

On the uniqueness of the infinite cluster and the cluster density in the Poisson driven random connection model*

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Abstract

We consider a random connection model (RCM) on a general space driven by a Poisson process whose intensity measure is scaled by a parameter $t \geq 0$. We say that the infinite clusters are deletion stable if the removal of a Poisson point cannot split a cluster in two or more infinite clusters. We prove that this stability together with a natural irreducibility assumption implies uniqueness of the infinite cluster. Conversely, if the infinite cluster is unique then this stability property holds. Several criteria for irreducibility will be established. We also study the analytic properties of expectations of functions of clusters as a function of t . In particular we show that the position dependent cluster density is differentiable. A significant part of this paper is devoted to the important case of a stationary marked RCM (in Euclidean space), containing the Boolean model with general compact grains and the so-called weighted RCM as special cases. In this case we establish differentiability and a convexity property of the cluster density $\kappa(t)$. These properties are crucial for our proof of deletion stability of the infinite clusters but are also of interest in their own right. It then follows that an irreducible stationary marked RCM can have at most one infinite cluster. This extends and unifies several results in the literature.

Keywords: random connection model; Poisson process; percolation; Margulis–Russo formula; cluster density; uniqueness of infinite cluster.

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1 Introduction

Let \mathbb{X} be a complete separable metric space, denote its Borel- σ -field by \mathcal{X} , and let λ be a locally finite and diffuse measure on \mathbb{X} . Let $t \in \mathbb{R}_+ := [0, \infty)$ be an intensity parameter and let η be a Poisson process on \mathbb{X} with intensity measure $t\lambda$, defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We often write \mathbb{P}_t instead of \mathbb{P} and \mathbb{E}_t for the associated expectation operator.

Let $\varphi: \mathbb{X}^2 \rightarrow [0, 1]$ be a measurable and symmetric function satisfying

$$D_\varphi(x) := \int \varphi(x, y) \lambda(dy) < \infty, \quad \lambda\text{-a.e. } x \in \mathbb{X}. \quad (1.1)$$

We refer to φ as *connection function*. The *random connection model* (RCM) is the random graph ξ whose vertices are the points of η and where a pair of distinct points $x, y \in \eta$ forms an edge with probability $\varphi(x, y)$, independently for different pairs. In an Euclidean setting the RCM was introduced in [37]; see [33] for a textbook treatment. It can be defined on point processes other than Poisson. The general Poisson version was studied in [26]. The RCM is a fundamental and versatile example of a spatial random graph. Of particular interest is the *stationary marked* case. In this case we have $\mathbb{X} = \mathbb{R}^d \times \mathbb{M}$ for some mark space \mathbb{M} and λ is proportional to the product of Lebesgue measure and a given mark distribution. Then the RCM becomes stationary and ergodic under shifts in the spatial coordinate. This model contains the Boolean model (see [27, 41]) with general compact grains and the so-called weighted RCM as special cases and keeps attracting a lot of attention; see e.g. [6, 11, 13, 19, 21, 29, 38].

Following common terminology of percolation theory we refer to a component of ξ as *cluster*. The RCM ξ *percolates*, if it has an infinite cluster, that is a component with infinitely many vertices. We say that the infinite clusters of ξ are *deletion stable* if the removal of a point cannot split a cluster in two or more infinite clusters. If the infinite cluster is unique, then it is easy to prove that ξ is deletion stable. In fact, ξ is then almost surely even *2-indivisible* in the sense of [35]; see Corollary 6.4. Our first main result (Theorem 6.1) says that deletion stability together with *irreducibility* implies (almost sure) uniqueness of the infinite cluster. We prove this by a peculiar addition and removal procedure, which seems to be new. Our method crucially relies on the properties of the underlying Poisson process. Irreducibility is a very natural assumption for uniqueness (see Remark 5.9) and will be discussed in Section 5. Theorem 11.1 shows that the infinite clusters of the stationary marked RCM are deletion stable. This is the second main result of this paper. Our proof transfers some of the beautiful ideas from the seminal paper [1] by Aizenman, Kesten and Newman to the continuum. To this end we significantly extend and complement the arguments in [25], where the methods from [1] were used to treat the Gilbert graph with deterministic balls. Taken together, Theorems 6.1 and 11.1 yield uniqueness of the infinite cluster of an irreducible stationary marked RCM; see Theorem 12.1. This extends and unifies several results in the literature. The stationary (unmarked) RCM was treated in [33] for an isotropic and norm-decreasing connection function; see also [4]. A special case of the marked RCM was treated in [23]. The uniqueness of the infinite cluster of the spherical Boolean model was proved in [32, 33]. As a consequence of we also obtain that an irreducible stationary marked RCM is 2-indivisible.

We also establish several analytic properties of cluster expectations, first in the general and then in the stationary marked case. Since clusters are not locally determined, the proof of these results requires some efforts. In particular we show that the *position dependent cluster density* (given by (8.6)) is, as a function of t , continuously differentiable; see Theorem 8.8. In the stationary marked case this is true for the *cluster density* $\kappa(t)$, defined by (4.6); see Theorem 10.1. Our proofs partially follow [7], where the

Boolean model with deterministic balls was considered. We also prove that $t\kappa(t) + d_\varphi t^2/2$ is a convex function of t , where d_φ is the expected degree of a typical vertex, given by (4.2). This remarkable property was established in [1, 14] for discrete percolation models and in [25] for the Boolean model with deterministic balls. This convexity is crucial for proving deletion stability of the infinite clusters in the stationary marked case and its proof heavily depends on the (amenability) properties of Euclidean space; see Remark 11.9.

With the exception of [25], all previous uniqueness proofs in continuum percolation seem to use the approach in [10]; see also [16]. It is often argued that this approach is more elegant than the one in [1]. However, our paper shows that the methods from [1] can be conveniently extended to the continuum, at least in the case of a Poisson driven RCM. Moreover, this approach provides a lot of additional information on the clusters, which are valid for all values of the intensity parameter t . And last but not least our general uniqueness theorem applies to a general state space \mathbb{X} , without any structural assumptions.

The paper is organized as follows. In Section 2 we give the formal definition of the RCM ξ , while Section 3 presents the RCM version of the multivariate Mecke equation and the Margulis–Russo formula. In Section 4 we discuss the stationary marked RCM, an important special case of the general RCM. In Section 5 we define a RCM to be irreducible if, roughly speaking, every pair of Poisson points has a positive probability of being in the same cluster. Without such a property one cannot expect the infinite cluster (if it exists) to be unique. For a stationary marked RCM Theorem 5.7 characterises irreducibility in terms of the symmetric function $\int \varphi((0, p), (x, q)) dx$, which is (up to the factor t) the density of the expected number of neighbours of $(0, p)$ with respect to the mark distribution. Theorems 5.11 and 5.12 provide sufficient conditions for irreducibility under more specific assumptions. In Section 6 we prove that deletion stability of infinite clusters and irreducibility together imply uniqueness of the infinite cluster; see Theorem 6.1. Section 7 presents a spatial Markov property. In Section 8 we establish differentiability of certain cluster expectations, while Section 9 rewrites the derivatives as Margulis–Russo type formulas. In Section 10 we show that the position dependent cluster density is continuously differentiable. In Section 11 we prove that the infinite clusters of the stationary marked RCM are deletion stable; see Theorem 11.1. The final Section 12 provides several examples of irreducible stationary marked RCMs.

For the reader’s convenience, we list below our main results separately for the general and the stationary marked cases.

Main results for the general RCM:

- Theorem 6.1 shows that an irreducible RCM with deletion stable infinite clusters can have at most one infinite cluster, while showing that deletion stability is necessary for uniqueness.
- Theorem 8.8 shows continuous differentiability of certain cluster expectations, while Theorem 9.4 and Remark 9.5 rewrite the derivative as a Margulis–Russo type formula.
- Theorem 10.1 shows continuous differentiability of the position dependent cluster density, while Theorem 10.7 shows that this remains true after some additional integration.

Main results for the stationary marked RCM:

- Theorem 5.7 characterises irreducibility while Theorems 5.11 and 5.12 provide sufficient conditions under more specific assumptions.

- Theorem 11.1 shows that the infinite clusters of a stationary marked RCM are deletion stable.
- Theorem 11.6 shows that the cluster density $\kappa(t)$ is continuously differentiable and that $t\kappa(t) + d_\varphi t^2/2$ is convex.

2 Formal definition of the RCM

It is convenient to model a RCM as a suitable point process. Let \mathbf{N} denote the space of all simple locally finite counting measures on \mathbb{X} , equipped with the standard σ -field, see e.g. [27]. A measure $\nu \in \mathbf{N}$ is identified with its support $\{x \in \mathbb{X} : \nu(\{x\}) = 1\}$ and describes the set of vertices of a (deterministic) graph. If $\nu(\{x\}) = 1$ we write $x \in \nu$. Using the Dirac measure δ_x at point $x \in \mathbb{X}$, any $\nu \in \mathbf{N}$ can be written as a finite or infinite sum $\nu = \delta_{x_1} + \delta_{x_2} + \dots$, where the x_i are pairwise distinct and do not accumulate in bounded sets. The space of (undirected) graphs with vertices from \mathbb{X} (and no loops) is described by the set \mathbf{G} of all counting measures μ on $\mathbb{X} \times \mathbf{N}$ with the following properties. First we assume that the measure $V(\mu) := \mu(\cdot \times \mathbf{N})$ is locally finite and simple, that is, an element of \mathbf{N} . Hence, if $x \in V(\mu)$ (that is $\mu(\{x\} \times \mathbf{N}) = 1$), then there is a unique $\psi_x \in \mathbf{N}$ such that $(x, \psi_x) \in \mu$. We assume that $x \notin \psi_x$. Finally, if $x \in V(\mu)$ and $y \in \psi_x$ then we assume that $(y, \psi_y) \in \mu$ and $x \in \psi_y$. Also \mathbf{G} is equipped with the standard σ -field. There is an edge between $x, y \in V(\mu)$ if $y \in \psi_x$ (and hence $x \in \psi_y$). If $\psi_x = 0$, then x is isolated.

We write $|\mu| := \mu(\mathbb{X} \times \mathbf{N})$ for the cardinality of $\mu \in \mathbf{G}$ and similarly for $\nu \in \mathbf{N}$. Hence $|\mu| = |V(\mu)|$. For $x, y \in V(\mu)$ we write $x \sim y$ (in μ) if there is an edge between x and y and $x \leftrightarrow y$ (in μ) if there is a path in μ leading from x to y . For $A \subset \mathbb{X}$ we write $x \sim A$ (in μ) if there exists $y \in A \cap V(\mu)$ such that $x \sim y$.

Let $\mu, \mu' \in \mathbf{G}$. Then μ is a *subgraph* of μ' if $V(\mu) \leq V(\mu')$ (as measures) and if $(x, \psi) \in \mu$ and $(x, \psi') \in \mu'$ together imply that $\psi \leq \psi'$. Note that this is not the same as $\mu \leq \mu'$.

Let χ be a simple point process on \mathbb{X} , that is a random element of \mathbf{N} . The reader should think of a Poisson process possibly augmented by additional (deterministic) points. By [27, Proposition 6.2] there exist random elements X_1, X_2, \dots of \mathbb{X} such that

$$\chi = \sum_{n=1}^{|\chi|} \delta_{X_n}, \quad (2.1)$$

where $X_m \neq X_n$ whenever $m \neq n$ and $m, n \leq |\chi|$. Let $(Z_{m,n})_{m,n \in \mathbb{N}}$ be a double sequence of random elements uniformly distributed on $[0, 1]$ such that $Z_{m,n} = Z_{n,m}$ for all $m, n \in \mathbb{N}$ and such that the $Z_{m,n}$, $m < n$, are independent. Then the RCM (based on χ) is the point process

$$\xi := \sum_{m=1}^{|\chi|} \delta_{(X_m, \Psi_m)}, \quad (2.2)$$

where

$$\Psi_m := \sum_{n=1}^{|\chi|} \mathbf{1}_{\{n \neq m, Z_{m,n} \leq \varphi(X_m, X_n)\}} \delta_{X_n}.$$

In this notation we suppress the dependence on the $Z_{m,n}$. While the definition of ξ depends on the ordering of the points of χ , its distribution does not.

Below we will work with a Poisson process η with a diffuse intensity measure λ . Then η is simple and can be identified with its support. Otherwise it is not obvious how to treat multiplicities. One way to proceed is described in the following remark.

Remark 2.1. Let λ be a possibly non-diffuse locally finite measure on \mathbb{X} and let η be a Poisson process with intensity measure λ ; see [27]. Then η is a random element of the space of all locally finite counting measures on \mathbb{X} whose atoms may have multiplicities. Let \mathbb{U} denote the uniform distribution on $[0, 1]$ and let $\hat{\eta}$ be a \mathbb{U} -marking of η . Then $\hat{\eta}$ is a Poisson process on $\hat{\mathbb{X}} := \mathbb{X} \times [0, 1]$ with intensity measure $\hat{\lambda} := \lambda \otimes \mathbb{U}$; see [27, Theorem 5.6]. Since $\hat{\lambda}$ is diffuse, $\hat{\eta}$ is simple and can be written as

$$\hat{\eta} = \sum_{n=1}^{\hat{\eta}(\mathbb{X})} \delta_{(X_n, U_n)}, \quad (2.3)$$

where $(X_m, U_m) \neq (X_n, U_n)$ whenever $m \neq n$. Given a connection function φ we can define a connection function $\hat{\varphi}: \hat{\mathbb{X}}^2 \rightarrow [0, 1]$. We set $\hat{\varphi}((x, u), (y, v)) := \varphi(x, y)$ for $u, v \in [0, 1]$ if $x \neq y$. If $x = y$ we set $\hat{\varphi}((x, u), (x, v)) := 1$. The resulting random connection model $\hat{\xi}$ admits the following interpretation. An atom of η with size $k \in \mathbb{N}$ is split into k atoms of size 1. A pair of atoms of size 1 sitting at different locations x and y are independently connected with probability $\varphi(x, y)$. A pair of atoms of size 1 sitting at the same location are always connected. By definition of $\hat{\varphi}$, we have $C^x := C^{(x, u)}(\xi_1)$ for each $u \in [0, 1]$ with $(x, u) \in \hat{\eta}$. Hence $\hat{\xi}$ induces a random graph ξ^* on the support $\text{supp } \eta$ of η with components C^x , $x \in \eta$. The infinite components of this random graph are in one-to-one correspondence to those of ξ .

Remark 2.2. Establish the setting of Remark 2.1 There are other ways to define a random connection model driven by $\hat{\eta}$. For instance we might set $\hat{\varphi}((x, u), (x, v)) := 0$, while (as before) $\hat{\varphi}((x, u), (y, v)) := \varphi(x, y)$ for $x \neq y$. Then a pair of atoms sitting at the same location is never connected. In our opinion the choice in Remark 2.1 is rather natural.

We now introduce some notation used throughout the paper. For $\mu, \mu' \in \mathbf{G}$ we often interpret $\mu + \mu'$ as the measure in \mathbf{G} with the same support as $\mu + \mu'$. A similar convention applies to $\nu, \nu' \in \mathbf{N}$. Let $\mu \in \mathbf{G}$. For $B \in \mathcal{X}$ we write $\mu(B) := \mu(B \times \mathbf{N})$. More generally, given a measurable function $f: \mathbb{X} \rightarrow \mathbb{R}$ we write $\int f(x) \mu(dx) := \int f(x) \mu(dx \times \mathbf{N})$. Similarly, given $x \in \mathbb{X}$, we write $x \in \mu$ instead of $x \in V(\mu) = \mu(\cdot \times \mathbf{N})$. In the same spirit we write $g(\mu) := g(V(\mu))$, whenever g is a mapping on \mathbf{N} . These (slightly abusing) conventions lighten the notation and should not cause any confusion. For $B \in \mathcal{X}$ we denote by $\mu[B] \in \mathbf{G}$ the restriction of μ to B , that is the graph with vertex set $V(\mu) \cap B$ which keeps only those edges from μ with both end points from B . In the same way we use the notation $\mu[\nu]$ for $\nu \in \mathbf{N}$. Similarly for a measure ν on \mathbb{X} (for instance for $\nu \in \mathbf{N}$) we denote by $\nu_B := \nu(B \cap \cdot)$ the restriction of ν to a set $B \in \mathcal{X}$. Assume now that $v \in V(\mu)$. For $n \in \mathbb{N}_0$ let $C_n^v(\mu) \in \mathbf{G}$ denote the graph restricted to the set of vertices $x \in V(\mu)$ with $d_\mu(v, x) = n$, where d_μ denotes distance within the graph μ . Note that $C_0^v(\mu)$ is just the isolated vertex v . Slightly abusing our notation we write $C_0^v(\mu) = \delta_v$. For $v \notin V(\mu)$ we set $C^v(\mu) := 0$, interpreted as an empty graph (a graph with no vertices). The cluster $C^v(\mu)$ of v in μ is the graph μ restricted to

$$\sum_{n=0}^{\infty} V(C_n^v(\mu)),$$

while $C_{\leq n}^v(\mu)$, $n \in \mathbb{N}_0$, is the graph μ restricted to $V(C_0^v(\mu)) + \dots + V(C_n^v(\mu))$. For later purposes it will be convenient to define $C_{\leq -1}^v(\mu) = C_{-1}^v(\mu) := 0$ as the zero measure. For $\mu \in \mathbf{G}$ and $x \in \mathbb{X}$ we denote by $\mu - \delta_x := \mu[V(\mu) - \delta_x]$ the graph resulting from μ by removing the point x . If $x \notin V(\mu)$ then $\mu - \delta_x = \mu$.

3 Basic properties of the RCM

Let ξ be a RCM based on a Poisson process η on \mathbb{X} with diffuse intensity measure λ . Our first crucial tool is a version of the Mecke equation (see [27, Chapter 4]) for ξ . Given $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{X}$ we denote $\eta^{x_1, \dots, x_n} := \eta + \delta_{x_1} + \dots + \delta_{x_n}$ (removing possible multiplicities) and let ξ^{x_1, \dots, x_n} denote a RCM based on η^{x_1, \dots, x_n} . It is useful to construct ξ^{x_1, \dots, x_n} in a specific way as follows. We connect x_1 with the points in η using independent connection decisions which are independent of ξ . We then proceed inductively finally connecting x_n to $\eta + \delta_{x_1} + \dots + \delta_{x_{n-1}}$. For $n \in \mathbb{N}$ and a measurable function $f: \mathbb{X}^n \times \mathbf{G} \rightarrow [0, \infty]$ the Mecke equation for ξ states that

$$\begin{aligned} \mathbb{E} \int f(x_1, \dots, x_n, \xi) \eta^{(n)}(d(x_1, \dots, x_n)) \\ = \mathbb{E} \int f(x_1, \dots, x_n, \xi^{x_1, \dots, x_n}) \lambda^n(d(x_1, \dots, x_n)), \end{aligned} \quad (3.1)$$

where integration with respect to the *factorial measure* $\eta^{(n)}$ of η means summation over all n -tuples of pairwise distinct points from η . A convenient way to prove this and related formulas is to introduce a probability kernel Γ from \mathbf{N} to \mathbf{G} , satisfying

$$\mathbb{P}((\eta, \xi) \in \cdot) = \mathbb{E} \int \mathbf{1}\{(\eta, \mu) \in \cdot\} \Gamma(\eta, d\mu). \quad (3.2)$$

The kernel Γ is just a regular version of the conditional distribution of ξ given η and can be defined explicitly; see Section 2. A crucial property of this kernel is

$$\mathbb{E} \Gamma(\eta^{x_1, \dots, x_n}, \cdot) = \mathbb{P}(\xi^{x_1, \dots, x_n} \in \cdot), \quad \lambda^n\text{-a.e. } (x_1, \dots, x_n) \in \mathbb{X}^n. \quad (3.3)$$

It follows from [27, Theorem 4.4] that the left-hand side of (3.1) is given by

$$\mathbb{E} \iint f(x_1, \dots, x_n, \mu) \Gamma(\eta^{x_1, \dots, x_n}, d\mu) \lambda^n(d(x_1, \dots, x_n)).$$

Therefore (3.1) follows from (3.3).

Given $v \in \mathbb{X}$ we sometimes use (3.1) in the form

$$\begin{aligned} \mathbb{E} \int f(x_1, \dots, x_n, \xi^v) \eta^{(n)}(d(x_1, \dots, x_n)) \\ = \mathbb{E} \int f(x_1, \dots, x_n, \xi^{v, x_1, \dots, x_n}) \lambda^n(d(x_1, \dots, x_n)). \end{aligned} \quad (3.4)$$

This can be derived from (3.1) as follows. We can write $\xi^v = h(\xi, v, U)$, where U is a random element of $[0, 1]^{\mathbb{N}}$ with independent and uniformly distributed components, independent of ξ ; see the proof of Lemma 6.2 for more detail. It remains to note that $h(\xi^{x_1, \dots, x_n}, v, U)$ has the same distribution as ξ^{v, x_1, \dots, x_n} , provided that v, x_1, \dots, x_n are pairwise distinct.

To state another useful version of (3.1) we recall the notation $\mu - \delta_x = \mu[V(\mu) - \delta_x]$ for $\mu \in \mathbf{G}$ and $x \in \mathbb{X}$. Given $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathbb{X}$ we define $\mu - \delta_{x_1} - \dots - \delta_{x_n}$ inductively. The kernel Γ has the property

$$\int \mathbf{1}\{\mu - \delta_{x_1} - \dots - \delta_{x_n} \in \cdot\} \Gamma(\nu, d\mu) = \Gamma(\nu - \delta_{x_1} - \dots - \delta_{x_n}, \cdot), \quad \nu \in \mathbf{N}.$$

Therefore we obtain from [27, Theorem 4.5] for each measurable $f: \mathbb{X}^n \times \mathbf{G} \rightarrow [0, \infty]$ that

$$\begin{aligned} \mathbb{E} \int f(x_1, \dots, x_n, \xi - \delta_{x_1} - \dots - \delta_{x_n}) \eta^{(n)}(d(x_1, \dots, x_n)) \\ = \mathbb{E} \int f(x_1, \dots, x_n, \xi) \lambda^n(d(x_1, \dots, x_n)). \end{aligned} \quad (3.5)$$

Given $v \in \mathbb{X}$ we also have

$$\mathbb{E} \int f(x, \xi^v - \delta_x) \eta(dx) = \mathbb{E} \int f(x, \xi^v) \lambda(dx). \quad (3.6)$$

This follows similarly as (3.4). Indeed, given η and $x \in \eta$ the random graph $h(\xi, v, U) - \delta_x$ has the same distribution as $h(\xi - \delta_x, v, U)$, provided that $v \neq x$.

Another quick consequence of the multivariate Mecke equation is the following *deletion tolerance* of ξ . Removing a finite number of points from η results in a random graph whose distribution is absolutely continuous with respect to the distribution of ξ . Deletion tolerance of point processes was studied in [22].

Proposition 3.1. *Let $A \subset \mathbf{G}$ be a measurable set such that $\mathbb{P}(\xi \in A) = 0$. Let η_0 be a point process such that $\mathbb{P}(\eta_0(\mathbb{X}) < \infty) = 1$ and $\mathbb{P}(\eta_0 \leq \eta) = 1$. Then $\mathbb{P}(\xi[\eta - \eta_0] \in A) = 0$.*

Proof. Let $n \in \mathbb{N}$. By the Mecke equation (3.5) we have

$$0 = \int \mathbb{P}(\xi \in A) \lambda^n(d(x_1, \dots, x_n)) = \mathbb{E} \int \mathbf{1}\{\xi - \delta_{x_1} - \dots - \delta_{x_n} \in A\} \eta^{(n)}(d(x_1, \dots, x_n)).$$

Since $\mathbb{P}(\eta_0(\mathbb{X}) < \infty) = 1$ it follows that

$$\begin{aligned} & \mathbb{P}(\xi[\eta - \eta_0] \in A) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E} \mathbf{1}\{\eta_0(\mathbb{X}) = n\} \int \mathbf{1}\{\xi - \delta_{x_1} - \dots - \delta_{x_n} \in A\} \eta_0^{(n)}(d(x_1, \dots, x_n)) = 0. \end{aligned}$$

The result follows. \square

We define $\bar{\varphi} := 1 - \varphi$ and for $x \in \mathbb{X}$, $\nu \in \mathbf{N}$

$$\bar{\varphi}(\nu, x) := \prod_{y \in \nu} \bar{\varphi}(x, y), \quad \varphi(\nu, x) := 1 - \bar{\varphi}(\nu, x), \quad \varphi_\lambda(\nu) := \int \varphi(\nu, x) \lambda(dx). \quad (3.7)$$

We recall our general convention $\varphi(\mu, x) := \varphi(V(\mu), x)$ and $\varphi_\lambda(\mu) := \varphi_\lambda(V(\mu))$ for $\mu \in \mathbf{G}$. Throughout we often abbreviate $C^v := C^v(\xi^v)$, $C_n^v := C_n^v(\xi^v)$ and $C_{\leq n}^v := C_{\leq n}^v(\xi^v)$. Moreover we write $C^{v!} := C^v - \delta_v$.

We shall need the following consequence of (3.1).

Lemma 3.2. *Let $v \in \mathbb{X}$ and $h: \mathbf{G} \rightarrow [0, \infty)$ be measurable. Then*

$$\mathbb{E} \int h(\xi^v - \delta_x) C^{v!}(dx) = \mathbb{E} h(\xi^v) \varphi_\lambda(C^v). \quad (3.8)$$

Proof. Let I denote the left-hand side of the asserted formula. Then

$$I = \mathbb{E} \int h(\xi^v - \delta_x) \mathbf{1}\{x \in C^v(\xi^v)\} \eta(dx) = \int \mathbb{E} h(\xi^{v,x} - \delta_x) \mathbf{1}\{x \in C^v(\xi^{v,x})\} \lambda(dx),$$

where we have used the Mecke equation (3.4) to obtain the second identity. By definition we have that $\xi^{v,x} - \delta_x = \xi^v$ for each $x \in \mathbb{X}$. Hence we obtain that

$$I = \int \mathbb{E} h(\xi^v) \mathbf{1}\{x \in C^v(\xi^{v,x})\} \lambda(dx) = \int \mathbb{E} h(\xi^v) \mathbb{P}(x \in C^v(\xi^{v,x}) \mid \xi^v) \lambda(dx).$$

By definition of $\xi^{v,x}$ we have $\mathbb{P}(x \in C^v(\xi^{v,x}) \mid \xi^v) = \varphi(x, C^v)$, concluding the proof. \square

Next we turn to the *Margulis–Russo formula*. Let λ_1 and λ_2 be two measures on \mathbb{X} , where λ_1 is locally finite and λ_2 is finite. Given $t \geq 0$ we consider a RCM driven by a Poisson process η with intensity measure $\lambda_1 + t\lambda_2$. The associated expectation operator is denoted by \mathbb{E}_t . Let $f: \mathbf{G} \rightarrow [-\infty, \infty]$ be a measurable function and assume that $\mathbb{E}_{t_0}|f(\xi)| < \infty$ for some $t_0 > 0$. From [27, Exercise 3.8] and (3.2) we then obtain that $\mathbb{E}_t|f(\xi)| < \infty$ for all $t \leq t_0$. We assert that

$$\frac{d}{dt}\mathbb{E}_t f(\xi) = \int \mathbb{E}_t[f(\xi^x) - f(\xi)] \lambda_2(dx), \quad t \in [0, t_0]. \quad (3.9)$$

Using the kernel Γ , this can be seen as follows. From [27, Theorem 19.3] we obtain that

$$\frac{d}{dt}\mathbb{E}_t f(\xi) = \int \mathbb{E}_t[\tilde{f}(\eta^x) - \tilde{f}(\eta)] \lambda_2(dx), \quad t \in [0, t_0],$$

where $\tilde{f}(\nu) := \int f(\mu) \Gamma(\nu, d\mu)$, $\nu \in \mathbf{N}$. Note that $\tilde{f}(\xi)$ is \mathbb{P}_t -a.s. well-defined.

Take $t \in [0, t_0]$. Theorem 19.3 in [27] shows that $\int \mathbb{E}_t[|\tilde{f}(\eta^x) - \tilde{f}(\eta)|] \lambda_2(dx) < \infty$. Furthermore we have

$$\int \mathbb{E}_t[|\tilde{f}(\eta)|] \lambda_2(dx) = \lambda_2(\mathbb{X})\mathbb{E}_t[|\tilde{f}(\eta)|] \leq \lambda_2(\mathbb{X})\mathbb{E}_t[|f(\xi)|] < \infty,$$

where we have used the triangle inequality and (3.2).

Therefore we also have $\int \mathbb{E}_t[|\tilde{f}(\eta^x)|] \lambda_2(dx) < \infty$. It follows that

$$\frac{d}{dt}\mathbb{E}_t f(\xi) = \int (\mathbb{E}_t \tilde{f}(\eta^x) - \mathbb{E}_t \tilde{f}(\eta)) \lambda_2(dx) = \int (\mathbb{E}_t f(\xi^x) - \mathbb{E}_t f(\xi)) \lambda_2(dx),$$

where we have used (3.3). Since the above right-hand side is finite we have $|\mathbb{E}_t f(\xi^x)| < \infty$ and hence also $\mathbb{E}_t|f(\xi^x)| < \infty$ for λ_2 -a.e. x . This implies (3.9).

4 The stationary marked RCM

In this section we introduce an important special case of the general RCM. The setting is that of [11, 13]. Special cases were studied in [12, 18, 19].

Let \mathbb{M} be a complete separable metric space equipped with a probability measure \mathbb{Q} . This is our *mark space*, while \mathbb{Q} is said to be the *mark distribution*. In this section we consider the space $\mathbb{X} = \mathbb{R}^d \times \mathbb{M}$ equipped with the product of the Borel σ -field $\mathcal{B}(\mathbb{R}^d)$ on \mathbb{R}^d and the Borel σ -field on \mathbb{M} . We assume that $\lambda = t\lambda_d \otimes \mathbb{Q}$, where $t \in \mathbb{R}_+$ and λ_d denotes Lebesgue measure on \mathbb{R}^d . If $(x, p) \in \mathbb{X}$ then we call x location of (x, p) and p the mark of x . Instead of \mathbf{N} we consider the (smaller set) $\mathbf{N}(\mathbb{X})$ of all counting measures χ on \mathbb{X} such that $\chi(\cdot \times \mathbb{M})$ is locally finite (w.r.t. the Euclidean metric) and simple. The symmetric connection function $\varphi: (\mathbb{R}^d \times \mathbb{M})^2 \rightarrow [0, 1]$ is assumed to satisfy

$$\varphi((x, p), (y, q)) = \varphi((0, p), (y - x, q)). \quad (4.1)$$

This allows us to write $\varphi(x, p, q) := \varphi((0, p), (x, q))$, where 0 denotes the origin in \mathbb{R}^d . We also assume that

$$d_\varphi := \iint \varphi(x, p, q) dx \mathbb{Q}^2(d(p, q)) < \infty, \quad (4.2)$$

referring to Remark 4.2 for some comments. Let $t > 0$ and let η be a Poisson process on \mathbb{X} with intensity measure $t\lambda$. We can and will assume that η is a random element of $\mathbf{N}(\mathbb{X})$. We consider a RCM ξ based on η and connection function φ .

The RCM ξ is *stationary* in the sense that $T_x \xi \stackrel{d}{=} \xi$, $x \in \mathbb{R}^d$, where for $\mu \in \mathbf{G}$, the measure $T_x \mu$ is (as usual) defined by

$$T_x \mu := \int \mathbf{1}\{(y-x, q, \nu) \in \cdot\} \mu(d(y, q, \nu)).$$

To see this, it is convenient to define ξ in a slightly different way, without changing its distribution. As at (2.1) we can write

$$\eta = \sum_{m=1}^{\infty} \delta_{(X_m, Q_m)}, \quad (4.3)$$

where X_1, X, \dots are pairwise distinct random elements of \mathbb{R}^d and Q_1, Q_2, \dots are random elements of \mathbb{M} . Let $Z'_{m,n}$, $m, n \in \mathbb{N}$, be independent random variables uniformly distributed on $[0, 1]$ and set $Z'_m := (Z'_{m,n})_{n \in \mathbb{N}}$, $m \in \mathbb{N}$. By the marking theorem (see [27, Theorem 5.6]),

$$\eta^* := \sum_{m=1}^{\infty} \delta_{(X_m, Q_m, Z'_m)} \quad (4.4)$$

is again a Poisson process. We then connect (X_m, Q_m) with (X_n, Q_n) if X_m is lexicographically smaller than X_n and $Z'_{m,\tau} \leq \varphi(X_n - X_m, Q_m, Q_n)$, where the \mathbb{N} -valued random variable τ is determined by the fact that X_n is the τ -th nearest neighbour of X_m in the set $\{X_k : k \neq m\}$, where we can use the lexicographic order to break ties. Then we have $\xi = F(\eta^*)$ for a well-defined measurable mapping F . Since the nearest neighbour relation is translation invariant it follows from (4.1) that F can be assumed to satisfy $T_x \xi = F(T_x \eta^*)$ for each $x \in \mathbb{R}^d$. Since $T_x \eta^* \stackrel{d}{=} \eta^*$ it follows that ξ is stationary. The same argument combined with [27, Exercise 10.1] shows that ξ is *ergodic*, i.e. we have $\mathbb{P}(\xi \in A) \in \{0, 1\}$ for each translation invariant measurable $A \subset \mathbf{G}$. If \mathbb{M} contains only one element, we identify \mathbb{X} with \mathbb{R}^d . In this case ξ is said to be a *stationary RCM*.

The following consequence of the Mecke equation will be often used to treat cluster expectations.

Lemma 4.1. *Let $B \in \mathcal{B}(\mathbb{R}^d)$ and $f: \mathbb{N} \rightarrow \mathbb{R}_+$. Then*

$$\mathbb{E}_t \int \mathbf{1}\{x \in B\} f(|C^{(x,p)}(\xi)|) \eta(d(x, p)) = t \lambda_d(B) \mathbb{E}_t \int f(|C^{(0,p)}|) \mathbb{Q}(dp). \quad (4.5)$$

Proof. By the Mecke equation (3.1) the left-hand side of (4.5) equals

$$\begin{aligned} & t \mathbb{E}_t \iint \mathbf{1}\{x \in B\} f(|C^{(x,p)}(\xi^{(x,p)})|) dx \mathbb{Q}(dp) \\ &= t \mathbb{E}_t \iint \mathbf{1}\{x \in B\} f(|C^{(0,p)}(T_x \xi^{(x,p)})|) dx \mathbb{Q}(dp), \end{aligned}$$

where we have used that $|C^{(x,p)}(\mu)| = |C^{(0,p)}(T_x \mu)|$ for all $\mu \in \mathbf{G}$. It follows from stationarity of ξ and definition of $\xi^{(x,p)}$, that $T_x \xi^{(x,p)} \stackrel{d}{=} \xi^{(0,p)}$ for $\lambda_d \otimes \mathbb{Q}$ -a.e. $(x, p) \in \mathbb{R}^d \times \mathbb{M}$. Therefore the result follows. \square

Let Q_0 be a random element of \mathbb{M} with distribution \mathbb{Q} which is independent of η^* given by (4.4). In accordance with Palm theory we refer to $C^{(0,Q_0)}(\xi^{(0,Q_0)})$ as cluster of the *typical vertex* (of ξ).

Remark 4.2. Let $p \in \mathbb{M}$. Then the degree D_p of $(0, p)$ (the origin marked with p) in $\xi^{(0,p)}$ has a Poisson distribution with parameter $t \int \varphi(x, p, q) dx \mathbb{Q}(dq)$. Our integrability assumption (4.2) means that $\int \mathbb{E} D_p \mathbb{Q}(dp) < \infty$. This means that the expected degree of the typical vertex is finite. Hence (4.2) excludes Pareto type degree distributions but is still much weaker than the integrability assumption made in [11].

The function

$$\kappa(t) := \int \mathbb{E}_t |C^{(0,p)}|^{-1} \mathbb{Q}(dp) = \mathbb{E}_t |C^{(0,Q_0)}|^{-1}, \quad t \in \mathbb{R}_+, \quad (4.6)$$

plays a crucial role in Section 11. To interpret it, we introduce a point process $\eta_c \leq \eta(\cdot \times \mathbb{M})$ modeling finite clusters as follows. Let $(x, p) \in \eta$. Then $x \in \eta_c$ if $|C^{(x,p)}(\xi)| < \infty$ and x is the lexicographically smallest spatial coordinate of the points in $C^{(x,p)}(\xi)$. Since ξ is stationary, it is easy to see that η_c is a stationary point process. The following result shows that $t\kappa(t)$ is the *intensity* of η_c , that is the density of finite clusters. With a slight abuse of language we refer to $\kappa(t)$ as *cluster density*. In the unmarked case this function is also called *free energy*; see [1, 7, 14].

Lemma 4.3. *For each $t \in \mathbb{R}_+$ we have that $t\kappa(t) = \mathbb{E}_t \eta_c([0, 1]^d)$.*

Proof. The result follows from [29, Proposition 3.1] upon taking there η as the projection of the point process $\{(x, p) \in \eta : |C^{(x,p)}(\xi)| < \infty\}$ onto \mathbb{R}^d and $\xi := \eta_c$. A direct proof can start with

$$t\kappa(t) = \mathbb{E}_t \int \mathbf{1}\{x \in [0, 1]^d\} |C^{(x,p)}(\xi)|^{-1} \eta(d(x, p)), \quad (4.7)$$

a consequence of (4.5). The right-hand side can be written as

$$\mathbb{E}_t \iint \mathbf{1}\{x \in [0, 1]^d\} |C^{(x,p)}(\xi)|^{-1} \mathbf{1}\{\tau(x, p) = y\} \eta(d(x, p)) \eta_c(dy),$$

where $\tau(x, p)$ is the lexicographic minimum of the spatial coordinates of $C^{(x,p)}(\xi)$. The key step is then the application of the refined Campbell theorem for η_c . \square

The cluster density can also be obtained as an ergodic limit:

Proposition 4.4. *Let $B_n \in \mathcal{B}(\mathbb{R}^d)$, $n \in \mathbb{N}$, be an increasing sequence of compact convex sets whose inradius diverges to ∞ . Then*

$$\lim_{n \rightarrow \infty} (\lambda_d(B_n))^{-1} \int \mathbf{1}\{x \in B_n\} |C^{(x,p)}(\xi)|^{-1} \eta(d(x, p)) = t\kappa(t), \quad \mathbb{P}_t\text{-a.s.}$$

Proof. For each $\mu \in \mathbf{G}$

$$M_\mu := \int \mathbf{1}\{x \in \cdot\} |C^{(x,p)}(\mu)|^{-1} \mu(d(x, p))$$

is a locally finite measure on \mathbb{R}^d . For $x, y \in \mathbb{R}^d$ and $\mu \in \mathbf{G}$ we have $C^{(x,p)}(T_y \mu) = T_y C^{(x+y,p)}(\mu)$. Therefore, we obtain for $B \in \mathcal{B}(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$

$$\begin{aligned} M_{T_y \mu}(B) &:= \int \mathbf{1}\{x \in B\} |C^{(x,p)}(\mu)|^{-1} T_y \mu(d(x, p)) \\ &= \int \mathbf{1}\{x - y \in B\} |C^{(x,p)}(\mu)|^{-1} \mu(d(x, p)). \end{aligned}$$

This means that $M_{T_y \mu} = T_y M_\mu$. Therefore M_ξ is a stationary and ergodic random measure. By (4.7) it has intensity $t\kappa(t)$. Hence the result follows from [36, Satz 3]; see also [24, Theorem 30.10]. \square

We continue with a basic fact from percolation theory. Define

$$\theta(t) := \mathbb{P}_t(|C^{(0,Q_0)}(\xi^{(0,Q_0)})| = \infty) = \int \mathbb{P}_t(|C^{(0,p)}| = \infty) \mathbb{Q}(dp), \quad t \geq 0, \quad (4.8)$$

as the probability that the cluster of a typical vertex has infinite size. Let C_∞ denote the set of all $\mu \in \mathbf{G}$ such that μ has an infinite cluster.

Proposition 4.5. *Let $t > 0$. Then $\theta(t) > 0$ if and only if $\mathbb{P}_t(\xi \in C_\infty) = 1$.*

Proof. Let $B \in \mathcal{B}(\mathbb{R}^d)$ be a Borel set with $\lambda_d(B) \in (0, \infty)$. By (4.5),

$$\theta(t) = (t\lambda_d(B))^{-1} \mathbb{E}_t \int \mathbf{1}\{x \in B\} \mathbf{1}\{|C^{(x,p)}(\xi)| = \infty\} \eta(d(x,p)).$$

Hence, if $\theta(t) > 0$, then the probability that there is some $(x,p) \in \eta$ with $|C^{(x,p)}(\xi)| = \infty$ must be positive. Since ξ is ergodic and C_∞ is translation invariant, we obtain $\mathbb{P}_t(\xi \in C_\infty) = 1$. If, on the other hand, $\theta(t) = 0$, then the probability that $|C^{(x,p)}(\xi)| = \infty$ for some $(x,p) \in \eta$ with $x \in B$ is zero. Letting $B \uparrow \mathbb{R}^d$ we obtain that $\mathbb{P}_t(\xi \in C_\infty) = 0$. \square

The *critical intensity* (percolation threshold) is defined by

$$t_c := \inf\{t > 0 : \theta(t) > 0\}. \quad (4.9)$$

If $t < t_c$ then $\mathbb{P}_t(\xi \in C_\infty) = 0$ and if $t > t_c$ then $\mathbb{P}_t(\xi \in C_\infty) = 1$. Under a natural irreducibility assumption our Theorem 12.1 will show that ξ can have at most one infinite cluster.

We finish this section with some examples.

Example 4.6. In the unmarked case the connection function φ is just a function on \mathbb{R}^d . Under the minimal assumption $d_\varphi \in (0, \infty)$ it was shown in [37] that $t_c \in (0, \infty)$.

Example 4.7. Assume that $\mathbb{M} = \mathbb{R}_+$ and $\varphi(x,p,q) = \mathbf{1}\{\|x\| \leq p+q\}$, where $\|x\|$ denotes the Euclidean norm of $x \in \mathbb{R}^d$. The RCM ξ is then said to be the *Gilbert graph* with radius distribution \mathbb{Q} ; see e.g. [27, Chapter 16] for more detail. The integrability assumption (4.2) is then equivalent with $\int r^d \mathbb{Q}(dr) < \infty$, which is the minimal assumption for having a reasonable model. Under the additional assumption $\mathbb{Q}\{0\} < 1$ it was proved in [17, 20] that $t_c \in (0, \infty)$.

Example 4.8. Suppose that \mathbb{M} equals the space \mathcal{C}^d of all non-empty compact subsets of \mathbb{R}^d , equipped with the Hausdorff metric cf. [27, 41]. Let $V : \mathcal{C}^d \cup \{\emptyset\} \rightarrow [0, \infty]$ be measurable and translation invariant with $V(\emptyset) = 0$. For instance, V could be the volume or, if \mathbb{Q} is concentrated on the convex bodies, a linear combination of the *intrinsic volumes*; see [41]. Assume that the connection function is given by

$$\varphi((x,K), (y,L)) = 1 - e^{-V((K+x) \cap (L+y))}, \quad (x,K), (y,L) \in \mathbb{R}^d \times \mathcal{C}^d.$$

Then (4.1) follows from translation invariance of V . The case of the Gilbert graph arises if \mathbb{Q} is concentrated on balls centered at the origin. A sufficient condition for (4.2) is

$$\int D(K)^d \mathbb{Q}(dK) < \infty,$$

where $D(K)$ is the radius of the smallest ball centered at the origin and containing K . This easily follows from

$$\varphi(x,K,L) \leq \mathbf{1}\{K \cap (L+x) \neq \emptyset\} \leq \mathbf{1}\{\|x\| \leq D(K) + D(L)\}.$$

The random closed set $\bigcup_{(x,K) \in \eta} K+x$ is known as the *Boolean model* and a fundamental model of stochastic geometry (see [27, 41]) and continuum percolation (see [33]). This model corresponds to the choice $V(K) = \infty \cdot \mathbf{1}\{K \neq \emptyset\}$. In that case and under some additional assumptions on \mathbb{Q} it was proved in [20] that $t_c \in (0, \infty)$. The present much more general model is taken from [6] and is partially motivated by statistical physics.

Example 4.9. Assume that $M = (0, 1)$ equipped with Lebesgue measure Q . Assume that

$$\varphi((x, p), (y, q)) = \rho(g(p, q)\|x - y\|^d),$$

for a decreasing function $\rho: [0, \infty) \rightarrow [0, 1]$ and a function $g: (0, 1) \times (0, 1) \rightarrow [0, \infty)$ which is increasing in both arguments. We assume that $m_\rho := \int \rho(\|x\|^d) dx$ is positive and finite. This model was studied in [19] under the name *weight-dependent random connection model*. A simple calculation shows that

$$d_\varphi = m_\rho \iint g(p, q)^{-1} dp dq.$$

To ensure (4.2) we have to assume that g^{-1} is integrable. This is the case in all examples studied in [19], where it is also asserted that $t_c < \infty$. Sufficient conditions for $t_c \in (0, \infty)$ can also be found in [11, 12].

5 Irreducibility

In this section, we first consider a general RCM ξ based on a Poisson process η on \mathbb{X} with diffuse intensity measure $t\lambda$. We fix the intensity parameter $t > 0$ and therefore drop the lower index t in P_t . To simplify the notation, we take $t = 1$. We say that ξ is *irreducible* if

$$P(x_1 \leftrightarrow x_2 \text{ in } \xi^{x_1, x_2}) > 0, \quad \lambda^2\text{-a.e. } (x_1, x_2) \in \mathbb{X}^2. \quad (5.1)$$

Given $k \in \mathbb{N}$ and random elements Y_1, \dots, Y_k of \mathbb{X} we let $\Xi[Y_1, \dots, Y_k]$ be a RCM based on the point process $\delta_{Y_1} + \dots + \delta_{Y_k}$. Of course we can allow here some of the Y_1, \dots, Y_k to be deterministic. Further we define for each $n \in \mathbb{N}$ a measurable function $\varphi^{(n)}: \mathbb{X}^n \rightarrow [0, \infty)$ recursively by $\varphi^{(1)} := \varphi$ and

$$\varphi^{(n+1)}(x_1, x_2) := \int \varphi^{(n)}(x_1, z) \varphi(z, x_2) \lambda(dz), \quad x_1, x_2 \in \mathbb{X}, n \in \mathbb{N}.$$

These functions are symmetric. It follows straight from the Mecke equation (3.1) that

$$\varphi^{(n)}(x_1, x_2) = \mathbb{E} \int \prod_{i=1}^n \mathbf{1}\{y_{i-1} \sim y_i \text{ in } \xi^{x_1, x_2}\} \eta^{(n-1)}(d(y_1, \dots, y_{n-1})), \quad (5.2)$$

where $y_0 := x_1$ and $y_n := x_2$. This is the expected number of paths of length n from x_1 to x_2 in ξ^{x_1, x_2} .

Proposition 5.1. *The following six statements are equivalent:*

- (i) ξ is irreducible.
- (ii) There exist for λ^2 -a.e. $(x_1, x_2) \in \mathbb{X}^2$ a set $B \in \mathcal{X}$ with $\lambda(B) \in (0, \infty)$, an $n \in \mathbb{N}_0$ and independent random variables Y_1, \dots, Y_n with distribution $\lambda_B/\lambda(B)$ such that

$$P(x_1 \leftrightarrow x_2 \text{ in } \Xi[x_1, x_2, Y_1, \dots, Y_n]) > 0. \quad (5.3)$$

- (iii) There exists for λ^2 -a.e. $(x_1, x_2) \in \mathbb{X}^2$ an $n \in \mathbb{N}_0$ such that

$$\int P(x_1 \leftrightarrow x_2 \text{ in } \Xi[x_1, x_2, y_1, \dots, y_n]) \lambda^n(d(y_1, \dots, y_n)) > 0. \quad (5.4)$$

- (iv) For λ^2 -a.e. $(x_1, x_2) \in \mathbb{X}^2$ it holds that

$$\sup_{n \geq 1} \varphi^{(n)}(x_1, x_2) > 0. \quad (5.5)$$

(v) For λ^2 -a.e. $(x_1, x_2) \in \mathbb{X}^2$ and for all $k \in \mathbb{N}$ we have

$$\sup_{n \geq k} \varphi^{(n)}(x_1, x_2) > 0. \quad (5.6)$$

(vi) There exist for λ^2 -a.e. $(x_1, x_2) \in \mathbb{X}^2$ a set $B \in \mathcal{X}$ with $\lambda(B) \in (0, \infty)$, an $n \in \mathbb{N}$ and independent random variables Y_1, \dots, Y_n with distribution $\lambda_B/\lambda(B)$ such that

$$\mathbb{P}(x_1 \leftrightarrow x_2 \text{ in } \Xi'[x_1, x_2, Y_1, \dots, Y_n]) > 0, \quad (5.7)$$

where $\Xi'[x_1, x_2, Y_1, \dots, Y_n]$ is the graph obtained from $\Xi[x_1, x_2, Y_1, \dots, Y_n]$ by removing the edge between x_1 and x_2 (if there is one).

Proof. Assume that (i) holds, and let $x_1, x_2 \in \mathbb{X}$ be such that $\mathbb{P}(x_1 \leftrightarrow x_2 \text{ in } \xi^{x_1, x_2}) > 0$. Let (B_m) be a sequence of measurable sets of finite λ -measure increasing towards \mathbb{X} . By monotone convergence there exists $m \in \mathbb{N}$ such that $x_1, x_2 \in B_m$ and

$$\mathbb{P}(x_1 \leftrightarrow x_2 \text{ in } \xi^{x_1, x_2}[B_m]) > 0. \quad (5.8)$$

Let $B := B_m$. Note that $\xi^{x_1, x_2}[B]$ is a RCM based on $\delta_{x_1} + \delta_{x_2} + \eta_B$ and $\eta_B \stackrel{d}{=} \sum_{k=0}^d Y_k$, where Y_1, Y_2, \dots are independent with distribution $\lambda_B/\lambda(B)$, independent of η_B . Splitting the event $\{x_1 \leftrightarrow x_2 \text{ in } \xi^{x_1, x_2}[B]\}$ according to the value of $\eta(B)$ yields (ii).

Assume that (ii) holds. Then we have for λ^2 -a.e. (x_1, x_2) that

$$\begin{aligned} & \mathbb{P}(x_1 \leftrightarrow x_2 \text{ in } \Xi[x_1, x_2, Y_1, \dots, Y_n]) \\ &= \lambda(B)^{-n} \int \mathbb{P}(x_1 \leftrightarrow x_2 \text{ in } \Xi[x_1, x_2, y_1, \dots, y_n]) \lambda_B^n(d(y_1, \dots, y_n)) > 0, \end{aligned}$$

which implies (iii).

Assume that $x_1, x_2 \in \mathbb{X}$ satisfy (5.4). If $x_1 \leftrightarrow x_2$ in $\Xi[x_1, x_2, y_1, \dots, y_n]$ then there exist $k \in \{0, \dots, n\}$ and pairwise distinct $i_1, \dots, i_k \in \{1, \dots, n\}$ with $x_1 \sim y_{i_1} \sim \dots \sim y_{i_k} \sim x_2$ in $\Xi[x_1, x_2, y_1, \dots, y_n]$. Therefore and by the symmetry of λ^n

$$\sum_{k=0}^n \int \prod_{i=1}^{k+1} \varphi(y_{i-1}, y_i) \lambda^k(d(y_1, \dots, y_k)) > 0,$$

where $y_0 := x_1$ and $y_{k+1} := x_2$. Hence (iv) follows.

Assume that (iv) holds, then for λ^2 -a.e. $(x_1, x_2) \in \mathbb{X}^2$

$$\begin{aligned} 0 &< \sum_{m,n=1}^{\infty} \varphi^{(m+n)}(x_1, x_2) = \sum_{m,n=1}^{\infty} \int \varphi^{(m)}(x_1, z) \varphi^{(n)}(z, x_2) \lambda(dz) \\ &= \int \left(\sum_{m=1}^{\infty} \varphi^{(m)}(x_1, z) \right) \left(\sum_{n=1}^{\infty} \varphi^{(n)}(z, x_2) \right) \lambda(dz). \end{aligned}$$

Therefore $\sup_{n \geq 2} \varphi^{(n)}(x_1, x_2) > 0$ for λ^2 -a.e. $(x_1, x_2) \in \mathbb{X}^2$. To obtain (5.6) for general $k \in \mathbb{N}$ one has to start with a k -fold summation instead of a double summation.

Assume now that $x_1, x_2 \in \mathbb{X}$ satisfy (5.6), then there exists $n \geq 2$ such that $\varphi^{(n)}(x_1, x_2) > 0$. Therefore there exists $B \in \mathcal{X}$ of finite λ -measure with

$$\int \prod_{i=1}^n \varphi(y_{i-1}, y_i) \lambda_B^{n-1}(d(y_1, \dots, y_{n-1})) > 0,$$

where $y_0 := x_1$ and $y_n := x_2$. This implies (vi).

Finally, note that for any $n \in \mathbb{N}$

$$\begin{aligned} \mathbb{P}(x_1 \leftrightarrow x_2 \text{ in } \Xi'[x_1, x_2, Y_1, \dots, Y_n]) &\leq \mathbb{P}(x_1 \leftrightarrow x_2 \text{ in } \Xi[x_1, x_2, Y_1, \dots, Y_n]) \\ &= \mathbb{P}(x_1 \leftrightarrow x_2 \text{ in } \xi^{x_1, x_2}[B] \mid \eta(B) = n). \end{aligned}$$

Therefore (vi) implies (i). \square

Remark 5.2. Proposition 5.1 shows that irreducibility does not depend on the intensity parameter t as long as it is positive.

Remark 5.3. Consider the setting of Remark 2.1, where λ is not assumed to be diffuse. In the diffuse case, the value of the connection function on the diagonal of \mathbb{X}^2 is irrelevant for the definition of the RCM. But now we assume that $\varphi(x, x) = 1$ for all $x \in \mathbb{X}$. For each $n \in \mathbb{N}$ we can define a measurable function $\hat{\varphi}^{(n)}: \hat{\mathbb{X}}^n \rightarrow [0, \infty)$ as before. Now it can easily be checked that

$$\hat{\varphi}^{(n)}((x, u), (y, v)) = \varphi^{(n)}(x, y), \quad (x, u), (y, v) \in \hat{\mathbb{X}},$$

where $\varphi^{(n)}$ is defined as before, non-withstanding the fact that λ might not be diffuse. By Proposition 5.1 we therefore have that the random connection model $\hat{\xi}$ is irreducible, if and only if (5.5) holds.

Example 5.4. Let \mathbb{Y} be a locally compact separable Hausdorff space and assume that \mathbb{X} is the class of closed subsets of \mathbb{Y} equipped with the Fell topology; see e.g. [41]. Let ν be a locally finite measure on \mathbb{Y} , \mathbb{Q} be a probability measure on $[0, \infty)$, such that $\int \nu(B(x, r)) \mathbb{Q}(dr) < \infty$ for all $x \in \mathbb{X}$ and all $r \geq 0$, where $B(x, r)$ denotes the closed ball with centre x and radius r . Assume that λ is given by

$$\lambda = \iint \mathbf{1}\{B(x, r) \in \cdot\} \nu(dx) \mathbb{Q}(dr). \quad (5.9)$$

It is easy to see, that λ is locally finite. Similarly as in Example 4.8 we take a measurable function $V: \mathbb{X} \rightarrow [0, \infty]$ and assume that $\varphi(K, L) = 1 - e^{-V(K \cap L)}$, $K, L \in \mathbb{X}$. Assume also that the function V is increasing w.r.t. set inclusion and that $V(K) > 0$ if $K \neq \emptyset$. If $\mathbb{Q}([\varepsilon, \infty)) > 0$ for some $\varepsilon > 0$ and

$$\int \mathbf{1}\{B(x_1, \varepsilon) \cap B(x_2, \varepsilon) \neq \emptyset\} \nu^2(d(x_1, x_2)) > 0,$$

then $\varphi(K, L) > 0$ for all $K, L \in \mathbb{X}$ and the RCM ξ is irreducible.

In the remainder of this section we consider the stationary marked RCM as discussed in Section 4. Recall that without loss of generality, we can take $t = 1$. It is easy to see that for all $n \in \mathbb{N}$ and all $(x, p), (y, q) \in \mathbb{X}$

$$\varphi^{(n)}((x, p), (y, q)) = \varphi^{(n)}((0, p), (y - x, q)) = \varphi^{(n)}((0, q), (x - y, p)) \quad (5.10)$$

We write $\varphi^{(n)}(x, p, q) := \varphi^{(n)}((0, p), (x, q))$ and note that $\varphi^{(n)}(x, p, q) = \varphi^{(n)}(-x, q, p)$. In the unmarked case we can identify \mathbb{X} with \mathbb{R}^d . In this case $\varphi^{(n)} = \varphi^{*n}$ is the n -fold convolution of φ , where φ is considered as function on \mathbb{R}^d .

Proposition 5.5. *The stationary (unmarked) RCM is irreducible.*

The proof of Proposition 5.5 is a quick consequence of the first part of the following lemma.

Lemma 5.6. *Assume that $f: \mathbb{R}^d \rightarrow [0, \infty)$ is a bounded measurable function with $0 < \int f(y) dy < \infty$ and $f(y) = f(-y)$ for all $y \in \mathbb{R}^d$. Let $R > 0$.*

- (i) *There exist $n \in \mathbb{N}$ and $\varepsilon > 0$ such that $f^{*n}(x) \geq \varepsilon$ whenever $\|x\| \leq R$.*

- (ii) Let $g: \mathbb{R}^d \rightarrow \mathbb{R}_+$ be another bounded measurable function with $\int g(y) dy > 0$ and let $x \in \mathbb{R}^d$. Then there exists $n \in \mathbb{N}$ such that $(f^{*n} * g)(x) > 0$.

Proof. (i) The convolution of an integrable and a bounded function is bounded and uniformly continuous; see [15, Proposition 8.8]. It follows that f^{*2} is bounded and uniformly continuous. Since $\int f^{*2}(x) dx = \left(\int f(x) dx\right)^2 > 0$ there exist a ball $B' \subset \mathbb{R}^d$ with positive radius and $\varepsilon' > 0$ such that $f^{*2} \geq \varepsilon'$ on B' . Since f is symmetric, f^{*2} is symmetric as well. Hence we can find a ball B with center 0 and positive radius and some $\varepsilon > 0$ such that $f^{*4} \geq \varepsilon$ on B . Finally we find $m \in \mathbb{N}$ and $\varepsilon > 0$ such that $f^{*4m}(x) \geq \varepsilon$ whenever $\|x\| \leq R$.

(ii) By assumption $\int g(x) \mathbf{1}\{g(x) \geq \varepsilon_0\} dx > 0$ for some $\varepsilon_0 > 0$. Set $C := \{g \geq \varepsilon_0\}$. Then we have for each $n \in \mathbb{N}$ that

$$(f^{*n} * g)(x) \geq \varepsilon_0 \int f^{*n}(x - z) \mathbf{1}\{z \in C\} dz.$$

Choose $R > 0$ so large that

$$\int \mathbf{1}\{\|x - z\| \leq R, z \in C\} dz > 0.$$

By the first part of the lemma we can find $n \in \mathbb{N}$ and $\varepsilon > 0$ such that $f^{*n}(y) \geq \varepsilon$ whenever $\|y\| \leq R$. It follows that

$$\begin{aligned} (f^{*n} * g)(x) &\geq \varepsilon_0 \int f^{*n}(x - z) \mathbf{1}\{\|x - z\| \leq R, z \in C\} dz \\ &\geq \varepsilon_0 \varepsilon \int \mathbf{1}\{\|x - z\| \leq R, z \in C\} dz. \end{aligned}$$

By the choice of R this is positive. \square

Proof of Proposition 5.5. We can use Lemma 5.6 (i) and condition (5.5) from Proposition 5.1 to conclude the proof. Indeed, given $x_1, x_2 \in \mathbb{X}$ we find an $n \in \mathbb{N}$ such that $\varphi^{(n)}(x_1, x_2) = \varphi^{*n}(x_2 - x_1) > 0$. \square

It is natural to characterize irreducibility of the stationary marked RCM in terms of the functions $d_\varphi^{(n)}: \mathbb{M}^2 \rightarrow [0, \infty]$, $n \in \mathbb{N}$, defined by

$$d_\varphi^{(n)}(p, q) := \int \varphi^{(n)}(x, p, q) dx, \quad p, q \in \mathbb{M}.$$

Similarly as at (5.2) we see that $\int d_\varphi^{(n)}(p, q) \mathbf{1}\{q \in A\} \mathbb{Q}(dq)$ is the expected number of paths of length n from $(0, p)$ to some location with mark in a measurable set $A \subset \mathbb{M}$. From the symmetry property of $\varphi^{(n)}$ we obtain that $d_\varphi^{(n)}$ is symmetric. Furthermore,

$$d_\varphi^{(n)}(p, q) = \int \prod_{i=1}^n d_\varphi(q_{i-1}, q_i) \mathbb{Q}^{n-1}(d(q_1, \dots, q_{n-1})),$$

where $d_\varphi(\cdot, \cdot) := d_\varphi^{(1)}(\cdot, \cdot)$, $q_0 := p$, $q_n := q$. Therefore

$$d_\varphi^{(m+n)}(p, q) = \int d_\varphi^{(m)}(p, r) d_\varphi^{(n)}(r, q) \mathbb{Q}(dr), \quad p, q \in \mathbb{M}, m, n \in \mathbb{N}. \quad (5.11)$$

We may interpret $q \mapsto \sum_{n=1}^\infty d_\varphi^{(n)}(p, q)$ as a mark occupation density of the RCM.

Theorem 5.7. Let ξ be a stationary marked RCM. Then ξ is irreducible if and only if

$$\sup_{n \geq 1} d_\varphi^{(n)}(p, q) > 0, \quad \mathbb{Q}^2\text{-a.e. } (p, q) \in \mathbb{M}^2. \quad (5.12)$$

Proof. Assume that ξ is irreducible and for the sake of contradiction that there exists some measurable set $B \subset \mathbb{M}^2$ satisfying $\mathbb{Q}^2(B) > 0$, and $d_\varphi^{(n)}(p, q) = 0$ for all $(p, q) \in B$ and for each $n \in \mathbb{N}$. But then $\varphi^{(n)}((0, p), (x, q)) = 0$ for all $(p, q) \in B$, $n \in \mathbb{N}$ and λ_d -a.e. $x \in \mathbb{R}^d$. This contradicts Proposition 5.1 (iv).

Now we assume that (5.12) holds. Then $d_\varphi > 0$. Indeed, otherwise we would have $\varphi(x, p, q) = 0$ and then also $\varphi^{(n)}(x, p, q) = 0$, $n \in \mathbb{N}$, for λ -a.e. $(x, p, q) \in \mathbb{R}^d \times \mathbb{M}^2$. Therefore there exist measurable sets $A \subset \mathbb{R}^d$ and $E, F \subset \mathbb{M}$ such that $\lambda_d(A) > 0$, $\mathbb{Q}(E) > 0$, $\mathbb{Q}(F) > 0$, and

$$\varepsilon_0 := \inf_{x \in A, p \in E, q \in F} \varphi(x, p, q) = \inf_{x \in -A, p \in E, q \in F} \varphi(x, q, p) > 0. \quad (5.13)$$

Set $\varepsilon(x) := \varepsilon_0 \mathbf{1}\{x \in A\}$. Then $\varphi(x, p, q) \geq \varepsilon(x)$ for each $(x, p, q) \in \mathbb{R}^d \times E \times F$. For all $x \in \mathbb{R}^d$ and all $r, s \in F$ we have

$$\begin{aligned} \varphi^{(2)}(x, r, s) &= \iint \varphi(z, r, q) \varphi(x - z, q, s) \mathbb{Q}(dq) dz \\ &= \iint \varphi(-z, q, r) \varphi(x - z, q, s) \mathbb{Q}(dq) dz \\ &\geq \mathbb{Q}(E) \int \varepsilon(-z) \varepsilon(x - z) dz = \mathbb{Q}(E) f(x), \end{aligned}$$

where f is the convolution of ε and $\varepsilon_0 \mathbf{1}_{-A}$. Note that f is symmetric and has a positive and finite integral. We further have

$$\varphi^{(4)}(x, r, s) \geq \iint \mathbf{1}\{q \in F\} \varphi^{(2)}(z, r, q) \varphi^{(2)}(x - z, q, s) \mathbb{Q}(dq) dz \geq \mathbb{Q}(E)^2 \mathbb{Q}(F) f^{*2}(x)$$

and, inductively,

$$\varphi^{(2n)}(x, r, s) \geq \mathbb{Q}(E)^n \mathbb{Q}(F)^{n-1} f^{*n}(x). \quad (5.14)$$

Our goal is to use Proposition 5.1 (iv). Let $p, q \in \mathbb{M}$. In view of this goal and assumption (5.12) we can assume that there exist $k, l \in \mathbb{N}$ such that

$$\int_F d_\varphi^{(k)}(p, r) \mathbb{Q}(dr) > 0 \quad \text{and} \quad \int_F d_\varphi^{(l)}(r, q) \mathbb{Q}(dr) > 0.$$

Then we obtain by (5.14) for each $x \in \mathbb{R}^d$ and each $n \in \mathbb{N}$ that

$$\begin{aligned} \varphi^{(k+2n+l)}(x, p, q) &= \iint \varphi^{(k)}(z, p, r) \varphi^{(2n)}(w, r, s) \varphi^{(l)}(x - z - w, s, q) \mathbb{Q}^2(d(r, s)) d(z, w) \\ &\geq \mathbb{Q}(E)^n \mathbb{Q}(F)^{n-1} \iint_{F^2} \varphi^{(k)}(z, p, r) f^{*n}(w) \varphi^{(l)}(x - z - w, s, q) \mathbb{Q}^2(d(r, s)) d(z, w) \\ &= \mathbb{Q}(E)^n \mathbb{Q}(F)^{n-1} (g * f^{*n} * h)(x) = \mathbb{Q}(E)^n \mathbb{Q}(F)^{n-1} (f^{*n} * g * h)(x), \end{aligned}$$

where $g(z) := \int_F \varphi^{(k)}(z, p, r) \mathbb{Q}(dr)$ and $h(z) := \int_F \varphi^{(l)}(z, r, q) \mathbb{Q}(dr)$. By the choice of k and l we have

$$\int g * h(x) dx = \int g(x) dx \int h(x) dx > 0.$$

Therefore, given $x \in \mathbb{R}^d$, we obtain from Lemma 5.6 that $(f^{*n} * g * h)(x)$ is positive for some sufficiently large n . Taking into the account translation invariance (5.10) we hence obtain from Proposition 5.1 (iv) that ξ is irreducible. \square

Remark 5.8. Condition (5.12) can be expressed as follows. For each $B \in \mathcal{X}$ with $\mathbb{Q}(B) > 0$ we have

$$\int_B \sum_{n=1}^{\infty} d_{\varphi}^{(n)}(p, q) > 0, \quad \mathbb{Q}\text{-a.e. } p. \quad (5.15)$$

Assume (for the sake of illustration) that $d(p) := \int d_{\varphi}(p, q) \mathbb{Q}(dq) > 0$ for each $p \in \mathbb{M}$ and let $(X_n)_{n \geq 0}$ be a Markov chain with transition kernel $p \mapsto d(p)^{-1} d_{\varphi}(p, q) \mathbb{Q}(dq)$. Then condition (5.15) means that

$$\mathbb{E} \left[\sum_{n=1}^{\infty} \mathbf{1}\{X_n \in B\} \middle| X_0 = p \right] > 0, \quad \mathbb{Q}\text{-a.e. } p, \quad (5.16)$$

which is slightly weaker than \mathbb{Q} -irreducibility of (X_n) , as studied in [34]. Without additional assumptions on φ , (5.12) seems to be both a natural and a minimal assumption. If there exists $A \in \mathcal{X}$ with $0 < \mathbb{Q}(A) < 1$ and $d_{\varphi}(p, q) = 0$ for $(p, q) \in A \times A^c$, then $d_{\varphi}^{(n)}(p, q) = 0$ for each $n \in \mathbb{N}$ and all $(p, q) \in A \times A^c$, so that (5.12) fails.

The next remark shows the relevance of irreducibility for the uniqueness of the infinite cluster.

Remark 5.9. Assume that \mathbb{M} is discrete and that $\mathbb{Q}\{p\} > 0$ for each $p \in \mathbb{M}$. Given $p, q \in \mathbb{M}$ we write $p \simeq q$ if either $p = q$ or $\sup_{n \geq 1} d_{\varphi}^{(n)}(p, q) > 0$. It follows from (5.11) that \simeq is an equivalence relation. Let $[p] := \{q \in \mathbb{M} : p \simeq q\}$ be the equivalence class of $p \in \mathbb{M}$. Then $\eta_{[p]} := \{(x, q) \in \eta : q \in [p]\}$ are for different equivalence classes independent Poisson processes with intensity measures $\lambda_d \otimes \mathbb{Q}([p] \cap \cdot)$. Assume now that there exist some marks $p, q \in \mathbb{M}$ such that $[p] \cap [q] = \emptyset$. We assert that $\xi[\eta_{[p]}]$ and $\xi[\eta_{[q]}]$ are vertex disjoint, that is, there is no edge in ξ between $\eta_{[p]}$ and $\eta_{[q]}$. To see this, we take a bounded Borel set $B \subset \mathbb{R}^d$ and let A denote the event that there exist $x \in B$ and $y \in \mathbb{R}^d$ such that $(x, p), (y, q) \in \eta$ and $(x, p) \leftrightarrow (y, q)$ in ξ . Similarly, as in previous calculations, we obtain

$$\mathbb{P}(A) \leq \sum_{n=1}^{\infty} \lambda_d(B) \mathbb{Q}\{p\} \mathbb{Q}\{q\} d_{\varphi}^{(n)}(p, q)$$

which comes to zero. If $\xi[\eta_{[p]}]$ and $\xi[\eta_{[q]}]$ both percolate, then ξ has at least two infinite clusters. Without the irreducibility condition, the number of infinite clusters might be any natural number or even infinity.

In the following we will discuss some consequences of Theorem 5.7. We start with the case, where \mathbb{Q} has an atom. This covers discrete (that is finite or countably infinite) mark spaces and generalizes Proposition 5.5.

Corollary 5.10. *Let ξ be a stationary marked RCM and assume that there exists $p_0 \in \mathbb{M}$ with $\mathbb{Q}\{p_0\} > 0$. Then ξ is irreducible if and only if*

$$\sup_{n \geq 1} d_{\varphi}^{(n)}(p_0, q) > 0, \quad \mathbb{Q}\text{-a.e. } q \in \mathbb{M}. \quad (5.17)$$

Proof. Take $p, q \in \mathbb{M}$ and $m, n \in \mathbb{N}$. By (5.11),

$$d_{\varphi}^{(m+n)}(p, q) \geq \mathbb{Q}\{p_0\} d_{\varphi}^{(m)}(p, p_0) d_{\varphi}^{(n)}(p_0, q) = \mathbb{Q}\{p_0\} d_{\varphi}^{(m)}(p_0, p) d_{\varphi}^{(n)}(p_0, q).$$

Hence, if (5.17) holds, then we obtain that $\sup_{k \geq 2} d_{\varphi}^{(k)}(p, q) > 0$ for \mathbb{Q}^2 -a.e. (p, q) , so that irreducibility follows from Theorem 5.7. The converse is obvious. \square

Under a suitable minorization assumption on the connection function we have the following version of Corollary 5.10 which does not assume p_0 to be an atom of \mathbb{Q} .

Theorem 5.11. Assume that there exist a measurable set $A \subset \mathbb{M}$ with $\mathbb{Q}(A) > 0$ and some $p_0 \in \mathbb{M}$ satisfying

$$d_\varphi(p, q) \geq d_\varphi(p, p_0), \quad \mathbb{Q}^2\text{-a.e. } (p, q) \in \mathbb{M} \times A. \quad (5.18)$$

Assume also that (5.17) holds. Then the RCM ξ is irreducible.

Proof. Taking $n = 1$ in (5.11) it follows by induction that

$$d_\varphi^{(m)}(p, q) \geq d_\varphi^{(m)}(p, p_0), \quad \mathbb{Q}^2\text{-a.e. } (p, q) \in \mathbb{M} \times A.$$

Just as in the proof of Corollary 5.10 we hence obtain for all $m, n \in \mathbb{N}$ and \mathbb{Q}^2 -a.e. (p, q) that

$$d_\varphi^{(m+n)}(p, q) \geq \mathbb{Q}(A) d_\varphi^{(m)}(p, p_0) d_\varphi^{(n)}(p_0, q).$$

Therefore we obtain that $\sup_{k \geq 2} d_\varphi^{(k)}(p, q) > 0$ for \mathbb{Q}^2 -a.e. (p, q) , so that irreducibility follows from Theorem 5.7. \square

A minimal assumption for irreducibility could be

$$\int d_\varphi(p, q) \mathbb{Q}(dq) > 0, \quad \mathbb{Q}\text{-a.e. } p \in \mathbb{M}. \quad (5.19)$$

Under suitable assumptions on \mathbb{Q} and φ we shall show with Theorem 5.12 that (5.19) is already sufficient for irreducibility.

In Theorem 5.12 we will consider a partial ordering \preceq on \mathbb{M} which is measurable, that is $\{(p, q) : p \preceq q\}$ is a measurable subset of \mathbb{M}^2 . Slightly generalizing [30] we say that \mathbb{M} is *partially ordered probability (POP) space*. A real-valued function f on \mathbb{M} is said to be non-decreasing if $x \preceq y$ implies $f(x) \leq f(y)$. The probability measure \mathbb{Q} is called (positively) *associated* if

$$\int f g d\mathbb{Q} \geq \int f d\mathbb{Q} \int g d\mathbb{Q} \quad (5.20)$$

for all non-decreasing measurable $f, g: \mathbb{M} \rightarrow \mathbb{R}$ for which the integrals make sense. Our next result provides assumptions on φ and \mathbb{Q} , under which the minimal assumption (5.19) implies irreducibility. Corollary 5.13 and Example 12.7 will demonstrate the usefulness of this result.

Theorem 5.12. Assume that \mathbb{M} is a POP space and that \mathbb{Q} is associated. Assume also that $d_\varphi(p, \cdot)$ is monotone (non-decreasing or non-increasing) for all $p \in \mathbb{M}$. Then the RCM ξ is irreducible if and only if (5.19) holds.

Proof. Assume that (5.12) holds but (5.19) fails. Then there exists a measurable set $B \subset \mathbb{M}$ with $\mathbb{Q}(B) > 0$ and $d_\varphi(p, q) = 0$ for \mathbb{Q}^2 -a.e. $(p, q) \in B \times \mathbb{M}$. This implies for all $n \in \mathbb{N}$, that $d_\varphi^{(n)}(p, q) = 0$ for \mathbb{Q}^2 -a.e. $(p, q) \in B \times \mathbb{M}$. The resulting contradiction shows that (5.12) implies (5.19). Let us assume the latter holds. Since \mathbb{Q} is associated we obtain for all $p, q \in \mathbb{M}$

$$d_\varphi^{(2)}(p, q) = \int d_\varphi(p, r) d_\varphi(r, q) \mathbb{Q}(dr) \geq \int d_\varphi(p, r) \mathbb{Q}(dr) \int d_\varphi(r, q) \mathbb{Q}(dr).$$

This implies (5.12) and hence the result. \square

Corollary 5.13. Assume that $\mathbb{M} \subset \mathbb{R}$ is an interval and that $d_\varphi(p, \cdot)$ is monotone for all $p \in \mathbb{M}$. Then the RCM ξ is irreducible if and only if (5.19) holds.

Proof. Since any probability measure on \mathbb{M} is associated (see e.g. [30]), the result follows from Theorem 5.12. \square

Remark 5.14. Let the assumptions of Theorem 5.12 be satisfied and assume moreover that $\iint \mathbf{1}\{p \preceq q\} \mathbb{Q}(dp) \mathbb{Q}(dq) > 0$. Let B be the increasing Borel set of all $p \in \mathbb{M}$ such that $\sup_{n \geq 1} d^{(n)}(p, q) > 0$ for \mathbb{Q} -a.e. $q \in \mathbb{M}$. Since ξ is irreducible (by Theorem 5.12) we obtain from Theorem 5.7 that $\mathbb{Q}(B) = 1$. We assert that there is some $p_0 \in B$ such that $\mathbb{Q}(C_{p_0}) > 0$, where $C_{p_0} := \{p \in \mathbb{X} : p_0 \preceq p\}$. Indeed, if this were not the case, then

$$\begin{aligned} 0 &= \iint \mathbf{1}\{p \preceq q, p \in B\} \mathbb{Q}(dp) \mathbb{Q}(dq) = \iint \mathbf{1}\{p \preceq q, p \in B, q \in B\} \mathbb{Q}(dp) \mathbb{Q}(dq) \\ &= \iint \mathbf{1}\{p \preceq q\} \mathbb{Q}(dp) \mathbb{Q}(dq), \end{aligned}$$

contradicting our assumption. It follows that (5.17) and (5.18) both hold with $A := C_{p_0}$. Therefore Theorem 5.11 applies. However, given a monotone d_φ and an associated \mathbb{Q} , Theorem 5.12 is much easier to apply. It only remains to check (5.19), which could be assumed without too much loss of generality; see Remark 5.15.

Remark 5.15. Assume that (5.19) fails and choose a measurable set $B \subset \mathbb{M}$ with $\mathbb{Q}(B) > 0$ and $d_\varphi(p, q) = 0$ for \mathbb{Q}^2 -a.e. $(p, q) \in B \times \mathbb{M}$. This easily implies that

$$|C^{(x,p)}| = 1, \quad (x, p) \in \eta_{\mathbb{R}^d \times B}, \quad \mathbb{P}\text{-a.s.}$$

Therefore Poisson points with a mark in B are isolated in ξ . In particular they do not contribute to infinite clusters.

6 Deletion stability and uniqueness

In this section, we consider a general RCM ξ based on a Poisson process η on \mathbb{X} with diffuse intensity measure λ . Given $(x, \mu) \in \mathbb{X} \times \mathbf{G}$ we let $N^\infty(x, \mu)$ denote the number of infinite clusters in $C^x(\mu) - \delta_x$. We say that the infinite clusters in ξ are *deletion stable* if

$$\mathbb{P}(N^\infty(x, \xi^x) \geq 2) = 0, \quad \lambda\text{-a.e. } x \in \mathbb{X}. \quad (6.1)$$

Using the Mecke equation it is not difficult to see that the infinite clusters in ξ are deletion stable if $N_{ds} = 0$ a.s., where

$$N_{ds} := \int \mathbf{1}\{N^\infty(x, \xi) \geq 2\} \eta(dx). \quad (6.2)$$

Theorem 6.1. *Assume that ξ is irreducible and that the infinite clusters of ξ are deletion stable. Then ξ has \mathbb{P} -almost surely at most one infinite cluster. If, conversely, the latter holds then the infinite cluster of ξ is deletion stable.*

The converse implication in Theorem 6.1 will be an easy consequence of the Mecke equation. The proof of the non-trivial implication will be based on two lemmas. Let Y_1, \dots, Y_n be random elements of \mathbb{X} , which are a.s. pairwise distinct. In accordance with Section 3 we define a random connection model ξ^{Y_1, \dots, Y_n} based on the point process $\eta + \delta_{Y_1} + \dots + \delta_{Y_n}$ as follows. We connect Y_1 with the points in η using independent connection decisions which are independent of ξ . We then proceed inductively finally connecting Y_n to $\eta + \delta_{Y_1} + \dots + \delta_{Y_{n-1}}$.

Lemma 6.2. *Suppose that $B \in \mathcal{X}$ with $\lambda(B) \in (0, \infty)$ and let Y_1, \dots, Y_n be independent random variables with distribution $\lambda_B/\lambda(B)$, independent of ξ . Assume that the infinite clusters of ξ are deletion stable, then*

$$\int \mathbb{P}(N^\infty(Y_n, \xi^{x_1, x_2, Y_1, \dots, Y_n}) \geq 2) \lambda^2(d(x_1, x_2)) = 0. \quad (6.3)$$

Proof. It is useful to add a point $x \in \mathbb{X}$ to a graph $\mu \in \mathbf{G}$ in the following explicit way. There are measurable mappings $\pi_n: \mathbf{N} \rightarrow \mathbb{R}^d$ such that $\mu(\cdot \times \mathbf{N}) = \sum_{n=1}^{|\mu|} \delta_{\pi_n(\mu)}$, for each $\mu \in \mathbf{G}$. Let $(\mu, x) \in \mathbf{G} \times \mathbb{X}$ and $u = (u_n)_{n \geq 1} \in [0, 1]^{\mathbb{N}}$. Define $\mu_u^x \in \mathbf{G}$ as the graph with vertex measure $V(\mu) + \delta_x$, edges from μ and further edges between $\pi_n(\mu)$ and x if $\varphi(\pi_n(\mu), x) \geq u_n$. Define $h(x, \mu, u) := \mathbf{1}\{N^\infty(x, \mu_u^x) \geq 2\}$. Assume that U is a random element of $[0, 1]^{\mathbb{N}}$ with independent and uniformly distributed components, independent of ξ . Then $\mathbf{1}\{N^\infty(x, \xi^x) \geq 2\}$ has the same distribution as $h(x, \xi, U)$ and deletion stability means that

$$\iiint h(x, \mu, u) \mathbb{P}(\xi \in d\mu) \lambda(dx) \mathbb{P}(U \in du) = 0. \quad (6.4)$$

Given $x_1, x_2 \in \mathbb{X}$ we also have

$$\mathbf{1}\{N^\infty(Y_n, \xi^{x_1, x_2, Y_1, \dots, Y_n}) \geq 2\} \stackrel{d}{=} h(Y_n, \xi^{x_1, x_2, Y_1, \dots, Y_{n-1}}, U_n),$$

where U_n is independent of the pair $(Y_n, \xi^{x_1, x_2, Y_1, \dots, Y_{n-1}})$ and has the same distribution as U . Therefore

$$\begin{aligned} & \iint \mathbb{P}(N^\infty(Y_n, \xi^{x_1, x_2, Y_1, \dots, Y_n}) \geq 2) \lambda^2(d(x_1, x_2)) \\ &= (\lambda(B))^{-1} \iiint \mathbb{E} h(y_n, \xi^{x_1, x_2, Y_1, \dots, Y_{n-1}}, u) \lambda_B(dy_n) \mathbb{P}(U \in du) \lambda^2(d(x_1, x_2)) \\ &= (\lambda(B))^{-n} \iiint \mathbb{E} h(y_n, \xi^{x_1, x_2, y_1, \dots, y_{n-1}}, u) \mathbb{P}(U \in du) \lambda_B^n(dy_1, \dots, y_n) \lambda^2(d(x_1, x_2)), \end{aligned}$$

where we have used the definition of $\xi^{x_1, x_2, Y_1, \dots, Y_{n-1}}$. From the Mecke equation we obtain that the above equals

$$(\lambda(B))^{-n} \mathbb{E} \iint h(y_n, \xi, u) \mathbf{1}\{y_1, \dots, y_{n-1} \in B\} \lambda_B(dy_n) \mathbb{P}(U \in du) \eta^{(n+1)}(d(x_1, x_2, y_1, \dots, y_{n-1})).$$

By (6.4), the integral $\iint h(y, \xi, u) \lambda_B(dy) \mathbb{P}(U \in du)$ does almost surely vanish. This concludes the proof. \square

For given $x_1, x_2 \in \mathbb{X}$ let $A(x_1, x_2)$ be the event that the clusters $C^{x_1}(\xi^{x_1, x_2})$ and $C^{x_2}(\xi^{x_1, x_2})$ are infinite and not connected. Further, for $n \in \mathbb{N}_0$ let $B_n(x_1, x_2)$ be the event that x_1 and x_2 are connected in $\xi^{x_1, x_2, Y_1, \dots, Y_n}$, where Y_1, \dots, Y_n are defined in Lemma 6.2.

Lemma 6.3. *Let the assumptions of Lemma 6.2 be in force. Then for a given $n \in \mathbb{N}_0$*

$$\int \mathbb{P}(A(x_1, x_2) \cap B_n(x_1, x_2)) \lambda^2(d(x_1, x_2)) = 0. \quad (6.5)$$

Proof. We can remove the points Y_n, \dots, Y_1 from $\xi^{x_1, x_2, Y_1, \dots, Y_n}$ one by one. Each time we can apply Lemma 6.2. Hence removing Y_i (for $i \leq n$) cannot split the cluster of Y_i in $\xi^{x_1, x_2, Y_1, \dots, Y_i}$ into more than one infinite cluster. Take $x_1, x_2 \in \mathbb{X}$ and $n \in \mathbb{N}_0$ such that $B_n(x_1, x_2)$ holds and assume for the sake of contradiction that $A(x_1, x_2)$ holds. In particular $C^{x_1}(\xi^{x_1, x_2})$ and $C^{x_2}(\xi^{x_1, x_2})$ are vertex disjoint, so that there must be an $i \in \{1, \dots, n\}$ such that x_1, x_2 are connected in $\xi^{x_1, x_2, Y_1, \dots, Y_i}$ but not in $\xi^{x_1, x_2, Y_1, \dots, Y_{i-1}}$. Hence, the removal of Y_i would split the cluster of Y_i in $\xi^{x_1, x_2, Y_1, \dots, Y_i}$ into two infinite clusters. This is a contradiction, showing that almost surely $B_n(x_1, x_2) \subset A^c(x_1, x_2)$ for λ^2 -a.e. $(x_1, x_2) \in \mathbb{X}^2$. \square

Proof of Theorem 6.1. Let us first assume that $\mathbb{P}(\xi \in A_\infty) = 0$, where A_∞ is the set of all $\mu \in \mathbf{G}$ such that μ has at least two infinite clusters. If $x \in \eta$ satisfies $N^\infty(x, \xi) \geq 2$ then $\xi - \delta_x \in A_\infty$. Therefore we obtain from the Mecke equation (3.1)

$$\mathbb{E} \int \mathbf{1}\{N^\infty(x, \xi) \geq 2\} \eta(dx) \leq \mathbb{E} \int \mathbf{1}\{\xi - \delta_x \in A_\infty\} \eta(dx) = \mathbb{E} \int \mathbf{1}\{\xi \in A_\infty\} \lambda(dx)$$

which comes to zero.

Let us now assume that ξ is irreducible and that the infinite clusters are deletion stable. We need to show that almost surely two points of η cannot belong to two different infinite clusters. By the Mecke equation (3.1) for $n = 2$ the latter is equivalent to

$$\int \mathbb{P}(A(x_1, x_2)) \lambda^2(d(x_1, x_2)) = 0. \quad (6.6)$$

The following arguments apply to λ^2 -a.e. $(x_1, x_2) \in \mathbb{X}^2$. By Proposition 5.1 (vi) there exist a set $B \in \mathcal{X}$ with $0 < \lambda(B) < \infty$, an $n \in \mathbb{N}$ and random variables Y_1, \dots, Y_n with distribution $\lambda_B/\lambda(B)$ such that $\mathbb{P}(B'_n(x_1, x_2)) > 0$, where

$$B'_n(x_1, x_2) := \{x_1 \leftrightarrow x_2 \text{ in } \Xi'[x_1, x_2, Y_1, \dots, Y_n]\}.$$

We can couple the random graphs $\xi^{x_1, x_2, Y_1, \dots, Y_n}$ and $\Xi'[x_1, x_2, Y_1, \dots, Y_n]$ in such a way that ξ^{x_1, x_2} and $\Xi'[x_1, x_2, Y_1, \dots, Y_n]$ are independent and every edge in the latter graph is also present in the former. Then $B'_n(x_1, x_2)$ implies $B_n(x_1, x_2)$ and we obtain from Lemma 6.3 that

$$\mathbb{P}(A(x_1, x_2) \cap B'_n(x_1, x_2)) = 0.$$

By the above coupling the events $A(x_1, x_2)$ and $B'_n(x_1, x_2)$ are independent. Hence $\mathbb{P}(A(x_1, x_2)) = 0$, as required. \square

Motivated by [35] we might call an infinite graph $\mu \in \mathbf{G}$ *2-indivisible* if the removal of a finite number of vertices results in at most one infinite connected component. The following corollary of Proposition 3.1 shows that ξ is almost surely 2-indivisible.

Corollary 6.4. Assume that ξ has almost surely at most one infinite cluster. Assume further that η_0 is a point process such that $\mathbb{P}(\eta_0(\mathbb{X}) < \infty) = 1$ and $\mathbb{P}(\eta_0 \leq \eta) = 1$. Then $\xi[\eta - \eta_0]$ has a.s. at most one infinite cluster.

Remark 6.5. Consider the setting of Remark 2.1. Assume that the infinite clusters of ξ are deletion stable. This means that reducing the size of an atom x of η by one (and removing the corresponding edges), cannot split the associate cluster into more than one infinite clusters. This is not an intrinsic property of the random graph ξ^* , as defined in Remark 2.1. Assume in addition that $\hat{\xi}$ is irreducible (characterized in Remark 5.3), so that Theorem 6.1 applies. By Corollary 6.4 we can then remove any finite number of points from $\hat{\eta}$ without splitting the infinite cluster (if existent) in two or more infinite clusters. In particular we can remove a finite number of points from the support of η , without splitting the infinite cluster of ξ^* in two or more infinite clusters.

Remark 6.6. In accordance with the physics literature (see e.g. [9]) we might call a point $x \in \eta$ *red*, if any doubly infinite path in ξ has to use x . If ξ has a unique infinite cluster, then Corollary 6.4 says in particular that ξ cannot have red points. More generally, we may call a subset of η *red*, if any doubly infinite path in ξ contains at least one point from this set. Corollary 6.4 says that ξ cannot have a finite red set.

Remark 6.7. The authors of [8] studied random connection models on finite point processes in an asymptotic setting. Under a natural irreducibility assumption (similar to Proposition 5.1 (iv)) they proved uniqueness of the giant component; see Theorem 3.6 and Example 4.9 in [8].

Remark 6.8. Consider Example 5.4 in the special case where \mathbb{Y} is the hyperbolic plane and $\nu = t\mu_2$, where μ_2 is the invariant measure on \mathbb{Y} and $t > 0$ is an intensity parameter. Let \mathbb{Q} be concentrated on a single positive radius and let ξ be RCM with connection function $\varphi(K, L) = \mathbf{1}\{K \cap L \neq \emptyset\}$. Then ξ describes a hyperbolic Boolean model of balls. Inspired by the seminal paper [5], it was shown in [42] that there are numbers $0 < t_c < t_u < \infty$ such that there is no percolation for $t \in (0, t_c]$, infinitely many infinite clusters for $t \in (t_c, t_u)$ and a unique infinite cluster for $t \geq t_u$. Corollary 6.4 shows that the unique infinite cluster cannot be destroyed by the removal of a finite number of Poisson points. A referee of this paper has asked whether it is possible to check (resp. to reject) deletion stability for certain values of t which are then (by Theorem 6.1) upper (resp. lower) bounds for t_u . We do not know whether this is possible; see also Remark 11.9.

7 A spatial Markov property

We again consider a general RCM ξ based on a Poisson process η on \mathbb{X} with diffuse intensity measure λ . Let $v \in \mathbb{X}$. In the next section we shall establish and exploit a useful explicit change of measure for the distribution of $C^v = C^v(\xi^v)$. This is possible since for $n \in \mathbb{N}_0$ the conditional distribution of C_{n+1}^v given $C_{\leq n}^v$ can be described in terms of a RCM driven by Poisson process with a thinned intensity measure. In this section we derive a general version of this *spatial Markov property*.

Let ν be a locally finite and diffuse measure on \mathbb{X} . Then we denote by Π_ν the distribution of a Poisson process with this intensity measure. We define two kernels from \mathbb{N} to \mathbb{X} and from $\mathbb{N} \times \mathbb{N}$ to \mathbb{X} (using the same notation K_ν for simplicity), by

$$K_\nu(\mu, dx) := \bar{\varphi}(\mu, x)\nu(dx), \quad K_\nu(\mu, \mu', dx) := \bar{\varphi}(\mu, x)\varphi(\mu', x)\nu(dx), \quad (7.1)$$

where we recall the definitions (3.7). Proposition 7.2 will provide an interpretation of this kernel. Denoting by 0 the zero measure, we note that

$$K_\nu(0, dx) = \nu(dx), \quad K_\nu(0, \mu', dx) = \varphi(\mu', x)\nu(dx), \quad K_\nu(\mu, 0, dx) = 0. \quad (7.2)$$

We write $K_\nu(\mu) := K_\nu(\mu, \cdot)$ and $K_\nu(\mu, \mu') := K_\nu(\mu, \mu', \cdot)$. Note that $K_\lambda(0, \mu, \mathbb{X}) = \varphi_\lambda(\mu)$; see (3.7).

For $n \in \mathbb{N}_0$, $\mu \in \mathbf{G}$ and $v \in \mathbb{X}$ let $\Gamma_n^v(\lambda, \mu, \cdot)$ denote the distribution of a random graph ξ_n defined as follows. Let ξ'_n be a RCM based on $\eta_n + C_n^v(\mu)$, where η_n is a Poisson process with intensity measure $K_\lambda(C_{\leq n-1}^v(\mu))$, and where we recall that $C_{\leq -1}^v := 0$. Remove in ξ'_n all edges between vertices from $C_n^v(\mu)$ to obtain a random graph ξ''_n . Finally set $\xi_n := C_{\leq n}^v(\mu) \oplus \xi''_n$, with an obvious definition of the operation \oplus . We set $C_{\leq 0}^v(\mu) := \delta_v$, which is the graph with vertex set $\{v\}$ and no edges.

Theorem 7.1. *Let $v \in \mathbb{X}$ and $n \in \mathbb{N}_0$. Then,*

$$\mathbb{P}(\xi^v \in \cdot \mid C_{\leq n}^v) = \Gamma_n^v(\lambda, C_{\leq n}^v, \cdot), \quad \mathbb{P}\text{-a.s.} \quad (7.3)$$

Proof. This follows from the proof of [21, Lemma 3.3]; see also Proposition 2 in [31]. Essentially the assertion is equivalent to equation (3.6) in this proof. The arguments given there apply to a RCM on a general state space \mathbb{X} and not only to \mathbb{R}^d . \square

A quick consequence of Theorem 7.1 is that $\{(V(C_{\leq n-1}^v), V(C_n^v))\}_{n \in \mathbb{N}_0}$ is a Markov process.

Proposition 7.2. *The sequence $\{(V(C_{\leq n-1}^v), V(C_n^v))\}_{n \in \mathbb{N}_0}$ is a Markov process with transition kernel*

$$(\mu, \mu') \mapsto \int \mathbf{1}\{(\mu + \mu', \psi) \in \cdot\} \Pi_{K_\lambda(\mu, \mu')}(d\psi).$$

We also note that

$$K_\lambda(\mu, \mu', \mathbb{X}) \leq \int \varphi(\mu', x) \lambda(dx) \leq \iint \varphi(y, x) \mu'(dy) \lambda(dx),$$

where we have used the Bernoulli inequality. Hence

$$K_\lambda(\mu, \mu', \mathbb{X}) \leq \int D_\varphi(y) \mu'(dy). \quad (7.4)$$

Corollary 7.3. *Let $n \in \mathbb{N}_0$. Then we have for λ -a.e. $v \in \mathbb{X}$ that $\mathbb{P}(|C_n^v| < \infty) = 1$.*

Proof. We can proceed by induction. For $n = 0$ the assertion is trivial. Assume that $\mathbb{P}(|C_n^v| < \infty) = 1$ for some $n \in \mathbb{N}_0$. From Proposition 7.2 we know that the conditional distribution of $V(C_{n+1}^v)$ given $(V(C_{\leq n-1}^v), V(C_n^v))$ is that of a Poisson process with intensity measure $K_\lambda(V(C_{\leq n-1}^v), V(C_n^v))$. By (7.4) we obtain that

$$\mathbb{E}[|C_{n+1}^v| \mid (V(C_{\leq n-1}^v), V(C_n^v))] \leq \int D_\varphi(y) C_n^v(dy)$$

which is for λ -a.e. $v \in \mathbb{X}$ a.s. finite by our general assumption (1.1) and induction hypothesis. \square

The following useful property of the kernel K_λ can easily be proved by induction.

Lemma 7.4. *Let $n \in \mathbb{N}$ and $\mu_0, \dots, \mu_n \in \mathbb{N}$. Then*

$$K_\lambda(0, \mu_0) + K_\lambda(\mu_0, \mu_1) + \dots + K_\lambda(\mu_0 + \dots + \mu_{n-1}, \mu_n) = K_\lambda(0, \mu_0 + \dots + \mu_n).$$

8 Perturbation formulas

In the next sections we vary the intensity measure λ and consider $t\lambda$ for $t \in \mathbb{R}_+$. We fix $v \in \mathbb{X}$ and let \mathbb{P}_t be a probability measure governing a RCM ξ based on η , where η is a Poisson process with intensity measure $t\lambda$. The associated expectation is denoted by \mathbb{E}_t . Recall the definition (3.7).

Lemma 8.1. *Let $\tilde{\xi}$ be a RCM based on a Poisson process $\tilde{\eta}$ with finite intensity measure ν . Let $f: \mathbf{G} \rightarrow [0, \infty)$. Then*

$$\mathbb{E}_t f(\tilde{\xi}) = \mathbb{E}_1 f(\tilde{\xi}) t^{|\tilde{\eta}|} e^{(1-t)\nu(\mathbb{X})}$$

Proof. It is well-known that

$$\Pi_{t\nu} = \int \mathbf{1}\{\mu \in \cdot\} t^{|\mu|} e^{(1-t)\nu(\mathbb{X})} \Pi_\nu(d\mu), \quad t \geq 0. \quad (8.1)$$

This follows, for instance from [27, Exercise 3.7] and an easy calculation. The assertion then follows by conditioning, using the kernel Γ in (3.2). \square

Proposition 8.2. *Let $v \in \mathbb{X}$, $t \in \mathbb{R}_+$, $n \in \mathbb{N}$ and $t_0 > 0$. Then*

$$\mathbb{P}_t(C_{\leq n}^v \in \cdot) = \mathbb{E}_{t_0} \mathbf{1}\{C_{\leq n}^v \in \cdot\} (t/t_0)^{|C_{\leq n}^v| - 1} e^{(t_0 - t)\varphi_\lambda(C_{\leq n-1}^v)}.$$

Proof. It is sufficient to consider the special case $t_0 = 1$. The general case can be proved similarly or can be derived from the special case. We omit the dependence on v in our notation by writing $C_n := C_n^v$, and $C_{\leq n} := C_{\leq n}^v$. Given $\mu \in \mathbf{G}$ we let $C_n^+(\mu)$ denote the graph $\mu[V(C_{n-1}(\mu)) + V(C_n(\mu))]$ with the edges between vertices of $C_{n-1}(\mu)$ removed.

Let $f: \mathbf{G} \rightarrow [0, \infty)$ be measurable. By Theorem 7.1,

$$\mathbb{E}_t f(C_{\leq n}) = \mathbb{E}_t \int f(C_{\leq n-1} \oplus C_n^+(\mu)) \Gamma_{n-1}(t\lambda, C_{\leq n-1}, d\mu).$$

By (7.4) we have

$$K_{t\lambda}(C_{\leq n-2}, C_{n-1}, \mathbb{X}) \leq t \int D_\varphi(y) C_{n-1}(dy)$$

which is almost surely finite by Corollary 7.3. By definition of Γ_n and the thinning properties of a Poisson process, the distribution of $C_n^+(\cdot)$ under $\Gamma_{n-1}(t\lambda, C_{\leq n-1}, \cdot)$ is that of a RCM driven by a Poisson process with intensity measure $K_{t\lambda}(C_{\leq n-2}, C_{n-1})$ with additional independent connections to $V(C_{n-1})$; see also Proposition 7.2. Therefore we obtain from Lemma 8.1 that

$$\mathbb{E}_t f(C_{\leq n}) = \mathbb{E}_t \int f(C_{\leq n-1} \oplus C_n^+(\mu)) e^{(1-t)K_\lambda(C_{\leq n-2}, C_{n-1}, \mathbb{X})} t^{|\mu|} \Gamma_{n-1}(\lambda, C_{\leq n-1}, d\mu).$$

Iterating this identity yields that the above equals

$$\begin{aligned} & \int \cdots \int f(C_1^+(\mu_1) \oplus \cdots \oplus C_n^+(\mu_n)) e^{(1-t)K_\lambda(\delta_v + \mu_1 + \cdots + \mu_{n-2}, \mu_{n-1}, \mathbb{X})} \cdots e^{(1-t)K_\lambda(\delta_v, \mu_1, \mathbb{X})} \\ & t^{|\mu_n|} \cdots t^{|\mu_1|} \Gamma_{n-1}(\lambda, C_1^+(\mu_1) \oplus \cdots \oplus C_{n-1}^+(\mu_{n-1}), d\mu_n) \cdots \Gamma_0(\lambda, \delta_v, d\mu_1). \end{aligned}$$

By Lemma 7.4 this equals

$$\begin{aligned} & \int \cdots \int f(C_1^+(\mu_1) \oplus \cdots \oplus C_n^+(\mu_n)) e^{(1-t)K_\lambda(0, \delta_v + \mu_1 + \cdots + \mu_{n-1}, \mathbb{X})} t^{|\mu_1| + \cdots + |\mu_n|} \\ & \Gamma_{n-1}(\lambda, C_1^+(\mu_1) \oplus \cdots \oplus C_{n-1}^+(\mu_{n-1}), d\mu_n) \cdots \Gamma_0(\lambda, [\delta_v], d\mu_1). \end{aligned}$$

By Theorem 7.1 we obtain

$$\mathbb{E}_t f(C_{\leq n}) = \mathbb{E}_1 f(C_{\leq n}) t^{|C_{\leq n}| - 1} e^{(1-t)\varphi_\lambda(C_{\leq n-1})}$$

and hence the assertion. \square

Theorem 8.3. Let $v \in \mathbb{X}$, $t \in \mathbb{R}_+$ and $t_0 > 0$. Then

$$\mathbb{P}_t(C^v \in \cdot, |C^v| < \infty) = \mathbb{E}_{t_0} \mathbf{1}\{C^v \in \cdot, |C^v| < \infty\} (t/t_0)^{|C^v| - 1} e^{(t_0 - t)\varphi_\lambda(C^v)}. \quad (8.2)$$

Proof. Again it is sufficient to consider the special case $t_0 = 1$. By Proposition 8.2 the distribution $\mathbb{P}_t(C_n^v \in \cdot)$ is absolutely continuous w.r.t. $\mathbb{P}_1(C_n^v \in \cdot)$ with Radon-Nikodym derivative $M_n^v := t^{|C_{\leq n}^v| - 1} e^{(1-t)\varphi_\lambda(C_{\leq n-1}^v)}$. In particular $\{M_n^v\}_{n \in \mathbb{N}_0}$ is a (non-negative) martingale with respect to $\{\sigma(C_{\leq n}^v)\}_{n \in \mathbb{N}_0}$ and therefore converges a.s. towards $M_\infty^v := \limsup_{n \rightarrow \infty} M_n^v$. By [40, Theorem VII.6.1] we have

$$\mathbb{P}_t(C^v \in \cdot) = \mathbb{E}_1 \mathbf{1}\{C^v \in \cdot\} M_\infty^v + \mathbb{E}_t \mathbf{1}\{C^v \in \cdot, M_\infty^v = \infty\}.$$

On the event $\{|C^v| < \infty\}$ we clearly have

$$M_\infty^v = t^{|C^v| - 1} e^{(1-t)\varphi_\lambda(C^v)}$$

which is finite. This concludes the assertion. \square

Let $f: \mathbf{G} \rightarrow \mathbb{R}$ be a measurable mapping. Define for $v \in \mathbb{X}$, $n \in \mathbb{N}$, and $t \in \mathbb{R}_+$

$$F_n(\xi^v) := f(C^v) \mathbf{1}\{|C^v| = n\}, \quad f_n^v(t) := \mathbb{E}_t F_n(\xi^v), \quad (8.3)$$

$$F_{\leq n}(\xi^v) := f(C^v) \mathbf{1}\{|C^v| \leq n\}, \quad f_{\leq n}^v(t) := \mathbb{E}_t F_{\leq n}(\xi^v), \quad (8.4)$$

$$F(\xi^v) := f(C^v) \mathbf{1}\{|C^v| < \infty\}, \quad f^v(t) := \mathbb{E}_t F(\xi^v). \quad (8.5)$$

We also write $|f|^v(t) := \mathbb{E}_t |F(\xi^v)|$ and define $|f|_n^v(t)$ and $|f|_{\leq n}^v(t)$ similarly. We are interested in the analytic properties of the function $f^v(t)$ under the assumption $|f|^v(t) < \infty$. A key example is the *position dependent cluster density*

$$\kappa^v(t) := \mathbb{E}_t |C^v|^{-1}, \quad t \in \mathbb{R}_+. \quad (8.6)$$

Our terminology is motivated by the stationary marked case (see Lemma 4.3) and also supported by the Mecke equation, implying

$$\int t \kappa^v(t) \lambda_B(dv) = \mathbb{E}_t \int |C^v(\xi)|^{-1} \eta_B(dv), \quad B \in \mathcal{X}.$$

Suppose that $|f|_n^v(t_0) < \infty$ for some $t_0 > 0$ and $n \in \mathbb{N}$, then by Theorem 8.3

$$f_n^v(t) = \left(\frac{t}{t_0}\right)^{n-1} \int_0^\infty e^{-tu} \nu_{f,n,t_0}(du), \quad (8.7)$$

where the signed measure ν_{f,n,t_0} is defined by

$$\nu_{f,n,t_0}(\cdot) := \mathbb{E}_{t_0} \mathbf{1}\{\varphi_\lambda(C^v) \in \cdot\} e^{t_0 \varphi_\lambda(C^v)} F_n(\xi^v). \quad (8.8)$$

By Corollary 7.3 this is a locally finite signed measure on \mathbb{R}_+ . It follows from Theorem 8.3 that the function $|f|_n^v(t)/t^{n-1}$ is monotone decreasing on $(0, \infty)$, so that

$$|f|_n^v(t) \leq \left(\frac{t}{t_0}\right)^{n-1} |f|_n^v(t_0), \quad t \geq t_0.$$

Lemma 8.4. *Let $v \in \mathbb{X}$, $n \in \mathbb{N}$ and $t_0 > 0$. If $|f|_n^v(t_0) < \infty$, then for $t \geq t_0$*

$$f_n^v(t) = \frac{t^n}{t_0^{n-1}} \int_0^\infty \nu_{f,n,t_0}[0, u] e^{-tu} du.$$

Proof. We obtain from (8.7) that

$$f_n^v(t) = \frac{t^n}{t_0^{n-1}} \iint \mathbf{1}\{u \leq s\} e^{-ts} ds \nu_{f,n,t_0}(du). \quad (8.9)$$

Since ν_{f,n,t_0} is locally finite, we can apply Fubini's theorem to obtain the assertion. \square

Lemma 8.5. *Let $v \in \mathbb{X}$, $n \in \mathbb{N}$ and $t_0 > 0$. If $|f|_n^v(t_0) < \infty$, then the function f_n^v is analytic on (t_0, ∞) and for $t > t_0$*

$$\frac{d}{dt} f_n^v(t) = \frac{nt^{n-1}}{t_0^{n-1}} \int_0^\infty \nu_{f,n,t_0}[0, u] e^{-tu} du - \frac{t^n}{t_0^{n-1}} \int_0^\infty u \nu_{f,n,t_0}[0, u] e^{-tu} du. \quad (8.10)$$

Proof. Let $\Omega_{t_0} := \{z \in \mathbb{C} : \Re(z) > t_0\}$, and extend f_n^v to Ω_{t_0} by setting

$$f_n^v(z) := \frac{z^n}{t_0^{n-1}} \int_0^\infty \nu_{f,n,t_0}[0, u] e^{-zu} du, \quad z \in \Omega_{t_0}.$$

By (8.8) we have

$$|\nu_{f,n,t_0}[0, u]| \leq e^{t_0 u} |f|_n^v(t_0).$$

Since $|f|_n^v(t_0) < \infty$ this implies that f_n^v is a complex analytic function on Ω_{t_0} . Since $(t_0, \infty) \subset \Omega_{t_0} \cap \mathbb{R}$, the restriction of this function to (t_0, ∞) is real analytic. The formula (8.10) follows from Lemma 8.4 the product rule of calculus and the Leibniz rule for differentiating integrals. The latter can be applied since for each $\varepsilon > 0$ and all $u > 0$

$$u|\nu_{f,n,t_0}[0, u]|e^{-tu} \leq ue^{(t_0-t)u} |f|_n^v(t_0) \leq ue^{-\varepsilon u} |f|_n^v(t_0)$$

uniformly for $t \geq t_0 + \varepsilon$. \square

To rewrite Lemma 8.5 in a different way, we define

$$M_t^v := |C^v| - 1 - t\varphi_\lambda(C^v), \quad t \in \mathbb{R}_+, v \in \mathbb{X}. \quad (8.11)$$

Lemma 8.6. *Let $v \in \mathbb{X}$, $n \in \mathbb{N}$ and $t_0 > 0$. If $|f|_n^v(t_0) < \infty$, then function f_n^v is analytic on (t_0, ∞) and for $t > t_0$*

$$\frac{d}{dt} f_n^v(t) = t^{-1} \mathbb{E}_t [M_t^v F_n(\xi^v)]. \quad (8.12)$$

Proof. By Theorem 8.3,

$$f_n^v(t) = \left(\frac{t}{t_0}\right)^{n-1} \mathbb{E}_{t_0} [F_n(\xi^v) e^{(t_0-t)\varphi_\lambda(C^v)}].$$

Hence the result follows from Lemma 8.5 and calculus, where the application of the Leibniz differentiation rule can be justified as in the proof of Lemma 8.5. \square

Lemma 8.7. *Let $v \in \mathbb{X}$, $n \in \mathbb{N}$ and $t_0 > 0$. If $|f|_{\leq n}^v(t_0) < \infty$, then function $f_{\leq n}^v$ is analytic on (t_0, ∞) and for $t > t_0$*

$$\frac{d}{dt} f_{\leq n}^v(t) = t^{-1} \mathbb{E}_t [M_t^v F_{\leq n}(\xi^v)]. \quad (8.13)$$

Proof. The result follows from the definition of $f_{\leq n}^v$ and Lemma 8.6, since $|f|_{\leq n}^v(t_0) = \sum_{k=1}^n |f|_k^v(t_0)$. \square

Theorem 8.8. *Let $0 < t_0 < t_1 < \infty$. Assume for each $t \in [t_0, t_1]$ that $|f|^v(t) < \infty$. Assume moreover that for each $\varepsilon > 0$,*

$$\lim_{n \rightarrow \infty} \sup_{t \in [t_0 + \varepsilon, t_1]} \left| \sum_{k > n} \frac{d}{dt} f_k^v(t) \right| = 0. \quad (8.14)$$

Then f^v is continuously differentiable on $(t_0, t_1]$ with derivative given by

$$\frac{d}{dt} f^v(t) = \lim_{n \rightarrow \infty} t^{-1} \frac{d}{dt} f_{\leq n}^v(t) = t^{-1} \sum_{n=1}^{\infty} \frac{d}{dt} f_n^v(t). \quad (8.15)$$

Proof. Let $n \in \mathbb{N}$. Since $|f|_{\leq n}^v(t_0) \leq |f|^v(t_0) < \infty$, we can apply Lemma 8.7 to obtain that the function $f_{\leq n}^v$ is analytic on (t_0, ∞) , with derivative

$$\frac{d}{dt} f_{\leq n}^v(t) = t^{-1} \sum_{k=1}^n \frac{d}{dt} f_k^v(t) = t^{-1} \mathbb{E}_t [M_t^v F_{\leq n}(\xi^v)], \quad t > t_0.$$

By dominated convergence

$$\lim_{n \rightarrow \infty} f_{\leq n}^v(t) = f^v(t).$$

Furthermore we have

$$\left| \frac{d}{dt} f^v(t) - \frac{d}{dt} f_{\leq n}^v(t) \right| = t^{-1} \left| \sum_{k > n} \frac{d}{dt} f_k^v(t) \right|.$$

By assumption (8.14) this tends to zero uniformly in $t \in [t_0 + \varepsilon, t_1]$ for each $\varepsilon > 0$. A standard result of analysis gives us that f^v is continuously differentiable on $(t_0, t_1]$ with derivative given by the right-hand side of (8.15). \square

Theorem 8.9. *Let $0 < t_0 < t_1 < \infty$. Assume for each $t \in [t_0, t_1]$ that $|f|^v(t) < \infty$. Assume moreover that for each $\varepsilon > 0$,*

$$\lim_{n \rightarrow \infty} \sup_{t \in [t_0 + \varepsilon, t_1]} \sum_{k > n} \mathbb{E}_t |M_t^v F_k(\xi^v)| = 0. \quad (8.16)$$

Then f^v is continuously differentiable on $(t_0, t_1]$ with derivative given by

$$\frac{d}{dt} f^v(t) = t^{-1} \mathbb{E}_t [M_t^v F(\xi^v)]. \quad (8.17)$$

Proof. Let $n \in \mathbb{N}$. Since $|f|_n^v(t_0) \leq |f|_{\leq n}^v(t_0) \leq |f|^v(t_0) < \infty$, then by Lemma 8.5 the function f_n^v is analytic on (t_0, ∞) , with derivative

$$\left| \frac{d}{dt} f_n^v(t) \right| = t^{-1} |\mathbb{E}_t [M_t^v F_n(\xi^v)]| \leq t^{-1} \mathbb{E}_t |M_t^v F_n(\xi^v)|, \quad t > t_0.$$

Hence

$$\left| \sum_{k > n} \frac{d}{dt} f_k^v(t) \right| \leq t^{-1} \sum_{k > n} \mathbb{E}_t |M_t^v F_k(\xi^v)|.$$

By assumption (8.16) this tends to zero uniformly in $t \in [t_0 + \varepsilon, t_1]$ for each $\varepsilon > 0$. Therefore by Theorem 8.8 the function $f^v(t)$ is continuously differentiable on $(t_0, t_1]$ with derivative

$$\frac{d}{dt} f^v(t) = t^{-1} \sum_{n=1}^{\infty} \frac{d}{dt} f_n^v(t) = t^{-1} \mathbb{E}_t [M_t^v F(\xi^v)],$$

where the last equality we get from Fubini's theorem, since by assumption (8.16) we have that for $t \in (t_0, t_1]$

$$\mathbb{E}_t |M_t^v F(\xi^v)| = \sum_{n \geq 1} \mathbb{E}_t |M_t^v F_n(\xi^v)| < \infty. \quad \square$$

The following theorem provides a large class of functions satisfying the assumptions of Theorem 8.9, covering the cluster density (8.6). We shall prove it in Section 10.

Theorem 8.10. *Let $f: \mathbf{G} \rightarrow \mathbb{R}$ be a measurable mapping satisfying $|f(\mu)| \leq |\tilde{f}(|V(\mu)|)|$ for each $\mu \in \mathbf{G}$, where $\tilde{f}: \mathbb{N} \rightarrow \mathbb{R}$ satisfies*

$$\lim_{n \rightarrow \infty} \tilde{f}(n) \sqrt{n \log n} = 0. \quad (8.18)$$

Then f^v is for each $v \in \mathbb{X}$ continuously differentiable on $(0, \infty)$ with derivative given by (8.17).

9 Difference operators

In this section we shall rewrite Theorem 8.8 and Theorem 8.9 in the form of a Margulis–Russo formula. Recall that $\mu - \delta_x := \mu[V(\mu) - \delta_x]$ is the graph resulting from μ by removing the point x (if $x \in V(\mu)$) along with all edges with vertex x for $\mu \in \mathbf{G}$ and $x \in \mathbb{X}$. Given a measurable function $f: \mathbf{G} \rightarrow \mathbb{R}$ and $x \in \mathbb{X}$ we define $\nabla_x f: \mathbf{G} \rightarrow \mathbb{R}$ by

$$\nabla_x f(\mu) := f(\mu) - f(\mu - \delta_x). \quad (9.1)$$

Theorem 9.1. *Let the assumptions of Theorem 8.8 be satisfied. Then the function f^v is continuously differentiable on $(t_0, t_1]$ with derivative given by*

$$\frac{d}{dt} f^v(t) = \lim_{n \rightarrow \infty} t^{-1} \mathbb{E}_t \int \nabla_x F_{\leq n}(\xi^v) C^{v!}(dx) = \sum_{n \geq 1} t^{-1} \mathbb{E}_t \int \nabla_x F_n(\xi^v) C^{v!}(dx). \quad (9.2)$$

We start the proof with the counterpart of Lemma 8.5 and Lemma 8.7.

Lemma 9.2. *Let $v \in \mathbb{X}$, $n \in \mathbb{N}$ and $t_0 > 0$. If $|f|_n^v(t_0) < \infty$, then function f_n^v is analytic on (t_0, ∞) and for $t > t_0$*

$$\frac{d}{dt} f_n^v(t) = t^{-1} \mathbb{E}_t \int \nabla_x F_n(\xi^v) C^{v!}(dx).$$

Proof. Let $t > t_0$. We wish to apply Lemma 8.6. By definition we have

$$\mathbb{E}_t |F_n(\xi^v)| |C^v| = n |f|_n^v(t) \leq n \left(\frac{t}{t_0} \right)^{n-1} |f|_n^v(t_0)$$

which is finite by assumption. Therefore we obtain from Theorem 8.3, (8.8) (with $|f|$ instead of f) and Fubini's theorem

$$\begin{aligned} \mathbb{E}_t |F_n(\xi^v)| \varphi_\lambda(C^v) &= \left(\frac{t}{t_0} \right)^{n-1} \int_0^\infty u e^{-tu} \nu_{|f|, n, t_0}(du) \\ &= \left(\frac{t}{t_0} \right)^{n-1} \int_0^\infty \int_u^\infty (ts - 1) e^{-ts} ds \nu_{|f|, n, t_0}(du) \\ &= \left(\frac{t}{t_0} \right)^{n-1} \int_0^\infty \nu_{|f|, n, t_0}[0, s] (ts - 1) e^{-ts} ds \\ &\leq |f|_n^v(t_0) \frac{t^n}{t_0^{n-1}} \int_0^\infty s e^{(t_0-t)s} ds < \infty, \end{aligned}$$

where we have used that $\nu_{|f|, n, t_0}[0, s] \leq e^{t_0 s} |f|_n^v(t_0)$. Hence we obtain from Lemma 3.2 that

$$t \mathbb{E}_t F_n(\xi^v) \varphi_\lambda(C^v) = \mathbb{E}_t \int F_n(\xi^v - \delta_x) C^{v!}(dx).$$

Now the assertion follows from Lemma 8.6. \square

Lemma 9.3. *Let $v \in \mathbb{X}$, $n \in \mathbb{N}$ and $t_0 > 0$. If $|f|_{\leq n}^v(t_0) < \infty$, then function $f_{\leq n}^v$ is analytic on (t_0, ∞) and for $t > t_0$*

$$\frac{d}{dt} f_{\leq n}^v(t) = t^{-1} \mathbb{E}_t \int \nabla_x F_{\leq n}(\xi^v) C^{v!}(dx).$$

Proof. The result follows from the definition of $f_{\leq n}^v$, Lemma 8.7 and Lemma 9.2. \square

Proof of Theorem 9.1. By Theorem 8.8 we have that f^v is continuously differentiable on $(t_0, t_1]$ with derivative given by (8.15). Hence we can apply Lemma 9.2 and Lemma 9.3 to obtain the assertion. \square

Theorem 9.4. *Let the assumptions of Theorem 8.9 be satisfied. Assume moreover that for each $t \in [t_0, t_1]$*

$$\mathbb{E}_t[|F(\xi^v)|(|C^v| + \varphi_\lambda(C^v))] < \infty. \quad (9.3)$$

Then f^v is continuously differentiable on $(t_0, t_1]$ with derivative given by

$$\frac{d}{dt}f^v(t) = t^{-1}\mathbb{E}_t \int \nabla_x F(\xi^v) C^{v!}(dx). \quad (9.4)$$

Proof. Let $t > t_0$. Theorem 8.9 states that f^v is continuously differentiable on $(t_0, t_1]$ with derivative given by (8.17). The assertion follows from (9.3) and Lemma 3.2, since splitting f into its negative and positive part we can apply Lemma 3.2 to get

$$t \mathbb{E}_t F(\xi^v) \varphi_\lambda(C^v) = \mathbb{E}_t \int F(\xi^v - \delta_x) C^{v!}(dx).$$

The result follows. \square

Remark 9.5. Let the assumptions of Theorem 9.4 be satisfied. By the Mecke equation (3.6) we have

$$\mathbb{E}_t \int \nabla_x F(\xi^v) C^{v!}(dx) = t \mathbb{E}_t \int (F(\xi^{v,x}) - F(\xi^v)) \mathbf{1}\{v \leftrightarrow x \text{ in } \xi^{v,x}\} \lambda(dx).$$

If v and x are not connected in $\xi^{v,x}$, then $F(\xi^{v,x}) = F(\xi^v)$. Therefore we can rewrite (9.4) as

$$\frac{d}{dt}f^v(t) = \mathbb{E}_t \int (F(\xi^{v,x}) - F(\xi^v)) \lambda(dx). \quad (9.5)$$

10 Differentiability of the cluster density

In this section we prove in particular that the position dependent cluster density (given by (8.6)) is continuously differentiable on $(0, \infty)$.

Theorem 10.1. *Suppose that $f: \mathbb{N} \rightarrow \mathbb{R}$ is a function satisfying*

$$\lim_{n \rightarrow \infty} f(n) \sqrt{n \log n} = 0. \quad (10.1)$$

Then $t \mapsto \mathbb{E}_t f(|C^v|)$ is for each $v \in \mathbb{X}$ continuously differentiable on $(0, \infty)$ with derivative given by (8.17).

We prove the theorem via some lemmas, partially following the proof of [7, (LP) (3.6)]. Let $v \in \mathbb{X}$. For $t > 0$ and $n \in \mathbb{N}$ we define

$$p_n^v(t) := \mathbb{P}_t(|C^v| = n).$$

Specializing definition (8.8) in the case $f \equiv 1$ we set

$$\nu_n^v(\cdot) := \mathbb{E}_1 \mathbf{1}\{\varphi_\lambda(C^v) \in \cdot\} \mathbf{1}\{|C^v| = n\} e^{\varphi_\lambda(C^v)}. \quad (10.2)$$

Then we obtain from (8.7) in the case $t_0 = 1$ that

$$p_n^v(t) = t^{n-1} \int_0^\infty e^{-tu} \nu_n^v(du). \quad (10.3)$$

Since $p_1^v(t) = e^{-tD_\varphi(v)}$ we have

$$\nu_1^v = \delta_{D_\varphi(v)}, \quad v \in \mathbb{X}. \quad (10.4)$$

Lemma 10.2. Let $v \in \mathbb{X}$, $n \in \mathbb{N}$ and $u > 0$. Then

$$\nu_n^v[0, u] \leq \left(\frac{eu}{n-1} \right)^{n-1}, \quad (10.5)$$

where the right-hand side has to be interpreted as 1 if $n = 1$.

Proof. In view of (10.4) we can assume that $n \geq 2$. Since $p_n^v(t) \leq 1$ for $t > 0$, we have that

$$t^{-n+1} \geq \int_0^u e^{-tu} \nu_n^v(du) \geq e^{-tu} \nu_n^v[0, u].$$

Optimizing over $t \in (0, \infty)$ yields the assertion. \square

Lemma 10.3. Let $v \in \mathbb{X}$, $n \in \mathbb{N}$ and $t > 0$. Then

$$p_n^v(t) = t^n \int_0^\infty \nu_n^v[0, u] e^{-tu} du.$$

Proof. The assertion follows from Lemma 8.4. \square

Lemma 10.4. Let $v \in \mathbb{X}$, $n \in \mathbb{N}$. Then $t \mapsto p_n^v(t)$ is analytic on $(0, \infty)$ with derivative given by

$$\frac{d}{dt} p_n^v(t) = nt^{n-1} \int_0^\infty \nu_n^v[0, u] e^{-tu} du - t^n \int_0^\infty u \nu_n^v[0, u] e^{-tu} du \quad (10.6)$$

Proof. The assertion follows from Lemma 8.5. \square

Lemma 10.4 implies

$$\left| \frac{d}{dt} p_n^v(t) \right| \leq \frac{n}{t} \int_0^\infty \nu_n^v[0, u] \left| 1 - \frac{ut}{n} \right| t^n e^{-tu} du. \quad (10.7)$$

The next lemma provides a bound for the above right-hand side.

Lemma 10.5. Let $v \in \mathbb{X}$, $n \geq 2$ and $\delta \in (0, 1)$. Then we have for all $t > 0$ that

$$\int_0^\infty \nu_n^v[0, u] \left| 1 - \frac{ut}{n} \right| t^n e^{-tu} du \leq \delta p_n^v(t) + (1 - \delta)^n e^{\delta n} + (1 + \delta)^n e^{-\delta n}. \quad (10.8)$$

Proof. By Lemma 10.3 we have

$$\begin{aligned} \int_0^\infty \nu_n^v[0, u] \left| 1 - \frac{ut}{n} \right| t^n e^{-tu} du &\leq \delta p_n^v(t) + \int_{|1 - \frac{tu}{n}| > \delta} \nu_n^v[0, u] t^n e^{-tu} \left| 1 - \frac{tu}{n} \right| du \\ &\leq \delta p_n^v(t) + \left(\frac{e}{n-1} \right)^{n-1} \int_{|1 - \frac{tu}{n}| > \delta} t^n u^{n-1} e^{-tu} \left| 1 - \frac{tu}{n} \right| du. \end{aligned}$$

Changing variables yields that the above equals

$$\delta p_n^v(t) + \left(\frac{e}{n-1} \right)^{n-1} \int_{|1 - \frac{u}{n}| > \delta} u^{n-1} e^{-u} \left| 1 - \frac{u}{n} \right| du.$$

Splitting the integral on the above right-hand side into two pieces corresponding to $tu < n(1 - \delta)$ and $tu > n(1 + \delta)$ yields

$$\begin{aligned} \int_0^{n(1-\delta)} u^{n-1} e^{-u} \left(1 - \frac{u}{n} \right) du &= n^{n-1} (1 - \delta)^n e^{-n(1-\delta)}, \\ \int_{n(1+\delta)}^\infty u^{n-1} e^{-u} \left(\frac{u}{n} - 1 \right) du &= n^{n-1} (1 + \delta)^n e^{-n(1+\delta)}. \end{aligned}$$

Since $(1 + 1/(n-1))^{n-1} < e$ for all $n \geq 2$, we obtain the assertion (10.8). \square

Let f be as in Theorem 10.1 and $v \in \mathbb{X}$. Then f is bounded and

$$\mathbb{E}_t f(|C^v|) = \mathbb{E}_t F(\xi^v) = \sum_{n=1}^{\infty} f(n) p_n^v(t).$$

In order to prove Theorem 10.1 we will check the condition (8.16) on $[t_0, \infty)$ for each $t_0 > 0$. This is achieved by the previous and the following lemma.

Lemma 10.6. Suppose that $f: \mathbb{N} \rightarrow \mathbb{R}$. Then

$$\mathbb{E}_t |M_t^v F_n(\xi^v)| \leq |f(n)| \int_0^\infty \nu_n^v[0, u] |n - ut| t^n e^{-tu} du. \quad (10.9)$$

Proof. It is easy to see the following identities which follow from integration by parts

$$\begin{aligned} \int_u^{(n-1)/t} (n - ts) e^{-ts} ds &= -t^{-1} (n - ts) e^{-ts} \Big|_u^{(n-1)/t} - \int_u^{(n-1)/t} e^{-ts} ds \\ &= t^{-1} \left((n - tu) e^{-tu} \pm e^{-(n-1)} - e^{-tu} \right) = t^{-1} (n - 1 - tu) e^{-tu}, \\ \int_u^\infty (n - ts) e^{-ts} ds &= t^{-1} (n - tu) e^{-tu} - \int_u^\infty e^{-ts} ds = t^{-1} (n - 1 - tu) e^{-tu}. \end{aligned}$$

Since ν_n^v is locally finite, we can apply Fubini's theorem to obtain that

$$\begin{aligned} &\mathbb{E}_t |M_t^v F_n(\xi^v)| \\ &= |f(n)| \mathbb{E}_t [|n - 1 - t\varphi_\lambda(C^v)| \mathbf{1}\{|C^v| = n\}] = |f(n)| t^{n-1} \int_0^\infty |n - 1 - tu| e^{-tu} \nu_n^v(du) \\ &= |f(n)| t^{n-1} \left(\int_0^{(n-1)/t} (n - 1 - tu) e^{-tu} \nu_n^v(du) - \int_{(n-1)/t}^\infty (n - 1 - tu) e^{-tu} \nu_n^v(du) \right) \\ &= |f(n)| t^n \left(\int_0^{(n-1)/t} \int_u^{(n-1)/t} (n - ts) e^{-ts} ds \nu_n^v(du) - \int_{(n-1)/t}^\infty \int_u^\infty (n - ts) e^{-ts} ds \nu_n^v(du) \right) \\ &= |f(n)| t^n \left(\int_0^{(n-1)/t} \nu_n^v[0, s] (n - ts) e^{-ts} ds - \int_{(n-1)/t}^\infty \nu_n^v[0, s] (n - ts) e^{-ts} ds \right) \\ &= |f(n)| \int_0^\infty \nu_n^v[0, s] |n - st| t^n e^{-ts} ds - 2|f(n)| \int_{(n-1)/t}^{n/t} \nu_n^v[0, s] (n - ts) t^n e^{-ts} ds. \quad \square \end{aligned}$$

Proof of Theorem 10.1. Let $v \in \mathbb{X}$, $t_0 > 0$ and $n \geq 2$. We need to check the condition (8.16). To do so, we start with inequality (10.9). In (10.8) we choose $\delta \equiv \delta_n$ by $\delta_n := \sqrt{9 \log n/n}$. We use the inequalities $(1 - r)e^r \leq e^{-r^2/2}$ which holds for all $r \geq 0$ and $(1 + r)e^{-r} \leq e^{-r^2/3}$ which holds for all $r \in [0, 1/3]$. Then we obtain for all sufficiently large $n \in \mathbb{N}$ that $(1 - \delta_n)^n e^{-n\delta_n} \leq n^{-9/2}$ and $(1 - \delta_n)^n e^{-n\delta_n} \leq n^{-3}$. Hence there exist $n_0 \in \mathbb{N}$ such that for each $t \geq t_0$

$$\sum_{n=n_0}^\infty t^{-1} |f(n)| \int_0^\infty \nu_n^v[0, u] |n - ut| t^n e^{-tu} du \leq \frac{\sqrt{9}}{t_0} \sum_{n=n_0}^\infty |f(n)| \sqrt{n \log n} p_n^v(t) + \frac{2}{t_0} \sum_{n=n_0}^\infty n^{-2}. \quad (10.10)$$

Let $\varepsilon > 0$ and choose $n_1 \geq n_0$ such that $|f(n)| \sqrt{n \log n} \leq \varepsilon$ for each $n \geq n_1$. Then

$$\sum_{n=n_1}^\infty |f(n)| \sqrt{n \log n} p_n^v(t) \leq \varepsilon,$$

finishing the proof. \square

Proof of Theorem 8.10. We check condition (8.16). By assumption (8.18) it suffices to show that

$$\lim_{n \rightarrow \infty} \sup_{t \geq t_0} \sum_{k > n} |\tilde{f}(k)| \mathbb{E}_t |M_t^v| \mathbf{1}\{|C^v| = k\} = 0,$$

for any $t_0 > 0$. This follows from (10.9) and the proof of Theorem 10.1. \square

Later we shall need the following integrated version of Theorem 10.1

Theorem 10.7. Assume that $\mathbb{X} = \mathbb{Y} \times \mathbb{M}$ is the product of two complete separable metric spaces and let \mathbb{Q} be a finite measure \mathbb{Q} on \mathbb{M} . Suppose that $f: \mathbb{N} \rightarrow \mathbb{R}$ is a function satisfying (10.1). Then $t \mapsto \int \mathbb{E}_t f(|C^{(y,q)}|) \mathbb{Q}(dq)$ is for each $y \in \mathbb{Y}$ continuously differentiable on $(0, \infty)$.

Proof. Let $y \in \mathbb{Y}$ and $t_0 > 0$. We know from Theorem 10.1 that $t \mapsto \mathbb{E}_t f(|C^{(y,q)}|)$ is for each $(y, q) \in \mathbb{Y} \times \mathbb{M}$ continuously differentiable. The assertion follows from the Leibniz differentiation rule once we can show that

$$\sum_{n=1}^{\infty} |f(n)| \int_0^{\infty} \nu_n^{(y,q)}[0, u] |n - ut| t^n e^{-tu} du \leq c, \quad t \geq t_0, q \in \mathbb{M}, \quad (10.11)$$

for some $c > 0$. Since $f(n)\sqrt{n \log n}$ is bounded, we see from (10.10) that the above series, starting from $n = n_0$ is bounded in $q \in \mathbb{M}$ and $t \geq t_0$. The remaining terms in the series can be bounded by (10.8). Similarly as in the proof of Lemma 8.5 one can show that $\int f_n^{(y,q)}(t) \mathbb{Q}(dq)$ is an analytic function on $(0, \infty)$. Therefore the continuity of the derivative follows from (10.10), since $\sum_{n \geq n_0} |f(n)| \sqrt{n \log n} \int p_n^{(y,q)}(t) \mathbb{Q}(dq) \rightarrow 0$ as $n_0 \rightarrow \infty$ uniformly in $t \in \mathbb{R}_+$. \square

Strengthening the assumption on the function f in Theorem 10.7, we can write the derivative as a Margulis–Russo type formula.

Theorem 10.8. Suppose that $f: \mathbb{N} \rightarrow \mathbb{R}$ is a function satisfying

$$\sup_{n \geq 1} |f(n)|n < \infty. \quad (10.12)$$

Then $t \mapsto \mathbb{E}_t f(|C^v|)$ is for each $v \in \mathbb{X}$ continuously differentiable on $(0, \infty)$ with derivative given by (9.4).

Proof. It is enough to check condition (9.3) on $[t_0, \infty)$ for each $t_0 > 0$. Condition (10.12) implies that

$$\mathbb{E}_t |f(|C^v|)| |C^v| < \infty.$$

It follows from Fubini's theorem and Lemma 10.2 that for $n \geq 2$

$$\begin{aligned} \mathbb{E}_t |F_n(\xi^v)| \varphi_\lambda(C^v) &= |f(n)| t^{n-1} \int_0^{\infty} u e^{-tu} \nu_n^v(du) = |f(n)| t^{n-1} \int_0^{\infty} \int_u^{\infty} (ts - 1) e^{-ts} ds \nu_n^v(du) \\ &= |f(n)| t^{n-1} \int_0^{\infty} \nu_n^v[0, s] (ts - 1) e^{-ts} ds < |f(n)| t^n \int_0^{\infty} \nu_n^v[0, s] s e^{-ts} ds \\ &\leq |f(n)| \left(2np_n^v(t) + t^n \int_{2n}^{\infty} \nu_n^v[0, s] s e^{-ts} ds \right) \\ &\leq |f(n)| \left(2np_n^v(t) + t^{-1} \left(\frac{e}{n-1} \right)^{n-1} \int_{2n}^{\infty} u^n e^{-u} du \right) \\ &= |f(n)| 2np_n^v(t) + t^{-1} |f(n)| \left(\frac{e}{n-1} \right)^{n-1} n! \mathbb{P}(X_n \leq n), \end{aligned}$$

where X_n has a Poisson distribution with parameter $2n$. By assumption (10.12) the sum over the first terms is converging. By a rather elementary concentration inequality (using a Chernoff bound argument) we have $\mathbb{P}(X_n \leq n) \leq (2/e)^n$ for each $n \in \mathbb{N}$. Therefore the sum over the second terms is converging too. \square

Remark 10.9. The position dependent cluster density satisfies the condition (10.12) and its derivative can be represented by (9.4), i.e.

$$\begin{aligned} \frac{d}{dt} \kappa^v(t) &= t^{-1} \left(\mathbb{P}_t(|C^v| < \infty) - \kappa^v(t) - \mathbb{E}_t \int |C^v - \delta_x|^{-1} C^v(dx) \right) \\ &= \mathbb{E}_t \int (|C^v(\xi^{v,x})|^{-1} - |C^v|^{-1}) \mathbf{1}\{x \in C^v(\xi^{v,x})\} \lambda(dx). \end{aligned}$$

11 Deletion stability of the stationary marked RCM

In this section we consider the stationary marked RCM as introduced in Section 4. Hence we take a Poisson process η on $\mathbb{R}^d \times \mathbb{M}$ with intensity measure $t\lambda = t\lambda_d \otimes \mathbb{Q}$ and consider the random connection model ξ based on η and a fixed connection function $\varphi: (\mathbb{R}^d \times \mathbb{M})^2 \rightarrow [0, 1]$ satisfying (4.1) and (4.2).

Theorem 11.1. *The infinite clusters of a stationary marked random connection model are deletion stable.*

Our proof of the theorem partially follows the seminal paper [1]. It requires a significant extension of some of the arguments in [25] treating the Gilbert graph with deterministic radii; see Example 4.8.

We need to introduce some further notation. For $\mu \in \mathbf{G}$ and $(x, p) \in V(\mu)$ we denote by $N^0(x, p, \mu)$ the number of clusters in $C^{(x,p)}(\mu) - \delta_{(x,p)}$. Hence $N^0(x, p, \mu)$ is the number of clusters in $\mu - \delta_{(x,p)}$ which are connected in μ with (x, p) . We then define $N^+(x, p, \mu)$ similarly to $N^0(x, p, \mu)$, except that at most one infinite cluster is counted, i.e.

$$N^+(x, p, \mu) := N^0(x, p, \mu) - \mathbf{1}\{N^\infty(x, p, \mu) \geq 1\}(N^\infty(x, p, \mu) - 1).$$

Given $B \in \mathcal{B}(\mathbb{R}^d)$ and a measure ν on $\mathbb{R}^d \times \mathbb{M}$ it will be convenient to write $\nu_B := \nu_{B \times \mathbb{M}}$ for the restriction of ν to $B \times \mathbb{M}$.

We fix some arbitrary $t_0 > 0$. It is then no restriction of generality to assume that $t \in (0, t_0]$. Let $(B_n)_{n \in \mathbb{N}}$ be an increasing sequence of convex and compact sets with union \mathbb{R}^d . Our proofs require a specific *coupling* of the RCM ξ with two random graphs $\xi_{n,0}$ and $\xi_{n,+}$, $n \in \mathbb{N}$, according two different boundary conditions: *free* and *wired*. To this end we let $\tilde{\xi}$ be a RCM based on a Poisson process $\tilde{\eta}$ with intensity measure $t_0\lambda$. We can assume without loss of generality that η is t/t_0 -thinning of $\tilde{\eta}$ (see [27, Corollary 5.9]) and that ξ is given as the restriction $\tilde{\xi}[\eta]$ of $\tilde{\xi}$ to the vertices from η . Let us first set $\xi_n := \tilde{\xi}[\eta_{B_n} + \tilde{\eta}_{B_n^c}]$. This is a RCM driven by the Poisson process $\eta_{B_n} + \tilde{\eta}_{B_n^c}$ which has intensity measure $t\lambda_{B_n} + t_0\lambda_{B_n^c}$. We define $\xi_{n,0}$ as the restriction $\xi[\eta_{B_n}] = \xi_n[\eta_{B_n}]$ of ξ to $B_n \times \mathbb{M}$. This is a RCM driven by η_{B_n} . We let $\xi_{n,+}$ be the random graph resulting from ξ_n by connecting all points from $\tilde{\eta}_{B_n^c}$ to each other. Then $\xi_{n,+}$ is a RCM driven by $\eta_{B_n} + \tilde{\eta}_{B_n^c}$ with a connection function which is equal to one for any pair of points from $(B_n^c \times \mathbb{M})^2$ and φ otherwise. The reader should keep in mind that $\xi_{n,+}$ is a very simple function of the RCM ξ_n . An important property of this coupling is that $\xi_{n,0}$ is a subgraph of ξ , while ξ is a subgraph of $\xi_{n,+}$ (in fact of ξ_n).

For $(x, p) \in \eta_{B_n}$ we define $C_{n,0}^{(x,p)} := C^{(x,p)}(\xi_{n,0})$ and $C_{n,+}^{(x,p)} := C^{(x,p)}(\xi_{n,+})$ noting that

$$V(C_{n,+}^{(x,p)}) = V(C_{n,0}^{(x,p)}) + \mathbf{1}\{(x, p) \leftrightarrow \tilde{\eta}_{B_n^c} \text{ in } \xi_n\} \tilde{\eta}_{B_n^c}.$$

Note also that if $(x, p) \notin \eta_{B_n}$, then $C_{n,0}^{(x,p)} = C_{n,+}^{(x,p)} = 0$. Note also that $C_{n,+}^{(x,p)}$ is infinite if and only if (x, p) is connected (in ξ_n) to $\tilde{\eta}_{B_n^c}$. Otherwise it is finite and coincides (by the coupling construction) with $C_{n,0}^{(x,p)} = C^{(x,p)}(\xi)$.

Lemma 11.2. *Let $(x, p) \in \mathbb{R}^d \times \mathbb{M}$. Then, almost surely, $V(C_{n,0}^{(x,p)}) \uparrow V(C^{(x,p)}(\xi))$ and $V(C_{n,+}^{(x,p)}) \downarrow V(C^{(x,p)}(\xi))$ as $n \rightarrow \infty$.*

Proof. Let $(x, p) \in \eta$ otherwise the statement is trivial. There exists $m \in \mathbb{N}$ such that $x \in B_m$. We shall always take $n \geq m$. The second assertion has to be interpreted this way. Clearly $C_{n,0}^{(x,p)}$ is a subgraph of $C^{(x,p)}(\xi)$. Assume that $(y, q) \in C^{(x,p)}(\xi)$. Then there exists n such that (x, p) is connected to (y, q) within $\xi_{n,0}$. This proves the first assertion. Next we note that $C^{(x,p)}(\xi)$ is a subgraph of $C_{n+1,+}^{(x,p)}$ while $C_{n+1,+}^{(x,p)}$ is a subgraph of $C_{n,+}^{(x,p)}$. Assume that $(y, q) \in C_{n,+}^{(x,p)}$ for each $n \geq m$. For large enough n we then have $y \in B_n$ and hence $(y, q) \in C^{(x,p)}(\xi)$. \square

For each $n \in \mathbb{N}$ we define

$$M_{n,\star} := \int |C_{n,\star}^{(x,p)}|^{-1} \eta_{B_n}(d(x, p)),$$

where we use a star to denote either 0 or +. A simple counting argument shows that $M_{n,0}$ is the number of clusters (finite) in $\xi_{n,0}$ while $M_{n,+}$ is the number of finite clusters in $\xi_{n,+}$. Moreover, we have that

$$M_{n,+} \leq M_n \leq M_{n,0}, \quad (11.1)$$

where (see also Proposition 4.4)

$$M_n := \int |C^{(x,p)}(\xi)|^{-1} \eta_{B_n}(d(x, p)). \quad (11.2)$$

Recalling the definition (4.6) of the cluster density $\kappa(t)$, we have the following lemma.

Lemma 11.3. *Let $t \in [0, t_0]$. Then $(\lambda_d(B_n))^{-1} \mathbb{E}_t M_{n,\star} \rightarrow t\kappa(t)$ as $n \rightarrow \infty$.*

Proof. By Lemma 4.1,

$$\mathbb{E}_t M_n = t \iint \mathbf{1}\{x \in B_n\} \mathbb{E}_t |C^{(x,p)}|^{-1} dx \mathbb{Q}(dp) = \lambda_d(B_n) t \kappa(t).$$

Almost surely $M_{n,0} - M_n$ is less than the number of clusters with points from η_{B_n} which are connected in ξ with $\eta_{B_n^c}$, and therefore less than the number of points from η_{B_n} which are directly connected in ξ with $\eta_{B_n^c}$. Analogously, $M_n - M_{n,+}$ is less than number of clusters with points from η_{B_n} which are connected in ξ_n with $\tilde{\eta}_{B_n^c}$, and therefore less than the number of points from η_{B_n} which are directly connected in ξ_n with $\tilde{\eta}_{B_n^c}$. Then with probability one, we have

$$\begin{aligned} M_{n,0} - \int \mathbf{1}\{(x, p) \sim \tilde{\eta}_{B_n^c} \text{ in } \xi_n\} \eta_{B_n}(d(x, p)) &\leq M_{n,0} - \int \mathbf{1}\{(x, p) \sim \eta_{B_n^c} \text{ in } \xi\} \eta_{B_n}(d(x, p)) \\ &\leq M_n \leq M_{n,+} + \int \mathbf{1}\{(x, p) \sim \tilde{\eta}_{B_n^c} \text{ in } \xi_n\} \eta_{B_n}(d(x, p)). \end{aligned}$$

By the Mecke equation, we have

$$\begin{aligned} &\mathbb{E}_t \int \mathbf{1}\{(x, p) \sim \tilde{\eta}_{B_n^c} \text{ in } \xi_n\} \eta_{B_n}(d(x, p)) \\ &= t \iint \mathbf{1}\{x \in B_n\} \left(1 - e^{-t_0 \int \mathbf{1}\{y \in B_n^c\} \varphi(y-x, p, q) dy \mathbb{Q}(dq)}\right) dx \mathbb{Q}(dp) \\ &\leq t_0 t \iiint \mathbf{1}\{x \in B_n, y \in B_n^c\} \varphi(y-x, p, q) dx dy \mathbb{Q}^2(d(p, q)) \\ &= t_0 t \iint \mathbf{1}\{x \in B_n, y \in B_n^c\} \psi(y-x) dx dy, \end{aligned}$$

where

$$\psi(x) := \int \varphi(x, p, q) \mathbb{Q}^2(d(p, q)), \quad x \in \mathbb{R}^d. \quad (11.3)$$

By assumption (4.2), ψ is integrable.

Let $\varepsilon > 0$ and choose $r > 0$ so large that $\int \mathbf{1}\{|z| > r\} \psi(z) dz \leq \varepsilon$. Then

$$\int \mathbf{1}\{x \in B_n, y \in B_n^c, |y - x| > r\} \psi(y - x) dx dy \leq \varepsilon \lambda_d(B_n).$$

Further

$$\frac{1}{\lambda_d(B_n)} \int \mathbf{1}\{x \in B_n, y \in B_n^c, |y - x| \leq r\} \psi(y - x) dx dy \leq d_\varphi \frac{\lambda_d((B_n)_{\ominus r})}{\lambda_d(B_n)} \xrightarrow{n \rightarrow \infty} 0, \quad (11.4)$$

where, for a bounded set $B \subset \mathbb{R}^d$, $B_{\ominus r} := \{x \in B : d(x, \partial B) \leq r\}$ and ∂B denotes the boundary of B . Therefore

$$\limsup_{n \rightarrow \infty} \frac{\mathbb{E}_t M_{n,0}}{\lambda_d(B_n)} - \varepsilon t_0 t \leq t \kappa(t) \leq \liminf_{n \rightarrow \infty} \frac{\mathbb{E}_t M_{n,+}}{\lambda_d(B_n)} + \varepsilon t_0 t$$

Taking into account (11.1), this yields the assertion. \square

Let $n \in \mathbb{N}$. We will now explore the derivatives of $t \mapsto \mathbb{E}_t M_{n,\star}$. For $(x, p) \in B_n \times \mathbb{M}$ we define $N_n^*(x, p) := N^0(x, p, \xi_{n,\star}^{(x,p)})$, the finite volume counterparts of $N^0(x, p, \xi^{(x,p)})$ and $N^+(x, p, \xi^{(x,p)})$. By this definition $N_n^0(x, p)$ is the number of (finite) clusters in $\xi_{n,0}$ which are connected to (x, p) in $\xi_{n,0}^{(x,p)}$, and $N_n^+(x, p)$ is the number of clusters (with at most one infinite) in $\xi_{n,+}$ which are connected to (x, p) in $\xi_{n,+}^{(x,p)}$.

Lemma 11.4. *For any $n \in \mathbb{N}$ and either choice of boundary conditions the function $t \mapsto \mathbb{E}_t M_{n,\star}$ is differentiable on $[0, t_0)$ and the derivative is given by*

$$\frac{d}{dt} \mathbb{E}_t M_{n,\star} = \lambda_d(B_n) - \mathbb{E}_t \iint \mathbf{1}\{x \in B_n\} N_n^*(x, p) dx \mathbb{Q}(dp).$$

Proof. Since $M_{n,\star} \leq \eta(B_n)$ we have $\mathbb{E}_t M_{n,\star} < \infty$ for all $t > 0$. We now apply the Margulis-Russo formula (3.9), where $\lambda_2 = (\lambda_d)_{B_n} \otimes \mathbb{Q}$ and $\lambda_1 = 0$ for the free boundary condition ($\star = 0$) and $\lambda_1 = t_0(\lambda_d)_{B_n^c} \otimes \mathbb{Q}$ for the wired boundary condition ($\star = +$). Hence $\mathbb{E}_t M_{n,\star}$ is a differentiable function of t and

$$\frac{d}{dt} \mathbb{E}_t M_{n,\star} = \mathbb{E}_t \iint \mathbf{1}\{x \in B_n\} (M_{n,\star}(\xi_{n,\star}^{(x,p)}) - M_{n,\star}(\xi_{n,\star})) dx \mathbb{Q}(dp).$$

Let $(x, p) \in B_n \times \mathbb{M}$. If $N_n^*(x, p) = 0$, then with probability one $M_{n,\star}(\xi_{n,\star}^{(x,p)}) - M_{n,\star}(\xi_{n,\star}) = 1$. Otherwise the removal of (x, p) from $\xi_{n,\star}^{(x,p)}$ results in $M_{n,\star}(\xi_{n,\star}^{(x,p)}) - M_{n,\star}(\xi_{n,\star}) = 1 - N_n^*(x, p)$ a.s., proving the result. \square

Lemma 11.5. *For any $n \in \mathbb{N}$ and either choice of boundary conditions $\mathbb{E}_t M_{n,\star} + \lambda_d(B_n) d_\varphi t^2 / 2$ is a convex function of t on $[0, t_0)$.*

Proof. For $(x, p) \in \mathbb{R}^d \times \mathbb{M}$ we let $\Psi(x, p)$ denote the point process of the *Poisson neighbours* of (x, p) in $\xi^{(x,p)}$, that is the points in η which are directly connected to (x, p) in $\xi^{(x,p)}$. For a Borel set $B \subset \mathbb{R}^d$ we let $\Psi_B(x, p)$ denote the restriction of $\Psi(x, p)$ to $B \times \mathbb{M}$.

We further denote $d_\varphi(p) := \int d(p, q) \mathbb{Q}(dq)$ so that $d_\varphi = \int d_\varphi(p) \mathbb{Q}(dp)$. By Lemma 11.4,

$$\begin{aligned} & \frac{d}{dt} \left[\mathbb{E}_t M_{n,\star} + \lambda_d(B_n) d_\varphi \frac{t^2}{2} \right] \\ &= \lambda_d(B_n) - \mathbb{E}_t \iint \mathbf{1}\{x \in B_n\} N_n^\star(x, p) dx \mathbb{Q}(dp) + t \lambda_d(B_n) d_\varphi \\ &= \lambda_d(B_n) + \iint \mathbf{1}\{x \in B_n\} (t d_\varphi(p) - \mathbb{E}_t N_n^\star(x, p)) dx \mathbb{Q}(dp) \\ &= \lambda_d(B_n) + \iint \mathbf{1}\{x \in B_n\} (\mathbb{E}_t |\Psi(x, p)| - \mathbb{E}_t N_n^\star(x, p)) dx \mathbb{Q}(dp) \\ &= \lambda_d(B_n) + \iint \mathbf{1}\{x \in B_n\} (\mathbb{E}_t |\Psi_{B_n^c}(x, p)| + \mathbb{E}_t (|\Psi_{B_n}(x, p)| - N_n^\star(x, p))) dx \mathbb{Q}(dp), \end{aligned}$$

Clearly $\mathbb{E}_t |\Psi_{B_n^c}(x, p)|$ is increasing in t . We shall now argue that $\mathbb{E}_t [\Psi_{B_n}(x, p) - N_n^\star(x, p)]$ is increasing in t . Applying the Margulis-Russo formula (3.9) similarly as in the proof of Lemma 11.4, we see that it is sufficient to check that $\Psi_{B_n}(x, p) - N_n^\star(x, p)$ cannot strictly decrease when adding a point $(y, q) \in B_n \times \mathbb{M}$ to η . Assume first that (y, q) is not directly connected to (x, p) . Then $\Psi_{B_n}(x, p)$ does not change while $N_n^\star(x, p)$ can only decrease (namely by connecting some of the clusters in $\xi_{n,\star}$ which are connected to (x, p) in $\xi_{n,\star}^{(x,p)}$). Assume now that (y, q) is directly connected to (x, p) , so that $\Psi_{B_n}(x, p)$ increases by one. In that case $N_n^\star(x, p)$ can increase by at most 1, namely if some of the clusters in $\xi_{n,\star}$ which are not connected to (x, p) in $\xi_{n,\star}^{(x,p)}$ get connected to the new point (y, q) while none of the clusters in $\xi_{n,\star}$ which are connected to (x, p) in $\xi_{n,\star}^{(x,p)}$ are connected by (y, q) . This proves the asserted monotonicity and hence the convexity assertion. \square

Now we are in the position to prove the first main result in this section.

Theorem 11.6. *The function $t \mapsto t\kappa(t) + d_\varphi t^2/2$ is continuously differentiable on $(0, \infty)$, convex on \mathbb{R}_+ and right differentiable at zero.*

Proof. The first assertion follows from Theorem 10.7 while the second follows from Lemmas 11.3 and 11.5 and the (elementary) fact that the limit of a sequence of convex functions is convex. The function is right differentiable at zero since κ is a monotone function. \square

In the final step of the proof of Theorem 11.1 we need to identify the limits of the derivatives in Lemma 11.4.

Lemma 11.7. *Let $t \in [0, t_0]$. Then*

$$\begin{aligned} \liminf_{n \rightarrow \infty} (\lambda_d(B_n))^{-1} \mathbb{E}_t \iint \mathbf{1}\{x \in B_n\} N_n^0(x, p) dx \mathbb{Q}(dp) &\geq \int \mathbb{E}_t N^0(0, p, \xi^{(0,p)}) \mathbb{Q}(dp), \\ \limsup_{n \rightarrow \infty} (\lambda_d(B_n))^{-1} \mathbb{E}_t \iint \mathbf{1}\{x \in B_n\} N_n^+(x, p) dx \mathbb{Q}(dp) &\leq \int \mathbb{E}_t N^+(0, p, \xi^{(0,p)}) \mathbb{Q}(dp). \end{aligned}$$

Proof. Similarly as in the proof of Proposition 4.5 by stationarity, we have

$$\int \mathbb{E}_t N^\star(0, p, \xi^{(0,p)}) \mathbb{Q}(dp) = (\lambda_d(B_n))^{-1} \mathbb{E}_t \iint \mathbf{1}\{x \in B_n\} N_n^\star(x, p, \xi^{(x,p)}) dx \mathbb{Q}(dp), \quad n \in \mathbb{N}.$$

Hence our task is to show that $N_n^\star(x, p, \xi^{(x,p)})$ is well approximated by $N_n^\star(x, p)$. For a given Borel set $B \subset \mathbb{R}^d$ and $(x, p) \in \mathbb{R}^d \times \mathbb{M}$ we denote by $N_B^0(x, p)$ the number of clusters in ξ to which the Poisson neighbors of (x, p) in $B \times \mathbb{M}$ belong. Note that a.s.

$$N_B^0(x, p) + N_{B^c}^0(x, p) \geq N^0(x, p, \xi^{(x,p)}).$$

It is, moreover, easy to see that $N_n^0(x, p) \geq N_{B_n}^0(x, p)$ a.s. for each $n \in \mathbb{N}$. It follows that a.s.

$$N_n^0(x, p) \geq N^0(x, p, \xi^{(x,p)}) - N_{B_n^c}^0(x, p). \quad (11.5)$$

Obviously $N_{B_n^c}^0(x, p)$ is dominated by the number of points from $\eta_{B_n^c}$ which are directly connected to (x, p) in $\xi^{(x,p)}$. Therefore

$$\mathbb{E}_t N_{B_n^c}^0(x, p) \leq t \iint \mathbf{1}\{y \in B_n^c\} \varphi(y - x, p, q) dy \mathbb{Q}(dq).$$

It now follows from (11.5) and exactly as in the proof of Lemma 11.3 that for each $\varepsilon > 0$

$$\liminf_{n \rightarrow \infty} (\lambda_d(B_n))^{-1} \mathbb{E}_t \iint \mathbf{1}\{x \in B_n\} N_n^0(x, p) dx \mathbb{Q}(dp) \geq \int \mathbb{E}_t N^0(0, p, \xi^{(0,p)}) \mathbb{Q}(dp) - \varepsilon t.$$

This implies the first asserted inequality. The second follows from $N_n^+(x, p) \leq N^+(x, p, \xi^{(x,p)})$ a.s.. \square

Proof of Theorem 11.1. The convex function in Theorem 11.6 is differentiable and approximated by the differentiable convex functions $(\lambda_d(B_n))^{-1} \mathbb{E}_t M_{n,\star} + d_\varphi t^2/2$; see Lemmas 11.3 and 11.4. A classical result from convex analysis (see [39, Theorem 25.7]) implies that

$$\lim_{n \rightarrow \infty} (\lambda_d(B_n))^{-1} \frac{d}{dt} \mathbb{E}_t M_{n,\star} = \frac{d}{dt} t\kappa(t).$$

Therefore we obtain from Lemma 11.4 that the limit inferior in Lemma 11.7 coincides with the limit superior. Hence Lemma 11.7 yields

$$\int \mathbb{E}_t N^0(0, p, \xi^{(0,p)}) \mathbb{Q}(dp) \leq \int \mathbb{E}_t N^+(0, p, \xi^{(0,p)}) \mathbb{Q}(dp),$$

or

$$\int \mathbb{E}_t (N^0(0, p, \xi^{(0,p)}) - N^+(0, p, \xi^{(0,p)})) \mathbb{Q}(dp) \leq 0.$$

Since

$$N^0(0, p, \xi^{(0,p)}) - N^+(0, p, \xi^{(0,p)}) = \mathbf{1}\{N^\infty(0, p, \xi^{(0,p)}) \geq 1\} (N^\infty(0, p, \xi^{(0,p)}) - 1),$$

we obtain

$$\int \mathbb{P}_t(N^\infty(0, p, \xi^{(0,p)}) \geq 2) \mathbb{Q}(dp) = 0. \quad (11.6)$$

Using stationarity as in the proof of Lemma 4.1 we see, that this is equivalent to the assertion. \square

The preceding proof yields the following corollary.

Corollary 11.8. *The cluster density is continuously differentiable function and*

$$\frac{d}{dt}(t\kappa(t)) = 1 - \int \mathbb{E}_t N^0(0, p, \xi^{(0,p)}) \mathbb{Q}(dp).$$

Remark 11.9. The convergence on the right-hand side of (11.4) is crucial for the proof of Lemma 11.3. This *amenability* property of Euclidean space is also important for Lemma 11.7. It might be possible to extend the methods of this section to establish deletion stability for other amenable homogeneous spaces. But we do not know, how to prove or disprove deletion stability in a non-amenable situation, like the hyperbolic space.

Remark 11.10. Assume that $\varphi(x, p, q) = \tilde{\varphi}(\|x\|, p, q)$ and $t = 1$, for a measurable function $\tilde{\varphi}: [0, \infty) \times \mathbb{M} \times \mathbb{M} \rightarrow [0, 1]$ which is decreasing and right-continuous in the first coordinate. Using the notation at (2.2) and (4.3) we define

$$W_{m,n} := \frac{\|X_m - X_n\|}{\tilde{\varphi}^{-1}(Z_{m,n}, Q_m, Q_n)}, \quad m, n \in \mathbb{N}, \quad (11.7)$$

where $\tilde{\varphi}^{-1}(s, p, q) := \inf\{r \geq 0 : \varphi(r, p, q) \leq s\}$, $(s, p, q) \in [0, 1] \times \mathbb{M} \times \mathbb{M}$. Given $r > 0$ we define a RCM ξ_r with vertex set η by connecting X_m with X_n if $W_{m,n} \leq r$. Note that $W_{m,n} \leq r$ if and only if

$$Z_{m,n} \leq \tilde{\varphi}(r^{-1}\|X_m - X_n\|, Q_m, Q_n).$$

Since $\sum_{n=1}^{\infty} \delta_{(r^{-1}X_n, Q_n)}$ is under \mathbb{P}_1 a Poisson process with intensity measure $r^d \lambda_d \otimes \mathbb{Q}$, we hence have

$$\mathbb{P}_1(\xi_r \in \cdot) = \mathbb{P}_{r^d}(\xi \in \cdot), \quad r > 0, \quad (11.8)$$

i.e. a joint coupling of the RCMs with different intensity parameters. In the unmarked case this construction can be found in [4, Example 1.3].

Remark 11.11. Consider the setting of Remark 11.10 and the complete graph with vertex set η . We can interpret the random variable (11.7) as *weight* of the edge between (X_m, Q_m) and (X_n, Q_n) . As in [3] we define the associated *minimal spanning forest* T as the forest (a graph without cycles) with vertex set η and an edge between (X_m, Q_m) and (X_n, Q_n) if there is no path between these points with weights strictly less than $W_{m,n}$. In special cases it was observed in [2, 3, 4, 7] that there is a close relationship between the RCM ξ_r and T . For instance it was proved in [3] that the trees (clusters) of T are all infinite and can only have one or two ends. Two-ended trees T can only occur if r equals the percolation threshold in which case T contains all points of the infinite clusters (should they exist). It would be interesting to explore the consequences of deletion stability of ξ_r for T .

12 The stationary marked RCM: Irreducibility and uniqueness

In this section we consider a stationary marked RCM ξ as introduced in Section 4. When combined with Theorem 11.1, Theorem 6.1 immediately yields the following result.

Theorem 12.1. *An irreducible stationary marked random connection model can almost surely have at most one infinite cluster.*

Remark 12.2. Theorem 12.1 and Corollary 6.4 show that an irreducible stationary marked RCM is 2-indivisible. In particular this holds at the critical intensity t_c . This provides some evidence for the absence of doubly-infinite paths at criticality. In fact, it is a common belief that in Euclidean space there is no infinite cluster in the critical phase.

We now present several examples, starting with the classical stationary RCM; see Example 4.6.

Example 12.3. By Theorem 12.1 and Proposition 5.5 the (unmarked) stationary RCM can have at most one infinite component. This generalizes [33, Theorem 6.3], where it is assumed that $\varphi(x) = \tilde{\varphi}(\|x\|)$, $x \in \mathbb{R}^d$, for a decreasing function $\tilde{\varphi}: [0, \infty) \rightarrow [0, 1]$. The proof there is based on an extension of the approach from [10] to the continuum and is very different from ours.

Next we treat the simple case, where the connection factorizes; see also [11, Section 1.2].

Example 12.4. Let $\psi: \mathbb{R}^d \rightarrow [0, 1]$ be a symmetric function with $0 < m_\psi := \int \psi(x) dx < \infty$ and let $K: \mathbb{M}^2 \rightarrow [0, 1]$ be measurable and symmetric. Assume that $\varphi(x, p, q) = \psi(x)K(p, q)$, $(x, p, q) \in \mathbb{R}^d \times \mathbb{M}^2$. Then $d_\varphi(p, q) = m_\psi K(p, q)$, $(p, q) \in \mathbb{M}^2$. By Theorem 5.7 ξ is irreducible if and only if

$$\sup_{n \geq 1} K^n(p, q) > 0, \quad \mathbb{Q}^2\text{-a.e. } (p, q) \in \mathbb{M}^2.$$

We continue with the examples from Section 4.

Example 12.5. Let us consider Example 4.8. Assume that there exists $x_0 \in \mathbb{R}^d$ (for instance $x_0 = 0$) such that $\mathbb{Q}(\{L : B(x_0, \varepsilon_0) \subset K\}) > 0$, where $B(x_0, \varepsilon)$ denotes the ball with center x_0 and radius ε . Assume also that the function V is increasing w.r.t. set inclusion and that $V(K) > 0$ if $K \neq \emptyset$. We show now how irreducibility of ξ follows from Corollary 5.10. As point p_0 in (5.17) we can take the ball $B(x_0, \varepsilon_0)$, while the set A is chosen as $\{L \in \mathcal{C}^d : B(x_0, \varepsilon_0) \subset L\}$. Then $\mathbb{Q}(A) > 0$ and (5.18) holds. To check (5.17) we take $K \in \mathcal{C}^d$. Then

$$d_\varphi(B(x_0, \varepsilon_0), K) = \int (1 - e^{-V(K \cap B(x, \varepsilon_0))}) dx.$$

By assumption on V this is positive, since

$$\int \mathbf{1}\{K \cap B(x, \varepsilon_0) \neq \emptyset\} dx = \int \mathbf{1}\{y \in B(x, \varepsilon_0)\} dx > 0,$$

where y is some point from K . We can now apply Theorem 12.1 to conclude that the infinite cluster is unique. For the spherical Boolean model this result can be found as Theorem 3.6 in [33]. For general Boolean models (i.e. $\varphi(x, K, L) = \mathbf{1}\{K \cap (L + x) \neq \emptyset\}$) the result seems to be new.

Example 12.6. The weighted RCM from Example 4.9 is irreducible by Corollary 5.13. Indeed we have $d_\varphi(p, q) = m_\rho g(p, q)^{-1}$, which is positive and monotone. By Theorem 12.1 the infinite cluster is unique. This was asserted in [19] without providing details of a proof. A more detailed proof in a special case (based on the approach in [10]) was given in [23].

Example 12.7. Consider a stationary marked RCM with \mathbb{M} as the space of all locally finite simple counting measures on \mathbb{R}^d . Let \mathbb{Q} be a distribution of a simple stationary point process χ satisfying $\mathbb{Q}\{0\} = 0$. For $x \in \mathbb{R}^d$ and $p \in \mathbb{M}$ let $d(x, p)$ be the distance between x and p . Similarly as in Example 12.6 we consider a connection function of the form

$$\varphi(x, p, q) = \rho(d(-x, p))^{-\alpha} d(x, q)^{-\alpha} \|x\|^d$$

for a decreasing function $\rho: [0, \infty) \rightarrow [0, 1]$ such that $m_\rho := \int \rho(\|x\|^d) dx$ is positive and finite and where $\alpha > 0$ is a fixed parameter. By stationarity,

$$\begin{aligned} \iint \varphi(x, p, q) \mathbb{Q}^2(d(p, q)) dx &= \iint \rho(d(0, p))^{-\alpha} d(0, q)^{-\alpha} \|x\|^d dx \mathbb{Q}^2(d(p, q)) \\ &= m_\rho \int \mathbf{1}\{d(0, p) < \infty, d(0, q) < \infty\} d(0, p)^\alpha d(0, q)^\alpha \mathbb{Q}^2(d(p, q)) \\ &= m_\rho \left(\int d(0, p)^\alpha \mathbb{Q}(dp) \right)^2. \end{aligned}$$

To ensure (4.2) we assume that $\int d(0, p)^\alpha \mathbb{Q}(dp) < \infty$, which is a rather weak assumption.

Next we check that $\int d_\varphi(p, q) \mathbb{Q}(dq) > 0$ for \mathbb{Q} -a.e. p , so that (5.19) holds. Fix $p \in \mathbb{M} \setminus \{0\}$ such that $0 \notin p$. As above we have

$$\iint \varphi(x, p, q) \mathbb{Q}(dq) dx = \iint \rho(d(x, p)^{-\alpha} d(0, q)^{-\alpha} \|x\|^d) dx \mathbb{Q}(dq).$$

Moreover, since $0 \notin p$ there exist $\varepsilon, c > 0$ such that $d(x, p) \geq c$ for $\|x\| \leq \varepsilon$. It follows that

$$\iint \varphi(x, p, q) dx \mathbb{Q}(dq) \geq \iint \mathbf{1}\{\|x\| \leq \varepsilon\} \rho(c^{-\alpha} d(0, q)^{-\alpha} \|x\|^d) dx \mathbb{Q}(dq).$$

Assume for the sake of contradiction that the above outer integral vanishes and take $q \in \mathbb{M} \setminus \{0\}$ such that $\int \mathbf{1}\{\|x\| \leq \varepsilon\} \rho(c_1^{-\alpha} d(0, q)^{-\alpha} \|x\|^d) dx = 0$. However, since $m_\rho > 0$ and ρ is decreasing, $\rho(r)$ is positive for sufficiently small $r > 0$. The resulting contradiction shows that $\int d_\varphi(p, q) \mathbb{Q}(dq) > 0$. The function $\varphi(x, p, \cdot)$ is for all $(x, p) \in \mathbb{R}^d \times \mathbb{M}$ non-decreasing with respect to the natural partial ordering on \mathbb{M} . Therefore, if \mathbb{Q} is associated, then Theorem 5.12 implies that ξ is irreducible. For instance we might take \mathbb{Q} as the distribution of a Poisson process; see e.g. [28]. Hence Theorem 12.1 applies.

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