the help of the discrete Gronwall lemma.

Proof of Theorem 1.2 for u_n given by (1.3). The error

$$e_n \coloneqq u(t_n) - u_n$$

satisfies the recursion formula

$$\begin{aligned}
e_{n+1} &= u(t_{n+1}) - \Phi_\tau(u(t_n)) + \Phi_\tau(u(t_n)) - \Phi_\tau(u_n) \\
&= u(t_{n+1}) - \Phi_\tau(u(t_n)) + e^{\mathrm{i}\tau\partial_x^2} e_n \\
&\quad - \mathrm{i}\tau\mu e^{\mathrm{i}\tau\partial_x^2} \Big((u(t_n))^2 \varphi_1(-2\mathrm{i}\tau\partial_x^2) \bar{u}(t_n) - (u_n)^2 \varphi_1(-2\mathrm{i}\tau\partial_x^2) \bar{u}_n \Big).
\end{aligned}$$

Lemma A.8 now implies that

$$\begin{aligned}
e_n &= \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k-1)\tau\partial_x^2} \Big(u(t_{k+1}) - \Phi_\tau(u(t_k)) \Big) \\
&\quad - \mathrm{i}\tau\mu \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2} \Big((u(t_k))^2 \varphi_1(-2\mathrm{i}\tau\partial_x^2) \bar{u}(t_k) - (u_k)^2 \varphi_1(-2\mathrm{i}\tau\partial_x^2) \bar{u}_k \Big),
\end{aligned}$$

exploiting that $e_0 = 0$. The term in brackets can be rewritten as

$$\begin{aligned}
&(u(t_k))^2 \varphi_1(-2\mathrm{i}\tau\partial_x^2) \bar{u}(t_k) - (u_k)^2 \varphi_1(-2\mathrm{i}\tau\partial_x^2) \bar{u}_k \\
&\qquad = (u(t_k))^2 \varphi_1(-2\mathrm{i}\tau\partial_x^2) \bar{e}_k + (u(t_k) + u_k) e_k \varphi_1(-2\mathrm{i}\tau\partial_x^2) \bar{u}_k.
\end{aligned}$$

Moreover, due to the definition (1.4), the operator $\varphi_1(-2\mathrm{i}\tau\partial_x^2)$ is clearly bounded uniformly in τ on all Sobolev spaces H^s with $s \geq 0$. From Lemma 2.5 and standard estimates we thus infer that

$$\|e_n\|_{L^2} \lesssim_{M_1,T} \tau + \tau \sum_{k=0}^{n-1} (1 + \|e_k\|_{H^{\frac{3}{4}}}^2)\|e_k\|_{L^2}$$

by means of the Sobolev embedding $H^{3/4} \hookrightarrow L^\infty$ and the representation $u_k = u(t_k) - e_k$. Theorem 1.1 with $r = 3/4$ and $\gamma = 1/4$ yields a time $\tau_0 > 0$ depending only on M_1 and T such that

$$\|e_n\|_{H^{\frac{3}{4}}} \leq 1$$

for all $\tau \in (0, \tau_0]$ and $n \in \mathbb{N}_0$ with $n\tau \leq T$. For such τ and n we thus derive that

$$\|e_n\|_{L^2} \lesssim_{M_1,T} \tau + \tau \sum_{k=0}^{n-1} \|e_k\|_{L^2}.$$

The discrete Gronwall inequality from Lemma A.9 then implies that

$$\|e_n\|_{L^2} \lesssim_{M_1,T} \tau.$$

□

2.2. Error analysis of the second-order low-regularity integrator

In this section, we prove Theorems 1.2 and 1.3 for the second-order low-regularity integrator (1.6). Our strategy is similar to that of Section 2.1. Compared to that, more error terms will appear, but the critical ones are roughly of the same structure as above. The proof of the first-order error bound under the H^1-assumption will only be sketched since it does not require new ideas. We first extend Proposition 2.3 to H^2-solutions.

Assumption 2.6. There exists a time $T > 0$ and a solution $u \in C([0,T], H^2) \cap C^1([0,T], L^2)$ to the nonlinear Schrödinger equation (1.1) with bound

$$M_2 := \|u\|_{L^\infty([0,T],H^2)}.$$

Remark 2.7. Similar as in Remark 2.2, the time T in Assumption 2.6 can actually be taken arbitrarily large.

Proposition 2.8. *Let u, T, and M_2 be given by Assumption 2.6. Then we have the estimate*

$$\|\partial_x^2 u\|_{L^4([0,T]\times\mathbb{T})} \lesssim_{M_2,T} 1.$$

Proof. Similar as in the proof of Proposition 2.3, we apply estimate (1.5) to Duhamel's formula and obtain

$$\|\partial_x^2 u\|_{L^4([0,T]\times\mathbb{T})} \lesssim_T \|\partial_x^2 u_0\|_{L^2} + \int_0^T \|\partial_x^2(|u|^2 u)(s)\|_{L^2}\,\mathrm{d}s$$

$$
\begin{aligned}
&\lesssim_{M_2} 1 + \|u\|^2_{L^2([0,T],L^\infty)} \|\partial_x^2 u\|_{L^\infty([0,T],L^2)} \\
&\quad + \|\partial_x u\|^2_{L^2([0,T],L^\infty)} \|u\|_{L^\infty([0,T],L^2)} \\
&\lesssim_{M_2,T} 1,
\end{aligned}
$$

exploiting also Minkowski's inequality and the Sobolev embedding $H^1 \hookrightarrow L^\infty$. □

The next lemma is the main reason why the scheme (1.6) is formally of second order.

Lemma 2.9. *Let X be a Banach space, $\tau > 0$, and $F \in C^2([0,\tau], X)$. We then have the representation*

$$
\begin{aligned}
F(s) - F(0) - \frac{s}{\tau}[F(\tau) - F(0)] &= \int_0^s F'(\sigma)\,\mathrm{d}\sigma - \frac{s}{\tau}\int_0^\tau F'(\sigma)\,\mathrm{d}\sigma \\
&= \int_0^s (s-\sigma) F''(\sigma)\,\mathrm{d}\sigma - \frac{s}{\tau}\int_0^\tau (\tau - \sigma) F''(\sigma)\,\mathrm{d}\sigma
\end{aligned}
$$

for all $s \in [0,\tau]$.

Proof. The fundamental theorem of calculus and Fubini's theorem yield

$$
\begin{aligned}
F(s) - F(0) - \frac{s}{\tau}[F(\tau) - F(0)] &= \int_0^s F'(\theta)\,\mathrm{d}\theta - \frac{s}{\tau}\int_0^\tau F'(\theta)\,\mathrm{d}\theta \\
&= \int_0^s [F'(\theta) - F'(0)]\,\mathrm{d}\theta - \frac{s}{\tau}\int_0^\tau [F'(\theta) - F'(0)]\,\mathrm{d}\theta \\
&= \int_0^s \int_0^\theta F''(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\theta - \frac{s}{\tau}\int_0^\tau \int_0^\theta F''(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\theta \\
&= \int_0^s (s-\sigma) F''(\sigma)\,\mathrm{d}\sigma - \frac{s}{\tau}\int_0^\tau (\tau-\sigma) F''(\sigma)\,\mathrm{d}\sigma.
\end{aligned}
$$ □

We can now compute the local error representation of (1.6).

Lemma 2.10. *Let u and T be given by Assumption 2.6. Then, the local error of* (1.6) *is given by*

$$
\begin{aligned}
u(\tau) - u_1 = \mu &\int_0^\tau \int_0^s (s-\sigma) e^{\mathrm{i}(\tau-\sigma)\partial_x^2} \tilde{D}(\sigma, s)\,\mathrm{d}\sigma\,\mathrm{d}s \\
+ \mu &\int_0^\tau \int_0^\tau \frac{s}{\tau}(\tau-\sigma) e^{\mathrm{i}(\tau-\sigma)\partial_x^2} \tilde{D}(\sigma,s)\,\mathrm{d}\sigma\,\mathrm{d}s + R(u,\tau),
\end{aligned}
$$

for $\tau \in (0,T]$. Here we define

$$
\tilde{D}(\sigma,s) = \tilde{D}_1(\sigma,s) + \tilde{D}_2(\sigma,s) + \tilde{D}_3(\sigma,s)
$$

for the terms

$$
\begin{aligned}
\tilde{D}_1(\sigma,s) &:= 4\mathrm{i}(\partial_x^2 u(\sigma))^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2} \bar{u}(\sigma), \\
\tilde{D}_2(\sigma,s) &:= 8\mathrm{i} u(\sigma) \partial_x^2 u(\sigma) e^{2\mathrm{i}(\sigma-s)\partial_x^2} \partial_x^2 \bar{u}(\sigma),
\end{aligned}
$$

$$\begin{aligned}\tilde{D}_3(\sigma,s) &\coloneqq [-\mathrm{i}\partial_x^2+\partial_1]D_1(\sigma,s)+6\mathrm{i}(\partial_x u(\sigma))^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x^2\bar{u}(\sigma)\\ &\quad+20\mathrm{i}\partial_x u(\sigma)\partial_x^2 u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar{u}(\sigma)+4\mathrm{i}\mu\partial_x u(\sigma)\partial_x(|u|^2u)(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar{u}(\sigma)\\ &\quad-2(\partial_x u(\sigma))^2e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_t\bar{u}(\sigma)-4\partial_t u(\sigma)\partial_x u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar{u}(\sigma)\\ &\quad+4\mathrm{i}\mu u(\sigma)\Big(\partial_x(|u|^2u)(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar{u}(\sigma)-\partial_x u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x(|u|^2\bar{u})(\sigma)\Big),\end{aligned}$$

with D_1 from Lemma 2.4. Moreover, the remainder $R(u,\tau)$ is given by

$$\begin{aligned}R(u,\tau) &\coloneqq -\mathrm{i}\mu\tau u(\tau)^2 e^{2\mathrm{i}\tau\partial_x^2}\varphi_1'(-2\mathrm{i}\tau\partial_x^2)\bar{u}(\tau)\\ &\quad+i\mu\tau(e^{\mathrm{i}\tau\partial_x^2}u_0)^2e^{\mathrm{i}\tau\partial_x^2}\varphi_1'(-2\mathrm{i}\tau\partial_x^2)\bar{u}_0+\frac{\tau^2}{2}e^{\mathrm{i}\tau\partial_x^2}(|u_0|^4u_0).\end{aligned}$$

If only Assumption 2.1 is satisfied, we can still write

$$u(\tau)-u_1=\mu\int_0^\tau\int_0^s e^{\mathrm{i}(\tau-\sigma)\partial_x^2}D(\sigma,s)\,\mathrm{d}\sigma\,\mathrm{d}s+\mu\int_0^\tau\int_0^\tau\frac{s}{\tau}e^{\mathrm{i}(\tau-\sigma)\partial_x^2}D(\sigma,s)\,\mathrm{d}\sigma\,\mathrm{d}s+R(u,\tau) \tag{2.5}$$

with the expression $D(s,\sigma)$ given by Lemma 2.4.

Proof. By the expression (1.7), we have

$$\tau\varphi_1'(-2\mathrm{i}\tau\partial_x^2)f=\int_0^\tau\frac{s}{\tau}e^{-2\mathrm{i}s\partial_x^2}f\,\mathrm{d}s$$

for all $f\in L^2$. Define u_1 by the second-order scheme (1.6). Using formula (2.1), the definition of N from (2.2), and Lemma 2.9, we then compute

$$\begin{aligned}u(\tau)-u_1 &= \mu e^{\mathrm{i}\tau\partial_x^2}\int_0^\tau\Big(N(s,s)-N(0,s)-\frac{s}{\tau}[N(\tau,s)-N(0,s)]\Big)\,\mathrm{d}s+R(u,\tau)\\ &= \mu e^{\mathrm{i}\tau\partial_x^2}\int_0^\tau\int_0^s\partial_1N(\sigma,s)\,\mathrm{d}\sigma\,\mathrm{d}s+\mu e^{\mathrm{i}\tau\partial_x^2}\int_0^\tau\int_0^\tau\frac{s}{\tau}\partial_1N(\sigma,s)\,\mathrm{d}\sigma\,\mathrm{d}s+R(u,\tau)\\ &= \mu e^{\mathrm{i}\tau\partial_x^2}\int_0^\tau\int_0^s(s-\sigma)\partial_1^2N(\sigma,s)\,\mathrm{d}\sigma\,\mathrm{d}s\\ &\quad+\mu e^{\mathrm{i}\tau\partial_x^2}\int_0^\tau\int_0^\tau\frac{s}{\tau}(\tau-\sigma)\partial_1^2N(\sigma,s)\,\mathrm{d}\sigma\,\mathrm{d}s+R(u,\tau).\end{aligned}$$

Under the weaker Assumption 2.1, the first derivative of $N(\cdot,s)$ was already computed in (2.3) as

$$\partial_1N(\sigma,s)=e^{-\mathrm{i}\sigma\partial_x^2}D(\sigma,s)$$

with $D=D_1+D_2+D_3$ from Lemma 2.4. Hence, formula (2.5) is true. Let now the stronger Assumption 2.6 hold. We still need to verify that in this case, $N(\cdot,s)$ indeed belongs to $C^2([0,\tau],H^{-1})$. Note that the term D_1 is harmless since it does not contain any derivatives of u. Similar as in the proof of Lemma 2.4, we compute

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\Big[e^{-\mathrm{i}\sigma\partial_x^2}D_2(\sigma,s)\Big]$$

$$
\begin{aligned}
&= e^{-\mathrm{i}\sigma\partial_x^2}\Big[-\mathrm{i}\partial_x^2\Big(-2(\partial_x u(\sigma))^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar{u}(\sigma)\Big) - 4\partial_x u(\sigma)\partial_{xt}u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar{u}(\sigma)\\
&\quad - 4\mathrm{i}(\partial_x u(\sigma))^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x^2\bar{u}(\sigma) - 2(\partial_x u(\sigma))^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_t\bar{u}(\sigma)\Big]\\
&= e^{-\mathrm{i}\sigma\partial_x^2}\Big[4\mathrm{i}\Big((\partial_x^2 u(\sigma))^2 + \partial_x u(\sigma)\partial_x^3 u(\sigma)\Big)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar{u}(\sigma) + 2\mathrm{i}(\partial_x u(\sigma))^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x^2\bar{u}(\sigma)\\
&\quad + 8\mathrm{i}\partial_x u(\sigma)\partial_x^2 u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar{u}(\sigma)\\
&\quad - 4\partial_x u(\sigma)\partial_x\Big(\mathrm{i}\partial_x^2 u(\sigma) - \mathrm{i}\mu(|u|^2u)(\sigma)\Big)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar{u}(\sigma)\\
&\quad - 4\mathrm{i}(\partial_x u(\sigma))^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x^2\bar{u}(\sigma) - 2(\partial_x u(\sigma))^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_t\bar{u}(\sigma)\Big]\\
&= e^{-\mathrm{i}\sigma\partial_x^2}\Big[4\mathrm{i}(\partial_x^2 u(\sigma))^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar{u}(\sigma) - 2\mathrm{i}(\partial_x u(\sigma))^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x^2\bar{u}(\sigma)\\
&\quad + 8\mathrm{i}\partial_x u(\sigma)\partial_x^2 u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar{u}(\sigma) + 4\mathrm{i}\mu\partial_x u(\sigma)\partial_x(|u|^2u)(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\bar{u}(\sigma)\\
&\quad - 2(\partial_x u(\sigma))^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_t\bar{u}(\sigma)\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\sigma}\Big[e^{-\mathrm{i}\sigma\partial_x^2}D_3(\sigma,s)\Big]\\
&= e^{-\mathrm{i}\sigma\partial_x^2}\Big[-\mathrm{i}\partial_x^2\Big(-4u(\sigma)\partial_x u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar{u}(\sigma)\Big)\\
&\quad - 4\partial_t u(\sigma)\partial_x u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar{u}(\sigma) - 4u(\sigma)\partial_{xt}u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar{u}(\sigma)\\
&\quad - 8\mathrm{i}u(\sigma)\partial_x u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x^3\bar{u}(\sigma) - 4u(\sigma)\partial_x u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_{xt}\bar{u}(\sigma)\Big]\\
&= e^{-\mathrm{i}\sigma\partial_x^2}\Big[4\mathrm{i}\partial_x^2 u(\sigma)\partial_x u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar{u}(\sigma) + 4\mathrm{i}u(\sigma)\partial_x^3 u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar{u}(\sigma)\\
&\quad + 4\mathrm{i}u(\sigma)\partial_x u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x^3\bar{u}(\sigma) + 8\mathrm{i}\partial_x u(\sigma)\partial_x^2 u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar{u}(\sigma)\\
&\quad + 8\mathrm{i}(\partial_x u(\sigma))^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x^2\bar{u}(\sigma) + 8\mathrm{i}u(\sigma)\partial_x^2 u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x^2\bar{u}(\sigma)\\
&\quad - 4\partial_t u(\sigma)\partial_x u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar{u}(\sigma) - 4u(\sigma)\partial_x\Big(\mathrm{i}\partial_x^2 u(\sigma) - \mathrm{i}\mu(|u|^2u)(\sigma)\Big)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar{u}(\sigma)\\
&\quad - 8\mathrm{i}u(\sigma)\partial_x u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x^3\bar{u}(\sigma)\\
&\quad - 4u(\sigma)\partial_x u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\Big(-\mathrm{i}\partial_x^2\bar{u}(\sigma) + \mathrm{i}\mu(|u|^2\bar{u})(\sigma)\Big)\Big]\\
&= e^{-\mathrm{i}\sigma\partial_x^2}\Big[12\mathrm{i}\partial_x^2 u(\sigma)\partial_x u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar{u}(\sigma) + 8\mathrm{i}(\partial_x u(\sigma))^2 e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x^2\bar{u}(\sigma)\\
&\quad + 8\mathrm{i}u(\sigma)\partial_x^2 u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x^2\bar{u}(\sigma) - 4\partial_t u(\sigma)\partial_x u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar{u}(\sigma)\\
&\quad + 4\mathrm{i}\mu u(\sigma)\partial_x(|u|^2u)(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x\bar{u}(\sigma) - 4\mathrm{i}\mu u(\sigma)\partial_x u(\sigma)e^{2\mathrm{i}(\sigma-s)\partial_x^2}\partial_x(|u|^2\bar{u})(\sigma)\Big],
\end{aligned}
$$

again exploiting the differential equation (1.1) and that all expressions are well-defined in H^{-1}. Note that all third-order derivatives of u are canceled. We thus obtain the assertion by summing up all the terms for the second derivative of $N(\cdot,s)$. □

The next lemma gives control on the term $\tilde{D}_3(\sigma,s)$ defined in Lemma 2.10.

Lemma 2.11. *Let u, T, and M_2 be given by Assumption 2.6. The term $\tilde{D}_3(\sigma, s)$ defined by Lemma 2.10 can then by estimated by*

$$\|\tilde{D}_3(\sigma, s)\|_{L^2} \lesssim_{M_2} 1$$

for all σ, $s \in [0, T]$.

Proof. All appearing terms are roughly of the form fgh with $f \in L^2$ and g, $h \in H^1$ and can thus be estimated using the Sobolev embedding $H^1 \hookrightarrow L^\infty$ via

$$\|fgh\|_{L^2} \leq \|f\|_{L^2}\|g\|_{L^\infty}\|h\|_{L^\infty} \lesssim \|f\|_{L^2}\|g\|_{H^1}\|h\|_{H^1}. \qquad \square$$

To control the remainder $R(u, \tau)$ defined in Lemma 2.10, we need two more preparatory lemmas.

Lemma 2.12. *The estimate*

$$\|e^{\mathrm{i}\tau\partial_x^2} f - f\|_{L^2} + \|\varphi_1'(-2\mathrm{i}\tau\partial_x^2)f - \tfrac{1}{2}f\|_{L^2} \lesssim \tau\|f\|_{H^2}$$

holds for all $f \in H^2$ and $\tau \neq 0$.

Proof. The estimate of $e^{\mathrm{i}\tau\partial_x^2} f - f$ is standard. The other term is then controlled using

$$\varphi_1'(-2\mathrm{i}\tau\partial_x^2)f - \frac{1}{2}f = \frac{1}{\tau^2}\int_0^\tau s(e^{-2\mathrm{i}s\partial_x^2} - I)f \,\mathrm{d}s. \qquad \square$$

Lemma 2.13. *Let u, T, and M_2 be given by Assumption 2.6. For all $\tau \in [0, T]$, we can then write*

$$u(\tau) = e^{\mathrm{i}\tau\partial_x^2} u_0 - \mathrm{i}\mu\tau|u_0|^2 u_0 + \tilde{R}(u, \tau).$$

The remainder $\tilde{R}(u, \tau)$ satisfies the inequality

$$\|\tilde{R}(u, \tau)\|_{L^2} \lesssim_{M_2} \tau^2.$$

If only Assumption 2.1 is satisfied, we still obtain

$$\|\tilde{R}(u, \tau)\|_{H^1} \lesssim_{M_1} \tau.$$

Proof. The H^1-estimate is a direct consequence of Duhamel's formula and the algebra property of H^1. For the inequality in L^2, we additionally exploit the fundamental theorem of calculus to write

$$\begin{aligned} &u(\tau) - e^{\mathrm{i}\tau\partial_x^2} u_0 + \mathrm{i}\mu\tau|u_0|^2 u_0 \\ &\quad = -\mathrm{i}\mu \int_0^\tau \int_0^s \frac{\mathrm{d}}{\mathrm{d}\sigma}\Big[e^{\mathrm{i}(\tau-\sigma)\partial_x^2}(|u|^2 u)(\sigma)\Big] \,\mathrm{d}\sigma \,\mathrm{d}s - \mathrm{i}\mu\tau(e^{\mathrm{i}\tau\partial_x^2} - I)(|u_0|^2 u_0). \end{aligned}$$

The assertion then follows from standard estimates and Lemma 2.12. $\square$

We can now give the estimates for $R(u, \tau)$ from Lemma 2.10.

Lemma 2.14. *Let u, T, and M_2 be given by Assumption 2.6. The remainder $R(u,\tau)$ defined by Lemma 2.10 can then by estimated by*

$$\|R(u,\tau)\|_{L^2} \lesssim_{M_2} \tau^3$$

for all $\tau \in (0,T]$. If only Assumption 2.1 is satisfied, we obtain

$$\|R(u,\tau)\|_{L^2} \lesssim_{M_1} \tau^2$$

for all $\tau \in (0,T]$.

Proof. We insert the expression

$$u(\tau) = e^{\mathrm{i}\tau\partial_x^2}u_0 - \mathrm{i}\mu\tau|u_0|^2u_0 + \tilde{R}(u,\tau)$$

from Lemma 2.13 and obtain

$$\begin{aligned}
&-\mathrm{i}\mu\tau u(\tau)^2 e^{2\mathrm{i}\tau\partial_x^2}\varphi_1'(-2\mathrm{i}\tau\partial_x^2)\bar{u}(\tau)\\
&\quad= -\mathrm{i}\mu\tau(e^{\mathrm{i}\tau\partial_x^2}u_0)^2 e^{\mathrm{i}\tau\partial_x^2}\varphi_1'(-2\mathrm{i}\tau\partial_x^2)\bar{u}_0 + 2(\mathrm{i}\mu\tau)^2(e^{\mathrm{i}\tau\partial_x^2}u_0)(|u_0|^2u_0)e^{\mathrm{i}\tau\partial_x^2}\varphi_1'(-2\mathrm{i}\tau\partial_x^2)\bar{u}_0\\
&\qquad - (\mathrm{i}\mu\tau)^2(e^{\mathrm{i}\tau\partial_x^2}u_0)^2 e^{2\mathrm{i}\tau\partial_x^2}\varphi_1'(-2\mathrm{i}\tau\partial_x^2)(|u_0|^2\bar{u}_0) + R_1(u,\tau)\\
&\quad= -\mathrm{i}\mu\tau(e^{\mathrm{i}\tau\partial_x^2}u_0)^2 e^{\mathrm{i}\tau\partial_x^2}\varphi_1'(-2\mathrm{i}\tau\partial_x^2)\bar{u}_0 - \frac{\tau^2}{2}e^{\mathrm{i}\tau\partial_x^2}(|u_0|^4u_0) + R_1(u,\tau) + R_2(u,\tau).
\end{aligned}$$

The appearing error terms are given by

$$\begin{aligned}
&R_1(u,\tau)\\
&\quad= -2\mathrm{i}\mu\tau(e^{\mathrm{i}\tau\partial_x^2}u_0)\tilde{R}(u,\tau)e^{\mathrm{i}\tau\partial_x^2}\varphi_1'(-2\mathrm{i}\tau\partial_x^2)\bar{u}_0 - \mathrm{i}\mu\tau(e^{\mathrm{i}\tau\partial_x^2}u_0)^2 e^{2\mathrm{i}\tau\partial_x^2}\varphi_1'(-2\mathrm{i}\tau\partial_x^2)\overline{\tilde{R}(u,\tau)}\\
&\qquad - 2\mathrm{i}\mu\tau(e^{\mathrm{i}\tau\partial_x^2}u_0)(-\mathrm{i}\mu\tau|u_0|^2u_0 + \tilde{R}(u,\tau))e^{2\mathrm{i}\tau\partial_x^2}\varphi_1'(-2\mathrm{i}\tau\partial_x^2)(\mathrm{i}\mu\tau|u_0|^2\bar{u}_0 + \overline{\tilde{R}(u,\tau)})\\
&\qquad - \mathrm{i}\mu\tau(-\mathrm{i}\mu\tau|u_0|^2u_0 + \tilde{R}(u,\tau))^2 e^{2\mathrm{i}\tau\partial_x^2}\varphi_1'(-2\mathrm{i}\tau\partial_x^2)\bar{u}(\tau)
\end{aligned}$$

and

$$\begin{aligned}
R_2(u,\tau) &= -2\tau^2(e^{\mathrm{i}\tau\partial_x^2}u_0)(|u_0|^2u_0)e^{\mathrm{i}\tau\partial_x^2}\varphi_1'(-2\mathrm{i}\tau\partial_x^2)\bar{u}_0\\
&\qquad + \tau^2(e^{\mathrm{i}\tau\partial_x^2}u_0)^2 e^{2\mathrm{i}\tau\partial_x^2}\varphi_1'(-2\mathrm{i}\tau\partial_x^2)(|u_0|^2\bar{u}_0) + \frac{\tau^2}{2}e^{\mathrm{i}\tau\partial_x^2}(|u_0|^4u_0)
\end{aligned}$$

and satisfy $R = R_1 + R_2$. Under Assumption 2.6, we deduce

$$\|R(u,\tau)\|_{L^2} \le \|R_1(u,\tau)\|_{L^2} + \|R_2(u,\tau)\|_{L^2} \lesssim_{M_2} \tau^3$$

by Lemma 2.13 and iterative application of Lemma 2.12, respectively. Here we also exploit that by the definition (1.7), the operator $\varphi_1'(-2\mathrm{i}\tau\partial_x^2)$ is bounded uniformly in τ on all Sobolev spaces H^s with $s \ge 0$. Similarly, Assumption 2.1 and Lemma 2.13 imply that

$$\|R(u,\tau)\|_{L^2} \le \|R_1(u,\tau)\|_{L^2} + \|R_2(u,\tau)\|_{L^2} \lesssim_{M_1} \tau^2. \qquad \square$$

The next auxiliary result bounds the sum of the local error terms. Here we will crucially exploit the L^4-bound from Proposition 2.8.

Lemma 2.15. *Let u, T, and M_2 be given by Assumption 2.6. Then we can bound the sum of local errors of* (1.6) *by*

$$\Big\| \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k-1)\tau\partial_x^2}\Big(u(t_{k+1}) - \tilde{\Phi}_\tau(u(t_k))\Big)\Big\|_{L^2} \lesssim_{M_2,T} \tau^2,$$
$$\Big\| \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k-1)\tau\partial_x^2}\Big(u(t_{k+1}) - \tilde{\Phi}_\tau(u(t_k))\Big)\Big\|_{H^1} \lesssim_{M_2,T} \tau,$$

for all $\tau \in (0,T]$ and $n \in \mathbb{N}_0$ with $n\tau \le T$. If only Assumption 2.1 is satisfied, we still obtain

$$\Big\| \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k-1)\tau\partial_x^2}\Big(u(t_{k+1}) - \tilde{\Phi}_\tau(u(t_k))\Big)\Big\|_{L^2} \lesssim_{M_1,T} \tau,$$
$$\Big\| \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k-1)\tau\partial_x^2}\Big(u(t_{k+1}) - \tilde{\Phi}_\tau(u(t_k))\Big)\Big\|_{H^{\frac{3}{4}}} \lesssim_{M_1,T} \tau^{\frac{1}{4}},$$

for all $\tau \in (0,T]$ and $n \in \mathbb{N}_0$ with $n\tau \le T$.

Proof. We only give the details for the estimates under the stronger Assumption 2.6, since the results under the weaker Assumption 2.1 are obtained analogously by using the first-order error representation (2.5), compare also the proof of Lemma 2.5. From Lemma 2.10 with $u(t_k + \cdot)$ instead of u, we deduce

$$\begin{aligned}
&\sum_{k=0}^{n-1} e^{\mathrm{i}(n-k-1)\tau\partial_x^2}\Big(u(t_{k+1}) - \tilde{\Phi}_\tau(u(t_k))\Big) \\
&\quad = \mu \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2} \int_0^\tau \int_0^s (s-\sigma) e^{-\mathrm{i}\sigma\partial_x^2} \tilde{D}(t_k+\sigma, t_k+s)\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\qquad + \mu \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2} \int_0^\tau \int_0^\tau \frac{s}{\tau}(\tau-\sigma) e^{-\mathrm{i}\sigma\partial_x^2} \tilde{D}(t_k+\sigma, t_k+s)\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\qquad + \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k-1)\tau\partial_x^2} R(u(t_k+\cdot),\tau).
\end{aligned} \tag{2.6}$$

First, Lemma 2.14 implies

$$\Big\| \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k-1)\tau\partial_x^2} R(u(t_k+\cdot),\tau)\Big\|_{L^2} \lesssim_{M_2} n\tau^3 \lesssim_T \tau^2.$$

We next use the decomposition $\tilde{D} = \tilde{D}_1 + \tilde{D}_2 + \tilde{D}_3$ from Lemma 2.10. The terms involving $\tilde{D}_3$ are controlled by means of Lemma 2.11, which gives

$$\Big\| \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2} \int_0^\tau \int_0^s (s-\sigma) e^{-\mathrm{i}\sigma\partial_x^2} \tilde{D}_3(t_k+\sigma, t_k+s)\,\mathrm{d}\sigma\,\mathrm{d}s\Big\|_{L^2} \lesssim_{M_2} n\tau^3 \lesssim_T \tau^2,$$

$$\Big\| \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2} \int_0^\tau \int_0^\tau \frac{s}{\tau}(\tau-\sigma) e^{-\mathrm{i}\sigma\partial_x^2} \tilde{D}_3(t_k+\sigma, t_k+s)\,\mathrm{d}\sigma\,\mathrm{d}s \Big\|_{L^2} \lesssim_{M_2,T} \tau^2.$$

The first terms in (2.6) including $\tilde{D}_1$ are bounded by

$$\begin{aligned}
&\Big\| \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2} \int_0^\tau \int_0^s (s-\sigma) e^{-\mathrm{i}\sigma\partial_x^2} \tilde{D}_1(t_k+\sigma, t_k+s)\,\mathrm{d}\sigma\,\mathrm{d}s \Big\|_{L^2} \\
&\quad\le \sum_{k=0}^{n-1} \int_0^\tau \int_0^s |s-\sigma| \|\tilde{D}_1(t_k+\sigma, t_k+s)\|_{L^2}\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\quad\lesssim \tau \sum_{k=0}^{n-1} \int_0^\tau \int_0^\tau \|(\partial_x^2 u(t_k+\sigma))^2\|_{L^2} \|e^{2\mathrm{i}(\sigma-s)\partial_x^2} \bar{u}(t_k+\sigma)\|_{L^\infty}\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\quad\lesssim \tau^2 \sum_{k=0}^{n-1} \int_0^\tau \|\partial_x^2 u(t_k+\sigma)\|_{L^4}^2 \|u(t_k+\sigma)\|_{H^1}\,\mathrm{d}\sigma \lesssim_{M_2} \tau^2 \|\partial_x^2 u\|_{L^2([0,T],L^4)}^2 \\
&\quad\lesssim_T \tau^2 \|\partial_x^2 u\|_{L^4([0,T]\times\mathbb{T})}^2 \lesssim_{M_2,T} \tau^2.
\end{aligned}$$

Here we used Hölder's inequality, the Sobolev embedding $H^1 \hookrightarrow L^\infty$, and Proposition 2.8. Similarly, we obtain

$$\Big\| \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2} \int_0^\tau \int_0^\tau \frac{s}{\tau}(\tau-\sigma) e^{-\mathrm{i}\sigma\partial_x^2} \tilde{D}_1(t_k+\sigma, t_k+s)\,\mathrm{d}\sigma\,\mathrm{d}s \Big\|_{L^2} \lesssim_{M_2,T} \tau^2.$$

The terms involving $\tilde{D}_2$ are first rewritten as

$$\begin{aligned}
&\sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2} \int_0^\tau \int_0^s (s-\sigma) e^{-\mathrm{i}\sigma\partial_x^2} \tilde{D}_2(t_k+\sigma, t_k+s)\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\quad= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^s (s-\sigma) e^{\mathrm{i}(n\tau-\sigma)\partial_x^2} \tilde{D}_2(\sigma, s)\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\quad= \int_0^{n\tau} e^{\mathrm{i}(n\tau-\sigma)\partial_x^2} \int_\sigma^{\lceil \frac{\sigma}{\tau} \rceil \tau} (s-\sigma) \tilde{D}_2(\sigma, s)\,\mathrm{d}s\,\mathrm{d}\sigma, \\
&\sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2} \int_0^\tau \int_0^\tau \frac{s}{\tau}(\tau-\sigma) e^{-\mathrm{i}\sigma\partial_x^2} \tilde{D}_2(t_k+\sigma, t_k+s)\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\quad= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_k}^s \frac{s-t_k}{\tau}(\tau-\sigma+t_k) e^{\mathrm{i}(n\tau-\sigma)\partial_x^2} \tilde{D}_2(\sigma, s)\,\mathrm{d}\sigma\,\mathrm{d}s \\
&\quad= \int_0^{n\tau} e^{\mathrm{i}(n\tau-\sigma)\partial_x^2} \int_{\lfloor \frac{\sigma}{\tau} \rfloor \tau}^{\lceil \frac{\sigma}{\tau} \rceil \tau} \tau\big(\tfrac{s}{\tau} - \lfloor \tfrac{\sigma}{\tau} \rfloor\big)\big(\lceil \tfrac{\sigma}{\tau} \rceil - \tfrac{\sigma}{\tau}\big) \tilde{D}_2(\sigma, s)\,\mathrm{d}s\,\mathrm{d}\sigma,
\end{aligned}$$

where the application of Fubini's theorem is again justified since the double integral converges absolutely in H^{-1}. We next apply the Strichartz estimate (2.4) to infer that

$$\Big\| \int_0^{n\tau} e^{\mathrm{i}(n\tau-\sigma)\partial_x^2} \int_\sigma^{\lceil \frac{\sigma}{\tau} \rceil \tau} (s-\sigma) \tilde{D}_2(\sigma, s)\,\mathrm{d}s\,\mathrm{d}\sigma \Big\|_{L^2}$$

$$
\begin{aligned}
&\lesssim_T \Big\| \sigma \mapsto \int_\sigma^{\lceil \frac{\sigma}{\tau} \rceil \tau} (s-\sigma) \tilde{D}_2(\sigma, s) \,\mathrm{d}s \Big\|_{L^{\frac{4}{3}}([0,T]\times\mathbb{T})} \\
&\lesssim \tau \Big\| \sigma \mapsto \|u(\sigma)\|_{L^\infty} \|\partial_x^2 u(\sigma)\|_{L^4} \int_\sigma^{\lceil \frac{\sigma}{\tau} \rceil \tau} \|e^{2\mathrm{i}(\sigma-s)\partial_x^2} \partial_x^2 \bar{u}(\sigma)\|_{L^2} \,\mathrm{d}s \Big\|_{L^{\frac{4}{3}}([0,T])} \\
&\lesssim_{M_2} \tau^2 \|\partial_x^2 u\|_{L^{\frac{4}{3}}([0,T],L^4)} \lesssim_T \tau^2 \|\partial_x^2 u\|_{L^4([0,T]\times\mathbb{T})} \lesssim_{M_2,T} \tau^2,
\end{aligned}
$$

using again Hölder's inequality, the Sobolev embedding $H^1 \hookrightarrow L^\infty$, and Proposition 2.8. Similarly, we obtain

$$
\begin{aligned}
\tau \Big\| \int_0^{n\tau} & e^{\mathrm{i}(n\tau-\sigma)\partial_x^2} \int_{\lfloor \frac{\sigma}{\tau} \rfloor \tau}^{\lceil \frac{\sigma}{\tau} \rceil \tau} (\tfrac{s}{\tau} - \lfloor \tfrac{\sigma}{\tau} \rfloor)(\lceil \tfrac{\sigma}{\tau} \rceil - \tfrac{\sigma}{\tau}) \tilde{D}_2(\sigma, s) \,\mathrm{d}s \,\mathrm{d}\sigma \Big\|_{L^2} \\
&\lesssim_T \tau \Big\| \sigma \mapsto \int_{\lfloor \frac{\sigma}{\tau} \rfloor \tau}^{\lceil \frac{\sigma}{\tau} \rceil \tau} (\tfrac{s}{\tau} - \lfloor \tfrac{\sigma}{\tau} \rfloor)(\lceil \tfrac{\sigma}{\tau} \rceil - \tfrac{\sigma}{\tau}) \tilde{D}_2(\sigma, s) \,\mathrm{d}s \Big\|_{L^{\frac{4}{3}}([0,T]\times\mathbb{T})} \\
&\lesssim \tau \Big\| \sigma \mapsto \|u(\sigma)\|_{L^\infty} \|\partial_x^2 u(\sigma)\|_{L^4} \int_{\lfloor \frac{\sigma}{\tau} \rfloor \tau}^{\lceil \frac{\sigma}{\tau} \rceil \tau} \|e^{2\mathrm{i}(\sigma-s)\partial_x^2} \partial_x^2 \bar{u}(\sigma)\|_{L^2} \,\mathrm{d}s \Big\|_{L^{\frac{4}{3}}([0,T])} \\
&\lesssim_{M_2} \tau^2 \|\partial_x^2 u\|_{L^{\frac{4}{3}}([0,T],L^4)} \lesssim_T \tau^2 \|\partial_x^2 u\|_{L^4([0,T]\times\mathbb{T})} \lesssim_{M_2,T} \tau^2.
\end{aligned}
$$

This finishes the proof of the L^2-estimate. To obtain the first-order bound for the H^1-norm, we first note that a rough estimate based on the algebra property of H^2 gives

$$
\Big\| \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k-1)\tau\partial_x^2} \Big(u(t_{k+1}) - \tilde{\Phi}_\tau(u(t_k)) \Big) \Big\|_{H^2} \lesssim_{M_2} n\tau \lesssim_T 1.
$$

The assertion then follows by interpolation. □

As in the last section, we conclude the proof of the global error bounds by means of the discrete Gronwall lemma.

Proof of Theorem 1.3. For the scheme (1.6), we define the error

$$
e_n := u(t_n) - u_n
$$

for $n \in \mathbb{N}_0$ and $\tau \in (0, 1]$ with $n\tau \leq T$. It satisfies the recursion formula

$$
\begin{aligned}
e_{n+1} &= u(t_{n+1}) - \tilde{\Phi}_\tau(u(t_n)) + \tilde{\Phi}_\tau(u(t_n)) - \tilde{\Phi}_\tau(u_n) \\
&= u(t_{n+1}) - \tilde{\Phi}_\tau(u(t_n)) + e^{\mathrm{i}\tau\partial_x^2} e_n \\
&\quad - \mathrm{i}\tau\mu e^{\mathrm{i}\tau\partial_x^2} \Big((u(t_n))^2 [\varphi_1(-2\mathrm{i}\tau\partial_x^2) - \varphi_1'(-2\mathrm{i}\tau\partial_x^2)] \bar{u}(t_n) \\
&\qquad\qquad - (u_n)^2 [\varphi_1(-2\mathrm{i}\tau\partial_x^2) - \varphi_1'(-2\mathrm{i}\tau\partial_x^2)] \bar{u}_n \Big) \\
&\quad - \mathrm{i}\tau\mu \Big((e^{\mathrm{i}\tau\partial_x^2} u(t_n))^2 e^{\mathrm{i}\tau\partial_x^2} \varphi_1'(-2\mathrm{i}\tau\partial_x^2) \bar{u}(t_n) \\
&\qquad\qquad - (e^{\mathrm{i}\tau\partial_x^2} u_n)^2 e^{\mathrm{i}\tau\partial_x^2} \varphi_1'(-2\mathrm{i}\tau\partial_x^2) \bar{u}_n \Big)
\end{aligned}
$$

$$- \frac{\tau^2}{2} e^{\mathrm{i}\tau\partial_x^2}\Big(|u(t_n)|^4 u(t_n) - |u_n|^4 u_n\Big).$$

By Lemma A.8, this formula and $e_0 = 0$ imply that

$$\begin{aligned} e_n = & \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k-1)\tau\partial_x^2}\Big(u(t_{k+1}) - \tilde{\Phi}_\tau(u(t_k))\Big) \\ & - \mathrm{i}\tau\mu \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2}\Big((u(t_k))^2[\varphi_1(-2\mathrm{i}\tau\partial_x^2) - \varphi_1'(-2\mathrm{i}\tau\partial_x^2)]\bar{u}(t_k) \\ & \qquad - (u_k)^2[\varphi_1(-2\mathrm{i}\tau\partial_x^2) - \varphi_1'(-2\mathrm{i}\tau\partial_x^2)]\bar{u}_k\Big) \\ & - \mathrm{i}\tau\mu \sum_{k=0}^{n-1} e^{\mathrm{i}(n-k-1)\tau\partial_x^2}\Big((e^{\mathrm{i}\tau\partial_x^2} u(t_k))^2 e^{\mathrm{i}\tau\partial_x^2}\varphi_1'(-2\mathrm{i}\tau\partial_x^2)\bar{u}(t_k) \\ & \qquad - (e^{\mathrm{i}\tau\partial_x^2} u_k)^2 e^{\mathrm{i}\tau\partial_x^2}\varphi_1'(-2\mathrm{i}\tau\partial_x^2)\bar{u}_k\Big) \\ & - \frac{\tau^2}{2}\sum_{k=0}^{n-1} e^{\mathrm{i}(n-k)\tau\partial_x^2}\Big(|u(t_k)|^4 u(t_k) - |u_k|^4 u_k\Big). \end{aligned}$$

From Lemma 2.15 and standard estimates, we now infer that

$$\|e_n\|_{L^2} \le c\tau^2 + c\tau \sum_{k=0}^{n-1}(1 + \|e_k\|_{H^1}^4)\|e_k\|_{L^2},$$
$$\|e_n\|_{H^1} \le c\tau + c\tau \sum_{k=0}^{n-1}(1 + \|e_k\|_{H^1}^4)\|e_k\|_{H^1}$$

with a constant $c > 0$ that only depends on M_2 and T. Here we exploited the elementary re-writings

$$\begin{aligned} f^2 v - g^2 w &= (f+g)v(f-g) + g^2(v-w), \\ |v|^4 v - |w|^4 w &= |v|^4(v-w) + |v|^2 vw(\bar{v} - \bar{w}) + |v|^2|w|^2(v-w) + v|w|^2 w(\bar{v} - \bar{w}) \\ &\quad + |w|^4(v-w), \end{aligned}$$

the Sobolev embedding $H^1 \hookrightarrow L^\infty$, the algebra property of H^1, and the representation $u_k = u(t_k) - e_k$. We then define the maximum step size

$$\tau_0 := (ce^{2cT})^{-1}.$$

Let $\tau \in (0, \tau_0]$. Using the discrete Gronwall lemma A.9, by induction on n we deduce

$$\|e_n\|_{H^1} \le ce^{2cn\tau}\tau \le 1$$

and thus also

$$\|e_n\|_{L^2} \le ce^{2cT}\tau^2$$

for all $n \in \mathbb{N}_0$ with $n\tau \le T$. □

Proof of Theorem 1.2 for u_n given by (1.6). This is done analogously to the preceding proof of Theorem 1.3. The H^1-estimates are replaced by estimates in $H^{3/4}$, compare also the proof of Theorem 1.2 for u_n given by the first-order scheme (1.3). □

2.3. Numerical experiment

The numerical behavior of the scheme (1.3) applied to the nonlinear Schrödinger equation (1.1) has already been extensively studied in the literature. See, e.g., [1, 54, 55] for numerical experiments including comparisons with other schemes. However, they do dot provide a clear picture concerning the convergence rate of the L^2 error in the situation of H^1 initial data. While the experiment in Figure 1 of [1] shows first-order convergence as proven in our Theorem 1.2, the experiment in Figure 1 of [54] suggests an order reduction down to 3/4. A possible explanation of this behavior is that an error bound of the form

$$\|u(n\tau) - u_n\|_{L^2(\mathbb{T})} \le c\tau^\beta$$

could hold for some $\beta \in (0,1)$, where one might have $c \ll C$, with C from Theorem 1.2, depending on the precise choice of initial data. Therefore, we provide a numerical example where a wider range of τ is considered than in [1, 54].

We solve the nonlinear Schrödinger equation (1.1) with $\mu = 1$ and $T = 1$ using the first-order low-regularity integrators (1.3) and the second-order scheme (1.6), called LRI1 and LRI2, respectively. To construct the initial datum u_0, we utilize the following standard procedure that was similarly used in [54], see also Section 8 for more details on this construction. We set $u_0 = \phi/\|\phi\|_{L^2(\mathbb{T})}$, where the function $\phi \in H^s(\mathbb{T})$ is defined by its Fourier coefficients

$$\hat{\phi}_k = (1+|k|^2)^{-\frac{1}{2}(s+\frac{1}{2}+\varepsilon)} r_k$$

for $k \in \{-K_0, \dots, K_0 - 1\}$, and $\hat{\phi}_k = 0$ elsewhere. Here we use the maximum frequency $K_0 = 2^{19}$, uniformly distributed numbers $r_k \in [-1,1] + \mathrm{i}[-1,1]$, and a very small parameter $\varepsilon > 0$. The results are similar if we set $r_k = 1$ (for all k) instead. The space is discretized by the standard Fourier pseudo-spectral method, where we choose $K = 2^{11}$ grid points. The reference solution is computed using the second-order low-regularity integrator (1.6) with $\tau_{\mathrm{ref}} = 10^{-7}$. Higher values of K and/or smaller values of τ_{ref} and/or a different reference integrator such as the Strang splitting did not change the outcome for the range of τ considered in the experiments below. Our Python code to reproduce the results is available at `https://doi.org/10.35097/v54nh3fcvy5u8my6`.

In Figure 2.1, we take H^1 initial data and plot the maximal errors of the first-order scheme (1.3) in the $L^2(\mathbb{T})$ norm against the step sizes τ. We observe a convergence rate of approximately 3/4 as in [54] for the greater values of τ in the range $(10^{-3}, 10^{-2})$. For smaller values of τ, the rate increases to approximately 9/10. In Table 2.1, we list the values of the step sizes τ_ℓ and L^2 errors e_ℓ of (1.3), where the index $\ell \in \{1, \dots, 10\}$ denotes the corresponding run of the experiment. Moreover, we compute the experimental order of convergence (EOC) by

$$\mathrm{EOC}_\ell = \frac{\log e_\ell - \log e_{\ell-1}}{\log \tau_\ell - \log \tau_{\ell-1}}. \tag{2.7}$$

The results in Table 2.1 indicate that the order of convergence is still increasing for very small values of τ. According to Theorem 1.2, the convergence rate 1 must show up for even smaller values of τ. Unfortunately, we were unable to make this visible

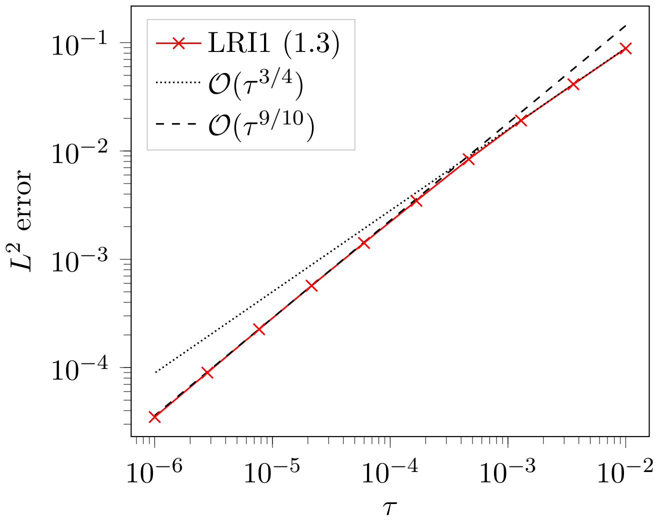

Figure 2.1.: L^2 errors of the first-order scheme (1.3) for $u_0 \in H^1$.

numerically since further experiments revealed that this would require a disproportionate computational effort.

Table 2.1.: Experimental order of convergence of (1.3) in Figure 2.1 according to (2.7).

τ	L^2 error	EOC
1e-02	8.84e-02	–
3.59e-03	4.13e-02	0.74
1.29e-03	1.91e-02	0.75
4.64e-04	8.40e-03	0.81
1.67e-04	3.47e-03	0.86
5.99e-05	1.42e-03	0.87
2.15e-05	5.71e-04	0.89
7.7e-06	2.26e-04	0.90
2.8e-06	8.98e-05	0.91
1e-06	3.50e-05	0.92

Interestingly, in the situation of H^1 initial data, the second-order scheme (1.6) behaves better. In Figure 2.2, we observe first-order convergence as predicted by our Theorem 1.2. This behavior has already been observed in Figure 2 (a) of [14].

When trying to numerically verify the second-order convergence predicted by Theorem 1.3 in the case $u_0 \in H^2$, we observe a similar pathology as for the first-order scheme above. The full convergence rate 2 is not visible for the range of τ considered in our experiment. Instead, we observe a convergence rate of approximately 1.88. In Figure 2.3, we take H^2 initial data and plot the maximal errors of (1.6) in the $L^2(\mathbb{T})$ norm against the step sizes τ. Moreover, in Table 2.2, we list the experimental order of convergence

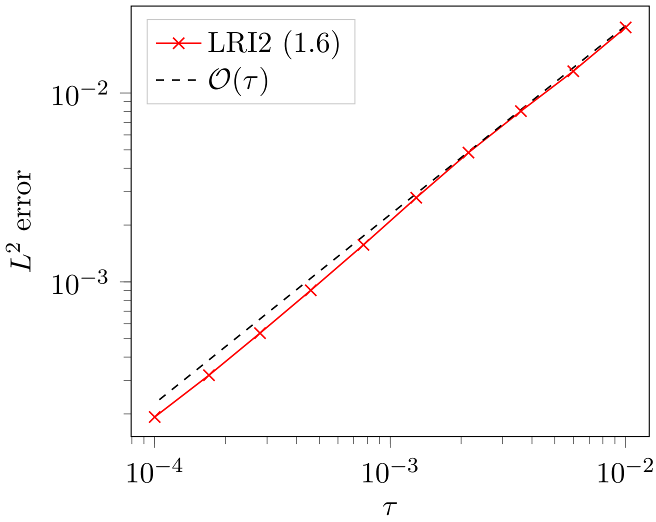

Figure 2.2.: L^2 errors of the second-order scheme (1.6) for $u_0 \in H^1$.

according to (2.7).

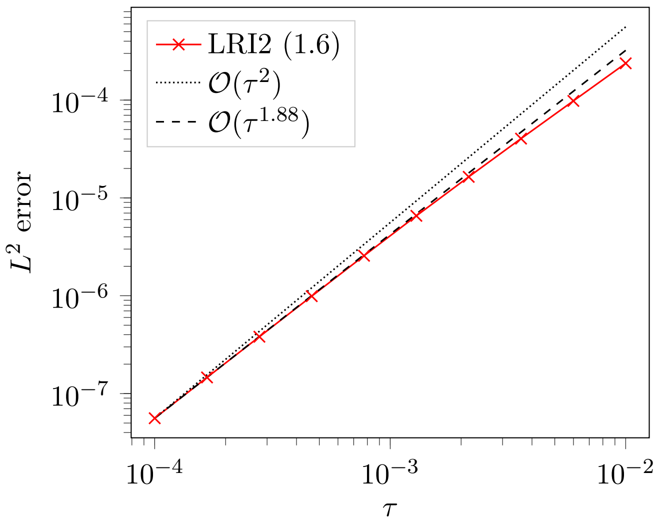

Figure 2.3.: L^2 errors of the second-order scheme (1.6) for $u_0 \in H^2$.

Table 2.2.: Experimental order of convergence of (1.6) in Figure 2.3 according to (2.7).

τ	L^2 error	EOC
1e-02	2.38e-04	–
5.99e-03	9.81e-05	1.73
3.59e-03	4.04e-05	1.73
2.15e-03	1.64e-05	1.76
1.29e-03	6.56e-06	1.79
7.74e-04	2.56e-06	1.84
4.64e-04	9.90e-07	1.86
2.78e-04	3.81e-07	1.87
1.67e-04	1.46e-07	1.87
1e-04	5.60e-08	1.88

3. The wave case

3.1. Proof of the null form estimate

In this section we show estimate (1.11). Therefore we define the bilinear form

$$Q(\phi, \psi) := \partial_t\phi\partial_t\psi - \partial_x\phi\partial_x\psi.$$

As a preparatory step, we treat the homogeneous problem.

Lemma 3.1. *Let ϕ and ψ solve the homogeneous wave equations*

$$\begin{aligned}\partial_t^2\phi - \partial_x^2\phi = 0, \quad \phi(0) = \phi_0, \quad \partial_t\phi(0) = \phi_1\\ \partial_t^2\psi - \partial_x^2\psi = 0, \quad \psi(0) = \psi_0, \quad \partial_t\psi(0) = \psi_1\end{aligned}$$

with Cauchy data ϕ_0, $\psi_0 \in H^1(\mathbb{T})$ and ϕ_1, $\psi_1 \in L^2(\mathbb{T})$. We then have the estimate

$$\|Q(\phi, \psi)\|_{L^2(\mathbb{T}\times\mathbb{T})} \lesssim (\|\partial_x\phi_0\|_{L^2} + \|\phi_1\|_{L^2})(\|\partial_x\psi_0\|_{L^2} + \|\psi_1\|_{L^2}).$$

Proof. The solution to the wave equation is given by d'Alembert's formula

$$\phi(t, x) = \frac{1}{2}(\phi_0(x + t) + \phi_0(x - t)) + \frac{1}{2}\int_{x-t}^{x+t} \phi_1(y)\,\mathrm{d}y,$$

which is stated in equation (8) of p. 68 of [19] for smooth initial data, and extends to our setting by density. Using the definitions

$$v_\phi = \frac{1}{2}(\partial_x\phi_0 + \phi_1), \quad w_\phi = \frac{1}{2}(\partial_x\phi_0 - \phi_1),$$

we can then write

$$\partial_t\phi(t, x) = v_\phi(x + t) - w_\phi(x - t), \quad \partial_x\phi(t, x) = v_\phi(x + t) + w_\phi(x - t),$$

and analogously for ψ. We compute

$$\begin{aligned}Q(\phi, \psi)(t, x) &= (v_\phi(x + t) - w_\phi(x - t))(v_\psi(x + t) - w_\psi(x - t))\\ &\quad - (v_\phi(x + t) + w_\phi(x - t))(v_\psi(x + t) + w_\psi(x - t))\\ &= -2v_\phi(x + t)w_\psi(x - t) - 2w_\phi(x - t)v_\psi(x + t).\end{aligned}$$

Note that the "parallel interactions" cancel (where one has twice "$x + t$" or twice "$x - t$") and only the "transverse interactions" remain (where one has once "$x + t$" and once

"$x - t$"). We refer to p. 293 of [71] for further explanations of this phenomenon that also apply to the higher dimensional cases.

To obtain the desired estimate, by substituting $x - t = y$ and $y + 2t = s$, we compute

$$\int_{\mathbb{T}} \int_{\mathbb{T}} |v(x+t)w(x-t)|^2 \,\mathrm{d}x\,\mathrm{d}t = \|v\|^2_{L^2(\mathbb{T})} \|w\|^2_{L^2(\mathbb{T})}$$

for general functions v, w. It follows that

$$\begin{aligned} \|Q(\phi,\psi)\|_{L^2(\mathbb{T}\times\mathbb{T})} &\lesssim \|v_\phi\|_{L^2}\|w_\psi\|_{L^2} + \|w_\phi\|_{L^2}\|v_\psi\|_{L^2} \\ &\lesssim (\|\partial_x\phi_0\|_{L^2} + \|\phi_1\|_{L^2})(\|\partial_x\psi_0\|_{L^2} + \|\psi_1\|_{L^2}). \end{aligned}$$ □

Now we give the proof of (1.11).

Proposition 3.2. *Let $T > 0$ and ϕ solve the inhomogeneous wave equation*

$$\partial_t^2\phi - \partial_x^2\phi = F, \quad \phi(0) = \phi_0, \quad \partial_t\phi(0) = \phi_1 \tag{3.1}$$

on $[0,T] \times \mathbb{T}$ with data $\phi_0 \in H^1(\mathbb{T})$, $\phi_1 \in L^2(\mathbb{T})$, and $F \in L^1([0,T], L^2(\mathbb{T}))$. Then we have the inequality

$$\|Q(\phi,\phi)\|_{L^2([0,T]\times\mathbb{T})} \lesssim_T \|\partial_x\phi_0\|^2_{L^2} + \|\phi_1\|^2_{L^2} + \|F\|^2_{L^1([0,T],L^2)}.$$

Proof. We decompose $\phi = \phi_{\mathrm{hom}} + \phi_{\mathrm{inh}}$, where ϕ_{hom} solves (3.1) with $F = 0$ and ϕ_{inh} solves (3.1) with $\phi_0 = \phi_1 = 0$. The estimate for $Q(\phi_{\mathrm{hom}}, \phi_{\mathrm{hom}})$ follows directly from Lemma 3.1 and the periodicity of $\partial_{t,x}\phi_{\mathrm{hom}}$ in time. To treat the inhomogeneous part, for almost all $s \in [0,T]$, we define ϕ^s to be the solution to the homogeneous equation

$$\partial_t^2\phi^s - \partial_x^2\phi^s = 0, \quad \phi^s(s) = 0, \quad \partial_t\phi^s(s) = F(s).$$

By Duhamel's formula, ϕ_{inh} is then given by

$$\phi_{\mathrm{inh}}(t) = \int_0^t \phi^s(t)\,\mathrm{d}s.$$

It follows that we can express the bilinear term as

$$Q(\phi_{\mathrm{inh}}, \phi_{\mathrm{inh}})(t) = \int_0^t \int_0^t Q(\phi^s, \phi^r)(t)\,\mathrm{d}s\,\mathrm{d}r.$$

Minkowski's inequality, Lemma 3.1, and the energy equality imply

$$\begin{aligned} &\|Q(\phi_{\mathrm{inh}}, \phi_{\mathrm{inh}})\|_{L^2([0,T]\times\mathbb{T})} \\ &\quad\le \int_0^T \int_0^T \|Q(\phi^s,\phi^r)\|_{L^2([0,T]\times\mathbb{T})}\,\mathrm{d}s\,\mathrm{d}r \\ &\quad\lesssim_T \int_0^T \int_0^T (\|\partial_x\phi^s(0)\|_{L^2} + \|\partial_t\phi^s(0)\|_{L^2})(\|\partial_x\phi^r(0)\|_{L^2} + \|\partial_t\phi^r(0)\|_{L^2})\,\mathrm{d}s\,\mathrm{d}r \\ &\quad\lesssim \|F\|^2_{L^1([0,T],L^2)}. \end{aligned}$$

The mixed term $Q(\phi_{\mathrm{hom}}, \phi_{\mathrm{inh}})$ is treated similarly. □

3.2. Error analysis of the corrected Lie splitting

In this section we carry out the proof of Theorem 1.6. It is convenient to work with the first-order reformulation of the nonlinear wave equation (1.2). With the definitions

$$U := \begin{pmatrix} u \\ v \end{pmatrix} \mathrel{\widehat{=}} \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \; A := \begin{pmatrix} 0 & I \\ \partial_x^2 & 0 \end{pmatrix}, \; G(U) := \begin{pmatrix} 0 \\ g(u) \end{pmatrix}, \; U_0 := \begin{pmatrix} u_0 \\ v_0 \end{pmatrix},$$

we obtain the differential equation

$$\begin{aligned} \partial_t U(t) &= AU(t) + G(U(t)), \quad t \in [0,T], \\ U(0) &= U_0. \end{aligned} \tag{3.2}$$

Clearly, $U = (u, \partial_t u) \in C([0,T], H^1 \times L^2) \cap C^1([0,T], L^2 \times H^{-1})$ solves (3.2) if and only if $u \in C([0,T], H^1) \cap C^1([0,T], L^2) \cap C^2([0,T], H^{-1})$ solves (1.2). The local wellposedness of (3.2) can be shown by a standard Duhamel fixed-point iteration in a closed ball of $C([0,b], H^1 \times L^2)$ for a suitable choice of $b > 0$.

Assumption 3.3. There exists a time $T > 0$ and a solution $U = (u, \partial_t u) \in C([0,T], H^1 \times L^2) \cap C^1([0,T], L^2 \times H^{-1})$ to the nonlinear wave equation (3.2) with bound

$$M := \|U\|_{L^\infty([0,T], H^1 \times L^2)}.$$

Since the nonlinearity g belongs to $C^2(\mathbb{R}, \mathbb{R})$, we can find an increasing function L such that g satisfies

$$|g(z)| + |g'(z)| + |g''(z)| \leq L(|z|) \tag{3.3}$$

for all $z \in \mathbb{R}$. In the following, we suppress the dependency on the function L from (3.3) in the $\lesssim$ notation. We now apply Proposition 3.2 to the solution u to the nonlinear problem (1.2).

Proposition 3.4. *Let u, T, and M be given by Assumption 3.3. Then we have the estimate*

$$\|(\partial_t u)^2 - (\partial_x u)^2\|_{L^2([0,T] \times \mathbb{T})} \lesssim_{M,T} 1.$$

Proof. By Proposition 3.2, the result follows from the inequality

$$\|g(u)\|_{L^1([0,T], L^2)} \lesssim_{M,T} 1,$$

which is a consequence of (3.3), Hölder's inequality, and the Sobolev embedding $H^1 \hookrightarrow L^\infty$. □

We now give a brief derivation of the corrected Lie splitting (1.8) proposed in [49]. It is based of the Lie splitting approximation for (3.2), which is a formally first-order scheme given by

$$U_{n+1}^{\text{Lie}} = e^{\tau A}[U_n^{\text{Lie}} + \tau G(U_n^{\text{Lie}})]. \tag{3.4}$$

By the Duhamel formulation of (3.2), the fundamental theorem of calculus, and Fubini's theorem, the local error of (3.4) can be represented as

$$U(\tau)-U_1^{\mathrm{Lie}} = e^{\tau A}\int_0^\tau [e^{-\sigma A}G(U(\sigma))-G(U_0)]\,\mathrm{d}\sigma = e^{\tau A}\int_0^\tau (\tau-s)e^{-sA}H(U(s))\,\mathrm{d}s. \quad (3.5)$$

Here we use the definition

$$H(U(s)) \coloneqq e^{sA}\frac{\mathrm{d}}{\mathrm{d}s}\Big[e^{-sA}G(U(s))\Big] = \begin{pmatrix} -g(u(s)) \\ g'(u(s))\partial_t u(s)\end{pmatrix}.$$

Similar as in the Schrödinger case, we do not insert the approximation $U(s) \approx e^{sA}U_0$ (which was used in [49]) in order to create better conditions for applying Proposition 3.4 later.

The construction of the low-regularity integrator depends on the following crucial observation. Since u solves (1.2) and $L^1(\mathbb{T}) \hookrightarrow H^{-1}(\mathbb{T})$, the map $H(U)$ satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}s}H(U(s)) = -AH(U(s)) + B(U(s)) \quad (3.6)$$

in $L^2(\mathbb{T}) \times H^{-1}(\mathbb{T})$, where the remainder

$$B(U) \coloneqq \begin{pmatrix} 0 \\ g''(u)[(\partial_t u)^2 - (\partial_x u)^2] + g'(u)g(u)\end{pmatrix} \quad (3.7)$$

only contains first-order derivatives of u. We plug the Duhamel approximation $H(U(s)) \approx e^{-sA}H(U_0)$ for (3.6) into (3.5) and exploit (1.9) to infer

$$U(\tau) - U_1^{\mathrm{Lie}} \approx e^{\tau A}\int_0^\tau (\tau-s)e^{-2sA}H(U_0)\,\mathrm{d}s = \tau^2 e^{\tau A}\varphi_2(-2\tau A)H(U_0).$$

Adding this term on the Lie splitting (3.4) gives the formally second-order *corrected Lie splitting*

$$U_{n+1} = \Psi_\tau(U_n) \coloneqq e^{\tau A}[U_n + \tau G(U_n) + \tau^2\varphi_2(-2\tau A)H(U_n)], \quad (3.8)$$

which corresponds to (1.8). From this derivation we immediately get the following representation of the local error. A related formula was derived in Lemma 6.2 of [63] in the 3D case.

Lemma 3.5. *Let U and T be given by Assumption* (3.3)*. Then the local error of the corrected Lie splitting* (3.8) *is given by*

$$U(\tau) - U_1 = e^{\tau A}\int_0^\tau (\tau-s)e^{-2sA}\int_0^s e^{\sigma A}B(U(\sigma))\,\mathrm{d}\sigma\,\mathrm{d}s$$

for all $\tau \in (0,T]$.

Proof. Follows directly from (3.5), (3.8), and the Duhamel formulation of (3.6). □

We can now bound the sum of local errors with the help of Proposition 3.4.

Lemma 3.6. *Let $U = (u, \partial_t u)$, T, and M be given by Assumption 3.3. Then we can estimate the sum of local errors of* (3.8) *by*

$$\Big\| \sum_{k=0}^{n-1} e^{(n-k-1)\tau A} \Big(U(t_{k+1}) - \Psi_\tau(U(t_k)) \Big) \Big\|_{H^1 \times L^2} \lesssim_{M,T} \tau^2,$$

for all $\tau \in (0, T]$ and $n \in \mathbb{N}_0$ with $n\tau \leq T$.

Proof. The triangle inequality and Lemma 3.5 with $U(t_k + \cdot)$ instead of U yield

$$\begin{aligned} &\Big\| \sum_{k=0}^{n-1} e^{(n-k-1)\tau A} \Big(U(t_{k+1}) - \Psi_\tau(U(t_k)) \Big) \Big\|_{H^1 \times L^2} \\ &\qquad \lesssim_T \tau^2 \sum_{k=0}^{n-1} \int_0^\tau \|B(U(t_k + \sigma))\|_{H^1 \times L^2} \, \mathrm{d}\sigma \leq \tau^2 \|B(U)\|_{L^1([0,T], H^1 \times L^2)}. \end{aligned}$$

We next insert the definition (3.7) of B and apply (3.3) and finally Proposition 3.4 to obtain

$$\begin{aligned} \|B(U)\|_{L^1([0,T], H^1 \times L^2)} &= \|g''(u)[(\partial_t u)^2 - (\partial_x u)^2] + g'(u) g(u)\|_{L^1([0,T], L^2)} \\ &\lesssim_{M,T} \|(\partial_t u)^2 - (\partial_x u)^2\|_{L^2([0,T] \times \mathbb{T})} + 1 \lesssim_{M,T} 1. \end{aligned}$$

□

As in the Schrödinger case, we conclude the proof of the global error bound by means of the discrete Gronwall lemma.

Proof of Theorem 1.6. We proceed similar as in the proof of Theorems 1.2 and 1.3. The error

$$E_n \coloneqq U(t_n) - U_n$$

of (3.8) satisfies the recursion formula

$$\begin{aligned} E_{n+1} &= U(t_{n+1}) - \Psi_\tau(U(t_n)) + \Psi_\tau(U(t_n)) - \Psi_\tau(U_n) \\ &= U(t_{n+1}) - \Psi_\tau(U(t_n)) + e^{\tau A} E_n + \tau e^{\tau A} (G(U(t_n)) - G(U_n)) \\ &\quad + \tau^2 e^{\tau A} \varphi_2(-2\tau A)(H(U(t_n)) - H(U_n)). \end{aligned}$$

Since $E_0 = 0$, Lemma (A.8) gives the representation

$$\begin{aligned} E_n &= \sum_{k=0}^{n-1} e^{(n-k-1)\tau A} \Big(U(t_{k+1}) - \Psi_\tau(U(t_k)) \Big) \\ &\quad + \tau \sum_{k=0}^{n-1} e^{(n-k)\tau A} \Big(G(U(t_k)) - G(U_k) \Big) \\ &\quad + \tau^2 \varphi_2(-2\tau A) \sum_{k=0}^{n-1} e^{(n-k)\tau A} \Big(H(U(t_k)) - H(U_k) \Big). \end{aligned}$$

By the bounds on g from (3.3), we have the inequality

$$|g(y) - g(z)| + |g'(y) - g'(z)| \lesssim L(|y| + |z|)|y - z|$$

for all y, $z \in \mathbb{R}$. Moreover, due to its definition (1.9), the operator $\varphi_2(-2\tau A)$ is bounded uniformly in τ on $H^1 \times L^2$. Lemma 3.6, the Sobolev embedding $H^1 \hookrightarrow L^\infty$, the relation $U_k = U(t_k) - E_k$, and standard estimates thus imply that

$$\|E_n\|_{H^1 \times L^2} \leq c\tau^2 + \tau \sum_{k=0}^{n-1} K(\|E_k\|_{H^1 \times L^2})\|E_k\|_{H^1 \times L^2}$$

with a constant $c > 0$ and an increasing function K, both depending on M, T, and L. We define the maximum step size

$$\tau_0 \coloneqq (ce^{K(1)T})^{-\frac{1}{2}}.$$

Using the discrete Gronwall lemma A.9, we deduce via induction on n

$$\|E_n\|_{H^1 \times L^2} \leq c\tau^2 e^{K(1)T} \leq 1$$

for all $\tau \in (0, \tau_0]$ and $n \in \mathbb{N}_0$ with $n\tau \leq T$. $\square$

Part II.

Error analysis of the Strang splitting for 3D semilinear wave equations with finite-energy solutions

4. Overview

4.1. Problem setting

We study the time integration of the semilinear wave equation with power-type nonlinearity

$$\begin{aligned} \partial_t^2 u - \Delta u + \mu |u|^{\alpha-1} u = 0, \quad (t,x) \in \overline{[0,T)} \times \Omega, \\ u(0) = u^0, \quad \partial_t u(0) = v^0, \end{aligned} \tag{4.1}$$

where the spatial domain Ω can be either $\mathbb{T}^3$ or $\mathbb{R}^3$. We allow for powers $\alpha \in [3,5]$ and both signs $\mu \in \{-1,1\}$. The initial data (u^0, v^0) are assumed to belong to the physically natural energy space $\mathcal{H}^1(\Omega) \times L^2(\Omega)$. Here, the Sobolev space $\mathcal{H}^s(\Omega)$ is defined as the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$ if $\Omega = \mathbb{R}^3$, and the standard Sobolev space $H^s(\mathbb{T}^3)$ if $\Omega = \mathbb{T}^3$, respectively. To avoid some technicalities, we require u^0 and v^0 to be real-valued, though we could also treat the complex-valued case.

It is well known that local wellposedness of (4.1) can be shown by a fixed-point argument. In the cubic case $\alpha = 3$, the nonlinear term can be controlled only using classical tools such as Sobolev embedding. In the case of higher powers $\alpha > 3$, one has to exploit the dispersive character of the wave equation. A particular useful tool are the Strichartz estimates, which control mixed space-time $L^p L^q$-norms of solutions to the linear wave equation in terms of the data. Thanks to the L^p-norm in time, one can choose the space integrability exponent q larger than predicted by a fixed-time Sobolev embedding. This makes it possible to show local wellposedness of (4.1) for powers up to the critical value $\alpha = 5$, see, e.g., the monographs [66, 71].

In this work we are interested in approximating the temporal evolution of (4.1). A natural choice for the time integration of such equations is the class of second-order trigonometric (or exponential) integrators, cf. Chapter XIII.2.2 of [27] for an overview. As explained in [11], these methods in one-step form can be interpreted as variants of the Strang splitting with additional filter functions in the nonlinear part. In the context of an ordinary differential equation with a globally Lipschitz continuous nonlinearity, error estimates for such schemes were derived in, e.g., [11, 21, 25, 27]. For the PDE (4.1) with pure power nonlinearity, an error analysis was first given in [22], but only on the one-dimensional torus $\mathbb{T}$. The proof uses a similar strategy as the earlier work [50] for the nonlinear Schrödinger equation. Under the finite-energy assumption $(u^0, v^0) \in H^1 \times L^2$, it was shown that trigonometric integrators converge with order two in $L^2 \times H^{-1}$ and with order one in the energy space $H^1 \times L^2$ itself, even if no filter functions are used. In [10], the same error bounds were shown in a more general setting (possibly including boundary conditions) which in particular allows for rough L^∞ coefficients in the nonlinear part. This made it necessary to equip the schemes with suitable filter functions to avoid numerical resonances for certain step-sizes. The higher dimensional cases $d \in \{2,3\}$ were also considered in [10], but only under the stronger regularity assumption $(u^0, v^0) \in H^2 \times H^1$.

In the proofs of the one-dimensional results in [10, 22], it was crucially exploited that the Sobolev space $H^1(\mathbb{T})$ forms an algebra. This is however not the case in higher dimensions, where the estimates for the nonlinear terms become more delicate. The local wellposedness theory suggests to exploit Strichartz estimates in numerical analysis. Starting from [37, 40], this was first done for the nonlinear Schrödinger equation. Subsequent works used discrete-time Strichartz estimates to show error bounds under low regularity assumptions, such as [17, 36, 53] for the nonlinear Schrödinger equation on $\mathbb{R}^d$ and [41, 54] in the case of the nonlinear Schrödinger equation on the torus $\mathbb{T}^d$. In the latter case, the authors further made use of discrete Bourgain spaces. For the nonlinear wave equation (4.1), no literature has been available in this context until recently. Based on discrete-time Strichartz estimates, in our paper [63] an error analysis of the Lie splitting for (4.1) on the full space $\mathbb{R}^3$ was given, notably including the scaling-critical power $\alpha = 5$. It was shown that under the assumption $(u^0, v^0) \in \mathcal{H}^1 \times L^2$, the scheme converges with optimal first order in $L^2 \times \mathcal{H}^{-1}$.

Recently, another class of methods to approximate the temporal evolution of nonlinear dispersive problems especially in low regularity gained a lot of attention, namely, the low-regularity integrators. See [59, 60] for an overview. Due to an improved local error structure, such schemes can produce higher convergence rates at low regularity than classical methods such as the Strang splitting. The authors in [49] proposed the corrected Lie splitting, which is a low-regularity integrator that can be applied to the nonlinear wave equation (4.1). It was shown that the corrected Lie splitting is second-order convergent in $H^1 \times L^2$ under the regularity condition $(u^0, v^0) \in H^{1+d/4} \times H^{d/4}$ for dimensions $d \in \{1,2,3\}$. If $d = 1$, the regularity condition for second-order convergence can even be relaxed to $(u^0, v^0) \in H^1 \times L^2$, see [62] and Part I. We also refer to [13] for an error analysis of the corrected Lie splitting in lower regularity with reduced convergence rates under a CFL-type condition. However, the analyses from [13, 49] do not apply to our problem (4.1) since they either require a global Lipschitz condition on the nonlinearity (which is not satisfied by the power-type nonlinearity $\pm|u|^{\alpha-1}u$), or higher-regularity solutions satisfying $u(t) \in H^s$ with $s > d/2$.

The cases $\mu = 1$ and $\mu = -1$ are called *defocusing* and *focusing*, respectively. In the defocusing situation $\mu = 1$, it is known that the solutions to (4.1) exist globally in time (if $\Omega = \mathbb{R}^3$ and $\alpha < 5$, this additionally requires that $u^0 \in L^{\alpha+1}$). Moreover, in the case of the full space $\Omega = \mathbb{R}^3$, so-called *scattering results* are known, meaning that the nonlinear solution u asymptotically behaves like a solution to the linear problem with possibly different initial value. Depending on the power α, such results sometimes require additional spatial decay of the initial data, cf. Section 3.6 of [71]. It is a natural question whether such techniques can be used to describe the long-time behavior of numerical approximations as well. A first step in this direction was done in [15], where the authors show a global-in-time convergence result for the Lie splitting applied to the energy-subcritical Schrödinger equation by transferring results from scattering theory to the discrete-time setting. In addition to the discrete-time Strichartz estimates from [36], they exploit the *pseudo-conformal conservation law* which requires initial data in the *conformal space*, that is a subspace of H^1 whose elements have suitable spatial decay at infinity.

The very recent preprint [42] considers splitting schemes for the cubic wave equation on the two-dimensional torus $\mathbb{T}^2$ for H^s solutions with $s > 1/4$. Two main results on error bounds are proven, where one relies on Sobolev embedding and the other one makes use of discrete Bourgain spaces. However, the convergence rates seem sub-optimal in the case when no uniform space-time L^∞ bound on the solution u is available, which corresponds to the case $s < 1$ in two dimensions. The losses in the convergence rates arise due to the the use of fixed-time Sobolev embeddings (that could be improved by means of Strichartz estimates) in the first case. On the other hand, in the Bourgain space framework a frequency-filtered scheme is used where already the approximation of the initial data costs regularity. We believe that the results in [42] for $s < 1$ could be improved by adapting our techniques to the two-dimensional setting.

4.2. Our contributions

The goal of the present part of this thesis is to prove optimal error bounds for a second-order scheme applied to (4.1) under the finite-energy condition. For powers α away from the critical value $\alpha = 5$, the convergence rates obtained here are higher than those from our previous paper [63]. In the important cubic case $\alpha = 3$, we even almost recover the optimal temporal second-order convergence in L^2. The treatment of the scaling-critical nonlinearity $\alpha = 5$ is based on Section 5 of our earlier work [63], where the Lie splitting was considered. As far as we know, this provided the first error analysis of a time discretization for a scaling-critical problem. We adapt this analysis to show first-order convergence in L^2 and convergence without rate in $\mathcal{H}^1$ also for the Strang splitting. In the case of the defocusing quintic nonlinearity on the full space (i.e., when $\Omega = \mathbb{R}^3$, $\mu = 1$, and $\alpha = 5$), our analysis is global in time and we include a scattering result. To our knowledge, such a global analysis was only done in [15] for the semilinear Schrödinger equation previously. However, we treat the energy-critical case which was not considered in [15], and as opposed to that paper, we do not need additional decay assumptions on the initial data. Indeed, our previous analysis from [63] can be extended to the global case with only minor adjustments. Finally, for the torus $\Omega = \mathbb{T}^3$, we treat the fully discrete setting (using the Fourier pseudo-spectral method) with optimal spatial convergence.

In our earlier work [63], the terms stemming from the local error were estimated using discrete-time Strichartz estimates. In combination with a non-optimal frequency filtering, this led to a loss of convergence order in the error analysis of the formally second-order corrected Lie splitting. Here, we use a more suitable filtering that was similarly proposed as "method $(\tilde{B})$" in the one-dimensional case in [22]. Moreover, we show that at least in the case of the Strang splitting, one can avoid the issues coming from the application of discrete-time Strichartz inequalities to the local error terms by using the continuous-time Strichartz estimates instead. In the cubic case $\alpha = 3$, it even turns out that we do not need any discrete Strichartz estimates to prove our semi-discrete error result at all (the continuous ones are still used). This is related to the fact that, as mentioned above, the wellposedness of (4.1) can be shown without using Strichartz estimates for $\alpha = 3$. In the case $\alpha \in (3, 5]$ however, we need the discrete-time Strichartz estimates to ensure the stability of the numerical scheme.

We establish various discrete-time Strichartz estimates in Section 5. Here one controls discrete-time points $(u(n\tau))_{n\in\mathbb{Z}}$ of the solutions to the linear problem in spaces like $\ell^p(\mathbb{Z}, L^q(\mathbb{R}^3))$ by L^2-based norms of the initial data, where $\tau \in (0, 1]$ is the time-step size. It is easy to see that a naive discrete-time version of results in continuous time fails, cf. Remark 5.7. As a remedy, we include frequency cut-offs π_K that truncate the high frequencies at level $K \geq 1$. The estimates then depend on $K\tau$, but are otherwise in complete analogy with the estimates in continuous time. Similar results for the Schrödinger equation have been obtained in [36, 53], see also [41, 54] for the case of periodic boundary conditions using Bourgain spaces. Moreover, Strichartz estimates for spatially discrete Schrödinger equations were treated in the seminal works [37, 40]. In contrast to the Schrödinger equation on $\mathbb{R}^n$, in the wave case one has to work with frequency-localized estimates and the Littlewood–Paley decomposition already for the basic Strichartz inequalities. In Theorem 5.12 we also derive local-in-time estimates at the forbidden endpoint $(p, q) = (2, \infty)$ with an additional logarithmic correction depending on K and the end-time T. Such an inequality was shown in [43] for continuous time. Moreover, by exploiting the finite propagation speed of the wave equation on the torus $\mathbb{T}^3$, we obtain locally in time the same Strichartz estimates as on the full space $\mathbb{R}^3$. This is in sharp contrast to the Schrödinger case (with infinite speed of propagation) where the Strichartz estimates on the torus are restricted compared to those on the full space, cf. [7].

Even though our nonlinearity is of power-type, we have to make use of a filter function inside the nonlinearity when estimating the terms resulting from the local error (compared to the one-dimensional case [22]). This is essentially caused by the fact that in 3D, the multiplication by an $\mathcal{H}^1 \cap L^\infty$ function is not a bounded operator on $\mathcal{H}^{-1}$. As a filter, we use the operator $\pi_{\tau^{-1}}$ already mentioned above, which is the Fourier multiplier for the characteristic function of the cube $[-\tau^{-1}, \tau^{-1}]^3$, where $\tau > 0$ denotes the time step size. This particular choice is made for several reasons. First, it enables us to use the summation by parts formula to exploit cancellations in the terms stemming from the local error. Second, a filter of this type is needed to obtain discrete-time Strichartz estimates (compare, e.g., [36, 41, 53, 54, 63]), which are necessary if $\alpha > 3$. Third, it fits well to the spatial discretization with the Fourier pseudo-spectral method. As a conceptual novelty, the proof of our error estimates combines the summation/integration by parts technique (as already used in, e.g., [10, 11]) with the use of Strichartz estimates.

While extending our results to the fully discrete setting, we face the difficulty that one cannot take advantage of negative-order Sobolev spaces when estimating the trigonometric interpolation error. We solve this issue by using an L^q estimate for the trigonometric interpolation error from [2, 35, 58].

4.3. Results in the semi-discrete setting

We analyze a variant of the Strang splitting scheme that computes approximations $U_n \approx (u(n\tau), \partial_t u(n\tau))$ for a step size $\tau > 0$ and $n \in \mathbb{N}_0$. With the notation $A(u,v) \coloneqq (v, \Delta u)$ for the wave operator and $G(u,v) \coloneqq (0, -\mu|u|^{\alpha-1}u)$ for the nonlinearity, the semi-discrete form of the scheme reads

$$\begin{aligned} U_{n+1/2} &= e^{\tau A}[U_n + \tfrac{\tau}{2} G(\Pi_{\tau^{-1}} U_n)], \\ U_{n+1} &= U_{n+1/2} + \tfrac{\tau}{2} G(\Pi_{\tau^{-1}} U_{n+1/2}), \\ U_0 &= (u^0, v^0). \end{aligned} \tag{4.2}$$

This scheme fits into the class of trigonometric integrators in one-step formulation as described in Section XIII.2.2 of [27], with “inner filter” $\Pi_{\tau^{-1}} = \operatorname{diag}(\pi_{\tau^{-1}}, \pi_{\tau^{-1}})$. It corresponds to a variant of “method $(\tilde{B})$” that was proposed and analyzed in the one-dimensional case in [22]. See (5.8) and (5.15) for the precise definition of the filter.

Our convergence result for the subcritical case $\alpha \in [3, 5)$ reads as follows. It is proved at the end of Sections 7.4 and 7.5, respectively.

Theorem 4.1. *Let $T \in (0, \infty)$ and $U = (u, \partial_t u) \in C([0,T], \mathcal{H}^1(\Omega) \times L^2(\Omega))$ solve the semilinear wave equation (4.1). Then there are a constant $C > 0$ and a maximum step size $\tau_0 > 0$, such that the approximations U_n obtained by the filtered Strang splitting scheme (4.2) satisfy the error bounds*

$$\begin{aligned} \|U(n\tau) - U_n\|_{\mathcal{H}^1 \times L^2} &\leq C\tau|\log \tau|, \\ \|U(n\tau) - U_n\|_{L^2 \times \mathcal{H}^{-1}} &\leq C\tau^2|\log \tau| \end{aligned}$$

for $\alpha = 3$, and

$$\begin{aligned} \|U(n\tau) - U_n\|_{\mathcal{H}^1 \times L^2} &\leq C\tau^{\frac{5-\alpha}{2}}, \\ \|U(n\tau) - U_n\|_{L^2 \times \mathcal{H}^{-1}} &\leq C\tau^{\frac{7-\alpha}{2}} \end{aligned}$$

for $\alpha \in (3, 5)$. Theses bounds are uniform in $\tau \in (0, \tau_0]$ and $n \in \mathbb{N}_0$ with $n\tau \leq T$. The numbers C and τ_0 only depend on T, α, and $\|U\|_{L^\infty([0,T], \mathcal{H}^1 \times L^2)}$.

Remark 4.2. The logarithm in the result of Theorem 4.1 for $\alpha = 3$ comes from the use of the endpoint Strichartz estimate for the $L^2 L^\infty$ norm that only holds with a logarithmic correction, cf. Corollary 5.16.

Remark 4.3. In the case of a slower growing nonlinearity with $\alpha \in [2, 3)$, an inspection of the proof of Theorem 4.1 shows that one obtains the same error bounds as in the cubic case $\alpha = 3$. If $\Omega = \mathbb{R}^3$, one has to assume an additional condition such as $u^0 \in L^2$ to deal with the case $\alpha < 3$ (due to issues with homogeneous Sobolev spaces).

We also provide an error analysis for the critical case $\alpha = 5$. Notably, the analysis is global in time in the defocusing case on the full space, i.e., if $\Omega = \mathbb{R}^3$ and $\mu = 1$. To obtain this result, it is crucial to work in homogeneous Sobolev spaces, since the wave group e^{tA} is not bounded uniformly in t on the standard (inhomogeneous) Sobolev spaces $H^r \times H^{r-1}$. The proof is given a the end of Section 7.6.

Theorem 4.4. *Let $\alpha = 5$ and $T \in (0, \infty)$. If $\Omega = \mathbb{R}^3$ and $\mu = 1$, we also allow $T = \infty$. Let $U = (u, \partial_t u) \in C(\overline{[0,T)}, \mathcal{H}^1(\Omega) \times L^2(\Omega))$ with $u \in L^4([0,T], L^{12}(\Omega))$ solve the semilinear wave equation (4.1). Then there are a constant $C > 0$ and a maximum step size $\tau_0 > 0$, such that the iterates U_n of the filtered Strang splitting scheme (4.2) satisfy the error bound*

$$\|U(n\tau) - U_n\|_{L^2 \times \mathcal{H}^{-1}} \leq C\tau$$

for all $\tau \in (0, \tau_0]$ and $n \in \mathbb{N}_0$ with $n\tau \in \overline{[0,T)}$. Moreover, we obtain the convergence

$$\|U(n\tau) - U_n\|_{\mathcal{H}^1 \times L^2} \to 0$$

as $\tau \to 0$, uniformly in $n \in \mathbb{N}_0$ with $n\tau \in \overline{[0,T)}$. In the case $\Omega = \mathbb{R}^3$, the number C only depends on $\|U\|_{L^\infty([0,T], \mathcal{H}^1 \times L^2(\Omega))}$ and $\|u\|_{L^4([0,T], L^{12}(\Omega))}$, whereas τ_0 only depends on u^0 and v^0. In the case $\Omega = \mathbb{T}^3$, the constants C and τ_0 additionally depend on T. Furthermore, if $\Omega = \mathbb{R}^3$ and $\mu = 1$, we obtain the scattering result

$$\lim_{\tau \to 0} \lim_{n \to \infty} \|U_n - e^{n\tau A} U_+\|_{\dot{H}^1 \times L^2} = 0$$

for some asymptotic state $U_+ \in \dot{H}^1 \times L^2$.

Remark 4.5. The more sophisticated analysis for $\alpha = 5$ is reflected by the dependence of the maximum step size τ_0 on the solution itself, rather than just on its norm. A similar behavior occurs in the wellposedness theory, see Section 6.1 and [71].

Remark 4.6. The global-in-time analysis does not apply to the subcritical case $\alpha < 5$ without further assumptions. The reasons are the same as those which prevent a straightforward scattering result in that case, cf. Remark 6.6.

Remark 4.7. We compare our 3D results to the known results in 1D. If $\alpha = 3$, our convergence rates are almost the same as those obtained in the one-dimensional cases in [10, 22]. For $\alpha > 3$, the Theorems 4.1 and 4.4 exhibit an order reduction. This reduction can be observed in our numerical experiment in Section 8.

Remark 4.8. In the defocusing case $\mu = 1$, energy conservation shows that the solutions to (4.1) exist globally in time (if $\Omega = \mathbb{R}^3$ and $\alpha < 5$, one needs the additional assumption $u^0 \in L^{\alpha+1}$ to ensure that the energy is finite). Moreover, the numbers C and τ_0 from Theorems 4.1, as well as the number C from Theorem 4.4, then only depend on T, α, $\|u^0\|_{\mathcal{H}^1(\Omega)}$, $\|v^0\|_{L^2(\Omega)}$, and possibly $\|u^0\|_{L^{\alpha+1}(\mathbb{R}^3)}$. See Remarks 6.4–6.6.

Remark 4.9. The integrability condition $u \in L^4([0,T], L^{12}(\Omega))$ is assumed in Theorem 4.4 since uniqueness of solutions to (4.1) for $\alpha = 5$ is in general not known without such a restriction, cf. Remark 6.2 d) and [57]. If $\mu = 1$ and $\Omega = \mathbb{R}^3$, one can always take $T = \infty$ in Theorem 4.4, cf. Remark 6.6.

Remark 4.10. With somewhat greater technical effort we could also treat the equation

$$\partial_t^2 u - \Delta u = g(u) \tag{4.3}$$

with a general nonlinearity $g \in C^2(\mathbb{R}, \mathbb{R})$ satisfying the bounds

$$\begin{aligned} |g(z)| &\lesssim 1 + |z|^\alpha, \\ |g'(z)| &\lesssim 1 + |z|^{\alpha-1}, \\ |g''(z)| &\lesssim 1 + |z|^{\alpha-2}, \end{aligned}$$

for $z \in \mathbb{R}$. This covers in particular the semilinear Klein–Gordon equation since the lower-order mass term can be moved into the nonlinearity. The global-in-time result of Theorem 4.4 will however not directly extend to a setting where the nonlinearity contains a lower-order part. Still, we expect that global results are possible in the Klein–Gordon case, for instance.

Remark 4.11. One might wonder if under our assumption $(u^0, v^0) \in \mathcal{H}^1 \times L^2$, a low-regularity integrator such as the corrected Lie splitting proposed in [49] can give higher convergence rates than classical schemes such as the Strang splitting. The authors in [49] show that this is possible in the one-dimensional case, see also Part I. However, in our 3D case we did not succeed to find such a result so far, cf. Remark 1.7.

4.4. The fully discrete scheme

In the case $\Omega = \mathbb{T}^3$, we can also provide error bounds for a full discretization. Denoting by $K \geq 1$ the spatial discretization parameter for the Fourier pseudo-spectral method, we consider the fully discrete scheme

$$\begin{aligned} U_{n+1/2} &= e^{\tau A}[U_n + \tfrac{\tau}{2}\mathcal{I}_K G(\Pi_{\tau^{-1}} U_n)] \\ U_{n+1} &= U_{n+1/2} + \tfrac{\tau}{2}\mathcal{I}_K G(\Pi_{\tau^{-1}} U_{n+1/2}) \\ U_0 &= \Pi_K(u^0, v^0). \end{aligned} \tag{4.4}$$

Here, we use the notation $\mathcal{I}_K = \operatorname{diag}(I_K, I_K)$ for the trigonometric interpolation operator I_K, cf. Definition 5.21. Note that (4.4) corresponds to the semi-discrete scheme (4.2) in the case $K = \infty$, where we define $\Pi_\infty = \mathcal{I}_\infty := I$.

In the subcritical range $\alpha \in [3, 5)$, we establish the following fully discrete convergence result.

Theorem 4.12. *Let $T \in (0, \infty)$ and $U = (u, \partial_t u) \in C([0, T], H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3))$ solve the semilinear wave equation (4.1). Then there are positive constants C, τ_0, and K_0, such that the approximations U_n obtained by the fully discrete filtered Strang algorithm (4.4) satisfy the error bounds*

$$\begin{aligned} \|U(n\tau) - U_n\|_{L^2 \times H^{-1}} &\leq C(\tau^2 |\log \tau| + K^{-1}), \quad \text{if } \alpha = 3, \\ \|U(n\tau) - U_n\|_{L^2 \times H^{-1}} &\leq C(\tau^{\frac{7-\alpha}{2}} + K^{-1}), \qquad \text{if } \alpha \in (3, 5), \end{aligned}$$

uniformly in $\tau \in (0, \tau_0]$, $K \geq K_0$, and $n \in \mathbb{N}_0$ with $n\tau \leq T$. We moreover obtain the convergence

$$\|U(n\tau) - U_n\|_{H^1 \times L^2} \to 0$$

as $\tau \to 0$ and $K \to \infty$, uniformly in $n \in \mathbb{N}_0$ with $n\tau \in [0,T]$. The numbers C, τ_0, and K_0 only depend on T, α, and $\|U\|_{L^\infty([0,T],H^1\times L^2)}$.

In the critical case $\alpha = 5$, our fully discrete result reads as follows.

Theorem 4.13. *Let $\alpha = 5$ and $T \in (0,\infty)$. Let $U = (u, \partial_t u) \in C([0,T], H^1(\mathbb{T}^3)\times L^2(\mathbb{T}^3))$ with $u \in L^4([0,T], L^{12}(\mathbb{T}^3))$ solve the semilinear wave equation* (4.1)*. Then there are positive constants C, τ_0, and K_0, such that the iterates U_n of the fully discrete filtered Strang splitting scheme* (4.4) *satisfy the error bound*

$$\|U(n\tau) - U_n\|_{L^2\times H^{-1}} \leq C(\tau + K^{-1})$$

for all $\tau \in (0, \tau_0]$, $K \geq K_0$, and $n \in \mathbb{N}_0$ with $n\tau \leq T$. We moreover obtain the convergence

$$\|U(n\tau) - U_n\|_{H^1\times L^2} \to 0$$

as $\tau \to 0$ and $K \to \infty$, uniformly in $n \in \mathbb{N}_0$ with $n\tau \in [0,T]$. The number C only depends on T, $\|U\|_{L^\infty([0,T],H^1\times L^2)}$ and $\|u\|_{L^4([0,T],L^{12})}$, whereas τ_0 and K_0 only depend on u^0 and v^0.

Remark 4.14. We emphasize that K and τ can be chosen independently in our results. Note that our fully discrete Theorems 4.12 and 4.13 formally imply their semi-discrete counterparts for the torus $\mathbb{T}^3$ from Theorems 4.1 and 4.4 in the limit $K \to \infty$.

Remark 4.15. In Theorems 4.12 and 4.13, the spatial order K^{-1} for the error in $L^2 \times H^{-1}$ is optimal. This can be seen by investigating the projection error

$$\|(I - \Pi_K)U(n\tau)\|_{L^2\times H^{-1}} \lesssim K^{-1}\|U(n\tau)\|_{H^1\times L^2},$$

cf. Lemma 5.19. Similarly, in the energy norm, the projection error satisfies

$$\|(I - \Pi_K)U(n\tau)\|_{H^1\times L^2} \to 0$$

for $K \to \infty$, but without rate in general, since we only assume the regularity $U(t) = (u, \partial_t u)(t) \in H^1 \times L^2$. Therefore, we can only expect spatial convergence without rate in the $H^1 \times L^2$-norm.

Remark 4.16. Here we could also treat the general equation (4.3). In view of the error bound for the trigonometric interpolation from Lemma 5.23, we then would additionally need the third derivative of g with bound

$$|g'''(z)| \lesssim 1 + |z|^{\alpha-3}.$$

Remark 4.17. In view of the error bounds of Theorem 4.12, it might be advantageous to choose the spatial resolution finer than the temporal one. In the cases $\alpha \in \{3,5\}$, the nonlinearity is a polynomial and thus respects the frequency localization up to a constant. In particular, if additionally $K > \alpha/\tau$, it turns out that the highest frequencies $(\Pi_K - \Pi_{\alpha/\tau})U_n$ in (4.4) are only influenced by the linear part e^{tA} and not by the nonlinear function G. Therefore, in that case, if one is only interested in the numerical approximation U_N for some $N \gg 1$, the high-frequency part can be computed directly from the initial data via $(\Pi_K - \Pi_{\alpha/\tau})U_N = e^{N\tau A}(\Pi_K - \Pi_{\alpha/\tau})(u^0, v^0)$, without time-stepping. This idea was also exploited in the recent paper [13].

Remark 4.18. The torus $\mathbb{T}^3$ corresponds to a cube with periodic boundary conditions. We can also treat the problem on a cube with homogeneous Dirichlet or Neumann boundary conditions by restricting the full Fourier basis to a sine or cosine basis, respectively, see Section 6.2. Similarly, the semi-discrete results from Section 4.3 naturally extend to (possibly irrational) tori $\mathbb{R}/(a_1\mathbb{Z})\times\mathbb{R}/(a_2\mathbb{Z})\times\mathbb{R}/(a_3\mathbb{Z})$ and cuboids $(0,a_1)\times(0,a_2)\times(0,a_3)$ with homogeneous Dirichlet or Neumann boundary conditions for some side-lengths a_1, a_2, $a_3 > 0$. To obtain fully discrete results also in that case, it remains to be checked whether the trigonometric interpolation error estimates from Lemma 5.23 hold true in that setting as well.

Remark 4.19. Our results do not cover the case that the differential equation (4.1) is posed on a general non-cuboidal domain with appropriate boundary conditions. In this setting the derivation of Strichartz estimates is considerably harder and the range of admissible exponents is restricted, see [6]. In the case when $K\tau = 1$ in the discrete Strichartz inequality Theorem 5.6 (which is sufficient for our applications), one can show discrete-time Strichartz estimates using the continuous ones as a “black box”, see Remark 5.9. Using this, we could transfer our semi-discrete results to the general domain case, where we would need to adjust the convergence rates for small α according to the restricted Strichartz estimates. However, it is still unclear how to involve space discretizations if spectral methods cannot be used and if it is possible to derive Strichartz estimates for, e.g., finite element approximations. For Schrödinger equations on the full space, Strichartz estimates for finite-difference Laplacians have been treated in, e.g., [32, 37, 40]. Nevertheless, the corresponding problem for a multi-dimensional wave equation seems to be unsolved even in this simple model case, cf. [16]. Moreover, since the previously mentioned works make essential use of Fourier techniques, it is still completely open how they extend to more general domains or non-equidistant meshes.

4.5. Notations

Let $p \in [1,\infty]$. The discrete p-norms on $\mathbb{R}^d$ are denoted by $|\cdot|_p$ and we simply write $|\cdot| = |\cdot|_2$ for the euclidean norm. The Hölder-conjugated index to p is denoted by $p' \in [1,\infty]$ and satisfies the relation $1/p + 1/p' = 1$. The set $B(x_0, r) = \{x \in \mathbb{R}^d : |x| \le r\}$ is the ball with radius $r \ge 0$ centered at some position $x_0 \in \mathbb{R}^d$. The identity operator is denoted by I. We write $\mathbb{1}_B$ for the indicator function of a set B and $\mathbb{1}$ for the function being constantly one.

We denote by $\mathcal{D}(\mathbb{T}^3)$ the space of 2π-periodic $C^\infty(\mathbb{R}^3)$-functions and by $\mathcal{D}'(\mathbb{T}^3)$ the space of distributions on the torus, cf. Definitions 1.23–1.24 of [64]. The k-th Fourier coefficient of a distribution $v \in \mathcal{D}'(\mathbb{T}^3)$ is defined by

$$\hat{v}_k = (2\pi)^{-\frac{3}{2}} \langle v, e^{-\mathrm{i}k\cdot x} \rangle_{\mathcal{D}'(\mathbb{T}^3)\times\mathcal{D}(\mathbb{T}^3)}, \quad k \in \mathbb{Z}^3.$$

For a real number $s \in \mathbb{R}$, the Sobolev spaces on $\mathbb{T}^3$ are given by

$$H^s(\mathbb{T}^3) = \{v \in \mathcal{D}'(\mathbb{T}^3) \colon \|v\|_{H^s(\mathbb{T}^3)} < \infty\}$$

with norm

$$\|v\|_{H^s(\mathbb{T}^3)}^2 = \sum_{k\in\mathbb{Z}^3} (1+|k|^2)^s |\hat{v}_k|^2. \tag{4.5}$$

We have $H^0(\mathbb{T}^3) = L^2(\mathbb{T}^3)$ by the Parseval equality.

We write $\mathcal{F}$ for the Fourier transform on $\mathbb{R}^3$ (and also on $\mathbb{T}^3$), using the convention with the prefactor $(2\pi)^{-3/2}$. We set $\hat{u} \coloneqq \mathcal{F}u$. We denote by $\mathcal{S}(\mathbb{R}^3)$ the Schwartz space and by $\mathcal{S}'(\mathbb{R}^3)$ the space of tempered distributions on $\mathbb{R}^3$. In the context of Fourier multipliers, we often simply write ξ instead of the map $\xi \mapsto \xi$. For $s \in \mathbb{R}$, we use the inhomogeneous and homogeneous Sobolev spaces

$$H^s(\mathbb{R}^3) = \{w \in \mathcal{S}'(\mathbb{R}^3) : \hat{w} \in L^1_{\mathrm{loc}}(\mathbb{R}^3) \text{ and } \|w\|_{H^s(\mathbb{R}^3)} < \infty\},$$
$$\dot{H}^s(\mathbb{R}^3) = \{w \in \mathcal{S}'(\mathbb{R}^3) : \hat{w} \in L^1_{\mathrm{loc}}(\mathbb{R}^3) \text{ and } \|w\|_{\dot{H}^s(\mathbb{R}^3)} < \infty\}$$

with norms

$$\|w\|_{H^s(\mathbb{R}^3)} = \|(1+|\xi|^2)^{\frac{s}{2}}\hat{w}\|_{L^2(\mathbb{R}^3)}, \quad \|w\|_{\dot{H}^s(\mathbb{R}^3)} = \||\xi|^s\hat{w}\|_{L^2(\mathbb{R}^3)}, \tag{4.6}$$

where $L^1_{\mathrm{loc}}(\mathbb{R}^3)$ denotes the space of locally integrable functions on $\mathbb{R}^3$. Plancherel's theorem yields $H^0(\mathbb{R}^3) = \dot{H}^0(\mathbb{R}^3) = L^2(\mathbb{R}^3)$. By Proposition 1.34 of [3], the homogeneous space $\dot{H}^s(\mathbb{R}^3)$ is complete if and only if $s < 3/2$. For $\Omega \in \{\mathbb{R}^3, \mathbb{T}^3\}$, we set $\mathcal{H}^s(\Omega) \coloneqq \dot{H}^s(\mathbb{R}^3)$ if $\Omega = \mathbb{R}^3$ and $\mathcal{H}^s(\Omega) \coloneqq H^s(\mathbb{T}^3)$ if $\Omega = \mathbb{T}^3$.

Let $h\colon \mathbb{R} \to \mathbb{C}$ be a measurable function. To denote the Fourier multiplication operator for the function $\xi \mapsto h(|\xi|)$ (on $\mathbb{R}^3$) and $k \mapsto h(|k|)$ (on $\mathbb{T}^3$), we will use the notation $h(|\nabla|)$ in both cases (see Definition A.4 with $m\colon \xi \mapsto h(|\xi|)$). It is clear from the definition of the Sobolev norms that if the function h is bounded, then the operator $h(|\nabla|)$ is uniformly bounded on all spaces $H^s(\mathbb{T}^3)$, $H^s(\mathbb{R}^3)$ and $\dot{H}^s(\mathbb{R}^3)$. Moreover, $\Delta = -|\nabla|^2$ is the Laplacian. Some more properties of Fourier multipliers and function spaces can be found in Appendix A.1.

Let $p \in [1,\infty]$, J be a time interval, X be a Banach space, and $\tau > 0$ be a time step size. In analogy to the continuous-time Bochner norm $\|\cdot\|_{L^p(J,X)}$, we also introduce the discrete-time norms

$$\|F\|_{\ell^p_\tau(J,X)} \coloneqq \Big(\tau \sum_{\substack{n\in\mathbb{Z}\\ n\tau\in J}} \|F_n\|_X^p\Big)^{\frac{1}{p}}$$

if $p < \infty$, and

$$\|F\|_{\ell^\infty_\tau(J,X)} \coloneqq \sup_{\substack{n\in\mathbb{Z}\\ n\tau\in J}} \|F_n\|_X.$$

To simplify notation we often write $\|F_n\|_{\ell^p_\tau(J,X)}$ instead of $\|(F_n)_{n\in\mathbb{Z}}\|_{\ell^p_\tau(J,X)}$, where a “free” variable n is assumed as the summation variable. In the case $J = [0,T]$, we abbreviate $L^p_T X = L^p([0,T],X)$ and $\ell^p_{\tau,T}X = \ell^p_\tau([0,T],X)$. We also use the short-hand notations $L^pX = L^p(\mathbb{R},X)$, $\ell^p_\tau X = \ell^p_\tau(\mathbb{R},X)$, and $\ell^pX = \ell^p_1(\mathbb{R},X) = \ell^p(\mathbb{Z},X)$.

5. Linear estimates

In the following two Sections 5.1 and 5.2 we only work on $\mathbb{R}^3$, therefore we abbreviate $L^q = L^q(\mathbb{R}^3)$ etc.

5.1. Strichartz estimates on the full space

A triple (p, q, γ) is called *admissible, $\mathcal{H}^1$-admissible* (for the wave equation in dimension three) if $p \in (2, \infty]$, $q \in [2, \infty)$, and

$$\frac{1}{p} + \frac{1}{q} \le \frac{1}{2}, \qquad \frac{1}{p} + \frac{3}{q} = \frac{3}{2} - \gamma. \tag{5.1}$$

One then has $\gamma \in [0, \frac{3}{2})$, and the equality in (5.1) is called *scaling condition.* The following theorem is well known, cf. Chapter IV.1 of [66].

Theorem 5.1. *Let (p, q, γ) be admissible. Then we have the estimate*

$$\|e^{\mathrm{i}t|\nabla|} f\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^3))} \lesssim_{p,q} \|f\|_{\dot{H}^\gamma}$$

for all $f \in \dot{H}^\gamma(\mathbb{R}^3)$.

Observe that for $p = \infty$, the estimate can also be deduced from homogeneous Sobolev embedding (cf. Theorem A.1) followed by Plancherel's theorem. If $p < \infty$ however, the scaling condition implies that we save $1/p$ derivatives compared to the fixed-time Sobolev embedding. Moreover, we then obtain some temporal decay at infinity since the estimate is global in time. The equality in (5.1) is necessary due to the scaling $f \mapsto f_\lambda$ with $f_\lambda(x) = f(\lambda x)$, which explains the terminology. The necessity of the inequality in (5.1) follows from the so-called *Knapp example*, see Exercise 2.43 of [71]. Theorem 5.1 remains true for triples (p, ∞, γ) with $p \in (2, \infty)$ that satisfy (5.1), see Proposition 0.1 of [47].

Remark 5.2. The transformation $f \mapsto \bar{f}$ shows that one can always replace $e^{\mathrm{i}t|\nabla|}$ with $e^{-\mathrm{i}t|\nabla|}$ in the estimates of this chapter.

The estimate from Theorem 5.1 can be applied to the wave equation in the following way. Let $\gamma \in [0, \frac{3}{2})$. For given initial data $f \in \dot{H}^\gamma(\mathbb{R}^3)$ and $g \in \dot{H}^{\gamma-1}(\mathbb{R}^3)$, we define the function

$$w(t) \coloneqq \cos(t|\nabla|) f + |\nabla|^{-1} \sin(t|\nabla|) g, \quad t \in \mathbb{R}.$$

Using the Fourier transform, one checks that $w \in C(\mathbb{R}, \dot{H}^\gamma)$ solves (in the sense of tempered distributions) the linear wave equation

$$\partial_{tt} w - \Delta w = 0, \quad w(0) = f, \quad \partial_t w(0) = g.$$

Applied to the decomposition

$$\sin(t|\nabla|) = \frac{1}{2\mathrm{i}}(e^{\mathrm{i}t|\nabla|} - e^{-\mathrm{i}t|\nabla|}), \qquad \cos(t|\nabla|) = \frac{1}{2}(e^{\mathrm{i}t|\nabla|} + e^{-\mathrm{i}t|\nabla|}), \tag{5.2}$$

Theorem 5.1 and Remark 5.2 yield the inequality

$$\|w\|_{L^pL^q} \lesssim_{p,q} \|f\|_{\dot{H}^\gamma} + \|g\|_{\dot{H}^{\gamma-1}},$$

whenever (p,q,γ) is admissible.

To obtain estimates for discrete time, we start with some standard definitions and results.

Definition 5.3. *Let $\chi \in C_c^\infty(\mathbb{R}^3,\mathbb{R})$ be a radial function with $\chi = 1$ on $B(0,1)$ and $\operatorname{supp}\chi \subseteq B(0,2)$. For $\xi \in \mathbb{R}^3$ and $j \in \mathbb{Z}$ we set*

$$\psi(\xi) \coloneqq \chi(\xi) - \chi(2\xi),$$
$$\psi_j(\xi) \coloneqq \psi\Big(\frac{\xi}{2^j}\Big), \quad P_j u \coloneqq \mathcal{F}^{-1}(\psi_j \hat{u})$$

for $u \in \mathcal{S}'$.

These definitions yield $\operatorname{supp}\psi_j \subseteq \{\xi \in \mathbb{R}^3 : 2^{j-1} \le |\xi| \le 2^{j+1}\}$ and the identity

$$\sum_{j\in\mathbb{Z}} \psi_j(\xi) = 1, \quad \xi \in \mathbb{R}^3 \setminus \{0\}.$$

By the convolution theorem, we can also write $P_j f = (2\pi)^{-3/2}\mathcal{F}^{-1}(\psi_j) * f$ (cf. Proposition A.6). Young's convolution inequality then shows that the "Littlewood–Paley projections" P_j are bounded in L^p uniformly in $j \in \mathbb{Z}$ and $p \in [1,\infty]$. Indeed, using the dilation operator D_a given by $(D_a f)(x) \coloneqq f(ax)$, we have

$$\begin{aligned}\|P_j f\|_{L^p} &\le \|\mathcal{F}^{-1}(D_{2^{-j}}\psi) * f\|_{L^p} \le \|\mathcal{F}^{-1}(D_{2^{-j}}\psi)\|_{L^1}\|f\|_{L^p} = 2^{3j}\|D_{2^j}\mathcal{F}^{-1}(\psi)\|_{L^1}\|f\|_{L^p} \\ &= \|\mathcal{F}^{-1}(\psi)\|_{L^1}\|f\|_{L^p} \lesssim \|f\|_{L^p},\end{aligned}$$

exploiting Lemma A.7.

From Proposition III.1.5 of [66], we recall the kernel bound

$$\|\mathcal{F}^{-1}(e^{\mathrm{i}t|\xi|}\psi)\|_{L^\infty} \lesssim (1+|t|)^{-1}, \quad t \in \mathbb{R}. \tag{5.3}$$

The proof of the Strichartz estimates is based on the following well-known frequency-localized dispersive inequality. We give the proof for convenience.

Lemma 5.4. *It holds*

$$\|e^{\mathrm{i}t|\nabla|}P_j f\|_{L^q} \lesssim 2^{3j(1-\frac{2}{q})}(1+2^j|t|)^{-(1-\frac{2}{q})}\|f\|_{L^{q'}}$$

for all $j \in \mathbb{Z}$, $t \in \mathbb{R}$, $q \in [2,\infty]$, and $f \in L^{q'}$.

Proof. Let first $f \in L^1$. Young's convolution inequality yields

$$\|e^{\mathrm{i}t|\nabla|}P_j f\|_{L^\infty} \lesssim \|\mathcal{F}^{-1}(e^{\mathrm{i}t|\xi|}\psi_j)\|_{L^\infty}\|f\|_{L^1}.$$

With the dilation operator D_a from above, we compute

$$\mathcal{F}^{-1}(e^{\mathrm{i}t|\xi|}\psi_j) = \mathcal{F}^{-1}(e^{\mathrm{i}t|\xi|}D_{2^{-j}}\psi) = 2^{3j}D_{2^j}\mathcal{F}^{-1}(e^{\mathrm{i}2^j t|\xi|}\psi).$$

Estimate (5.3) now gives

$$\|e^{\mathrm{i}t|\nabla|}P_j f\|_{L^\infty} \lesssim 2^{3j}(1+2^j|t|)^{-1}\|f\|_{L^1}.$$

The assertion follows from the Riesz–Thorin theorem by interpolation with the L^2-bound $\|e^{\mathrm{i}t|\nabla|}P_j f\|_{L^2} \lesssim \|f\|_{L^2}$. □

Now we turn our attention to the discrete-time setting. In the following $\ell^p L^q$ estimates, the ℓ^p-summation is always taken over the variable n. We start with frequency-localized inequalities.

Lemma 5.5. *Let (p, q, γ) be admissible. Then the estimates*

$$\Big\|\sum_{k\in\mathbb{Z}} e^{\mathrm{i}(n-k)|\nabla|}P_j F_k\Big\|_{\ell^p L^q} \lesssim_{p,q} 2^{2j\gamma}(2^{\frac{2j}{p}}+1)\|F\|_{\ell^{p'}L^{q'}}, \tag{5.4}$$

$$\Big\|\sum_{k\in\mathbb{Z}} e^{-\mathrm{i}k|\nabla|}P_j F_k\Big\|_{L^2} \lesssim_{p,q} 2^{j\gamma}(2^{\frac{j}{p}}+1)\|F\|_{\ell^{p'}L^{q'}}, \tag{5.5}$$

$$\|P_j e^{\mathrm{i}n|\nabla|}f\|_{\ell^p L^q} \lesssim_{p,q} 2^{j\gamma}(2^{\frac{j}{p}}+1)\|P_j f\|_{L^2} \tag{5.6}$$

hold for all $F \in \ell^{p'}L^{q'}$, $f \in L^2$, and $j \in \mathbb{Z}$.

Proof. We first deduce from Lemma 5.4 the estimate

$$\begin{aligned}
\Big\|\sum_{k\in\mathbb{Z}} e^{\mathrm{i}(n-k)|\nabla|}P_j F_k\Big\|_{\ell^p L^q} &\le \Big\|\sum_{k\in\mathbb{Z}} \|e^{\mathrm{i}(n-k)|\nabla|}P_j F_k\|_{L^q}\Big\|_{\ell^p} \\
&\lesssim 2^{3j(1-\frac{2}{q})}\Big\|\sum_{k\in\mathbb{Z}} \frac{\|F_k\|_{L^{q'}}}{(1+2^j|n-k|)^{1-\frac{2}{q}}}\Big\|_{\ell^p} \\
&\le 2^{2j(\frac{1}{p}+\gamma)}\Big\|\sum_{k\in\mathbb{Z}} \frac{\|F_k\|_{L^{q'}}}{(1+2^j|n-k|)^{\frac{2}{p}}}\Big\|_{\ell^p},
\end{aligned}$$

where the last inequality follows from the admissibility conditions (5.1). The first assertion for $p=\infty$ is now clear. For $p<\infty$ we compute

$$\begin{aligned}
2^{2j(\frac{1}{p}+\gamma)}\Big\|\sum_{k\in\mathbb{Z}} \frac{\|F_k\|_{L^{q'}}}{(1+2^j|n-k|)^{\frac{2}{p}}}\Big\|_{\ell^p} &\le 2^{2j(\frac{1}{p}+\gamma)}\Big(\Big\|\|F_n\|_{L^{q'}}\Big\|_{\ell^p} + \Big\|\sum_{\substack{k\in\mathbb{Z}\\ k\neq n}} \frac{\|F_k\|_{L^{q'}}}{(1+2^j|n-k|)^{\frac{2}{p}}}\Big\|_{\ell^p}\Big) \\
&\le 2^{2j\gamma}\Big(2^{\frac{2j}{p}}\|F\|_{\ell^{p'}L^{q'}} + \Big\|\sum_{\substack{k\in\mathbb{Z}\\ k\neq n}} \frac{\|F_k\|_{L^{q'}}}{|n-k|^{\frac{2}{p}}}\Big\|_{\ell^p}\Big)
\end{aligned}$$

$$\lesssim_{p,q} 2^{2j\gamma}(2^{\frac{2j}{p}}+1)\|F\|_{\ell^{p'}L^{q'}}$$

with the help of the discrete Hardy–Littlewood–Sobolev inequality (see Proposition (a) in [67]). This proves (5.4). We note that in the case $n=k$ the factor $2^{2j/p}$ does not cancel. This is the main difference to the continuous case, where such a term does not appear in the continuous Hardy–Littlewood–Sobolev inequality.

The other two claims follow by a TT^* *argument*, exploiting the duality of $\ell^p L^q$ and $\ell^{p'}L^{q'}$. Let first $F \in c_{00}(\mathbb{Z}, L^{q'})$ be a finitely supported sequence, i.e., there exists $N \in \mathbb{N}$ such that $F_n = 0$ for all $|n| \geq N$. From (5.4) we derive

$$\begin{aligned}\Big\|\sum_{k\in\mathbb{Z}} e^{-\mathrm{i}k|\nabla|}P_jF_k\Big\|_{L^2}^2 &= \sum_{n\in\mathbb{Z}}\Big\langle \sum_{k\in\mathbb{Z}} e^{\mathrm{i}(n-k)|\nabla|}P_jF_k, P_jF_n\Big\rangle \\ &\leq \Big\|\sum_{k\in\mathbb{Z}} e^{\mathrm{i}(n-k)|\nabla|}P_jF_k\Big\|_{\ell^pL^q}\|P_jF_n\|_{\ell^{p'}L^{q'}} \lesssim_{p,q} 2^{2j\gamma}(2^{\frac{2j}{p}}+1)\|F\|^2_{\ell^{p'}L^{q'}}.\end{aligned}$$

Here we write $\langle\cdot,\cdot\rangle$ for the L^2-inner product and exploit that the adjoint operator in L^2 is given by $(e^{\mathrm{i}n|\nabla|})^* = e^{-\mathrm{i}n|\nabla|}$, and that by Bernstein's inequality from Lemma A.3, P_j maps $L^{q'}$ to L^2. The assertion (5.5) for general $F \in \ell^{p'}L^{q'}$ then follows from the density of $c_{00}(\mathbb{Z}, L^{q'})$ in $\ell^{p'}L^{q'}$. By duality, (5.5) implies that

$$\|P_je^{\mathrm{i}n|\nabla|}f\|_{\ell^pL^q} \lesssim_{p,q} 2^{j\gamma}(2^{\frac{j}{p}}+1)\|f\|_{L^2} \tag{5.7}$$

for all $f \in L^2$, using that P_j is self-adjoint in L^2. To recover P_j on the right-hand side, we use the fattened Littlewood–Paley projection $\widetilde{P}_j \coloneqq P_{j-1}+P_j+P_{j+1}$ for $j\in\mathbb{Z}$, noting that $\widetilde{P}_jP_j = P_j$. Clearly, (5.7) also holds with $\widetilde{P}_j$ instead of P_j. In this inequality, we then replace f by P_jf to obtain the last assertion (5.6). □

To obtain discrete-time Strichartz estimates, it is necessary to include a suitable filter operator. This was first observed in case of the Schrödinger equation, cf. [36]. The filter will be exploited to deal with the factor $2^{j/p}+1$ in (5.6). As filter, we choose a frequency cut-off. For each $K \geq 1$ we define the Fourier multiplication operator

$$\pi_K \coloneqq \mathcal{F}^{-1}\mathbb{1}_{\{|\xi|_\infty\leq K\}}\mathcal{F}. \tag{5.8}$$

By Plancherel's theorem, the operators π_K are clearly bounded uniformly in K on every L^2-based Sobolev space.

We can now show the desired discrete Strichartz estimates. We stress that these estimates fail without the cut-off if $p<\infty$. For instance, take a function $f \in \dot{H}^\gamma \setminus L^q$ in Theorem 5.6, cf. Remark 5.7. On the other hand, for $f \in \dot{H}^\gamma$ the map $\pi_K f$ belongs to all L^r with $r \geq q_0$ and $3/2-\gamma = 3/q_0$ by Sobolev's embedding (Theorem A.1) and since its Fourier transform belongs to L^1.

Theorem 5.6. *Let (p,q,γ) be admissible. Then we have the estimate*

$$\|\pi_Ke^{\mathrm{i}n\tau|\nabla|}f\|_{\ell^p_\tau L^q} \lesssim_{p,q} (K\tau)^{\frac{1}{p}}\|f\|_{\dot{H}^\gamma},$$

for all $\tau\in(0,1]$, $K \geq \tau^{-1}$, and $f \in \dot{H}^\gamma$.

Proof. We first prove the theorem in the case $\tau = 1$. By means of the Littlewood–Paley square function estimate (see Theorem 6.1.2 of [24]), Minkowski's inequality, and Lemma 5.5, we compute

$$\begin{aligned}
\|\pi_K e^{\mathrm{i}n|\nabla|} f\|_{\ell^p L^q} &\lesssim_q \Big\| \Big(\sum_{j\in\mathbb{Z}} |P_j \pi_K e^{\mathrm{i}n|\nabla|} f|^2 \Big)^{\frac12} \Big\|_{\ell^p L^q} \le \Big(\sum_{j\in\mathbb{Z}} \|P_j e^{\mathrm{i}n|\nabla|} \pi_K f\|_{\ell^p L^q}^2 \Big)^{\frac12} \\
&\lesssim_{p,q} \Big(\sum_{j\in\mathbb{Z}} \|2^{j\gamma}(2^{\frac{j}{p}} + 1) P_j \pi_K f\|_{L^2}^2 \Big)^{\frac12} \lesssim K^{\frac1p} \Big(\sum_{j\in\mathbb{Z}} \|2^{j\gamma} P_j f\|_{L^2}^2 \Big)^{\frac12} \\
&\lesssim K^{\frac1p} \|f\|_{\dot H^\gamma},
\end{aligned}$$

also using that $P_j \pi_K = 0$ for $K \lesssim 2^j$. The assertion for general $\tau \in (0, 1]$ then follows from a scaling argument. Indeed, we can write

$$\pi_K e^{\mathrm{i}t|\nabla|} f = D_{\tau^{-1}} \pi_{K\tau} e^{\mathrm{i}\frac{t}{\tau}|\nabla|} D_\tau f, \tag{5.9}$$

where the spatial dilation operator D_a is given by $(D_a f)(x) = f(ax)$. Thus, we get the general estimate

$$\begin{aligned}
\|\pi_K e^{\mathrm{i}n\tau|\nabla|} f\|_{\ell^p_\tau L^q} &= \tau^{\frac1p} \|D_{\tau^{-1}} \pi_{K\tau} e^{\mathrm{i}n|\nabla|} D_\tau f\|_{\ell^p L^q} = \tau^{\frac1p + \frac3q} \|\pi_{K\tau} e^{\mathrm{i}n|\nabla|} D_\tau f\|_{\ell^p L^q} \\
&\lesssim_{p,q} \tau^{\frac1p + \frac3q} (K\tau)^{\frac1p} \|D_\tau f\|_{\dot H^\gamma} = \tau^{\frac1p + \frac3q - \frac32 + \gamma} (K\tau)^{\frac1p} \|f\|_{\dot H^\gamma} = (K\tau)^{\frac1p} \|f\|_{\dot H^\gamma}
\end{aligned}$$

by the scaling condition in (5.1). □

Remark 5.7. The estimate from Theorem 5.6 is optimal in the following sense. If we only consider the term with $n = 0$ in the left-hand side of Theorem 5.6, we obtain the frequency-localized Sobolev embedding

$$\|\pi_K f\|_{L^q} \lesssim_{p,q} K^{\frac1p} \|f\|_{\dot H^\gamma},$$

which in general is sharp by scaling and (5.1) (cf. Theorem A.1 and Lemma A.3).

Remark 5.8. Later on, we will always set $K = \tau^{-1}$, since this choice optimizes the global error for the Strang splitting scheme. If we allowed for higher frequencies $K > \tau^{-1}$, the factor $(K\tau)^{1/p}$ in the discrete Strichartz estimates would grow. Nevertheless, such a choice could be interesting for further applications (for example for an error analysis of the corrected Lie splitting proposed in [49], a method with a different local error structure). However, we were not able to show better convergence rates for the corrected Lie splitting than for the Strang splitting in 3D so far, cf. Remarks 1.7 and 4.11. See also [53] for a related discussion in case of Schrödinger equations.

Remark 5.9. There exists an alternative (simpler) approach to discrete-time Strichartz estimates, which uses the well-known continuous estimates just as a "black box". In the context of Schrödinger equations, it was used in Lemma 2.6 of the recent preprint [73], see also Lemma 2.1 of [70] for a similar technique. But this approach yields a weaker estimate compared to Theorem 5.6 in the case when $K > \tau^{-1}$. We give the details.

Let (p, q, γ) be admissible with $p < \infty$. We first let $\tau = 1$ and $K > 0$ be arbitrary. For a function $f \in \dot{H}^\gamma$, we compute

$$\begin{aligned}\|e^{\mathrm{i}n|\nabla|}\pi_K f\|_{\ell^p L^q}^p &= \sum_{n\in\mathbb{Z}} \int_{n-1}^{n} \|e^{\mathrm{i}n|\nabla|}\pi_K f\|_{L^q}^p \,\mathrm{d}t \\ &\lesssim \sum_{n\in\mathbb{Z}} \int_{n-1}^{n} \|(e^{\mathrm{i}n|\nabla|} - e^{\mathrm{i}t|\nabla|})\pi_K f\|_{L^q}^p \,\mathrm{d}t + \sum_{n\in\mathbb{Z}} \int_{n-1}^{n} \|e^{\mathrm{i}t|\nabla|}\pi_K f\|_{L^q}^p \,\mathrm{d}t.\end{aligned}$$

Note that the last term is equal to $\|e^{\mathrm{i}t|\nabla|}\pi_K f\|_{L^p L^q}^p$, therefore it can be treated directly by the continuous Strichartz estimate from Theorem 5.1. The first term is estimated by

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \int_{n-1}^{n} \|(e^{\mathrm{i}n|\nabla|} - e^{\mathrm{i}t|\nabla|})\pi_K f\|_{L^q}^p \,\mathrm{d}t &= \sum_{n\in\mathbb{Z}} \int_{n-1}^{n} \Big\| \int_t^n \mathrm{i}|\nabla| e^{\mathrm{i}\sigma|\nabla|}\pi_K f \,\mathrm{d}\sigma \Big\|_{L^q}^p \,\mathrm{d}t \\ &\le \sum_{n\in\mathbb{Z}} \int_{n-1}^{n} \int_{n-1}^{n} \||\nabla| e^{\mathrm{i}\sigma|\nabla|}\pi_K f\|_{L^q}^p \,\mathrm{d}\sigma \,\mathrm{d}t \\ &= \|e^{\mathrm{i}t|\nabla|}\pi_K |\nabla| f\|_{L^p L^q}^p \lesssim_{p,q} \|\pi_K |\nabla| f\|_{\dot{H}^\gamma}^p \\ &\lesssim K^p \|f\|_{\dot{H}^\gamma}^p,\end{aligned}$$

where we used Theorem 5.1 and finally Bernstein's inequality from Lemma A.3. Altogether, this gives the estimate

$$\|\pi_K e^{\mathrm{i}n|\nabla|} f\|_{\ell^p L^q} \lesssim_{p,q} (1 + K)\|f\|_{\dot{H}^\gamma}.$$

The scaling argument from the proof of Theorem 5.6 then yields the estimate

$$\|\pi_K e^{\mathrm{i}n\tau|\nabla|} f\|_{\ell^p_\tau L^q} \lesssim_{p,q} (1 + K\tau)\|f\|_{\dot{H}^\gamma},$$

for general $\tau > 0$. We see that this estimate is inferior to Theorem 5.6 if $K > \tau^{-1}$, but for $K = \tau^{-1}$ they are the same.

5.2. Endpoint estimates with logarithmic loss

The estimates of Theorem 5.1 and 5.6 fail at the so-called "double endpoint" $(p, q, \gamma) = (2, \infty, 1)$, see [70] or Exercise 2.44 in [71] for a discussion. However, estimates with logarithmic corrections in time and frequency are available. For instance, from Proposition 6.3 of [43], one can deduce that the inequality

$$\|\pi_K e^{\mathrm{i}t|\nabla|} f\|_{L^2_T L^\infty} \lesssim (\log(1 + KT))^{\frac{1}{2}} \|f\|_{\dot{H}^1} \tag{5.10}$$

holds for all $f \in \dot{H}^1(\mathbb{R}^3)$, $K \ge 1$, and $T \ge 0$. In this section, we deduce the discrete-time analogue of (5.10) by adapting the approach from Section 8 of [43]. First, we need two lemmas with basic estimates. The first one is contained in the proof of Lemma 8.1 in [43].

Lemma 5.10. *The function*

$$M(\lambda, z) := \int_{B(0,1)} |\xi|^{-2} \cos(\lambda|\xi|) e^{\mathrm{i}z\cdot\xi} \,\mathrm{d}\xi$$

satisfies the decay estimate

$$|M(\lambda, z)| \lesssim \frac{1}{1+|\lambda|}$$

for all $\lambda \in \mathbb{R}$ and $z \in \mathbb{R}^3$.

Proof. We note that M is rotation invariant in z, hence we can take $z = (0, 0, |z|)$ without loss of generality. The application of polar coordinates then yields

$$\begin{aligned} M(\lambda, z) &= 2\pi \int_0^1 \int_{-\pi/2}^{\pi/2} \cos(\lambda r) e^{\mathrm{i}r|z|\sin(\theta)} \cos(\theta) \,\mathrm{d}\theta \,\mathrm{d}r = 2\pi \int_0^1 \int_{-1}^1 \cos(\lambda r) e^{\mathrm{i}r|z|\omega} \,\mathrm{d}\omega \,\mathrm{d}r \\ &= \pi \int_{-1}^1 \int_{-1}^1 \cos(\lambda r) e^{\mathrm{i}r|z|\omega} \,\mathrm{d}\omega \,\mathrm{d}r, \end{aligned}$$

since the r-integral from 0 to 1 coincides with that from -1 to 0 by symmetry. We clearly have

$$|M(\lambda, z)| \leq 4\pi. \tag{5.11}$$

On the other hand, the r-integral can be calculated as

$$\begin{aligned} M(\lambda, z) &= \frac{\pi}{2} \int_{-1}^1 \int_{-1}^1 (e^{\mathrm{i}\lambda r} + e^{-\mathrm{i}\lambda r}) e^{\mathrm{i}r|z|\omega} \,\mathrm{d}r \,\mathrm{d}\omega \\ &= \frac{\pi}{2} \int_{-1}^1 \int_{-1}^1 \Big(e^{\mathrm{i}r(|z|\omega+\lambda)} + e^{\mathrm{i}r(|z|\omega-\lambda)} \Big) \,\mathrm{d}r \,\mathrm{d}\omega \\ &= \frac{\pi}{2\mathrm{i}} \int_{-1}^1 \Big(\frac{e^{\mathrm{i}(|z|\omega+\lambda)} - e^{-\mathrm{i}(|z|\omega+\lambda)}}{|z|\omega+\lambda} + \frac{e^{\mathrm{i}(|z|\omega-\lambda)} - e^{-\mathrm{i}(|z|\omega-\lambda)}}{|z|\omega-\lambda} \Big) \,\mathrm{d}\omega \\ &= \pi \int_{-1}^1 \Big(\operatorname{sinc}(|z|\omega+\lambda) + \operatorname{sinc}(|z|\omega-\lambda) \Big) \,\mathrm{d}\omega, \end{aligned}$$

where $\operatorname{sinc}(\lambda) = (\sin\lambda)/\lambda$. If $|\lambda| \geq 2|z| > 0$, it follows that

$$\begin{aligned} |M(\lambda, z)| &\leq \pi \int_{-1}^1 \Big(\frac{1}{||z|\omega+\lambda|} + \frac{1}{||z|\omega-\lambda|} \Big) \,\mathrm{d}\omega \leq 2\pi \int_{-1}^1 \frac{1}{|\lambda| - |z||\omega|} \,\mathrm{d}\omega \leq \frac{4\pi}{|\lambda| - |z|} \\ &\leq \frac{8\pi}{|\lambda|}. \end{aligned} \tag{5.12}$$

Moreover, for arbitrary $\lambda \in \mathbb{R}$ and $z \neq 0$, using the symmetry of sinc, we can compute

$$\begin{aligned} M(\lambda, z) &= \pi \int_{-1}^1 \Big(\operatorname{sinc}(|z|\omega+\lambda) + \operatorname{sinc}(|z|\omega-\lambda) \Big) \,\mathrm{d}\omega \\ &= \frac{\pi}{|z|} \Big(\operatorname{Si}(|z|+\lambda) - \operatorname{Si}(-|z|+\lambda) + \operatorname{Si}(|z|-\lambda) - \operatorname{Si}(-|z|-\lambda) \Big) \end{aligned}$$

$$= \frac{2\pi}{|z|}\Big(\operatorname{Si}(|z|+\lambda) + \operatorname{Si}(|z|-\lambda)\Big).$$

Since the function $\operatorname{Si}(t) = \int_0^t \operatorname{sinc}(s)\,\mathrm{d}s$ satisfies the global bound $|\operatorname{Si}(t)| \leq 2$, this implies

$$|M(\lambda,z)| \leq \frac{8\pi}{|z|}.$$

We combine this estimate with (5.11) and (5.12) to obtain the assertion. □

Lemma 5.11. *The function*

$$A(\beta,n,j) \coloneqq \Big(\frac{1}{1+\beta|n-j|} + \frac{1}{1+\beta(n+j)}\Big)$$

satisfies the estimate

$$\max_{j=0,\dots,N} \sum_{n=0}^{N} A(\beta,n,j) \lesssim 1 + \beta^{-1}\log(1+N\beta)$$

for all $N \in \mathbb{N}_0$ and $\beta > 0$.

Proof. Let $j \in \{0,\dots,N\}$. We have

$$\begin{aligned}
\sum_{n=0}^{N} \frac{1}{1+\beta(n+j)} &\leq \sum_{n=0}^{N} \frac{1}{1+\beta n} = 1 + \sum_{n=1}^{N} \frac{1}{1+\beta n} \leq 1 + \beta^{-1}\int_0^{N\beta} \frac{1}{1+t}\,\mathrm{d}t \\
&= 1 + \beta^{-1}\log(1+N\beta), \\
\sum_{n=0}^{N} \frac{1}{1+\beta|n-j|} &= 1 + \sum_{n=0}^{j-1} \frac{1}{1+\beta(j-n)} + \sum_{n=j+1}^{N} \frac{1}{1+\beta(n-j)} \\
&= 1 + \sum_{n=1}^{j} \frac{1}{1+\beta n} + \sum_{n=1}^{N-j} \frac{1}{1+\beta n} \leq 1 + 2\sum_{n=1}^{N} \frac{1}{1+\beta n} \\
&\leq 1 + 2\beta^{-1}\log(1+N\beta).
\end{aligned}$$

□

Now we show the announced discrete-time endpoint estimates with logarithmic loss.

Theorem 5.12. *The estimate*

$$\|\pi_K e^{\mathrm{i}n\tau|\nabla|} f\|_{\ell^2_{\tau,T}L^\infty} \lesssim (K\tau + \log(1+KT))^{\frac{1}{2}} \|f\|_{\dot{H}^1}$$

holds for all $\tau \in (0,1]$, $K > 0$, $T \geq 0$, and $f \in \dot{H}^1(\mathbb{R}^3)$.

Proof. Due to a scaling argument, it is enough to show that

$$\|\tilde{\pi}_1 e^{\mathrm{i}n\beta|\nabla|} f\|_{\ell^2_{1,N}L^\infty} \lesssim (1 + \beta^{-1}\log(1+N\beta))^{\frac{1}{2}} \|f\|_{\dot{H}^1} \tag{5.13}$$

for any $\beta > 0$ and $N \in \mathbb{N}_0$, where $\tilde{\pi}_K \coloneqq \mathcal{F}^{-1}\mathbb{1}_{B(0,K)}\mathcal{F}$. Indeed, this estimate and (5.9) imply that

$$\begin{aligned}
&\|\tilde{\pi}_K e^{in\tau|\nabla|} f\|_{\ell^2_{\tau,T}L^\infty} \\
&\quad = \tau^{\frac{1}{2}} \|D_K \tilde{\pi}_1 e^{inK\tau|\nabla|} D_{K^{-1}} f\|_{\ell^2_{1,\lfloor T/\tau\rfloor}L^\infty} = \tau^{\frac{1}{2}} \|\tilde{\pi}_1 e^{inK\tau|\nabla|} D_{K^{-1}} f\|_{\ell^2_{1,\lfloor T/\tau\rfloor}L^\infty} \\
&\quad \lesssim \tau^{\frac{1}{2}} (1 + (K\tau)^{-1}\log(1+KT))^{\frac{1}{2}} \|D_{K^{-1}} f\|_{\dot{H}^1} \\
&\quad = (K\tau)^{\frac{1}{2}} (1 + (K\tau)^{-1}\log(1+KT))^{\frac{1}{2}} \|f\|_{\dot{H}^1} = (K\tau + \log(1+KT))^{\frac{1}{2}} \|f\|_{\dot{H}^1}
\end{aligned}$$

and thus

$$\begin{aligned}
\|\pi_K e^{in\tau|\nabla|} f\|_{\ell^2_{\tau,T}L^\infty} &= \|\tilde{\pi}_{\sqrt{3}K} e^{in\tau|\nabla|} \pi_K f\|_{\ell^2_{\tau,T}L^\infty} \lesssim (K\tau + \log(1+KT))^{\frac{1}{2}} \|\pi_K f\|_{\dot{H}^1} \\
&\lesssim (K\tau + \log(1+KT))^{\frac{1}{2}} \|f\|_{\dot{H}^1}.
\end{aligned}$$

Inequality (5.13) is shown via the dual estimate

$$\Big\| \sum_{n=0}^{N} \tilde{\pi}_1 e^{-in\beta|\nabla|} F_n \Big\|_{\dot{H}^{-1}} \lesssim (1 + \beta^{-1}\log(1+N\beta))^{\frac{1}{2}} \|F\|_{\ell^2_{1,N}L^1} \tag{5.14}$$

for $F \in \ell^2 L^1$. Instead of the exponential, we treat sine and cosine. From the definition of the $\dot{H}^1$-norm and Fubini's theorem, we deduce

$$\begin{aligned}
&\Big\| \sum_{n=0}^{N} \tilde{\pi}_1 \sin(n\beta|\nabla|) F_n \Big\|_{\dot{H}^{-1}}^2 = \Big\| |\xi|^{-1} \sum_{n=0}^{N} \mathbb{1}_{B(0,1)} \sin(n\beta|\xi|) \hat{F}_n(\xi) \Big\|_{L^2}^2 \\
&= \int_{B(0,1)} |\xi|^{-2} \sum_{n,j=0}^{N} \sin(n\beta|\xi|)\sin(j\beta|\xi|) \hat{F}_n(\xi)\overline{\hat{F}_j(\xi)} \,\mathrm{d}\xi \\
&= (2\pi)^{-3} \int_{B(0,1)} |\xi|^{-2} \sum_{n,j=0}^{N} \sin(n\beta|\xi|)\sin(j\beta|\xi|) \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{i(y-x)\cdot\xi} F_n(x)\overline{F_j(y)} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}\xi \\
&= (2\pi)^{-3} \sum_{n,j=0}^{N} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G_-(n\beta, j\beta, y-x) F_n(x)\overline{F_j(y)} \,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

where

$$G_-(a,b,z) \coloneqq \int_{B(0,1)} |\xi|^{-2} \sin(a|\xi|)\sin(b|\xi|) e^{iz\cdot\xi} \,\mathrm{d}\xi.$$

Analogously, we obtain

$$\Big\| \sum_{n=0}^{N} \pi_1 \cos(n\beta|\nabla|) F_n \Big\|_{\dot{H}^{-1}}^2 = (2\pi)^{-3} \sum_{n,j=0}^{N} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G_+(n\beta, j\beta, y-x) F_n(x)\overline{F_j(y)} \,\mathrm{d}x\,\mathrm{d}y,$$

with

$$G_+(a,b,z) \coloneqq \int_{B(0,1)} |\xi|^{-2} \cos(a|\xi|)\cos(b|\xi|) e^{iz\cdot\xi} \,\mathrm{d}\xi.$$

Next, we use the identities $2\cos(t)\cos(s) = \cos(t-s) + \cos(t+s)$ and $2\sin(t)\sin(s) = \cos(t-s) - \cos(t+s)$ to write

$$G_{\pm}(a,b,z) = \frac{1}{2}\Big(M(a-b,z) \pm M(a+b,z)\Big)$$

with $M(\lambda, z)$ from Lemma 5.10. Combined with this lemma, the above equations lead to

$$\begin{aligned}
&\Big\| \sum_{n=0}^{N} \tilde{\pi}_1 e^{-\mathrm{i}n\beta|\nabla|} F_n \Big\|_{\dot{H}^{-1}}^2 \\
&\lesssim \sum_{n,j=0}^{N} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Big(|M(\beta(n-j), y-x)| + |M(\beta(n+j), y-x)| \Big) |F_n(x)||F_j(y)| \,\mathrm{d}x\,\mathrm{d}y \\
&\lesssim \sum_{n,j=0}^{N} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \Big(\frac{1}{1+\beta|n-j|} + \frac{1}{1+\beta|n+j|} \Big) |F_n(x)||F_j(y)| \,\mathrm{d}x\,\mathrm{d}y \\
&= \sum_{n,j=0}^{N} A(\beta,n,j) \|F_n\|_{L^1} \|F_j\|_{L^1},
\end{aligned}$$

with A from Lemma 5.11. We next apply Cauchy–Schwarz twice and Lemma 5.11. Also noting that $A(\beta,n,j)$ is symmetric in n and j, we estimate

$$\begin{aligned}
&\sum_{n,j=0}^{N} A(\beta,n,j) \|F_n\|_{L^1} \|F_j\|_{L^1} \\
&\qquad \le \|F\|_{\ell^2_{1,N} L^1} \Big[\sum_{n=0}^{N} \Big(\sum_{j=0}^{N} A(\beta,j,n) \|F_j\|_{L^1} \Big)^2 \Big]^{\frac{1}{2}} \\
&\qquad \le \|F\|_{\ell^2_{1,N} L^1} \Big[\sum_{n=0}^{N} \Big(\sum_{j=0}^{N} A(\beta,j,n) \Big) \Big(\sum_{j=0}^{N} A(\beta,n,j) \|F_j\|_{L^1}^2 \Big) \Big]^{\frac{1}{2}} \\
&\qquad \le \Big(\max_{n=0,\dots,N} \sum_{j=0}^{N} A(\beta,n,j) \Big)^{\frac{1}{2}} \Big(\max_{j=0,\dots,N} \sum_{n=0}^{N} A(\beta,n,j) \Big)^{\frac{1}{2}} \|F\|_{\ell^2_{1,N} L^1}^2 \\
&\qquad \lesssim \Big(1 + \beta^{-1} \log(1+N\beta) \Big) \|F\|_{\ell^2_{1,N} L^1}^2,
\end{aligned}$$

which shows (5.14). □

5.3. Strichartz estimates on the torus

Thanks to the finite speed of propagation for the wave equation, one expects that locally in time, one has the same Strichartz estimates on the torus $\mathbb{T}^3$ as on the full space $\mathbb{R}^3$. For continuous time, this has been carried out in, e.g., [45] by using suitable extension and cut-off operators. We will follow the same strategy to prove corresponding versions of the discrete-time Theorems 5.6 and 5.12 for the torus.

Let $E\colon \mathcal{D}'(\mathbb{T}^3) \to \mathcal{S}'(\mathbb{R}^3)$ denote the periodic extension operator (where we interpret $\mathbb{T}^3 = [-\pi,\pi]^3$ as above). Note that for $f \in C^\infty(\mathbb{T}^3)$ we have $Ef \in C^\infty(\mathbb{R}^3)$ with periodic partial derivatives. The next lemma shows that an extended Sobolev function multiplied with a smooth cut-off function belongs to the corresponding Sobolev space on $\mathbb{R}^3$.

Lemma 5.13. *Let $\eta \in C_c^\infty(\mathbb{R}^3)$ and $s \in \mathbb{R}$. Then the estimate*

$$\|\eta E f\|_{H^s(\mathbb{R}^3)} \lesssim_{\eta,s} \|f\|_{H^s(\mathbb{T}^3)}$$

is true for any $f \in H^s(\mathbb{T}^3)$.

Proof. By approximation, it suffices to consider smooth f. The statement is clear if $s = 0$, and inductively extends to all $s \in \mathbb{N}$. By interpolation, we then infer the assertion for all $s \geq 0$. The case $s < 0$ is handled via duality. Let $(\phi_j)_{j\in\mathbb{N}}$ be a smooth partition of unity such that $\sum_{j\in\mathbb{N}} \phi_j = \mathbb{1}$ and $\phi_j \in C_c^\infty(\mathbb{R}^3)$ with $\operatorname{supp}\phi_j \subseteq \{y_j\} + (-\pi,\pi)^3$ for all $j \in \mathbb{N}$ and some $y_j \in \mathbb{R}^3$. We compute

$$\begin{aligned}
\|\eta E f\|_{H^s(\mathbb{R}^3)} &= \sup_{\|g\|_{H^{-s}(\mathbb{R}^3)}=1} \Big| \int_{\mathbb{R}^3} \eta E f \cdot g \,\mathrm{d}x \Big| \\
&= \sup_{\|g\|_{H^{-s}(\mathbb{R}^3)}=1} \Big| \sum_{j\in\mathbb{N}} \int_{\{y_j\}+(-\pi,\pi)^3} E f \cdot \eta g \phi_j \,\mathrm{d}x \Big| \\
&\leq \sup_{\|g\|_{H^{-s}(\mathbb{R}^3)}=1} \|f\|_{H^s(\mathbb{T}^3)} \sum_{j\in\mathbb{N}} \|\eta g \phi_j\|_{H^{-s}(\mathbb{R}^3)} \lesssim_{\eta,s} \|f\|_{H^s(\mathbb{T}^3)},
\end{aligned}$$

where the supremum is taken over smooth g. Here we use that we can consider $(\eta g\phi_j)(y_j + \cdot)$ as a test function on $\mathbb{T}^3$, and that the sum is actually finite thanks to the compact support of η. □

For the discrete-time Strichartz estimates, we need to introduce a Fourier cut-off on the torus (similar as (5.8) on $\mathbb{R}^3$). For $f \in \mathcal{D}'(\mathbb{T}^3)$ and $K \geq 1$ we define the cube frequency cut-off operator π_K via the truncated Fourier series

$$(\pi_K f)(x) := (2\pi)^{-\frac{3}{2}} \sum_{|k|_\infty \leq K} \hat{f}_k e^{\mathrm{i}k\cdot x}, \quad x \in \mathbb{T}^3. \tag{5.15}$$

Here, the sum is taken over all $k \in \mathbb{Z}^3$ with $|k|_\infty = \max_{j=1,2,3} |k_j| \leq K$, and $\hat{f}_k$ denotes the k-th Fourier coefficient of f. From the definition of the Sobolev norm (4.5), it follows that π_K is bounded on all spaces $H^s(\mathbb{T}^3)$, uniformly in $s \in \mathbb{R}$ and $K \geq 1$.

Theorem 5.14. *Let (p,q,γ) be admissible and $T \geq 0$. We then have the estimates*

$$\begin{aligned}
\|e^{\mathrm{i}n\tau|\nabla|}\pi_K f\|_{\ell^p_{\tau,T} L^q(\mathbb{T}^3)} &\lesssim_{p,q,T} (1 + K\tau)^{\frac{1}{p}} \|f\|_{H^\gamma(\mathbb{T}^3)}, \\
\|e^{\mathrm{i}n\tau|\nabla|}\pi_K h\|_{\ell^2_{\tau,T} L^\infty(\mathbb{T}^3)} &\lesssim_T (K\tau + \log K)^{\frac{1}{2}} \|h\|_{H^1(\mathbb{T}^3)},
\end{aligned}$$

for all $f \in H^\gamma(\mathbb{T}^3)$, $h \in H^1(\mathbb{T}^3)$, $\tau \in (0,1]$, and $K \geq 1$.

Proof. We only give the proof for the first estimate, since the second one can be shown in the same way, using Theorem 5.12 instead of Theorem 5.6. We define the function $v(t) := e^{\mathrm{i}t|\nabla|}\pi_K f$ for $t \in \mathbb{R}$. Since

$$v(t) = e^{\mathrm{i}t|\nabla|}\pi_K f = \cos(t|\nabla|)\pi_K f + \mathrm{i}|\nabla|^{-1}\sin(t|\nabla|)|\nabla|\pi_K f$$

is the smooth solution to the linear homogeneous wave equation on $\mathbb{R} \times \mathbb{T}^3$ with initial data $(\pi_K f, \mathrm{i}|\nabla|\pi_K f)$, the extended function Ev solves the corresponding problem on $\mathbb{R} \times \mathbb{R}^3$ with extended initial data $(E\pi_K f, \mathrm{i}E|\nabla|\pi_K f)$, i.e.,

$$(\partial_{tt} - \Delta)Ev = 0, \quad Ev(0) = E\pi_K f, \quad \partial_t Ev(0) = \mathrm{i}E|\nabla|\pi_K f.$$

Let $\eta \in C_c^\infty(\mathbb{R}^3)$ be a cut-off function such that $\eta = 1$ on $B(0, \pi + T)$. The function

$$w(t) := \cos(t|\nabla|)(\eta E\pi_K f) + \mathrm{i}|\nabla|^{-1}\sin(t|\nabla|)(\eta E|\nabla|\pi_K f), \quad t \in \mathbb{R},$$

solves the same full space wave equation with truncated initial data. Finite speed of propagation (see, e.g., Theorem 6 on p. 84 of [19]) yields $Ev(t,x) = w(t,x)$ for all $(t,x) \in \mathbb{R}^{1+3}$ with $|t| + |x| \le \pi + T$. Since this condition is satisfied if $(t,x) \in [0,T] \times (-\pi,\pi)^3$, we obtain

$$\begin{aligned}
&\|v(n\tau)\|_{\ell^p_{\tau,T}L^q(\mathbb{T}^3)} \qquad (5.16)\\
&\quad = \|Ev(n\tau)\|_{\ell^p_{\tau,T}L^q((-\pi,\pi)^3)} = \|w(n\tau)\|_{\ell^p_{\tau,T}L^q((-\pi,\pi)^3)} \le \|w(n\tau)\|_{\ell^p_{\tau,T}L^q(\mathbb{R}^3)}\\
&\quad \le \|\cos(n\tau|\nabla|)(\eta E\pi_K f)\|_{\ell^p_{\tau,T}L^q(\mathbb{R}^3)} + \||\nabla|^{-1}\sin(n\tau|\nabla|)(\eta E|\nabla|\pi_K f)\|_{\ell^p_{\tau,T}L^q(\mathbb{R}^3)}.
\end{aligned}$$

We decompose the cosine-term in (5.16) as

$$\begin{aligned}
&\|\cos(n\tau|\nabla|)(\eta E\pi_K f)\|_{\ell^p_{\tau,T}L^q(\mathbb{R}^3)} \qquad (5.17)\\
&\quad \le \|\pi_{2K}e^{\pm \mathrm{i}n\tau|\nabla|}(\eta E\pi_K f)\|_{\ell^p_{\tau,T}L^q(\mathbb{R}^3)} + \|(I - \pi_{2K})e^{\pm \mathrm{i}n\tau|\nabla|}(\eta E\pi_K f)\|_{\ell^p_{\tau,T}L^q(\mathbb{R}^3)}.
\end{aligned}$$

The first term of (5.17) is estimated using Theorem 5.6 and Lemma 5.13, which gives

$$\begin{aligned}
\|\pi_{2K}e^{\pm \mathrm{i}n\tau|\nabla|}(\eta E\pi_K f)\|_{\ell^p_{\tau,T}L^q(\mathbb{R}^3)} &\lesssim_{p,q} (1 + K\tau)^{\frac{1}{p}}\|\eta E\pi_K f\|_{\dot{H}^\gamma(\mathbb{R}^3)}\\
&\lesssim_T (1 + K\tau)^{\frac{1}{p}}\|f\|_{H^\gamma(\mathbb{T}^3)}.
\end{aligned}$$

For the second term of (5.17), we first compute the Fourier transform

$$(2\pi)^{\frac{3}{2}}\mathcal{F}(\eta E\pi_K f) = \hat{\eta} * \mathcal{F}(E\pi_K f) = \hat{\eta} * \sum_{|k|_\infty \le K}\hat{f}_k\delta_k = \sum_{|k|_\infty \le K}\hat{f}_k\hat{\eta}(\cdot - k),$$

where δ_k denotes the Dirac delta at $k \in \mathbb{Z}^3$. The Hausdorff–Young inequality then implies

$$\begin{aligned}
\|(I - \pi_{2K})e^{\pm \mathrm{i}n\tau|\nabla|}(\eta E\pi_K f)\|_{\ell^p_{\tau,T}L^q(\mathbb{R}^3)} &\le \|\mathbb{1}_{\{|\xi|_\infty \ge 2K\}}e^{\pm \mathrm{i}n\tau|\xi|}\mathcal{F}(\eta E\pi_K f)\|_{\ell^p_{\tau,T}L^{q'}(\mathbb{R}^3)}\\
&\lesssim_T \Big\|\mathbb{1}_{\{|\xi|_\infty \ge 2K\}}\sum_{|k|_\infty \le K}|\hat{f}_k||\hat{\eta}(\cdot - k)|\Big\|_{L^{q'}(\mathbb{R}^3)}.
\end{aligned}$$

For $|\xi|_\infty \geq 2K \geq 2|k|$, we obtain

$$\sum_{|k|_\infty \leq K} |\hat{f}_k| \lesssim K^{\frac{3}{2}} \Big(\sum_{|k|_\infty \leq K} |\hat{f}_k|^2 \Big)^{\frac{1}{2}} \lesssim |\xi|^{\frac{3}{2}} \|f\|_{L^2(\mathbb{T}^3)}$$

thanks to the Cauchy–Schwarz estimate. Moreover, we have

$$|\hat{\eta}(\xi - k)| \lesssim_\eta |\xi - k|^{-5} \leq (|\xi| - |k|)^{-5} \lesssim |\xi|^{-5}$$

since $\hat{\eta}$ is a Schwartz function. These inequalities result in

$$\begin{aligned} \|(I - \pi_{2K}) e^{\pm i n\tau|\nabla|}(\eta E \pi_K f)\|_{\ell^p_{\tau,T} L^q(\mathbb{R}^3)} &\lesssim_T \|\mathbb{1}_{\{|\xi|_\infty \geq 2K\}} |\xi|^{-\frac{7}{2}} \|_{L^{q'}(\mathbb{R}^3)} \|f\|_{L^2(\mathbb{T}^3)} \\ &\lesssim \|f\|_{L^2(\mathbb{T}^3)}. \end{aligned}$$

For the sine term in (5.16), we treat the low frequencies separately to avoid problems coming from the homogeneous anti-derivative $|\nabla|^{-1}$. We decompose

$$\begin{aligned} &\||\nabla|^{-1} \sin(n\tau|\nabla|)(\eta E |\nabla| \pi_K f)\|_{\ell^p_{\tau,T} L^q(\mathbb{R}^3)} \\ &\quad \leq \|\pi_1 |\nabla|^{-1} \sin(n\tau|\nabla|)(\eta E |\nabla| \pi_K f)\|_{\ell^p_{\tau,T} L^q(\mathbb{R}^3)} \\ &\qquad + \|(I - \pi_1)\pi_{2K} |\nabla|^{-1} e^{\pm i n\tau|\nabla|}(\eta E |\nabla| \pi_K f)\|_{\ell^p_{\tau,T} L^q(\mathbb{R}^3)} \\ &\qquad + \|(I - \pi_{2K}) |\nabla|^{-1} e^{\pm i n\tau|\nabla|}(\eta E |\nabla| \pi_K f)\|_{\ell^p_{\tau,T} L^q(\mathbb{R}^3)}. \end{aligned}$$

For the low frequencies, we use Bernstein's inequality Lemma A.3 in space, Hölder's inequality in time, the boundedness of $x \mapsto \frac{1}{x}\sin x$ and finally Lemma 5.13 to obtain

$$\begin{aligned} &\|\pi_1 |\nabla|^{-1} \sin(n\tau|\nabla|)(\eta E |\nabla| \pi_K f)\|_{\ell^p_{\tau,T} L^q(\mathbb{R}^3)} \\ &\quad \lesssim_T \|\pi_1 |\nabla|^{-1} \sin(n\tau|\nabla|)(\eta E |\nabla| \pi_K f)\|_{\ell^\infty_{\tau,T} L^2(\mathbb{R}^3)} \\ &\quad \lesssim_T \|\pi_1(\eta E |\nabla| \pi_K f)\|_{L^2(\mathbb{R}^3)} \lesssim \|\eta E |\nabla| \pi_K f\|_{H^{\gamma-1}(\mathbb{R}^3)} \lesssim_T \||\nabla| \pi_K f\|_{H^{\gamma-1}(\mathbb{T}^3)} \\ &\quad \leq \|f\|_{H^\gamma(\mathbb{T}^3)}. \end{aligned}$$

The medium and high frequency terms are treated as the cosine-term. Theorem 5.6 and Lemma 5.13 yield

$$\begin{aligned} &\|(I - \pi_1)\pi_{2K} |\nabla|^{-1} e^{\pm i n\tau|\nabla|}(\eta E |\nabla| \pi_K f)\|_{\ell^p_{\tau,T} L^q(\mathbb{R}^3)} \\ &\quad \lesssim_{p,q} (1 + K\tau)^{\frac{1}{p}} \|(I - \pi_1)|\nabla|^{-1}(\eta E |\nabla| \pi_K f)\|_{\dot{H}^\gamma(\mathbb{R}^3)} \\ &\quad \lesssim (1 + K\tau)^{\frac{1}{p}} \|\eta E |\nabla| \pi_K f\|_{H^{\gamma-1}(\mathbb{R}^3)} \lesssim_T (1 + K\tau)^{\frac{1}{p}} \|f\|_{H^\gamma(\mathbb{T}^3)}, \end{aligned}$$

where the operator $I - \pi_1$ was used to replace the homogeneous by the inhomogeneous Sobolev norm. Finally, we get as above

$$\|(I - \pi_{2K}) |\nabla|^{-1} e^{\pm i n\tau|\nabla|}(\eta E |\nabla| \pi_K f)\|_{\ell^p_{\tau,T} L^q(\mathbb{R}^3)}$$

$$
\begin{aligned}
&\leq \|\mathbb{1}_{\{|\xi|_\infty \geq 2K\}} |\xi|^{-1} e^{\pm \mathrm{i} n\tau|\xi|} \mathcal{F}(\eta E |\nabla| \pi_K f)\|_{\ell^p_{\tau,T} L^{q'}(\mathbb{R}^3)} \\
&\lesssim_T \Big\| \mathbb{1}_{\{|\xi|_\infty \geq 2K\}} |\xi|^{-1} \sum_{|k|_\infty \leq K} |k \hat{f}_k \hat{\eta}(\cdot - k)| \Big\|_{L^{q'}(\mathbb{R}^3)} \\
&\lesssim \Big\| \mathbb{1}_{\{|\xi|_\infty \geq 2K\}} \sum_{|k|_\infty \leq K} |\hat{f}_k \hat{\eta}(\cdot - k)| \Big\|_{L^{q'}(\mathbb{R}^3)} \lesssim_T \|f\|_{L^2(\mathbb{T}^3)}.
\end{aligned}
$$

The assertion now follows from (5.16) and the above estimates. □

We now show that the discrete-time Strichartz estimates imply the ones in continuous time, using an argument from Theorem 1.3 of [70]. The estimates could also be deduced from the full space inequalities reasoning as in Theorem 5.14, cf. [45].

Corollary 5.15. *Let (p, q, γ) be admissible and $T > 0$. We then have the estimates*

$$
\begin{aligned}
\|e^{\mathrm{i}t|\nabla|} f\|_{L^p_T L^q(\mathbb{T}^3)} &\lesssim_{p,q,T} \|f\|_{H^\gamma(\mathbb{T}^3)}, \\
\|e^{\mathrm{i}t|\nabla|} \pi_K h\|_{L^2_T L^\infty(\mathbb{T}^3)} &\lesssim_T (1 + \log K)^{\frac{1}{2}} \|h\|_{H^1(\mathbb{T}^3)},
\end{aligned}
$$

for all $f \in H^\gamma(\mathbb{T}^3)$, $h \in H^1(\mathbb{T}^3)$, and $K \geq 1$.

Proof. We only give the proof for the second estimate, since it is somewhat non-standard. The first one can be proven in the same way, where one uses the density of functions having compact Fourier support in $H^\gamma(\mathbb{T}^3)$ to get rid of the projection π_K. From Theorem 5.14 we get

$$
\tau \sum_{n=0}^{N} \|\pi_K e^{\mathrm{i}n\tau|\nabla|} h\|^2_{L^\infty(\mathbb{T}^3)} \lesssim_T (K\tau + \log K) \|h\|^2_{H^1(\mathbb{T}^3)}
$$

for all $\tau \in [1/K, 1]$ and $N \in \mathbb{N}_0$ with $N\tau \leq T$. We now replace h with $e^{\mathrm{i}\theta|\nabla|} h$ and integrate from 0 to τ to obtain

$$
\int_0^\tau \sum_{n=0}^{N} \|\pi_K e^{\mathrm{i}(n\tau+\theta)|\nabla|} h\|^2_{L^\infty(\mathbb{T}^3)} \, \mathrm{d}\theta \lesssim_T (K\tau + \log K) \|h\|^2_{H^1(\mathbb{T}^3)},
$$

which implies the assertion if we set $\tau = 1/K$. □

5.4. Application to the wave equation

From now on we will treat the cases $\Omega \in \{\mathbb{R}^3, \mathbb{T}^3\}$ simultaneously whenever possible. We will abbreviate $L^q = L^q(\Omega)$ and $\mathcal{H}^s = \mathcal{H}^s(\Omega)$, where we recall that $\mathcal{H}^s(\Omega) = \dot{H}^s(\mathbb{R}^3)$ if $\Omega = \mathbb{R}^3$ and $\mathcal{H}^s(\Omega) = H^s(\mathbb{T}^3)$ if $\Omega = \mathbb{T}^3$. Moreover, we will only use admissible triples (p, q, γ) with derivative loss $\gamma = 1$. We call a pair (p, q) $\mathcal{H}^1$*-admissible* if $(p, q, 1)$ is admissible in the sense of (5.1).

Corollary 5.16. *Let $T \in (0,\infty)$, $f \in \mathcal{H}^1$, $g \in L^2$, $F \in L^1_T L^2$, and $w \in C([0,T],\mathcal{H}^1)$ be the solution to the inhomogeneous wave equation*

$$\partial_{tt} w - \Delta w = F, \quad w(0) = f, \quad \partial_t w(0) = g.$$

Let moreover (p,q) be $\mathcal{H}^1$-admissible. Then w satisfies the estimates

$$\|w\|_{L^p_T L^q} + \|\pi_N w\|_{L^p_T L^q} + \|\pi_N w(n\tau)\|_{\ell^p_{\tau,T} L^q} \lesssim_{p,q,T} \|f\|_{\mathcal{H}^1} + \|g\|_{L^2} + \|F\|_{L^1_T L^2} \tag{5.18}$$

and

$$\|\pi_N w\|_{L^2_T L^\infty} + \|\pi_N w(n\tau)\|_{\ell^2_{\tau,T} L^\infty} \lesssim_T (1+\log N)^{\frac{1}{2}} \Big(\|f\|_{\mathcal{H}^1} + \|g\|_{L^2} + \|F\|_{L^1_T L^2} \Big),$$

for all $N \geq 1$ and $\tau \in (0, 1/N]$. In the case $\Omega = \mathbb{R}^3$, the constant in the first inequality (5.18) *is independent of T and we can thus take $T = \infty$.*

Proof. We only give the details for the discrete-time estimate in (5.18), since the others are obtained similarly, also using Theorems 5.1 and 5.12, estimate (5.10), and Corollary 5.15 in place of Theorems 5.6 and 5.14. Let first $\Omega = \mathbb{R}^3$. By Duhamel's formula, w is given by

$$w(t) = \cos(t|\nabla|) f + |\nabla|^{-1} \sin(t|\nabla|) g + \int_0^t |\nabla|^{-1} \sin((t-s)|\nabla|) F(s) \, \mathrm{d}s,$$

for $t \in [0,T]$. By a direct application of Theorem 5.6 and (5.2),

$$\begin{aligned} \|\pi_N w(n\tau)\|_{\ell^p_{\tau,T} L^q(\mathbb{R}^3)} &\lesssim_{p,q} \|f\|_{\dot{H}^1(\mathbb{R}^3)} + \||\nabla|^{-1} g\|_{\dot{H}^1(\mathbb{R}^3)} + \int_0^T \|e^{\pm \mathrm{i}s|\nabla|} |\nabla|^{-1} F(s)\|_{\dot{H}^1(\mathbb{R}^3)} \, \mathrm{d}s \\ &= \|f\|_{\dot{H}^1(\mathbb{R}^3)} + \|g\|_{L^2(\mathbb{R}^3)} + \|F\|_{L^1_T L^2(\mathbb{R}^3)}. \end{aligned}$$

In the case $\Omega = \mathbb{T}^3$, we need to treat the zero-th Fourier coefficient separately, since the operator $|\nabla|^{-1}$ is in that case only well-defined for mean value free functions. We rewrite the Duhamel formula as

$$w(t) = \cos(t|\nabla|) f + t \operatorname{sinc}(t|\nabla|) g + \int_0^t (t-s) \operatorname{sinc}((t-s)|\nabla|) F(s) \, \mathrm{d}s,$$

for $t \in [0,T]$. Theorem 5.14 yields

$$\|\pi_N \cos(n\tau|\nabla|) f\|_{\ell^p_{\tau,T} L^q} \leq \|\pi_N e^{\pm \mathrm{i}n\tau|\nabla|} f\|_{\ell^p_{\tau,T} L^q} \lesssim_{p,q,T} \|f\|_{H^1}.$$

For the other terms we separate the zero-th Fourier coefficient before using Theorem 5.14. So we estimate

$$\begin{aligned} \|\pi_N n\tau \operatorname{sinc}(n\tau|\nabla|) g\|_{\ell^p_{\tau,T} L^q} &\lesssim \|\pi_N e^{\pm \mathrm{i}n\tau|\nabla|} |\nabla|^{-1} (g - \hat{g}_0)\|_{\ell^p_{\tau,T} L^q} + T|\hat{g}_0| \\ &\lesssim_{p,q,T} \||\nabla|^{-1}(g - \hat{g}_0)\|_{H^1} + \|g\|_{L^2} \lesssim \|g\|_{L^2} \end{aligned}$$

and similarly

$$\begin{aligned}
&\Big\|\pi_N \int_0^{n\tau} (n\tau - s)\operatorname{sinc}((n\tau - s)|\nabla|)F(s)\,\mathrm{d}s\Big\|_{\ell^p_{\tau,T}L^q} \\
&\quad \le \int_0^T \|\pi_N (n\tau - s)\operatorname{sinc}((n\tau - s)|\nabla|)F(s)\|_{\ell^p_{\tau,T}L^q}\,\mathrm{d}s \\
&\quad \lesssim \int_0^T \|\pi_N e^{\mathrm{i}(n\tau - s)|\nabla|}|\nabla|^{-1}(F(s) - \hat{F}_0(s))\|_{\ell^p_{\tau,T}L^q}\,\mathrm{d}s + T\int_0^T |\hat{F}_0(s)|\,\mathrm{d}s \\
&\quad \lesssim_{p,q,T} \int_0^T \|e^{-\mathrm{i}s|\nabla|}|\nabla|^{-1}(F(s) - \hat{F}_0(s))\|_{H^1}\,\mathrm{d}s + \int_0^T \|F(s)\|_{L^2}\,\mathrm{d}s \lesssim \|F\|_{L^1_T L^2}. \quad \square
\end{aligned}$$

It is often convenient to work with the wave equation in first-order formulation. We therefore define the operator

$$A := \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \tag{5.19}$$

which maps continuously $H^{r+1} \times H^r \to H^r \times H^{r-1}$ and generates the strongly continuous group of operators

$$e^{tA} := \begin{pmatrix} \cos(t|\nabla|) & t\operatorname{sinc}(t|\nabla|) \\ -|\nabla|\sin(t|\nabla|) & \cos(t|\nabla|) \end{pmatrix} \tag{5.20}$$

on $H^r \times H^{r-1}$, for all $r \in \mathbb{R}$. In the case $\Omega = \mathbb{R}^3$, the Fourier transform shows that $A\colon \dot{H}^{r+1} \times \dot{H}^r \to \dot{H}^r \times \dot{H}^{r-1}$ and $e^{tA}\colon \dot{H}^r \times \dot{H}^{r-1} \to \dot{H}^r \times \dot{H}^{r-1}$ then continuously map between the homogeneous Sobolev spaces as well. Moreover, the operators e^{tA} even form a strongly continuous group of unitary operators on $\dot{H}^r \times \dot{H}^{r-1}$.

Corollary 5.17. *Let $f \in \mathcal{H}^1$, $g \in L^2$, and $F \in \ell^1 L^2$. For $\tau \in (0,1]$ and $n \in \mathbb{N}$, we define*

$$W_n := e^{n\tau A}(f,g) + \tau \sum_{k=0}^{n} e^{(n-k)\tau A}\begin{pmatrix} 0 \\ F_k \end{pmatrix}.$$

Let w_n be the first component of W_n. For $T \in [0,\infty)$ and $\mathcal{H}^1$-admissible (p,q) we then get the estimate

$$\|\pi_N w_n\|_{\ell^p_{\tau,T}L^q} \lesssim_{p,q,T} \|f\|_{\mathcal{H}^1} + \|g\|_{L^2} + \|F\|_{\ell^1_{\tau,T}L^2}$$

for all $\tau \in (0,1]$ and $N \in [1, 1/\tau]$. In the case $\Omega = \mathbb{R}^3$, the implicit constant is independent of T and we can thus take $T = \infty$.

Proof. The estimate for the summand containing (f,g) is already contained in Corollary 5.16. The term containing F is treated in the same manner as the corresponding integral in the proof of Corollary 5.16. $\square$

Concerning the inhomogeneity F, there are also variants involving $L^{\tilde{p}'}_T L^{\tilde{q}'}$-norms instead of the $L^1_T L^2$-norm on the right-hand side of Corollary 5.16, and similarly for the discrete-time estimate in Corollary 5.17. Since we do not use them, we omit them for simplicity. However, the following dual Strichartz estimates will be needed.

Corollary 5.18. *Let (p,q) be $\mathcal{H}^1$-admissible and $T \in (0,\infty)$. Then we have the estimates*

$$\Big\| \int_0^T e^{-sA} \begin{pmatrix} 0 \\ F(s) \end{pmatrix} \mathrm{d}s \Big\|_{L^2 \times \mathcal{H}^{-1}} \lesssim_{p,q,T} \|F\|_{L_T^{p'} L^{q'}}, \tag{5.21}$$

$$\Big\| \int_0^T e^{-sA} \begin{pmatrix} 0 \\ \pi_K G(s) \end{pmatrix} \mathrm{d}s \Big\|_{L^2 \times \mathcal{H}^{-1}} \lesssim_T (1+\log K)^{\frac12} \|G\|_{L_T^2 L^1},$$

for all $F \in L_T^{p'} L^{q'}$, $G \in L_T^2 L^1$ and $K \geq 1$. In the case $\Omega = \mathbb{R}^3$, the constant in the first inequality (5.21) *is independent of T and we can thus take $T = \infty$.*

Proof. These estimates follow from the dual versions of Theorem 5.1, inequality (5.10), and Corollary 5.15 for $\gamma = 1$, which are given by

$$\Big\| \int_0^T e^{-\mathrm{i}s|\nabla|} F(s) \,\mathrm{d}s \Big\|_{\mathcal{H}^{-1}} \lesssim_{p,q,T} \|F\|_{L_T^{p'} L^{q'}},$$

$$\Big\| \int_0^T \pi_K e^{-\mathrm{i}s|\nabla|} G(s) \,\mathrm{d}s \Big\|_{\mathcal{H}^{-1}} \lesssim_T (1+\log K)^{\frac12} \|G\|_{L_T^2 L^1}. \tag{5.22}$$

We give the details for the term containing G. We split

$$\begin{aligned} &\Big\| \int_0^T e^{-sA} \begin{pmatrix} 0 \\ \pi_K G(s) \end{pmatrix} \mathrm{d}s \Big\|_{L^2 \times \mathcal{H}^{-1}} \\ &\quad \lesssim \Big\| \int_0^T \pi_K s \operatorname{sinc}(-s|\nabla|) G(s) \,\mathrm{d}s \Big\|_{L^2} + \Big\| \int_0^T \pi_K \cos(-s|\nabla|) G(s) \,\mathrm{d}s \Big\|_{\mathcal{H}^{-1}}. \end{aligned}$$

The cosine term is estimated directly using (5.22). For the sine term we compute as before

$$\begin{aligned} &\Big\| \int_0^T \pi_K s \operatorname{sinc}(-s|\nabla|) G(s) \,\mathrm{d}s \Big\|_{L^2} \\ &\quad \lesssim \Big\| \int_0^T \pi_K e^{\pm \mathrm{i}s|\nabla|} (G(s) - \hat{G}_0(s)) \,\mathrm{d}s \Big\|_{\mathcal{H}^{-1}} + \Big| \int_0^T s \hat{G}_0(s) \,\mathrm{d}s \Big| \\ &\quad \lesssim_T (1+\log K)^{\frac12} \|G - \hat{G}_0\|_{L_T^2 L^1} + \|G\|_{L_T^1 L^1} \lesssim_T (1+\log K)^{\frac12} \|G\|_{L_T^2 L^1}, \end{aligned}$$

using (5.22) and $|\hat{G}_0(s)| \leq \|G(s)\|_{L^1}$. □

5.5. Some properties of the filter operator π_K

The following lemma quantifies the convergence $\pi_K \to I$ as $K \to \infty$, and will be used to control the error terms that arise from the insertion of the filter into the numerical scheme (4.4).

Lemma 5.19. *For all $K \geq 1$ and $s > 0$, we can write*

$$I - \pi_K = (K^{-1} |\nabla| \phi_K)^s,$$

for a Fourier multiplication operator ϕ_K that is bounded uniformly in K on all Sobolev spaces $\mathcal{H}^r$, $r \in \mathbb{R}$. We moreover get the estimate

$$\|(I - \pi_K)f\|_{\mathcal{H}^\gamma} \leq K^{\gamma - r}\|f\|_{\mathcal{H}^r}$$

for all $K \geq 1$, $r \in \mathbb{R}$, $\gamma \leq r$, and $f \in \mathcal{H}^r$.

Proof. We work in Fourier space. The frequency variable is denoted by ξ regardless of $\Omega \in \{\mathbb{R}^3, \mathbb{T}^3\}$. Let ϕ_K be the Fourier multiplier for the function $\mathbb{1}_{\{|\xi|_\infty > K\}} K/|\xi|$, which is bounded by 1. The estimate then follows from

$$\|(I - \pi_K)f\|_{\mathcal{H}^\gamma} = K^{\gamma - r}\|(|\nabla|\phi_K)^{r-\gamma} f\|_{\mathcal{H}^\gamma} \leq K^{\gamma - r}\|f\|_{\mathcal{H}^r}. \qquad \square$$

The next lemma will be crucially exploited in the error analysis of (4.4) when using the summation by parts formula. This strategy is inspired by [10], cf. Property (OF4) in Theorem 3.14 there. Roughly speaking, the idea is the following. Let $v \in \mathcal{H}^r \times \mathcal{H}^{r-1}$. Using the Fourier transform, one can deduce that the integral

$$\int_0^T e^{tA} v \,\mathrm{d}t \in \mathcal{H}^{r+1} \times \mathcal{H}^r$$

is an element of the domain of A; and

$$A \int_0^T e^{tA} v \,\mathrm{d}t = (e^{TA} - I)v.$$

We would like to exploit something similar in the discrete setting, namely, that

$$\tau A \sum_{k=0}^{n-1} e^{k\tau A}$$

is a bounded operator on $\mathcal{H}^r \times \mathcal{H}^{r-1}$, uniformly in $\tau \in (0, 1]$ and $N \in \mathbb{N}$ with $N\tau \leq T$. If we formally insert the geometric sum formula, we obtain

$$\tau A \sum_{k=0}^{n-1} e^{k\tau A} = \tau A \frac{e^{n\tau A} - I}{e^{\tau A} - I}.$$

But this does not lead anywhere since the operator $e^{\tau A} - I$ might not be invertible for certain “resonant” step-sizes τ. However, if we introduce the filter operator

$$\Pi_N \coloneqq \operatorname{diag}(\pi_N, \pi_N)$$

and apply the assertion of the following Lemma 5.20, we get

$$\Pi_N \tau A \sum_{k=0}^{n-1} e^{k\tau A} = \Psi_{\tau,N}(e^{n\tau A} - I),$$

which indeed is a bounded operator on $\mathcal{H}^r \times \mathcal{H}^{r-1}$ as desired.

Lemma 5.20. *For all $\tau \in (0,1]$ and $N \in [1, \tau^{-1}]$, we can write*

$$\tau A \Pi_N = (e^{\tau A} - I)\Psi_{\tau,N},$$

where the operator $\Psi_{\tau,N}$ is bounded uniformly in τ and N on all Sobolev spaces $\mathcal{H}^r \times \mathcal{H}^{r-1}$, $r \in \mathbb{R}$.

Proof. One checks that the equality holds for

$$\Psi_{\tau,N} := -\frac{\tau}{2}\Pi_N \begin{pmatrix} \dfrac{|\nabla|\sin(\tau|\nabla|)}{\cos(\tau|\nabla|) - I} & I \\ \Delta & \dfrac{|\nabla|\sin(\tau|\nabla|)}{\cos(\tau|\nabla|) - I} \end{pmatrix}.$$

This operator is uniformly bounded in τ and N thanks to the presence of Π_N, which ensures that we only need to consider the Fourier modes with $\tau|k| \leq \sqrt{3}N^{-1}|k|_\infty \leq \sqrt{3}$. Therefore, we can exploit that the function

$$x \mapsto \frac{x \sin x}{\cos x - 1}$$

is bounded on $[0, \sqrt{3}]$. □

5.6. Trigonometric interpolation

In this section we only work on $\Omega = \mathbb{T}^3$.

Definition 5.21. *Let $N \in \mathbb{N}$ and $f \in C(\mathbb{T}^3)$. We define the trigonometric interpolation $I_N f$ as the trigonometric polynomial*

$$(I_N f)(x) := (2\pi)^{-\frac{3}{2}} \sum_{|k|_\infty \leq N} \tilde{f}_{k,N} e^{\mathrm{i}k\cdot x}, \qquad x \in \mathbb{T}^3,$$

where the coefficients $\tilde{f}_{k,N}$ are given by the discrete Fourier transform

$$\tilde{f}_{k,N} := (2\pi)^{\frac{3}{2}}(2N+1)^{-3} \sum_{|j|_\infty \leq N} f\Big(\frac{2\pi j}{2N+1}\Big) e^{-\mathrm{i}\frac{2\pi j}{2N+1}\cdot k}.$$

We moreover set $\mathcal{I}_\mathcal{N} := \mathrm{diag}(I_N, I_N)$.

We need the following well-known generalization of Bernstein's inequality to the L^q setting, see, e.g., inequality (5.2) in [26].

Lemma 5.22. *The estimate*

$$\|\pi_K f\|_{W^{1,q}} \lesssim K \|\pi_K f\|_{L^q}$$

holds for all $q \in [1, \infty]$, $f \in \mathcal{D}'(\mathbb{T}^3)$, and $K \in \mathbb{N}$.

We further need an estimate for the trigonometric interpolation error. The L^2 case is standard, see, e.g., Lemma 2.4 and expression (5.5) of [26]. The estimate in L^q is more involved. For a proof, we refer to see Corollary 3 of [35], Theorem 1 of [2], and Lemma 3 of [58].

Lemma 5.23. *Let $q \in (1, \infty)$. We then have the inequality*

$$\|(I - I_K)f\|_{L^q} \lesssim_q \sum_{m=1}^{3} K^{-m} \|f\|_{W^{m,q}}$$

for all $f \in W^{3,q}$ and $K \in \mathbb{N}$. For $q = 2$, we have the stronger result

$$\|(I - I_K)h\|_{L^2} \lesssim_s K^{-s} \||\nabla|^s h\|_{L^2}$$

for all $s > 3/2$, $h \in H^s$, and $K \in \mathbb{N}$.

The two preceding lemmas can be combined to the following estimates, which are used below with $\beta = \alpha \in \{3, 5\}$.

Lemma 5.24. *Let $q \in (1, \infty)$ and $\beta \geq 1$. Then the estimates*

$$\|(I - I_K)\pi_{\beta K} f\|_{L^q} \lesssim_{q,\beta} K^{-1} \|\pi_{\beta K} f\|_{W^{1,q}},$$
$$\|I_K \pi_{\beta K} f\|_{L^q} \lesssim_{q,\beta} \|\pi_{\beta K} f\|_{L^q},$$

hold for all $f \in \mathcal{D}'(\mathbb{T}^3)$ and $K \geq 1$.

6. The semilinear wave equation

6.1. Review of wellposedness theory

The wellposedness theory for (4.1) is well known, therefore we only address the most important points. See, e.g., the monographs [3, 66, 71] for more details. Note that, thanks to finite propagation speed, the local theory is essentially identical regardless of $\Omega \in \{\mathbb{R}^3, \mathbb{T}^3\}$. We first reformulate the equation (4.1) as a first-order system in time. Using the wave operator A from (5.19) and the notation

$$g(u) \coloneqq -\mu |u|^{\alpha-1} u, \qquad G(u,v) \coloneqq (0, g(u))$$

for the nonlinearity, one obtains the equivalent system

$$\begin{aligned} \partial_t U(t) &= AU(t) + G(U(t)), \quad t \in [0,T], \\ U(0) &= (u^0, v^0) \end{aligned} \tag{6.1}$$

for the new variable $U \mathrel{\widehat{=}} (u, \partial_t u)$. The local wellposedness is shown by a classical fixed point argument based on the Duhamel formula

$$U(t) = e^{tA}(u^0, v^0) + \int_0^t e^{(t-s)A} G(U(s)) \, \mathrm{d}s \tag{6.2}$$

for (6.1). In the case $\alpha = 3$, the Sobolev embedding $\mathcal{H}^1 \hookrightarrow L^6$ from Theorem A.1 implies that the nonlinearity G leaves the space $\mathcal{H}^1 \times L^2$ invariant. Therefore, the fixed point space for U can be chosen as a closed ball in $C([0,b], \mathcal{H}^1 \times L^2)$ for some $b > 0$ small enough. If $\alpha > 3$, one needs to involve a Strichartz space for u in the fixed point space.

Let $\alpha \in [3,5]$. We define the exponent $p_\alpha \in [4, \infty]$ such that $(p_\alpha, 3(\alpha-1))$ are $\mathcal{H}^1$-admissible, i.e.,

$$p_\alpha \coloneqq \frac{2(\alpha-1)}{\alpha-3}, \tag{6.3}$$

see (5.1) with $\gamma = 1$. One then obtains the following existence and uniqueness theorem for the nonlinear wave equation (4.1).

Theorem 6.1. *Let $(u^0, v^0) \in \mathcal{H}^1 \times L^2$. Then there exists a time $b > 0$ and a unique function $U = (u, \partial_t u)$ satisfying (6.2) such that $U \in C([0,b], \mathcal{H}^1 \times L^2)$ and $u \in L^{p_\alpha}([0,b], L^{3(\alpha-1)})$. If $\alpha < 5$, the time b only depends on $\|(u^0, v^0)\|_{\mathcal{H}^1 \times L^2}$ and α.*

Remark 6.2. a) Since $g(u) \in C([0,b], L^{6/\alpha}) \hookrightarrow C([0,b], H^{-1})$ and $\Delta u \in C([0,b], \mathcal{H}^{-1}) \hookrightarrow C([0,b], H^{-1})$, one can deduce from (6.2) that $\partial_t^2 u$ belongs to $C([0,b], H^{-1})$ and that the differential equation in (4.1) holds in this space. Thus, the equation (6.1) holds in $C([0,b], L^2 \times H^{-1})$.

b) If $\Omega = \mathbb{R}^3$, the equation (4.1) enjoys the following scaling symmetry. Let u be a solution to (4.1). Then, for each $\lambda > 0$, the resealed function $u_\lambda(t,x) := \lambda^{2/(\alpha-1)} u(\lambda t, \lambda x)$ also solves (4.1) with initial data $u^0_\lambda(x) := \lambda^{2/(\alpha-1)} u^0(\lambda x)$ and $v^0_\lambda(x) := \lambda^{2/(\alpha-1)+1} v^0(\lambda x)$. The map $(u^0, v^0) \mapsto (u^0_\lambda, v^0_\lambda)$ with $\lambda \neq 1$ leaves the $\dot{H}^1 \times L^2$ norm invariant if and only if $\alpha = 5$. This explains why this case is referred to as *scaling-critical*. Correspondingly, the situations $\alpha < 5$ and $\alpha > 5$ are called *sub-* and *supercritical*, respectively. See Principle 3.1 of [71] for some heuristics on the behavior of solutions in these cases.

c) See Proposition 7.15 for a more precise statement regarding a lower bound for the time b in the critical case $\alpha = 5$. A result concerning the continuous dependence on the initial data in that case can also be found there. Analogous results on continuous dependence are available in the subcritical range $\alpha \in [3,5)$ as well.

d) In the subcritical case $\alpha < 5$, one has uniqueness of solutions $U = (u, \partial_t u)$ to (4.1) in the energy class $C([0,b], \mathcal{H}^1 \times L^2)$ without the requirement that $u \in L^{p_\alpha}([0,b], L^{3(\alpha-1)})$, cf. [57].

From now on we will always assume the existence of a solution on a fixed interval $\overline{[0,T)}$.

Assumption 6.3. There exists a time $T \in (0,\infty)$ and a solution $U = (u, \partial_t u)$ of the nonlinear equation (4.1) such that $U \in C(\overline{[0,T)}, \mathcal{H}^1 \times L^2)$ and $u \in L^{p_\alpha}([0,T], L^{3(\alpha-1)})$. If $\alpha = 5$, $\mu = 1$, and $\Omega = \mathbb{R}^3$, we also admit $T = \infty$. We define the bound

$$M := \max\{\|U\|_{L^\infty_T(\mathcal{H}^1 \times L^2)}, \|u\|_{L^{p_\alpha}_T L^{3(\alpha-1)}}\}. \tag{6.4}$$

Remark 6.4. If $\alpha < 5$, the quantity M in fact only depends on $\|U\|_{L^\infty_T(\mathcal{H}^1 \times L^2)}$. Indeed, the "minimal" existence time b and the number M for $T = b$ are controlled by $\|(u^0, v^0)\|_{\mathcal{H}^1 \times L^2}$ in Theorem 6.1, compare p. 143 of [71]. Hence, we can divide the interval $[0,T]$ into a finite number of smaller subintervals such that the Strichartz norm of u is bounded on each of them.

The following two remarks concern the long-time behavior of (4.1).

Remark 6.5. It is well known that the *energy*

$$E[U(t)] := \frac{1}{2}\|\nabla u(t)\|^2_{L^2} + \frac{1}{2}\|\partial_t u(t)\|^2_{L^2} + \frac{\mu}{\alpha+1}\|u(t)\|^{\alpha+1}_{L^{\alpha+1}}$$

is constant in t along solutions $U = (u, \partial_t u)$ to (4.1). This conservation law gives hope for a global wellposedness result in the defocusing case $\mu = 1$. Indeed, the works [44, 65] established the global-in-time existence of solutions to 4.1 in the critical case $\alpha = 5$ with $\mu = 1$, i.e., the time b in Theorem 6.1 can be taken arbitrarily large in that case. The subcritical range $\alpha < 5$ is much easier (though if $\Omega = \mathbb{R}^3$, the global existence for $\mu = 1$ requires the additional condition $u^0 \in L^{\alpha+1}$ to ensure that the energy is finite, compare Theorem 8.41 of [3]). In the focusing case $\mu = -1$ however, the energy might become negative and solutions which blow up in finite time are known to exist. In the supercritical focusing case $\alpha > 5$ and $\mu = -1$, one can even construct solutions with arbitrary small initial data that blow up in arbitrary short time, implying illposedness of (4.1) in that case, compare p. 142 of [66].

Remark 6.6. Let $\Omega = \mathbb{R}^3$, $\alpha = 5$, and $\mu = 1$. In this case we can take $T = \infty$ in the definition (6.4) of M and obtain $M \leq C(\|u^0\|_{\dot{H}^1}, \|v^0\|_{L^2})$, see [72] for more details on C. The global L^4L^{12} bound is known to imply *scattering* in the energy space as follows. Define the *asymptotic state* $U_+ \in \dot{H}^1 \times L^2$ by

$$U_+ := (u^0, v^0) + \int_0^\infty e^{-sA} G(U(s))\,\mathrm{d}s.$$

The integral converges absolutely in $\dot{H}^1 \times L^2$ since

$$\||u|^4 u\|_{L^1([0,\infty),L^2)} \leq \|u\|^4_{L^4([0,\infty),L^{12})} \|u\|_{L^\infty([0,\infty),L^6)} \lesssim_M 1 \tag{6.5}$$

by Hölder's inequality and Sobolev's embedding. We then obtain the scattering result

$$\|U(t) - e^{tA}U_+\|_{\dot{H}^1 \times L^2} = \|e^{-tA}U(t) - U_+\|_{\dot{H}^1 \times L^2} \to 0$$

as $t \to \infty$, using the unitarity of e^{tA} in $\dot{H}^1 \times L^2$, the Duhamel formula (6.2), and the definition of U_+. This means that the nonlinear solution $U(t)$ behaves like the linear solution $e^{tA}U_+$ as $t \to \infty$. Note that these arguments do *not* (without further assumptions) imply scattering in the case $\alpha < 5$. Indeed, we cannot afford to use Hölder's inequality in time in (6.5) due to the unbounded interval. Another idea would be the application of an inhomogeneous Sobolev embedding in (6.5) to "waste regularity", but this does not not work either since we do not have a uniform bound on $\|u(t)\|_{L^2}$ as $t \to \infty$.

6.2. Treatment of boundary conditions

Let $Q := (0, \pi)^3$. In this section, we shortly explain how the differential equation (4.1) on the periodic domain $\Omega = \mathbb{T}^3$ already contains the cases of homogeneous Dirichlet or Neumann conditions on Q as special cases. We define the Dirichlet Laplacian

$$\Delta_D\colon H_0^1(Q) \to H^{-1}(Q), \qquad \langle \Delta_D f, g\rangle_{H^{-1}(Q)\times H_0^1(Q)} := -\int_Q \nabla f \cdot \nabla g \,\mathrm{d}x,$$

for f, $g \in H_0^1(Q)$. As usual in the literature, the space $H_0^1(Q)$ denotes the closure of $C_c^\infty(Q)$ in $H^1(Q)$, and $H^{-1}(Q)$ is its dual space.

Proposition 6.7. *Let $u^0 \in H_0^1(Q)$ and $v^0 \in L^2(Q)$. The nonlinear wave equation on Q with homogeneous Dirichlet boundary conditions*

$$\begin{aligned} \partial_t^2 u - \Delta_D u = g(u), \quad (t,x) \in [0,T] \times Q, \\ u(0) = u^0, \quad \partial_t u(0) = v^0 \end{aligned} \tag{6.6}$$

can be reduced to that one on $\mathbb{T}^3$ *by the following transformation. We extend the initial data* (u^0, v^0) *oddly and then periodically to* $H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)$*, solve the corresponding problem* (4.1) *on* $\mathbb{T}^3$*, and afterwards restrict the solution to* Q *again.*

Proof. It is known that the eigenfunctions of Δ_D, given by

$$e_k(x) := \Big(\frac{2}{\pi}\Big)^{\frac{3}{2}} \sin(k_1 x_1)\sin(k_2 x_2)\sin(k_3 x_3), \qquad x \in Q,\ k \in \mathbb{N}^3,$$

form an orthonormal basis of $L^2(Q)$ and an orthogonal basis of $H_0^1(Q)$. Therefore, any function $f \in L^2(Q)$ can be expanded in the Fourier sine series

$$f = \sum_{k \in \mathbb{N}^3} \tilde{f}_k e_k,$$

where the coefficients $\tilde{f}_k$ are given by the $L^2(Q)$ scalar product of f and e_k, for each $k \in \mathbb{N}^3$. The functions e_k naturally extend to functions on the torus $\mathbb{T}^3$ and are then also eigenfunctions of the Laplacian $\Delta\colon H^1(\mathbb{T}^3) \to H^{-1}(\mathbb{T}^3)$. Observe that the Fourier coefficients satisfy

$$\hat{f}_k = \mathrm{i}\,\mathrm{sgn}(k_1 k_2 k_3)\tilde{f}_{(|k_1|,|k_2|,|k_3|)}, \qquad k \in \mathbb{Z}^3, \tag{6.7}$$

with $\mathrm{sgn}(0) = 0$. By comparing coefficients, we see that the function f also extends to a function in $L^2(\mathbb{T}^3)$. This procedure corresponds to the odd extension (in every coordinate direction) of f. Similar considerations apply if we replace $L^2(G)$ by $H_0^1(Q)$. In this sense, we can identify $H_0^1(Q) \times L^2(Q)$ as a closed subspace of $H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)$, and the operator Δ_D as a restriction of Δ. Using the relation (6.7), one checks that the group e^{tA} on $H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)$ leaves $H_0^1(Q) \times L^2(Q)$ invariant. Moreover, the nonlinearity $g\colon H_0^1(Q) \cap L^{3(\alpha-1)}(\mathbb{T}^3) \to L^2(\mathbb{T}^3)$ maps odd functions to odd functions. Note that the oddness property (almost everywhere) is preserved by an L^2 limes, too. It follows that if we extend the initial data (u^0, v^0) oddly to $H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)$, the fixed point iteration based on (6.2) also converges in $C([0,T], H_0^1(Q) \times L^2(Q)) \hookrightarrow C([0,T], H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3))$, and yields a solution to (6.6) after restricting it to Q again. □

Similar considerations also apply to the homogeneous Neumann boundary problem. The Neumann Laplacian is defined as

$$\Delta_N\colon H^1(Q) \to H_0^{-1}(Q), \qquad \langle \Delta_D f, g\rangle_{H_0^{-1}(Q)\times H^1(Q)} := -\int_Q \nabla f \cdot \nabla g \,\mathrm{d}x,$$

where $H_0^{-1}(Q)$ is the dual space of $H^1(Q)$.

Proposition 6.8. *Let* $u^0 \in H^1(Q)$ *and* $v^0 \in L^2(Q)$*. The nonlinear wave equation on* Q *with homogeneous Neumann boundary conditions*

$$\begin{aligned} \partial_t^2 u - \Delta_N u &= g(u), \quad (t,x) \in [0,T] \times Q, \\ u(0) &= u^0, \quad \partial_t u(0) = v^0 \end{aligned}$$

can be reduced to that one on $\mathbb{T}^3$ *by the following transformation. We extend the initial data* (u^0, v^0) *evenly and then periodically to* $H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)$*, solve the corresponding problem* (4.1) *on* $\mathbb{T}^3$*, and afterwards restrict the solution to* Q *again.*

Proof. The Neumann Laplacian Δ_N has the basis of eigenfunctions $\tilde{e}_0 := \pi^{-3/2}\mathbb{1}$ and

$$\tilde{e}_k(x) := \Big(\frac{2}{\pi}\Big)^{\frac{3}{2}} \cos(k_1x_1)\cos(k_2x_2)\cos(k_3x_3), \qquad x \in Q,\ k \in \mathbb{N}_0^3 \setminus \{0\},$$

which form a basis that is orthonormal in $L^2(Q)$ and orthogonal in $H^1(Q)$. In contrast to the Dirichlet case, we now use the even extension (in every coordinate direction) to identify $H^1(Q) \times L^2(Q)$ as a closed subspace of $H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)$. The condition on the Fourier coefficients is now given by $\hat{f}_0 = 2^{3/2}\tilde{f}_0$ and

$$\hat{f}_k = \tilde{f}_{(|k_1|,|k_2|,|k_3|)}, \qquad k \in \mathbb{Z}^3 \setminus \{0\},$$

where the coefficients $\tilde{f}_k$ now come from the Fourier cosine series expansion using $\tilde{e}_k$. Again, Δ_N is a restriction of Δ. Moreover, the group e^{tA} leaves $H^1(Q) \times L^2(Q)$ invariant and the nonlinearity $g\colon H^1(Q) \cap L^{3(\alpha-1)}(\mathbb{T}^3) \to L^2(\mathbb{T}^3)$ also maps even functions to even functions. Hence, for given initial data $(u^0, v^0) \in H^1(Q) \times L^2(Q)$, we can solve the Neumann problem by extending (u^0, v^0) evenly, solving the corresponding periodic problem on $\mathbb{T}^3$, and afterwards restricting it to Q again. □

6.3. Nonlinear estimates

We derive some important estimates for u from Assumption 6.3 that will be used later. First, we extend the $L_T^{p_\alpha} L^{3(\alpha-1)}$-bound from the definition (6.4) of M to other $\mathcal{H}^1$-admissible Strichartz pairs (p, q) and also to discrete time. We frequently exploit the endpoint Sobolev embedding from Theorem A.1 b) in the following.

Proposition 6.9. *Let u, T, and M be given by Assumption 6.3 and let (p, q) be $\mathcal{H}^1$-admissible. Then we have the estimate*

$$\|u\|_{L_T^p L^q} + \|\pi_N u\|_{L_T^p L^q} + \|\pi_N u(n\tau)\|_{\ell_\tau^p([0,T],L^q)} \lesssim_{p,q,M,T} 1, \tag{6.8}$$

and if $T < \infty$ additionally

$$\|\pi_N u\|_{L_T^2 L^\infty} + \|\pi_N u(n\tau)\|_{\ell_\tau^2([0,T],L^\infty)} \lesssim_{M,T} (1 + \log N)^{\frac{1}{2}},$$

for all $N \geq 1$ and $\tau \in (0, 1/N]$. The implicit constant in (6.8) is independent of T if $\Omega = \mathbb{R}^3$ and $\alpha = 5$.

Proof. Sobolev and Hölder inequalities yield that

$$\|g(u)\|_{L_T^1 L^2} \leq \||u|^{\alpha-1}\|_{L_T^1 L^3} \|u\|_{L_T^\infty L^6} \lesssim_T \|u\|_{L_T^{p_\alpha} L^{3(\alpha-1)}}^{\alpha-1} \|u\|_{L_T^\infty \mathcal{H}^1} \lesssim_{M,T} 1. \tag{6.9}$$

Since $p_\alpha \geq \alpha - 1$, the result then follows from Corollary 5.16. Note that if $\alpha = 5$, we do not need Hölder's inequality in time and hence, the implicit constant in (6.9) is independent of T. □

In the next lemma we give convergence rates for the difference between $g(u)$ and $g(\pi_K u)$. We will often use the elementary Lipschitz bound

$$|g(v) - g(w)| \lesssim (|v|^{\alpha-1} + |w|^{\alpha-1})|v - w| \tag{6.10}$$

for the nonlinearity g. Moreover, in addition to $(p_\alpha, 3(\alpha-1))$ from (6.3), for $\alpha \in (3,5]$ we will use the $\mathcal{H}^1$-admissible pair $(\alpha - 1, q_\alpha)$, where

$$q_\alpha := \frac{6(\alpha-1)}{\alpha-3}. \tag{6.11}$$

Note that $(p_\alpha, 3(\alpha-1)) = (\alpha-1, q_\alpha)$ in the scaling-critical situation $\alpha = 5$. Moreover, if $\alpha = 3$, the pair $(\alpha-1, q_\alpha)$ corresponds to the "forbidden endpoint" $(2, \infty)$.

Lemma 6.10. *Let u, T, and M be given by Assumption 6.3. Then we have the estimate*

$$\|g(u) - g(\pi_K u)\|_{L^1_T \mathcal{H}^{-1}} \lesssim_{M,T} K^{-1}.$$

Moreover, we obtain

$$\|g(u) - g(\pi_K u)\|_{L^1_T L^2} \lesssim_{M,T} K^{-1}(1 + \log K),$$

if $\alpha = 3$ and

$$\|g(u) - g(\pi_K u)\|_{L^1_T L^2} \lesssim_{M,T,\alpha} K^{-\frac{5-\alpha}{2}}$$

for $\alpha \in (3,5]$. These inequalities are uniform in $K \geq 1$. The implicit constants are independent of T if $\Omega = \mathbb{R}^3$ and $\alpha = 5$.

Proof. We first compute

$$\begin{aligned}
\|g(u) - g(\pi_K u)\|_{L^1_T \mathcal{H}^{-1}} &\lesssim \|(|u|^{\alpha-1} + |\pi_K u|^{\alpha-1})(I - \pi_K)u\|_{L^1_T L^{\frac{6}{5}}} \\
&\lesssim (\|u\|^{\alpha-1}_{L^{\alpha-1}_T L^{3(\alpha-1)}} + \|\pi_K u\|^{\alpha-1}_{L^{\alpha-1}_T L^{3(\alpha-1)}})\|(I - \pi_K)u\|_{L^\infty_T L^2} \\
&\lesssim_{M,T} K^{-1}\|u\|_{L^\infty_T \mathcal{H}^1} \lesssim_M K^{-1},
\end{aligned}$$

using the dual Sobolev embedding $L^{\frac{6}{5}} \hookrightarrow \mathcal{H}^{-1}$, estimate (6.10), Hölder's inequality, Proposition 6.9, and Lemma 5.19. Note that Hölder's inequality in time is not needed if $\alpha = 5$, again. Similarly, for $\alpha \in (3,5]$ it follows that

$$\begin{aligned}
\|g(u) - g(\pi_K u)\|_{L^1_T L^2} &\lesssim \|(|u|^{\alpha-1} + |\pi_K u|^{\alpha-1})(I - \pi_K)u\|_{L^1_T L^2} \\
&\lesssim \|(|u|^{\alpha-1} + |\pi_K u|^{\alpha-1})\|_{L^1_T L^{\frac{6}{\alpha-3}}} \|(I - \pi_K)u\|_{L^\infty_T L^{\frac{6}{6-\alpha}}} \\
&\lesssim (\|u\|^{\alpha-1}_{L^{\alpha-1}_T L^{q_\alpha}} + \|\pi_K u\|^{\alpha-1}_{L^{\alpha-1}_T L^{q_\alpha}})\|(I - \pi_K)u\|_{L^\infty_T \mathcal{H}^{\frac{\alpha-3}{2}}} \\
&\lesssim_{M,T,\alpha} K^{-\frac{5-\alpha}{2}}\|u\|_{L^\infty_T \mathcal{H}^1} \lesssim_M K^{-\frac{5-\alpha}{2}},
\end{aligned}$$

by means of Hölder's inequality with $\frac{1}{2} = \frac{\alpha-3}{6} + \frac{6-\alpha}{6}$, the Sobolev embedding $\mathcal{H}^{\frac{\alpha-3}{2}} \hookrightarrow L^{\frac{6}{6-\alpha}}$, and Proposition 6.9 with $(p,q) = (\alpha-1, q_\alpha)$ from (6.11).

Finally, let $\alpha = 3$. Then we decompose

$$\|g(u) - g(\pi_K u)\|_{L^1_T L^2} \le \|g(u) - g(\pi_{K^2} u)\|_{L^1_T L^2} + \|g(\pi_{K^2} u) - g(\pi_K u)\|_{L^1_T L^2}.$$

Proceeding as above, we obtain

$$\begin{aligned}
\|g(u) - g(\pi_{K^2} u)\|_{L^1_T L^2} &\lesssim \|(|u|^2 + |\pi_{K^2} u|^2)(I - \pi_{K^2})u\|_{L^1_T L^2} \\
&\lesssim \|(|u|^2 + |\pi_{K^2} u|^2)\|_{L^1_T L^6} \|(I - \pi_{K^2})u\|_{L^\infty_T L^3} \\
&\lesssim_T (\|u\|^2_{L^4_T L^{12}} + \|\pi_{K^2} u\|^2_{L^4_T L^{12}}) \|(I - \pi_{K^2})u\|_{L^\infty_T \mathcal{H}^{\frac{1}{2}}} \\
&\lesssim_{M,T} K^{-1} \|u\|_{L^\infty_T \mathcal{H}^1} \lesssim_M K^{-1},
\end{aligned}$$

using $(p, q) = (4, 12)$, and

$$\begin{aligned}
\|g(\pi_{K^2} u) - g(\pi_K u)\|_{L^1_T L^2} &\lesssim \|(|\pi_{K^2} u|^2 + |\pi_K u|^2)\pi_{K^2}(I - \pi_K)u\|_{L^1_T L^2} \\
&\lesssim \|(|\pi_{K^2} u|^2 + |\pi_K u|^2)\|_{L^1_T L^\infty} \|\pi_{K^2}(I - \pi_K)u\|_{L^\infty_T L^2} \\
&\lesssim (\|\pi_{K^2} u\|^2_{L^2_T L^\infty} + \|\pi_K u\|^2_{L^2_T L^\infty}) \|(I - \pi_K)u\|_{L^\infty_T L^2} \\
&\lesssim_{M,T} K^{-1}(1 + \log K) \|u\|_{L^\infty_T \mathcal{H}^1} \lesssim_M K^{-1}(1 + \log K).
\end{aligned}$$

Here, the logarithmic estimate for the $L^2_T L^\infty$ norm from Proposition 6.9 was applied. □

7. Error analysis

We now start with the error analysis of the splitting scheme. We directly treat the fully discrete algorithm (4.4), given by

$$\begin{aligned} U_{n+1/2} &= e^{\tau A}[U_n + \tfrac{\tau}{2}\mathcal{I}_K G(\Pi_{\tau^{-1}} U_n)], \\ U_{n+1} &= U_{n+1/2} + \tfrac{\tau}{2}\mathcal{I}_K G(\Pi_{\tau^{-1}} U_{n+1/2}), \\ U_0 &= \Pi_K(u^0, v^0), \end{aligned}$$

which contains the semi-discrete one (4.2) in the special case $K = \infty$. In the case $\Omega = \mathbb{R}^3$, we always set $K = \infty$ since our full discretization only makes sense on the torus $\Omega = \mathbb{T}^3$.

Note that since $G(u, v) = (0, g(u))$ we have $G(\Pi_{\tau^{-1}} U_{n+1/2}) = G(\Pi_{\tau^{-1}} U_{n+1})$ due to (4.4). Moreover, since $\Pi_K \mathcal{I}_K = \mathcal{I}_K$, it inductively follows that U_n defined by (4.4) satisfies the frequency localization $U_n = \Pi_K U_n$. Thus, we can also state the scheme (4.4) in the more compact form

$$U_{n+1} = e^{\tau A} U_n + \frac{\tau}{2}\Big(e^{\tau A}\mathcal{I}_K G(\Pi_N U_n) + \mathcal{I}_K G(\Pi_N U_{n+1})\Big), \tag{7.1}$$

where $N \coloneqq \min\{\tau^{-1}, K\}$. In view of a later iteration argument, we allow here for general initial values $U_0 \in \mathcal{H}^1 \times L^2$ that are not necessarily equal to $\Pi_K U(0)$. We often denote the discrete times by $t_n \coloneqq n\tau$.

7.1. Error recursion

We first establish a discrete Duhamel formula for U_n given by (4.4). We introduce the notation $S(t)$ for the first line of e^{tA}, i.e.,

$$S(t)(f, v) \coloneqq \cos(t|\nabla|) f + t \operatorname{sinc}(t|\nabla|) v.$$

Lemma 7.1. *The iterates U_n given by* (7.1) *satisfy the formulas*

$$U_n = e^{n\tau A} U_0 + \tau \sum_{k=0}^{n} c_{k,n} e^{(n-k)\tau A} \mathcal{I}_K G(\Pi_N U_k) \tag{7.2}$$

and

$$u_n = S(t_n) U_0 + \tau \sum_{k=0}^{n-1} c_{k,n} t_{n-k} \operatorname{sinc}(t_{n-k}|\nabla|) I_K g(\pi_N u_k) \tag{7.3}$$

for all $n \in \mathbb{N}_0$, $\tau \in (0, 1]$, and $K \in \mathbb{N} \cup \{\infty\}$, where u_n denotes the first component of U_n. Here we define

$$c_{0,0} \coloneqq 0, \quad c_{0,n} = c_{n,n} \coloneqq \frac{1}{2}, \quad c_{k,n} \coloneqq 1 \tag{7.4}$$

for $k \in \{1, \dots, n-1\}$ and $n \geq 1$.

Proof. Starting from (7.1) and Lemma A.8, we compute

$$\begin{aligned} U_n &= e^{n\tau A}U_0 + \frac{\tau}{2}\sum_{k=0}^{n-1} e^{(n-k-1)\tau A}\Big(e^{\tau A}\mathcal{I}_K G(\Pi_N U_k) + \mathcal{I}_K G(\Pi_N U_{k+1})\Big) \\ &= e^{n\tau A}U_0 + \frac{\tau}{2}\sum_{k=0}^{n-1} e^{(n-k)\tau A}\mathcal{I}_K G(\Pi_N U_k) + \frac{\tau}{2}\sum_{k=1}^{n} e^{(n-k)\tau A}\mathcal{I}_K G(\Pi_N U_k) \\ &= e^{n\tau A}U_0 + \tau \sum_{k=0}^{n} c_{k,n} e^{(n-k)\tau A}\mathcal{I}_K G(\Pi_N U_k). \end{aligned}$$

Formula (7.3) for the first component is obtained by inserting $G = (0, g)$ and the expression (5.20) for e^{tA}. □

Now we derive a useful decomposition of the error. Recall that for the sake of notational simplicity, we assume that the initial data (u^0, v^0) are real-valued, which is inherited by the solution U and the approximations U_n, cf. Lemma A.5. For the nonlinearity $g(u) = -\mu|u|^{\alpha-1}u$, we then obtain

$$\begin{aligned} g'(u) &= -\mu\alpha|u|^{\alpha-1}, \\ g''(u) &= -\mu\alpha(\alpha-1)|u|^{\alpha-3}u. \end{aligned}$$

Proposition 7.2. *Let the solution $U = (u, \partial_t u)$ satisfy Assumption 6.3 and the approximations U_n be given by* (7.1)*. Define the (projected) error E_n by*

$$E_n := \Pi_K U(t_n) - U_n. \tag{7.5}$$

We then have

$$\begin{aligned} E_n &= e^{n\tau A}E_0 + \Pi_K B(n\tau) + \Pi_K D_n + \Pi_K Q_n + \Pi_K H_n \qquad (7.6)\\ &= e^{n\tau A}E_0 + \Pi_K B(n\tau) + \Pi_K D_n + \tilde{Q}_n + \Pi_K \tilde{H}_n \qquad (7.7) \end{aligned}$$

for all $\tau \in (0,1]$, $K \in \mathbb{N} \cup \{\infty\}$, and $n \in \mathbb{N}_0$ with $t_n \in \overline{[0,T)}$. The appearing terms are given by

$$\begin{aligned} B(t) &:= \int_0^t e^{(t-s)A}[G(U(s)) - G(\Pi_N U(s))]\,\mathrm{d}s, \\ D_n &:= \tfrac{\tau^2}{2}\int_0^{t_n} e^{(n\tau-s)A}(\lfloor \tfrac{s}{\tau} \rfloor - \tfrac{s}{\tau})(\lceil \tfrac{s}{\tau} \rceil - \tfrac{s}{\tau}) \begin{pmatrix} d_1(s) \\ d_2(s) + d_3(s) + d_4(s) \end{pmatrix} \mathrm{d}s, \\ Q_n &:= \tau \sum_{k=0}^{n} c_{k,n} e^{(n-k)\tau A}[G(\Pi_N U(t_k)) - G(\Pi_N U_k)], \\ H_n &:= \tau \sum_{k=0}^{n} c_{k,n} e^{(n-k)\tau A}(I - \mathcal{I}_K)G(\Pi_N U_k), \end{aligned} \tag{7.8}$$

and

$$\tilde{Q}_n := \tau \sum_{k=0}^{n} c_{k,n} e^{(n-k)\tau A} \mathcal{I}_K [G(\Pi_N U(t_k)) - G(\Pi_N U_k)],$$
$$\tilde{H}_n := \tau \sum_{k=0}^{n} c_{k,n} e^{(n-k)\tau A} (I - \mathcal{I}_K) G(\Pi_N U(t_k)), \tag{7.9}$$

for $c_{k,n}$ *from* (7.4), $N := \min\{\tau^{-1}, K\}$, *and*

$$\begin{aligned} d_1(t) &:= -2g'(\pi_N u(t))\pi_N \partial_t u(t), \\ d_2(t) &:= g''(\pi_N u(t))\big[|\nabla \pi_N u(t)|^2 + (\pi_N \partial_t u(t))^2\big], \\ d_3(t) &:= g'(\pi_N u(t))\pi_N g(u(t)), \\ d_4(t) &:= 2g'(\pi_N u(t))\pi_N \Delta u(t). \end{aligned} \tag{7.10}$$

We can alternatively write

$$D_n = \tfrac{\tau^2}{2} \int_0^{\tau} \tfrac{s}{\tau}(\tfrac{s}{\tau} - 1) \sum_{k=0}^{n-1} e^{((n-k)\tau - s)A} \begin{pmatrix} d_1 \\ d_2 + d_3 + d_4 \end{pmatrix} (t_k + s)\, \mathrm{d}s, \tag{7.11}$$

and also

$$D_n = \tau \int_0^{t_n} e^{(n\tau - s)A} (\tfrac{1}{2} + \lfloor \tfrac{s}{\tau} \rfloor - \tfrac{s}{\tau}) \begin{pmatrix} -g(\pi_N u(s)) \\ g'(\pi_N u(s))\pi_N \partial_t u(s) \end{pmatrix} \mathrm{d}s. \tag{7.12}$$

Proof. We subtract the discrete Duhamel formula (7.2) from its continuous analogue

$$\Pi_K U(n\tau) = e^{n\tau A}\Pi_K U(0) + \Pi_K \int_0^{n\tau} e^{(n\tau - s)A} G(U(s))\, \mathrm{d}s$$

(see (6.2)) to obtain

$$\begin{aligned} E_n = e^{n\tau A} E_0 + \Pi_K B(n\tau) + \Pi_K \int_0^{t_n} e^{(n\tau - s)A} G(\Pi_N U(s))\, \mathrm{d}s \\ - \tau \Pi_K \sum_{k=0}^{n} c_{k,n} e^{(n-k)\tau A} G(\Pi_N U(t_k)) + \Pi_K Q_n + \Pi_K H_n \end{aligned}$$

where we exploit that $\mathcal{I}_K = \Pi_K \mathcal{I}_K$. To get the desired formulas for D_n, we use the error representation of the trapezoidal sum (in second and first order)

$$\begin{aligned} \int_0^{t_n} F(s)\, \mathrm{d}s - \tau \sum_{k=0}^{n} c_{k,n} F(t_k) &= \frac{1}{2} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (s - t_k)(s - t_{k+1}) F''(s)\, \mathrm{d}s \\ &= \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} (\tfrac{\tau}{2} + t_k - s) F'(s)\, \mathrm{d}s, \end{aligned} \tag{7.13}$$

where we set $F(s) := e^{(n\tau - s)A} G(\Pi_N U(s))$. We compute

$$F'(s) = e^{(n\tau - s)A} \Bigg[- \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} 0 \\ g(\pi_N u(s)) \end{pmatrix} + \frac{\mathrm{d}}{\mathrm{d}s} \begin{pmatrix} 0 \\ g(\pi_N u(s)) \end{pmatrix} \Bigg]$$

$$
\begin{aligned}
&= e^{(n\tau-s)A}\begin{pmatrix} -g(\pi_N u(s)) \\ g'(\pi_N u(s))\pi_N\partial_t u(s)\end{pmatrix}, \\
F''(s) &= e^{(n\tau-s)A}\Bigg[-\begin{pmatrix} 0 & I \\ \Delta & 0\end{pmatrix}\begin{pmatrix} -g(\pi_N u(s)) \\ g'(\pi_N u(s))\pi_N\partial_t u(s)\end{pmatrix} + \frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} -g(\pi_N u(s)) \\ g'(\pi_N u(s))\pi_N\partial_t u(s)\end{pmatrix}\Bigg] \\
&= e^{(n\tau-s)A}\begin{pmatrix} d_1(s) \\ d_2(s)+d_3(s)+d_4(s)\end{pmatrix},
\end{aligned}
$$

using that $\Delta[g(w)] = g''(w)|\nabla w|^2 + g'(w)\Delta w$ and the differential equation (4.1). Since

$$
\frac{1}{2}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}(s-t_k)(s-t_{k+1})F''(s)\,\mathrm{d}s = \tfrac{\tau^2}{2}\int_0^{t_n}(\lfloor\tfrac{s}{\tau}\rfloor-\tfrac{s}{\tau})(\lceil\tfrac{s}{\tau}\rceil-\tfrac{s}{\tau})F''(s)\,\mathrm{d}s = D_n,
$$

from (7.13) we deduce the second-order formula (7.6). The substitution $\tilde{s} = s - t_k$ yields the alternative representation (7.11). Similarly, the second line of (7.13) leads to the first-order representation (7.12). Since $\Pi_K Q_n + \Pi_K H_n = \tilde{Q}_n + \Pi_K \tilde{H}_n$, we also obtain the other recursion formula (7.7). □

Remark 7.3. The first component of $\tilde{Q}_n$ defined by (7.9) satisfies

$$
[\tilde{Q}_n]_1 = \tau\sum_{k=0}^{n-1}\Big[e^{(n-k)\tau A}c_{k,n}\mathcal{I}_K[G(\Pi_N U(t_k)) - G(\Pi_N U_k)]\Big]_1,
$$

and similarly for Q_n from (7.8). Here, the notation $[\cdot]_1$ means that we take the first component of the vector. The n-th term in the sum vanishes since the first component of the nonlinearity G is zero.

7.2. Estimates for error terms resulting from the filter

We now deal with the term B in (7.6) that results from the introduction of the filter function Π_N. Here we face the following difficulty. If we move the $L^2 \times \mathcal{H}^{-1}$ norm inside the integral and apply Lemma 5.19 with $s = 2$, we end up with a term roughly of the form

$$
\|g'(u)(I-\pi_N)u\|_{L^1_T\mathcal{H}^{-1}} \approx N^{-2}\|g'(u)\Delta u\|_{L^1_T\mathcal{H}^{-1}}.
$$

Now we would like to use a nonlinear product estimate, but we do not have enough regularity available to obtain an optimal error bound if $\alpha < 5$. For example, consider $\alpha = 3$ so that $g'(u) \approx u^2$ and we are aiming for an almost second-order error bound in time. Recall that $N = \min\{\tau^{-1}, K\}$, thus we cannot afford a loss in the product estimate. Assumption 6.3 yields $u \in L^\infty_T\mathcal{H}^1$ and hence $\Delta u \in L^\infty_T\mathcal{H}^{-1}$. Moreover, thanks to Proposition 6.9 we almost have $u \in L^2_T L^\infty$. But a product estimate of the form $\|vw\|_{\mathcal{H}^{-1}} \lesssim \|v\|_{\mathcal{H}^1\cap L^\infty}\|w\|_{\mathcal{H}^{-1}}$ is wrong, because in 3D, one only has

$$
\|vw\|_{\mathcal{H}^{-1}} \lesssim \|v\|_{W^{1,3}\cap L^\infty}\|w\|_{\mathcal{H}^{-1}}
$$

in general, which would require additional integrability.

To solve this problem, we follow a different strategy. We do not move the $L^2 \times \mathcal{H}^{-1}$ norm into the integral at first. Instead, we involve integration by parts in time, which helps to "move regularity to the right position". This technique was used previously in, e.g., [10] in a context without Strichartz estimates.

Lemma 7.4. *Let $U = (u, \partial_t u)$, T, and M be given by Assumption 6.3, and B by (7.8). We then have*

$$\|B(t)\|_{\mathcal{H}^1 \times L^2} \lesssim_{M,T} N^{-1}(1 + \log N),$$
$$\|B(t)\|_{L^2 \times \mathcal{H}^{-1}} \lesssim_{M,T} N^{-2}(1 + \log N)$$

if $\alpha = 3$, and

$$\|B(t)\|_{\mathcal{H}^1 \times L^2} \lesssim_{M,T,\alpha} N^{-\frac{5-\alpha}{2}},$$
$$\|B(t)\|_{L^2 \times \mathcal{H}^{-1}} \lesssim_{M,T,\alpha} N^{-\frac{7-\alpha}{2}}$$

for $\alpha \in (3, 5]$, uniformly in $N \geq 1$ and $t \in \overline{[0, T)}$. The implicit constants are independent of T if $\Omega = \mathbb{R}^3$ and $\alpha = 5$.

Proof. Since

$$\|B(t)\|_{\mathcal{H}^1 \times L^2} \lesssim_T \|g(u) - g(\pi_N u)\|_{L^1_T L^2},$$

the bounds for the energy norm follow directly from Lemma 6.10. Similarly, using that

$$\|B(t)\|_{L^2 \times \mathcal{H}^{-1}} \lesssim_T \|g(u) - g(\pi_N u)\|_{L^1_T \mathcal{H}^{-1}},$$

we obtain the bound for the case $\alpha = 5$, where the constant is independent of T if $\Omega = \mathbb{R}^3$. For the remaining estimates in the $L^2 \times \mathcal{H}^{-1}$ norm, we use a decomposition. We first split[1]

$$\begin{aligned} B(t) &= \int_0^t e^{(t-s)A}[G(U(s)) - G(\Pi_{N^2} U(s))] \, \mathrm{d}s \\ &\quad + \int_0^t e^{(t-s)A}[G(\Pi_{N^2} U(s)) - G(\Pi_N U(s))] \, \mathrm{d}s \\ &=: I_1 + I_2. \end{aligned}$$

Lemma 6.10 also yields

$$\|I_1\|_{L^2 \times \mathcal{H}^{-1}} \lesssim_T \|g(u) - g(\pi_{N^2} u)\|_{L^1_T \mathcal{H}^{-1}} \lesssim_{M,T} N^{-2}.$$

The second term is reformulated as

$$G(\Pi_{N^2} U(s)) - G(\Pi_N U(s)) = \int_0^1 G'(U_{N,\theta}(s)) \Pi_{N^2}(I - \Pi_N) U(s) \, \mathrm{d}\theta.$$

[1]Similar as in the proof of Lemma 6.10, this first decomposition is in principle only necessary for $\alpha = 3$.

Here we use the notation

$$U_{N,\theta}(s) \coloneqq \theta \pi_{N^2} U(s) + (1-\theta)\pi_N U(s)$$

and write $u_{N,\theta}$ for the first component of $U_{N,\theta}$ so that

$$G'(U_{N,\theta}(s)) = \begin{pmatrix} 0 & 0 \\ g'(u_{N,\theta}(s)) & 0 \end{pmatrix}.$$

Moreover, in order to gain a negative power of N, we insert the equality

$$(I - \Pi_N) = (N^{-1}\mathcal{D}\Phi_N)^{\frac{7-\alpha}{2}}$$

from Lemma 5.19 with $\mathcal{D} \coloneqq \operatorname{diag}(|\nabla|, |\nabla|)$ and $\Phi_N \coloneqq \operatorname{diag}(\phi_N, \phi_N)$. This leads to the representation

$$I_2 = N^{-\frac{7-\alpha}{2}} \int_0^1 \int_0^t e^{(t-s)A} G'(U_{N,\theta}(s)) \Pi_{N^2} \Phi_N^{\frac{7-\alpha}{2}} \mathcal{D}^{\frac{7-\alpha}{2}} U(s) \, \mathrm{d}s \, \mathrm{d}\theta.$$

Next, observe that $\mathcal{D} = JA$ for the operator

$$J \coloneqq \begin{pmatrix} 0 & -|\nabla|^{-1} \\ |\nabla| & 0 \end{pmatrix}.$$

If $\Omega = \mathbb{T}^3$, here we define the zero-th Fourier coefficient of $|\nabla|^{-1} f$ to be zero, for arbitrary functions $f \in H^r$. To simplify notation, we set $\widetilde{\Phi}_N \coloneqq \Pi_{N^2} \Phi_N^{(7-\alpha)/2} J$, which is a bounded operator on $\mathcal{H}^r \times \mathcal{H}^{r-1}$ for all $r < 3/2$, uniformly in $N \geq 1$. Altogether it follows that

$$I_2 = N^{-\frac{7-\alpha}{2}} \int_0^1 \int_0^t e^{(t-s)A} G'(U_{N,\theta}(s)) \widetilde{\Phi}_N \mathcal{D}^{\frac{5-\alpha}{2}} A U(s) \, \mathrm{d}s \, \mathrm{d}\theta.$$

Recall that the differential equation $AU = \partial_t U - G(U)$ from (6.1) holds in $C([0,T], L^2 \times H^{-1})$. Inserting it, we split I_2 into

$$\begin{aligned}
I_2 &= N^{-\frac{7-\alpha}{2}} \int_0^1 \int_0^t e^{(t-s)A} G'(U_{N,\theta}(s)) \widetilde{\Phi}_N \mathcal{D}^{\frac{5-\alpha}{2}} \partial_t U(s) \, \mathrm{d}s \, \mathrm{d}\theta \\
&\quad - N^{-\frac{7-\alpha}{2}} \int_0^1 \int_0^t e^{(t-s)A} G'(U_{N,\theta}(s)) \widetilde{\Phi}_N \mathcal{D}^{\frac{5-\alpha}{2}} G(U(s)) \, \mathrm{d}s \, \mathrm{d}\theta \\
&\eqqcolon N^{-\frac{7-\alpha}{2}} (I_{2,1} - I_{2,2}).
\end{aligned}$$

Using Hölder's inequality with $\frac{5}{6} = \frac{1}{3} + \frac{1}{2}$ and the Sobolev embedding $L^{\frac{6}{\alpha}} \hookrightarrow \mathcal{H}^{\frac{3-\alpha}{2}}$, the term with $G(U)$ is estimated by

$$\begin{aligned}
\|I_{2,2}\|_{L^2 \times \mathcal{H}^{-1}} &\lesssim_T \sup_{\theta \in [0,1]} \|g'(u_{N,\theta}) [\widetilde{\Phi}_N \mathcal{D}^{\frac{5-\alpha}{2}} G(U)]_1\|_{L^1_T L^{\frac{6}{5}}} \\
&\leq \sup_{\theta \in [0,1]} \|g'(u_{N,\theta})\|_{L^1_T L^3} \||\nabla|^{\frac{5-\alpha}{2}} g(u)\|_{L^\infty_T \mathcal{H}^{-1}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \||\pi_{N^2}u|^{\alpha-1} + |\pi_N u|^{\alpha-1}\|_{L^1_T L^3} \||u|^\alpha\|_{L^\infty_T L^{\frac{6}{\alpha}}} \\
&\lesssim \Big(\|\pi_{N^2}u\|^{\alpha-1}_{L^{\alpha-1}L^{3(\alpha-1)}} + \|\pi_N u\|^{\alpha-1}_{L^{\alpha-1}L^{3(\alpha-1)}}\Big)\|u\|^\alpha_{L^\infty_T L^6} \lesssim_{M,T} 1,
\end{aligned}
$$

where the estimate in the last line follows from Hölder's inequality in time, $p_\alpha \geq \alpha - 1$, Proposition 6.9, and Assumption 6.3. Recall that the notation $[\cdot]_1$ means that we take the first component of the vector.

The summand with $\partial_t U$ is integrated by parts in time, which gives

$$
\begin{aligned}
I_{2,1} &= \int_0^1 \Big[e^{(t-s)A} G'(U_{N,\theta}(s))\widetilde{\Phi}_N \mathcal{D}^{\frac{5-\alpha}{2}} U(s)\Big]_{s=0}^t \,\mathrm{d}\theta \\
&\quad + \int_0^1\int_0^t A e^{(t-s)A} G'(U_{N,\theta}(s))\widetilde{\Phi}_N \mathcal{D}^{\frac{5-\alpha}{2}} U(s)\,\mathrm{d}s\,\mathrm{d}\theta \\
&\quad - \int_0^1\int_0^t e^{(t-s)A} \frac{\mathrm{d}}{\mathrm{d}s} G'(U_{N,\theta}(s))\widetilde{\Phi}_N \mathcal{D}^{\frac{5-\alpha}{2}} U(s)\,\mathrm{d}s\,\mathrm{d}\theta \\
&=: I_{2,1,1} + I_{2,1,2} - I_{2,1,3}.
\end{aligned}
$$

The boundary terms $I_{2,1,1}$ can be estimated by means of Sobolev and Hölder inequalities. For $\theta \in [0,1]$ and $s \in \{0,t\}$, we infer

$$
\begin{aligned}
\|I_{2,1,1}\|_{L^2\times\mathcal{H}^{-1}} &\lesssim_T \|g'(u_{N,\theta}(s))[\mathcal{D}^{\frac{5-\alpha}{2}}\widetilde{\Phi}_N U(s)]_1\|_{L^{\frac{6}{5}}} \\
&\leq \|g'(u_{N,\theta}(s))\|_{L^{\frac{6}{\alpha-1}}} \|[\mathcal{D}^{\frac{5-\alpha}{2}}\widetilde{\Phi}_N U(s)]_1\|_{L^{\frac{6}{6-\alpha}}} \\
&\lesssim \Big(\|\pi_{N^2}u(s)\|^{\alpha-1}_{L^6} + \|\pi_N u(s)\|^{\alpha-1}_{L^6}\Big)\|U(s)\|_{\mathcal{H}^1\times L^2} \lesssim_M 1,
\end{aligned}
$$

using $\mathcal{H}^{\frac{\alpha-3}{2}} \hookrightarrow L^{\frac{6}{6-\alpha}}$. Similarly the contribution with A is estimated by

$$
\begin{aligned}
\|I_{2,1,2}\|_{L^2\times\mathcal{H}^{-1}} &\lesssim_T \|g'(u_{N,\theta})[\mathcal{D}^{\frac{5-\alpha}{2}}\widetilde{\Phi}_N U]_1\|_{L^1_T L^2} \\
&\leq \|g'(u_{N,\theta})\|_{L^1_T L^{\frac{6}{\alpha-3}}} \|[\mathcal{D}^{\frac{5-\alpha}{2}}\widetilde{\Phi}_N U]_1\|_{L^\infty_T L^{\frac{6}{6-\alpha}}} \\
&\lesssim_M \|\pi_{N^2}u\|^{\alpha-1}_{L^{\alpha-1}_T L^{q_\alpha}} + \|\pi_N u\|^{\alpha-1}_{L^{\alpha-1}_T L^{q_\alpha}}.
\end{aligned}
$$

If $\alpha \in (3,5)$, the right-hand side is uniformly bounded by a constant depending on M, T, and α, using Proposition 6.9 with $(p,q) = (\alpha-1, q_\alpha)$ from (6.11), and hence, $\|I_{2,1,2}\|_{L^2\times\mathcal{H}^{-1}} \lesssim_{M,T,\alpha} 1$. If $\alpha = 3$, we instead use the logarithmic endpoint estimate from Proposition 6.9 for the $L^2_T L^\infty$ norm, which gives $\|I_{2,1,2}\|_{L^2\times\mathcal{H}^{-1}} \lesssim_{M,T} 1 + \log N$. Finally, to get the estimate for $I_{2,1,3}$, we observe that

$$
\frac{\mathrm{d}}{\mathrm{d}s} g'(u_{N,\theta}(s)) = g''(u_{N,\theta}(s))\partial_s u_{N,\theta}(s) = g''(u_{N,\theta}(s))\Big(\theta\pi_{N^2}\partial_t u(s) + (1-\theta)\pi_N \partial_t u(s)\Big).
$$

If $\alpha \in (3,5)$, the dual Strichartz estimate from Corollary 5.18 with $(p,q) = (\alpha-1, q_\alpha)$ implies

$$
\|I_{2,1,3}\|_{L^2\times\mathcal{H}^{-1}} \lesssim_T \|g''(u_{N,\theta})\partial_s u_{N,\theta}[\mathcal{D}^{\frac{5-\alpha}{2}}\widetilde{\Phi}_N U]_1\|_{L_T^{\frac{\alpha-1}{\alpha-2}} L^{q'_\alpha}}
$$

$$
\begin{aligned}
&\leq \|g''(u_{N,\theta})\|_{L_T^{\frac{\alpha-1}{\alpha-2}} L^{\frac{6(\alpha-1)}{(\alpha-3)(\alpha-2)}}} \|\partial_s u_{N,\theta}\|_{L_T^\infty L^2} \|[\mathcal{D}^{\frac{5-\alpha}{2}} \widetilde{\Phi}_N U]_1\|_{L_T^\infty L^{\frac{6}{6-\alpha}}} \\
&\lesssim \Big(\|\pi_{N^2} u\|^{\alpha-2}_{L_T^{\alpha-1} L^{\frac{6(\alpha-1)}{\alpha-3}}} + \|\pi_N u\|^{\alpha-2}_{L_T^{\alpha-1} L^{\frac{6(\alpha-1)}{\alpha-3}}} \Big) \|\partial_t u\|_{L_T^\infty L^2} \|U\|_{L_T^\infty(\mathcal{H}^1 \times L^2)} \\
&\lesssim_{M,T,\alpha} 1,
\end{aligned}
$$

using Hölder's inequality with

$$
\frac{1}{q'_\alpha} = \frac{5\alpha-3}{6(\alpha-1)} = \frac{(\alpha-3)(\alpha-2)}{6(\alpha-1)} + \frac{6-\alpha}{6} + \frac{1}{2} = \frac{(\alpha-3)(\alpha-2)}{6(\alpha-1)} + \frac{9-\alpha}{6} = \frac{3(\alpha-1)}{\alpha} + \frac{1}{2}, \tag{7.14}
$$

the Sobolev embedding $\mathcal{H}^{\frac{\alpha-3}{2}} \hookrightarrow L^{\frac{6}{6-\alpha}}$, and Proposition 6.9. In the case $\alpha = 3$, we exploit that the polynomial $g(u) = -\mu u^3$ keeps the frequency localization π_{N^2} up to a factor 3. This means that $I_{2,1,3} = \Pi_{3N^2} I_{2,1,3}$.[2] Hence, we can apply the dual endpoint logarithmic Strichartz estimate from Corollary 5.18 to conclude

$$
\begin{aligned}
&\|\Pi_{3N^2} I_{2,1,3}\|_{L^2 \times \mathcal{H}^{-1}} \\
&\qquad \lesssim_T (1+\log N)^{\frac{1}{2}} \|g''(u_{N,\theta}) \partial_s u_{N,\theta} [\mathcal{D}\widetilde{\Phi}_N U]_1\|_{L_T^2 L^1} \\
&\qquad \leq (1+\log N)^{\frac{1}{2}} \|g''(u_{N,\theta})\|_{L_T^2 L^\infty} \|\partial_s u_{N,\theta}\|_{L_T^\infty L^2} \|[\mathcal{D}\widetilde{\Phi}_N U]_1\|_{L_T^\infty L^2} \\
&\qquad \lesssim (1+\log N)^{\frac{1}{2}} \Big(\|\pi_{N^2} u\|_{L_T^2 L^\infty} + \|\pi_N u\|_{L_T^2 L^\infty} \Big) \|\partial_t u\|_{L_T^\infty L^2} \|U\|_{L_T^\infty(\mathcal{H}^1 \times L^2)} \\
&\qquad \lesssim_{M,T} (1+\log N),
\end{aligned}
$$

where in the end we again apply the logarithmic endpoint estimate for u from Proposition 6.9. □

7.3. Estimates for local error terms

Next, we treat the term D_n from (7.8) that includes the local error terms, starting with d_1, d_2, and d_3. In the following, we will not always explicitly mention the repeated use of the Sobolev embeddings $\mathcal{H}^{(\alpha-3)/2} \hookrightarrow L^{6/(6-\alpha)}$ and $L^{6/5} \hookrightarrow \mathcal{H}^{-1}$, Hölder's inequality with $\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$ and $\frac{1}{2} = \frac{\alpha-3}{6} + \frac{6-\alpha}{6}$, and Proposition 6.9 with admissible exponents $(p,q) \in \{(p_\alpha, 3(\alpha-1)), (\alpha-1, q_\alpha)\}$.

Lemma 7.5. *Let $U = (u, \partial_t u)$, T, and M be given by Assumption 6.3. The third term in* (7.10) *satisfies*

$$
\|d_3\|_{L_T^1 \mathcal{H}^{-1}} \lesssim_{M,T} 1
$$

if $\alpha \in [3,4]$, and

$$
\|d_3\|_{L_T^1 \mathcal{H}^{-1}} \lesssim_{M,T} N^{\alpha-4}
$$

[2] In view of Remark 4.10, one could avoid to use the special form of g, by involving another triangle inequality with $\Pi_{N^2} I_{2,1,3}$, for instance.

for $\alpha \in (4,5]$*. For the first two terms in* (7.10)*, we have*

$$\|d_1\|_{L^1_T L^2} \lesssim_{M,T} (1+\log N),$$
$$\|d_2\|_{L^2_T L^1} \lesssim_{M,T} (1+\log N)^{\frac{1}{2}}$$

for $\alpha = 3$*, and*

$$\|d_1\|_{L^1_T L^2} + \|d_2\|_{L_T^{\frac{\alpha-1}{\alpha-2}} L^{q'_\alpha}} \lesssim_{M,T,\alpha} N^{\frac{\alpha-3}{2}}$$

for $\alpha \in (3,5]$ *and* q_α *from* (6.11)*. All estimates are uniform in* $N \geq 1$*.*

Proof. First, Sobolev and Hölder inequalities yield

$$\begin{aligned}
\|d_3\|_{L^1_T \mathcal{H}^{-1}} &\lesssim \|g'(\pi_N u)\pi_N g(u)\|_{L^1_T L^{\frac{6}{5}}} \leq \|g'(\pi_N u)\|_{L_T^{\frac{2}{\alpha-3}} L^3} \|\pi_N g(u)\|_{L_T^{\frac{2}{5-\alpha}} L^2} \\
&\lesssim \|\pi_N u\|^{\alpha-1}_{L^{p_\alpha}_T L^{3(\alpha-1)}} \|\pi_N g(u)\|_{L_T^{\frac{2}{5-\alpha}} L^2} \lesssim_M \|\pi_N g(u)\|_{L_T^{\frac{2}{5-\alpha}} L^2}.
\end{aligned}$$

For $\alpha \in [3,4]$, we get

$$\|\pi_N g(u)\|_{L_T^{\frac{2}{5-\alpha}} L^2} \lesssim \|u\|^\alpha_{L_T^{\frac{2\alpha}{5-\alpha}} L^{2\alpha}} \lesssim_T \|u\|^\alpha_{L_T^{\frac{2\alpha}{\alpha-3}} L^{2\alpha}} \lesssim_{M,T} 1$$

by Hölder's inequality in time (using $\alpha \leq 4$) and Proposition 6.9 with $(p,q) = (\frac{2\alpha}{\alpha-3}, 2\alpha)$. If $\alpha \in (4,5]$, we need to apply Bernstein's Lemma A.3 to obtain first

$$\begin{aligned}
\|\pi_N g(u)\|_{L_T^{\frac{2}{5-\alpha}} L^2} &\lesssim N^{\alpha-4} \|g(u)\|_{L_T^{\frac{2}{5-\alpha}} \mathcal{H}^{4-\alpha}} \lesssim N^{\alpha-4} \|g(u)\|_{L_T^{\frac{2}{5-\alpha}} L^{\frac{6}{2\alpha-5}}} \\
&\leq N^{\alpha-4} \|u\|^\alpha_{L_T^{\frac{2\alpha}{5-\alpha}} L^{\frac{6\alpha}{2\alpha-5}}} \lesssim_{M,T} N^{\alpha-4},
\end{aligned}$$

exploiting the dual Sobolev embedding $L^{\frac{6}{2\alpha-5}} \hookrightarrow \mathcal{H}^{4-\alpha}$ and Proposition 6.9 with $(p,q) = (\frac{2\alpha}{5-\alpha}, \frac{6\alpha}{2\alpha-5})$.

Let now $\alpha = 3$. We then derive

$$\begin{aligned}
\|d_1\|_{L^1_T L^2} &\lesssim \|g'(\pi_N u)\pi_N \partial_t u\|_{L^1_T L^2} \leq \|g'(\pi_N u)\|_{L^1_T L^\infty} \|\pi_N \partial_t u\|_{L^\infty_T L^2} \\
&\lesssim_M \|\pi_N u\|^2_{L^2_T L^\infty} \lesssim_{M,T} 1 + \log N, \\
\|d_2\|_{L^2_T L^1} &\lesssim \|g''(\pi_N u)\,[|\nabla \pi_N u|^2 + (\pi_N \partial_t u)^2]\,\|_{L^2_T L^1} \\
&\leq \|g''(\pi_N u)\|_{L^2_T L^\infty} \||\nabla \pi_N u|^2 + (\pi_N \partial_t u)^2\|_{L^\infty_T L^1} \\
&\lesssim \|\pi_N u\|_{L^2_T L^\infty} \Big(\|\nabla u\|^2_{L^\infty_T L^2} + \|\partial_t u\|^2_{L^\infty_T L^2}\Big) \lesssim_{M,T} (1+\log N)^{\frac{1}{2}},
\end{aligned}$$

using the logarithmic endpoint estimate from Proposition 6.9. Similarly, for $\alpha \in (3,5]$, these terms are bounded by

$$\|d_1\|_{L^1_T L^2} \lesssim \|g'(\pi_N u)\|_{L^1_T L^{\frac{6}{\alpha-3}}} \|\pi_N \partial_t u\|_{L^\infty_T L^{\frac{6}{6-\alpha}}}$$

$$
\begin{aligned}
&\lesssim \|\pi_N u\|_{L_T^{\alpha-1} L^{q_\alpha}}^{\alpha-1} \|\pi_N \partial_t u\|_{L_T^\infty \mathcal{H}^{\frac{\alpha-3}{2}}} \lesssim_{M,T,\alpha} N^{\frac{\alpha-3}{2}} \|\partial_t u\|_{L_T^\infty L^2} \lesssim_M N^{\frac{\alpha-3}{2}}, \\
\|d_2\|_{L_T^{\frac{\alpha-1}{\alpha-2}} L^{q'_\alpha}} &\lesssim \|g''(\pi_N u)\|_{L_T^{\frac{\alpha-1}{\alpha-2}} L^{\frac{6(\alpha-1)}{(\alpha-3)(\alpha-2)}}} \||\nabla \pi_N u|^2 + (\pi_N \partial_t u)^2\|_{L_T^\infty L^{\frac{6}{9-\alpha}}} \\
&\lesssim \|\pi_N u\|_{L_T^{\alpha-1} L^{q_\alpha}}^{\alpha-2} \Big(\|\nabla \pi_N u\|^2_{L_T^\infty L^{\frac{12}{9-\alpha}}} + \|\pi_N \partial_t u\|^2_{L_T^\infty L^{\frac{12}{9-\alpha}}} \Big) \\
&\lesssim_{M,T,\alpha} N^{\frac{\alpha-3}{2}} \Big(\|\nabla u\|^2_{L_T^\infty L^2} + \|\partial_t u\|^2_{L_T^\infty L^2} \Big) \lesssim_M N^{\frac{\alpha-3}{2}},
\end{aligned}
$$

employing the relation 7.14 and Proposition 6.9 with $(p,q) = (\alpha-1, q_\alpha)$. The loss of $N^{(\alpha-3)/2}$ comes from the Bernstein inequality in Lemma A.3. □

The term d_4 from (7.10) is the most difficult one, because it involves second partial derivatives of u. Therefore, we follow the same strategy as in Lemma 7.4. However, since the situation is now more "discrete" in time, we apply summation by parts instead of integration by parts. Roughly speaking, this transforms the term containing d_4 into terms that can be estimated in the same way as d_1 and d_2 in Lemma 7.5. To use summation by parts, we need the filter $\Pi_{\tau^{-1}}$, cf. the discussion before Lemma 5.20. As noted before, such a strategy was already used in [10] in a situation without Strichartz estimates.

Lemma 7.6. *Let $U = (u, \partial_t u)$, T, and M be given by Assumption 6.3. Then the term D_n from* (7.8)*. is estimated by*

$$
\begin{aligned}
\|D_n\|_{\mathcal{H}^1 \times L^2} &\lesssim_{M,T} \tau(1 + |\log \tau|), \\
\|D_n\|_{L^2 \times \mathcal{H}^{-1}} &\lesssim_{M,T} \tau^2(1 + |\log \tau|)
\end{aligned}
$$

if $\alpha = 3$, and

$$
\begin{aligned}
\|D_n\|_{\mathcal{H}^1 \times L^2} &\lesssim_{M,T,\alpha} \tau^{\frac{5-\alpha}{2}}, \\
\|D_n\|_{L^2 \times \mathcal{H}^{-1}} &\lesssim_{M,T,\alpha} \tau^{\frac{7-\alpha}{2}}
\end{aligned}
$$

for $\alpha \in (3,5]$, uniformly in $\tau \in (0,1]$, $K \in \mathbb{N} \cup \{\infty\}$, and $n \in \mathbb{N}_0$ with $n\tau \in \overline{[0,T)}$. The implicit constants are independent of T if $\Omega = \mathbb{R}^3$ and $\alpha = 5$.

Proof. We first show the estimates for the energy norm. Recall that $N = \min\{\tau^{-1}, K\}$. The first-order representation (7.12) implies

$$
\begin{aligned}
\|D_n\|_{\mathcal{H}^1 \times L^2} &\lesssim_T \tau \Big(\|g(\pi_N u)\|_{L_T^1 \mathcal{H}^1} + \|g'(\pi_N u) \pi_N \partial_t u\|_{L_T^1 L^2} \Big) \\
&\lesssim \tau \||\pi_N u|^{\alpha-1}\|_{L_T^1 L^{\frac{6}{\alpha-3}}} \Big(\|\pi_N \tilde{\nabla} u\|_{L_T^\infty L^{\frac{6}{6-\alpha}}} + \|\pi_N \partial_t u\|_{L_T^\infty L^{\frac{6}{6-\alpha}}} \Big) \\
&\lesssim \tau^{1-\frac{\alpha-3}{2}} \|\pi_N u\|_{L_T^{\alpha-1} L^{q_\alpha}}^{\alpha-1} \Big(\|\pi_N u\|_{L_T^\infty \mathcal{H}^1} + \|\pi_N \partial_t u\|_{L_T^\infty L^2} \Big) \lesssim_{M,T,\alpha} \tau^{\frac{5-\alpha}{2}},
\end{aligned}
$$

for $\alpha \in (3,5]$, also taking into account Bernstein's inequality from Lemma A.3 and Proposition 6.9, where $\tilde{\nabla} = \nabla$ on $\mathbb{R}^3$ and $\tilde{\nabla} = I + \nabla$ on $\mathbb{T}^3$. Similarly, for $\alpha = 3$ we derive the inequality

$$
\|D_n\|_{\mathcal{H}^1 \times L^2} \lesssim_{M,T} \tau(1 + |\log \tau|),
$$

by means of the logarithmic endpoint estimate from Proposition 6.9. For $\alpha = 5$, we get

$$\begin{aligned}\|D_n\|_{L^2\times\mathcal{H}^{-1}} &\lesssim_T \tau\Big(\|g(\pi_N u)\|_{L^1_T L^2} + \|g'(\pi_N u)\pi_N \partial_t u\|_{L^1_T \mathcal{H}^{-1}}\Big)\\ &\lesssim \tau \|\pi_N u\|^4_{L^4_T L^{12}}\Big(\|\pi_N u\|_{L^\infty_T L^6} + \|\pi_N \partial_t u\|_{L^\infty_T L^2}\Big) \lesssim_M \tau\end{aligned}$$

using again Proposition 6.9, where the constant is independent of T if $\Omega = \mathbb{R}^3$.

We now show the remaining bounds for the $L^2 \times \mathcal{H}^{-1}$ norm. Using Corollary 5.18 for the term involving d_2, we start with the inequality

$$\|D_n\|_{L^2\times\mathcal{H}^{-1}} \lesssim_T \tau^2\Big(\|d_1\|_{L^1_T L^2} + \tilde{d}_2 + \|d_3\|_{L^1_T\mathcal{H}^{-1}} + \|D_{n,4}\|_{L^2\times\mathcal{H}^{-1}}),$$

where $\tilde{d}_2 := (1 + |\log\tau|)^{1/2}\|d_2\|_{L^2_T L^1}$ for $\alpha = 3$ and $\tilde{d}_2 := \|d_2\|_{L_T^{\frac{\alpha-1}{\alpha-2}} L^{q'_\alpha}}$ for $\alpha \in (3,5)$, as well as

$$D_{n,4} := \int_0^{t_n} e^{(n\tau - s)A}(\lfloor \tfrac{s}{\tau}\rfloor - \tfrac{s}{\tau})(\lceil \tfrac{s}{\tau}\rceil - \tfrac{s}{\tau}) \begin{pmatrix} 0 \\ d_4(s)\end{pmatrix} \,\mathrm{d}s$$

independent of α. The terms containing d_1, d_2, and d_3 are estimated by Lemma 7.5 (using that $\alpha \le 5$ for d_3). We still need to deal with the term $D_{n,4}$. As in (7.11), the remaining summand is given by

$$D_{n,4} = \int_0^\tau \tfrac{s}{\tau}(\tfrac{s}{\tau} - 1)\sum_{k=0}^{n-1} e^{((n-k)\tau - s)A}\begin{pmatrix} 0 \\ d_4(t_k + s)\end{pmatrix} \,\mathrm{d}s.$$

Since

$$\begin{aligned}\begin{pmatrix} 0 \\ d_4(t_k+s)\end{pmatrix} &= 2G'(\Pi_N U(t_k+s))A^2\Pi_N U(t_k+s),\\ G'(\Pi_N U(t_k+s)) &= \begin{pmatrix} 0 & 0 \\ g'(\pi_N u(t_k+s)) & 0\end{pmatrix},\end{aligned}$$

we can write

$$D_{n,4} = 2\int_0^\tau \tfrac{s}{\tau}(\tfrac{s}{\tau} - 1)\sum_{k=0}^{n-1} e^{((n-k)\tau - s)A} G'(\Pi_N U(t_k+s))A^2\Pi_N U(t_k+s)\,\mathrm{d}s.$$

Recall the summation by parts formula

$$\sum_{k=0}^{n-1} a_k b_k = a_{n-1}b_{n-1} + a_{n-1}\sum_{k=0}^{n-2} b_k + \sum_{k=0}^{n-2}(a_k - a_{k+1})\sum_{j=0}^{k} b_j,$$

with $a_k = e^{(n-k)\tau A}G'(\Pi_N U(t_k+s))$ and $b_k = A^2\Pi_N U(t_k+s)$. It yields

$$D_{n,4} = 2(I_1 + I_2 + I_3)$$

with

$$I_1 := \int_0^\tau \tfrac{s}{\tau}(\tfrac{s}{\tau}-1)e^{(\tau-s)A}G'(\Pi_N U(t_{n-1}+s))A^2\Pi_N U(t_{n-1}+s)\,\mathrm{d}s,$$

$$I_2 := \int_0^\tau \tfrac{s}{\tau}(\tfrac{s}{\tau}-1)e^{(\tau-s)A}G'(\Pi_N U(t_{n-1}+s))\sum_{k=0}^{n-2}A^2\Pi_N U(t_k+s)\,\mathrm{d}s,$$

$$\begin{aligned}I_3 := \int_0^\tau \tfrac{s}{\tau}(\tfrac{s}{\tau}-1)\sum_{k=0}^{n-2}e^{((n-k)\tau-s)A}\Big(&G'(\Pi_N U(t_k+s))\\ &- e^{-\tau A}G'(\Pi_N U(t_{k+1}+s))\Big)\sum_{j=0}^{k}A^2\Pi_N U(t_j+s)\,\mathrm{d}s.\end{aligned}$$

Next, we insert the equality

$$\tau A\Pi_N = (e^{\tau A}-I)\Psi_{\tau,N}$$

from Lemma 5.20, where the operator $\Psi_{\tau,N}$ is bounded on $\mathcal{H}^r\times\mathcal{H}^{r-1}$ for all $r\in\mathbb{R}$, uniformly in $\tau\in(0,1]$ and $N\in[1,1/\tau]$. The term I_1 is controlled by Sobolev, Hölder, and Bernstein inequalities via

$$\begin{aligned}\|I_1\|_{L^2\times\mathcal{H}^{-1}} &\lesssim \sup_{s\in[0,\tau]}\|g'(\pi_N u(t_{n-1}+s))[(e^{\tau A}-I)\Psi_{\tau,N}AU(t_{n-1}+s)]_1\|_{L^{\frac{6}{5}}}\\ &\lesssim \sup_{s\in[0,\tau]}\|g'(\pi_N u(t_{n-1}+s))\|_{L^3}\|AU(t_{n-1}+s)\|_{L^2\times\mathcal{H}^{-1}}\\ &\lesssim_M \|\pi_N u\|^{\alpha-1}_{L^\infty_T L^{3(\alpha-1)}} \lesssim \tau^{-\frac{\alpha-3}{2}}\|u\|^{\alpha-1}_{L^\infty_T\mathcal{H}^1} \lesssim_M \tau^{-\frac{\alpha-3}{2}}.\end{aligned}$$

Next, for $j\in\{0,\dots,n-2\}$ and $s\in[0,\tau]$ we define the sum

$$\tilde{S}(\tau,j,s) := \tau\sum_{k=0}^{j}A^2\Pi_N U(t_k+s) = \sum_{k=0}^{j}(e^{\tau A}-I)\Psi_{\tau,N}AU(t_k+s). \tag{7.15}$$

A shifted version of Duhamel's formula (6.2) yields

$$\begin{aligned}\tilde{S}(\tau,j,s) &= \sum_{k=0}^{j}\Psi_{\tau,N}A[U(t_{k+1}+s)-U(t_k+s)]\\ &\quad - \sum_{k=0}^{j}\int_0^\tau \Psi_{\tau,N}Ae^{(\tau-\sigma)A}G(U(t_k+s+\sigma))\,\mathrm{d}\sigma.\end{aligned}$$

We exploit this telescopic and (6.9) sum to conclude that

$$\begin{aligned}\|\tilde{S}(\tau,j,s)\|_{L^2\times\mathcal{H}^{-1}} &\lesssim_T \|U(t_{j+1}+s)-U(s)\|_{\mathcal{H}^1\times L^2} + \sum_{k=0}^{j}\int_0^\tau\|g(u(t_k+s+\sigma))\|_{L^2}\,\mathrm{d}\sigma\\ &\lesssim \|U\|_{L^\infty_T(\mathcal{H}^1\times L^2)} + \|g(u)\|_{L^1_T L^2} \lesssim_{M,T} 1,\end{aligned} \tag{7.16}$$

uniformly in $j \in \{0, \dots, n-2\}$, $\tau \in (0,1]$ and $s \in [0,\tau]$. Hence, the summand I_2 is estimated similar as I_1 by

$$\begin{aligned}\|I_2\|_{L^2\times\mathcal{H}^{-1}} &\lesssim \sup_{s\in[0,\tau]} \|g'(\pi_N u(t_{n-1}+s))[\tilde{S}(\tau,n-2,s)]_1\|_{L^{\frac{6}{5}}} \\ &\lesssim \sup_{s\in[0,\tau]} \|g'(\pi_N u(t_{n-1}+s))\|_{L^3} \|\tilde{S}(\tau,n-2,s)\|_{L^2\times\mathcal{H}^{-1}} \lesssim_{M,T} \tau^{-\frac{\alpha-3}{2}}.\end{aligned}$$

The term I_3 is treated by means of another decomposition. Involving (7.15), we split it as

$$I_3 = I_{3,1} + I_{3,2},$$

with the expressions

$$\begin{aligned} I_{3,1} &:= \tfrac{1}{\tau}\int_0^\tau \tfrac{s}{\tau}(\tfrac{s}{\tau}-1)\sum_{k=0}^{n-2} e^{((n-k)\tau-s)A}(I-e^{-\tau A})G'(\Pi_N U(t_k+s))\tilde{S}(\tau,k,s)\,\mathrm{d}s, \\ I_{3,2} &:= \tfrac{1}{\tau}\int_0^\tau \tfrac{s}{\tau}(\tfrac{s}{\tau}-1)\sum_{k=0}^{n-2} e^{((n-k-1)\tau-s)A} \\ &\qquad \cdot \Big(G'(\Pi_N U(t_k+s)) - G'(\Pi_N U(t_{k+1}+s))\Big)\tilde{S}(\tau,k,s)\,\mathrm{d}s. \end{aligned}$$

For $I_{3,1}$, observe that

$$I - e^{-\tau A} = \tau A \varphi_1(-\tau A),$$

where $\varphi_1(z) := (e^z-1)/z$. Since the function φ_1 is bounded on $\mathrm{i}\mathbb{R}$, the operator $\varphi_1(-\tau A)$ is bounded on $\mathcal{H}^1 \times L^2$, uniformly in $\tau \in (0,1]$. For $\alpha = 3$, we then derive

$$\begin{aligned}\|I_{3,1}\|_{L^2\times\mathcal{H}^{-1}} &\lesssim_T \int_0^\tau \sum_{k=0}^{n-2} \|G'(\Pi_N U(t_k+s))\tilde{S}(\tau,k,s)\|_{\mathcal{H}^1\times L^2}\,\mathrm{d}s \\ &\le \int_0^\tau \sum_{k=0}^{n-2} \|g'(\pi_N u(t_k+s))\|_{L^\infty}\|\tilde{S}(\tau,k,s)\|_{L^2\times\mathcal{H}^{-1}}\,\mathrm{d}s \\ &\lesssim_{M,T} \|\pi_N u\|^2_{L^2_T L^\infty} \lesssim_{M,T} 1 + |\log\tau|,\end{aligned}$$

using estimate (7.16) and the logarithmic endpoint estimate for u from Proposition 6.9. Similarly, for $\alpha \in (3,5)$ one obtains

$$\begin{aligned}\|I_{3,1}\|_{L^2\times\mathcal{H}^{-1}} &\lesssim_T \int_0^\tau \sum_{k=0}^{n-2} \|g'(\pi_N u(t_k+s))\|_{L^{\frac{6}{\alpha-3}}} \|[\tilde{S}(\tau,k,s)]_1\|_{L^{\frac{6}{6-\alpha}}}\,\mathrm{d}s \\ &\lesssim \|\pi_N u\|^{\alpha-1}_{L^{\alpha-1}_T L^{q_\alpha}} \tau^{-\frac{\alpha-3}{2}} \sup_{\substack{k\in\{0,\dots,n-2\}\\ s\in[0,\tau]}} \|\tilde{S}(\tau,k,s)\|_{L^2\times\mathcal{H}^{-1}} \lesssim_{M,T,\alpha} \tau^{-\frac{\alpha-3}{2}}\end{aligned}$$

due to Bernstein's inequality and Proposition 6.9. For the term $I_{3,2}$, we first substitute $\tilde{s} = s + t_k$ to get

$$I_{3,2} = \tfrac{1}{\tau}\int_0^{t_{n-1}} e^{((n-1)\tau-s)A}(\tfrac{s}{\tau} - \lfloor\tfrac{s}{\tau}\rfloor)(\tfrac{s}{\tau} - \lceil\tfrac{s}{\tau}\rceil)$$

$$\cdot \Big(G'(\Pi_N U(s)) - G'(\Pi_N U(\tau+s))\Big)\tilde{S}(\tau, \lfloor \tfrac{s}{\tau} \rfloor, s - \tau \lfloor \tfrac{s}{\tau} \rfloor)\, \mathrm{d}s.$$

We first consider the case $\alpha = 3$. By the polynomial structure of g, we have the frequency localization $I_{3,2} = \Pi_{3\tau^{-1}} I_{3,2}$. The dual logarithmic endpoint Strichartz estimate from Corollary 5.18 and the bound (7.16) for $\tilde{S}$ thus yield

$$\begin{aligned} \|I_{3,2}\|_{L^2 \times \mathcal{H}^{-1}} &\lesssim_T \tfrac{1}{\tau}\Big((1 + |\log \tau|) \int_0^{t_{n-1}} \|(g'(\pi_N u(s)) - g'(\pi_N u(\tau+s))) \\ &\qquad \cdot [\tilde{S}(\tau, \lfloor \tfrac{s}{\tau} \rfloor, s - \tau \lfloor \tfrac{s}{\tau} \rfloor)]_1\|_{L^1}^2 \,\mathrm{d}s\Big)^{\frac{1}{2}} \\ &\lesssim_{M,T} \tfrac{1}{\tau}(1 + |\log \tau|)^{\frac{1}{2}} \|g'(\pi_N u) - g'(\pi_N u(\tau + \cdot))\|_{L^2_{t_{n-1}} L^2}. \end{aligned} \tag{7.17}$$

To conclude, from the equation

$$u(s) - u(\tau + s) = -\int_0^\tau \partial_t u(s + \sigma)\, \mathrm{d}\sigma$$

we deduce

$$\begin{aligned} &\|g'(\pi_N u) - g'(\pi_N u(\tau + \cdot))\|_{L^2_{t_{n-1}} L^2} \\ &\quad \lesssim \||g''(\pi_N u)| + |g''(\pi_N u(\tau + \cdot))|\|_{L^2_{t_{n-1}} L^\infty} \sup_{s \in [0, t_{n-1}]} \int_0^\tau \|\partial_t u(s+\sigma)\|_{L^2}\, \mathrm{d}\sigma \\ &\quad \lesssim_{M,T} \|\pi_N u\|_{L^2_T L^\infty} \tau \|\partial_t u\|_{L^\infty_T L^2} \lesssim_{M,T} (1 + |\log \tau|)^{\frac{1}{2}} \tau. \end{aligned}$$

Together with (7.17), this implies the desired bound for $I_{3,2}$.

If $\alpha \in (3, 5]$, we follow a similar strategy to obtain

$$\begin{aligned} \|I_{3,2}\|_{L^2 \times \mathcal{H}^{-1}} &\lesssim_T \tfrac{1}{\tau}\Big(\int_0^{t_{n-1}} \|(g'(\pi_N u(s)) - g'(\pi_N u(\tau+s))) \\ &\qquad \cdot [\tilde{S}(\tau, \lfloor \tfrac{s}{\tau} \rfloor, s - \tau \lfloor \tfrac{s}{\tau} \rfloor)]_1\|_{L^{q'_\alpha}}^{\frac{\alpha-1}{\alpha-2}} \,\mathrm{d}s\Big)^{\frac{\alpha-2}{\alpha-1}} \\ &\lesssim_{M,T} \tfrac{1}{\tau} \|g'(\pi_N u) - g'(\pi_N u(\tau + \cdot))\|_{L^{\frac{\alpha-1}{\alpha-2}}_{t_{n-1}} L^{\frac{3(\alpha-1)}{\alpha}}} \\ &\lesssim \|g''(\pi_N u)\|_{L_T^{\frac{\alpha-1}{\alpha-2}} L^{\frac{6(\alpha-1)}{(\alpha-3)(\alpha-2)}}} \|\pi_N \partial_t u\|_{L^\infty_T L^{\frac{6}{6-\alpha}}} \\ &\lesssim_{M,T} \tau^{-\frac{\alpha-3}{2}} \|\pi_N u\|_{L_T^{\alpha-1} L^{q_\alpha}}^{\alpha-1} \|\partial_t u\|_{L^\infty_T L^2} \lesssim_{M,T} \tau^{-\frac{\alpha-3}{2}}, \end{aligned}$$

exploiting the relation (7.14), which concludes the proof. □

7.4. Proof of the global error bounds for $\alpha = 3$

We give different proofs of the global error bounds depending on $\alpha \in [3, 5]$. We start with the proof for $\alpha = 3$, which is somewhat simpler.

In the cases $\alpha \in \{3,5\}$, the nonlinearity g is a polynomial. This makes the trigonometric interpolation I_K operator quite harmless due to Lemma 5.24. Therefore, in these cases we use the recursion formula (7.7), instead of formula (7.6) that is employed if $\alpha \in (3,5)$. The next lemma deals with the term $\tilde{H}_n$ from (7.9) and will be exploited in both cases $\alpha \in \{3,5\}$.

Lemma 7.7. *Let $U = (u, \partial_t u)$, T, and M be given by Assumption 6.3 with $\Omega = \mathbb{T}^3$ and define $\tilde{H}_n$ by* (7.9)*. We then obtain*

$$\|\tilde{H}_n\|_{H^1 \times L^2} \lesssim_{M,T} K^{-1}(1 + \log K)$$

if $\alpha = 3$, and

$$\|\tilde{H}_n\|_{L^2 \times H^{-1}} \lesssim_{M,T} K^{-1}$$

for $\alpha \in \{3,5\}$, uniformly in $\tau \in (0,1]$, $K \in \mathbb{N} \cup \{\infty\}$, and $n \in \mathbb{N}_0$ with $n\tau \in [0,T]$.

Proof. For $\alpha = 3$, Lemma 5.24 with $\beta = 3$ and Proposition 6.9 yield

$$\begin{aligned}
\|\tilde{H}_n\|_{\ell^\infty_\tau([0,T],H^1 \times L^2)} &\lesssim_T \|(I - I_K)g(\pi_N u(t_n))\|_{\ell^1_\tau([0,T],L^2)} \lesssim K^{-1}\|g(\pi_N u(t_n))\|_{\ell^1_\tau([0,T],H^1)} \\
&\lesssim K^{-1}\|g'(\pi_N u(t_n))\|_{\ell^1_\tau([0,T],L^\infty)}\|\pi_N u(t_n)\|_{\ell^\infty_\tau([0,T],H^1)} \\
&\lesssim K^{-1}\|\pi_N u(t_n)\|^2_{\ell^2_\tau([0,T],L^\infty)}\|u(t_n)\|_{\ell^\infty_\tau([0,T],H^1)} \\
&\lesssim_{M,T} K^{-1}(1 + \log K).
\end{aligned}$$

Similarly, for $\alpha \in \{3,5\}$ we obtain

$$\begin{aligned}
\|\tilde{H}_n\|_{\ell^\infty_\tau([0,T],L^2 \times H^{-1})} &\lesssim_T \|(I - I_K)g(\pi_N u(t_n))\|_{\ell^1_\tau([0,T],L^{\frac{6}{5}})} \\
&\lesssim K^{-1}\|g(\pi_N u(t_n))\|_{\ell^1_\tau([0,T],W^{1,\frac{6}{5}})} \\
&\lesssim K^{-1}\|g'(\pi_N u(t_n))\|_{\ell^1_\tau([0,T],L^3)}\|\pi_N u(t_n)\|_{\ell^\infty_\tau([0,T],H^1)} \\
&\lesssim K^{-1}\|\pi_N u(t_n)\|^{\alpha-1}_{\ell^{\alpha-1}_\tau([0,T],L^{3(\alpha-1)})} \lesssim_{M,T} K^{-1}.
\end{aligned}$$

Here it is important to use the interpolation error estimate from Lemma 5.24 with $q = 6/5$. If we stuck to L^2-based estimates, we could for the $L^2 \times H^{-1}$ norm only reach a sub-optimal estimate (the same as above for the energy norm, which would not yield convergence for $\alpha = 5$), since optimal error bounds for trigonometric interpolation in negative Sobolev spaces are not available. □

We still need to deal with the term $\tilde{Q}_n$ from (7.6). For $\alpha = 3$ it turns out that it is enough to use Sobolev and Hölder inequalities. We write u_n for the first component of U_n, as well as e_n for the first component of E_n.

Lemma 7.8. *Let $U = (u, \partial_t u)$, T, and M be given by Assumption 6.3 with $\alpha = 3$. Define the error E_n by* (4.4) *and* (7.5)*. We then have the estimates*

$$\|I_K(g(\pi_N u(t_n)) - g(\pi_N u_n))\|_{L^2} \lesssim_M \Big(1 + \|e_n\|^2_{\mathcal{H}^1}\Big)\|e_n\|_{\mathcal{H}^1},$$

$$\|I_K(g(\pi_N u(t_n)) - g(\pi_N u_n))\|_{\mathcal{H}^{-1}} \lesssim_M \Big(1 + \|e_n\|^2_{\mathcal{H}^1}\Big)\|e_n\|_{L^2}$$

for all $\tau \in (0,1]$, $K \in \mathbb{N} \cup \{\infty\}$, and $n \in \mathbb{N}$ with $n\tau \le T$, where we set $N = \min\{\tau^{-1}, K\}$.

Proof. From (6.10), we deduce that

$$\begin{aligned}
\|I_K(g(\pi_N u(t_n)) - g(\pi_N u_n))\|_{L^2} &\lesssim_M \Big(1 + \|u_n\|_{\mathcal{H}^1}^2\Big)\|e_n\|_{\mathcal{H}^1},\\
\|I_K(g(\pi_N u(t_n)) - g(\pi_N u_n))\|_{\mathcal{H}^{-1}} &\lesssim_M \Big(1 + \|u_n\|_{\mathcal{H}^1}^2\Big)\|e_n\|_{L^2}.
\end{aligned}$$

Here we use the Sobolev embeddings $\mathcal{H}^1 \hookrightarrow L^6$ and $L^{6/5} \hookrightarrow \mathcal{H}^{-1}$, Lemma 5.24 with $q \in \{6/5, 2\}$ and $\beta = 3$, as well as Hölder's inequality with $\frac{1}{2} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$ and $\frac{5}{6} = \frac{1}{6} + \frac{1}{6} + \frac{1}{2}$. The assertion follows from this by inserting $u_n = \pi_K u(t_n) - e_n$. □

We can now give the proof of the global error bound for $\alpha = 3$. We use a standard procedure based on the discrete Gronwall inequality. The error bound for e_n in the $\mathcal{H}^1$-norm is inductively exploited to get a uniform control on the numerical solution u_n in $\mathcal{H}^1$, which is also essential to obtain the error bound in the L^2 norm. This strategy goes back to [50].

Theorem 7.9. *Let U, T, and M be given by Assumption 6.3 with $\alpha = 3$. Define the iterates U_n by the Strang splitting (4.4) and the (projected) error $E_n = \Pi_K U(t_n) - U_n$ by Proposition 7.2. Then there are positive constants $\tau_0 \in (1, e^{-1}]$ and $K_0 \geq 3$ that only depend on M and T, such that we have the estimates*

$$\begin{aligned}
\|E_n\|_{\mathcal{H}^1 \times L^2} &\lesssim_{M,T} \tau|\log\tau| + K^{-1}\log K,\\
\|E_n\|_{L^2 \times \mathcal{H}^{-1}} &\lesssim_{M,T} \tau^2|\log\tau| + K^{-1},
\end{aligned}$$

for all $\tau \in (0, \tau_0]$, $K \in \mathbb{N} \cup \{\infty\}$ with $K \geq K_0$, and $n \in \mathbb{N}_0$ with $n\tau \leq T$.

Proof. We apply Lemmas 7.4, 7.6, 7.7, and 7.8 to the formula (7.7), which gives

$$\begin{aligned}
\|e_n\|_{\mathcal{H}^1} &\lesssim \|B(n\tau)\|_{\mathcal{H}^1 \times L^2} + \|D_n\|_{\mathcal{H}^1 \times L^2} + \|\tilde{H}_n\|_{\mathcal{H}^1 \times L^2} + \|[\tilde{Q}_n]_1\|_{\mathcal{H}^1}\\
&\lesssim_{M,T} \tau|\log\tau| + K^{-1}\log K + \tau\sum_{k=1}^{n-1}\Big(1 + \|e_k\|_{\mathcal{H}^1}^2\Big)\|e_k\|_{\mathcal{H}^1}, \qquad (7.18)\\
\|E_n\|_{\mathcal{H}^1 \times L^2} &\lesssim \|B(n\tau)\|_{\mathcal{H}^1 \times L^2} + \|D_n\|_{\mathcal{H}^1 \times L^2} + \|\tilde{H}_n\|_{\mathcal{H}^1 \times L^2} + \|\tilde{Q}_n\|_{\mathcal{H}^1 \times L^2}\\
&\lesssim_{M,T} \tau|\log\tau| + K^{-1}\log K + \tau\sum_{k=1}^{n}\Big(1 + \|e_k\|_{\mathcal{H}^1}^2\Big)\|e_k\|_{\mathcal{H}^1} \qquad (7.19)
\end{aligned}$$

and similarly

$$\begin{aligned}
\|E_n\|_{L^2 \times \mathcal{H}^{-1}} &\lesssim \|B(n\tau)\|_{L^2 \times \mathcal{H}^{-1}} + \|D_n\|_{L^2 \times \mathcal{H}^{-1}} + \|\tilde{H}_n\|_{L^2 \times \mathcal{H}^{-1}} + \|\tilde{Q}_n\|_{L^2 \times \mathcal{H}^{-1}}\\
&\lesssim_{M,T} \tau^2|\log\tau| + K^{-1} + \tau\sum_{k=1}^{n}\Big(1 + \|e_k\|_{\mathcal{H}^1}^2\Big)\|e_k\|_{L^2} \qquad (7.20)
\end{aligned}$$

for all $\tau \in (0, e^{-1}]$, $K \in \mathbb{N} \cup \{\infty\}$ with $K \geq 3$, and $n \in \mathbb{N}_0$ with $n\tau \leq T$. Here we use Remark 7.3 for the estimate on e_n, and we exploit that $E_0 = 0$. Let $c = c(M, T) > 0$ be maximum of the implicit constants in (7.18), (7.19), and (7.20). We then define the final error constant

$$C := 2ce^{4cT}$$

and choose the maximum step size $\tau_0 \in (0, e^{-1}]$ such that

$$4c\tau_0 \le 1, \qquad 2\tau_0 |\log \tau_0| C \le 1.$$

Moreover, the space discretization restriction parameter $K_0 \ge 3$ should satisfy the condition

$$2K_0^{-1} \log(K_0) C \le 1.$$

Let $\tau \in (0, \tau_0]$ and $K \in \mathbb{N} \cup \{\infty\}$ with $K \ge K_0$. We first show that the bound

$$\|e_n\|_{\mathcal{H}^1} \le 1 \tag{7.21}$$

holds for all $n \in \mathbb{N}_0$ with $n\tau \le T$. For $n = 0$ this is clear since $e_0 = 0$. Take now $n \in \mathbb{N}$ with $n\tau \le T$. We assume that we already have

$$\|e_k\|_{\mathcal{H}^1} \le 1$$

for all $k \in \{0, \dots, n-1\}$. Inequality (7.18) thus yields

$$\|e_n\|_{\mathcal{H}^1} \le c(\tau |\log \tau| + K^{-1} \log K) + 2c\tau \sum_{k=1}^{n-1} \|e_k\|_{\mathcal{H}^1}.$$

The discrete Gronwall lemma A.9 then implies that

$$\|e_n\|_{\mathcal{H}^1} \le ce^{2cn\tau}(\tau |\log \tau| + K^{-1} \log K) \le C(\tau |\log \tau| + K^{-1} \log K) \le 1,$$

using the restrictions $\tau \le \tau_0$ and $K \ge K_0$. Hence, (7.21) is true. From (7.19) we can now infer that

$$\|E_n\|_{\mathcal{H}^1 \times L^2} \le c(\tau |\log \tau| + K^{-1} \log K) + 2c\tau \sum_{k=1}^{n} \|e_k\|_{\mathcal{H}^1},$$
$$\|E_n\|_{\mathcal{H}^1 \times L^2} \le 2c(\tau |\log \tau| + K^{-1} \log K) + 4c\tau \sum_{k=1}^{n-1} \|E_k\|_{\mathcal{H}^1 \times L^2},$$

where we use the step size restriction $2c\tau \le \frac{1}{2}$ to absorb the n-th term in the above sum. The discrete Gronwall inequality then gives the desired bound

$$\|E_n\|_{\mathcal{H}^1 \times L^2} \le 2ce^{4cn\tau}(\tau |\log \tau| + K^{-1} \log K) \le C(\tau |\log \tau| + K^{-1} \log K).$$

Similarly, starting from (7.20) we establish the inequality

$$\|E_n\|_{L^2 \times \mathcal{H}^{-1}} \le C(\tau^2 |\log \tau| + K^{-1}),$$

again using (7.21) and the discrete Gronwall lemma. If we deal with homogeneous Sobolev norms on the full space $\Omega = \mathbb{R}^3$, we have to check that $\|e_n\|_{L^2}$ is finite, in order to absorb it above. This is done inductively starting from $E_0 = 0$, using the inequality

$$\|e_n\|_{L^2} \lesssim_{M,T} \tau^2|\log\tau| + K^{-1} + \tau\sum_{k=1}^{n-1}\Big(1 + \|e_k\|_{\mathcal{H}^1}^2\Big)\|e_k\|_{L^2}$$

that follows similar to (7.18) and (7.20) from Remark 7.3. □

Proof of Theorems 4.1 and 4.12 for $\alpha = 3$. The semi-discrete Theorem 4.1 directly follows from Theorem 7.9 with $K = \infty$. For the fully discrete Theorem 4.12, we take τ_0 and K_0 from Theorem 7.9 with $\Omega = \mathbb{T}^3$ and obtain

$$\|U(t_n) - U_n\|_{L^2\times H^{-1}} \le \|(I - \Pi_K)U(t_n)\|_{L^2\times H^{-1}} + \|E_n\|_{L^2\times H^{-1}} \lesssim_{M,T} \tau^2|\log\tau| + K^{-1},$$

for $\tau \in (0,\tau_0]$, $K \ge K_0$, and $n \in \mathbb{N}_0$ with $n\tau \le T$, where the projection error is bounded by Lemma 5.19. Similarly, for the energy norm we have

$$\|U(t_n) - U_n\|_{H^1\times L^2} \lesssim_{M,T,\alpha} \|(I - \Pi_K)U(t_n)\|_{H^1\times L^2} + \tau|\log\tau| + K^{-1}\log K \to 0$$

as $\tau \to 0$ and $K \to \infty$, uniformly in $n \in \mathbb{N}_0$ with $n\tau \le T$. Here we used dominated convergence for the projection error. □

7.5. Proof of the global error bounds for $\alpha \in (3,5)$

For $\alpha \in (3,5)$, we rely on the recursion formula (7.6). It turns out that estimates in discrete Strichartz norms are needed to control the terms Q_n and H_n in (7.6). The bound on the solution u is already contained in Proposition 6.9. However, we will also need a corresponding inequality for the approximation u_n which is not clear a priori. To this aim, we first show a "discrete local wellposedness result" for the scheme (7.1). It is the discrete counterpart to Theorem 6.1.

Lemma 7.10. *Let $R > 0$ and $\alpha \in [3,5)$. Then there is a time $b_0 = b_0(R,\alpha) \in (0,1]$ such that for all $U_0 \in \mathcal{H}^1 \times L^2$ with $\|U_0\|_{\mathcal{H}^1\times L^2} \le R$, and all $\mathcal{H}^1$-admissible pairs (p,q), the sequence (U_n) defined by (7.1) satisfies the estimate*

$$\|u_n\|_{\ell^\infty_\tau([0,b_0],\mathcal{H}^1)} + \|\pi_N u_n\|_{\ell^p_\tau([0,b_0],L^q)} \lesssim_{p,q,R} 1$$

for all $\tau \in (0,b_0]$ and $K \in \mathbb{N}\cup\{\infty\}$, where $N = \min\{\tau^{-1},K\}$.

Proof. Let $j \in \mathbb{N}_0$ with $t_{j+1} \le 1$. The discrete Duhamel formula (7.3) and the discrete Strichartz estimate from Corollary 5.17 imply

$$\max\{\|u_n\|_{\ell^\infty_\tau([0,t_{j+1}],\mathcal{H}^1)}, \|\pi_N u_n\|_{\ell^p_\tau([0,t_{j+1}],L^q)}\} \lesssim_{p,q} \|U_0\|_{\mathcal{H}^1\times L^2} + \|I_K g(\pi_N u_n)\|_{\ell^1_\tau([0,t_j],L^2)}. \tag{7.22}$$

By means of Hölder's inequality in space and time with $\frac{1}{2} = \frac{1}{3} + \frac{1}{6}$ and $\frac{1}{\alpha-1} = \frac{\alpha-3}{2(\alpha-1)} + \frac{5-\alpha}{2(\alpha-1)}$, respectively, and Sobolev's embedding $\mathcal{H}^1 \hookrightarrow L^6$, we estimate

$$\begin{aligned}\|g(\pi_N u_n)\|_{\ell^1_\tau([0,t_j],L^2)} &\leq \|\pi_N u_n\|^{\alpha-1}_{\ell^{\alpha-1}_\tau([0,t_j],L^{3(\alpha-1)})} \|\pi_N u_n\|_{\ell^\infty_\tau([0,t_j],L^6)} \\ &\lesssim t_{j+1}^{\frac{5-\alpha}{2}} \|\pi_N u_n\|^{\alpha-1}_{\ell^{p_\alpha}_\tau([0,t_j],L^{3(\alpha-1)})} \|u_n\|_{\ell^\infty_\tau([0,t_j],\mathcal{H}^1)}.\end{aligned}$$

The interpolation error satisfies

$$\begin{aligned}\|(I - I_K)g(\pi_N u_n)\|_{\ell^1_\tau([0,t_j],L^2)} &\lesssim K^{-2} \|\Delta[g(\pi_N u_n)]\|_{\ell^1_\tau([0,t_j],L^2)} \\ &\leq K^{-2} \|g'(\pi_N u_n)\pi_N \Delta u_n\|_{\ell^1_\tau([0,t_j],L^2)} \\ &\quad + K^{-2} \|g''(\pi_N u_n)|\nabla \pi_N u_n|^2\|_{\ell^1_\tau([0,t_j],L^2)}\end{aligned}$$

by Lemma 5.23. Similar as before, these terms are controlled by

$$\begin{aligned}\|g'(\pi_N u_n)\pi_N \Delta u_n\|_{\ell^1_\tau([0,t_j],L^2)} &\lesssim \|\pi_N u_n\|^{\alpha-1}_{\ell^{\alpha-1}_\tau([0,t_j],L^{3(\alpha-1)})} \|\pi_N \Delta u_n\|_{\ell^\infty_\tau([0,t_j],L^6)} \\ &\lesssim t_{j+1}^{\frac{5-\alpha}{2}} \|\pi_N u_n\|^{\alpha-1}_{\ell^{p_\alpha}_\tau([0,t_j],L^{3(\alpha-1)})} K^2 \|u_n\|_{\ell^\infty_\tau([0,t_j],\mathcal{H}^1)}\end{aligned}$$

and

$$\begin{aligned}&\|g''(\pi_N u_n)|\nabla \pi_N u_n|^2\|_{\ell^1_\tau([0,t_j],L^2)} \\ &\quad\lesssim \||\pi_N u_n|^{\alpha-2}\|_{\ell_\tau^{\frac{\alpha-1}{\alpha-2}}([0,t_j],L^{\frac{3(\alpha-1)}{\alpha-2}})} \||\nabla \pi_N u_n|^2\|_{\ell^{\alpha-1}_\tau([0,t_j],L^{\frac{6(\alpha-1)}{\alpha+1}})} \\ &\quad\leq \|\pi_N u_n\|^{\alpha-2}_{\ell^{\alpha-1}_\tau([0,t_j],L^{3(\alpha-1)})} \|\nabla \pi_N u_n\|_{\ell^\infty_\tau([0,t_j],L^6)} \|\nabla \pi_N u_n\|_{\ell^{\alpha-1}_\tau([0,t_j],L^{3(\alpha-1)})} \\ &\quad\lesssim t_{j+1}^{\frac{5-\alpha}{2}} K^2 \|\pi_N u_n\|^{\alpha-1}_{\ell^{p_\alpha}_\tau([0,t_j],L^{3(\alpha-1)})} \|u_n\|_{\ell^\infty_\tau([0,t_j],\mathcal{H}^1)},\end{aligned}$$

using the inverse inequality from Lemma 5.22 twice. Plugging all this into (7.22), we derive

$$\begin{aligned}&\max\{\|u_n\|_{\ell^\infty_\tau([0,t_{j+1}],\mathcal{H}^1)}, \|\pi_N u_n\|_{\ell^p_\tau([0,t_{j+1}],L^q)}\} \\ &\quad\lesssim_{p,q} \|U_0\|_{\mathcal{H}^1 \times L^2} + t_{j+1}^{\frac{5-\alpha}{2}} \|\pi_N u_n\|^{\alpha-1}_{\ell^{p_\alpha}_\tau([0,t_j],L^{3(\alpha-1)})} \|u_n\|_{\ell^\infty_\tau([0,t_j],\mathcal{H}^1)}.\end{aligned} \tag{7.23}$$

Also, the Bernstein inequality from Lemma A.3 and the admissibility (5.1) yield

$$\tau^{\frac{1}{p}} \|\pi_N u_0\|_{L^q} \lesssim \|u_0\|_{\mathcal{H}^1}. \tag{7.24}$$

Let C be the maximum of 1 and the implicit constants in (7.23) and (7.24) with $(p, q) = (p_\alpha, 3(\alpha - 1))$. Since $\alpha < 5$, we can choose the time $b_0 \in (0, 1]$ such that

$$b_0^{\frac{5-\alpha}{2}} (2C)^\alpha R^{\alpha-1} \leq 1. \tag{7.25}$$

We next show via induction that

$$\max\{\|u_n\|_{\ell^\infty_\tau([0,t_j],\mathcal{H}^1)}, \|\pi_N u_n\|_{\ell^{p_\alpha}_\tau([0,t_j],L^{3(\alpha-1)})}\} \leq 2CR \tag{7.26}$$

for all $j \in \mathbb{N}_0$ with $t_j \le b_0$. For $j = 0$, the claim follows from (7.24). Assume now that (7.26) holds for some $j \in \mathbb{N}_0$ with $t_{j+1} \le b_0$. Estimate (7.23) and (7.25) then imply

$$\max\{\|u_n\|_{\ell^\infty_\tau([0,t_{j+1}],\mathcal{H}^1)}, \|\pi_N u_n\|_{\ell^{p_\alpha}_\tau([0,t_{j+1}],L^{3(\alpha-1)})}\} \le C[R + b_0^{\frac{5-\alpha}{2}}(2CR)^\alpha] \le 2CR$$

for all $\tau \in (0, b_0]$, which ends the proof of (7.26). The assertion for general $\mathcal{H}^1$-admissible (p, q) now follows from (7.23). ☐

Using the previous lemma, we can now give an estimate for Q_n on a possibly small time interval of fixed size, depending on the $\mathcal{H}^1 \times L^2$ norm of the initial value U_0 of the numerical scheme.

Lemma 7.11. *Let U, T, and M be given by Assumption 6.3 with $\alpha \in [3, 5)$. Let moreover $R > 0$ and $U_0 \in \mathcal{H}^1 \times L^2$ with $\|U_0\|_{\mathcal{H}^1\times L^2} \le R$. Define U_n by (7.1), E_n and Q_n by Proposition 7.2, and $b_0 = b_0(R, \alpha) \in (0, 1]$ by Lemma 7.10. Then for any time $b > 0$ with $b \le \min\{b_0, T\}$, we obtain*

$$\|Q_n\|_{\ell^\infty_\tau([0,b],\mathcal{H}^1\times L^2)} \lesssim_{M,T,R} b^{\frac{5-\alpha}{2}} \|e_n\|_{\ell^\infty_\tau([0,b],\mathcal{H}^1)},$$
$$\|Q_n\|_{\ell^\infty_\tau([0,b],L^2\times\mathcal{H}^{-1})} \lesssim_{M,T,R} b^{\frac{5-\alpha}{2}} \|e_n\|_{\ell^\infty_\tau([0,b],L^2)},$$

for all $\tau \in (0, b]$ and $K \in \mathbb{N} \cup \{\infty\}$.

Proof. We estimate

$$\begin{aligned}
&\|Q_n\|_{\ell^\infty_\tau([0,b],\mathcal{H}^1\times L^2)} \\
&\quad\lesssim_T \|g(\pi_N u(t_n)) - g(\pi_N u_n)\|_{\ell^1_\tau([0,b],L^2)} \\
&\quad\lesssim \||g'(\pi_N u(t_n))| + |g'(\pi_N u_n)|\|_{\ell^1_\tau([0,b],L^3)} \|\pi_N u(t_n) - \pi_N u_n\|_{\ell^\infty_\tau([0,b],L^6)} \\
&\quad\lesssim b^{\frac{5-\alpha}{2}} \Big(\|\pi_N u(t_n)\|^{\alpha-1}_{\ell^{p_\alpha}_\tau([0,b],L^{3(\alpha-1)})} + \|\pi_N u_n\|^{\alpha-1}_{\ell^{p_\alpha}_\tau([0,b],L^{3(\alpha-1)})}\Big) \|\pi_N e_n\|_{\ell^\infty_\tau([0,b],\mathcal{H}^1)} \\
&\quad\lesssim_{M,T,R} b^{\frac{5-\alpha}{2}} \|e_n\|_{\ell^\infty_\tau([0,b],\mathcal{H}^1)},
\end{aligned}$$

using Hölder and Sobolev inequalities as in the proof of Lemma 7.10, as well as the estimates from Proposition 6.9 and Lemma 7.10. The other claim follows similarly via

$$\begin{aligned}
&\|Q_n\|_{\ell^\infty_\tau([0,b],L^2\times\mathcal{H}^{-1})} \\
&\quad\lesssim_T \|g(\pi_N u(t_n)) - g(\pi_N u_n)\|_{\ell^1_\tau([0,b],L^{\frac{6}{5}})} \\
&\quad\lesssim \||g'(\pi_N u(t_n))| + |g'(\pi_N u_n)|\|_{\ell^1_\tau([0,b],L^3)} \|\pi_N u(t_n) - \pi_N u_n\|_{\ell^\infty_\tau([0,b],L^2)} \\
&\quad\lesssim b^{\frac{5-\alpha}{2}} \Big(\|\pi_N u(t_n)\|^{\alpha-1}_{\ell^{p_\alpha}_\tau([0,b],L^{3(\alpha-1)})} + \|\pi_N u_n\|^{\alpha-1}_{\ell^{p_\alpha}_\tau([0,b],L^{3(\alpha-1)})}\Big) \|\pi_N e_n\|_{\ell^\infty_\tau([0,b],L^2)} \\
&\quad\lesssim_{M,T,R} b^{\frac{5-\alpha}{2}} \|e_n\|_{\ell^\infty_\tau([0,b],L^2)}.
\end{aligned}$$

☐

The next lemma deal with H_n from (7.8). Since g is not a polynomial for $\alpha \in (3, 5)$, we cannot use Lemma 5.24. Instead, we need to write out the $W^{k,q}$-norm appearing in Lemma 5.23 using product and chain rule, apply Hölder's inequality, and only afterwards exploit Lemma 5.22. Hence, the proof is more involved than Lemma 7.7 for $\alpha \in \{3, 5\}$, though the general procedure stays the same.

Lemma 7.12. *Let $\alpha \in (3,5)$, $R > 0$, and $U_0 \in \mathcal{H}^1 \times L^2$ with $\|U_0\|_{\mathcal{H}^1 \times L^2} \leq R$. Define U_n by (7.1), H_n by (7.8), and $b_0 = b_0(R,\alpha) \in (0,1]$ by Lemma 7.10. Then for any time $b \in (0, b_0]$, we obtain the estimates*

$$\|H_n\|_{H^1\times L^2} \lesssim_{R,\alpha} K^{-\frac{5-\alpha}{2}},$$
$$\|H_n\|_{L^2\times H^{-1}} \lesssim_{R} K^{-1},$$

uniformly in $\tau \in (0,1]$, $K \in \mathbb{N} \cup \{\infty\}$, and $n \in \mathbb{N}_0$ with $n\tau \in [0,b]$.

Proof. For the energy norm, the interpolation error estimate from Lemma 5.23 yields

$$\begin{aligned}\|H_n\|_{H^1\times L^2} &\lesssim \|(I-I_K)g(\pi_N u_n)\|_{\ell^1_\tau([0,b],L^2)} \lesssim K^{-2}\|\Delta[g(\pi_N u_n)]\|_{\ell^1_\tau([0,b],L^2)} \\ &\leq K^{-2}\|g'(\pi_N u_n)\pi_N \Delta u_n\|_{\ell^1_\tau([0,b],L^2)} + K^{-2}\|g''(\pi_N u_n)|\nabla \pi_N u_n|^2\|_{\ell^1_\tau([0,b],L^2)}.\end{aligned}$$

The appearing terms are estimated by

$$\begin{aligned}\|g'(\pi_N u_n)\pi_N\Delta u_n\|_{\ell^1_\tau([0,b],L^2)} &\lesssim \||\pi_N u_n|^{\alpha-1}\|_{\ell^1_\tau([0,b],L^{\frac{6}{\alpha-3}})}\|\pi_N\Delta u_n\|_{\ell^\infty_\tau([0,b],L^{\frac{6}{6-\alpha}})} \\ &\lesssim \|\pi_N u_n\|^{\alpha-1}_{\ell^{\alpha-1}_\tau([0,b],L^{q_\alpha})}\|\pi_N u_n\|_{\ell^\infty_\tau([0,b],H^{\frac{1+\alpha}{2}})} \\ &\lesssim K^{\frac{\alpha-1}{2}}\|\pi_N u_n\|^{\alpha-1}_{\ell^{\alpha-1}_\tau([0,b],L^{q_\alpha})}\|u_n\|_{\ell^\infty_\tau([0,b],H^1)}\end{aligned}$$

and

$$\begin{aligned}&\|g''(\pi_N u_n)|\nabla\pi_N u_n|^2\|_{\ell^1_\tau([0,b],L^2)} \\ &\quad\lesssim \||\pi_N u_n|^{\alpha-2}\|_{\ell_\tau^{\frac{\alpha-1}{\alpha-2}}([0,b],L^{\frac{6(\alpha-1)}{(\alpha-2)(\alpha-3)}})}\|\nabla\pi_N u_n\|_{\ell^{\alpha-1}_\tau([0,b],L^{q_\alpha})}\|\nabla\pi_N u_n\|_{\ell^\infty_\tau([0,b],L^{\frac{6}{6-\alpha}})} \\ &\quad\lesssim \|\pi_N u_n\|^{\alpha-2}_{\ell^{\alpha-1}_\tau([0,b],L^{q_\alpha})}K\|\pi_N u_n\|_{\ell^{\alpha-1}_\tau([0,b],L^{q_\alpha})}\|\nabla\pi_N u_n\|_{\ell^\infty_\tau([0,b],H^{\frac{\alpha-3}{2}})} \\ &\quad\lesssim K^{\frac{\alpha-1}{2}}\|\pi_N u_n\|^{\alpha-1}_{\ell^{\alpha-1}_\tau([0,b],L^{q_\alpha})}\|u_n\|_{\ell^\infty_\tau([0,b],H^1)},\end{aligned}$$

using Hölder and Sobolev inequalities in a canonical way, as well as the inverse estimate from Lemma 5.22. Altogether, Lemma 7.10 leads to

$$\|H_n\|_{H^1\times L^2} \lesssim K^{-\frac{5-\alpha}{2}}\|\pi_N u_n\|^{\alpha-1}_{\ell^{\alpha-1}_\tau([0,b],L^{q_\alpha})}\|u_n\|_{\ell^\infty_\tau([0,b],H^1)} \lesssim_{R,\alpha} K^{-\frac{5-\alpha}{2}},$$

which concludes the proof of the bound for the energy norm.

For the estimate in $L^2 \times H^{-1}$, we follow a similar strategy, starting with the Sobolev embedding $L^{\frac{6}{5}} \hookrightarrow H^{-1}$ and the interpolation error estimate from Lemma 5.23 with $q = 6/5$. This gives

$$\begin{aligned}\|H_n\|_{L^2\times H^{-1}} &\lesssim \|(I-I_K)g(\pi_N u_n)\|_{\ell^1_\tau([0,b],L^{\frac{6}{5}})} \lesssim \sum_{m=1}^{3} K^{-m}\|g(\pi_N u_n)\|_{\ell^1_\tau([0,b],W^{m,\frac{6}{5}})} \\ &\lesssim K^{-1}\|g'(\pi_N u_n)\nabla\pi_N u_n\|_{\ell^1_\tau([0,b],L^{\frac{6}{5}})} + K^{-2}\|g''(\pi_N u_n)|\nabla\pi_N u_n|^2\|_{\ell^1_\tau([0,b],L^{\frac{6}{5}})} \\ &\quad + K^{-2}\|g'(\pi_N u_n)\nabla^2\pi_N u_n\|_{\ell^1_\tau([0,b],L^{\frac{6}{5}})}\end{aligned}$$

$$
\begin{aligned}
&+ K^{-3}\|g'''(\pi_N u_n)|\nabla\pi_N u_n|^3\|_{\ell^1_\tau([0,b],L^{\frac{6}{5}})} \\
&+ K^{-3}\|g''(\pi_N u_n)\nabla^2\pi_N u_n|\nabla\pi_N u_n|\|_{\ell^1_\tau([0,b],L^{\frac{6}{5}})} \\
&+ K^{-3}\|g'(\pi_N u_n)\nabla^3\pi_N u_n\|_{\ell^1_\tau([0,b],L^{\frac{6}{5}})}.
\end{aligned}
$$

The appearing terms are again treated by Hölder's inequality and the inverse estimate from Lemma 5.22, which results in

$$
\begin{aligned}
\|g'(\pi_N u_n)\nabla\pi_N u_n\|_{\ell^1_\tau([0,b],L^{\frac{6}{5}})} &\lesssim \|g'(\pi_N u_n)\|_{\ell^1_\tau([0,b],L^3)}\|\nabla\pi_N u_n\|_{\ell^\infty_\tau([0,b],L^2)} \\
&\lesssim \|\pi_N u_n\|^{\alpha-1}_{\ell^{\alpha-1}_\tau([0,b],L^{3(\alpha-1)})}\|\nabla\pi_N u_n\|_{\ell^\infty_\tau([0,b],L^2)} \\
&\lesssim \|\pi_N u_n\|^{\alpha-1}_{\ell^{p_\alpha}_\tau([0,b],L^{3(\alpha-1)})}\|u_n\|_{\ell^\infty_\tau([0,b],H^1)},
\end{aligned}
$$

$$
\begin{aligned}
&\|g''(\pi_N u_n)|\nabla\pi_N u_n|^2\|_{\ell^1_\tau([0,b],L^{\frac{6}{5}})} \\
&\quad\lesssim \||\pi_N u_n|^{\alpha-2}\|_{\ell_\tau^{\frac{\alpha-1}{\alpha-2}}([0,b],L^{\frac{3(\alpha-1)}{\alpha-2}})}\|\nabla\pi_N u_n\|_{\ell^{\alpha-1}_\tau([0,b],L^{3(\alpha-1)})}\|\nabla\pi_N u_n\|_{\ell^\infty_\tau([0,b],L^2)} \\
&\quad\lesssim K\|\pi_N u_n\|^{\alpha-1}_{\ell^{p_\alpha}_\tau([0,b],L^{3(\alpha-1)})}\|u_n\|_{\ell^\infty_\tau([0,b],H^1)},
\end{aligned}
$$

$$
\begin{aligned}
\|g'(\pi_N u_n)\nabla^2\pi_N u_n\|_{\ell^1_\tau([0,b],L^{\frac{6}{5}})} &\lesssim \|g'(\pi_N u_n)\|_{\ell^1_\tau([0,b],L^3)}\|\nabla^2\pi_N u_n\|_{\ell^\infty_\tau([0,b],L^2)} \\
&\lesssim K\|\pi_N u_n\|^{\alpha-1}_{\ell^{\alpha-1}_\tau([0,b],L^{3(\alpha-1)})}\|\nabla\pi_N u_n\|_{\ell^\infty_\tau([0,b],L^2)} \\
&\lesssim K\|\pi_N u_n\|^{\alpha-1}_{\ell^{p_\alpha}_\tau([0,b],L^{3(\alpha-1)})}\|u_n\|_{\ell^\infty_\tau([0,b],H^1)},
\end{aligned}
$$

$$
\begin{aligned}
&\|g'''(\pi_N u_n)|\nabla\pi_N u_n|^3\|_{\ell^1_\tau([0,b],L^{\frac{6}{5}})} \\
&\quad\lesssim \||\pi_N u_n|^{\alpha-3}\|_{\ell_\tau^{\frac{\alpha-1}{\alpha-3}}([0,b],L^{\frac{3(\alpha-1)}{\alpha-3}})}\||\nabla\pi_N u_n|^2\|_{\ell_\tau^{\frac{\alpha-1}{2}}([0,b],L^{\frac{3(\alpha-1)}{2}})}\|\nabla\pi_N u_n\|_{\ell^\infty_\tau([0,b],L^2)} \\
&\quad\lesssim K^2\|\pi_N u_n\|^{\alpha-1}_{\ell^{p_\alpha}_\tau([0,b],L^{3(\alpha-1)})}\|u_n\|_{\ell^\infty_\tau([0,b],H^1)},
\end{aligned}
$$

$$
\begin{aligned}
&\|g''(\pi_N u_n)\nabla^2\pi_N u_n|\nabla\pi_N u_n|\|_{\ell^1_\tau([0,b],L^{\frac{6}{5}})} \\
&\quad\lesssim \||\pi_N u_n|^{\alpha-2}\|_{\ell_\tau^{\frac{\alpha-1}{\alpha-2}}([0,b],L^{\frac{3(\alpha-1)}{\alpha-2}})}\|\nabla^2\pi_N u_n\|_{\ell^{\alpha-1}_\tau([0,b],L^{3(\alpha-1)})}\|\nabla\pi_N u_n\|_{\ell^\infty_\tau([0,b],L^2)} \\
&\quad\lesssim K^2\|\pi_N u_n\|^{\alpha-1}_{\ell^{p_\alpha}_\tau([0,b],L^{3(\alpha-1)})}\|u_n\|_{\ell^\infty_\tau([0,b],H^1)},
\end{aligned}
$$

$$
\begin{aligned}
\|g'(\pi_N u_n)\nabla^3\pi_N u_n\|_{\ell^1_\tau([0,b],L^{\frac{6}{5}})} &\lesssim \|g'(\pi_N u_n)\|_{\ell^1_\tau([0,b],L^3)}\|\nabla^3\pi_N u_n\|_{\ell^\infty_\tau([0,b],L^2)} \\
&\lesssim K^2\|\pi_N u_n\|^{\alpha-1}_{\ell^{\alpha-1}_\tau([0,b],L^{3(\alpha-1)})}\|\nabla\pi_N u_n\|_{\ell^\infty_\tau([0,b],L^2)} \\
&\lesssim K^2\|\pi_N u_n\|^{\alpha-1}_{\ell^{p_\alpha}_\tau([0,b],L^{3(\alpha-1)})}\|u_n\|_{\ell^\infty_\tau([0,b],H^1)}.
\end{aligned}
$$

Combining these bounds with Lemma 7.10, we end up with

$$
\|H_n\|_{L^2\times H^{-1}} \lesssim K^{-1}\|\pi_N u_n\|^{\alpha-1}_{\ell^{p_\alpha}_\tau([0,b],L^{3(\alpha-1)})}\|u_n\|_{\ell^\infty_\tau([0,b],H^1)} \lesssim_R K^{-1}. \qquad \square
$$

We now show the global error bound for $\alpha \in (3,5)$. Unlike as for the case $\alpha = 3$, it is not enough to use the discrete Gronwall lemma. Instead, we need to apply Lemmas 7.11 and 7.12 iteratively on the possibly small intervals $[0,T_1]$, $[T_1, 2T_1]$ and so on, where we reach the final time T after finitely many iterations. To apply Lemmas 7.11 and 7.12, we need the uniform boundedness of the numerical approximations U_n in $\mathcal{H}^1 \times L^2$. Similar as for $\alpha = 3$, this boundedness follows from the error estimate after imposing a restriction on the discretization parameters τ and K. This method goes back to [17, 36, 53] in the context of Schrödinger equations.

Theorem 7.13. *Let U, T, and M be given by Assumption 6.3 with $\alpha \in (3,5)$. Define the iterates U_n by the Strang splitting (4.4) and the (projected) error $E_n = \Pi_K U(t_n) - U_n$ by Proposition 7.2. Then there are positive constants τ_0 and K_0 that depend on M, T, and α, such that we have the estimates*

$$\|E_n\|_{\mathcal{H}^1 \times L^2} \lesssim_{M,T,\alpha} \tau^{\frac{5-\alpha}{2}} + K^{-\frac{5-\alpha}{2}},$$
$$\|E_n\|_{L^2 \times \mathcal{H}^{-1}} \lesssim_{M,T,\alpha} \tau^{\frac{7-\alpha}{2}} + K^{-1},$$

for all $\tau \in (0,\tau_0]$, $K \in \mathbb{N} \cup \{\infty\}$ with $K \geq K_0$, and $n \in \mathbb{N}_0$ with $n\tau \leq T$.

Proof. We set $R := M+1$ and define $b_0 = b_0(R,\alpha) \in (0,1]$ from Lemma 7.10. Formula (7.6) and Lemmas 7.4, 7.6, 7.11, and 7.12 yield

$$\|E_n\|_{\ell^\infty_\tau([t_j,t_j+b],\mathcal{H}^1 \times L^2)} \lesssim_{M,\alpha} \|E_j\|_{\mathcal{H}^1 \times L^2} + N^{-\frac{5-\alpha}{2}} + b^{\frac{5-\alpha}{2}} \|e_n\|_{\ell^\infty_\tau([t_j,t_j+b],\mathcal{H}^1)}, \tag{7.27}$$
$$\|E_n\|_{\ell^\infty_\tau([t_j,t_j+b],L^2 \times \mathcal{H}^{-1})} \lesssim_{M,\alpha} \|E_j\|_{L^2 \times \mathcal{H}^{-1}} + \tau^{\frac{7-\alpha}{2}} + K^{-1} + b^{\frac{5-\alpha}{2}} \|e_n\|_{\ell^\infty_\tau([t_j,t_j+b],L^2)} \tag{7.28}$$

for all $j \in \mathbb{N}_0$ and $b \in (0,b_0]$ which satisfy $\|U_j\|_{\mathcal{H}^1 \times L^2} \leq R$ and $j\tau + b \leq T$. Recall that $N = \min\{\tau^{-1}, K\}$. Let $c = c(M,\alpha)$ be the maximum of 1 and the implicit constants from (7.27) and (7.28). Since $\alpha < 5$, we can define the time $T_1 \in (0,T]$ by

$$T_1 := \min\{T, b_0, (2c)^{-\frac{2}{5-\alpha}}\}. \tag{7.29}$$

Moreover, we set $L := \lceil \frac{2T}{T_1} \rceil \in \mathbb{N}$ and define the final error constant

$$C := 2(2c)^{L+1}. \tag{7.30}$$

We then choose parameters $\tau_0 \in (0,T_1]$ and $K_0 > 0$ such that

$$N^{\frac{5-\alpha}{2}} \geq C \tag{7.31}$$

holds for all $\tau \in (0,\tau_0]$ and $K \geq K_0$, again exploiting that $\alpha < 5$.

Take now $\tau \in (0,\tau_0]$ and $K \in \mathbb{N} \cup \{\infty\}$ with $K \geq K_0$. We define the indices $\nu := \lfloor T/\tau \rfloor \in \mathbb{N}$, $\nu_1 := \lfloor T_1/\tau \rfloor \in \{1,\dots,N\}$, and $\nu_m := m\nu_1$ for all $m \in \mathbb{N}_0$. These definitions imply that $\nu \leq \nu_L$. In addition, we set $\ell := \lfloor \nu/\nu_1 \rfloor \in \{1,\dots,L\}$. This gives the decomposition

$$[0,t_\nu] = \bigcup_{m=0}^{\ell-1} [t_{\nu_m}, t_{\nu_{m+1}}] \cup [t_{\nu_\ell}, t_\nu] =: \bigcup_{m=0}^{\ell} J_m,$$

where each interval J_m has length less or equal T_1. To measure the error in J_m, we set $\mathrm{Err}_m := \|E_n\|_{\ell^\infty_\tau(J_m,\mathcal{H}^1\times L^2)}$ for $m \in \{-1,\dots,\ell\}$, where $J_{-1} := \{0\}$. We aim to show the recursion formula

$$\mathrm{Err}_m \le 2c(\mathrm{Err}_{m-1} + N^{-\frac{5-\alpha}{2}}). \tag{7.32}$$

Note that once (7.32) is proved for all indices in $\{0,\dots,m\}$, one can derive the absolute bound

$$\mathrm{Err}_m \le 2cN^{-\frac{5-\alpha}{2}} \sum_{k=0}^{m} (2c)^k = 2cN^{-\frac{5-\alpha}{2}} \frac{(2c)^{m+1}-1}{2c-1} \le 2(2c)^{L+1} N^{-\frac{5-\alpha}{2}} = CN^{-\frac{5-\alpha}{2}} \le 1, \tag{7.33}$$

using $\mathrm{Err}_{-1} = 0$, the definition 7.30 of C, and the discretization parameter restriction from (7.31).

Let us now fix an index $m \in \{0,\dots,\ell\}$. If $m > 0$ we assume that the inequality (7.32) holds for all indices in $\{0,\dots,m-1\}$. From (7.33) we obtain $\mathrm{Err}_{m-1} \le 1$, and thus

$$\|U_{\nu_m}\|_{\mathcal{H}^1\times L^2} \le \|\Pi_K U(t_{\nu_m})\|_{\mathcal{H}^1\times L^2} + \|E_{\nu_m}\|_{\mathcal{H}^1\times L^2} \le M + \mathrm{Err}_{m-1} \le M + 1 = R.$$

Estimate (7.27) and the definition (7.29) of T_1 then imply

$$\mathrm{Err}_m \le c\|E_{\nu_m}\|_{\mathcal{H}^1\times L^2} + cN^{-\frac{5-\alpha}{2}} + cT_1^{\frac{5-\alpha}{2}} \mathrm{Err}_m \le c\mathrm{Err}_{m-1} + cN^{-\frac{5-\alpha}{2}} + \frac{1}{2}\mathrm{Err}_m.$$

Hence, the recursion (7.32) and the bound (7.33) are true for all $m \in \{0,\dots,\ell\}$. This finishes the proof of the bound in the energy norm.

Similarly, starting from (7.28) we obtain the recursion formula

$$\|E_n\|_{\ell^\infty_\tau(J_m,L^2\times\mathcal{H}^{-1})} \le 2c(\|E_n\|_{\ell^\infty_\tau(J_{m-1},L^2\times\mathcal{H}^{-1})} + \tau^{\frac{7-\alpha}{2}} + K^{-1})$$

for all $m \in \{0,\dots,\ell\}$, which yields the estimate

$$\|E_n\|_{\ell^\infty_\tau([0,T],L^2\times\mathcal{H}^{-1})} \le C(\tau^{\frac{7-\alpha}{2}} + K^{-1})$$

as in (7.33). If we deal with homogeneous Sobolev norms on the full space $\Omega = \mathbb{R}^3$, the finiteness of $\|e_n\|_{L^2}$ (in order to absorb it) is inductively checked using Remark 7.3, starting from $E_0 = 0$ (as in the proof of Theorems 7.9 and 4.4 for $\alpha \in \{3,5\}$). ☐

Proof of Theorems 4.1 and 4.12 for $\alpha \in (3,5)$. We first note that the $L^{p_\alpha}L^{3(\alpha-1)}$-norm in the definition (6.4) of M only depends on $\|U\|_{L^\infty_T(\mathcal{H}^1\times L^2)}$, as explained in Remark 6.4. The assertions now follow from Theorem 7.13 by the same arguments as in the case $\alpha = 3$ at the end of Section 7.4. ☐

7.6. The critical case $\alpha = 5$

In the case $\alpha = 5$, the approach from the previous Section 7.5 needs to be modified, since two new difficulties arise. First, it is no longer possible to obtain smallness on small intervals by an application of Hölder's inequality in time as in Lemma 7.11. Second, we can no longer show a convergence rate for the scheme in the energy norm (the rate $(5-\alpha)/2$ from Theorem 7.13 approaches zero as $\alpha \to 5$). On the other hand, for the defocusing equation on the full space (i.e., $\Omega = \mathbb{R}^3$ and $\mu = 1$) we can treat the global case $T = \infty$ in this section.

To deal with the smallness issue, as a first auxiliary result we establish that the discrete-time Strichartz norm converges to the continuous-time Strichartz norm. This will be done for the homogeneous part of the evolution, i.e.,

$$S(t)(f,v) = \cos(t|\nabla|)f + t\,\mathrm{sinc}(t|\nabla|)v,$$

on an interval J.

Lemma 7.14. *Let (p,q,γ) be admissible with $p<\infty$, $f \in \mathcal{H}^\gamma$, $v \in \mathcal{H}^{\gamma-1}$, and $J \subseteq \mathbb{R}$ be a bounded interval. Then we have the limit*

$$\|S(n\tau)\Pi_N(f,v)\|_{\ell^p_\tau(J,L^q)} \to \|S(t)(f,v)\|_{L^p_J L^q},$$

as $\tau \to 0$ and $K \to \infty$, where $N = \min\{\tau^{-1}, K\}$. If $\Omega = \mathbb{R}^3$, the assertion also holds for the unbounded interval $J = [0,\infty)$.

Proof. Let $\varepsilon > 0$. We choose Schwartz functions φ, $\psi \in \mathcal{S}$ such that $\operatorname{supp}\hat{\varphi}$, $\operatorname{supp}\hat{\psi}$ are compact and $\|f-\varphi\|_{\mathcal{H}^\gamma}$, $\|v-\psi\|_{\mathcal{H}^{\gamma-1}} \le \varepsilon$ (note that $\mathcal{S}(\mathbb{T}^3) = \mathcal{D}(\mathbb{T}^3) = C^\infty(\mathbb{T}^3)$).

Let first J be bounded. The function $t \mapsto S(t)(\varphi,\psi)$ is then continuous with values in L^q, since the Hausdorff–Young inequality and dominated convergence yield

$$\begin{aligned}
&\|S(s)(\varphi,\psi) - S(t)(\varphi,\psi)\|_{L^q} \\
&\quad \le \|(\cos(s|\nabla|) - \cos(t|\nabla|))\varphi\|_{L^q} + \|(s\,\mathrm{sinc}(s|\nabla|) - t\,\mathrm{sinc}(t|\nabla|))\psi\|_{L^q} \\
&\quad \le \|(\cos(s|\xi|) - \cos(t|\xi|))\hat{\varphi}\|_{L^{q'}} + \|(s\,\mathrm{sinc}(s|\xi|) - t\,\mathrm{sinc}(t|\xi|))\hat{\psi}\|_{L^{q'}} \to 0
\end{aligned}$$

as $s \to t$, where the Fourier variable is denoted by ξ regardless of $\Omega \in \{\mathbb{R}^3, \mathbb{T}^3\}$, and $L^{q'}$ actually is $\ell^{q'}$ if $\Omega = \mathbb{T}^3$. We also have $\Pi_N(\varphi,\psi) = (\varphi,\psi)$ for $(\tau, 1/K)$ small enough, because φ and ψ have compact Fourier support. It directly follows that

$$\|S(n\tau)\Pi_N(\varphi,\psi)\|_{\ell^p_\tau(J,L^q)} \to \|S(t)(\varphi,\psi)\|_{L^p_J L^q} \tag{7.34}$$

as $(\tau, 1/K) \to 0$, as there are essentially Riemann sums on the left-hand side.

In the case $\Omega = \mathbb{R}^3$ and $J = [0,\infty)$ we use a different argument, exploiting that in this case,

$$\|S(n\tau)(\varphi,\psi)\|_{\ell^p_\tau(J,L^q)} = \|S(\lfloor \tfrac{t}{\tau} \rfloor \tau)(\varphi,\psi)\|_{L^p_J L^q}.$$

Observe that

$$\begin{aligned}
&\Big| \|S(t)(\varphi,\psi)\|_{L^p_J L^q} - \|S(\lfloor \tfrac{t}{\tau} \rfloor \tau)(\varphi,\psi)\|_{L^p_J L^q} \Big| \\
&\quad \le \|(S(t) - S(\lfloor \tfrac{t}{\tau} \rfloor \tau))(\varphi,\psi)\|_{L^p_{\mathbb{R}} L^q} \\
&\quad \lesssim \|(e^{\pm \mathrm{i} t|\nabla|} - e^{\pm \mathrm{i}\lfloor t/\tau \rfloor \tau |\nabla|})\varphi\|_{L^p_{\mathbb{R}} L^q} + \|(e^{\pm \mathrm{i} t|\nabla|} - e^{\pm \mathrm{i}\lfloor t/\tau \rfloor \tau |\nabla|})|\nabla|^{-1}\psi\|_{L^p_{\mathbb{R}} L^q}.
\end{aligned}$$

With the help of Hölder's inequality, Fubini's theorem, and Theorem 5.1, we derive that

$$\begin{aligned}\|(e^{\pm \mathrm{i}t|\nabla|} - e^{\pm \mathrm{i}\lfloor t/\tau \rfloor \tau |\nabla|})\varphi\|_{L^p_{\mathbb{R}} L^q} &= \Big\| \int_{\lfloor t/\tau \rfloor \tau}^t \mathrm{i}|\nabla| e^{\pm \mathrm{i}s|\nabla|} \varphi \,\mathrm{d}s \Big\|_{L^p_{\mathbb{R}} L^q} \\ &\leq \Big(\tau^{p-1} \int_{\mathbb{R}} \int_s^{\lceil s/\tau \rceil \tau} \|e^{\pm \mathrm{i}s|\nabla|} |\nabla| \varphi\|_{L^q}^p \,\mathrm{d}t \,\mathrm{d}s \Big)^{\frac{1}{p}} \\ &\leq \tau \|e^{\pm \mathrm{i}t|\nabla|} |\nabla| \varphi\|_{L^p_{\mathbb{R}} L^q} \lesssim_{p,q} \tau \|\varphi\|_{\dot{H}^{\gamma+1}} \to 0\end{aligned}$$

as $\tau \to 0$, and similarly with φ replaced by $|\nabla|^{-1}\psi$. Thus, (7.34) is also true for $\Omega = \mathbb{R}^3$ and $J = [0, \infty)$.

Putting things together, we conclude that

$$\begin{aligned}&\Big| \|S(n\tau)\Pi_N(f,v)\|_{\ell^p_\tau(J,L^q)} - \|S(t)(f,v)\|_{L^p_J L^q} \Big| \\ &\quad \leq \|S(n\tau)\Pi_N(f-\varphi, v-\psi)\|_{\ell^p_\tau(J,L^q)} + \Big| \|S(n\tau)\Pi_N(\varphi,\psi)\|_{\ell^p_\tau(J,L^q)} - \|S(t)(\varphi,\psi)\|_{L^p_J L^q} \Big| \\ &\qquad + \|S(t)(\varphi - f, \psi - v)\|_{L^p_J L^q} \\ &\quad \lesssim_{p,q} \|f-\varphi\|_{\mathcal{H}^\gamma} + \|v-\psi\|_{\mathcal{H}^{\gamma-1}} + \varepsilon \lesssim \varepsilon\end{aligned}$$

for τ small enough, using the reverse triangle inequality and the Strichartz estimates from Corollaries 5.16 and 5.17, where the Strichartz constants additionally depend on the length of J in the case $\Omega = \mathbb{T}^3$. □

In the critical case $\alpha = 5$, we use a regularization argument to obtain convergence without rate of the error in the energy norm. This type of argument was already used in the proof of Theorem 1.6 in [17] in the context of nonlinear Schrödinger equations, but only in the easier energy-subcritical case. We first establish first-order convergence of the scheme (4.4) in the $\mathcal{H}^1 \times L^2$-norm under the assumption that the initial data comes from the smaller space $H^2 \times H^1$. To exploit this result in the general case of $\mathcal{H}^1 \times L^2$-data, we need the continuous dependence on the initial data, both for the equation (6.1) and the scheme (4.4). We show these auxiliary results under a smallness condition on a Strichartz norm of the orbit, which can always be fulfilled by choosing a small end time b, see Corollary 5.16. We note that the time b in Proposition 7.15 corresponds to a lower bound for the existence time b in Theorem 6.1, cf. [66, 71].

Proposition 7.15. *Let $\bar{b} \in (0,\infty)$ if $\Omega = \mathbb{T}^3$ and set $\bar{b} := \infty$ if $\Omega = \mathbb{R}^3$. Let $R > 0$. Then there is a radius $\delta_0 = \delta_0(R, \bar{b}) > 0$ such that for any $\delta \in (0, \delta_0]$ the following is true. For all $W_0 \in \mathcal{H}^1 \times L^2$ with $\|W_0\|_{\mathcal{H}^1 \times L^2} \leq R$ and every $b \in (0, \bar{b}]$ with $\|S(\cdot)W_0\|_{L^4_b L^{12}} \leq \delta$, there is a time step $\bar{\tau} = \bar{\tau}(\delta, W_0, b) > 0$ such that the next assertions hold.*

a) For every $Y_0, Z_0 \in \overline{B}_{\mathcal{H}^1 \times L^2}(W_0, \delta)$, the solutions Y and Z of (6.1) with $\alpha = 5$ and initial values Y_0 and Z_0, respectively, exist on $[0, b]$. Moreover, we then have the estimates

$$\|\pi_N y(t_n)\|_{\ell^4_\tau([0,b],L^{12})} \lesssim_{\bar{b}} \delta, \tag{7.35}$$

$$\|\pi_N y_n\|_{\ell^4_\tau([0,b],L^{12})} \lesssim_{\bar{b}} \delta, \tag{7.36}$$

$$\|Y - Z\|_{L^\infty([0,b],\mathcal{H}^1 \times L^2)} \lesssim_{\bar{b}} \|Y_0 - Z_0\|_{\mathcal{H}^1 \times L^2}, \tag{7.37}$$

$$\|Y_n - Z_n\|_{\ell^\infty_\tau([0,b],\mathcal{H}^1\times L^2)} \lesssim_{\bar{b}} \|Y_0 - Z_0\|_{\mathcal{H}^1\times L^2}, \tag{7.38}$$

for all $\tau \in (0,\bar{\tau}]$ *and* $K \in \mathbb{N}\cup\{\infty\}$ *with* $K \geq (\bar{\tau})^{-1}$, *where* $N = \min\{K, \tau^{-1}\}$, Y_n *and* Z_n *are the iterates of* (7.1) *for initial values* Y_0 *and* Z_0, *and* y_n *and* y *are the first components of* Y_n *and* Y, *respectively.*

b) If $Y_0 \in H^2 \times H^1$ *satisfies* $\|Y_0 - W_0\|_{\mathcal{H}^1\times L^2} \leq \delta/2$, *then there is a constant* $C = C(\|Y_0\|_{H^2\times H^1}, \bar{b}) > 0$ *such that the error bound*

$$\|Y(t_n) - Y_n\|_{\ell^\infty_\tau([0,b],\mathcal{H}^1\times L^2)} \leq C(\tau + K^{-1}) \tag{7.39}$$

holds for all $\tau \in (0,\bar{\tau}]$ *and* $K \in \mathbb{N}\cup\{\infty\}$ *with* $K \geq (\bar{\tau})^{-1}$.

Proof. Step 1. Let $c = c(\bar{b})$ be the maximum of 1, the constants from Corollaries 5.16 and 5.17 with exponents $(p,q) \in \{(\infty,6),(4,12)\}$ and end time $T = \bar{b}$, and the constant from Lemma 5.24 with $q = 2$ and $\beta = 5$. We define

$$\delta_0 \coloneqq \min\Big\{R, (3c^4(3+c)^4R)^{-\frac{1}{3}}, (10c^4(3+c)^4)^{-\frac{1}{4}}\Big\}. \tag{7.40}$$

Let $\delta \in (0,\delta_0]$. Since by assumption $\|S(\cdot)W_0\|_{L^4_{\bar{b}}L^{12}} \leq \delta$, Lemma 7.14 yields a step size $\bar{\tau} > 0$ such that

$$\|S(t_n)\Pi_N W_0\|_{\ell^4_\tau([0,b],L^{12})} \leq 2\delta \tag{7.41}$$

for all $\tau \in (0,\bar{\tau}]$ and $K \in \mathbb{N}\cup\{\infty\}$ with $K \geq (\bar{\tau})^{-1}$. We first show

$$\|y_n\|_{\ell^\infty_\tau([0,t_j],\mathcal{H}^1)} \leq 3cR, \qquad \|\pi_N y_n\|_{\ell^4_\tau([0,t_j],L^{12})} \leq (3+c)\delta \tag{7.42}$$

for all $\tau \in (0,\bar{\tau}]$, $K \in \mathbb{N}\cup\{\infty\}$ with $K \geq (\bar{\tau})^{-1}$, and $j \in \mathbb{N}_0$ with $j\tau \leq b$. In particular, this shows the inequality (7.36).

We proceed by induction on j. For $j = 0$, we clearly have

$$\|Y_0\|_{\mathcal{H}^1\times L^2} \leq \|Y_0 - W_0\|_{\mathcal{H}^1\times L^2} + \|W_0\|_{\mathcal{H}^1\times L^2} \leq \delta + R \leq 2R \tag{7.43}$$

since $\delta \leq R$. Corollary 5.17 and (7.41) further imply

$$\begin{aligned}
\|\pi_N y_n\|_{\ell^4_\tau(\{0\},L^{12})} &= \tau^{\frac{1}{4}}\|S(0)\Pi_N Y_0\|_{L^{12}} \\
&\leq \tau^{\frac{1}{4}}\|S(0)\Pi_N(Y_0 - W_0)\|_{L^{12}} + \tau^{\frac{1}{4}}\|S(0)\Pi_N W_0\|_{L^{12}} \\
&\leq c\|Y_0 - W_0\|_{\mathcal{H}^1\times L^2} + 2\delta \leq c\delta + 2\delta = (2+c)\delta.
\end{aligned}$$

For the induction step $j \rightsquigarrow j+1$, we assume that (7.42) holds for some $j \in \mathbb{N}_0$ with $(j+1)\tau \leq b$. We compute

$$\begin{aligned}
&\|y_n\|_{\ell^\infty_\tau([0,t_{j+1}],\mathcal{H}^1)} \\
&\leq \|S(n\tau)Y_0\|_{\ell^\infty_\tau([0,t_{j+1}],\mathcal{H}^1)} + \tau\Big\|\sum_{k=0}^{n-1} c_{k,n} t_{n-k}\,\mathrm{sinc}(t_{n-k}|\nabla|) I_K g(\pi_N y_k)\Big\|_{\ell^\infty_\tau([0,t_{j+1}],\mathcal{H}^1)} \\
&\leq c\|Y_0\|_{\mathcal{H}^1\times L^2} + c^2\|g(\pi_N y_n)\|_{\ell^1_\tau([0,t_j],L^2)}
\end{aligned}$$

$$\leq 2cR + c^2\|\pi_N y_n\|^4_{\ell^4_\tau([0,t_j],L^{12})}\|\pi_N y_n\|_{\ell^\infty_\tau([0,t_j],L^6)} \leq 2cR + 3c^4(3+c)^4\delta^4 R \leq 3cR,$$

by means of the discrete Duhamel formula (7.3), (7.43), Lemma 5.24, Hölder's inequality, the induction assumption (7.42), and the definition of δ from (7.40). Note that Lemma 5.24 is applicable since for $\alpha = 5$, g is a polynomial and therefore $g(\pi_N y_k) = \pi_{5N} g(\pi_N y_k)$. Similarly, using (7.3) and Corollary 5.17, we estimate

$$\begin{aligned}
\|\pi_N y_n\|_{\ell^4_\tau([0,t_{j+1}],L^{12})} &\leq \|S(t_n)\Pi_N Y_0\|_{\ell^4_\tau([0,t_{j+1}],L^{12})} \\
&\quad + \tau\Big\|\sum_{k=0}^{n-1} c_{k,n} t_{n-k}\operatorname{sinc}(t_{n-k}|\nabla|)\pi_N I_K g(\pi_N y_k)\Big\|_{\ell^4_\tau([0,t_{j+1}],L^{12})} \\
&\leq \|S(t_n)\Pi_N(Y_0 - W_0)\|_{\ell^4_\tau([0,t_{j+1}],L^{12})} \\
&\quad + \|S(t_n)\Pi_N W_0\|_{\ell^4_\tau([0,t_{j+1}],L^{12})} + c^2\|g(\pi_N y_n)\|_{\ell^1_\tau([0,t_j],L^2)} \\
&\leq c\delta + 2\delta + 3c^4(3+c)^4\delta^4 R \leq (3+c)\delta.
\end{aligned}$$

Hence, the claim (7.42) is true for all $j\tau \leq b$.

Step 2. Estimate (7.38) is shown by an analogous argument starting from (7.2). Using also (7.42) for z_n, we deduce the inequality

$$\begin{aligned}
&\|Y_n - Z_n\|_{\ell^\infty_\tau([0,b],\mathcal{H}^1\times L^2)} \\
&\quad\leq c\|Y_0 - Z_0\|_{\mathcal{H}^1\times L^2} + c\|I_K(g(\pi_N y_n) - g(\pi_N z_n))\|_{\ell^1_\tau([0,b],L^2)} \\
&\quad\leq c\|Y_0 - Z_0\|_{\mathcal{H}^1\times L^2} + \frac{5}{2}c^2\|(|\pi_N y_n|^4 + |\pi_N z_n|^4)|\pi_N y_n - \pi_N z_n|\|_{\ell^1_\tau([0,b],L^2)} \\
&\quad\leq c\|Y_0 - Z_0\|_{\mathcal{H}^1\times L^2} \\
&\qquad + \frac{5}{2}c^2\Big(\|\pi_N y_n\|^4_{\ell^4_\tau([0,b],L^{12})} + \|\pi_N z_n\|^4_{\ell^4_\tau([0,b],L^{12})}\Big)\|\pi_N y_n - \pi_N z_n\|_{\ell^\infty_\tau([0,b],L^6)} \\
&\quad\leq c\|Y_0 - Z_0\|_{\mathcal{H}^1\times L^2} + 5c^3(3+c)^4\delta^4\|Y_n - Z_n\|_{\ell^\infty_\tau([0,b],\mathcal{H}^1\times L^2)} \\
&\quad\leq c\|Y_0 - Z_0\|_{\mathcal{H}^1\times L^2} + \frac{1}{2}\|Y_n - Z_n\|_{\ell^\infty_\tau([0,b],\mathcal{H}^1\times L^2)},
\end{aligned}$$

which in turn implies

$$\|Y_n - Z_n\|_{\ell^\infty_\tau([0,b],\mathcal{H}^1\times L^2)} \leq 2c\|Y_0 - Z_0\|_{\mathcal{H}^1\times L^2},$$

as desired.

Step 3. The existence of the continuous solutions Y and Z until time b as well as the estimate (7.37) are part of the known local wellposedness theory of (6.1), cf. Theorem 6.1 and Chapter 5.1 of [71]. We therefore omit the proof. To carry it out, one can proceed analogously to Step 1 and 2, replacing the discrete norms by the continuous ones and the induction by a fixed point argument. The estimate (7.35) in discrete Strichartz norm can be shown in a similar way as (7.36). More precisely, one obtains the inequality

$$\max\{\|y\|_{L^4_b L^{12}}, \|\pi_N y\|_{L^4_b L^{12}}, \|\pi_N y(t_n)\|_{\ell^4_\tau([0,b],L^{12})}\} \leq (3+c)\delta \tag{7.44}$$

analogously to (7.42) (possibly modyfying δ_0). So part a) is shown.

Step 4. Now we want to show the error bound (7.39) for better data $Y_0 \in H^2 \times H^1$ with $\|Y_0 - W_0\|_{\mathcal{H}^1 \times L^2} \leq \delta/2$. Since the nonlinearity G leaves the space $H^2 \times H^1$ invariant and is Lipschitz continuous on bounded subsets, the standard Duhamel iteration yields a unique solution $\tilde{Y} \in C([0, T_{\max}), H^2 \times H^1)$ of (6.1) with initial data Y_0 on a maximal existence interval $[0, T_{\max})$. By Sobolev's embedding, the integrability condition $\tilde{y} \in L^4_{\mathrm{loc}}([0, T_{\max}), L^{12})$ is satisfied. Hence, $\tilde{Y}$ coincides with the $\mathcal{H}^1 \times L^2$-solution Y on $[0, b]$ by uniqueness, as long as they are both defined.

In the following, we show that $[0, T_{\max}) \supseteq \overline{[0, b)}$. By a standard blow-up criterion, it suffices to show that $\|Y(t)\|_{H^2 \times H^1}$ is finite for all $t \in \overline{[0, b)}$. First, note that $(y, \partial_t y)$ belongs to $C([0, b], \mathcal{H}^1 \times L^2)$ and that the L^2 norm of y stays bounded since

$$\|y(t)\|_{L^2} \leq \|y(0)\|_{L^2} + \int_0^t \|\partial_t y(s)\|_{L^2} \,\mathrm{d}s \leq \|Y_0\|_{H^2 \times H^1} + t\|\partial_t y\|_{L^\infty([0,t], L^2)} < \infty$$

for all $t \in \overline{[0, b)}$.

For the boundedness of $\nabla^2 y$ and $\nabla \partial_t y$ in L^2, we use that the Sobolev norm of a function can be expressed by bounds on the norms of difference quotients. For any $h \in \mathbb{R}^3$, we introduce the spatial translation operator $\mathcal{T}_h$ by $(\mathcal{T}_h(f, g))(x) := (f(x + h), g(x + h))$, where f and g are functions on Ω. By Proposition 9.3 of [8] (which is only stated for $\Omega = \mathbb{R}^3$, but the same proof works for $\Omega = \mathbb{T}^3$), we have

$$\|\mathcal{T}_h Y_0 - Y_0\|_{\mathcal{H}^1 \times L^2} \lesssim |h| \|Y_0\|_{\mathcal{H}^2 \times \mathcal{H}^1}.$$

Therefore, there is a number $h_0 > 0$ with $\|\mathcal{T}_h Y_0 - Y_0\|_{\mathcal{H}^1 \times L^2} \leq \delta/2$ for all $|h| \leq h_0$. From now on we assume that $|h| \leq h_0$. The triangle inequality yields $\|\mathcal{T}_h Y_0 - W_0\|_{\mathcal{H}^1 \times L^2} \leq \delta$. Since $\mathcal{T}_h Y$ solves (6.1) with initial value $\mathcal{T}_h Y_0$, from (7.37) we can deduce that

$$\|\mathcal{T}_h Y - Y\|_{L^\infty([0,b], \mathcal{H}^1 \times L^2)} \lesssim_{\bar{b}} \|\mathcal{T}_h Y_0 - Y_0\|_{\mathcal{H}^1 \times L^2} \lesssim |h| \|Y_0\|_{\mathcal{H}^2 \times \mathcal{H}^1}.$$

Proposition 9.3 of [8] now shows that $Y(t)$ belongs to $H^2 \times H^1$ for $t \in [0, b]$ and

$$\|Y\|_{L^\infty([0,b], \mathcal{H}^2 \times \mathcal{H}^1)} \lesssim \|Y_0\|_{\mathcal{H}^2 \times \mathcal{H}^1}. \tag{7.45}$$

Thus, $[0, T_{\max}) \supseteq \overline{[0, b)}$.

Step 5. We can now estimate the error $\|Y(t_n) - Y_n\|_{\ell^\infty_\tau([0,b], \mathcal{H}^1 \times L^2)}$. Let $\tau \in (0, \bar{\tau}]$. For the rest of this proof, we allow our implicit constants to depend on $\|Y_0\|_{H^2 \times H^1}$. We use the expressions from Proposition 7.2 to write

$$Y(t_n) - Y_n = B(t_n) + D_n + \tilde{H}_n + \tilde{Q}_n, \tag{7.46}$$

where the terms on the right hand side now include Y instead of U. The terms $B(t_n)$ and D_n can be bounded similarly as in the Lemmas 7.4 and 7.6, respectively. Using Lemma 5.19, we infer that

$$\begin{aligned}\|B\|_{L^\infty([0,b], \mathcal{H}^1 \times L^2)} &\lesssim_{\bar{b}} \||y|^4 y - |\pi_N y|^4 \pi_N y\|_{L^1_b L^2} \lesssim \|(|y|^4 + |\pi_N y|^4)|y - \pi_N y|\|_{L^1_b L^2} \\ &\leq \Big(\|y\|^4_{L^4_b L^{12}} + \|\pi_N y\|^4_{L^4_b L^{12}}\Big)\|(I - \pi_N) y\|_{L^\infty_b L^6}\end{aligned}$$

$$\lesssim_{\bar{b}} \|(I-\pi_N)y\|_{L^\infty_b \mathcal{H}^1} \lesssim N^{-1}\|y\|_{L^\infty_b \mathcal{H}^2} \lesssim \tau + K^{-1}.$$

Here, the bounds for the $L^4_b L^{12}$-norm follow from (7.44). By (7.12) and Proposition 6.9, the second summand in (7.46) is estimated by

$$\begin{aligned}\|D_n\|_{\ell^\infty_\tau([0,b],\mathcal{H}^1\times L^2)} &\lesssim_{\bar{b}} \tau \left\| \begin{pmatrix} -g(\pi_N y) \\ g'(\pi_N y)\pi_N \partial_t y \end{pmatrix} \right\|_{L^1([0,b],\mathcal{H}^1\times L^2)} \\ &\lesssim \tau \|\pi_N y\|^4_{L^4_b L^{12}} \Big(\|\nabla \pi_N y\|_{L^\infty_b L^6} + \|\pi_N \partial_t y\|_{L^\infty_b L^6} \Big) \lesssim \tau.\end{aligned}$$

The term $\tilde{H}_n$ in (7.46) is controlled similarly by

$$\begin{aligned}\|\tilde{H}_n\|_{\ell^\infty_\tau([0,b],H^1\times L^2)} &\lesssim_{\bar{b}} \|(I-I_K)g(\pi_N y(t_n))\|_{\ell^1_\tau([0,b],L^2)} \lesssim K^{-1}\|g(\pi_N y(t_n))\|_{\ell^1_\tau([0,b],H^1)} \\ &\lesssim K^{-1}\|\pi_N y(t_n)\|^4_{\ell^4_\tau([0,b],L^{12})}\|y(t_n)\|_{\ell^\infty_\tau([0,b],H^2)} \lesssim K^{-1},\end{aligned}$$

by means of Lemma 5.24 and (7.44). Finally, we estimate the last part of (7.46) by

$$\begin{aligned}&\|Q_n\|_{\ell^\infty_\tau([0,b],\mathcal{H}^1\times L^2)} \\ &\le c^2 \||\pi_N y(t_n)|^4 \pi_N y(t_n) - |\pi_N y_n|^4 \pi_N y_n\|_{\ell^1_\tau([0,b],L^2)} \\ &\le \frac{5}{2}c^2 \Big(\|\pi_N y(t_n)\|^4_{\ell^4_\tau([0,b],L^{12})} + \|\pi_N y_n\|^4_{\ell^4_\tau([0,b],L^{12})} \Big) \|\pi_N y(t_n) - \pi_N y_n\|_{\ell^\infty_\tau([0,b],L^6)} \\ &\le 5c^3(3+c)^4\delta^4 \|Y(t_n) - Y_n\|_{\ell^\infty_\tau([0,b],\mathcal{H}^1\times L^2)} \le \frac{1}{2}\|Y(t_n)-Y_n\|_{\ell^\infty_\tau([0,b],\mathcal{H}^1\times L^2)},\end{aligned}$$

using (7.44), (7.42), and the definition of δ_0 in (7.40). This term can be absorbed by the left-hand side of (7.46) and we obtain (7.39). □

Proposition 7.15 only gives a local statement on a possibly small time interval $[0, b]$. Since we want to show a global error bound on the potentially much larger interval $[0, T)$, we need to apply Proposition 7.15 recursively. To this aim, we first have to iterate the smallness condition in $L^4_b L^{12}$.

Lemma 7.16. *Let $U = (u, \partial_t u)$, T, and M be given by Assumption 6.3 with $\alpha = 5$ and let $\delta > 0$. Then there are a number $L \in \mathbb{N}$ and times $0 = T_0 < T_1 < \cdots < T_L = T$ such that the inequality*

$$\|S(\cdot)U(T_m)\|_{L^4_{b_m} L^{12}} \le \delta$$

holds for all $m \in \{0, \dots, L-1\}$, where we set $b_m \coloneqq T_{m+1} - T_m > 0$. The number $L \in \mathbb{N}$ only depends on δ and M, and additionally on T in the case $\Omega = \mathbb{T}^3$.

Proof. Let C be the constant of the Strichartz estimates from Corollary 5.16 with respect to the exponents $(p, q) \in \{(\infty, 6), (4, 12)\}$ (and end time T if $\Omega = \mathbb{T}^3$). We define

$$r \coloneqq \min\Big\{\frac{\delta}{2}, \Big(\frac{\delta}{2C^2 M}\Big)^{\frac{1}{4}}\Big\}. \tag{7.47}$$

Since $\|u\|_{L^4_T L^{12}} \le M$ is finite, we can find times $0 = T_0 < T_1 < \cdots < T_L = T$, such that the inequality

$$\|u\|_{L^4([T_m, T_{m+1}], L^{12})} \le r$$

holds for all $m \in \{0, \dots, L-1\}$. Here we can choose $L = \lceil \|u\|_{L_T^4 L^{12}}^4 / r^4 \rceil$. Let $m \in \{0, \dots, L-1\}$ and $b_m \coloneqq T_{m+1} - T_m > 0$. Starting from (6.2), Corollary 5.16 and (7.47) imply that

$$\begin{aligned}
&\|S(\cdot)U(T_m)\|_{L_{b_m}^4 L^{12}} \\
&\le \|u(T_m + \cdot)\|_{L_{b_m}^4 L^{12}} + \Big\| \int_0^t (t-s)\operatorname{sinc}((t-s)|\nabla|) g(u(T_m+s))\,\mathrm{d}s \Big\|_{L_{b_m}^4 L^{12}} \\
&\le r + C\||u|^4 u\|_{L^1([T_m,T_{m+1}],L^2)} \le r + C^2 M \|u\|_{L^4([T_m,T_{m+1}],L^{12})}^4 \le r + C^2 M r^4 \le \delta. \square
\end{aligned}$$

We now show the global error bound for the critical case. The proof will be divided in three steps. In the first step, we define the needed variables and divide the interval $[0,T]$ into a finite number of subintervals, which have the property that we can apply Proposition 7.15 on each of them. In the second step, we first prove the convergence of the scheme in the $\mathcal{H}^1 \times L^2$-norm without any convergence rate. This fact ensures that the discrete approximation stays close to the solution in the $\mathcal{H}^1 \times L^2$-norm if τ is small enough. We can then apply Proposition 7.15 iteratively. Finally, in the last step, we estimate the error in the $L^2 \times \mathcal{H}^{-1}$-norm to obtain the convergence of order one. In contrast to Theorems 4.1 and 4.12, the maximum step size τ_0 will now not only depend on the size M of the solution u and on the end time T, but also on further properties of the solution u to (4.1).

Proof of Theorems 4.4 and 4.13. Step 1. Let $R \coloneqq \|U\|_{L^\infty([0,T],\mathcal{H}^1 \times L^2)} \le M < \infty$. Set $\overline{b} \coloneqq T$ in the case $\Omega = \mathbb{T}^3$ and $\overline{b} \coloneqq \infty$ in the case $\Omega = \mathbb{R}^3$. We take $\delta_0 = \delta_0(R, \overline{b})$ given by Proposition 7.15. Let moreover c be the maximum of 2 and the constants from Corollary 5.16 (with $(p,q) = (\infty, 6)$), Proposition 7.15, Lemma 7.4, Lemma 7.6, Lemma 7.7, and Lemma 5.24 (with $q = \frac{6}{5}$ and $\beta = 5$). Note that δ_0 and c only depend on M in the case $\Omega = \mathbb{R}^3$ and $T = \infty$. We define

$$\delta \coloneqq \min\Big\{\delta_0, (10c^7)^{-\frac{1}{4}}\Big\}. \tag{7.48}$$

Lemma 7.16 provides a number $L \in \mathbb{N}$ and times $0 = T_0 < T_1 < \dots < T_L = T$ such that

$$\|S(\cdot)U(T_m)\|_{L_{b_m}^4 L^{12}} \le \delta \tag{7.49}$$

holds for all $m \in \{0, \dots, L-1\}$, where $b_m \coloneqq T_{m+1} - T_m > 0$. Here, the number $L \in \mathbb{N}$ only depends on M and T. We now choose a parameter $\varepsilon > 0$ such that

$$\varepsilon \le \frac{\delta}{5c^{L+1}}. \tag{7.50}$$

By continuity of U, there is a number $\rho > 0$ such that

$$\|U(T_m) - U(t)\|_{\mathcal{H}^1 \times L^2} \le \varepsilon \tag{7.51}$$

for all $m \in \{1, \dots, L\}$ and $t \in [0,T]$ with $|T_m - t| \le \rho$. We pick functions $Y^0, \dots, Y^L \in H^2 \times H^1$ with

$$\|Y^m - U(T_m)\|_{\mathcal{H}^1 \times L^2} \le \varepsilon \le \frac{\delta}{2} \tag{7.52}$$

for all $m \in \{0, \dots, L\}$. Due to Corollary 5.16, we find a time $b_L > 0$ such that

$$\|S(\cdot)U(T)\|_{L^4_{b_L} L^{12}} \le \delta. \tag{7.53}$$

We define the maximal step size $\tau_0 > 0$ by

$$\tau_0 \coloneqq \min\Big\{\frac{\rho}{L}, \frac{b_L}{L}, \min_{m=0,\dots,L} \frac{c\varepsilon}{2C(Y^m)}, \min_{m=0,\dots,L} \bar\tau(\delta, U(T_m), b_m)\Big\} \tag{7.54}$$

and set $K_0 \coloneqq 1/\tau_0$. Here, the numbers $C(Y^m) = C(\|Y^m\|_{H^2\times H^1})$ and $\bar\tau(\delta, U(T_m), b_m)$ are taken from Proposition 7.15. In the case $T = \infty$, condition (7.53) is not needed, thus we then take $b_L = 0$ and replace L by $L-1$ in (7.51), (7.52), (7.54), and in the following.

Let $\tau \in (0, \tau_0]$ and $K \in \mathbb{N} \cup \{\infty\}$ with $K \ge K_0$. To decompose the interval, for any $m \in \{0, \dots, L\}$ we set

$$\nu_m \coloneqq \sum_{j=0}^{m-1} \Big\lfloor \frac{b_j}{\tau} \Big\rfloor \in \mathbb{N}_0.$$

The intervals J_m are defined as $J_m \coloneqq [t_{\nu_m}, t_{\nu_{m+1}}]$ if $m \in \{0, \dots, L-1\}$ and $J_L \coloneqq [t_{\nu_L}, T]$. Hence, we have

$$[0, T] = \bigcup_{m=0}^{L} J_m.$$

By construction, each subinterval J_m is of length less or equal b_m. This also holds for the last interval J_L, because of

$$T - t_{\nu_L} = \sum_{m=0}^{L-1} \Big(b_m - \Big\lfloor \frac{b_m}{\tau} \Big\rfloor \tau\Big) \le \sum_{m=0}^{L-1} \Big(b_m - \Big(\frac{b_m}{\tau} - 1\Big)\tau\Big) = L\tau \le b_L, \tag{7.55}$$

where we use (7.54).

Step 2. We prove the second assertions of Theorems 4.4 and 4.13, namely the convergence in the $\mathcal{H}^1 \times L^2$-norm

$$\|U(t_n) - U_n\|_{\ell^\infty_\tau([0,T], \mathcal{H}^1 \times L^2)} \to 0 \tag{7.56}$$

as $\tau \to 0$ and $K \to \infty$ (without any rate). To measure the error in each subinterval J_m, we define the error norms Err_m by $\mathrm{Err}_{-1} \coloneqq 0$ and

$$\mathrm{Err}_m \coloneqq \|U(t_n) - U_n\|_{\ell^\infty_\tau(J_m, \mathcal{H}^1 \times L^2)}, \quad m \in \{0, \dots, L\}.$$

Next, we show the recursion formula

$$\mathrm{Err}_m \le c\mathrm{Err}_{m-1} + 5c\varepsilon, \quad m \in \{0, \dots, L\} \tag{7.57}$$

via induction on m. First, let $m = 0$. We introduce the notations $U(t, W^0) \coloneqq W(t)$, where W is the solution of (6.1) with initial value W^0, and $\Phi^n_\tau(W^0)$ for the n-th iterate of the Strang splitting scheme (4.4) with initial value W^0. We get

$$\mathrm{Err}_0 = \|U(t_n) - U_n\|_{\ell^\infty_\tau(J_0, \mathcal{H}^1 \times L^2)}$$

$$\begin{aligned}
&\leq \|U(t_n, U^0) - U(t_n, Y^0)\|_{\ell^\infty_\tau(J_0,\mathcal{H}^1\times L^2)} + \|U(t_n, Y^0) - \Phi^n_\tau(Y^0)\|_{\ell^\infty_\tau(J_0,\mathcal{H}^1\times L^2)} \\
&\quad + \|\Phi^n_\tau(Y^0) - \Phi^n_\tau(U^0)\|_{\ell^\infty_\tau(J_0,\mathcal{H}^1\times L^2)} \\
&\leq c\|U^0 - Y^0\|_{\mathcal{H}^1\times L^2} + C(Y^0)(\tau + K^{-1}) + c\|U^0 - Y^0\|_{\mathcal{H}^1\times L^2} \leq 3c\varepsilon,
\end{aligned}$$

using the estimates from Proposition 7.15 and the relations (7.49), (7.52), (7.54) as well as $K^{-1} \leq \tau_0$.

For the induction step $m-1 \rightsquigarrow m$, we first deduce from the induction assumption (7.57) the inequality

$$\|U(t_{\nu_m}) - U_{\nu_m}\|_{\mathcal{H}^1\times L^2} \leq \mathrm{Err}_{m-1} \leq 5c\varepsilon \sum_{k=0}^{m-1} c^k = 5c\varepsilon \frac{c^m - 1}{c-1} \leq 5c\varepsilon(c^L - 1).$$

As in (7.55), we obtain

$$|T_m - t_{\nu_m}| = \sum_{j=0}^{m-1} \Big(b_j - \Big\lfloor \frac{b_j}{\tau} \Big\rfloor \tau\Big) \leq m\tau \leq L\tau \leq \rho, \tag{7.58}$$

using also (7.54). Hence, (7.51) and (7.50) imply

$$\begin{aligned}
\|U(T_m) - U_{\nu_m}\|_{\mathcal{H}^1\times L^2} &\leq \|U(T_m) - U(t_{\nu_m})\|_{\mathcal{H}^1\times L^2} + \|U(t_{\nu_m}) - U_{\nu_m}\|_{\mathcal{H}^1\times L^2} \\
&\leq \varepsilon + 5c\varepsilon(c^L - 1) \leq 5\varepsilon c^{L+1} \leq \delta.
\end{aligned} \tag{7.59}$$

So we can apply Proposition 7.15 (with $W^0 = U(T_m)$). Furthermore, we write

$$\mathrm{Err_m} = \|U(t_n) - U_n\|_{\ell^\infty_\tau(J_m,\mathcal{H}^1\times L^2)} = \|U(t_n, U(t_{\nu_m})) - \Phi^n_\tau(U_{\nu_m})\|_{\ell^\infty_\tau([0,b_m],\mathcal{H}^1\times L^2)}.$$

In the case $m = L$, we would have to replace the interval $[0, b_m]$ with the interval $[0, T - t_{\nu_L}]$ (which is smaller by (7.55)), but for simplicity we keep this abuse of notation. Using also (7.51), (7.52), and (7.58), we can now proceed similar as for $m = 0$ and conclude

$$\begin{aligned}
\mathrm{Err_m} &= \|U(t_n, U(t_{\nu_m})) - \Phi^n_\tau(U_{\nu_m})\|_{\ell^\infty_\tau([0,b_m],\mathcal{H}^1\times L^2)} \\
&\leq \|U(t_n, U(t_{\nu_m})) - U(t_n, Y^m)\|_{\ell^\infty_\tau([0,b_m],\mathcal{H}^1\times L^2)} \\
&\quad + \|U(t_n, Y^m) - \Phi^n_\tau(Y^m)\|_{\ell^\infty_\tau([0,b_m],\mathcal{H}^1\times L^2)} \\
&\quad + \|\Phi^n_\tau(Y^m) - \Phi^n_\tau(U_{\nu_m})\|_{\ell^\infty_\tau([0,b_m],\mathcal{H}^1\times L^2)} \\
&\leq c\|U(t_{\nu_m}) - Y^m\|_{\mathcal{H}^1\times L^2} + C(Y^m)(\tau + K^{-1}) + c\|Y^m - U_{\nu_m}\|_{\mathcal{H}^1\times L^2} \\
&\leq c\|U(t_{\nu_m}) - Y^m\|_{\mathcal{H}^1\times L^2} + c\varepsilon + c\|Y^m - U(t_{\nu_m})\|_{\mathcal{H}^1\times L^2} + c\|U(t_{\nu_m}) - U_{\nu_m}\|_{\mathcal{H}^1\times L^2} \\
&\leq 2c\|U(t_{\nu_m}) - U(T_m)\|_{\mathcal{H}^1\times L^2} + 2c\|U(T_m) - Y^m\|_{\mathcal{H}^1\times L^2} + c\varepsilon + c\mathrm{Err}_{m-1} \\
&\leq 5c\varepsilon + c\mathrm{Err}_{m-1}.
\end{aligned}$$

Therefore, (7.57) is true. It follows that

$$\mathrm{Err}_m \leq 5c\varepsilon \sum_{k=0}^{m} c^k = 5c\varepsilon \frac{c^{m+1} - 1}{c-1} \leq 5\varepsilon c^{L+2}$$

for all $m \in \{0, \dots, L\}$. This shows the convergence in the $\mathcal{H}^1 \times L^2$-norm as stated in (7.56), since L is independent of ε, which could be chosen arbitrarily small in (7.50). To complete the proof of Theorem 4.4, we will actually not need the full statement of (7.56). It is enough to know that $\|U(T_m) - U_{\nu_m}\|_{\mathcal{H}^1 \times L^2} \le \delta$ for all $m \in \{0, \dots, L\}$, as noted in (7.59).

Step 3. We show the first assertions of Theorems 4.4 and 4.13, which are the first-order convergences of the scheme in the $L^2 \times \mathcal{H}^{-1}$-norm. By Lemma 5.19, it is enough to bound $E_n = \Pi_K U(t_n) - U_n$. Let $m \in \{0, \dots, L\}$. We use the recursion formula (7.7) and estimate similar as in the proof of Theorem 7.13 to obtain

$$\begin{aligned}
&\|E_n\|_{\ell^\infty_\tau(J_m, L^2 \times \mathcal{H}^{-1})} \\
&\quad \le c\|E_{\nu_m}\|_{L^2 \times \mathcal{H}^{-1}} + 3cN^{-1} + c^2 \|I_K \pi_{5N}(g(\pi_N u(t_n)) - g(\pi_N u_n))\|_{\ell^1_\tau(J_m, L^{\frac{6}{5}})} \\
&\quad \le c\|E_{\nu_m}\|_{L^2 \times \mathcal{H}^{-1}} + 3cN^{-1} + \frac{5}{2}c^3 \Big(\|\pi_N u(t_n)\|^4_{\ell^4_\tau(J_m, L^{12})} + \|\pi_N u_n\|^4_{\ell^4_\tau(J_m, L^{12})} \Big) \\
&\qquad\qquad \cdot \|u(t_n) - u_n\|_{\ell^\infty_\tau(J_m, L^2)} \\
&\quad \le c\|E_{\nu_m}\|_{L^2 \times \mathcal{H}^{-1}} + 3cN^{-1} + 5c^3 c^4 \delta^4 \|e_n\|_{\ell^\infty_\tau(J_m, L^2)} \\
&\quad \le c\|E_{\nu_m}\|_{L^2 \times \mathcal{H}^{-1}} + 3cN^{-1} + \frac{1}{2}\|e_n\|_{\ell^\infty_\tau(J_m, L^2)}.
\end{aligned} \tag{7.60}$$

Here we use Lemmas 7.4, 7.6, 7.7, and 5.24, the bounds from Proposition 7.15 and the definition of δ in (7.48). We can apply Proposition 7.15 thanks to the estimates on $U(T_m)$ in (7.49) and (7.53), on $U_{\nu_m} - U(T_m)$ in (7.59), and on τ and K in (7.54). If we deal with homogeneous Sobolev norms on the full space $\Omega = \mathbb{R}^3$, we have to check that $\|e_j\|_{L^2}$ is finite for all $j\tau \in \overline{[0, T)}$, in order to absorb it in (7.60). This can be verified via induction on j, based on the inequality

$$\|e_j\|_{L^2} \le 3cN^{-1} + \frac{5}{2}c\Big(\|\pi_N u(t_n)\|^4_{\ell^4_\tau([0, t_{j-1}], L^{12})} + \|\pi_N u_n\|^4_{\ell^4_\tau([0, t_{j-1}], L^{12})} \Big) \|e_n\|_{\ell^\infty_\tau([0, t_{j-1}], L^2)}$$

that follows similarly to (7.60) from Remark 7.3, where we exploit that $E_0 = 0$. Inequality (7.60) now leads to

$$\|E_n\|_{\ell^\infty_\tau(J_m, L^2 \times \mathcal{H}^{-1})} \le 2c\|E_{\nu_m}\|_{L^2 \times \mathcal{H}^{-1}} + 6cN^{-1} \le 2c\|E_n\|_{\ell^\infty_\tau(J_{m-1}, L^2 \times \mathcal{H}^{-1})} + 6cN^{-1}$$

if $m > 0$. Since $E_0 = 0$, this recursion formula yields the global bound

$$\|E_n\|_{\ell^\infty_\tau([0, T], L^2 \times \mathcal{H}^{-1})} \le 6cN^{-1} \sum_{k=0}^{L} (2c)^k \le 2(2c)^{L+1}(\tau + K^{-1}),$$

which is the asserted first-order convergence in $L^2 \times \mathcal{H}^{-1}$.

Step 4. It remains to show the numerical scattering result of Theorem 4.4. Let $\Omega = \mathbb{R}^3$, $\mu = 1$, and $T = K = \infty$. Take the asymptotic state $U_+ \in \dot{H}^1 \times L^2$ from Remark 6.6. Let $n \in \mathbb{N}$ and $\tau \in (0, \tau_0]$. Since

$$\|U_n - e^{n\tau A} U_+\|_{\dot{H}^1 \times L^2} \le \|U_n - U(n\tau)\|_{\dot{H}^1 \times L^2} + \|U(n\tau) - e^{n\tau A} U_+\|_{\dot{H}^1 \times L^2},$$

from (7.56) and Remark 6.6 we obtain the convergence

$$\lim_{\tau\to 0}\limsup_{n\to\infty} \|U_n - e^{n\tau A}U_+\|_{\dot{H}^1\times L^2} = 0.$$

To complete the proof of Theorem 4.4, we still need to show that the limit

$$\lim_{n\to\infty} \|U_n - e^{n\tau A}U_+\|_{\dot{H}^1\times L^2}$$

exists for fixed $\tau \in (0, \tau_0]$. This is equivalent to the existence of $\lim_{n\to\infty} e^{-n\tau A}U_n$ in $\dot{H}^1 \times L^2$. By means of the discrete Duhamel formula from Lemma 7.1, we can write

$$e^{-n\tau A}U_n = U_0 + \tau\sum_{k=0}^{n} e^{-k\tau A}G(\Pi_N U_k) - \frac{1}{2}G(\Pi_N U_0) - \frac{1}{2}e^{-n\tau A}G(\Pi_N U_n). \tag{7.61}$$

By (7.56), (7.59) and Proposition 7.15, we already know that

$$\begin{aligned}\|g(\pi_N u_n)\|_{\ell^1_\tau([0,\infty),L^2)} &\lesssim \|u_n\|_{\ell^\infty_\tau([0,\infty),\dot{H}^1)}\|\pi_N u_n\|^4_{\ell^4_\tau([0,\infty),L^{12})} \lesssim_M \sum_{m=0}^{L-1}\|\pi_N u_n\|^4_{\ell^4_\tau(J_m,L^{12})}\\ &\le Lc^4\delta^4\end{aligned}$$

is finite. Thus, the series

$$\sum_{k=0}^{\infty} e^{-k\tau A}G(\Pi_N U_k)$$

converges absolutely in $\dot{H}^1 \times L^2$. This in particular implies that $e^{-n\tau A}G(\Pi_N U_n) \to 0$ as $n \to \infty$. Hence, the limit of (7.61) exists in $\dot{H}^1 \times L^2$ as $n \to \infty$. □

8. Numerical experiment

Let $\Omega = \mathbb{T}^3$. For the numerical tests we need initial data (u^0, v^0) which lie in $H^1 \times L^2$ but do not have higher regularity. The standard approach to obtain initial in a Sobolev space $H^s(\mathbb{T}^3)$ is to take Fourier coefficients of the form

$$(1 + |k|^2)^{-\frac{1}{2}(\frac{3}{2}+s+\varepsilon)} r_k, \qquad k \in \mathbb{Z}^3, \tag{8.1}$$

for some numbers $r \in \ell^\infty(\mathbb{Z}^3)$ and small $\varepsilon > 0$. Therefore, one most commonly uses r_k uniformly distributed in $[-1, 1] + \mathrm{i}[-1, 1]$. This approach is well suited to precisely obtain the desired differentiability of order s. However, it is known that such random initial data does not only belong to H^s, but also to all L^q-based Sobolev spaces $H^{s,q}$ for $1 \le q < \infty$ with probability one. This can be exploited to obtain an improved local wellposedness theory for the nonlinear wave equation (4.1) with random initial data compared to the deterministic setting, cf. [12]. Since our error bounds are purely deterministic and heavily use L^q-based inequalities such as Sobolev and Strichartz estimates, it is crucial to numerically work with initial data which do not have higher integrability than predicted by Sobolev embedding. The following lemma shows that this can be achieved by simply taking $r_k = 1$ in (8.1).

Lemma 8.1. *Let $s \in \mathbb{R}$. We define a distribution $f \in \mathcal{D}'(\mathbb{T}^d)$ by its Fourier coefficients*

$$\hat{f}_k \coloneqq (1 + |k|^2)^{-\frac{1}{2}(\frac{d}{2}+s)}, \qquad k \in \mathbb{Z}^d. \tag{8.2}$$

Then, for all $\varepsilon > 0$, the following assertions hold.

a) $f \in H^{s-\varepsilon}(\mathbb{T}^d)$, but $f \notin H^s(\mathbb{T}^d)$.
b) If $-d/2 \le s < d/2$, then $f \notin L^{\frac{2d}{d-2s}+\varepsilon}(\mathbb{T}^d)$.
c) If $s > 0$, then $f \in L^2(\mathbb{T}^d)$ is real-valued.

Proof. a) We have

$$\|f\|^2_{H^{s-\varepsilon}} = \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^{s-\varepsilon} |\hat{f}_k|^2 = \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^{-\frac{d}{2}-\varepsilon} \lesssim \Big(1 + \int_{\mathbb{R}^d \setminus B(0,1)} |x|^{-d-2\varepsilon} \,\mathrm{d}x\Big) < \infty,$$

but $\|f\|^2_{H^s} \eqsim 1 + \int_{\mathbb{R}^d \setminus B(0,1)} |x|^{-d} \,\mathrm{d}x = \infty$.

b) For $N \in \mathbb{N}$, we consider the truncated Fourier series

$$\pi_N f(x) = (2\pi)^{-\frac{d}{2}} \sum_{|k|_\infty \le N} \hat{f}_k e^{\mathrm{i}k \cdot x}, \qquad x \in \mathbb{T}^d. \tag{8.3}$$

Let $a > 0$ such that $\cos(z) \geq 1/2$ for all $z \in [-a, a]$. For all $x \in \mathbb{T}^d$ with $|x|_1 = \sum_{j=1}^d |x_j| \leq a/N$, we then infer that

$$\begin{aligned} \pi_N f(x) &\eqsim \sum_{|k|_\infty \leq N} (1 + |k|^2)^{-\frac{1}{2}(\frac{d}{2}+s)} e^{\mathrm{i}k\cdot x} = \sum_{|k|_\infty \leq N} (1 + |k|^2)^{-\frac{1}{2}(\frac{d}{2}+s)} \cos(k \cdot x) \\ &\gtrsim \sum_{1\leq|k|_\infty \leq N} |k|^{-\frac{d}{2}-s}. \end{aligned}$$

Hence, for $q \in (1, \infty)$ and $s < d/2$,

$$\begin{aligned} \|\pi_N f\|_{L^q} &\gtrsim N^{-\frac{d}{q}} \sum_{1\leq|k|_\infty \leq N} |k|^{-\frac{d}{2}-s} \eqsim N^{-\frac{d}{q}} \int_{1\leq|x|\leq N} |x|^{-\frac{d}{2}-s} \,\mathrm{d}x \\ &\eqsim N^{-\frac{d}{q}} \int_1^N \rho^{d-1-\frac{d}{2}-s} \,\mathrm{d}\rho = \frac{2}{d-2s} N^{-\frac{d}{q}+\frac{d}{2}-s}, \end{aligned}$$

which is unbounded as $N \to \infty$ if $d/q < d/2 - s$. By Theorem 4.1.8 in [24], $f \in L^q(\mathbb{T}^d)$ would imply that $\pi_N f \to f$ in L^q. Thus, if $d/q < d/2 - s$, f cannot belong to $L^q(\mathbb{T}^d)$.

c) By symmetry, (8.3) is real-valued. This property is inherited by f since $f_N \to f$ in L^2 for $s > 0$. □

We illustrate our error bounds by a numerical experiment for the nonlinear wave equation (4.1) on $\mathbb{T}^3$ with $\mu = 1$ and powers $\alpha \in \{3, 4, 5\}$. We focus on the error of the time integration. The initial data (u^0, v^0) are defined using (8.2) with $d = 3$ and $s = 1+\varepsilon$ or $s = \varepsilon$, for a very small $\varepsilon > 0$. We use a scaling such that $\|u^0\|_{H^1} = \|v^0\|_{L^2} \approx 3$. We apply the scheme (4.4) with spatial discretization parameters $K \in \{2^4, 2^5, 2^6\}$. For the implementation we identify $\mathbb{T}^3 = [0, 1]^3$ such that the spatial resolution (distance of the collocation points) is $h = (2K+1)^{-1}$. We compare the errors in the $\ell^\infty_\tau([0, 1/4], L^2 \times H^{-1})$ and $\ell^\infty_\tau([0, 1/4], H^1 \times L^2)$ norms for various step sizes τ, where the reference solution is computed using (4.4) with the same K and $\tau_{\mathrm{ref}} = 2^{-12}$. In the plots only the temporal error is visible since the reference solution has the same spatial accuracy. Our Python code to reproduce the results is available at `https://doi.org/10.35097/2zvaw7qyvy6ymuu2`.

For the cubic equation with $\alpha = 3$, in Figure 8.1 we numerically observe temporal convergence rates of order 2 in the $L^2 \times H^{-1}$ norm and order 1 in the $H^1 \times L^2$ norm, uniformly in the spatial discretization parameter K. These observations are in accordance with Theorem 4.1. If τ is small compared to the spatial resolution, the error is of second order even in $H^1 \times L^2$, however with deteriorating error constant as $K \to \infty$. This behavior was already observed in the one-dimensional case in [22].

In the case $\alpha = 4$, we observe in Figure 8.2 that the convergence rates which are uniform in K have reduced to 3/2 for the $L^2 \times H^{-1}$ norm and order 1/2 for the energy norm, again in accordance with Theorem 4.1.

We also did the experiment for the critical power $\alpha = 5$, see Figure 8.3. Here it turned out that we get temporal convergence of order 1 in the $L^2 \times H^{-1}$ norm, uniformly in K. Moreover, we cannot observe a clear convergence order for the error in the $H^1 \times L^2$ norm if τ is not small compared to the spatial resolution. This behavior fits to Theorem 4.4.

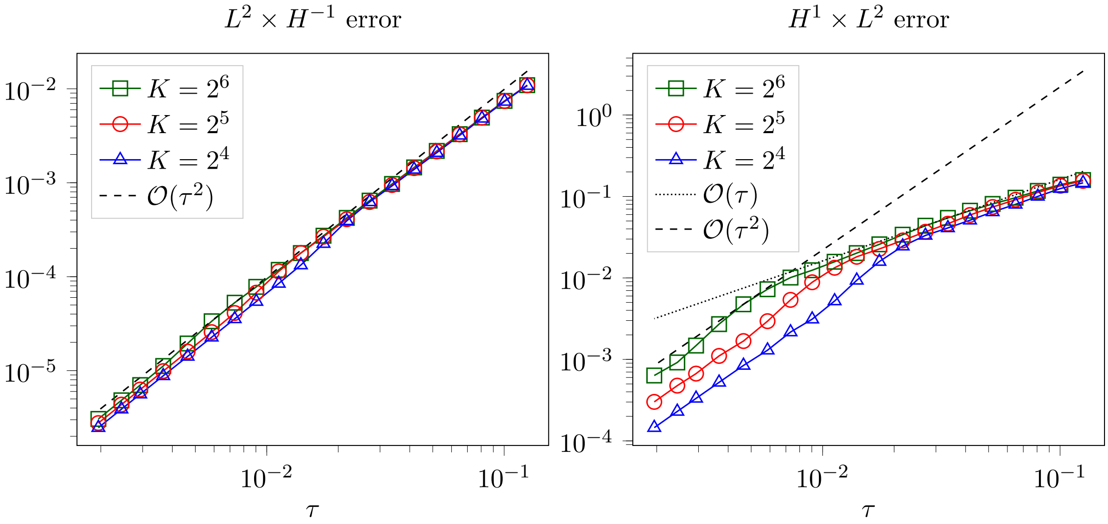

Figure 8.1.: Errors for $\alpha = 3$

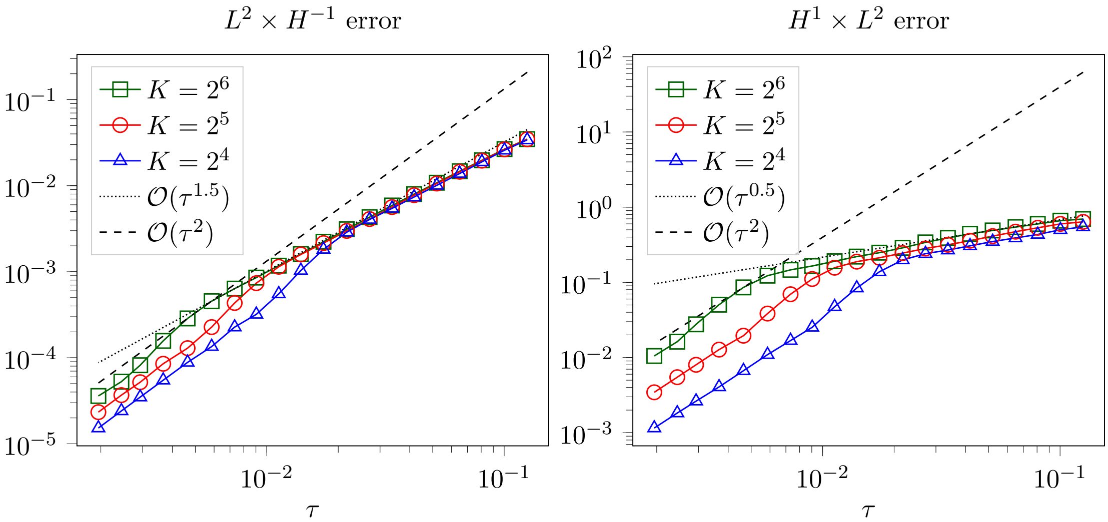

Figure 8.2.: Errors for $\alpha = 4$

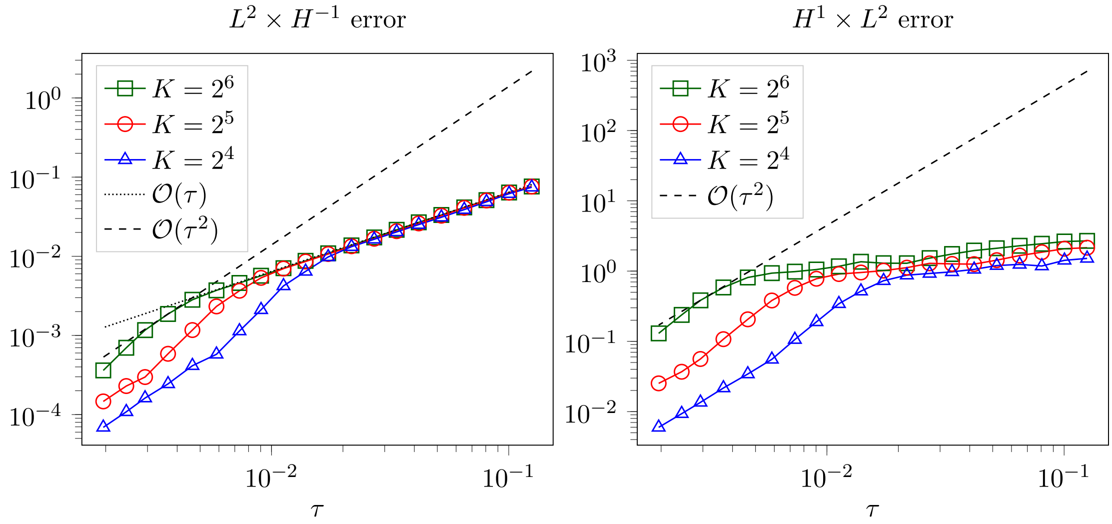

Figure 8.3.: Errors for $\alpha = 5$

A. Appendix

A.1. Function spaces and Fourier multipliers

The following *Sobolev embeddings* are used throughout this thesis. We recall that for $\Omega \in \{\mathbb{R}^d, \mathbb{T}^d\}$, we set $\mathcal{H}^s(\Omega) \coloneqq \dot{H}^s(\mathbb{R}^d)$ if $\Omega = \mathbb{R}^d$ and $\mathcal{H}^s(\Omega) \coloneqq H^s(\mathbb{T}^d)$ if $\Omega = \mathbb{T}^d$.

Theorem A.1. *Let $\Omega \in \{\mathbb{R}^d, \mathbb{T}^d\}$.*

a) Let $s > d/2$. Then $H^s(\Omega) \hookrightarrow L^\infty(\Omega)$.
b) Let $s \in [0, d/2)$. Then $\mathcal{H}^s(\Omega) \hookrightarrow L^{\frac{2d}{d-2s}}(\Omega)$.

Proof. Part a) is a direct consequence of the definition of the Sobolev norm via the Fourier transform since

$$\|f\|_{L^\infty} \lesssim \|\hat{f}\|_{L^1} \leq \|(1+|\xi|^2)^{-\frac{s}{2}}\|_{L^2} \|(1+|\xi|^2)^{\frac{s}{2}} \hat{f}\|_{L^2} \lesssim_{s,d} \|f\|_{H^s},$$

where the Fourier variable is denoted by ξ regardless of $\Omega \in \{\mathbb{R}^d, \mathbb{T}^d\}$, and L^1 and L^2 actually are ℓ^1 and ℓ^2 if $\Omega = \mathbb{T}^3$. For part b), we refer to Theorem 1.38 of [3] and Corollary 1.2 of [4] for the full space and torus cases, respectively. □

If $s > d/2$, the Sobolev space H^s even forms an algebra. Proofs can be found in Lemma A.8 of [71] and Proposition 1 of [5].

Lemma A.2. *Let $\Omega \in \{\mathbb{R}^d, \mathbb{T}^d\}$ and $s > d/2$. We then have the inequality*

$$\|fg\|_{H^s} \lesssim_{s,d} \|f\|_{H^s} \|g\|_{H^s}$$

for all f, $g \in H^s(\Omega)$.

The following *Bernstein inequalities* quantify the smoothness of functions whose Fourier transforms are supported in a ball.

Lemma A.3. *Let $\Omega \in \{\mathbb{R}^d, \mathbb{T}^d\}$, $r \in \mathbb{R}$, $s \geq 0$ and $q \in [2, \infty]$. We then have the estimates*

$$\|h\|_{\mathcal{H}^{s+r}} \lesssim K^s \|h\|_{\mathcal{H}^r},$$
$$\|f\|_{L^q} \lesssim K^{d(\frac{1}{2}-\frac{1}{q})} \|f\|_{L^2}$$

for all $f \in L^2(\Omega)$ and $h \in \mathcal{H}^r(\Omega)$ with $\operatorname{supp} \hat{f}$, $\operatorname{supp} \hat{h} \subseteq B(0, K)$ for some $K \geq 1$.

Proof. The first estimate follows directly from the representations of the Sobolev norm (4.5) and (4.6). The second one is a consequence of the Hausdorff–Young and Hölder inequalities, since

$$\|f\|_{L^q} \lesssim \|\hat{f}\|_{L^{q'}} \lesssim K^{d(\frac{1}{q'}-\frac{1}{2})}\|\hat{f}\|_{L^2} = K^{d(\frac{1}{2}-\frac{1}{q})}\|f\|_{L^2},$$

where the $L^{q'}$ and L^2 norms in Fourier space actually are $\ell^{q'}$ and ℓ^2 in the case $\Omega = \mathbb{T}^d$. □

We also make extensive use of Fourier multipliers in this thesis.

Definition A.4. *Let $m\colon \mathbb{R}^d \to \mathbb{C}$ be a locally integrable function that is polynomially bounded at infinity. We define the* Fourier multiplication operator $T_m\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ *by*

$$T_m f \coloneqq \mathcal{F}^{-1}(m\hat{f}). \tag{A.1}$$

On the torus, $T_m\colon \mathcal{D}(\mathbb{T}^d) \to \mathcal{D}'(\mathbb{T}^d)$ is also defined by (A.1)*, where one restricts m to $\mathbb{Z}^d$.*

It is clear from the definition that all Fourier multiplication operators formally commute, since $T_{m_1}T_{m_2} = T_{m_1m_2}$. For the complex conjugation, one has the following result that follows from Proposition 2.2.11 of [24].

Lemma A.5. *The operator T_m from Definition A.4 satisfies the property*

$$\overline{T_m f} = T_{\tilde{m}}\overline{f} \tag{A.2}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$ or $f \in \mathcal{D}(\mathbb{T}^d)$, where we set $\tilde{m}(x) \coloneqq \overline{m(-x)}$. In particular, T_m maps real-valued functions to real-valued functions in the case $m = \tilde{m}$.

One can often extend T_m and property (A.2) to larger subspaces of $\mathcal{S}'(\mathbb{R}^d)$ and $\mathcal{D}'(\mathbb{T}^d)$ by density.

The *convolution theorem* connects multiplication and convolution via the Fourier transform. The following assertions are a consequence of Propositions 2.2.11 and 2.3.22 of [24].

Proposition A.6. *We have*

$$\mathcal{F}(u * \varphi) = (2\pi)^{\frac{d}{2}}\hat{u}\hat{\varphi}, \quad \mathcal{F}(u\phi) = (2\pi)^{-\frac{d}{2}}\hat{u} * \hat{\varphi}$$

for all $u \in \mathcal{S}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. The first equality also holds if $u \in L^2(\mathbb{R}^d)$ and $\varphi \in L^1(\mathbb{R}^d)$.

The next lemma concerns basic properties of dilations that can be shown by means of elementary integral transformations and the duality between $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$.

Lemma A.7. *Let $a > 0$. The* dilation operator

$$D_a\colon \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d), \quad (D_a\varphi)(x) \coloneqq \varphi(ax),$$

extends to tempered distributions by

$$(D_a u)(\varphi) := a^{-d} u(D_{\frac{1}{a}}\varphi), \quad u \in \mathcal{S}'(\mathbb{R}^d),\ \varphi \in \mathcal{S}(\mathbb{R}^d).$$

It can be related to the Fourier transform via

$$\mathcal{F}(D_a u) = a^{-d} D_{\frac{1}{a}} \hat{u}, \quad \mathcal{F}^{-1}(D_a u) = a^{-d} D_{\frac{1}{a}} \mathcal{F}^{-1}(u), \quad u \in \mathcal{S}'(\mathbb{R}^d).$$

For the L^p-norms we further have

$$\|D_a f\|_{L^p} = a^{-\frac{d}{p}} \|f\|_{L^p}, \quad f \in L^p(\mathbb{R}^d),\ p \in [1,\infty].$$

A.2. Some discrete formulas

In this section, we collect elementary discrete formulas which are often used in the literature, however there does not seem to exist a standard reference for them. The following lemma is a discrete variant of Duhamel's formula.

Lemma A.8. *Let V be a vector space, $A\colon V \to V$ be a linear operator, and $(x_n)_{n=0}^N$, $(b_n)_{n=0}^N$ be finite sequences in V. Assume that*

$$x_{n+1} = Ax_n + b_n \tag{A.3}$$

holds for all $n \in \{0,\dots,N-1\}$. Then, the vectors satisfy

$$x_n = A^n x_0 + \sum_{k=0}^{n-1} A^{n-k-1} b_k \tag{A.4}$$

for all $n \in \{0,\dots,N\}$. If $V = \mathbb{R}$ and $A \geq 0$, we can replace "$=$" by "$\leq$" in (A.3) *and* (A.4).

Proof. The assertion is clear for $n = 0$. Assume now that (A.4) is true for some $n \in \{0,\dots,N-1\}$. Equation (A.3) then implies

$$x_{n+1} = Ax_n + b_n = A\Big(A^n x_0 + \sum_{k=0}^{n-1} A^{n-k-1} b_k\Big) + b_n = A^{n+1} x_0 + \sum_{k=0}^{n} A^{n-k} b_k,$$

and thus (A.4) also holds with n replaced by $n+1$. The addendum is shown in the same way. ☐

The next lemma is a discrete variant of Gronwall's inequality. It is often used to conclude the proof of a numerical convergence result.

Lemma A.9. *Let $b \geq 0$, $c > 0$, and let $(x_n)_{n=0}^N$ be a sequence of non-negative numbers such that the inequality*

$$x_n \leq b + c \sum_{k=0}^{n-1} x_k \tag{A.5}$$

holds for all $n \in \{0, \dots, N\}$*. We then obtain*

$$x_n \leq b(1+c)^n \leq be^{cn} \tag{A.6}$$

for all $n \in \{0, \dots, N\}$*.*

Proof. The second inequality in (A.6) is clear. If $n = 0$, the assertion follows directly from (A.5). Let now $n \in \{1, \dots, N\}$ and assume that

$$x_k \leq b(1+c)^k$$

holds for all $k \in \{0, \dots, n-1\}$. We then infer by (A.5) and the geometric sum formula that

$$x_n \leq b + c \sum_{k=0}^{n-1} x_k \leq b + cb \sum_{k=0}^{n-1} (1+c)^k = b + cb \frac{(1+c)^n - 1}{1+c-1} = b(1+c)^n. \qquad \square$$

Glossary

$[\cdot]_1$ first component of vector 82
$|\cdot|_p$ discrete ℓ^p norm on $\mathbb{R}^d$ 49
$|\cdot|$ euclidean norm on $\mathbb{R}^d$ 49
$|\nabla|$ homogeneous derivative $\mathcal{F}^{-1}|\xi|\mathcal{F}$ 50
$\mathbb{1}_B$ indicator function for set B 49
$\mathbb{1}$ the function being constantly one 49
A wave operator 35, 45
admissible, $\mathcal{H}^1$-admissible allowed parameters for Strichartz 51, 64
α power of nonlinearity 41
$B(x_0, r)$ ball with radius r centered at x_0 49
$\mathcal{D}(\mathbb{T}^d)$ space of smooth functions on the torus 49
$\mathcal{D}'(\mathbb{T}^d)$ space of distributions on the torus 49
E periodic extension operator 61
$\mathcal{F}$ Fourier transform operator 50
$\hat{f}$ Fourier transform of f 49, 50
g, G nonlinearity 9, 35, 45, 71
H^s L^2-based Sobolev space 6, 49, 50
$\dot{H}^s$ homogeneous Sobolev space 50
$\mathcal{H}^s$ Sobolev space $\dot{H}^s(\mathbb{R}^3)$ or $H^s(\mathbb{T}^3)$ 41
I identity operator 49
$I_K, \mathcal{I}_K$ trigonometric interpolation operator 69
L^p Lebesgue space 6
$L^p_T X$, $L^p X$ Bochner space $L^p([0,T], X)$ or $L^p(\mathbb{R}, X)$ 50
$\ell^p_\tau(J, X)$, $\ell^p_{\tau,T}X$, $\ell^p_\tau X$, $\ell^p X$ discrete-time Bochner space 50
μ focusing/defocusing parameter 9, 41
p' Hölder-conjucate index to p 49
p_α parameter such that $(p_\alpha, 3(\alpha-1))$ are $\mathcal{H}^1$-admissible 71
$\varphi_1(z) = (e^z - 1)/z$ 9
$\varphi_1'(z) = (ze^z - e^z + 1)/z^2$ 11
$\varphi_2(z) = (e^z - z - 1)/z^2$ 12
Φ_τ First-order low-regularity integrator 9
$\tilde{\Phi}_\tau$ Second-order low-regularity integrator 11
Ψ_τ Corrected Lie splitting 36
π_K, Π_K frequency cut-off operator 45, 54, 61
P_j Littlewood–Paley projection 52
$Q(\cdot,\cdot)$ null form 33
q_α parameter such that $(\alpha-1, q_\alpha)$ are $\mathcal{H}^1$-admissible 76
$\mathcal{S}(\mathbb{R}^d)$ Schwartz space 50
$\mathcal{S}'(\mathbb{R}^d)$ space of tempered distributions 50
$S(t)$ first line of wave group e^{tA} 79
$\mathbb{T}^d$ d-dimensional torus 6
t_n discrete times $n\tau$ 6
$W^{k,p}$ L^p-based Sobolev space 6
ξ Fourier variable 50

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