Hyperuniformity and diffraction of substitution tilings

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1 Introduction

In this thesis, we study the properties of point processes arising in the field of aperiodic order. Specifically, we study point processes arising from *substitution rules*. We obtain bounds for the decay of their diffraction around the origin, and hence derive a sufficient criterion for their hyperuniformity.

We rigorously prove a conjecture by Oğuz, Socolar, Steinhardt, and Torquato [47], and extend the work of Baake and Grimm [6] to a wider class of examples. We are able to prove hyperuniformity for most substitution rules on the plane with rigid symmetries: we are able to do this even when the diffraction has a singular continuous component, a case that had been hard to handle until now.

1.1 Hyperuniformity

In order to discuss hyperuniformity, we introduce some basic probabilistic language. First, we set down some notation. Let $d \in \mathbb{N}$, and let $m_{\mathbb{R}^d}$ be the Lebesgue measure on $\mathbb{R}^d$. Given $x \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}^\times$, we denote by $T_x : \mathbb{R}^d \to \mathbb{R}^d$ and $D_\lambda : \mathbb{R}^d \to \mathbb{R}^d$ the translation by x and dilation by λ, respectively. Translations and dilations act on subsets, functions and measures in the obvious way (cf. Definition 2.2). Given $R > 0$, we denote by B_R the closed Euclidean ball of radius R around the origin.

For any set S, let $\#S$ be its cardinality if S is finite, and $\#S = \infty$ if S is infinite. We say that a subset $Z \subset \mathbb{R}^d$ is *locally finite* if $\#(Z \cap B) < \infty$ for all relatively compact $B \subset \mathbb{R}^d$ and denote by $\mathcal{P}_{LF}(\mathbb{R}^d)$ the set of all locally finite subsets of $\mathbb{R}^d$. We equip $\mathcal{P}_{LF}(\mathbb{R}^d)$ with the smallest σ-algebra such that the map

$$p_B : \mathcal{P}_{LF}(\mathbb{R}^d) \to \mathbb{N} \cup \{\infty\}, \quad Z \mapsto \#(Z \cap B)$$

is measurable for every Borel set $B \subset \mathbb{R}^d$.

Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a random variable $\Phi : \Omega \to \mathcal{P}_{LF}(\mathbb{R}^d)$ is called a *point process on* $\mathbb{R}^d$ with *distribution measure* $\Phi_*\mathbb{P} = \mathbb{P} \circ \Phi^{-1}$. We think of a point process as a random locally finite subset of $\mathbb{R}^d$. If Φ, Φ' are two point processes, we write $\Phi \overset{d}{=} \Phi'$ if they have the same distribution.

A point process Φ is called *locally square integrable* if $\mathbb{E}[\#(\Phi \cap B)^2] < \infty$ for all relatively compact $B \subset \mathbb{R}^d$; it is called *stationary* if $T_x\Phi \overset{d}{=} \Phi$ for all $x \in \mathbb{R}^d$.

For the rest of this introduction, all point processes will be assumed to be locally square integrable and stationary.

Two prototypical examples of stationary point processes are the *homogeneous Poisson process* and the *shifted integer lattice*.

Example 1.1. The *homogeneous Poisson process* [39] on $\mathbb{R}^d$ with intensity 1 is the unique stationary point process with the following two properties.

(i) For every Borel set $B \subset \mathbb{R}^d$, the expectation of $\#(\Phi \cap B)$ is given by $\mathbb{E}[\#(\Phi \cap B)] = m_{\mathbb{R}^d}(B)$

(ii) For any disjoint Borel sets $B, B' \subset \mathbb{R}^d$, the random variables $\#(\Phi \cap B)$ and $\#(\Phi \cap B')$ are independent.

Example 1.2. Let $\Gamma = \mathbb{Z}^d$. Then the *randomly shifted integer lattice* is the random set $T_x\Gamma$, where $x \in [0,1)^d$ is a uniformly distributed random vector.

If Φ is a stationary point process, then there exists a unique number $\iota \geq 0$, called the *intensity of* Φ, such that

$$\mathbb{E}[\#(\Phi \cap B)] = \iota \cdot m_{\mathbb{R}^d}(B)$$

for all Borel $B \subset \mathbb{R}^d$, i.e. the expected number of points of Φ that fall inside B is proportional to the volume of B.

The variance $\operatorname{Var}(\#(\Phi \cap B_R))$ of the number of points of Φ that fall inside B_R is called the *number variance* of Φ. For the two examples we have introduced so far, the number variance is given as follows.

- For the Poisson point process, we have $\operatorname{Var}(\#(\Phi \cap B_R)) = m_{\mathbb{R}^d}(B_R)$. In particular, $\operatorname{Var}(\#(\Phi \cap B_R))$ is proportional to the volume of B_R. See Figure 1.1.
- For the shifted lattice, one can prove that $\operatorname{Var}(\#(\Phi \cap B_R)) = O(R^{d-1})$ as $R \to \infty$. That is, $\operatorname{Var}(\#(\Phi \cap B_R))$ is asymptotically bounded by the surface area of B_R. See Figure 1.2.

In fact, the bound one gets for the shifted lattice is optimal in the sense that there exists no stationary point process Φ such that $\operatorname{Var}(\#(\Phi \cap B_R)) = O(R^{d-k})$ for any $k > 1$. This is a consequence of Beck's Theorem. (Note that our point processes are locally square integrable).

Theorem 1.3 (Beck's Theorem [15, Theorem 5.1])**.** *Let Φ be a stationary point process on $\mathbb{R}^d$. Then, for every $R_0 > 0$ there exists a $C > 0$ such that*

$$\frac{1}{R}\int_0^R \operatorname{Var}(\#(\Phi \cap B_r))\,\mathrm{d}r \geq CR^{d-1}, \quad \textit{for all } R \geq R_0$$

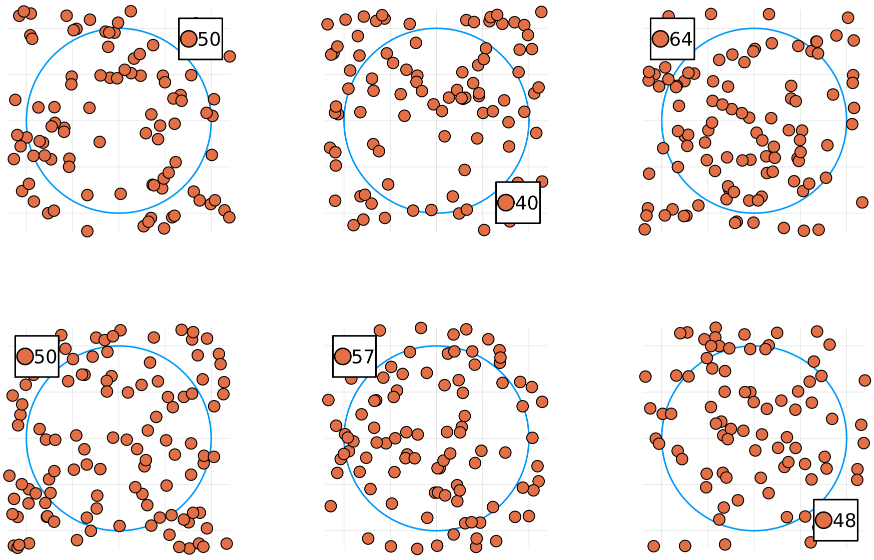


Figure 1.1: Six samples from a homogeneous Poisson process on $\mathbb{R}^2$, with the boundary of a ball B_R marked: for each sample, we count the amount of points that fall inside B_R. The radius R is chosen such that $\mathbb{E}[\#(\Phi \cap B_R)] = 50$.

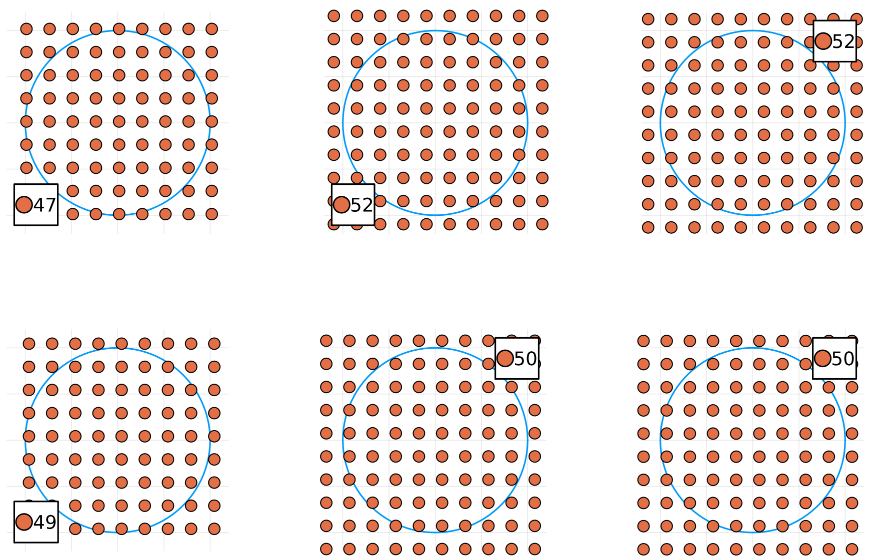


Figure 1.2: Six samples from a randomly shifted lattice on $\mathbb{R}^2$, with the boundary of a ball B_R marked. Again, the radius R is chosen such that $\mathbb{E}[\#(\Phi \cap B_R)] = 50$: we observe that the amount of points tends to be closer to the average than in the Poisson case.

Processes with low asymptotic number variance (such as shifted lattices) are "more uniform" than Poisson processes; this motivated Torquato and Stillinger [59] to introduce the following definition.

Definition 1.4 (Torquato and Stillinger [59]). Let Φ be a stationary point process on $\mathbb{R}^d$.

(i) We say that Φ is *hyperuniform* if

$$\lim_{R\to\infty} \frac{\operatorname{Var}(\#(\Phi \cap B_R))}{R^d} = 0.$$

(ii) We say that Φ is *Class I hyperuniform* if

$$\operatorname{Var}(\#(\Phi \cap B_R)) = O(R^{d-1})$$

as $R \to \infty$.

Thus randomly shifted lattices are Class I hyperuniform, whereas the homogeneous Poisson process is not hyperuniform.

Examples of hyperuniform systems include lattices, crystals, certain quasicrystals, but also many disordered random processes [16, 58, 59]. Such systems have attracted the attention of scientists in the last two decades, as such systems seem to crop up in nature: for example, the pattern of photoreceptors in some birds' eyes seems to be hyperuniform. See the survey by Torquato [58] for this perspective. Pure mathematicians have also been interested in hyperuniformity, proving or disproving it for a variety of point processes and relating it to other concepts such as invariant transports and rigidity [15, 16, 27, 35, 36].

Note that, if Z is a locally finite subset of $\mathbb{R}^d$, it defines a measure μ_Z by $\mu_Z(B) = \#(Z \cap B)$: therefore, every point process defines a *random measure*. Stationarity and hyperuniformity can be defined for random measures, just like in the point process case.

1.2 Point processes from substitution rules

We will be interested in the question of hyperuniformity for a class of point processes which are closely related to *substitution tilings* and arise in the theory of aperiodic order.

Tilings and multisets

Rather than starting from the formal definition of a tiling, we start with an example. Figure 1.4 is a picture of a *chair tiling*. It is a *tiling* of $\mathbb{R}^2$, i.e. a set of subsets of $\mathbb{R}^2$ whose union is $\mathbb{R}^2$ and whose intersections have measure 0. Every tile in the tiling is a

Figure 1.3: The four L-shaped prototiles $\tau_1, \tau_2, \tau_3, \tau_4$ used for the chair tiling. The origin of the coordinate system is marked by a black dot.

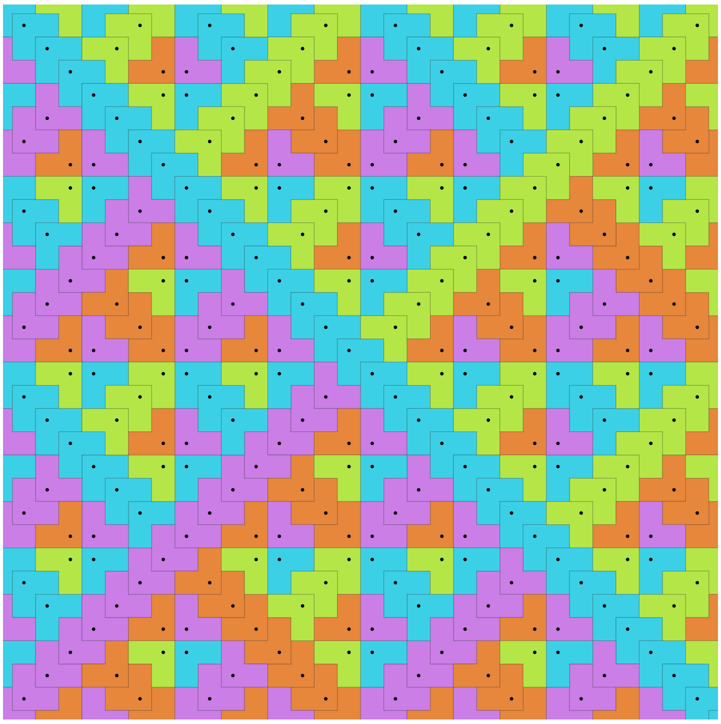


Figure 1.4: A chair tiling of the plane by the prototiles $\tau_1, \tau_2, \tau_3, \tau_4$. The Delone sets Λ_1, Λ_2, Λ_3, Λ_4 which define the tiling are pictured by black dots.

translate of one of the four *prototiles* $\tau_1, \tau_2, \tau_3, \tau_4 \subset \mathbb{R}^2$ depicted in Figure 1.3. In other words, there exist locally finite sets $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ such that the tiling is given by

$$\{T_x \tau_j \mid j \in [4], x \in \Lambda_j\}.$$

In Figure 1.4, the points in the set $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4$ are marked by black dots. Given the prototiles $\tau_1, \tau_2, \tau_3, \tau_4$, the tiling is uniquely determined by the collection $\mathbf{\Lambda} = (\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)$. Moreover, the sets Λ_j as well as their union are *Delone sets* in the following sense:

Definition 1.5. A discrete subset $\Lambda \subset \mathbb{R}^d$ is *Delone* if the following properties hold.

(i) Λ is *uniformly discrete*, i.e. there exists $r > 0$ such that the distance between any two distinct points in Λ is at least r.

(ii) Λ is *relatively dense*, i.e. there exists $R > 0$ such that for every $x \in \mathbb{R}^d$ there exists $y \in \Lambda$ such that $d(x, y) \leq R$.

Remark 1.6. Note that, when representing a tiling using Delone sets as above, we are making a choice of prototiles. We could represent the same tiling by using different translates of the prototiles in Figure 1.3: then the Delone sets $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4$ would have to be translated appropriately.

Now we formalize the above situation. Let $\ell \in \mathbb{N}$, and define $[\ell] = \{1, \cdots, \ell\}$.

We define a *multiset in* $\mathbb{R}^d$ *with* ℓ *colors*, or ℓ-multiset, as an ℓ-tuple $\mathbf{\Lambda} = (\Lambda_j)_{j \in [\ell]}$, where each Λ_j is a subset of $\mathbb{R}^d$: the Λ_j are the *components* of $\mathbf{\Lambda}$. We will always write multisets in **bold** to distinguish them from subsets of $\mathbb{R}^d$. We say $\mathbf{\Lambda}$ is a Delone multiset if Λ_j is Delone for all $j \in [\ell]$ and $\bigcup_{j \in [\ell]} \Lambda_j$ is Delone (cf. Section 4.1.1).

Remark 1.7. The term "multiset" is not standard: other authors use it to refer to sets with multiplicities instead [38]. We use the same definition as Lee, Moody, and Solomyak [40].

Definition 1.8. Let $\tau_1, \cdots, \tau_\ell$ be closed subsets of $\mathbb{R}^d$ and $\mathbf{\Lambda} = (\Lambda_1, \cdots, \Lambda_\ell)$ be a multiset in $\mathbb{R}^d$ with ℓ colors.

(i) We say $\mathbf{\Lambda}$ *patches* $\mathbb{R}^d$ *with prototiles* $\tau_1, \cdots, \tau_\ell$ if, for all $j, k \in [\ell], x \in \Lambda_j, y \in \Lambda_k$ such that $(x, j) \neq (y, k)$, $T_x \tau_j \cap T_y \tau_k$ has measure zero. In this case, we say the set $\{T_x \tau_j \mid j \in [\ell], x \in \Lambda_j\}$ is a *patch*.

(ii) If $\mathbf{\Lambda}$ patches $\mathbb{R}^d$ with prototiles $\tau_1, \cdots, \tau_\ell$, its *support* is the union $\operatorname{supp} \mathbf{\Lambda} = \bigcup_{j \in [\ell], x \in \Lambda_j} T_x \tau_j$.

(iii) We say $\mathbf{\Lambda}$ *tiles* $S \subset \mathbb{R}^d$ with prototiles $\tau_1, \cdots, \tau_\ell$ if it patches $\mathbb{R}^d$ and $\operatorname{supp} \mathbf{\Lambda} = S$. Then the set $\{T_x \tau_j \mid j \in [\ell], x \in \Lambda_j\}$ is a *tiling of* S by the prototiles $\tau_1, \cdots, \tau_\ell$.

With this terminology, the Delone multiset $\mathbf{\Lambda} = (\Lambda_1, \ldots, \Lambda_4)$ corresponding to the chair tiling in Figure 1.4 tiles $\mathbb{R}^2$ with prototiles $\tau_1, \cdots, \tau_4$ as above.

If $\mathbf{\Lambda} = (\Lambda_1, \cdots, \Lambda_\ell)$ is a multiset and $K \subset \mathbb{R}^d$, then we define the restriction $\mathbf{\Lambda} \cap K := (\Lambda_1 \cap K, \ldots, \Lambda_\ell \cap K)$. Similarly, dilations, translations and inclusion of multisets are defined componentwise. For $j \in [\ell]$, we define the *colored point at the origin* as $\boldsymbol{o}_j := (\emptyset, \ldots, \emptyset, \{0\}, \emptyset, \ldots, \emptyset)$, where the singleton $\{0\}$ is placed in the jth component. An ℓ-multiset of the form $T_x\boldsymbol{o}_j$ for some $x \in \mathbb{R}^d$ and $j \in [\ell]$ is called a *colored point.*

Substitution rules and substitution spaces

The chair tiling from Figure 1.4 is just one of many in a space of chair tilings (with the same prototiles) which can be generated from a *substitution rule* in the following sense. Here, $\mathcal{P}_{\text{fin}}(\mathbb{R}^d)$ denotes the collection of all finite subsets of $\mathbb{R}^d$.

Definition 1.9. A *substitution rule on* $\mathbb{R}^d$ *with* ℓ *colors* is a pair $\mathcal{S} = (\lambda, \mathbf{\Delta})$ where

- $\lambda > 1$ is the *scaling constant of* $\mathcal{S}$, and
- $\mathbf{\Delta} \in \mathcal{P}_{\text{fin}}(\mathbb{R}^d)^{\ell\times\ell}$ is the *displacement matrix of* $\mathcal{S}$.

Its associated *substitution map* $\varrho : \mathcal{P}(\mathbb{R}^d)^\ell \to \mathcal{P}(\mathbb{R}^d)^\ell$ maps $\mathbf{\Lambda} = (\Lambda_j)_{j\in[\ell]}$ to $\varrho(\mathbf{\Lambda}) = (\Lambda'_j)_{j\in[\ell]}$, where

$$\Lambda'_j = \bigcup_{k=1}^{\ell} \bigcup_{x\in\Delta_{jk}} T_x D_\lambda \Lambda_k$$

for all $j \in [\ell]$.

Informally, the substitution map acts by *dilating* the multiset and then "decomposing" it.

Definition 1.10. Let $\mathcal{S}$ be a substitution rule on $\mathbb{R}^d$ with ℓ colors. We say $\mathcal{S}$ is *stone* if there exist subsets $\tau_1, \cdots, \tau_\ell \subset \mathbb{R}^d$ with the following properties.

- For all $j \in [\ell]$, τ_j is compact and has positive Lebesgue measure.
- For all $j \in [\ell]$, $\varrho(\boldsymbol{o}_j)$ tiles $D_\lambda\tau_j$ with prototiles $\tau_1, \cdots, \tau_\ell$: i.e. we have

$$D_\lambda\tau_j = \bigcup_{k=1}^{\ell} \bigcup_{x\in\Delta_{kj}} T_x\tau_k$$

  and for all $x, y \in \mathbb{R}^d$, $k, k' \in [\ell]$ such that $(x, k) \neq (y, k')$, the set $T_x\tau_k \cap T_y\tau_{k'}$ has measure 0.

Example 1.11. The *chair substitution rule* $\mathcal{S}_{\text{chair}}$ is the following stone substitution rule with prototiles $\tau_1, \ldots, \tau_4$ as in Figure 1.5:

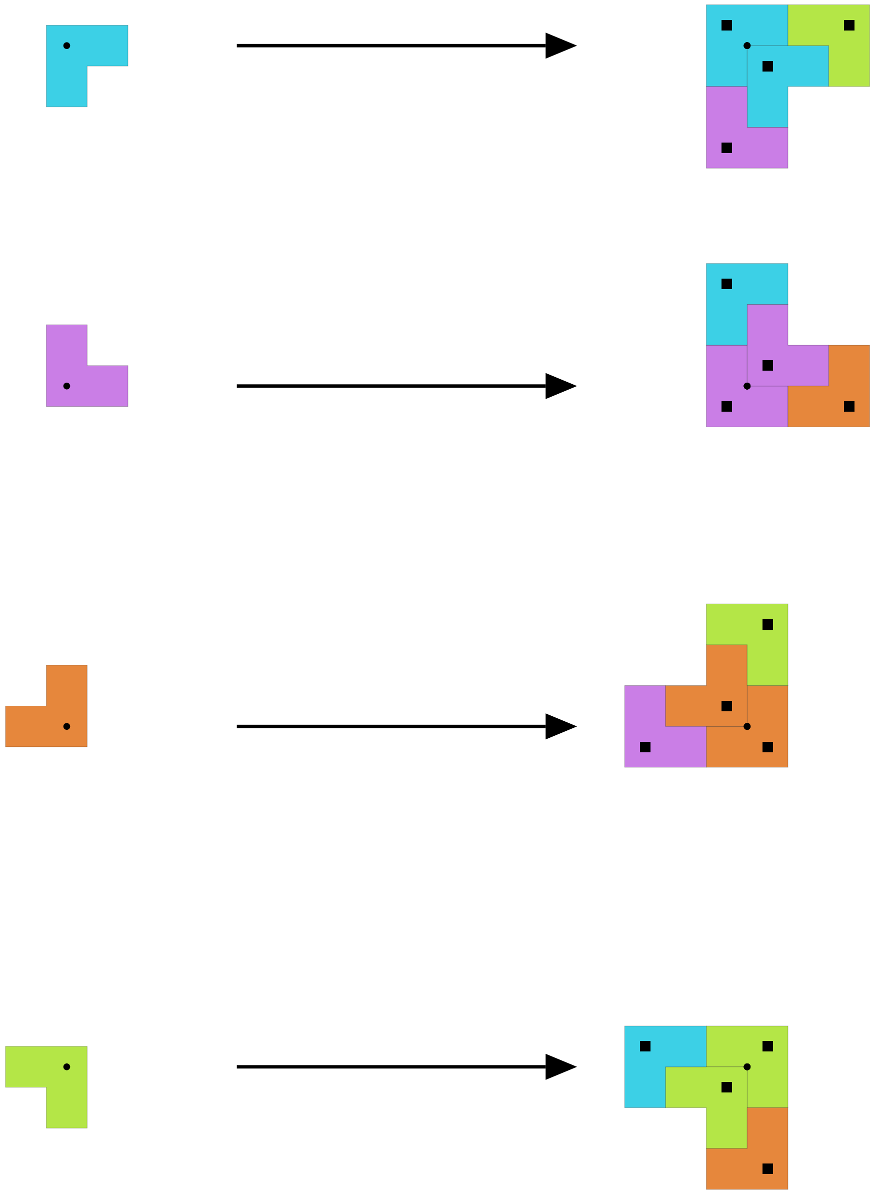


Figure 1.5: The chair substitution rule. In the left column, we see the prototiles $\tau_1, \tau_2, \tau_3, \tau_4$, with the origin marked by a black dot: the long sides of the prototiles have length 4. On the right, the tilings of $D_2\tau_j$ defined by the chair substitution rule are shown: the origin of the coordinate system is marked by a black dot, while the displacements of each tile are marked by black squares.

- The scaling constant is $\lambda = 2$.
- The displacements which give rise to the displacement matrix $\boldsymbol{\Delta}$ are depicted in Figure 1.5: For example, in the first column we see that $\varrho(\boldsymbol{o}_1)$ has two points of color 1, corresponding to the light blue tiles: the corresponding entry in the displacement matrix is

$$\Delta_{11} = \{(-1, 1)^\top, (1, -1)^\top\}.$$

Remark 1.12. Note that the chair substitution has rotational symmetry: the prototiles are rotations of each other, and the displacements are compatible with this rotation. Most substitution rules on $\mathbb{R}^2$ have some kind of rotation or reflection symmetry, which will become important later.

Definition 1.13. Let $\mathcal{S}$ be a substitution rule on $\mathbb{R}^d$ with ℓ colors.

(i) An ℓ-multiset $\boldsymbol{\Lambda}$ in $\mathbb{R}^d$ is *$\mathcal{S}$-legal* if the following holds: for every finite subset $\boldsymbol{p} \subset \boldsymbol{\Lambda}$ there exist $x \in \mathbb{R}^d$, $j \in [\ell]$ and $N \in \mathbb{N}$ such that $\boldsymbol{p} \subset \varrho^N(T_x \boldsymbol{o}_j)$.

(ii) The *substitution space* $\Omega_{\mathcal{S}}$ is the set of all $\mathcal{S}$-legal Delone ℓ-multisets.

Example 1.14. The multiset $(\Lambda_1, \ldots, \Lambda_4)$ constructed from Figure 1.4 is $\mathcal{S}_{\text{chair}}$-legal, where $\mathcal{S}_{\text{chair}}$ is the substitution rule from Example 1.11. It thus defines a point in the substitution space $\Omega_{\mathcal{S}_{\text{chair}}}$.

The substitution matrix and primitive substitution

Definition 1.15. The *full substitution matrix* of a substitution rule $\mathcal{S}$ with displacement matrix $\boldsymbol{\Delta} = (\Delta_{jk})_{j,k \in [\ell]}$ is the matrix $M_{\text{full}} \in \mathbb{N}_0^{\ell \times \ell}$ with entries

$$(M_{\text{full}})_{jk} = \#\Delta_{jk}.$$

A substitution rule is *primitive* if its full substitution matrix M_{full} is primitive, i.e. there exists $N \in \mathbb{N}$ such that every coefficient of M_{full}^N is strictly positive.

The full substitution matrix can be seen as a simplified version of the displacement matrix, where we forget the specific translations; we call it *full substitution matrix* to distinguish it from the *spherical substitution matrix*, which we will define later.

Example 1.16. By counting the tiles of each color in Figure 1.5, we see that the full substitution matrix of the chair rule is

$$M_{\text{full}} = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}.$$

One can check that every entry of M_{full}^2 is positive, so that the chair substitution rule is primitive.

Theorem 1.17 ([38, 41])**.** *Let $\mathcal{S}$ be a primitive, stone substitution rule on $\mathbb{R}^d$ with prototiles $\tau_1, \cdots, \tau_\ell$. Then:*

(i) There exists a $\mathcal{S}$-legal Delone multiset: therefore, the substitution space $\Omega_{\mathcal{S}}$ is nonempty.

(ii) Every $\mathbf{\Lambda} \in \Omega_{\mathcal{S}}$ tiles $\mathbb{R}^d$ with prototiles $\tau_1, \cdots, \tau_\ell$.

In the case of the chair substitution rule, we have already seen that (i) holds; (ii) means that every element in the chair substitution space corresponds to a tiling of $\mathbb{R}^d$ by translates of the four prototiles from Figure 1.3.

Measures on substitution spaces

Rather than just looking at individual legal Delone sets, we want to look at the whole substitution space. In order to do this, we restrict ourselves to the case of *finite local complexity.*

Definition 1.18. Let $\mathbf{\Lambda}$ be a Delone ℓ-multiset in $\mathbb{R}^d$. We say $\mathbf{\Lambda}$ is *FLC* (or *has finite local complexity*) if the following holds: for all compact $K \subset \mathbb{R}^d$, the set of K-patterns

$$\{T_x \mathbf{\Lambda} \cap K \mid x \in \mathbb{R}^d\}$$

has finitely many equivalence classes under translation.

We say a primitive, stone substitution rule $\mathcal{S}$ is *FLC* if every $\mathbf{\Lambda} \in \Omega_{\mathcal{S}}$ is FLC. Most substitution rules considered in the literature, including the chair substitution rule, are FLC. Under the FLC assumption, we define a topology on the substitution space.

Definition 1.19. Let $\mathcal{S}$ be a primitive, stone, FLC substitution rule. Then the *local topology* on its substitution space $\Omega_{\mathcal{S}}$ is the topology generated by the open sets

$$U_{K,V}(\mathbf{\Lambda}) = \{\mathbf{\Lambda}' \in \Omega_{\mathcal{S}} \mid \exists x \in V : T_x \mathbf{\Lambda}' \cap K = \mathbf{\Lambda} \cap K\}.$$

where $K \subset \mathbb{R}^d$ is compact, $V \subset \mathbb{R}^d$ is open, and $\mathbf{\Lambda} \in \Omega_{\mathcal{S}}$.

Every translate of a $\mathcal{S}$-legal multiset is still $\mathcal{S}$-legal, so $\mathbb{R}^d$ acts on the space $\Omega_{\mathcal{S}}$ by translation. This makes $\Omega_{\mathcal{S}}$ into a dynamical system which has the following nice properties.

Theorem 1.20 ([48, 54])**.** *Let $\mathcal{S}$ be a primitive, stone, FLC substitution rule. Then the following properties hold.*

(i) The substitution space $\Omega_{\mathcal{S}}$ is compact.

(ii) The space $\Omega_{\mathcal{S}}$ is minimal, i.e. every $\mathbb{R}^d$-orbit is dense.

(iii) The space $\Omega_{\mathcal{S}}$ is uniquely ergodic, i.e. there exists a unique invariant probability measure $\mathbb{P}$ on $\Omega_{\mathcal{S}}$.

(iv) The substitution map restricts to a map $\varrho : \Omega_{\mathcal{S}} \to \Omega_{\mathcal{S}}$, which is continuous and surjective.

(v) The substitution map preserves the unique invariant probability measure $\mathbb{P}$.

From now on we assume that $\mathcal{S}$ is a primitive, stone, FLC substitution rule and denote by $\mathbb{P}$ the unique invariant probability measure on $\Omega_{\mathcal{S}}$. Then, for every $j \in [\ell]$, we obtain a stationary point process

$$\Phi_j \colon \Omega_{\mathcal{S}} \longrightarrow \mathcal{P}_{\mathrm{LF}}(\mathbb{R}^d), \qquad (\Lambda_k)_{k\in[\ell]} \mapsto \Lambda_j.$$

As we did for Delone sets, it makes sense to consider the point processes $\Phi_1, \cdots, \Phi_\ell$ to be a single object.

Definition 1.21. The *vector point process associated to $\mathcal{S}$* is the tuple $\mathbf{\Phi} = (\Phi_j)_{j\in[\ell]}$, where the Φ_j are the point processes $\Phi_j : \Omega_{\mathcal{S}} \to \mathcal{P}_{\mathrm{LF}}(\mathbb{R}^d)$ defined above.

Remark 1.22. As ϱ preserves the invariant probability measure $\mathbb{P}$, $\Phi_j \overset{d}{=} \Phi_j \circ \varrho$ for all $j \in [\ell]$. We say the point processes Φ_j are *self-similar*. This property will be important later.

If $w \in \mathbb{C}^\ell$, we can consider the random measure $\langle \mathbf{\Phi}, w \rangle := \sum_{j=1}^{\ell} \Phi_j \overline{w}_j$: this is a pure point measure.

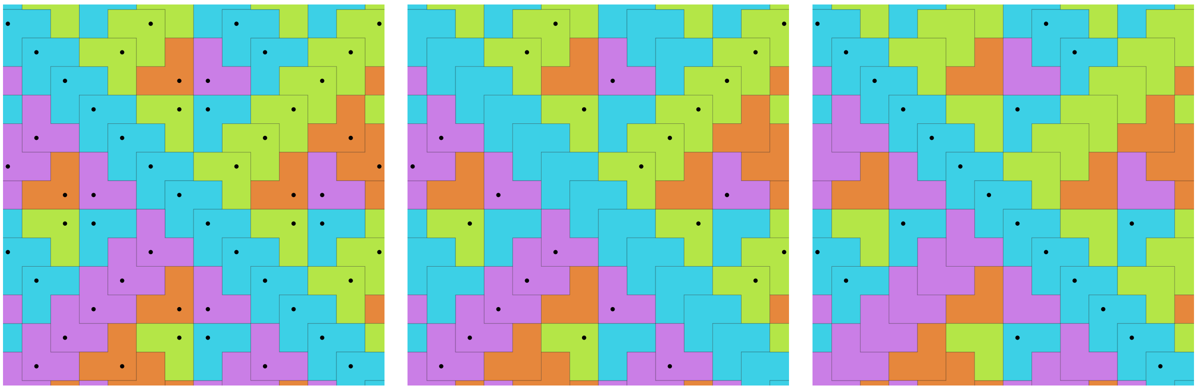
Example 1.23. Let $\mathcal{S}_{\text{chair}}$ be the chair substitution rule and $\mathbf{\Phi}$ be its associated point process.

(i) If $w = (1, 1, 1, 1)^\top$, $\langle \mathbf{\Phi}, w \rangle = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4$ is the union $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4$, where $(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)$ is a random element of $\Omega_{\mathcal{S}}$.

(ii) If $w = (0, 1, 0, 1)^\top$, $\langle \mathbf{\Phi}, w \rangle = \Phi_2 + \Phi_4$ is the union $\Lambda_2 \cup \Lambda_4$, where $(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)$ is a random element of $\Omega_{\mathcal{S}}$.

(iii) If $w = (1,0,0,0)^\top$, $\langle \mathbf{\Phi}, w \rangle = \Phi_1$ is simply Λ_1, where $(\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)$ is a random element of $\Omega_{\mathcal{S}}$.

The point processes $\langle \mathbf{\Phi}, w \rangle$ for these different choices of w are depicted in Figure 1.6.



(a) $w = (1,1,1,1)^\top$ **(b)** $w = (0,1,0,1)^\top$ **(c)** $w = (1,0,0,0)^\top$

Figure 1.6: The point processes $\langle \mathbf{\Phi}, w \rangle$ marked as black dots for different choices of $w \in \mathbb{C}^4$.

The central question of this thesis is the following.

Problem 1.24. *Let $\mathcal{S}$ be a primitive, stone, FLC substitution rule with ℓ colors and $\mathbf{\Phi}$ its associated vector point process. For which $w \in \mathbb{C}^\ell$ is $\langle \mathbf{\Phi}, w \rangle$ hyperuniform?*

Definition 1.25. Let $w \in \mathbb{C}^\ell$. We say $\mathcal{S}$ is *hyperuniform for weights w* if the random measure $\langle \mathbf{\Phi}, w \rangle$ is hyperuniform.

1.3 Hyperuniformity of substitution point processes

Let $\mathcal{S}$ be a primitive, stone, FLC substitution rule on $\mathbb{R}^d$ with ℓ colors. Recall that the substitution matrix M_{full} is the matrix given by counting the elements of each coefficient of the displacement matrix. Then its PF eigenvalue is $\lambda_{PF} = \lambda^d$, and a left PF eigenvector is given by the volumes of the prototiles.

It turns out hyperuniformity is related to the other eigenvalues and eigenvectors of the matrix. In particular, there is the following criterion for hyperuniformity, proposed by Oğuz, Socolar, Steinhardt, and Torquato [47] and Baake and Grimm [6]. They focus on the case where the substitution rule has "pure point diffraction": we will define this later.

Theorem 1.26 ([6, 47])**.** *Let $\mathcal{S}$ be a primitive, stone, FLC substitution rule in $\mathbb{R}^d$ with substitution matrix M_{full} and scaling constant λ. Assume $\mathcal{S}$ has pure point diffraction, and let μ_2 be the second largest eigenvalue of M_{full} in absolute value. Then, if $|\mu_2| < \lambda^{\frac{d}{2}}$, the substitution rule is hyperuniform for any choice of weights.*

However, this criterion alone is usually inadequate to prove hyperuniformity beyond dimension 1. Indeed, the substitution matrix M_{full} of the chair rule has the eigenvectors

$$v_1 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \quad v_2 = \begin{pmatrix}1\\i\\-1\\-i\end{pmatrix}, \quad v_3 = \begin{pmatrix}1\\-i\\-1\\i\end{pmatrix}, \quad v_4 = \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}.$$

with eigenvalues $4, 2, 2, 0$ respectively: in particular $\operatorname{Spec} M_{\text{full}} = \{4, 2, 0\}$. The second largest eigenvalue of M_{full} is $\mu_2 = 2$: therefore, $|\mu_2|$ is not strictly smaller than $\lambda^{\frac{d}{2}}$. This means the above criterion does not prove hyperuniformity for the chair substitution rule.

In this thesis, we will prove the following sufficient condition for hyperuniformity, which is robust enough to consider examples such as the chair substitution rule.

Theorem A. *Let $\mathcal{S}$ be a primitive, stone, FLC substitution rule in $\mathbb{R}^2$ with ℓ prototiles, substitution matrix M_{full} and scaling constant λ. Let $w \in \mathbb{C}^\ell$ be a vector, and let $S(w)$ be the set of eigenvalues μ of M_{full} with the following properties:*

- $\mu \neq \lambda^2$
- *The generalized eigenspace E_μ is not orthogonal to the vector w.*

Then, if $|\mu| < \lambda$ for all $\mu \in S(w)$, the substitution rule is hyperuniform for weights w.

The key point here is that, depending on the weight vector w, we are able to exclude certain eigenvalues of M_{full}; this is similar to a criterion for bounded displacement to a lattice due to Solomon [53]. For example, for the chair rule, we obtain the following.

Example 1.27. Let $w \in \operatorname{span}\{v_1, v_4\}$. Then the chair substitution rule is hyperuniform for weights w.

Proof. If we choose $w \in \operatorname{span}\{v_1, v_4\}$, w is orthogonal to the eigenvectors v_2 and v_3 with eigenvalue 2. Then $S(w)$ does not contain the eigenvalue 2, so $S(w) = \{0\}$ or $S(w) = \emptyset$. In either case, $|\mu| < 2$ for all $\mu \in S(w)$: therefore, by Theorem A, $\mathcal{S}_{\text{chair}}$ is hyperuniform for weights w. □

Symmetry

As we mentioned before, the chair substitution rule has rotational symmetry. Many substitution rules have some kind of rotational or reflectional symmetry, as follows.

Definition 1.28. Let $\mathcal{S}$ be a substitution rule on $\mathbb{R}^d$ with ℓ colors, and let $G < O(d)$ be a finite subgroup equipped with an action on $[\ell]$, in addition to its usual action on $\mathbb{R}^d$.

We say $\mathcal{S}$ is a *G-symmetric substitution rule* if the displacement matrix $\boldsymbol{\Delta} = (\Delta_{j,k})_{j,k\in[\ell]}$ satisfies

$$\Delta_{g\cdot j, g\cdot k} = g \cdot \Delta_{j,k}$$

for all $g \in G$, $j, k \in [\ell]$.

For example, the chair rule is C_4-symmetric, where a generator $R \in C_4$ acts on $\mathbb{R}^2$ by rotating by $\frac{\pi}{2}$ counterclockwise, and on [4] by the permutation (1234). This can be observed in Figure 1.5. (In fact, the chair rule is even D_4-symmetric, but we will not need this fact here.)

If $\mathcal{S}$ is G-symmetric, we can consider its *spherical substitution matrix* M_{sph}, which is the matrix given by identifying prototiles that are the same under the action of G: see Definition 7.16 for the concrete definition. For example, for the chair substitution rule, all the prototiles are rotations of one another, and the substitution of each prototile has 4 tiles, hence the spherical substitution matrix is $M_{\mathrm{sph}} = (4)$.

In general, given a substitution rule, determining the spherical substitution matrix is much easier than determining the full substitution matrix. We can use it to state a very simple sufficient criterion for hyperuniformity, at least for certain choices of weights.

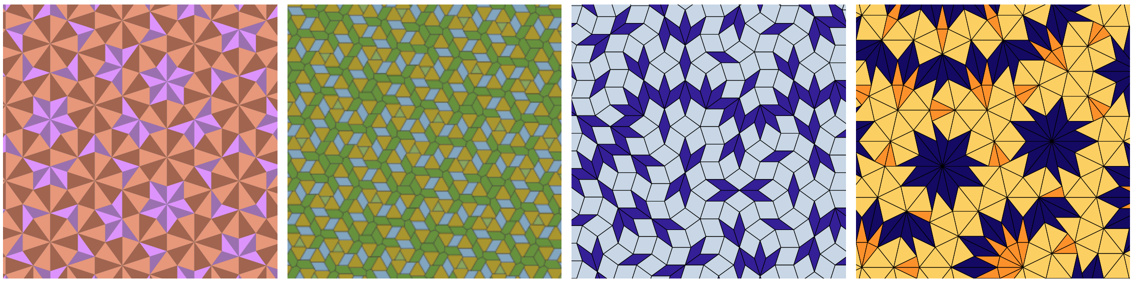
Theorem B. *Let $\mathcal{S}$ be a G-symmetric, primitive, stone, FLC substitution rule on $\mathbb{R}^2$. Then the following holds: if $|\mu| < \lambda$ for all $\mu \in \operatorname{Spec} M_{\mathrm{sph}} \setminus \{\lambda^2\}$, then $\mathcal{S}$ is hyperuniform for constant weights.*

Proof. The group G acts on $\mathbb{C}^\ell$ by permuting the basis vectors: as the substitution rule is G-symmetric, the matrix M_{full} commutes with the action of G, and the spherical matrix M_{sph} is the transformation matrix of the restriction of M_{full} to the space $V_{\mathrm{sph}} \subset \mathbb{C}^\ell$ of G-invariant vectors. Therefore, one can show that, for every eigenvalue $\mu \in \operatorname{Spec} M_{\mathrm{full}} \setminus \operatorname{Spec} M_{\mathrm{sph}}$, the corresponding eigenspace E_μ is orthogonal to V_{sph}, and in particular it is orthogonal to the constant vector $(1, 1, \cdots)^\top \in V_{\mathrm{sph}}$. Then the claim follows by applying Theorem A: the set $S(w)$ of the theorem contains only eigenvalues of M_{sph}. □

This criterion is powerful enough to prove hyperuniformity not only for the chair rule, but for many other substitution rules as well. The *Tilings Encyclopedia* [24] is a large compendium of interesting tilings including many coming from symmetric substitution rules. Armed with Theorem B, we can prove hyperuniformity for many substitution rules from the encyclopedia.

Corollary 1.29. *The following substitution rules are hyperuniform for constant weights.*

- *The Penrose substitution rule.*
- *The CAP substitution rule.*



(a) A Penrose tiling **(b)** A CAP tiling **(c)** A GLB tiling **(d)** Danzer 7-fold tiling

Figure 1.7: The tilings obtained from the substitution rules considered in Corollary 1.29. Figures (b), (c) and (d) by Frettlöh, Harriss, and Gähler [24] licensed under CC BY-NC-SA 2.0.

- *The Ammann substitution rule.*
- *The Godréche-Lançon-Billard substitution rule.*
- *Danzer's 7-fold substitution rule.*

See Figure 1.7 for pictures of the tilings obtained from the rules in Corollary 1.29.

1.4 Proof methods

Diffraction

One of the core tools in the study of substitution rules is *diffraction*. If $f \in L^1(\mathbb{R}^d)$, denote its Fourier transform by $\hat{f}$. See Chapter 2 for the definition of complex measures.

Theorem 1.30 ([15, 20])**.** *Let Φ be a stationary, locally square integrable random measure. Then there exist unique Radon measures $\hat{\eta}$ and $\hat{\eta}^+$ on $\mathbb{R}^d$*

$$\hat{\eta}(|\hat{f}|^2) = \mathrm{Var}(\Phi(f)) \quad \text{and} \quad \hat{\eta}^+(|\hat{f}|^2) = \mathbb{E}[|\Phi(f)|^2]$$

for all $f \in C_c(\mathbb{R}^d)$. Moreover, these measures are related by

$$\hat{\eta}^+ = \hat{\eta} + |\iota|^2 \delta_0$$

where ι is the intensity of the random measure Φ.

Both measures are known under the name “diffraction” in the literature. We refer to $\hat{\eta}$ and $\hat{\eta}^+$ as the *centered diffraction* and *uncentered diffraction* of Φ respectively. $\hat{\eta}$ is also known under the name *Bartlett spectral measure* in the probabilistic literature.

From now on, let $\mathcal{S}$ be a primitive, stone, FLC substitution rule on $\mathbb{R}^d$ with ℓ colors, and $\mathbf{\Phi}$ its associated vector point process. For every $w \in \mathbb{C}^\ell$ we denote by $\hat{\eta}_w$ the centered diffraction of $\langle \mathbf{\Phi}, w \rangle$, which we refer to as the *(centered) diffraction* $\hat{\eta}_w$ *of* $\mathcal{S}$ *with weights* w.

Theorem 1.31. *There exists a matrix of measures* $\hat{\boldsymbol{H}} = (\hat{H}_{jk})_{j,k\in[\ell]}$*, the* diffraction matrix of $\mathcal{S}$*, with the following property: for all* $w \in \mathbb{C}^\ell$*, we have*

$$\hat{\eta}_w = \sum_{j,k\in[\ell]} \hat{H}_{jk}\overline{w}_j w_k$$

We multiply matrices of measures with vectors by the usual formula, so we write

$$\langle \hat{\boldsymbol{H}} w, w \rangle = \sum_{j,k\in[\ell]} \hat{H}_{jk}\overline{w}_j w_k.$$

Remark 1.32. In the literature of aperiodic order, diffraction is usually defined using an ergodic average average due to Hof [34], instead of using the stochastic definition above. As pointed out by Baake, Birkner, and Moody [9], Baake, Birkner, and Grimm [10], both definitions coincide in the uniquely ergodic setting.

Specifically, given $\mathbf{\Lambda} = (\Lambda_j)_{j\in[\ell]} \in \Omega_{\mathcal{S}}$ and $w \in \mathbb{C}^\ell$, we define

$$\mu_{\mathbf{\Lambda},w} := \sum_{j=1}^{\ell} \sum_{x\in\Lambda_j} \delta_x \overline{w}_j$$

By the ergodic theorem, the limit

$$\eta_w^+ = \lim_{R\to\infty} \frac{1}{m_{\mathbb{R}^d}(B_R)} \left(\mu_{\mathbf{\Lambda},w}|_{B_R} * \widetilde{\mu_{\mathbf{\Lambda},w}|_{B_R}}\right)$$

exists and is independent of the choice of $\mathbf{\Lambda}$. Its Fourier transform is precisely the (uncentered) diffraction of $\mathcal{S}$ with weights w. Here $\widetilde{\mu}$ denotes the adjoint of a complex measure (cf. Chapter 2).

The above formula for the diffraction has a physical meaning: $\hat{\eta}_w$ describes the outcome of an X-ray diffraction experiment with diffractor $\mu_{\mathbf{\Lambda},w}$; see Hof [34].

From now on, we will only use the centered diffraction, since we are interested in the variance of the point processes rather than the squared expectation.

Hyperuniformity via diffraction

The reason we are concerned with diffraction is that a point process is hyperuniform if and only if its diffraction decays fast around the origin. This has been known in many forms since the discovery of hyperuniformity, but usually under some assumption on the type of diffraction, such as assuming that the diffraction measure is absolutely continuous. We specifically use the following criterion due to Björklund and Hartnick [16], which does not have this restriction.

Theorem 1.33 ([16]). *Let Φ be a locally square integrable stationary point process on $\mathbb{R}^d$, and let $\hat{\eta}$ be its (centered) diffraction measure. Then:*

(i) Φ is hyperuniform if and only if $\lim_{r\to 0} \frac{\hat{\eta}(B_r)}{r^d} = 0$.

(ii) Φ is Class I hyperuniform if and only if $\hat{\eta}(B_r) = O(r^{d+1})$ as $r \to 0$.

Beyond just determining when a process is or is not hyperuniform, the behavior of the diffraction around the origin encodes interesting information about the point process, such as *rigidity phenomena* [28, 37]. Physicists have studied this topic extensively, heuristically and with numerical experiments [25, 33, 47, 58], and there have also been results in this direction in probability theory: see Coste [21] for a survey.

However, when it comes to the point processes coming from substitution rules specifically, the mathematical literature seems to be lacking: the only rigorous results seem to be those of Baake and Grimm [6] for particular examples of one-dimensional self-similar tilings. In this thesis, we will extend the work of [6] to a wider class of self-similar tilings and prove a general bound for the diffraction of a substitution rule around the origin: in particular, this will allow us to prove Theorem A.

Remark 1.34. Any regular Borel measure μ can be decomposed as $\mu = \mu_{pp} + \mu_{ac} + \mu_{sc}$, where μ_{pp} is its *pure point* part, μ_{ac} its *absolutely continuous* part, and μ_{sc} its *singular continuous part.*

The diffraction of a substitution rule is often pure point, but not always: there even exist substitution rules which have singular continuous diffraction, such as the *Godréche-Lançon-Billard* rule. Our methods will be robust enough so that we will not need to make any assumptions on the type of the diffraction.

Renormalisation

Let $\mathcal{S}$ be a primitive, stone, FLC substitution rule on $\mathbb{R}^d$ with ℓ colors. Let λ be its scaling factor, $\mathbf{\Delta} = (\Delta_{jk})_{j,k\in[\ell]}$ be its displacement matrix, M_{full} be its full substitution matrix, and $\mathbf{\Phi} = (\Phi_j)_{j\in[\ell]}$ be its associated point process.

In order to prove Theorem A, we will use *renormalisation relations* associated to a substitution. These were originally introduced by Baake and Gähler [2] in the context of substitution rules, but we will define them in greater generality, using the point process associated to a substitution rule.

In particular, recall that for all $j \in [\ell]$, the point process Φ_j is *self-similar*: that is, $\Phi_j \overset{d}{=} \Phi_j \circ \varrho$, where ϱ is the substitution map. If we write out the definition of ϱ and Φ_j, we obtain the following.

Theorem 1.35. *The vector point process* $\mathbf{\Phi} = (\Phi_j)_{j\in[\ell]}$ *associated to* $\mathcal{S}$ *satisfies*

$$\Phi_k \overset{d}{=} \sum_{j=1}^{\ell} \sum_{x\in\Delta_{kj}} T_x D_\lambda \Phi_j$$

for all $k \in [\ell]$*. We call these equations the* renormalisation relations *for* $\mathbf{\Phi}$*.*

As both sides of the renormalisation relations have the same distribution, they have the same diffraction matrix, hence we obtain the following renormalisation relation for the diffraction matrix, originally due to Baake, Gähler, and Mañibo [11]. We denote the Hermitian adjoint of a matrix $A \in \mathbb{C}^{\ell\times\ell}$ by A^*.

Theorem 1.36 ([11])**.** *Let* $\mathcal{S}$ *be a primitive, FLC substitution rule in* $\mathbb{R}^d$ *with* ℓ *prototiles,* λ *its scaling constant, and* M_{full} *its substitution matrix. Let* $\hat{\boldsymbol{H}}$ *be its diffraction matrix. Then there exists a smooth, matrix-valued function* $\boldsymbol{A} : \mathbb{R}^d \to \mathbb{C}^{\ell\times\ell}$ *which satisfies* $\boldsymbol{A}(0) = \frac{1}{\lambda^d} M_{\text{full}}$*, called the* normalized Fourier matrix *of the substitution rule, such that the following holds:*

$$\hat{\boldsymbol{H}} \;=\; \boldsymbol{A}(D_{\lambda^{-1}}\hat{\boldsymbol{H}})\,\boldsymbol{A}^*,$$

Baake, Gähler, and Mañibo [11] used the above renormalisation relation to study the pure point part and the absolutely continuous part of the diffraction. In these cases, there is a natural way to write $\hat{\boldsymbol{H}}$ using a density function (with respect to either the counting measure or the Lebesgue measure, respectively), which then satisfies a recurrence relation. We improve on their methods by defining a *self-similar density* for $\hat{\boldsymbol{H}}$, which exists even if $\hat{\boldsymbol{H}}$ has a singular continuous component. Let $B_R^\times := B_R \setminus \{0\}$ be the punctured ball of radius R around the origin.

Definition 1.37. Let $R > 0$.

(i) Let ν be a positive measure supported on $B_R^\times$. We say ν is λ-*dilation invariant* if $D_\lambda\nu|_{B_R^\times} = \nu$.

(ii) A *self-similar density* of the diffraction matrix $\hat{\boldsymbol{H}}$ on $B_R^\times$ is a pair $(\boldsymbol{h}, \nu)$ where

- ν is a λ-dilation invariant measure on $B_R^\times$, and
- $\boldsymbol{h} : B_R^\times \to \mathbb{C}^{\ell\times\ell}$ is a ν-integrable, matrix-valued function

such that $\hat{\boldsymbol{H}}|_{B_R^\times} = \boldsymbol{h}\nu$.

Theorem 1.38. *Let* $R > 0$*. Then there exists a self-similar density of the diffraction matrix* $\hat{\boldsymbol{H}}$ *on* $B_R^\times$*.*

Theorem 1.39. *Let* $(\boldsymbol{h}, \nu)$ *be a self-similar density of the diffraction matrix* $\hat{\boldsymbol{H}}$ *on* $B_R^\times$. *Then the following renormalisation relation holds:*

$$\boldsymbol{h}(\xi) = \boldsymbol{A}(\xi)\boldsymbol{h}(\lambda\xi)\boldsymbol{A}(\xi)^*$$

for all $\xi \in B_{\lambda^{-1}R}^\times$.

Linear cocycles

By Theorem 1.39, if we want to understand the diffraction around the origin, we need to understand what happens when one multiplies repeatedly by the normalized Fourier matrix.

Definition 1.40. The *Fourier cocycle* of the substitution rule $\mathcal{S}$ is the matrix function given by the product

$$\boldsymbol{A}^{(N)}(\xi) = \boldsymbol{A}(\lambda^{-N+1}\xi)\boldsymbol{A}(\lambda^{-N+2}\xi)\cdots\boldsymbol{A}(\xi)$$

for $\xi \in \mathbb{R}^d$ and $N \in \mathbb{N}$.

This is analogous to the *internal cocycle* considered by Baake and Grimm [7] or the *spectral cocycle* considered by Solomyak and Treviño [57].

Then, applying Theorem 1.39 repeatedly, we get the following expression for the self-similar density around the origin.

Theorem 1.41. *Let* $(\boldsymbol{h}, \nu)$ *be a self-similar density on* $B_R^\times$ *for the diffraction matrix* $\hat{\boldsymbol{H}}$. *Then the self-similar density satisfies*

$$\boldsymbol{h}(\lambda^{-N}\xi) = \boldsymbol{A}^{(N)}(\lambda^{-1}\xi)\boldsymbol{h}(\xi)\boldsymbol{A}^{(N)}(\lambda^{-1}\xi)^*$$

for all $\xi \in B_R^\times$, $N \in \mathbb{N}$.

The normalized Fourier matrix $\boldsymbol{A}$ is smooth, therefore $\boldsymbol{A}(\xi) = \boldsymbol{A}(0) + O(\|\xi\|)$ as $\xi \to 0$. Then, if N is large, one would intuitively expect $\boldsymbol{A}^{(N)}(\xi)$ to be similar to the matrix power $\boldsymbol{A}(0)^N$, as most of the matrices in the product

$$\boldsymbol{A}^{(N)}(\xi) = \boldsymbol{A}(\lambda^{-N+1}\xi)\boldsymbol{A}(\lambda^{-N+2}\xi)\cdots\boldsymbol{A}(\xi)$$

are close to $\boldsymbol{A}(0)$. We will prove the following theorem to this effect: this similar to a theorem about products of converging matrices due to Dubiner [22]. See Section 2.4 for the definition of the asymptotic symbols $\lessapprox$ and $\approx$.

Theorem C. *Let* $\chi_1 > \cdots > \chi_l$ *be the distinct values of* $\{\log|\mu| \mid \mu \in \operatorname{Spec} \boldsymbol{A}(0), \mu \neq 0\}$ *and let* $\chi_{l+1} = -\infty$*. Let* $E_j := \oplus\{E_\mu \mid \mu \in \operatorname{Spec} \boldsymbol{A}(0), \log|\mu| = \chi_j\}$ *be the space of generalized eigenvectors associated to* χ_j *for* $j \in [l+1]$*. Then there exist uniquely defined idempotent operators* $P_j : \mathbb{C}^\ell \to \mathbb{C}^\ell$ *for* $j \in [l+1]$ *such that* $\operatorname{Im} P_j = E_j$ *and* $\sum_{j=1}^{l+1} P_j = I$ *and* $R > 0$ *such that the following hold.*

For all $\xi \in B_R$*, there exist linear maps* $P_j(\xi) : \mathbb{C}^\ell \to \mathbb{C}^\ell$ *for all* $j \in [l+1]$ *such that* $\sum_{j=1}^{l+1} P_j(\xi) = I$ *and the following asymptotic inequalities hold:*

(i) For all $j \in [l+1]$ *we have*

$$\|\boldsymbol{A}^{(N)}(\xi) P_j(\xi) x\| \approx e^{\chi_j N} \|P_j(\xi) x\| \tag{1.1}$$

as $N \to \infty$*.*

(ii) For all $j, k \in [l+1]$ *we have*

$$\|P_k \boldsymbol{A}^{(N)}(\xi) P_j(\xi) x\| \lesssim \lambda^{-N} \|\boldsymbol{A}^{(N)}(\xi) P_j(\xi) x\| \quad \text{if } j > k \tag{1.2}$$

$$\|P_k \boldsymbol{A}^{(N)}(\xi) P_j(\xi) x\| \lessapprox \max(e^{\chi_k - \chi_j}, \lambda^{-1})^N \|\boldsymbol{A}^{(N)}(\xi) P_j(\xi) x\| \quad \text{if } j < k \tag{1.3}$$

as $N \to \infty$*.*

(This is a simplified version of the theorem, see Theorem 5.3 for the full statement).

Proof sketch. For simplicity, we assume $\boldsymbol{A}(0)$ has an orthogonal basis of eigenvectors: this assumption is not necessary in general.

Recall that the matrix M_{full} is primitive with PF eigenvalue λ^d, which means $\boldsymbol{A}(0) = \lambda^{-d} M_{\text{full}}$ is primitive with PF eigenvalue $1 = e^{\chi_0}$; let w_{PF} be its left PF eigenvector. For $\xi \in \mathbb{R}^d$, define

$$w(\xi) := \lim_{N \to \infty} (\boldsymbol{A}^{(N)}(\xi))^* w_{PF}$$

One can prove the vector $w(\xi)$ exists and is nonzero for small enough ξ. Define the space

$$Y_1(\xi) := \operatorname{span} w(\xi).$$

Then the following dichotomy holds.

- If $x \notin Y_1(\xi)$, $\|\boldsymbol{A}^{(N)}(\xi) x\| \approx e^{\chi_1 N} \|x\|$ as $N \to \infty$.
- If $x \in Y_1(\xi)$, $\|\boldsymbol{A}^{(N)}(\xi) x\| \lessapprox e^{\chi_2 N} \|x\|$ as $N \to \infty$.

Now, if $B \in \mathbb{C}^{\ell \times \ell}$ and $k \in \mathbb{N}$, we define its *k-th exterior power* $B^{\wedge k}$, which is an operator on the k-th exterior power on $\mathbb{C}^\ell$ (cf. Section 5.2). Applying an argument similar to the above to the exterior powers of the normalized Fourier matrix, one can define a filtration

$$\mathbb{C}^\ell = Y_0(\xi) \supsetneq Y_1(\xi) \supsetneq Y_2(\xi) \supsetneq \cdots \supsetneq Y_{l+1}(\xi) \supseteq \{0\}$$

such that, for all $j \in [l+1]$ and $x \in Y_{j-1}(\xi) \setminus Y_j(\xi)$, we have

$$\|\boldsymbol{A}^{(N)}(\xi)x\| \approx e^{\chi_j N}\|x\|$$

Then the projections $P_j(\xi)$ can be taken to be the orthogonal projections onto the spaces $Y_{j-1}(\xi) \cap Y_{j-1}(\xi)^\perp$: this proves part (i). We can prove part (ii) by using the explicit formulas for the spaces. $\square$

Using Theorem C and Theorem 1.41, we obtain the following bound on the diffraction of a substitution rule.

Theorem D. *Let $\mathcal{S}$ be an FLC, primitive, stone substitution rule in $\mathbb{R}^d$, and $\hat{\eta}_w$ be its diffraction with weights $w \in \mathbb{C}^\ell$. Define the constants:*

$$\begin{aligned}
\beta_\parallel(w) &:= d - \max\left\{\log_\lambda|\mu| \;\middle|\; \mu \in \operatorname{Spec} M_{\text{full}} \setminus \{\lambda^d\},\ w \notin E_\mu^\perp\right\},\\
\beta_\perp(w) &:= d + 1 - \max\left\{\log_\lambda|\mu| \;\middle|\; \mu \in \operatorname{Spec} M_{\text{full}} \setminus \{\lambda^d\},\ w \in E_\mu^\perp\right\},\\
\beta(w) &:= \min\left(\beta_\perp(w), \beta_\parallel(w)\right).
\end{aligned}$$

Then we have

$$\hat{\eta}_w(B_r) \lessapprox r^{2\beta(w)}$$

as $r \to 0$.

This bound implies Theorem A (hence Theorem B) which is the criterion for hyperuniformity we stated before: furthermore, we will often be able to prove Class I hyperuniformity and obtain more precise bounds on $\hat{\eta}_w(B_r)$.

For $w = (1, 1, \cdots)^\top$, this criterion is analogous to a condition for *bounded displacement* to a lattice due to Solomon [53]. We say two discrete subsets $\Gamma, \Gamma' \subset \mathbb{R}^d$ are *bounded displacement equivalent* if there exists a bijection $\phi : \Gamma \to \Gamma'$ such that $\sup_{x\in\Gamma}\|\phi(x) - x\| < \infty$.

Theorem 1.42 ([53])**.** *Let $\mathcal{S}$ be a primitive, stone substitution rule on $\mathbb{R}^d$, and assume that $|\mu| < \lambda^{\frac{d-1}{d}}$ for all $\mu \in \operatorname{Spec} M_{\text{full}} \setminus \{\lambda^d\}$ such that $(1, 1, \cdots)^\top \notin E_\mu^\perp$.*

Then, for every self-similar $\boldsymbol{\Lambda}$, there exists $c > 0$ such that the union $\Lambda = \bigcup_{j=1}^{\ell} \Lambda_j$ is bounded displacement equivalent to the lattice $c\mathbb{Z}^d$.

1.5 Structure of the thesis

This thesis is structured as follows:

- Chapter 2 contains the basic notation used in this thesis, including operations in $\mathbb{R}^d$, probabilistic notions and asymptotic inequalities.
- Chapter 3 introduces the theory of stationary random vector measures and their autocorrelation and diffraction matrices. Vector measures are a straightforward generalization of stationary random measures, and their autocorrelation and diffraction are extensions of the scalar case. We also define hyperuniformity and its characterization in terms of diffraction. This chapter is mainly expository.
- Chapter 4 introduces the theory of substitution rules and self-similar tilings. We show that every appropriate substitution rule gives rise to a stationary random vector measure, hence we can ask about their hyperuniformity, diffraction and obtain renormalisation relations. This chapter is also expository, except for Section 4.3.2, where we define and prove the existence of self-similar densities in general.
- In Chapter 5 we prove a novel theorem about linear cocycles, using the theory of exterior powers.
- In Chapter 6, we state and prove the main result of this thesis, which provides sufficient criteria for hyperuniformity of a substitution rule in terms of its substitution matrix.
- In Chapter 7 we apply the results of Chapter 6 to a wide variety of examples, including both known and new examples of hyperuniform substitution rules.

2 Preliminaries

2.1 Notation and basics

In this thesis, we write $\mathbb{N}$ for the natural numbers without 0 and $\mathbb{N}_0$ for the natural numbers including 0. We write $\mathbb{R}$ for the real numbers and $\mathbb{R}_{\geq 0}$ for the non-negative reals. For $n \in \mathbb{N}$ we let $[n] = \{1, \cdots, n\}$. We denote the number of elements in a finite set S by $\#S$. We let $B_R \subset \mathbb{R}^d$ be the closed ball of radius R around the origin, and $B_R^\times = B_R \setminus \{0\}$ be the punctured ball.

Our inner products are always conjugate-linear in the second argument, as is standard in the mathematical literature. We denote the conjugate-transpose of a matrix $A \in \mathbb{C}^{\ell_1 \times \ell_2}$ by A^*. We let $\wedge$ be the exterior product of vectors, and $A^{\wedge q}$ be the q-th exterior power of a matrix A: see Section 5.2 for details.

For $x \in \mathbb{R}^d$ and $\lambda > 0$, we write T_x for the translation operator on $\mathbb{R}^d$ and D_λ for the dilation operator: these can also be applied to subsets of $\mathbb{R}^d$ as usual. If f is a function on $\mathbb{R}^d$, we define functions $T_x f$ and $D_\lambda f$ by $(T_x f)(y) = f(y - x)$ and $(D_\lambda f)(y) = f(\lambda^{-1} y)$ respectively. We also let $m_{\mathbb{R}^d}$ be the Lebesgue measure on $\mathbb{R}^d$.

If f is a complex valued function on $\mathbb{R}^d$, we denote its complex conjugate by $\overline{f}$. We also define the function $\tilde{f}$ by $\tilde{f}(x) = \overline{f(-x)}$. If $f \in L^1(\mathbb{R}^d)$ we denote its Fourier transform by $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle} \, \mathrm{d}x$ or $\mathcal{F}(f)$, and its inverse Fourier transform by $\check{f}$ or $\mathcal{F}^{-1}(f)$.

Later we will define vector and matrix valued measures, as well as colored subsets of $\mathbb{R}^d$ (see Section 3.1 and Section 4.1.1). We will define translation, dilation and other notions for these objects as well. In order to distinguish them from the scalar case, we will denote vector and matrix functions and measures in **bold**, and we will do the same for colored subsets of $\mathbb{R}^d$.

2.2 Complex measures

We define the basic properties of complex measures: see [14, 18] for a more detailed treatment and for proofs of the stated facts.

Let X be a locally compact Hausdorff space, and let $C_c(X)$ be the space of (complex-valued) continuous functions on X with compact support; we equip it with its usual inductive limit topology. A *(complex) measure* on X is, as usual in the literature of aperiodic order and harmonic analysis, a continuous linear functional $\mu : C_c(X) \to \mathbb{C}$. Let $C_c^+(X) \subset C_c(X)$ be the subspace of nonnegative functions: a measure μ is *positive* if $\mu(f) \geq 0$ for all $f \in C_c^+(X)$. By the Riesz representation theorem, a positive complex measure (using this definition of a complex measure) is the same as a regular Radon positive measure on X (using the classical definition of a measure as a function defined on a σ-algebra). We let $\mathcal{M}(X)$ be the space of complex measures on X and $\mathcal{M}_+(X)$ be the space of positive measures on X. We equip $\mathcal{M}(X)$ with the weak-* topology.

If μ is a complex measure on X, there exists an associated positive measure $|\mu|$, its *total variation measure*, which is the unique positive measure such that $|\mu|(f) := \sup_{|g|\leq f} |\mu(g)|$ for all $f \in C_c^+(X)$. Then one has $C_c(X) \subset L^1(|\mu|)$, so one can extend μ to $L^1(|\mu|)$ by continuity. In particular, if A is a bounded Borel set, we can define $\mu(A) := \mu(\mathbb{1}_A)$, where $\mathbb{1}_A$ is the characteristic function of A. The theorems of Fubini and Radon–Nikodym hold for complex measures as well.

Remark 2.1. This is the way Bourbaki [18] defines a measure, and it is ubiquitous in the fields of harmonic analysis and aperiodic order [14, 34, 45]. Note that, using this definition, the map $\mu : C_c(\mathbb{R}) \to \mathbb{C}$ given by $\phi \mapsto \int_{-\infty}^{\infty} \phi(x) \sin(x) \,\mathrm{d}x$ is a complex measure even though it is not a "signed measure", as $\mu(A)$ is not well-defined for unbounded Borel sets A. This is why we need to assume A is bounded when defining $\mu(A)$.

Now we focus on the case $X = \mathbb{R}^d$ for some $d \in \mathbb{N}$. We define the following operations on complex measures:

Definition 2.2. Let μ be a complex measure on X.

(i) The *conjugate* of μ is the measure $\overline{\mu}$ defined by $\overline{\mu}(f) = \overline{\mu(\overline{f})}$ for all $f \in C_c(X)$.

(ii) The *adjoint* of μ is the measure $\tilde{\mu}$ defined by $\tilde{\mu}(f) = \overline{\mu(\tilde{f})}$ for all $f \in C_c(X)$.

(iii) If g is a locally integrable function on X, the *product* of g and μ is the measure $g\mu$ defined by $(g\mu)(f) = \mu(gf)$ for all $f \in C_c(X)$.

(iv) If $X = \mathbb{R}^d$, the *translation* of μ by $x \in \mathbb{R}^d$ is the measure $T_x\mu$ defined by $(T_x\mu)(f) = \mu(T_{-x}f)$ for all $f \in C_c(\mathbb{R}^d)$, where $T_{-x}f(y) = f(y + x)$.

(v) If $X = \mathbb{R}^d$, the *dilation* of μ by $\lambda > 0$ is the measure $D_\lambda\mu$ defined by $(D_\lambda\mu)(f) = \mu(D_{\lambda^{-1}}f)$ for all $f \in C_c(\mathbb{R}^d)$.

In addition, one can define the Fourier transform of a complex measure on $\mathbb{R}^d$. Recall that, if $f \in L^1(\mathbb{R}^d)$, its *Fourier transform* $\hat{f}$ is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i\langle x,\xi\rangle} \,\mathrm{d}x$. We also define the *convolution* of two functions $f, g \in L^1(\mathbb{R}^d)$ by $(f * g)(x) = \int_{\mathbb{R}^d} f(y)g(x - y) \,\mathrm{d}y$.

Definition 2.3. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$.

(i) We say μ is *translation bounded* if the following holds: for every $f \in C_c^+(\mathbb{R}^d)$, the set

$$\{\, |\mu|(T_x f) \mid x \in \mathbb{R}^d \,\}$$

is bounded.

(ii) We say μ is *positive definite* if the following holds: for every $f \in C_c(\mathbb{R}^d)$, we have

$$\mu(f * \tilde{f}) \geq 0.$$

Definition 2.4. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a translation bounded measure. We say $\hat{\mu} \in \mathcal{M}(\mathbb{R}^d)$ is a *Fourier transform* of μ if

(i) for all $f \in C_c(\mathbb{R}^d)$, $|\hat{f}|^2 \in L^1(|\hat{\mu}|)$, and

(ii) $\mu(f * \tilde{f}) = \hat{\mu}(|\hat{f}|^2)$.

We say μ is *Fourier transformable* if it has a unique Fourier transform.

Theorem 2.5 ([14])**.** *Let μ be a positive definite measure on $\mathbb{R}^d$. Then it is Fourier transformable.*

2.3 Random variables and stochastic notation

In this thesis we will make extensive use of random variables. For the reader's convenience, we define the basic notions and notation we will use.

Recall that a *probability space* is a set Ω equipped with a σ-algebra $\mathcal{F}$ and a probability measure $\mathbb{P}$. Given a measurable space X, an *X-valued random variable* on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a measurable function $x : \Omega \to X$. If $\omega \in \Omega$, we let x_ω be the value of x at ω, and refer to it as a *sample* of the random object x. In line with common practice in probability theory, we will often say x is a random variable without specifying the underlying probability space: then, it is understood that all random objects being considered are defined over the same probability space Ω.

If $f : X \to Y$ is a measurable function between measurable spaces and $x : \Omega \to X$ is a random variable with probability space Ω, we let $f(x) : \Omega \to Y$ be the random variable defined by $\omega \mapsto f(x_\omega)$. Two random variables x, y are *equidistributed* if $x_*\mathbb{P} = y_*\mathbb{P}'$, where $\mathbb{P}$ and $\mathbb{P}'$ are the probability measures on their underlying probability spaces: in this case we write $x \overset{d}{=} y$.

Now let x, y be complex valued random variables. We say x is *integrable* (resp. *square integrable*) if the integrals $\int_\Omega |x_\omega| \, \mathrm{d}\mathbb{P}(\omega)$ and $\int_\Omega |x_\omega|^2 \, \mathrm{d}\mathbb{P}(\omega)$ are finite respectively. Then, if x, y are integrable and square integrable, we define the *expectation* and *variance* by

$\mathbb{E}[x] = \int_\Omega x_\omega \,\mathrm{d}\mathbb{P}(\omega)$ and $\mathrm{Var}[x] = \mathbb{E}\left[|x - \mathbb{E}[x]|^2\right]$ respectively. We also define the *covariance* of x and y by $\mathrm{Cov}(x, y) = \mathbb{E}\left[(x - \mathbb{E}[x])\,\overline{(y - \mathbb{E}[y])}\right]$.

2.4 Asymptotic notation

We will be concerned with the asymptotic behaviour of functions. In this section we will define the asymptotic notation we need. In particular, we will be interested in asymptotic inequalities where the constants are independent of a certain second parameter. For the reader's convenience, we define this notion.

Definition 2.6. Let X be a topological space, Y a set, $x_0 \in X$ and $f, g : X \setminus \{x_0\} \times Y \to \mathbb{R}_{\geq 0}$ be two functions.

- We write $f(x, y) \lesssim g(x, y)$ or $f(x, y) = O(g(x, y))$ *as* $x \to x_0$ *uniformly for* $y \in Y$ if there exists a neighbourhood U of x_0 and a constant $C > 0$ such that $f(x, y) \leq Cg(x, y)$ for all $x \in U$, $y \in Y$. We also write $f(x, y) \gtrsim g(x, y)$ if $g(x, y) \lesssim f(x, y)$, and $\asymp$ if both $\lesssim$ and $\gtrsim$ hold.
- We write $f(x, y) = o(g(x, y))$ *as* $x \to x_0$ *uniformly for* $y \in Y$ if for every $\epsilon > 0$ there exists a neighbourhood U of x_0 such that $f(x, y) \leq \epsilon\, g(x, y)$ for all $x \in U$, $y \in Y$.

If $X = \mathbb{N}$, we define these asymptotic symbols as $N \to \infty$ in the usual way. We can also define these symbols for functions $f, g : X \to \mathbb{R}_{\geq 0}$ as $x \to x_0$, by letting Y be the set with one element: in this way one recovers the usual meaning of O, o, $\lesssim$ and $\gtrsim$.

We also define a new asymptotic notation, which is weaker than the usual asymptotic inequalities: intuitively, it measures decay only "up to subexponential factors".

Definition 2.7. Let Y be a set.

(i) Let $f, g : \mathbb{N} \times Y \to \mathbb{R}_{\geq 0}$. We write $f(N, y) \lessapprox g(N, y)$ *as* $N \to \infty$ *uniformly for* $y \in Y$ if $\limsup_{N\to\infty} \frac{1}{N} \log \sup_{y\in Y} f(N, y) \leq \limsup_{N\to\infty} \frac{1}{N} \log \sup_{y\in Y} g(N, y)$.

(ii) Let $f, g : \mathbb{R}_{\geq 0} \times Y \to \mathbb{R}_{\geq 0}$. We write $f(r, y) \lessapprox g(r, y)$ *as* $r \to 0$ *uniformly for* $y \in Y$ if $\limsup_{r\to 0} \frac{1}{r} \log \sup_{y\in Y} f(r, y) \leq \limsup_{r\to 0} \frac{1}{r} \log \sup_{y\in Y} g(r, y)$.

We define $\gtrapprox$ analogously, using $\liminf$ instead of $\limsup$, and write $\approx$ if both $\lessapprox$ and $\gtrapprox$ hold. We can also define these notions without the second argument y, in which case we write $f(N) \lessapprox g(N)$ as $N \to \infty$. Note that we set $\log 0 = -\infty$.

Remark 2.8. In particular cases, the relation $\lessapprox$ can be expressed as follows:

- For $\alpha \in \mathbb{R}$, $f(r) \lessapprox r^\alpha$ as $r \to 0$ if and only if, for all $\epsilon > 0$, $f(r) = O(r^{\alpha+\epsilon})$ as $r \to 0$.

- $f(r) \lessapprox 0$ as $r \to 0$ if and only if, for all $k \in \mathbb{R}$, $f(r) = O(r^k)$ as $r \to 0$.

For example, $r\log(r) \lessapprox r$ as $r \to 0$ even though $r\log(r)$ is not $\lesssim r$, and $r^{-\log r} \lessapprox 0$ as $r \to 0$. Similar statements hold for functions of $\mathbb{N}$ as $N \to \infty$.

3 Diffraction of random vector measures

In this chapter, we will introduce the basics of stationary random vector measures and their diffraction. Random vector measures are a generalization of random measures, so the theory here will be a straightforward generalization of the theory of diffraction of stationary random measures. See [9, 10, 20, 39] for the classical case.

3.1 Vector measures

Definition 3.1. Let $\mathcal{A}$ be a finite set. A $\mathbb{C}^{\mathcal{A}}$*-measure* $\boldsymbol{\mu}$ *on* X is a vector of complex measures $\boldsymbol{\mu} = (\mu_a)_{a\in\mathcal{A}}$. We say μ_a is the *a-th entry* or *a-th component* of $\boldsymbol{\mu}$.

We denote by $\mathcal{M}(X, \mathbb{C}^{\mathcal{A}}) := \mathcal{M}(X)^{\mathcal{A}}$ the space of $\mathbb{C}^{\mathcal{A}}$-measures on X. We equip it with the product topology, where each $\mathcal{M}(X)$ is equipped with the weak-* topology, also called the vague topology.

In all examples we consider, $\mathcal{A}$ will be either $\mathcal{A} = [\ell]$ or $\mathcal{A} = [\ell_1] \times [\ell_2]$ for some $\ell, \ell_1, \ell_2 \in \mathbb{N}$, in which case $\boldsymbol{\mu}$ is a $\mathbb{C}^{\ell}$-measure or a $\mathbb{C}^{\ell_1\times\ell_2}$-measure respectively. We refer to the former as *vector measures* and the latter as *matrix measures*.

If $\boldsymbol{\mu} = (\mu_a)_{a\in\mathcal{A}}$ is a $\mathbb{C}^{\mathcal{A}}$-measure on X and $f \in C_c(X)$, we define a vector $\boldsymbol{\mu}(f) \in \mathbb{C}^{\mathcal{A}}$ with entries $\boldsymbol{\mu}(f) = (\mu_a(f))_{a\in\mathcal{A}}$: this defines a continuous linear map $\boldsymbol{\mu} : C_c(X) \to \mathbb{C}^{\mathcal{A}}$ which uniquely determines $\boldsymbol{\mu}$. When viewed as linear maps, the topology on $\mathcal{M}(X, \mathbb{C}^{\mathcal{A}})$ is the *strong operator topology*: a net $(\Phi_j)_{j\in J}$ converges to Φ if $\Phi_j(f) \to \Phi(f)$ in norm for all $f \in C_c(X)$. We use integral notation to evaluate vector and matrix measures, so $\int_X f(x)\,\mathrm{d}\boldsymbol{\mu}(x) := \boldsymbol{\mu}(f)$.

We can define the total variation measure of a vector or matrix measure, in analogy to the complex case. The proof that this is a measure can be found in Bourbaki [18].

Definition 3.2. Let $\mathcal{A}$ be a finite set and $\boldsymbol{\mu} = (\mu_a)_{a\in\mathcal{A}}$ be a $\mathbb{C}^{\mathcal{A}}$-measure on X. The *total variation measure* $\|\boldsymbol{\mu}\|$ is the unique positive measure on X such that

$$\|\boldsymbol{\mu}\|(f) = \sup_{g\in C_c(X), |g|\leq f} |\boldsymbol{\mu}(g)| \quad \text{for all } f \in C_c^+(X)$$

As for the case of complex measures, we can extend $\boldsymbol{\mu}$ to $L^1(\|\boldsymbol{\mu}\|)$ by continuity. Then, for every $f \in L^1(\|\boldsymbol{\mu}\|)$, we have the inequality $|\int_X f \,\mathrm{d}\boldsymbol{\mu}| \le \int_X |f| \,\mathrm{d}\|\boldsymbol{\mu}\|$.

We also have an analogue of the Radon–Nikodym theorem for stationary random measures.

Theorem 3.3 (Radon–Nikodym Theorem)**.** *Let $\mathcal{A}$ be a finite set and $\boldsymbol{\mu} = (\mu_a)_{a\in\mathcal{A}}$ be a $\mathbb{C}^{\mathcal{A}}$-measure on X. Let ν be a positive measure on X such that $\boldsymbol{\mu}(f) = 0$ for all $f \in C_c^+(X)$ with $\nu(f) = 0$. Then there exists a unique vector of functions $\boldsymbol{h} = (h_a)_{a\in\mathcal{A}}$ such that $\mu_a = h_a \nu$ for all $a \in \mathcal{A}$. We call $\boldsymbol{h}$ the* Radon–Nikodym derivative *of $\boldsymbol{\mu}$ with respect to ν.*

Furthermore, for vector measures, we define the following operations.

Definition 3.4. Let $\ell, \ell_1, \ell_2 \in \mathbb{N}$.

(i) Let $\boldsymbol{\mu} = (\mu_j)_{j=1}^{\ell}$ be a $\mathbb{C}^{\ell}$-measure on X and $w \in \mathbb{C}^{\ell}$. We define the complex measure $\langle \boldsymbol{\mu}, w\rangle$ by $\langle \boldsymbol{\mu}, w\rangle = \sum_{j=1}^{\ell} \overline{w}_j \mu_j$.

(ii) Let $\boldsymbol{M} = (\mu_{ij})_{i\in[\ell_1], j\in[\ell_2]}$ be a $\mathbb{C}^{\ell_1\times\ell_2}$-measure on X and $w \in \mathbb{C}^{\ell_2}$. We define the $\mathbb{C}^{\ell_1}$-measure $\boldsymbol{M}w$ by the entries $(\boldsymbol{M}w)_i = \sum_{j=1}^{\ell_2} \mu_{ij} w_j$ for all $i \in [\ell_1]$.

(iii) Let $\boldsymbol{\mu}$ be a $\mathbb{C}^{\ell_2}$-measure on X and $\boldsymbol{A} \in \mathbb{C}^{\ell_1\times\ell_2}$ be a matrix. We define the $\mathbb{C}^{\ell_1}$-measure $\boldsymbol{A}\boldsymbol{\mu}$ by the entries $(\boldsymbol{A}\boldsymbol{\mu})_i = \sum_{j=1}^{\ell_2} A_{ij}\mu_j$ for all $i \in [\ell_1]$.

(iv) Let $\boldsymbol{\mu} = (\mu_j)_{j=1}^{\ell}$ be a $\mathbb{C}^{\ell}$-measure on $X = \mathbb{R}^d$ and let $\boldsymbol{x} = (x_j)_{j=1}^{\ell}$ be an ℓ-tuple of elements of $\mathbb{R}^d$. Then we define the *translation* of $\boldsymbol{\mu}$ by $\boldsymbol{x}$ as the $\mathbb{C}^{\ell}$-measure $T_{\boldsymbol{x}}\boldsymbol{\mu}$ with entries $(T_{\boldsymbol{x}}\boldsymbol{\mu})_j = T_{x_j}\mu_j$ for all $j \in [\ell]$. If $x \in \mathbb{R}^d$ is a single vector, we define $T_x\boldsymbol{\mu} = T_{(x,x,\cdots,x)}\boldsymbol{\mu}$.

(v) Let $\boldsymbol{\mu}$ be a $\mathbb{C}^{\ell}$-measure on $X = \mathbb{R}^d$ and let $\lambda > 0$. Then we define the $\mathbb{C}^{\ell}$-measure $D_\lambda\boldsymbol{\mu}$ (the *dilation* of $\boldsymbol{\mu}$ by λ) with entries $(D_\lambda\boldsymbol{\mu})_j = D_\lambda\mu_j$ for all $j \in [\ell]$.

3.2 Stationary random vector measures

Let $d, \ell \in \mathbb{N}$. Recall that $\mathcal{M}(\mathbb{R}^d, \mathbb{C}^\ell)$ is equipped with the strong operator topology, which induces a Borel σ-algebra on $\mathcal{M}(\mathbb{R}^d, \mathbb{C}^\ell)$. Therefore it makes sense to talk about random $\mathbb{C}^\ell$-measures: explicitly, a random $\mathbb{C}^\ell$-measure on $\mathbb{R}^d$ is a measurable function $\boldsymbol{\mu} : \Omega \to \mathcal{M}(\mathbb{R}^d, \mathbb{C}^\ell)$. Equivalently, we can characterize a random $\mathbb{C}^\ell$-measure as an ℓ-tuple $\boldsymbol{\mu} = (\mu_j)_{j\in[\ell]}$, where each μ_j is a random complex measure over the same underlying probability space Ω.

Definition 3.5. A random $\mathbb{C}^\ell$-measure $\boldsymbol{\Phi}$ is *stationary* if the following holds: for all $x \in \mathbb{R}^d$, $T_x\boldsymbol{\Phi} \overset{d}{=} \boldsymbol{\Phi}$.

Note that a random $\mathbb{C}^\ell$-measure is the same as an ℓ-tuple of random complex measures over the same probability space.

Remark 3.6. The equality $T_x \mathbf{\Phi} \overset{d}{=} \mathbf{\Phi}$ has the following consequence: for all $f \in C_c(\mathbb{R}^d)$, $\mathbf{\Phi}(f) \overset{d}{=} \mathbf{\Phi}(T_{-x} f)$ as $\mathbb{C}^\ell$-valued random variables. A random measure with this property is called *wide-sense stationary*, and this is actually enough to define the autocorrelation measure.

Definition 3.7. Let $\mathbf{\Phi}$ be a stationary random $\mathbb{C}^\ell$-measure on $\mathbb{R}^d$. We say it is *locally square integrable* if, for all $f \in C_c(\mathbb{R}^d)$, we have $\mathbb{E}[\|\mathbf{\Phi}(f)\|^2] < \infty$.

From now on we assume all stationary random measures are locally square integrable. Note that every locally square integrable stationary random $\mathbb{C}^\ell$-measure is also locally integrable, i.e. $\mathbb{E}[\|\mathbf{\Phi}(f)\|] < \infty$.

The literature on stationary measures heavily focuses on *point processes*, i.e. stationary random measures Φ such that Φ is almost surely a countable sum of Dirac measures. Every example we consider in this thesis will be a linear combination of point processes.

Definition 3.8. An *ℓ-colored point process* is a stationary random $\mathbb{C}^\ell$-measure $\mathbf{\Phi} = (\Phi_1, \cdots, \Phi_\ell)$ such that Φ_j is almost surely a pure point measure: that is, it can be written as

$$\Phi_j = \sum_{y \in Z_j} \delta_y$$

for some discrete subset $Z_j \subset \mathbb{R}^d$. We also refer to $\mathbf{\Phi}$ as a *vector point process*, or just a *point process* if there is no potential for confusion with the scalar case.

Example 3.9. Let $\Gamma = \mathbb{Z}^2$ and $\Omega = \{T_x \Gamma \mid x \in \mathbb{R}^2\}$. We have $\Omega = \mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$: using the uniform distribution on Ω, we can define a stationary random measure by

$$\Phi : \Omega \to \mathcal{M}(\mathbb{R}^2), \quad \Gamma' \mapsto \Phi_{\Gamma'} = \sum_{y \in \Gamma'} \delta_y$$

Example 3.10. The *homogeneous Poisson process on $\mathbb{R}^d$ with intensity λ* [39] is the unique stationary point process with the two following properties:

(i) The expectation of Φ is given by $\mathbb{E}[\Phi(f)] = \lambda m_{\mathbb{R}^d}(f)$

(ii) For any $f, f' \in C_c(\mathbb{R}^d)$ such that f and f' have disjoint support, $\Phi(f)$ and $\Phi(f')$ are independent random variables.

3.3 Moments and autocorrelation of stationary random complex measures

Now we define the first and second moments of stationary random $\mathbb{C}^\ell$-measures. Due to the stationarity, they can be defined in a simplified form.

Definition 3.11. Let $\boldsymbol{\Phi}$ be a stationary random $\mathbb{C}^\ell$-measure on $\mathbb{R}^d$. The *intensity* of $\boldsymbol{\Phi}$ is the unique vector $\boldsymbol{\iota} \in \mathbb{C}^\ell$ satisfying

$$\mathbb{E}[\boldsymbol{\Phi}(f)] = \boldsymbol{\iota} m_{\mathbb{R}^d}(f)$$

for all $f \in C_c(\mathbb{R}^d)$.

Lemma 3.12. *Let $\boldsymbol{\Phi}$ be a stationary random $\mathbb{C}^\ell$-measure on $\mathbb{R}^d$ and let $f \in C_c(\mathbb{R}^d)$ with $m_{\mathbb{R}^d}(f) = 1$. Then $\boldsymbol{\Phi}$ has a unique intensity $\boldsymbol{\iota}$ given by $\boldsymbol{\iota} = \mathbb{E}[\boldsymbol{\Phi}(f)]$*

Proof. Consider the map $\boldsymbol{L} : C_c(\mathbb{R}^d) \to \mathbb{C}^\ell$ given by $f \mapsto \mathbb{E}[\boldsymbol{\Phi}(f)]$. As $\boldsymbol{\Phi}$ is locally square integrable, $\boldsymbol{L}$ is a $\mathbb{C}^\ell$-valued measure with entries L_j given by $L_j(f) = \mathbb{E}[\boldsymbol{\Phi}_j(f)]$: this follows from Campbell's formula for stationary random measures.

Furthermore, as $\boldsymbol{\Phi}$ is stationary, for all $x \in \mathbb{R}^d$, $f \in C_c(\mathbb{R}^d)$ we have:

$$\boldsymbol{L}(T_x f) = \mathbb{E}[\boldsymbol{\Phi}(T_x f)] = \mathbb{E}[\boldsymbol{\Phi}(f)] = \boldsymbol{L}(f)$$

Therefore, for all $j \in [\ell]$, there exists a unique $\iota_j \in \mathbb{C}$ such that $L_j(f) = \iota_j m_{\mathbb{R}^d}(f)$ [19, Chapter 7]. Therefore, $\boldsymbol{L} = \boldsymbol{\iota} m_{\mathbb{R}^d}$ for $\boldsymbol{\iota} = (\iota_j)_{j=1}^{\ell}$, which means $\boldsymbol{\iota}$ is the unique intensity of $\boldsymbol{\Phi}$. As $m_{\mathbb{R}^d}(f) = 1$, we have $\boldsymbol{\iota} = \mathbb{E}[\boldsymbol{\Phi}(f)]$. □

Example 3.13. Before we define the "homogeneuous Poisson process on $\mathbb{R}^d$ with intensity λ", which satisfies $\mathbb{E}[\Phi(f)] = \lambda m_{\mathbb{R}^d}(f)$. Its intensity in the sense of Definition 3.11 is therefore $\iota = \lambda$, justifying the name.

A complex-valued stationary random measure Φ on $\mathbb{R}^d$ always has an *autocorrelation measure* as we will now define.

Definition 3.14. Let Φ be a stationary random measure on $\mathbb{R}^d$. The *(centered) autocorrelation measure* η of Φ is the unique measure on $\mathbb{R}^d$ such that

$$\mathrm{Var}(\Phi(f)) = \eta(f * \tilde{f})$$

for all $f \in C_c(\mathbb{R}^d)$.

Remark 3.15. One can also define an *uncentered autocorrelation* η^+ of Φ by requiring

$$\eta^+(f * \tilde{f}) = \mathbb{E}[|\Phi(f)|^2]$$

for all $f \in C_c(\mathbb{R}^d)$ instead The uncentered autocorrelation exists if and only if the centered autocorrelation exists, and they are related by the following formula [16, Section 2.1]:

$$\eta^+(f) = \eta(f) + |\iota|^2 m_{\mathbb{R}^d}(f).$$

In the literature, the term "autocorrelation" may be used to refer to either the centered or the uncentered autocorrelation: we will always use the centered autocorrelation unless otherwise stated.

The existence and uniqueness of such measures is well known in the point process case [9, 10], sometimes defined via other related quantities such as the *reduced second factorial moment measure* [15, 20, 39]. The name "autocorrelation" is typical in the literature of aperiodic order. The existing literature only handles the case where Φ is positive: as we want to make sure they also exists in the complex valued case, we provide a proof, but it is essentially the same as the classical case, which can be found in the references above.

Recall that a measure is *positive definite* if $\eta(f * \tilde{f}) \geq 0$ for all $f \in C_c(\mathbb{R}^d)$.

Theorem 3.16. *Let Φ be a stationary random measure on $\mathbb{R}^d$. Then Φ has a unique autocorrelation measure η, and it is positive definite.*

To prove this we need the following lemma first:

Lemma 3.17. *Let η, η' be two measures on $\mathbb{R}^d$ such that $\eta(f * \tilde{f}) = \eta'(f * \tilde{f})$ for all $f \in C_c(\mathbb{R}^d)$. Then $\eta = \eta'$*

Proof. For all $f, g \in C_c(\mathbb{R}^d)$, the following *polarization identity* holds:

$$f * \tilde{g} = \frac{1}{4} \sum_{t=0}^{3} i^t (f + i^t g) * \widetilde{f + i^t g} \tag{3.1}$$

Therefore we can conclude that $\eta(f * \tilde{g}) = \eta'(f * \tilde{g})$ for all $f, g \in C_c(\mathbb{R}^d)$.

Now let $(g_\alpha)_{\alpha \in A}$ be an approximate identity, i.e. a net in $C_c(\mathbb{R}^d)$ such that $f * \tilde{g}_\alpha \to f$ in the strong topology (see Moody and Strungaru [45, Definition 4.7.5]). Then

$$\eta(f) = \lim_\alpha \eta(f * \tilde{g}_\alpha) = \lim_\alpha \eta'(f * \tilde{g}_\alpha) = \eta'(f)$$

for all $f \in C_c(\mathbb{R}^d)$, which concludes the proof. □

Proof of Theorem 3.16. By Lemma 3.17, any two autocorrelation measures η, η' are equal, as we have $\eta(f * \tilde{f}) = \eta'(f * \tilde{f}) = \operatorname{Var}\Phi(f)$: this means the autocorrelation measure is unique if it exists. Furthermore, we have $\eta(f * \tilde{f}) = \operatorname{Var}\Phi(f) \geq 0$ for all $f \in C_c(\mathbb{R}^d)$, so any η defined this way is positive definite.

We need to show that an autocorrelation measure η exists. To do this, pick any $\rho \in C_c(\mathbb{R}^d)$ such that $m_{\mathbb{R}^d}(\rho) = 1$: then we define the measure η on $C_c(\mathbb{R}^d)$ by the following formula:

$$\eta(f) = \mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y-x)\rho(x)\,\mathrm{d}\Phi(x)\mathrm{d}\overline{\Phi}(y)\right] - |\iota|^2 m_{\mathbb{R}^d}(f)$$

(This is the same formula as in Björklund and Byléhn [15, Definition 2.7], except for the different setting and notation). This is a complex measure on $\mathbb{R}^d$: we need to check that this is the autocorrelation measure of Φ.

So let $f \in C_c(\mathbb{R}^d)$. Then we have

$$(f * \tilde{f})(y-x) = \int_{\mathbb{R}^d} f(x-z)\overline{f(y-z)}\,\mathrm{d}z \tag{*}$$

for all $x, y \in \mathbb{R}^d$. Therefore

$$\begin{aligned}
\eta(f * \tilde{f}) &= \mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f * \tilde{f})(x-y)\rho(x)\,\mathrm{d}\Phi(x)\mathrm{d}\overline{\Phi}(y)\right] - |\iota|^2 m_{\mathbb{R}^d}(f * \tilde{f}) \\
&= \mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-z)\overline{f(y-z)}\rho(x)\,\mathrm{d}\Phi(x)\mathrm{d}\overline{\Phi}(y)\,\mathrm{d}z\right] - |\iota|^2 m_{\mathbb{R}^d}(f * \tilde{f}) \qquad (*) \\
&= \int_{\mathbb{R}^d}\mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x-z)\overline{f(y-z)}\rho(x)\,\mathrm{d}\Phi(x)\mathrm{d}\overline{\Phi}(y)\right]\mathrm{d}z - |\iota|^2 m_{\mathbb{R}^d}(f * \tilde{f}) \\
&\qquad\qquad \text{(Fubini)} \\
&= \int_{\mathbb{R}^d}\mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)\overline{f(y)}\rho(x+z)\,\mathrm{d}\Phi(x)\mathrm{d}\overline{\Phi}(y)\right]\mathrm{d}z - |\iota|^2 m_{\mathbb{R}^d}(f * \tilde{f}) \\
&\qquad\qquad (\Phi \text{ stationary}) \\
&= \mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)\overline{f(y)}\int_{\mathbb{R}^d}\rho(x+z)\,\mathrm{d}z\,\mathrm{d}\Phi(x)\mathrm{d}\overline{\Phi}(y)\right] - |\iota|^2 m_{\mathbb{R}^d}(f * \tilde{f}) \quad \text{(Fubini)} \\
&= m_{\mathbb{R}^d}(\rho)\mathbb{E}\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x)\overline{f(y)}\,\mathrm{d}\Phi(x)\mathrm{d}\overline{\Phi}(y)\right] - |\iota|^2 m_{\mathbb{R}^d}(f * \tilde{f}) \\
&= \mathbb{E}\left[\Phi(f)\overline{\Phi(f)}\right] - |\mathbb{E}[\Phi(f)]|^2 \\
&= \operatorname{Var}(\Phi(f))
\end{aligned}$$

Fubini is applicable because the term inside the integrals is continuous and compactly supported as a function of x, y, z, and Φ is assumed to be locally square integrable.

Therefore, $\eta(f * \tilde{f}) = \operatorname{Var}(\Phi(f))$ for all $f \in C_c(\mathbb{R}^d)$, which shows that η is indeed the autocorrelation measure of Φ. □

If $\mathbf{\Phi} = (\Phi_j)_{j=1}^{\ell}$ is a stationary random $\mathbb{C}^\ell$-measure, for every $w \in \mathbb{C}^\ell$ we obtain a stationary random measure $\langle \mathbf{\Phi}, w\rangle = \sum_{j=1}^{\ell} \overline{w}_j \Phi_j$. For example, if $w = (1, \cdots, 1)^\top$ is the constant vector, then $\langle \mathbf{\Phi}, w\rangle = \sum_{j=1}^{\ell} \Phi_j$, while if $w = e_j$ for some $j \in [\ell]$, then $\langle \mathbf{\Phi}, w\rangle = \Phi_j$. Each one of these has its own autocorrelation measure: we define the *autocorrelation matrix* of $\mathbf{\Phi}$, which contains the information of all these autocorrelation measures. This is analogous to the "pair autocorrelation matrix" in [11].

Definition 3.18. Let $\mathbf{\Phi}$ be a stationary random $\mathbb{C}^\ell$-measure on $\mathbb{R}^d$. Its *(centered) autocorrelation matrix* $\boldsymbol{H}$, if it exists, is the unique $\mathbb{C}^{\ell\times\ell}$-valued measure with the following property: for all $w \in \mathbb{C}^\ell$, $\langle \boldsymbol{H} w, w\rangle$ is the autocorrelation measure of $\langle \mathbf{\Phi}, w\rangle$.

Theorem 3.19. *Let $\mathbf{\Phi} = (\Phi_j)_{j=1}^{\ell}$ be a stationary random $\mathbb{C}^\ell$-measure on $\mathbb{R}^d$. Then $\mathbf{\Phi}$ has an autocorrelation matrix $\boldsymbol{H} = (H_{jk})_{j,k=1}^{\ell}$, whose entries are determined by the following formula:*

$$H_{jk}(f * \tilde{f}) = \operatorname{Cov}(\Phi_j(f), \Phi_k(f))$$

for all $f \in C_c(\mathbb{R}^d)$, $j, k \in [\ell]$.

Furthermore, every coefficient of H is a linear combination of positive definite measures.

Proof. For every $w \in \mathbb{C}^\ell$, the random measure $\langle \mathbf{\Phi}, w\rangle$ has a positive definite autocorrelation measure η_w, which by definition satisfies:

$$\operatorname{Var}(\langle \mathbf{\Phi}(f), w\rangle) = \eta_w(f * \tilde{f})$$

For $j, k \in [\ell]$ and $t \in \{0, 1, 2, 3\}$, let $w_{j,k,t} = e_j + i^t e_k$ and let $\eta_{j,k,t}$ be the corresponding autocorrelation measure. Then we define a matrix measure H by setting

$$H_{jk} := \frac{1}{4} \sum_{t=0}^{3} i^t \eta_{j,k,t}$$

for $j, k \in [\ell]$. Using the polarization identity from before (3.1),

$$\begin{aligned} H_{jk}(f * \tilde{f}) &= \frac{1}{4} \sum_{t=0}^{3} i^t \eta_{j,k,t}(f * \tilde{f}) \\ &= \frac{1}{4} \sum_{t=0}^{3} i^t \operatorname{Var}(\Phi_j(f) + i^t \Phi_k(f)) \\ &= \operatorname{Cov}(\Phi_j(f), \Phi_k(f)) \end{aligned}$$

and for any $w = (w_j)_{j=1}^{\ell} \in \mathbb{C}^\ell$, we have:

$$\begin{aligned} \langle \boldsymbol{H}(f * \tilde{f}) w, w \rangle &= \sum_{j,k=1}^{\ell} w_j \overline{w_k} H_{jk}(f * \tilde{f}) \\ &= \sum_{j,k=1}^{\ell} w_j \overline{w_k} \operatorname{Cov}(\Phi_j(f), \Phi_k(f)) \\ &= \operatorname{Var}\left(\sum_{j=1}^{\ell} w_j \Phi_j(f) \right) \\ &= \operatorname{Var}(\langle \boldsymbol{\Phi}(f), w \rangle) \end{aligned}$$

Therefore the matrix of measures $\boldsymbol{H}$ we just defined is in fact the autocorrelation matrix of $\boldsymbol{\Phi}$, and its coefficients are given as in the statement of the theorem.

Finally, by definition of the autocorrelation measure of a point process, $\eta_{j,k,t}$ is positive definite for all $j, k \in [\ell]$ and $t \in \{0, 1, 2, 3\}$, so its coefficients H_{jk} are linear combinations of positive definite measures. □

Note that, in the above proof, we are forced to consider complex measures to use the polarization identity, even if the original random measures are positive. This is one of the reasons we need to work with complex measures.

3.4 Ergodicity

If a stationary random measure $\boldsymbol{\Phi}$ is ergodic, then one gets a formula for the autocorrelation based on spatial averages. In this section we will prove a formula of this form assuming the stationary random measure $\boldsymbol{\Phi}$ is *uniquely ergodic*. This property is not usually defined in probability theory textbooks, as it requires putting a topology on the state space Ω: however, it will be satisfied for the point processes we consider later.

Definition 3.20. Let Ω be a compact metric space equipped with a continuous $\mathbb{R}^d$-action T. We say Ω is *uniquely ergodic* if it has a unique T-invariant probability measure.

Definition 3.21. A *uniquely ergodic stationary random $\mathbb{C}^\ell$-measure on $\mathbb{R}^d$* is a tuple $(\Omega, \mathbf{\Phi})$ such that

(i) Ω is a compact metric space equipped with a continuous $\mathbb{R}^d$-action which makes it uniquely ergodic, and

(ii) $\mathbf{\Phi} : \Omega \to \mathcal{M}(X, \mathbb{C}^\ell)$ is a continuous $\mathbb{R}^d$-equivariant map, i.e. for all $x \in \mathbb{R}^d$ and $\omega \in \Omega$, we have $T_x\mathbf{\Phi}(\omega) = \mathbf{\Phi}(T_x\omega)$.

As Ω has a unique $\mathbb{R}^d$-invariant probability measure, $\mathbf{\Phi}$ is a random measure: thanks to the equivariance we see that it is in fact a stationary random $\mathbb{C}^\ell$-measure. As Ω is compact, $\mathbf{\Phi}$ is locally square integrable. As usual, we say $\mathbf{\Phi}$ is a uniquely ergodic stationary random measure, leaving the underlying space Ω implicit.

If $\mathbf{\Phi}$ is a uniquely ergodic stationary random $\mathbb{C}^\ell$-measure on $\mathbb{R}^d$ and $w \in \mathbb{C}^\ell$, the complex-valued stationary random measure $\langle \mathbf{\Phi}, w \rangle$ is also uniquely ergodic: for the rest of this section we will focus on complex measures.

The following pointwise ergodic theorem holds for uniquely ergodic $\mathbb{R}^d$-actions. It is analogous to the classical pointwise ergodic theorem for uniquely ergodic $\mathbb{N}$-actions. As it is hard to find a reference for this result, we provide a proof for convenience.

Theorem 3.22 (Pointwise ergodic theorem for uniquely ergodic $\mathbb{R}^d$-actions)**.** *Let $F \in C(\Omega)$, and $\omega \in \Omega$. Then*

$$\lim_{R\to\infty} \frac{1}{m_{\mathbb{R}^d}(B_R)} \int_{B_R} F(T_x\omega)\,\mathrm{d}x = \int_\Omega F(\omega')\,\mathrm{d}\mu(\omega')$$

Proof. For $R > 0$, define the probability measure μ_R on Ω by

$$\mu_R(F) := \frac{1}{m_{\mathbb{R}^d}(B_R)} \int_{B_R} F(T_x\omega)\,\mathrm{d}x$$

for $F \in C(\Omega)$.

By the Banach-Alaoglu theorem, the space of probability measures on Ω is compact in the weak topology, so the net $(\mu_R)_{R>0}$ has an accumulation point.

Now we show every accumulation point is translation invariant: so let $(R_n)_{n\in\mathbb{N}}$ be a sequence of radii such that $R_n \to \infty$ and a limit $\mu_\infty = \lim_{n\to\infty} \mu_{R_n}$ exists in the weak topology. Then, for every $F \in C(\Omega)$ and $x \in \mathbb{R}^d$, we have

$$\begin{aligned}
|\mu_{R_n}(T_x F) - \mu_{R_n}(F)| &= |\frac{1}{m_{\mathbb{R}^d}(B_{R_n})}\int_{B_{R_n}} T_x F(T_y\omega) - F(T_y\omega)\,\mathrm{d}y| \\
&= \frac{1}{m_{\mathbb{R}^d}(B_{R_n})}|\int_{T_x B_{R_n}} F(T_y\omega)\,\mathrm{d}y - \int_{B_{R_n}} F(T_y\omega)\,\mathrm{d}y| \\
&\le \frac{\|F\|_\infty}{m_{\mathbb{R}^d}(B_{R_n})}(m_{\mathbb{R}^d}(T_x B_{R_n} \setminus B_{R_n}) + m_{\mathbb{R}^d}(B_{R_n} \setminus T_x B_{R_n})) \\
&\to 0
\end{aligned}$$

Therefore $\mu_\infty(T_x F) = \mu_\infty(F)$ for all $x \in \mathbb{R}^d$ and $F \in C(\Omega)$, which means μ_∞ is translation invariant. This means μ_∞ must be equal to the unique $\mathbb{R}^d$-invariant probability measure μ on Ω, so in fact $\mu_R \to \mu$ in the weak topology as $R \to \infty$.

Unwinding the definitions for any $F \in C(\Omega)$ yields

$$\lim_{R\to\infty} \mu_R(F) = \int_\Omega F(\omega')\,\mathrm{d}\mu(\omega'),$$

as claimed. □

We can use this to characterize the intensity and autocorrelation of a uniquely ergodic stationary random measure. As the proofs are similar, we only provide the proof for the autocorrelation, which is harder.

Corollary 3.23. *Let Φ be a uniquely ergodic stationary random measure on $\mathbb{R}^d$ with underlying space Ω, and $\omega \in \Omega$. Then the intensity ι of Φ is given by the formula*

$$\iota = \lim_{R\to\infty} \frac{1}{m_{\mathbb{R}^d}(B_R)}\int_{B_R} \Phi_\omega(T_x f)\,\mathrm{d}x$$

for any $f \in C_c(\mathbb{R}^d)$ with $m_{\mathbb{R}^d}(f) = 1$.

Corollary 3.24. *Let Φ be a uniquely ergodic stationary random measure on $\mathbb{R}^d$ with underlying space Ω, and $\omega \in \Omega$.*

Then the autocorrelation η of Φ is given by the formula

$$\eta(f) + |\iota|^2 m_{\mathbb{R}^d}(f) = \lim_{R\to\infty} \frac{1}{m_{\mathbb{R}^d}(B_R)}\int_{B_R}\int_{\mathbb{R}^d} f(x-y)\,\mathrm{d}\Phi_\omega(x)\,\mathrm{d}\overline{\Phi}_\omega(y)$$

for all $f \in C_c(\mathbb{R}^d)$.

Proof. It suffices to consider functions of the form $f = g * \tilde{g}$ for $g \in C_c(\mathbb{R}^d)$, as these are dense in $C_c(\mathbb{R}^d)$. Then we have

$$\eta(g * \tilde{g}) + |\iota|^2 m_{\mathbb{R}^d}(f) = \mathbb{E}[|\Phi(g)|^2]$$

Applying the pointwise ergodic theorem to the right hand side, we obtain

$$\begin{aligned}\eta(g * \tilde{g}) + |\iota|^2 m_{\mathbb{R}^d}(f) &= \lim_{R\to\infty} \frac{1}{m_{\mathbb{R}^d}(B_R)} \int_{B_R} |T_z \Phi_\omega(g)|^2 \,\mathrm{d}z \\ &= \lim_{R\to\infty} \frac{1}{m_{\mathbb{R}^d}(B_R)} \int_{B_R} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(x+z)\overline{g(y+z)} \,\mathrm{d}\Phi_\omega(x) \,\mathrm{d}\overline{\Phi}_\omega(y) \,\mathrm{d}z\end{aligned}$$

The function $(z, y, x) \mapsto g(x+z)\overline{g(y+z)}$ is in $C_c(B_R \times \mathbb{R}^d \times \mathbb{R}^d)$, so we can apply Fubini to change the order of integration and apply a change of variables on $z \mapsto z - y$ to obtain:

$$\begin{aligned}\eta(g * \tilde{g}) + |\iota|^2 m_{\mathbb{R}^d}(f) &= \lim_{R\to\infty} \frac{1}{m_{\mathbb{R}^d}(B_R)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{B_R} g(x+z)\overline{g(y+z)} \,\mathrm{d}z \,\mathrm{d}\Phi_\omega(x) \,\mathrm{d}\overline{\Phi}_\omega(y) \\ &= \lim_{R\to\infty} \frac{1}{m_{\mathbb{R}^d}(B_R)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{T_y B_R} g(x-y+z)\overline{g(z)} \,\mathrm{d}z \,\mathrm{d}\Phi_\omega(x) \,\mathrm{d}\overline{\Phi}_\omega(y)\end{aligned}$$

Now let $R_0 > 0$ be such that g is supported in B_{R_0}. Then the term under the integral can only be nonzero if $y \in B_{R+R_0}$, as otherwise $g(z) = 0$ for all $z \in T_y B_R$. Therefore, the integral remains the same if we integrate y over B_{R+R_0} and z over $\mathbb{R}^d$, and the function $(y, x, z) \mapsto g(x-y+z)\overline{g(z)}$ is in $C_c(B_{R+R_0} \times \mathbb{R}^d \times \mathbb{R}^d)$, so we can apply Fubini again:

$$\begin{aligned}\eta(g * \tilde{g}) + |\iota|^2 m_{\mathbb{R}^d}(f) &= \lim_{R\to\infty} \frac{1}{m_{\mathbb{R}^d}(B_R)} \int_{B_{R+R_0}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(x-y+z)\overline{g(z)} \,\mathrm{d}z \,\mathrm{d}\Phi_\omega(x) \,\mathrm{d}\overline{\Phi}_\omega(y) \\ &= \lim_{R\to\infty} \frac{1}{m_{\mathbb{R}^d}(B_R)} \int_{B_{R+R_0}} \int_{\mathbb{R}^d} (g * \tilde{g})(x-y) \,\mathrm{d}\Phi_\omega(x) \,\mathrm{d}\overline{\Phi}_\omega(y) \\ &= \lim_{R\to\infty} \frac{1}{m_{\mathbb{R}^d}(B_{R-R_0})} \int_{B_R} \int_{\mathbb{R}^d} (g * \tilde{g})(x-y) \,\mathrm{d}\Phi_\omega(x) \,\mathrm{d}\overline{\Phi}_\omega(y) \\ &= \lim_{R\to\infty} \frac{1}{m_{\mathbb{R}^d}(B_R)} \int_{B_R} \int_{\mathbb{R}^d} (g * \tilde{g})(x-y) \,\mathrm{d}\Phi_\omega(x) \,\mathrm{d}\overline{\Phi}_\omega(y)\end{aligned}$$

Where the last step holds because $\lim_{R\to\infty} \frac{m_{\mathbb{R}^d}(B_{R-R_0})}{m_{\mathbb{R}^d}(B_R)} = 1$. This concludes the proof. □

Note that, by standard approximation arguments, the formulas in the above corollaries also hold when $f = \mathbb{1}_{B_R}$.

3.5 Diffraction of stationary random vector measures

Recall that the autocorrelation measure is always positive definite. Therefore, by Theorem 3.16, we can define the following:

Definition 3.25. Let Φ be a stationary random measure on $\mathbb{R}^d$ with autocorrelation measure η. The *(centered) diffraction measure* $\hat{\eta}$ of Φ is the Fourier transform of its autocorrelation measure η.

Remark 3.26. The diffraction measure $\hat{\eta}$ of Φ satisfies

$$\mathrm{Var}(\Phi(f)) = \hat{\eta}(|\hat{f}|^2)$$

for all $f \in C_c(\mathbb{R}^d)$. A measure satisfying this property is also called the *Bartlett spectral measure* or *structure factor* [15, 20, 21].

Remark 3.27. Recall that, as was the case for the autocorrelation, one could define the *uncentered diffraction* $\hat{\eta}^+$ which satisfies

$$\mathbb{E}[|\Phi(f)|^2] = \hat{\eta}^+(|\hat{f}|^2)$$

for all $f \in C_c(\mathbb{R}^d)$ instead. In this case, $\hat{\eta}^+ = \hat{\eta} + |\iota|^2\delta_0$, where ι is the intensity of Φ. The term "diffraction" is often used to refer to the uncentered version, particularly in the literature of aperiodic order: we will however always use the centered version unless otherwise specified.

Example 3.28.

(i) Using the Poisson summation formula, we see that the lattice process has (centered) autocorrelation $\eta = \delta_{\mathbb{Z}^2} - m_{\mathbb{R}^d}$, hence it has (centered) diffraction $\hat{\eta} = \delta_{\mathbb{Z}^2\setminus\{0\}}$. It has intensity $\iota = 1$.

(ii) The Poisson process with intensity $\lambda > 0$ has (centered) autocorrelation $\eta = \lambda\delta_0$, hence it has diffraction $\hat{\eta} = \lambda m_{\mathbb{R}^d}$.

Analogously, in the vector-valued case, we define the diffraction matrix as follows:

Definition 3.29. Let $\boldsymbol{\Phi}$ be a stationary random $\mathbb{C}^\ell$-measure on $\mathbb{R}^d$. The *diffraction matrix* $\hat{\boldsymbol{H}}$ of $\boldsymbol{\Phi}$ is the componentwise Fourier transform of its autocorrelation matrix $\boldsymbol{H}$.

The following lemma is a direct consequence of Theorem 3.19 and Theorem 2.5, together with the polarization identity Equation (3.1).

Lemma 3.30. *Let* $\boldsymbol{\Phi} = (\Phi_j)_{j=1}^{\ell}$ *be a stationary random* $\mathbb{C}^{\ell}$*-measure on* $\mathbb{R}^d$ *with autocorrelation matrix* $\boldsymbol{H} = (H_{jk})_{j,k=1}^{\ell}$*. Then the following hold:*

(i) The entries of $\hat{\boldsymbol{H}}$ *are given by*

$$\hat{H}_{jk}(f\overline{g}) = \operatorname{Cov}(\Phi_j(f), \Phi_k(g))$$

for all $f, g \in C_c(\mathbb{R}^d)$ *and* $j, k \in [\ell]$*.*

(ii) For all $w \in \mathbb{C}^{\ell}$*,* $\langle \hat{\boldsymbol{H}} w, w\rangle$ *is the diffraction measure of* $\langle \boldsymbol{\Phi}, w\rangle$*. That is, for all* $f \in C_c(\mathbb{R}^d)$ *we have the identity*

$$\operatorname{Var}(\langle \boldsymbol{\Phi}, w\rangle(f)) = \langle \hat{\boldsymbol{H}} w, w\rangle(|\hat{f}|^2)$$

In the scalar case, the Fourier transform of a positive definite measure is a positive measure. Here, an analogous result holds.

Lemma 3.31. *For all* $w \in \mathbb{C}^{\ell}$*,* $\langle \hat{\boldsymbol{H}} w, w\rangle$ *is a positive measure.*

Proof. Let $f \in C_c(\mathbb{R}^d)$. Then, for all $w \in \mathbb{C}^{\ell}$, we have $\langle \hat{\boldsymbol{H}} w, w\rangle(|\hat{f}|^2) \geq 0$. As $\langle \hat{\boldsymbol{H}} w, w\rangle$ is the diffraction measure of $\langle \boldsymbol{\Phi}, w\rangle$, it is positive. □

In the uniquely ergodic case, the diffraction matrix vanishes at zero. Note that this happens because we are using the centered autocorrelation measure: otherwise we would get an atom at the origin related to the intensity (compare with [5, Proposition 9.2]).

Lemma 3.32. *Assume* $\boldsymbol{\Phi}$ *is uniquely ergodic. Then* $\hat{\boldsymbol{H}}(\{0\}) = 0$*.*

Proof. Björklund and Hartnick [16] prove this in the case where $\boldsymbol{\Phi}$ is positive: the same proof works when $\boldsymbol{\Phi}$ is complex. Then, if $\boldsymbol{\Phi}$ is a $\mathbb{C}^{\ell}$-measure, for all $w \in \mathbb{C}^{\ell}$, $\langle \hat{\boldsymbol{H}}(\{0\}) w, w\rangle = 0$, as $\langle \hat{\boldsymbol{H}} w, w\rangle$ is the diffraction of the uniquely ergodic stationary random complex measure $\langle \boldsymbol{\Phi}, w\rangle$. This means that $\hat{\boldsymbol{H}}(\{0\})$ is the zero matrix. □

3.6 Hyperuniformity

Now we define the notion of *hyperuniformity*, introduced by Torquato and Stillinger [59] in the context of point processes. Intuitively, hyperuniformity indicates a certain degree of "order", in the sense that the variance of the measure on large sets is less than one would expect from random chance.

We will define hyperuniformity for complex-valued measures first, which is the classical definition, and then we will extend it to vector-valued measures by using weights.

Definition 3.33. Let Φ be a stationary random (complex) measure on $\mathbb{R}^d$.

(i) We say Φ is *hyperuniform* if

$$\lim_{R\to\infty} \frac{\operatorname{Var}\Phi(B_R)}{R^d} = 0$$

(ii) We say Φ is *Class I hyperuniform* if

$$\operatorname{Var}\Phi(B_R) = O(R^{d-1})$$

as $R \to \infty$.

The following examples justify the name "hyperuniformity": informally, for a hyperuniform random measure, the *number variance* $\operatorname{Var}\Phi(B_R)$ grows more slowly than it would for a Poisson process.

Example 3.34.

(i) Let Φ be the Poisson process on $\mathbb{R}^d$ with intensity λ. Then the variance is given by $\operatorname{Var}\Phi(B_R) = \lambda m_{\mathbb{R}^d}(B_R)$, hence $\frac{\operatorname{Var}\Phi(B_R)}{R^d} = \lambda m_{\mathbb{R}^d}(B_1)$ for all $R > 0$. This means Φ is *not* hyperuniform.

(ii) Let $\Phi_{\mathbb{Z}^d}$ be the point process associated to the integer lattice on $\mathbb{R}^d$. Then one can compute that $\operatorname{Var}\Phi(B_R) = O(R^{d-1})$ as $R \to \infty$: in particular, Φ is Class I hyperuniform.

The bound we get for the integer lattice is the "best possible" in the following sense: there exists no stationary random measure Φ such that $\operatorname{Var}\Phi(B_R) = O(R^{d-k})$ as $R \to \infty$ for $k > 1$. This follows from the following theorem.

Theorem 3.35 (Beck's Theorem [15, Theorem 5.1])**.** *Let Φ be a stationary point process on $\mathbb{R}^d$. Then, for every $R_0 > 0$ there exists a $C > 0$ such that*

$$\frac{1}{R}\int_0^R \operatorname{Var}(\#\Phi \cap B_r)\,\mathrm{d}r \geq CR^{d-1}, \quad R \geq R_0$$

In particular, there exists no stationary point process Φ such that $\operatorname{Var}(\#\Phi \cap B_R) = O(R^{d-k})$ for any $k > 1$.

One of the most important properties of hyperuniformity is that it can be characterized in terms of the diffraction measure: in particular, hyperuniformity is equivalent to fast decay of the diffraction measure at the origin. While this has been known at least experimentally since its invention, most existing proofs of this fact focus on the case where the diffraction measure is absolutely continuous, as this is the most commonly considered case in materials science and point process theory. For our purposes, we will need the following criteria due to Björklund and Hartnick [16], which are valid without making any assumptions on the type of the diffraction measure.

Theorem 3.36 ([16])**.** *Let Φ be a stationary random measure on $\mathbb{R}^d$ and $\hat{\eta}$ its diffraction measure.*

(i) *For $\alpha \in [0,1]$, we have $\mathrm{Var}(\Phi(B_R)) = O(R^{d-\alpha})$ as $R \to \infty$ if and only if $\hat{\eta}(B_r) = O(r^{d+\alpha})$ as $r \to 0$.*

(ii) *For $\alpha \in [0,1)$, we have $\mathrm{Var}(\Phi(B_R)) = o(R^{d-\alpha})$ as $R \to \infty$ if and only if $\hat{\eta}(B_r) = o(r^{d+\alpha})$ as $r \to 0$.*

(iii) *Φ is hyperuniform if and only if $\hat{\eta}(B_r) = o(r^d)$ as $r \to 0$.*

(iv) *Φ is Class I hyperuniform if and only if $\hat{\eta}(B_r) = O(r^{d-1})$ as $r \to 0$.*

Note that bounds in the above theorem only hold for $\alpha \leq 1$, as we have already seen that $\mathrm{Var}\,\Phi(B_R)$ grows at least as R^{d-1}. However, the diffraction $\hat{\eta}$ can decay faster than r^{d-1}: indeed, if Φ is the point process associated to the integer lattice, then $\hat{\eta}(B_r) = 0$ for all $r < 1$.

Now let $\mathbf{\Phi}$ be a stationary random $\mathbb{C}^\ell$-measure on $\mathbb{R}^d$. Recall that, for every $w \in \mathbb{C}^\ell$, we obtain a stationary random complex measure $\langle \mathbf{\Phi}, w \rangle$. Then we define hyperuniformity for $\mathbf{\Phi}$ depending on the weights $w \in \mathbb{C}^\ell$.

Definition 3.37. Let $\mathbf{\Phi}$ be a stationary $\mathbb{C}^\ell$-valued measure on $\mathbb{R}^d$, and $w \in \mathbb{C}^\ell$.

(i) We say $\mathbf{\Phi}$ is *hyperuniform for weights w* if $\langle \mathbf{\Phi}, w \rangle$ is hyperuniform.

(ii) We say $\mathbf{\Phi}$ is *Class I hyperuniform for weights w* if $\langle \mathbf{\Phi}, w \rangle$ is Class I hyperuniform.

(iii) We say $\mathbf{\Phi}$ is *(Class I) hyperuniform for constant weights* if it is (Class I) hyperuniform for weights $w = (1, \dots, 1)^\top$.

4 From substitutions to point processes

In this section, we introduce the basics of multi-color point sets, tilings and substitutions. We show that every well-behaved substitution rule gives rise to a stationary vector measure, and hence we are able to ask about its diffraction and hyperuniformity properties. Furthermore, we prove that the diffraction matrix of a substitution rule satisfies a *renormalisation relation*, and express it in term of a *self-similar density*.

There exists a large literature on substitution rules, substitution rules and tilings: see [5, 23, 48] for some surveys on the topic. However, the terminology, notation and particular formalism are not standardized, so we will define the terms in the way that is most convenient for us.

There is a discrepancy we need to deal with: most of the literature on substitution rules concerns itself with substitutions of tilings, but diffraction and hyperuniformity are properties of point sets and point processes. Given a tiling, one can produce a set of points by choosing a center in each tile, but this choice is not canonical.

We will follow the approach of Lagarias and Wang [38], as well as Lee, Moody, and Solomyak [41]. They defined substitution rules acting not on tilings, but families of discrete sets, and showed that self-similar families exist if and only if they can be associated with a self-similar tiling: therefore, both theories are equivalent.

4.1 Substitution rules

4.1.1 Multi-color sets

We denote the powerset of $\mathbb{R}^d$ by $\mathcal{P}(\mathbb{R}^d)$, and the set of finite subsets of $\mathbb{R}^d$ by $\mathcal{P}_{\text{fin}}(\mathbb{R}^d)$.

Definition 4.1. A *multiset in* $\mathbb{R}^d$ *with* ℓ *colors*, or ℓ*-multiset in* $\mathbb{R}^d$, is a multiset in $\mathbb{R}^d$ with alphabet $[\ell]$: that is, an ℓ-tuple $\mathbf{\Lambda} = (\Lambda_1, \cdots, \Lambda_\ell)$ of subsets of $\mathbb{R}^d$.

Remark 4.2. This terminology is not standard: some authors [40] use "multiset" to refer to our definition, while others [38] use it to refer to sets with multiplicity. We stick to our definition, as we will not need to consider sets with multiplicity.

Notation 4.3. Let $\mathbf{\Lambda} = (\Lambda_j)_{j\in[\ell]}, \mathbf{\Lambda}' = (\Lambda'_j)_{j\in[\ell]}$ be multisets in $\mathbb{R}^d$ with ℓ colors.

- If $B \subset \mathbb{R}^d$, we define the *restriction of* $\mathbf{\Lambda}$ *to* B, written $\mathbf{\Lambda} \cap B$, as the multiset given by $(\mathbf{\Lambda} \cap B) = (\Lambda_j \cap B)_{j\in[\ell]}$.
- For $x \in \mathbb{R}^d$, we define the *translate of* $\mathbf{\Lambda}$ *by* x, written $T_x\mathbf{\Lambda}$, as the multiset given by $(T_x\mathbf{\Lambda})_j = T_x\Lambda_j$ for all $j \in [\ell]$.
- For $\lambda > 0$, we define the *dilate of* $\mathbf{\Lambda}$ *by* λ, written $D_\lambda\mathbf{\Lambda}$, as the multiset given by $(D_\lambda\mathbf{\Lambda})_j = D_\lambda\Lambda_j$ for all $j \in [\ell]$.
- For $j \in [\ell]$, we let $\boldsymbol{o}_j$ be the multiset which has $\{0\}$ in the j-th color and is empty for all other colors: that is, $\boldsymbol{o}_j = (o_{j,k})_{k\in[\ell]}$ where

$$o_{j,k} = \begin{cases} \{0\} & \text{if } j = k, \\ \emptyset & \text{otherwise.} \end{cases}$$

- A *colored point* is a multiset of the form $T_x\boldsymbol{o}_j$ for some $x \in \mathbb{R}^d$ and $j \in [\ell]$.
- We say $\mathbf{\Lambda}$ is a *subset of* $\mathbf{\Lambda}'$, written $\mathbf{\Lambda} \subset \mathbf{\Lambda}'$, if $\Lambda_j \subset \Lambda'_j$ for all $j \in [\ell]$. If $\boldsymbol{p}$ is a colored point and $\boldsymbol{p} \subset \mathbf{\Lambda}$, we write $\boldsymbol{p} \in \mathbf{\Lambda}$.
- We let $\#\mathbf{\Lambda} \in \mathbb{C}^\ell$ be the vector given by $\#\mathbf{\Lambda} = (\#\Lambda_j)_{j\in[\ell]}$.
- We say $\mathbf{\Lambda}$ is *r-uniformly discrete* if the union $\bigcup_{j\in[\ell]} \Lambda_j$ is r-uniformly discrete, i.e. the distance between any two distinct points in $\bigcup_{j\in[\ell]} \Lambda_j$ is at least r.
- We say $\mathbf{\Lambda}$ is *R-relatively dense* with radius R if each Λ_j is R-relatively dense for all $j \in [\ell]$, i.e. for every $x \in \mathbb{R}^d$ and $j \in [\ell]$, there exists $y \in \Lambda_j$ such that $d(x, y) < R$.
- We say $\mathbf{\Lambda}$ is *Delone* if it is both uniformly discrete and relatively dense.

Remark 4.4. Other sources define multi-color sets as subsets of $\mathcal{P}(\mathbb{R}^d \times [\ell])$ instead: that is, sets of tuples (x, j) where $x \in \mathbb{R}^d$ is a point and $j \in [\ell]$ is a color. There is a canonical bijection $\mathcal{P}(\mathbb{R}^d)^\ell \cong \mathcal{P}(\mathbb{R}^d \times [\ell])$, so we can also think of multi-color sets as subsets of $\mathbb{R}^d$ where every point is labeled with a color. It is useful to keep both viewpoints in mind.

Remark 4.5. There is nothing, in principle, stopping us from defining multisets with infinitely many colors. There are some important examples of this kind: possibly the most famous is the *pinwheel tiling*, which contains infinitely many rotations of the same basic tiles [5]. There also has been work on the case where one uses a compact space of labels instead of a finite set [42]. With some effort, the definitions in this section could be extended to this case, but we will not do so here.

4.1.2 Substitution rules and self-similar sets

Definition 4.6. A *substitution rule on* $\mathbb{R}^d$ *with* ℓ *colors* is a pair $\mathcal{S} = (\lambda, \boldsymbol{\Delta})$ where

- $\lambda > 1$ is the *scaling constant of* $\mathcal{S}$, and
- $\boldsymbol{\Delta} \in \mathcal{P}_{\text{fin}}(\mathbb{R}^d)^{\ell\times\ell}$ is the *displacement matrix of* $\mathcal{S}$.

Definition 4.7. Let $\mathcal{S} = (\lambda, \boldsymbol{\Delta})$ be a substitution rule with ℓ colors, and let $\boldsymbol{\Delta} = (\Delta_{jk})_{j,k\in[\ell]}$ be the entries of the displacement matrix. Then we define its associated *substitution map* $\varrho : \mathcal{P}(\mathbb{R}^d)^\ell \to \mathcal{P}(\mathbb{R}^d)^\ell$ as follows: if $\boldsymbol{\Lambda} = (\Lambda_j)_{j\in[\ell]}$, its image $\varrho(\boldsymbol{\Lambda}) = (\Lambda'_j)_{j\in[\ell]}$ is given by

$$\Lambda'_j = \bigcup_{k=1}^{\ell} \bigcup_{x\in\Delta_{jk}} T_x D_\lambda \Lambda_k$$

for all $j \in [\ell]$.

Note that we can also write this as

$$\varrho(\boldsymbol{\Lambda}) = \bigcup_{\boldsymbol{p}\in\boldsymbol{\Lambda}} \varrho(\boldsymbol{p})$$

where the union runs over all colored points $\boldsymbol{p}$ in $\boldsymbol{\Lambda}$.

Substitution rules are important because they can be used to define and construct *self-similar multisets.*

Definition 4.8. Let $\mathcal{S}$ be a substitution rule with ℓ colors. A multiset $\boldsymbol{\Lambda} \in \mathcal{P}(\mathbb{R}^d)^\ell$ is *self-similar (with rule* $\mathcal{S}$*)* if there exists $N \in \mathbb{N}$ such that

(i) there exists a colored point $\boldsymbol{p}_{\boldsymbol{\Lambda}} \in \boldsymbol{\Lambda}$ such that

$$\boldsymbol{\Lambda} = \bigcup_{n=0}^{\infty} \varrho^{nN}(\boldsymbol{p}_{\boldsymbol{\Lambda}}), \text{ and}$$

(ii) For all $n \in [N]$, the union

$$\varrho^n(\boldsymbol{p}_{\boldsymbol{\Lambda}}) = \dot{\bigcup}_{\boldsymbol{q}\in\varrho^{n-1}(\boldsymbol{\Lambda})} \varrho(\boldsymbol{q})$$

is disjoint.

We call $\boldsymbol{p}_{\boldsymbol{\Lambda}}$ the *seed point* of $\boldsymbol{\Lambda}$. We will leave out the reference to $\mathcal{S}$ when it is clear from the context.

Remark 4.9. The above definition requires some explanation, as it does not exactly match the usual definitions. First, condition *(i)* guarantees that $\varrho^N(\boldsymbol{\Lambda}) = \boldsymbol{\Lambda}$, is an "irreducible Delone set satisfying an inflation functional equation" in the sense of Lagarias Lagarias and Wang [38]. Condition *(ii)* guarantees that the substitution map does not send two points to the same point: therefore $\varrho^N(\mathbf{\Lambda}) = \mathbf{\Lambda}$ also holds if one counts multiplicities. If one defines substitution rules acting on tilings instead of multi-color sets, condition *(ii)* automatically holds, and condition *(i)* implies that $\mathbf{\Lambda}$ is $\mathcal{S}$-legal (see Definition 4.31), a condition which is usually included in the definition of a self-similar tiling.

Central to the study of substitution rules is the *substitution matrix* M, a matrix which encodes many of its properties.

Definition 4.10. The *(full) substitution matrix* of $\mathcal{S}$ is the matrix $M \in \mathbb{N}^{\ell\times\ell}$ given by

$$M_{jk} = \#\Delta_{jk}$$

for $j, k \in [\ell]$.

In Chapter 7, we will define a *spherical substitution matrix* $M_{\text{sph}} \in \mathbb{N}^{\ell\times\ell}$, so later we will call M the *full substitution matrix* to avoid confusion: in the literature, the term "substitution matrix" or "inflation matrix" can refer to either one. For now we will call it "substitution matrix".

Definition 4.11. We say $\mathcal{S}$ is *primitive* if the substitution matrix M is a primitive matrix: that is, if there exists $N \in \mathbb{N}$ such that all of the entries of M^N are positive.

Primitivity of a substitution rule has the following geometric interpretation: if $\mathcal{S}$ is primitive, then there exists $N_0 \in \mathbb{N}$ such that for all $j \in [\ell]$, $\varrho^N(\boldsymbol{o}_j)$ contains a point of every color. This implies that every image of ϱ^N contains a point of every color.

Furthermore, if $\mathcal{S}$ is primitive, the substitution matrix satisfies the well-known *Perron–Frobenius theorem*:

Lemma 4.12 (Perron–Frobenius Theorem, [51])**.** *Let M be a primitive matrix. Then:*

(i) M has a positive eigenvalue λ_{PF}, its Perron–Frobenius eigenvalue *(or* PF eigenvalue *for short), such that every other eigenvalue has a strictly smaller absolute value.*

(ii) λ_{PF} has algebraic and geometric multiplicity 1*.*

(iii) The PF eigenvalue λ_{PF} has an eigenvector with strictly positive entries.

(iv) Every nonnegative eigenvector of M is an eigenvector of λ_{PF}.

An eigenvector of λ_{PF} with strictly positive entries is called a Perron–Frobenius eigenvector*, or* PF eigenvector *for short: it may be a left or right PF eigenvector.*

We still have not proven that self-similar multisets exist (for nice substitution rules). A good way to prove $\mathcal{S}$ admits a self-similar multiset is to show that it is *stone*. A stone substitution rule is a substitution rule that produces tilings of the plane, as we will now explain.

Definition 4.13. Let $\tau_1, \cdots, \tau_\ell$ be closed subsets of $\mathbb{R}^d$ and $\mathbf{\Lambda} = (\Lambda_1, \cdots, \Lambda_\ell)$ be a multiset in $\mathbb{R}^d$ with ℓ colors.

(i) We say $\mathbf{\Lambda}$ *patches* $\mathbb{R}^d$ *with prototiles* $\tau_1, \cdots, \tau_\ell$ if, for all $j, k \in [\ell], x \in \Lambda_j, y \in \Lambda_k$ such that $(x, j) \neq (y, k)$, $T_x\tau_j \cap T_y\tau_k$ has measure zero. In this case, we say the set $\{T_x\tau_j \mid j \in [\ell], x \in \Lambda_j\}$ is a *patch*.

(ii) If $\mathbf{\Lambda}$ patches $\mathbb{R}^d$ with prototiles $\tau_1, \cdots, \tau_\ell$, its *support* is the union $\operatorname{supp} \mathbf{\Lambda} = \bigcup_{j\in[\ell], x\in\Lambda_j} T_x\tau_j$.

(iii) We say $\mathbf{\Lambda}$ *tiles* $S \subset \mathbb{R}^d$ with prototiles $\tau_1, \cdots, \tau_\ell$ if it patches $\mathbb{R}^d$ and $\operatorname{supp} \mathbf{\Lambda} = S$. Then the set $\{T_x\tau_j \mid j \in [\ell], x \in \Lambda_j\}$ is a *tiling of* S by the prototiles $\tau_1, \cdots, \tau_\ell$.

Remark 4.14. Sometimes, tilings are defined using a topological condition instead, specifically requiring that tiles intersect only at their boundaries. For sufficiently nice prototiles, this is equivalent to the above definition. We follow the definition in [38].

Definition 4.15. Let $\mathcal{S}$ be a substitution rule with ℓ colors. We say the nonempty, compact subsets $\tau_1, \cdots, \tau_\ell \subset \mathbb{R}^d$ are the *canonical prototiles of* $\mathcal{S}$ if they satisfy the following equation:

$$D_\lambda \tau_j = \bigcup_{k=1}^{\ell} \bigcup_{x\in\Delta_{kj}} T_x\tau_k$$

for all $j \in [\ell]$.

The above equation is called the *multi-tile functional equation* by [38].

The space of all compact subsets of $\mathbb{R}^d$ is a complete metric space with respect to the Hausdorff metric. Then Banach's fixed point theorem guarantees the following.

Theorem 4.16 ([52, Theorem 4.89])**.** *Let $\mathcal{S}$ be a substitution rule. Then there exist canonical prototiles $\tau_1, \cdots, \tau_\ell$ of $\mathcal{S}$, and they are unique.*

In general, the canonical prototiles could have measure zero or be otherwise pathological. We will want to assume this is not the case.

Definition 4.17. Let $\mathcal{S}$ be a substitution rule. λ the scaling constant and λ_{PF} the PF eigenvalue of the substitution matrix M. We say $\mathcal{S}$ is a *stone substitution rule* if the following properties hold:

(i) All of the canonical prototiles $\tau_1, \cdots, \tau_\ell$ have positive Lebesgue measure.

(ii) For all $j \in [\ell]$, $\varrho(\boldsymbol{o}_j)$ tiles $D_\lambda \tau_j$ with the canonical prototiles.

Theorem 4.18. *Let $\mathcal{S}$ be a primitive, stone substitution rule, λ its scaling constant, M its substitution matrix, and λ_{PF} the PF eigenvalue of its substitution matrix. Then the following hold.*

(i) The vector $(m_{\mathbb{R}^d}(\tau_j))_{j\in[\ell]}$ is a left PF eigenvector of M.

(ii) The PF eigenvalue of M is $\lambda_{PF} = \lambda^d$

Proof. By definition of the canonical prototiles, we have

$$D_\lambda \tau_j = \bigcup_{k=1}^{\ell} \bigcup_{x\in\Delta_{kj}} T_x \tau_k \tag{*}$$

for all $j \in [\ell]$. Furthermore, Condition (*ii*) of the definition of a stone substitution rule, the intersection $T_x\tau_k \cap T_y\tau_{k'}$ has Lebesgue measure 0. Then, by taking the measure on both sides of $(*)$, we obtain

$$\lambda^d m_{\mathbb{R}^d}(\tau_j) = \sum_{k=1}^{\ell} \sum_{x\in\Delta_{kj}} m_{\mathbb{R}^d}(\tau_k)$$

In other words, if we set $v_{PF} = (m_{\mathbb{R}^d}(\tau_j))_{j\in[\ell]}$, we have $\lambda^d v_{PF} = M^T v_{PF}$. This means v_{PF} is a nonnegative eigenvector of M with eigenvalue λ^d: by the Perron–Frobenius theorem, this is only possible if v_{PF} is a PF eigenvector and λ^d is the PF eigenvalue. This concludes the proof. □

Then, Lagarias and Wang [38] proved that being stone is equivalent to admitting a self-similar Delone multiset.

Theorem 4.19 ([38, Theorem 2.4])**.** *Let $\mathcal{S}$ be a primitive substitution rule. Then the following conditions are equivalent.*

(i) There exists a self-similar Delone $\boldsymbol{\Lambda}$.

(ii) There exists a self-similar Delone $\boldsymbol{\Lambda}$ which tiles $\mathbb{R}^d$ with the canonical prototiles $\tau_1, \cdots, \tau_\ell$.

(iii) $\mathcal{S}$ is a stone substitution rule.

Therefore, if we want to prove a primitive substitution rule admits a self-similar Delone multiset, it suffices to prove it is stone, which is something one can do by drawing pictures of the prototiles: see [5, Section 6] or Section 7.2.3 for examples. Furthermore, it means the definition of substitution rules using point sets is equivalent to the definition using tilings (such as in [5, 48]), as every substitution rule with a Delone self-similar set is a stone substitution rule.

Remark 4.20. Note that statement of Lagarias and Wang [38, Theorem 2.4] only says that $\varrho^N(\boldsymbol{\Lambda}) = \boldsymbol{\Lambda}$, which is weaker than our definition of self-similar multisets, which requires the existence of a seed point $\boldsymbol{p}_{\boldsymbol{\Lambda}}$. However, the $\boldsymbol{\Lambda}$ given in the proof is constructed from a seed point, so it is also self-similar in our sense.

Remark 4.21 (Recentering)**.** Let $\mathcal{S}$ be a stone substitution rule and $\tau_1, \cdots, \tau_\ell$. Then, for every tuples of vectors $\boldsymbol{z} = (z_1, \cdots, z_\ell) \in \mathbb{R}^d$ we can define a *recentered substitution rule* $T_{\boldsymbol{z}}\mathcal{S}$ which has the same scaling constant and is a stone substitution rule with canonical prototiles $\tau'_j = T_{z_j}\tau_j$ for all $j \in [\ell]$. Its displacement matrix $\boldsymbol{\Delta}' = (\Delta'_{jk})_{j,k\in[\ell]}$ is given by

$$\Delta'_{jk} = \{x + \lambda z_j - z_k \mid x \in \Delta_{jk}\},$$

where Δ_{jk} are the entries of the displacement matrix of $\mathcal{S}$.

The recentered substitution rule $T_{\boldsymbol{z}}\mathcal{S}$ produces the same self-similar tilings as $\mathcal{S}$, so one would like whatever properties we study to be invariant under recentering. In fact, the criteria for hyperuniformity we will prove in Chapter 6 will only depend on the substitution matrix M, hence they will not depend on the choice of recentering.

4.2 Substitution spaces and counting processes

In this section, we associate a vector point process to each primitive, stone, FLC substitution rule, and study its properties. For this, we will first define the *substitution space* of a substitution rule, and will use it to define a vector point process.

4.2.1 Hulls of FLC sets

In this section, we define FLC multisets, their hulls, and their basic properties. All results in this section are well-known for FLC sets and tilings [40, 48, 50] and their extension to multisets is straightforward.

Definition 4.22. Let $\boldsymbol{\Lambda}$ be a Delone ℓ-multiset in $\mathbb{R}^d$. We say $\boldsymbol{\Lambda}$ is *FLC* (or *has finite local complexity*) if the following holds: for all compact $K \subset \mathbb{R}^d$, the set of K-patterns

$$\{T_x\boldsymbol{\Lambda} \cap K \mid x \in \mathbb{R}^d\}$$

has finitely many equivalence classes under translation.

We denote the set of FLC ℓ-multisets in $\mathbb{R}^d$ by $\mathcal{P}^\ell_{FLC}(\mathbb{R}^d)$. If $\mathbf{\Lambda}$ is a Delone ℓ-multiset which tiles $\mathbb{R}^d$, there is a convenient way to check whether it's FLC: intuitively, $\mathbf{\Lambda}$ is FLC if there are only finitely many ways of putting tiles next to each other in the tiling given by $\mathbf{\Lambda}$.

Lemma 4.23 ([48])**.** *Let $\mathbf{\Lambda}$ be a Delone ℓ-multiset in $\mathbb{R}^d$ which tiles $\mathbb{R}^d$ with prototiles $\tau_1, \cdots, \tau_\ell$. Assume the set of* 2-patches

$$\mathcal{T}^{(2)} := \{\boldsymbol{p}, \boldsymbol{q} \in \mathbf{\Lambda} \mid \operatorname{supp} \boldsymbol{p} \cap \operatorname{supp} \boldsymbol{q} \neq \emptyset\}$$

has finitely many equivalence classes under translation. Then $\mathbf{\Lambda}$ is FLC.

In particular, in $\mathbb{R}^2$, the above condition is satisfied if the tiling defined is *edge-to-edge*, i.e. the tiles are polygons and their intersections are sides or corners. More generally, $\mathbf{\Lambda}$ is FLC if it defines a tiling which is *sibling edge-to-edge* as defined by Goodman-Strauss [32].

Definition 4.24. Let $\mathbf{\Lambda}$ be an FLC ℓ-multiset in $\mathbb{R}^d$, $K \subset \mathbb{R}^d$ be compact and $V \subset \mathbb{R}^d$ be open. We define the *cylinder set*

$$U_{K,V}(\mathbf{\Lambda}) = \{\mathbf{\Lambda}' \in \mathcal{P}^\ell_{disc}(\mathbb{R}^d) \mid \exists x \in V : T_x\mathbf{\Lambda}' \cap K = \mathbf{\Lambda} \cap K\}.$$

The *local topology* on $\mathcal{P}^\ell_{disc}(\mathbb{R}^d)$ is the topology generated by the cylinder sets $U_{K,V}(\mathbf{\Lambda})$ for all compact $K \subset \mathbb{R}^d$, open $V \subset \mathbb{R}^d$, and $\mathbf{\Lambda} \in \mathcal{P}^\ell_{disc}(\mathbb{R}^d)$.

In fact, these sets can be used to define a uniformity for the space Ω [17, 44, 50, 60].

Theorem 4.25 ([50])**.** *Let $\mathbf{\Lambda}$ be a Delone multiset in $\mathbb{R}^d$. Then the following are equivalent:*

(i) $\mathbf{\Lambda}$ is FLC.

(ii) The set $\{T_x\mathbf{\Lambda} \mid x \in \mathbb{R}^d\}$ is relatively compact in the local topology.

Remark 4.26. If $\mathbf{\Lambda}$ is not FLC, one can use a coarser topology, such as the *local rubber topology*, to get similar results. In the FLC case, this is equivalent to the local topology [8].

Definition 4.27. Let $\mathbf{\Lambda}$ be an FLC multiset in $\mathbb{R}^d$. The *hull* of $\mathbf{\Lambda}$ is the set

$$\Omega_{\mathbf{\Lambda}} = \overline{\{T_x\mathbf{\Lambda} \mid x \in \mathbb{R}^d\}}$$

where the closure is taken in the local topology.

Definition 4.28. Let $\mathbf{\Lambda}$ be an FLC multiset in $\mathbb{R}^d$.

- $\mathbf{\Lambda}$ is *repetitive* if, for every finite $\boldsymbol{p} \subset \mathbf{\Lambda}$, the set

$$\{x \in \mathbb{R}^d \mid T_x \boldsymbol{p} \subset \mathbf{\Lambda}\}$$

  is relatively dense.

- $\mathbf{\Lambda}$ *has uniform cluster frequencies* if, for all finite $\boldsymbol{p} \subset \mathbf{\Lambda}$, the limit

$$\lim_{R\to\infty} \frac{1}{m_{\mathbb{R}^d}(B_R)} \#\{x \in B_R : T_x \boldsymbol{p} \subset T_y \mathbf{\Lambda}\}$$

  exists uniformly for $y \in \mathbb{R}^d$.

Theorem 4.29 ([40, 50])**.** *Let $\mathbf{\Lambda}$ be an FLC multiset in $\mathbb{R}^d$. Then*

(i) $\mathbf{\Lambda}$ is repetitive if and only if $\Omega_{\mathbf{\Lambda}}$ is a minimal dynamical system, and

(ii) $\mathbf{\Lambda}$ has uniform cluster frequencies if and only if $\Omega_{\mathbf{\Lambda}}$ is uniquely ergodic.

Remark 4.30. Theorem 4.29 has the following consequence: for all $\mathbf{\Lambda}_1, \mathbf{\Lambda}_2 \in \Omega_{\mathbf{\Lambda}}$ and all compact $K \subset \mathbb{R}^d$, there exists a translation $x \in \mathbb{R}^d$ such that $T_x \mathbf{\Lambda}_1 \cap K = \mathbf{\Lambda}_2 \cap K$: one says $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$ are "locally isomorphic" [48].

Now we turn to the specific case of Delone multisets coming from substitution rules. In the last section, we already considered some multisets associated to substitution rules, namely the self-similar sets. In order to associate a dynamical system to a substitution rule, we will relax this notion as follows.

Definition 4.31. Let $\mathcal{S}$ be a substitution rule on $\mathbb{R}^d$ with ℓ colors.

(i) An ℓ-multiset $\mathbf{\Lambda}$ in $\mathbb{R}^d$ is $\mathcal{S}$*-legal* if the following holds: for every finite subset $\boldsymbol{p} \subset \mathbf{\Lambda}$ there exists a colored point $\boldsymbol{q}$ and $\mathbb{N} \in \mathbb{N}$ such that $\boldsymbol{p} \subset \varrho^N(\boldsymbol{p})$. In other words, there exists $j \in [\ell]$ and $N \in \mathbb{N}$ such that $\varrho^N(\boldsymbol{o}_j)$ contains a translate of $\boldsymbol{p}$.

(ii) The *substitution space* $\Omega_{\mathcal{S}}$ is the set of all $\mathcal{S}$-legal Delone ℓ-multisets.

Remark 4.32. Note that any self-similar multiset $\mathbf{\Lambda}$, using our definition, is $\mathcal{S}$-legal. If $\boldsymbol{p}_{\mathbf{\Lambda}}$ is the seed point of $\mathbf{\Lambda}$, we have $\mathbf{\Lambda} = \bigcup_{n=0}^{\infty} \varrho^{nN}(\boldsymbol{p}_{\mathbf{\Lambda}})$ by definition, hence every finite subset of $\mathbf{\Lambda}$ is contained in $\varrho^{nN}(\boldsymbol{p}_{\mathbf{\Lambda}})$ for some $n \in \mathbb{N}$. In particular, if $\mathcal{S}$ is primitive and stone, $\Omega_{\mathcal{S}}$ is not empty.

Here, the fact that $\mathbf{\Lambda}$ is constructed from a seed point is important. If $\mathbf{\Lambda}$ is a Delone multiset which decomposes as $\mathbf{\Lambda} = \mathbf{\Lambda}_1 \dot\cup \mathbf{\Lambda}_2$ with $\varrho(\mathbf{\Lambda}_1) = \mathbf{\Lambda}_1$ and $\varrho(\mathbf{\Lambda}_2) = \mathbf{\Lambda}_2$, $\mathbf{\Lambda}$ may not be legal, as a finite subset $\boldsymbol{p} \subset \mathbf{\Lambda}$ that contains points from both $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ may not necessarily be contained in $\varrho^N(\boldsymbol{q})$ for any colored point $\boldsymbol{q}$, $N \in \mathbb{N}$: see [5, Example 4.2]. This is why most sources include legality in the definition of self-similarity for sets or tilings.

Let $\mathcal{S}$ be a substitution rule. We say $\mathcal{S}$ is *FLC* if it admits an FLC self-similar multiset $\mathbf{\Lambda}$.

Theorem 4.33 ([48])**.** *Let $\mathcal{S}$ be a primitive, stone, FLC substitution rule on $\mathbb{R}^d$ with ℓ colors. Then:*

(i) Every legal multiset $\mathbf{\Lambda}$ is FLC, repetitive and has uniform cluster frequencies.

(ii) For every legal $\mathbf{\Lambda}$, $\Omega_{\mathbf{\Lambda}} = \Omega_{\mathcal{S}}$.

(iii) Every $\mathbf{\Lambda} \in \Omega_{\mathcal{S}}$ tiles $\mathbb{R}^d$ with the canonical prototiles $\tau_1, \cdots, \tau_\ell$.

(iv) The substitution space $\Omega_{\mathcal{S}}$ is compact, minimal and uniquely ergodic.

Furthermore, the substitution map acts on the substitution space, as follows.

Theorem 4.34 ([56])**.** *The substitution map ϱ has the following properties:*

(i) The restricted map $\varrho : \Omega_{\mathcal{S}} \to \Omega_{\mathcal{S}}$ is a continuous surjection.

(ii) For all $x \in \mathbb{R}^d$ and $\mathbf{\Lambda} \in \Omega_{\mathcal{S}}$, $\varrho(T_x\mathbf{\Lambda}) = T_{\lambda x}\varrho(\mathbf{\Lambda})$.

Remark 4.35. One may ask if ϱ is an homeomorphism. In fact [55], this is the case if and only if $\mathcal{S}$ is *aperiodic*, i.e. for all $\mathbf{\Lambda} \in \Omega_{\mathcal{S}}$ there exists no $x \in \mathbb{R}^d \setminus \{0\}$ such that $T_x\mathbf{\Lambda} = \mathbf{\Lambda}$.

4.2.2 The vector point process associated to a substitution rule

In this section, we will define the vector point process associated to an FLC set or substitution rule, and use it to define its autocorrelation, diffraction and hyperuniformity. We will also show that this definition is equivalent to the ergodic definition of autocorrelation due to Hof [34], other than the fact that we use the centered autocorrelation.

Definition 4.36. (i) Let $\mathbf{\Lambda}$ an FLC ℓ-multiset in $\mathbb{R}^d$ which is repetitive and has uniform cluster frequencies. Then the *vector point process* $\mathbf{\Phi} = (\Phi_1, \cdots \Phi_\ell)$ *associated to* $\mathbf{\Lambda}$ is the ℓ-point process with components

$$\Phi_j \colon \Omega_{\mathbf{\Lambda}} \longrightarrow \mathcal{M}(\mathbb{R}^d), \qquad (\Lambda_k)_{k\in[\ell]} \mapsto \sum_{x\in\Lambda_j} \delta_x$$

(ii) If $\mathcal{S}$ is a primitive, stone, FLC substitution rule, the *vector point process associated to* $\mathcal{S}$ is the vector point process associated to any $\mathbf{\Lambda} \in \Omega_{\mathcal{S}}$.

Note that the latter is well-defined: for any $\mathbf{\Lambda} \in \Omega_{\mathcal{S}}$, we obtain the same hull $\Omega_{\mathbf{\Lambda}} = \Omega_{\mathcal{S}}$, hence the same vector point process $\mathbf{\Phi}$.

Theorem 4.37. *Let* $\mathbf{\Phi}$ *be the vector point process associated to a FLC multiset* $\mathbf{\Lambda}$ *which is repetitive and has uniform cluster frequencies, or the vector point process associated to a primitive, stone, FLC substitution rule* $\mathcal{S}$*. Then* $\mathbf{\Phi}$ *is a uniquely ergodic vector point process on* $\mathbb{R}^d$*.*

Proof. It suffices to show the theorem when $\mathbf{\Phi}$ is the vector point process associated to an FLC multiset $\mathbf{\Lambda}$, as the vector point process associated to a substitution rule $\mathcal{S}$ is also of this form.

As seen in Remark 4.30, all elements of $\Omega_{\mathbf{\Lambda}}$ are locally isomorphic, hence there exists $r > 0$ such that every $\mathbf{\Lambda} \in \Omega_{\mathcal{S}}$ is r-uniformly discrete. This implies there exists a constant $C > 0$ such that, for all $j \in [\ell]$,

$$|\Phi_\omega(f)_j| = \sum_{x \in \Lambda_j} f(x) \leq C\|f\|_\infty m_{\mathbb{R}^d}(\operatorname{supp} f)$$

which means that $\mathbf{\Phi}$ is a continuous linear map $\Omega_{\mathcal{S}} \to M(\mathbb{R}^d, \mathbb{C}^\ell)$.

By construction, $\mathbf{\Phi}$ is equivariant under the action of $\mathbb{R}^d$: as $\Omega_{\mathcal{S}}$ is a uniquely ergodic dynamical system, this means $\mathbf{\Phi}$ is a uniquely ergodic stationary random measure on $\mathbb{R}^d$. This concludes the proof. □

Remark 4.38. Here we are implicitly using the unique ergodicity of $\Omega_{\mathcal{S}}$, as otherwise we would need to choose a specific probability measure μ on $\Omega_{\mathbf{\Lambda}}$. In general, every ergodic component of $\Omega_{\mathbf{\Lambda}}$ would give rise to a different ergodic point process: then the theorems from Section 3.4 would apply not to every element of $\Omega_{\mathbf{\Lambda}}$, but only to a set of *generic* multisets.

Now we can extend all properties of point processes we defined in Chapter 3 to substitution rules.

Definition 4.39. Let $\mathcal{S}$ be a primitive, stone, FLC substitution rule, and $\mathbf{\Phi}$ be its associated vector point process.

(i) The *autocorrelation measure* $\hat{\eta}_w$ *of* $\mathcal{S}$ *with weights* $w \in \mathbb{C}^\ell$ is the autocorrelation measure of the random stationary measure $\langle \mathbf{\Phi}, w \rangle$.

(ii) The *intensity* $\boldsymbol{\iota}$ *of* $\mathcal{S}$ is the intensity of $\mathbf{\Phi}$.

(iii) The *autocorrelation matrix of* $\mathcal{S}$ is the autocorrelation matrix of $\mathbf{\Phi}$.

(iv) The *diffraction matrix of* $\mathcal{S}$ is the diffraction matrix of $\mathbf{\Phi}$.

(v) We say $\mathcal{S}$ is *hyperuniform for weights* w if $\mathbf{\Phi}$ is hyperuniform for weights w.

(vi) We say $\mathcal{S}$ is *Class I hyperuniform for weights* w if $\mathbf{\Phi}$ is Class I hyperuniform for weights w.

So let $\mathcal{S}$ be a primitive, stone, FLC substitution rule, and $\mathbf{\Phi}$ be its associated vector point process. Thanks to the unique ergodicity we get the following expression the autocorrelation of $\langle \mathbf{\Phi}, w \rangle$ as a weighted spatial average, which follows from Corollary 3.24.

Corollary 4.40. *Let $w \in \mathbb{C}^\ell$. Then the autocorrelation of $\langle \mathbf{\Phi}, w \rangle$ satisfies*

$$\eta(f) + |\langle \boldsymbol{\iota}, w \rangle|^2 m_{\mathbb{R}^d}(f) = \lim_{R \to \infty} \frac{1}{m_{\mathbb{R}^d}(B_R)} \sum_{j,k \in [\ell]} w_j \overline{w_k} \sum_{x \in \Lambda_j \cap B_R} \sum_{y \in \Lambda_k} f(y - x)$$

In the literature of aperiodic order, it is customary to define the autocorrelation of an FLC set via a formula like the right hand side of the equation in Corollary 4.40: this approach was introduced by Hof [34]. This means our definition of the autocorrelation coincides with the usual definition for FLC, minimal, uniquely ergodic sets, other than the fact that we are using the centered autocorrelation instead of the uncentered version, which explains the extra $|\langle \boldsymbol{\iota}, w \rangle|^2 m_{\mathbb{R}^d}(f)$ term (see Remark 3.15).

By the ergodicity of $\mathbf{\Phi}$, we also get a similar result for the intensity of $\mathbf{\Phi}$: namely, $\boldsymbol{\iota} = (\mathrm{freq}_{\mathbf{\Lambda}}(\boldsymbol{o}_j))_{j \in [\ell]}$, where

$$\mathrm{freq}_{\mathbf{\Lambda}}(\boldsymbol{o}_j) = \lim_{R \to \infty} \frac{1}{m_{\mathbb{R}^d}(B_R)} \#\{x \in B_R : T_x \boldsymbol{o}_j \in \mathbf{\Lambda}\}$$

is the *relative frequency* of the color j in $\mathbf{\Lambda}$.

The above formula for the autocorrelation can be used to get a criterion for hyperuniformity which depends only on a single sample. This highlights the fact that, even though hyperuniformity is most easily defined in a probabilistic way, it is also a statement about the geometry of particular $\mathcal{S}$-legal sets.

Corollary 4.41. *Let $w \in \mathbb{C}^\ell$. Then $\mathcal{S}$ is hyperuniform for weights $w \in \mathbb{C}^\ell$ if and only if, for some $\omega \in \Omega_{\mathcal{S}}$, we have*

$$\lim_{R_1, R_2 \to \infty} \frac{1}{m_{\mathbb{R}^d}(B_{R_1})} \left(\sum_{j,k \in [\ell]} \sum_{x \in \Lambda_j \cap B_{R_1}} \sum_{y \in \Lambda_k} w_j \overline{w}_k \mathbb{1}_{B_{R_2}}(x - y) - |\langle \boldsymbol{\iota}, w \rangle|^2 m_{\mathbb{R}^d}(B_{R_1}) \right) = 0$$

Proof. One has to show, with standard arguments in ergodic theory, that the formula for the autocorrelation in Corollary 4.40 holds with $f = 1_{B_{R_2}}$, □

4.3 Renormalisation relations

4.3.1 Renormalisation measures and the normalized Fourier matrix

Let $\mathcal{S}$ be an FLC, primitive, stone substitution rule with scaling constant λ and displacement matrix $\mathbf{\Delta} = (\Delta_{jk})_{j,k\in[\ell]}$.

Theorem 4.42 (Renormalisation of the vector point process associated to a substitution rule)**.** *The vector point process $\mathbf{\Phi} = (\Phi_j)_{j\in[\ell]}$ associated to $\mathcal{S}$ satisfies the following relations*

$$\Phi_k \stackrel{d}{=} \sum_{j=1}^{\ell} \sum_{x\in\Delta_{kj}} T_x D_\lambda \Phi_j$$

for all $k \in [\ell]$. We call these the renormalisation relations *for $\mathbf{\Phi}$.*

Proof. Let μ be the unique translation invariant probability measure on the hull $\Omega_{\mathcal{S}}$. As $\varrho(T_x\mathbf{\Lambda}) = T_{\lambda x}\varrho(\mathbf{\Lambda})$, one can check $\varrho^*\mu = \mu \circ \varrho$ is also a translation invariant probability measure on $\Omega_{\mathcal{S}}$: by unique ergodicity this means $\varrho^*\mu = \mu$.

Let $\varrho^*\mathbf{\Phi} = (\Phi'_j)_{j\in[\ell]}$ be the random vector measure defined by concatenating $\varrho : \Omega_{\mathcal{S}} \to \Omega_{\mathcal{S}}$ and $\mathbf{\Phi} : \Omega_{\mathcal{S}} \to M(\mathbb{R}^d, \mathbb{C}^\ell)$. Explicitly, $\varrho^*\mathbf{\Phi}$ is given by

$$\Phi'_k = \sum_{j=1}^{\ell} \sum_{x\in\Delta_{kj}} T_x D_\lambda \Phi_j$$

for all $k \in [\ell]$. But we have $\varrho \circ \mu = \mu$, hence $\varrho^*\mathbf{\Phi} \stackrel{d}{=} \mathbf{\Phi}$. Then, by comparing the k-th components, we obtain

$$\Phi_k \stackrel{d}{=} \Phi'_k = \sum_{j=1}^{\ell} \sum_{x\in\Delta_{kj}} T_x D_\lambda \Phi_j$$

□

If we consider the intensity of both sides of the renormalisation relations, we obtain the following well known result.

Corollary 4.43. *The intensity $\boldsymbol{\iota}$ of $\mathcal{S}$ satisfies*

$$\boldsymbol{\iota} = \lambda^{-d} M \boldsymbol{\iota}.$$

Therefore, $\boldsymbol{\iota}$ is a right PF eigenvector of the substitution matrix M.

Now we will use Theorem 4.42 to obtain a similar result for the diffraction matrix $\hat{\boldsymbol{H}}$: this will be the *renormalisation relation* for the diffraction matrix, originally introduced by Baake, Gähler, and Mañibo [11].

Definition 4.44. The *normalized Fourier matrix* of $\mathcal{S}$ is the matrix function $\boldsymbol{A} : \mathbb{R}^d \to \mathbb{C}^{\ell\times\ell}$ with entries $A_{jk}(\xi)$ given by

$$A_{jk}(\xi) = \lambda^{-d} \sum_{x\in\Delta_{jk}} e^{2\pi i\langle x,\xi\rangle}.$$

Remark 4.45. Let M be the (full) substitution matrix of $\mathcal{S}$. Then we have:

$$A(0) = \lambda^{-d} M$$

Theorem 4.46. *The diffraction matrix measure* $\hat{\boldsymbol{H}}$ *satisfies*

$$\hat{\boldsymbol{H}} = \boldsymbol{A}(D_{\lambda^{-1}}\hat{\boldsymbol{H}})\boldsymbol{A}^*$$

where the product of matrix functions with matrix measures is defined by the matrix product formula, as in Definition 3.4.

In order to prove this theorem, recall the following property of the Fourier transform of functions.

Lemma 4.47. *Let* $f \in L^2(\mathbb{R}^d)$ *be a Fourier transformable function on* $\mathbb{R}^d$*,* $x \in \mathbb{R}^d$*. Then*

$$\mathcal{F}(D_{\lambda^{-1}}T_{-x}f)(\xi) = \lambda^{-d}e^{2\pi i\langle x,\xi\rangle}D_\lambda\mathcal{F}(f)(\xi)$$

Proof. By density, it suffices to assume $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. Then

$$\begin{aligned}
\mathcal{F}(D_{\lambda^{-1}}T_{-x}f)(\xi) &= \int_{\mathbb{R}^d} f(\lambda(z+x))e^{-2\pi i\langle z,\xi\rangle}\,\mathrm{d}z \\
&= \int_{\mathbb{R}^d} f(u)e^{-2\pi i\langle \lambda u-x,\xi\rangle}\,\lambda^{-d}\mathrm{d}u \qquad \text{(Substitute } u=\lambda(z+x)) \\
&= \lambda^{-d}e^{2\pi i\langle x,\xi\rangle}\int_{\mathbb{R}^d} f(u)e^{-2\pi i\langle u,\lambda^{-1}\xi\rangle}\,\mathrm{d}u \\
&= \lambda^{-d}e^{2\pi i\langle x,\xi\rangle}D_\lambda\hat{f}(\xi)
\end{aligned}$$

□

Proof of Theorem 4.46. For $x \in \mathbb{R}^d$, let $c(x) : \mathbb{R}^d \to \mathbb{C}^\ell$ be the function defined by $c(x)(\xi) = e^{2\pi i\langle x,\xi\rangle}$.

Componentwise, the equation in the statement of the theorem reads

$$\hat{H}_{mn} = \lambda^{-2d} \sum_{j,k\in[\ell]} \sum_{\substack{x\in\Delta_{mj}\\ y\in\Delta_{nk}}} c(x-y) D_{\lambda^{-1}} \hat{H}_{jk}.$$

It suffices to check the equation holds for all functions of the form $|\hat{f}|^2$ for $f \in C_c(\mathbb{R}^d)$, as this uniquely determines the Fourier transform of a measure. So let $f \in C_c(\mathbb{R}^d)$. Then, using the properties of the diffraction matrix and Lemma 4.47, we get

$$\begin{aligned}
\hat{H}_{mn}(|\hat{f}|^2) &= \operatorname{Cov}(\Phi_m(f), \Phi_n(f)) && \text{(Definition of diffraction measure)}\\
&= \operatorname{Cov}\left(\sum_{j=1}^{\ell} \sum_{x\in\Delta_{mj}} T_x D_\lambda \Phi_j(f), \sum_{k=1}^{\ell} \sum_{y\in\Delta_{nk}} T_y D_\lambda \Phi_k(f) \right) && \text{(Self-similarity of } \boldsymbol{\Phi})\\
&= \operatorname{Cov}\left(\sum_{j=1}^{\ell} \sum_{x\in\Delta_{mj}} \Phi_j\left(D_{\lambda^{-1}} T_{-x} f\right), \sum_{k=1}^{\ell} \sum_{y\in\Delta_{nk}} \Phi_k\left(D_{\lambda^{-1}} T_{-y} f\right) \right)\\
&= \sum_{j,k=1}^{\ell} \sum_{\substack{x\in\Delta_{mj}\\ y\in\Delta_{nk}}} \hat{H}_{jk}\left((\lambda^{-d} c(x) D_\lambda \hat{f}) \overline{(\lambda^{-d} c(y) D_\lambda \hat{f})} \right)\\
&= \sum_{j,k=1}^{\ell} \sum_{\substack{x\in\Delta_{mj}\\ y\in\Delta_{nk}}} \hat{H}_{jk}(\lambda^{-2d} c(x-y) |D_\lambda \hat{f}|^2)\\
&= \lambda^{-2d} \sum_{j,k=1}^{\ell} \sum_{\substack{x\in\Delta_{mj}\\ y\in\Delta_{nk}}} c(x-y) D_{\lambda^{-1}} \hat{H}_{jk}(|\hat{f}|^2).
\end{aligned}$$

This completes the proof. □

4.3.2 Renormalisation via density functions

In this section, we define *self-similar densities* as a tool to study the diffraction measure $\hat{\boldsymbol{H}}$.

Definition 4.48. Let $\boldsymbol{M} \in \mathcal{M}(\mathbb{R}^d, \mathbb{C}^{\ell\times\ell})$ be a matrix valued measure on $\mathbb{R}^d$. A *density* of $\boldsymbol{M}$ is a tuple $(\boldsymbol{h}, \nu)$ where

(i) ν is a positive (scalar, and not necessarily σ-finite) measure on $\mathbb{R}^d$, and

(ii) $\boldsymbol{h}$ is a locally integrable $\mathbb{C}^{\ell\times\ell}$-valued function on $\mathbb{R}^d$ such that $\boldsymbol{M} = \boldsymbol{h}\nu$.

We call $\boldsymbol{h}$ the *density function of* $\boldsymbol{M}$ *with respect to the base* ν.

Example 4.49.

- A $\mathbb{C}^{\ell\times\ell}$-measure $\boldsymbol{M}$ on $\mathbb{R}^d$ is pure point if and only if the function $\boldsymbol{h} : \mathbb{R}^d \to \mathbb{C}^\ell$ given by $\boldsymbol{h}(\xi) = \boldsymbol{M}(\{\xi\})$ is a density function of $\boldsymbol{M}$ with respect to the counting measure ν. (Note that the counting measure is not a measure in the Bourbaki sense, as it is not σ-finite, but $\boldsymbol{h}\nu$ still makes sense as a matrix valued measure since $\boldsymbol{h}$ is locally integrable with respect to ν).
- A $\mathbb{C}^{\ell\times\ell}$-measure $\boldsymbol{M}$ on $\mathbb{R}^d$ is absolutely continuous if and only if it has a density with respect to the Lebesgue measure $m_{\mathbb{R}^d}$.
- Let $\boldsymbol{M}$ be a $\mathbb{C}^{\ell\times\ell}$-measure. Recall that there exists a *total variation measure* $\nu = \|\boldsymbol{M}\|$, as defined in Section 3.1, such that every coefficient of $\boldsymbol{H}$ is absolutely continuous with respect to ν. Then, by Radon–Nikodym for the coefficients, $\boldsymbol{M}$ has a density with respect to ν. This means every $\mathbb{C}^{\ell\times\ell}$-measure has a density function with respect to some appropriate base.

We will want to use densities to study the diffraction measure $\hat{\boldsymbol{H}}$. The first important fact we will need is that the density of $\hat{\boldsymbol{H}}$ is a positive semidefinite matrix almost everywhere.

Lemma 4.50. *Let $(\boldsymbol{h}, \nu)$ be a density of $\hat{\boldsymbol{H}}$. Then $\boldsymbol{h}(\xi)$ is a positive semidefinite matrix for ν-almost all ξ.*

Proof. For any vector $w \in \mathbb{C}^\ell$, consider the measure M_w defined by $M_w(f) = \langle \boldsymbol{M}(f)w, w\rangle$ for $f \in C_c(\mathbb{R}^d)$. By Lemma Lemma 3.31, M_w is a positive measure for all $w \in \mathbb{C}^\ell$.

Furthermore, we have:

$$M_w(f) = \langle \hat{\boldsymbol{H}}(f)w, w\rangle = \left\langle \left(\int_{\mathbb{R}^d} f(\xi)\boldsymbol{h}(\xi)\, d\nu(\xi) \right) w, w \right\rangle = \int_{\mathbb{R}^d} f(\xi)\langle \boldsymbol{h}(\xi)w, w\rangle\, d\nu(\xi)$$

This shows that $\langle \boldsymbol{h}(\cdot)w, w\rangle$ is a density function for the positive measure M_w with base ν: as M_w is positive, this means $\langle \boldsymbol{h}(\xi)w, w\rangle \geq 0$ for ν-almost every $\xi \in \mathbb{R}^d$.

Now let S be a dense, countable subset of $\{w \in \mathbb{C}^\ell \mid \|w\| = 1\}$: for every $w \in S$, let A_w be the set of ξ such that $\langle \boldsymbol{h}(\xi)w, w\rangle$ is not positive semidefinite. Then A_w is a null set, hence the union $A_S = \bigcup_{w\in S} A_w$ also is.

Now let $\xi \notin A_S$. This means $\langle \boldsymbol{h}(\xi)w, w\rangle \geq 0$ for all $w \in S$. But S is a dense subset of the unit complex sphere, hence this means $\langle \boldsymbol{h}(\xi)w, w\rangle \geq 0$ for all $w \in \mathbb{C}^\ell$ with norm 1: therefore $\boldsymbol{h}(\xi)$ is positive semidefinite. As A_S is a null set, this means $\boldsymbol{h}(\xi)$ is positive semidefinite for ν-almost all ξ, which concludes the proof. □

In order to study the diffraction measure $\hat{\boldsymbol{H}}$, we want to use a density $(\boldsymbol{h}, \nu)$ which is compatible with the self-similarity of $\hat{\boldsymbol{H}}$. Specifically, we want to find a density $(\boldsymbol{h}, \nu)$ such that the density function $\boldsymbol{h}$ satisfies a renormalisation relation analogous to the one satisfied by $\hat{\boldsymbol{H}}$.

In order to find this density function, we need to choose the base ν appropriately. It will suffice to choose ν to be *dilation invariant*, in the sense we will define now. Recall that, for $R > 0$, $B_R^\times = B_R \setminus \{0\}$ is the punctured ball of radius R around 0.

Definition 4.51. Let $R > 0$, and let ν be a positive measure on $B_R^\times$. We say ν is λ-*dilation invariant* if $D_\lambda \nu|_{B_R^\times} = \nu$.

We will want to find a density function $\boldsymbol{h}$ for $\hat{\boldsymbol{H}}|_{B_R}$ such that the base ν is λ-dilation invariant. Under our assumptions, $\hat{\boldsymbol{H}}(\{0\}) = 0$, so it suffices to find a density for $\hat{\boldsymbol{H}}|_{B_R^\times}$. If $\hat{\boldsymbol{H}}$ is pure point, we can simply use the counting measure on $B_R^\times$ as a base. If $\hat{\boldsymbol{H}}$ is absolutely continuous, we get the following.

Example 4.52. Assume $\hat{\boldsymbol{H}}$ is an absolutely continuous measure on $B_R^\times$: that is, it has some density $(\boldsymbol{h}, m_{\mathbb{R}^d})$. Here the Lebesgue measure satisfies $D_\lambda m_{\mathbb{R}^d} = \lambda^{-d} m_{\mathbb{R}^d}$, so this is not a dilation invariant density. One way to fix this would be to define a base and a density function by

$$\nu'(f) = \int_{B_R^\times} \frac{f(\xi)}{\|\xi\|} \, m_{\mathbb{R}^d}(\mathrm{d}\xi)$$

$$\boldsymbol{h}'(\xi) = \|\xi\| \boldsymbol{h}(\xi)$$

for all $f \in C_c(b_R^\times)$, $\xi \in B_R^\times$. This clearly satisfies $\boldsymbol{h}'\nu' = \boldsymbol{h} m_{\mathbb{R}^d} = \hat{\boldsymbol{H}}$, and $D_\lambda \nu' = \nu'$. This is, implicitly, how Baake, Gähler, and Mañibo [11] analyzed the absolutely continuous part of the diffraction measure of substitution rules.

A second way to do this is the following: let $L = B_R \setminus B_{\lambda^{-1}R}$ and define

$$\rho(\xi) = \sum_{n=0}^{\infty} \lambda^{nd} \mathbb{1}\{D_{\lambda^{-n}} L\}(\xi)$$

$$\nu'' = \rho m_{\mathbb{R}^d}$$

$$\boldsymbol{h}'' = \frac{\boldsymbol{h}}{\rho}$$

for all $f \in C_c(b_R^\times)$, $\xi \in B_R^\times$. Then $\boldsymbol{h}''$ is a density function of $\hat{\boldsymbol{H}}$ with base ν'', and ν'' is λ-dilation invariant. Our general proof will use a similar construction to this second variant.

Theorem 4.53. *For all $R > 0$, $\hat{\boldsymbol{H}}|_{B_R^\times}$ has a λ-dilation invariant density.*

Proof. In the following proof, if ν_1, ν_2 are two positive measures on a space X, we will write $\nu_1 \ll \nu_2$ if ν_1 is absolutely continuous with respect to ν_2, i.e. $\nu_1(S) = 0$ for all Borel sets S such that $\nu_2(S) = 0$. Our goal will be to construct a σ-finite, λ-dilation invariant measure ν on B_R such that $\|\hat{\boldsymbol{H}}\||_{B_R} \ll \nu$, as then the Radon–Nikodym theorem implies that $\hat{\boldsymbol{H}}|_{B_R}$ has a density function with respect to ν. (See Section 3.1 for the matrix version of the Radon–Nikodym theorem.)

By Theorem 4.46, we have $\|\hat{\boldsymbol{H}}\| = \|\boldsymbol{A}(D_{\lambda^{-1}}\hat{\boldsymbol{H}})\boldsymbol{A}^*\| \leq \|\boldsymbol{A}\|^2\|D_{\lambda^{-1}}\hat{\boldsymbol{H}}\|$, where by $\|\boldsymbol{A}\|$ we mean the positive function $\xi \mapsto \|\boldsymbol{A}(\xi)\|$, and by $\|\hat{\boldsymbol{H}}\|$ and $\|D_{\lambda^{-1}}\hat{\boldsymbol{H}}\|$ we mean the total variation measures of $\hat{\boldsymbol{H}}$ and $D_{\lambda^{-1}}\hat{\boldsymbol{H}}$.

Therefore $\|\hat{\boldsymbol{H}}\| \ll \|D_{\lambda^{-1}}\hat{\boldsymbol{H}}\|$. By iterating this process, we deduce $\|\hat{\boldsymbol{H}}\| \ll \|D_{\lambda^{-n}}\hat{\boldsymbol{H}}\|$ for all $n \in \mathbb{N}$.

Now let let $L = B_R \setminus B_{\lambda^{-1}R}$. We define the positive measure $\nu_0 := \|\hat{\boldsymbol{H}}\||_L$, to be the total variation of $\hat{\boldsymbol{H}}$ restricted to L, and we define a positive measure ν by

$$\nu = \sum_{n=0}^{\infty} D_{\lambda^{-n}}\nu_0$$

As $\|\hat{\boldsymbol{H}}\|$ is a positive, σ-finite measure, so is ν. By construction, it is clear that $(D_\lambda \nu|_{\operatorname{supp}\nu}) = \nu$. We want to show that $\|\hat{\boldsymbol{H}}\||_{B_R} \ll \nu$, as this implies the existence of a density function by the Radon–Nikodym theorem.

To see this, let $S \subset B_R$ be a Borel set such that $\nu(S) = 0$ and write $S_n := S \cap D_{\lambda^{-n}}L$: as $\operatorname{supp}\hat{\boldsymbol{H}} \subset B_R$, we have $S \cap \operatorname{supp}\hat{\boldsymbol{H}} \subset \bigcup_{n=0}^{\infty} S_n$: as ν is positive, we have $\nu(S_n) = 0$ for all S. But we have $\|\hat{\boldsymbol{H}}\| \ll D_{\lambda^{-n}}\|\hat{\boldsymbol{H}}\|$, hence $\nu(S_n) = D_{\lambda^{-n}}\nu_0(S_n) = 0$ implies $\|\hat{\boldsymbol{H}}\|(S_n) = 0$. Therefore, $\|\hat{\boldsymbol{H}}\|(S) = \sum_{n=0}^{\infty}\|\hat{\boldsymbol{H}}\|(S_n) = 0$. □

Remark 4.54. Note that the base ν in the proof satisfies $\nu(B_r^\times) = \infty$ for all $r > 0$, so it does not extend to a measure on B_R in the sense of Bourbaki. This is not as problem for us, as $\hat{\boldsymbol{H}}\nu$ will still make sense as a matrix valued measure. One way to see this is that, even though ν is not a Radon measure on B_R, it is a Radon measure on the punctured ball $B_R^\times$, as every compact subset of $B_R^\times$ is bounded away from 0.

We define the following:

Definition 4.55. A *self-similar density* $\boldsymbol{h}$ *of* $\hat{\boldsymbol{H}}$ *on* $B_R^\otimes$ is a density function of $\hat{\boldsymbol{H}}|_{B_R^\times}$ such that the base ν is λ-dilation invariant.

By Theorem 4.53, a self-similar density exists for every $R > 0$. Now we show that self-similar densities satisfy a renormalisation relation.

Theorem 4.56. *Let* $(\boldsymbol{h}, \nu)$ *be a self-similar density of* $\hat{\boldsymbol{H}}$ *on* $B_R^\times$. *Then the density function* $\boldsymbol{h}$ *satisfies*

$$\boldsymbol{h}(\xi) = A(\xi)\boldsymbol{h}(\lambda\xi)A(\xi)^*$$

for all $\xi \in B_{\lambda^{-1}R}$.

Proof. Let $f \in C_c(B_{\lambda^{-1}R})$ be a test function. Then we have

$$\begin{aligned} D_{\lambda^{-1}}\hat{\boldsymbol{H}}(f) &= \int_{B_R} f(\lambda^{-1}\xi)\boldsymbol{h}(\xi)\,\nu'(\mathrm{d}\xi) \\ &= \int_{B_{\lambda^{-1}R}} f(\xi)\boldsymbol{h}(\lambda\xi)\,D_\lambda\nu'(\mathrm{d}\xi) \\ &= \int_{B_{\lambda^{-1}R}} f(\xi)\boldsymbol{h}(\lambda\xi)\,\nu'(\mathrm{d}\xi) \end{aligned}$$

Hence the density of $D_{\lambda^{-1}}\hat{\boldsymbol{H}}$ with respect to ν' is given by $\xi \mapsto \boldsymbol{h}(\lambda\xi)$. We know $\hat{\boldsymbol{H}}$ satisfies $\hat{\boldsymbol{H}} = \boldsymbol{A}(\cdot)D_{\lambda^{-1}}\hat{\boldsymbol{H}}\boldsymbol{A}(\cdot)$, so comparing the densities of both sides with respect to ν' give

$$\boldsymbol{h}(\xi) = A(\xi)\boldsymbol{h}(\lambda\xi)A(\xi)^*$$

for all $\xi \in B_{\lambda^{-1}R}$, which completes the proof. □

We will want to use this to study the decay of the diffraction measures of $\mathcal{S}$ around the origin. This is given by the following lemma.

Theorem 4.57. *Let $\mathcal{S}$ be a primitive, stone, FLC substitution rule, $\lambda > 0$ its scaling constant, and $\hat{\boldsymbol{H}}$ its diffraction matrix. Let $R > 0$ and let $(\boldsymbol{h}, \nu)$ be a self-similar density of $\hat{\boldsymbol{H}}$ on $B_R^\times$.*

Let $w \in \mathbb{C}^\ell$, and let $\hat{\eta}_w = \langle \boldsymbol{H}w, w\rangle$ be the diffraction of $\mathcal{S}$ with weights w.

(i) Let $\beta > 0$, and assume

$$\langle \boldsymbol{h}(\lambda^{-N}\xi)w, w\rangle \lesssim \|\boldsymbol{h}(\xi)\|\lambda^{-\beta N}$$

as $N \to \infty$ uniformly for ν-almost every $\xi \in B_R$. Then we have $\hat{\eta}_w(B_r) \lesssim r^{2\beta}$.

(ii) Let $\beta > 0$ and $A \subset B_R$ be a Borel set such that $\int_A \|\boldsymbol{h}(\xi)\|\,\nu(\mathrm{d}\xi) > 0$ and

$$\langle \boldsymbol{h}(\lambda^{-N}\xi)w, w\rangle \gtrsim \|\boldsymbol{h}(\xi)\|\lambda^{-\beta N}$$

as $N \to \infty$ uniformly for ν-almost every $\xi \in A$. Then we have $\hat{\eta}_w(B_r) \gtrsim r^{2\beta}$.

The same statements hold if we replace $\lesssim$ with $\lessapprox$ and $\gtrsim$ with $\gtrapprox$ (recall the definitions of $\lessapprox$ and $\gtrapprox$ from Section 2.4).

Proof. We prove the statements for $\lesssim$ and $\gtrsim$: the proof for $\lessapprox$ and $\gtrapprox$ is analogous. By definition of a density, we have:

$$\hat{\eta}_w(B_r) = \int_{B_r} \langle \boldsymbol{h}(\xi)w, w\rangle\,\nu(\mathrm{d}\xi)$$

Note that, in order to bound $\hat{\eta}_w(B_r)$ as $r \to \infty$, it suffices to bound $\hat{\eta}_w(B_{\lambda^{-N}})$ as $N \to \infty$: to see this, $r \to \hat{\eta}_w(B_r)$ is an increasing function, and that the sequence $a_n = \lambda^{-n}$ satisfies $|\frac{a_{n-1}}{a_n}|, |\frac{a_n}{a_{n-1}}| < \infty$.

Let $S := \int_{B_r} \|\boldsymbol{h}(\xi)\|^2 \, \mathrm{d}\nu(\xi)$: as $\boldsymbol{h} \in L^2_{loc}(\mathbb{R}^d, \mathbb{C}^{\ell\times\ell}; \nu)$, we have $S < \infty$.

For part *(i)*, we have

$$\begin{aligned}
\hat{\eta}_{\langle w, \boldsymbol{\Phi}\rangle}(B_{\lambda^{-N}r}) &= \int_{B_{\lambda^{-N}r}} \langle \boldsymbol{h}(\xi)w, w\rangle \, \nu(\mathrm{d}\xi) \\
&= \int_{B_R} \langle \boldsymbol{h}(\lambda^{-N}\xi)w, w\rangle \, \nu(\mathrm{d}\xi) && (D_\lambda \nu = \nu) \\
&\le S \sup_{\xi \in B_R} \langle \boldsymbol{h}(\lambda^{-N}\xi)w, w\rangle \\
&\lesssim \lambda^{-\beta N}
\end{aligned}$$

as $N \to \infty$. This proves *(i)*.

To prove *(ii)*, note that we have

$$\begin{aligned}
\hat{\eta}_{\langle w, \boldsymbol{\Phi}\rangle}(B_{\lambda^{-N}r}) &= \int_{B_{\lambda^{-N}r}} \langle \boldsymbol{h}(\xi)w, w\rangle \, d\nu_k(\xi) \\
&\ge \int_{B_{\lambda^{-N}r} \cap \lambda^{-N}A} \langle \boldsymbol{h}(\xi)w, w\rangle \, \nu(\mathrm{d}\xi) \\
&= \int_{B_r \cap A} \langle \boldsymbol{h}(\lambda^{-N}\xi)w, w\rangle \nu(\mathrm{d}\xi) && (D_\lambda \nu = \nu) \\
&\ge \lambda^{-\beta N} \int_A \|\boldsymbol{h}(\xi)\| \, \nu(\mathrm{d}\xi)
\end{aligned}$$

as $N \to \infty$. □

We will also need the following fact.

Lemma 4.58. *Let $\mathcal{S}$ be a primitive, stone, FLC substitution rule and $\hat{\boldsymbol{H}}$ be its diffraction matrix. Let $R > 0$, and let $(\boldsymbol{h}, \nu)$ be a self-similar density of $\hat{\boldsymbol{H}}$ on $B_R^\times$. Then we have*

$$\lim_{N\to\infty} \boldsymbol{h}(\lambda^{-N}\xi) = 0$$

for ν-almost all $\xi \in B_R^\times$.

Proof. Assume, towards a contradiction, that there exists some Borel set $A \subset B_R^\times$ of positive measure such that $\lim_{N\to\infty} \boldsymbol{h}(\lambda^{-N}\xi) \neq 0$. Without loss of generality, we may assume that there exists some $w \in \mathbb{C}^\ell$ and $\epsilon > 0$ such that $\mathrm{Re}\langle \boldsymbol{h}(\lambda^{-N}\xi)w, w\rangle \ge \epsilon$ for all

$\xi \in A$, $N \in \mathbb{N}$. But then we have

$$\begin{aligned}\hat{\eta}_{\langle w, \boldsymbol{\Phi} \rangle}(B_{\lambda^{-N}R}) &= \int_{B_{\lambda^{-N}R}} \langle \boldsymbol{h}(\xi) w, w \rangle \, \nu(\mathrm{d}\xi) \\ &\geq \int_{A \cap B_{\lambda^{-N}R}} \langle \boldsymbol{h}(\xi) w, w \rangle \, \nu(\mathrm{d}\xi) \\ &\geq \int_A \epsilon \, \nu(\mathrm{d}\xi) \geq \epsilon \nu(A) > 0\end{aligned}$$

However, by Lemma 3.32, we have $\lim_{N \to \infty} \hat{\eta}_{\langle w, \boldsymbol{\Phi} \rangle}(B_{\lambda^{-N}R}) = \hat{\eta}_{\langle w, \boldsymbol{\Phi} \rangle}(\{0\}) = 0$, which contradicts the previous inequality. □

Note that this depends on the fact that $\boldsymbol{h}$ is chosen to be self-similar: if $\hat{\boldsymbol{H}}$ is absolutely continuous and we use the Lebesgue measure $\nu = m_{\mathbb{R}^d}$ as a base, the density $\boldsymbol{h}$ does not need to vanish at the origin.

By the previous lemma, we are interested in the behavior of $\boldsymbol{h}(\lambda^{-N}\xi)$ as $N \to \infty$. We want to study this using the renormalisation relation for the diffraction density. This motivates the following definition:

Definition 4.59. The *Fourier cocycle* of the substitution rule $\mathcal{S}$ is the matrix function given by

$$\boldsymbol{A}^{(N)}(\xi) = \boldsymbol{A}(\lambda^{-N+1}\xi)\boldsymbol{A}(\lambda^{-N+2}\xi) \cdots \boldsymbol{A}(\xi)$$

for $\xi \in \mathbb{R}^d$ and $N \in \mathbb{N}$.

Corollary 4.60. *Let $(\boldsymbol{h}, \nu)$ be a self-similar density of $\hat{\boldsymbol{H}}$ on $B_R^{\times}$ with respect to the substitution rule $\mathcal{S} = (\tau_*, \lambda, \boldsymbol{\Delta}_*)$. Then the density function $\boldsymbol{h}$ satisfies*

$$\boldsymbol{h}(\lambda^{-N}\xi) = A^{(N)}(\lambda^{-1}\xi)\boldsymbol{h}(\xi)A^{(N)}(\lambda^{-1}\xi)^*$$

for all $\xi \in B_R^{\times}$, $N \in \mathbb{N}$.

5 Linear cocycles around the origin

In this chapter, we will prove Theorem 5.3, which controls the behavior of linear cocycles such as the ones arising from the renormalisation relations of self-similar tilings.

5.1 Linear cocycles around the origin

In this section, let $\|\cdot\|$ be any norm on $\mathbb{C}^\ell$, and fix $\lambda > 0$. Let $A : \mathbb{R}^d \to \mathbb{C}^{\ell\times\ell}$ be a matrix-valued function: unlike in the previous chapters, here we do not write it in bold, in order to ease the notation. For all $\xi \in \mathbb{R}^d$ and $N \in \mathbb{N}$ we define the cocycle associated to A by the product $A^{(N)}(\xi) = A(\lambda^{-N+1}\xi)\cdots A(\xi)$. In this section we will prove Theorem 5.3, which is a theorem about the asymptotic behavior of the linear cocycle. This theorem will hold under the following assumption.

Assumption 5.1. *Let $A : \mathbb{R}^d \to \mathbb{C}^{\ell\times\ell}$ be a matrix-valued function. We assume A is Lipschitz at 0: that is, $A(\xi) = A(0) + O(\|\xi\|)$ as $\xi \to 0$.*

Note that this assumption is satisfied if A is smooth, as it is the case for the linear cocycles arising from renormalisation relations of self-similar tilings.

Let A be a matrix function that satisfies Assumption 5.1. Let $\chi_1 > \cdots > \chi_l$ be the distinct values of $\{\log|\mu| \mid \mu \in \operatorname{Spec} A(0) \setminus \{0\}\}$ and let $\chi_{l+1} = -\infty$: we call these the *Lyapunov exponents* of $A(0)$. For $\mu \in \operatorname{Spec} A(0)$, denote its generalized eigenspace by E_μ, and let $E_j := \oplus\{E_\mu \mid \mu \in \operatorname{Spec} A(0), \log|\mu| = \chi_j\}$ be the space of generalized eigenvectors associated to χ_j for $j \in [l+1]$. Then there exist uniquely defined idempotent operators $P_j : \mathbb{C}^\ell \to \mathbb{C}^\ell$ for $j \in [l+1]$ such that $\operatorname{img} P_j = E_j$ and $\sum_{j=1}^{l+1} P_j = I$: we call these the *projections associated to the spaces E_j*. Note that we are not assuming that the P_j are orthogonal projections. Also define $m_j := \dim E_j$.

We would like to know how $A^{(N)}(\xi)x$ behaves as $N \to \infty$. If we restrict ourselves to the case of matrix powers, we can obtain the following result by elementary linear algebra.

Example 5.2. Assume $B \in \mathbb{C}^{\ell\times\ell}$ is unitarily diagonalizable, let $\chi_1, \cdots, \chi_l$ be the finitely many values of $\{\log|\mu| \mid \mu \in \operatorname{Spec} B\}$, and let $E_1, \cdots, E_l$ the corresponding sums of eigenspaces. Then, if $x \in E_j$, we have

$$\|B^N x\| = e^{\chi_j N}\|x\| \tag{5.1}$$
$$B^N x \in E_j \tag{5.2}$$

for all $N \in \mathbb{N}$.

Our goal will be to prove an analogous result for the linear cocycle $A^{(N)}(\xi)$, under Assumption 5.1.

Theorem 5.3. *Let A be a matrix valued function satisfying Assumption 5.1.*

Then there exists $R > 0$ such that the following holds: for all $\xi \in B_R$, there exist projections $P_j(\xi)$ for all $j \in [l+1]$ such that $\sum_{j=1}^{l+1} P_j(\xi) = I$ and the following asymptotic inequalities hold:

(i) For all $j \in [l+1]$ we have

$$\|A^{(N)}(\xi)P_j(\xi)x\| \approx e^{\chi_j N}\|P_j(\xi)x\| \tag{5.3}$$

as $N \to \infty$ uniformly for $\xi \in B_R, x \in \mathbb{C}^\ell$.

(ii) For all $j, k \in [l+1]$ we have

$$\|P_k A^{(N)}(\xi)P_j(\xi)x\| \lesssim \lambda^{-N}\|A^{(N)}(\xi)P_j(\xi)x\| \qquad \text{if } j > k \tag{5.4}$$
$$\|P_k A^{(N)}(\xi)P_j(\xi)x\| \lessapprox \max(e^{\chi_k - \chi_j}, \lambda^{-1})^N\|A^{(N)}(\xi)P_j(\xi)x\| \qquad \text{if } j < k \tag{5.5}$$

as $N \to \infty$ uniformly for $\xi \in B_R, x \in \mathbb{C}^\ell$.

(iii) For all $j \in [l]$ such that $m_j = 1$:

$$\frac{\|A^{(N)}(\xi)P_j(\xi)x\|}{\|P_j(\xi)x\|} \gtrsim e^{\chi_j N} \tag{5.6}$$

as $N \to \infty$ uniformly for $\xi \in B_R, x \in \mathbb{C}^\ell$.

(iv) If $A(0)$ is diagonalizable, for all $j \in [l]$:

$$\frac{\|A^{(N)}(\xi)P_j(\xi)x\|}{\|P_j(\xi)x\|} \asymp e^{\chi_j N} \tag{5.7}$$

as $N \to \infty$ uniformly for $\xi \in B_R, x \in \mathbb{C}^\ell$.

Remark 5.4. Dubiner [22] proved a similar theorem for the case of a sequence of matrices A_n converging to a limit matrix A, without assumptions on the speed of convergence.

In this setting he was able to show Inequality (5.3) and weaker versions of Inequalities (5.5) and (5.4). However, for our application, the stronger bounds on the speed of decay in Inequalities (5.5) and (5.4) will be crucial, as will be the uniformity for ξ.

Remark 5.5. In general, if $(A_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of matrices and $A^{(N)} = A_N \cdots A_1$ is its associated linear cocycle, one asks for the possible values of the set

$$\{\lim_{N\to\infty} \frac{1}{N} \log\|A^{(N)}x\| \mid x \in \mathbb{C}^\ell\},$$

as well as the spaces where these values are attained. These are called the *Lyapunov exponents* and *Lyapunov spaces* of the sequence, respectively: therefore, our theorem characterizes the Lyapunov exponents and spaces of the sequence $(A(\lambda^{-n}\xi))_{N\in\mathbb{N}}$ in terms of the Lyapunov exponents and spaces of the limit matrix $A(0)$.

There exists a large literature on the theory of Lyapunov exponents and spaces, particularly in the setting of cocycles arising from dynamical systems, or their continuous analogues: see [1, 13]. However, note that our case does not fit into the most common setting of Lyapunov theory: the map $x \mapsto \lambda^{-1}x$ is not measure preserving, so our cocycle does not satisfy the conditions of Oseledet's Theorem. This is why we will not make use of the existing literature on Lyapunov theory, instead providing an elementary proof of Theorem 5.3.

Remark 5.6. If we equip $\mathbb{C}^\ell$ with an inner product such that the projections P_j and $P_j(\xi)$ are orthogonal projections for all $j \in [l+1]$, we can interpret the fractions in the theorem by the formula

$$\frac{\|P_k A^{(N)}(\xi) P_j(\xi)x\|}{\|P_j(\xi)x\|} = \cos\angle(E_k, A^{(N)}P_j(\xi)x),$$

where the angle between a subspace U and a vector v is given by $\angle(U,v) = \min_{u\in U}\angle(u,v)$. Therefore, Equations (5.4) and (5.5) tell us that $A^{(N)}(\xi)P_j(\xi)x$ is "asymptotically orthogonal" to E_k whenever $k \neq j$: the intuition is that $A^{(N)}(\xi)P_j(\xi)x$ is close to being in E_j, which is orthogonal to the other eigenspaces. In the subsequent sections we will mostly reason using norms and inner products instead of angles, but it is still useful to keep this image in mind.

5.2 Exterior powers

In order to prove Theorem 5.3, we will use the formalism of exterior products in order to represent linear subspaces. Here we will spell out some basic facts about exterior products which we will need: everything in this section is either known or elementary, but we were not able to find a convenient reference for all of them.

If V is a finite dimensional vector space over $\mathbb{C}$, we denote by $V^{\wedge q}$ the q-th exterior power of V, which can be defined as the dual of the space of alternating multilinear maps from V^q to $\mathbb{C}$, or as an appropriate quotient of the q-th tensor power $V^{\otimes q}$: see Michler and Kowalsky [43] for a more detailed definition. For $q_1, q_2 \in \mathbb{N}$ we denote the *wedge product* by $\wedge : V^{\wedge q_1} \times V^{\wedge q_2} \to V^{\wedge (q_1+q_2)}$. This has the following property: for all $q \in \mathbb{N}$, the map $(v_1, \cdots, v_q) \mapsto v_1 \wedge \cdots \wedge v_q$ is an alternating multilinear map.

For the rest of this chapter, let $\ell, q \in \mathbb{N}$ and fix an inner product on $\mathbb{C}^\ell$. We will consider the exterior power $(\mathbb{C}^\ell)^{\wedge q}$, which we denote as $\mathbb{C}^{\ell \wedge q}$ to ease the notation. Then we can define a corresponding inner product on $(\mathbb{C}^\ell)^{\wedge q}$ with the following property: if $\alpha = v_1 \wedge \cdots \wedge v_q, \beta = w_1 \wedge \cdots \wedge w_q$,

$$\langle \alpha, \beta \rangle = \det J(\alpha, \beta)$$

where $J(\alpha, \beta) \in \mathbb{C}^{\ell \times \ell}$ is the matrix with entries $J_{i,k} = \langle v_i, w_k \rangle$. This inner product makes the wedge product continuous.

We will take a closer look at what happens when one takes the wedge product of q vectors.

Lemma 5.7. *Let $\alpha \in \mathbb{C}^{\ell \wedge q}$ and $a_1, \cdots, a_q \in \mathbb{C}^\ell$ be linearly independent. The following are equivalent:*

(i) There is some $c \in \mathbb{C}$ such that $c\alpha = a_1 \wedge \cdots \wedge a_q$

(ii) The space $H = \{v \in \mathbb{C}^\ell \mid \alpha \wedge v = 0\}$ is spanned by the a_j.

If either of these are true, we have $\{v \in \mathbb{C}^\ell \mid \alpha \wedge v = 0\} = \operatorname{span}(a_1, \cdots, a_q)$. In particular, the span of the vectors a_j is independent of the particular choice of vectors to represent α.

Proof. To show *(i)* $\implies$ *(ii)*, note that $c\alpha = a_1 \wedge \cdots \wedge a_q$ implies $\alpha \wedge a_j = 0$ for all $j \in [q]$, so they are all in the space H. Furthermore, if we pick $a_{q+1}, \cdots, a_\ell$ such that $a_1, \cdots, a_\ell$ is a basis, we have $\|\alpha \wedge a_{q+1} \wedge \cdots \wedge a_\ell\| = \det(a_1 \cdots a_\ell) \neq 0$, so the $a_{q+1}, \cdots, a_\ell$ are $\ell - q$ linearly independent vectors not in H. Therefore $H = \operatorname{span}(a_1, \cdots, a_q)$, and in particular $\dim H = q$.

For the opposite direction, let $a_1, \cdots, a_q$ be a basis of H and complete it to a basis of $\mathbb{C}^\ell$ as before. Now let $\mathcal{J} = \{(j_1, \cdots, j_q) \in [\ell]^q \mid j_1 < \cdots < j_q\}$ and $a_J = a_{j_1} \wedge \cdots \wedge a_{j_q}$ for $J \in \mathcal{J}$. Then the set $\{a_J \mid J \in \mathcal{J}\}$ is a basis of $\mathbb{C}^{\ell \wedge q}$. Furthermore, for every $j \in [\ell]$ and $J \in \mathcal{J}$, we have $a_J \wedge a_j = 0$ if and only if $j \in J$, and the set $\{a_J \wedge a_j \mid J \in \mathcal{J}, j \notin J\}$ is linearly independent.

Write $\alpha = \sum_{J \in \mathcal{J}} c_J a_J$, for coefficients $c_J \in \mathbb{C}$: by the above two properties, $\alpha \wedge a_j = 0$ implies $c_J = 0$ for all $J \in \mathcal{J}$ such that $j \notin J$. As $\alpha \wedge a_j = 0$ for all $j \in [q]$, we must have $c_J = 0$ for every $J \in \mathcal{J}$ except for $J = (1, \cdots, q)$: this concludes the proof. □

Due to this lemma, the following is well-defined.

Definition 5.8. If $\alpha \in \mathbb{C}^{\ell \wedge q}$ can be written as $\alpha = a_1 \wedge \cdots \wedge a_q$ for linearly independent vectors $a_1, \cdots, a_q \in \mathbb{C}^\ell$, we say α is a *q-blade* or *decomposable q-vector*, and let its *span* be defined by $\operatorname{span} \alpha = \operatorname{span}(a_1, \cdots, a_q)$, where the a_j are any q vectors such that $\alpha = a_1 \wedge \cdots \wedge a_q$. (By Lemma 5.7, this quantity does not depend on the choice of the a_j.) We also let π_α be the orthogonal projection onto the subspace $\operatorname{span} \alpha \subset \mathbb{C}^\ell$.

Definition 5.9. Let $q_1, q_2 \in \mathbb{N}$, $\alpha \in \mathbb{C}^{\ell \wedge q_1}$, and $\beta \in \mathbb{C}^{\ell \wedge q_2}$. We write $\alpha \subset \beta$ if there exists some γ such that $\beta = \alpha \wedge \gamma$. If α, β are blades, this is the case if and only if $\operatorname{span} \alpha \subset \operatorname{span} \beta$.

We will need certain geometric properties of the exterior powers.

Lemma 5.10. *Let $q, q' \in \mathbb{N}$, α, β be q-blades, α', β' be q'-blades.*

(i) Assume $\operatorname{span} \alpha \perp \operatorname{span} \alpha'$. *Then we have* $\langle \alpha \wedge \alpha', \alpha \wedge \beta' \rangle = \|\alpha\|^2 \langle \alpha', \beta' \rangle$.

(ii) Assume $\operatorname{span} \alpha \perp \operatorname{span} \alpha'$. *Then we have* $\|\alpha \wedge \alpha'\| = \|\alpha\| \|\alpha'\|$

(iii) For all $x \in \mathbb{C}^\ell$, we have $\|\alpha \wedge x\| = \|\alpha\| \|(I - \pi_\alpha) x\|$.

Proof. First we prove *(i)*. Let $\alpha = a_1 \wedge \cdots \wedge a_q$, $\alpha' = a'_1 \wedge \cdots \wedge a'_{q'}$ and $\beta' = b'_1 \wedge \cdots \wedge b'_{q'}$. Let A be the matrix with columns $a_1, \cdots, a_q$, B be the matrix with columns $b'_1, \cdots, b'_{q'}$, and A' be the matrix with columns $a'_1, \cdots, a'_{q'}$. By assumption, we have $\operatorname{span} \alpha \perp \operatorname{span} \alpha'$: this means that $\langle a_i, a'_j \rangle = 0$ for all i, j, so $A'^* A = 0$. Therefore, we obtain

$$\langle \alpha \wedge \alpha', \alpha \wedge \beta' \rangle = \begin{vmatrix} A^*A & A^*B \\ A'^*A & A'^*B \end{vmatrix} = \begin{vmatrix} A^*A & A^*B \\ 0 & A'^*B \end{vmatrix} = \det(A^*A) \det(A'^*B) = \|\alpha\|^2 \langle \alpha', \beta' \rangle$$

This proves *(i)*. *(ii)* follows immediately from *(i)*.

To show *(iii)*, note that $\pi_\alpha x \in \operatorname{span} \alpha$, and therefore $\alpha \wedge \pi_\alpha x = 0$; and also $\operatorname{span} \alpha \perp \operatorname{span}(1 - \pi_\alpha) x$. Then using *(i)*, we have

$$\|\alpha \wedge x\| = \|\alpha \wedge (1 - \pi_\alpha) x\| = \|\alpha\| \|(1 - \pi_\alpha) x\|$$

which concludes the proof. □

Lemma 5.11. *Let $B_q(\mathbb{C}^\ell) \subset \mathbb{C}^{\ell \wedge q}$ be the set of q-blades in $\mathbb{C}^{\ell \wedge q}$. Then its closure in the norm topology of $\mathbb{C}^{\ell \wedge q}$ is $B_q(\mathbb{C}^\ell) \cup \{0\}$.*

Proof. To see 0 is in the closure of $B_q(\mathbb{C}^\ell)$, let $\alpha \in B_q(\mathbb{C}^\ell)$: then $\lim_{n\to\infty} \frac{1}{n}\alpha = 0$.

Now let $(\alpha^n)_{n\in\mathbb{N}}$ be a sequence of q-blades in $\mathbb{C}^{\ell\wedge q}$ converging to some $\alpha \in \mathbb{C}^{\ell\wedge q} \setminus \{0\}$. We want to show that α is a q-blade.

For every $n \in \mathbb{N}$, choose an orthonormal basis $a_1^n, \cdots, a_q^n$ of $\operatorname{span} \alpha^n$ and $c_n \in \mathbb{C}$ such that $\alpha^n = c_n a_1^n \wedge \cdots \wedge a_q^n$. The space of orthonormal sets of q vectors in $\mathbb{C}^\ell$ is compact, so by passing to a subsequence we can assume that there exists an orthonormal set $a_1, \cdots, a_q$ such that $a_j^n \to a_j$ as $n \to \infty$ for all $j \in [q]$. Then there exists some $c \in \mathbb{C} \setminus \{0\}$ such that $\lim_{n\to\infty} c_n = c$ and

$$\alpha = \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} c_n(a_1^n \wedge \cdots \wedge a_q^n) = ca_1 \wedge \cdots \wedge a_q$$

so α is a q-blade. □

Lemma 5.12. *For all q-blades α, β we have*

$$\|\pi_\alpha - \pi_\beta\| \leq 2\frac{\|\alpha - \beta\|}{\min(\|\alpha\|, \|\beta\|)}$$

Proof. As π_α and π_β are orthogonal projections, it suffices to show that

$$\|(I - \pi_\beta)\pi_\alpha x\| \leq \frac{\|\alpha - \beta\|\|\pi_\alpha x\|}{\|\beta\|} \text{ and}$$
$$\|(I - \pi_\alpha)\pi_\beta x\| \leq \frac{\|\alpha - \beta\|\|\pi_\beta x\|}{\|\alpha\|}$$

for all $x \in \mathbb{C}^\ell$, as we have $(\pi_\alpha - \pi_\beta)x = 0$ for all x not in $\operatorname{span} \alpha + \operatorname{span} \beta$. We show only the first inequality, as the second follows by swapping α and β.

Let $x \in \mathbb{C}^\ell$. Then, using Lemma 5.10, we have

$$\|\beta \wedge \pi_\alpha x\| = \|\beta\|\|(I - \pi_\beta)\pi_\alpha x\| \implies \|(I - \pi_\beta)\pi_\alpha x\| = \frac{\|\beta \wedge \pi_\alpha x\|}{\|\beta\|}.$$

On the other hand, we can write $\beta \wedge \pi_\alpha x = (\beta - \alpha) \wedge \pi_\alpha x$, as $\pi_\alpha x \in \operatorname{span} \alpha$. Therefore

$$\|(I - \pi_\beta)\pi_\alpha x\| = \frac{\|(\beta - \alpha) \wedge \pi_\alpha x\|}{\|\beta\|} \leq \frac{\|\alpha - \beta\|\|\pi_\alpha x\|}{\|\beta\|}.$$

This concludes the proof. □

Now let $A \in \mathbb{C}^{\ell\times\ell}$. Then we define the *$q$-th exterior power of A* as the linear map $A^{\wedge q} : \mathbb{C}^{\ell\wedge q} \to \mathbb{C}^{\ell\wedge q}$ by

$$A^{\wedge q}(v_1 \wedge \cdots \wedge v_q) = Av_1 \wedge \cdots \wedge Av_q$$

for all $v_1, \cdots, v_q \in \mathbb{C}^\ell$.

Then the generalized eigenvalues of $A^{\wedge q}$ can be given in terms of the generalized eigenvalues of A.

Lemma 5.13. *Let $A \in \mathbb{C}^{\ell\times\ell}$ and let $B = (v_1, \cdots, v_\ell)$ be an ordered basis of $\mathbb{C}^\ell$ such that A is in upper Jordan normal form with respect to this basis. Let $\mu_1, \cdots, \mu_\ell$ be the corresponding diagonal entries of A, which are the generalized eigenvalues of A repeated according to their multiplicity.*

Let

$$B' = \{v_{j_1} \wedge \cdots \wedge v_{j_q} \mid j_1, \cdots, j_q \in [\ell], j_1 < \cdots < j_q\}$$

be the corresponding basis of $\mathbb{C}^{\ell\wedge q}$, ordered lexicographically on the indices. Let $(a_{JK})_{J,K}$ be the coefficients of the transformation matrix of $A^{\wedge q}$ with respect to the basis B', indexed by ordered q-tuples of indices J, K. Then $A^{\wedge q}$ has the following properties:

(i) For $J = (j_1, \cdots, j_q)$, the corresponding diagonal entry of $A^{\wedge q}$ is given by $a_{JJ} = \mu_{j_1} \cdots \mu_{j_q}$.

(ii) For $J = (j_1, \cdots, j_q)$, $K = (k_1, \cdots, k_q)$ with $J < K$, we have

- $a_{KJ} = 0$
- $a_{JK} \neq 0 \implies a_{JJ} = a_{KK}$.

(iii) $A^{\wedge q}$ is upper triangular with respect to the basis B', and its generalized eigenvalues with multiplicity are given by $\{\mu_{j_1} \cdots \mu_{j_q} \mid 1 \leq j_1 < \cdots < j_q \leq \ell\}$.

Proof. By assumption, A is in Jordan normal form with respect to the basis B. Therefore, for every $j \in [\ell]$, one of the following holds:

- v_j is an eigenvector of A with eigenvalue μ_j: that is, $Av_j = \mu_j v_j$.
- v_j is a generalized eigenvector of A with eigenvalue μ_j: that is, $Av_j = \mu_j v_j + v_k$ for some $k < j$ where $\mu_j = \mu_k$.

Using this, one can deduce (i) and (ii) by computing $A^{\wedge q}(v_{j_1} \wedge \cdots \wedge v_{j_q})$ for $j_1 < \cdots < j_q$: one gets the diagonal entry $\mu_1 \cdots \mu_q$, and one gets off-diagonal entries if some of the vectors are generalized eigenvectors, but they stay within the same Jordan block. Statement (iii) follows immediatly from (i) and (ii). □

Lemma 5.14. *For all $A, B \in \mathbb{C}^{\ell\times\ell}$, the inequality*

$$\|A^{\wedge q} - B^{\wedge q}\| \leq q \max(\|A\|, \|B\|)^{q-1} \|A - B\|$$

holds.

Proof. This follows by induction on q. For $q = 1$ it holds trivially. Furthermore, if it holds for $q \in \mathbb{N}$, we have

$$\begin{aligned}\|A^{\wedge(q+1)} - B^{\wedge(q+1)}\| &= \|(A^{\wedge q} - B^{\wedge q}) \wedge A + B^{\wedge q} \wedge (A - B)\| \\ &\leq q \max(\|A\|, \|B\|)^{q-1} \|A - B\| \|A\| + \|B\|^q \|A - B\| \\ &\leq (q+1) \max(\|A\|, \|B\|)^q \|A - B\|\end{aligned}$$

where at the second line we use the induction hypothesis. This concludes the proof. □

5.3 Proof of Theorem 5.3

Now we set out to prove Theorem 5.3. The proof is morally analogous to the QR algorithm for computing eigenvalues, which has been generalized to compute Lyapunov exponents [29, 46].

5.3.1 The power method

For this section, equip $\mathbb{C}^\ell$ with an arbitrary inner product and its corresponding norm: in the next section, we will make a specific choice of inner product.

Lemma 5.15. *Let $(a_m)_{m \in \mathbb{N}}$ be a sequence of positive real numbers. Then, for all $M \in \mathbb{N}$, the following inequality holds*

$$\prod_{m=1}^{M} a_m \leq \exp\left(\sum_{m=1}^{M} (a_m - 1)\right)$$

Proof. Take the logarithm of both sides of the inequality and use $\log x = \log(1 + (x-1)) \leq x - 1$. □

Lemma 5.16. *Let $A : \mathbb{R}^d \to \mathbb{C}^{\ell \times \ell}$ be a matrix function satisfying Assumption 5.1, and $R > 0$.*

(i) For all $k \in \mathbb{N}$ such that $A(0)^k \neq 0$, the matrix cocycle satisfies

$$\|A^{(N)}(\xi)\| \lesssim \|A(0)^k\|^{\frac{N}{k}}$$

as $N \to \infty$ uniformly for $\xi \in B_R$.

(ii) The matrix cocycle satisfies

$$\|A^{(N)}(\xi)\| \lessapprox \rho(A(0))^N$$

as $N \to \infty$ *uniformly for* $\xi \in B_R$*, where* $\rho(A(0)) = \max\{|\mu| \mid \mu \in \operatorname{Spec} A(0)\}$ *is the spectral radius of* $A(0)$*.*

Proof. We prove both parts separately:

(i) For the first part, let $N \in \mathbb{N}$: we can always write it as $N = kM + r$ for some $M \in \mathbb{N}$, $r \in \{0, \cdots, k-1\}$. Then taking the limit as $N \to \infty$ is the same as taking the limit as $M \to \infty$ for all $r \in \{0, \cdots, k-1\}$.

Recall that, for $N \in \mathbb{N}$, the linear cocycle is given by $A^{(N)}(\xi) = A(\lambda^{-N+1}\xi) \cdots A(\xi)$. Then we can decompose it as

$$A^{(N)}(\xi) = A^{(r)}(\lambda^{-kM}\xi) \prod_{m=0}^{M-1} A^{(k)}(\lambda^{-km}\xi)$$

(where the iterated product is taken from right to left) because

$$\begin{aligned} A^{(r)}(\lambda^{-kM}\xi) &= A(\lambda^{-(kM+r)+1}\xi) \cdots A(\lambda^{-kM}\xi) \\ A^{(k)}(\lambda^{-km}\xi) &= A(\lambda^{-k(m+1)+1}\xi) \cdots A(\lambda^{-km}\xi) \end{aligned}$$

Therefore, using the submultiplicativity of the matrix norm, we obtain

$$\frac{\|A^{(N)}(\xi)\|}{\|A(0)^k\|^M} \leq \|A^{(r)}(\lambda^{-kM}\xi)\| \prod_{m=0}^{M-1} \frac{\|A^{(k)}(\lambda^{-km}\xi)\|}{\|A(0)^k\|}$$

Using $A(\xi) = A(0) + O(\|\xi\|)$ as $\xi \to 0$ and Lemma 5.14, we can bound the two factors as follows.

- We have $\sum_{m=0}^{M-1} \left(1 - \frac{\|A^{(k)}(\lambda^{-km}\xi)\|}{\|A(0)^k\|}\right) \lesssim \|\xi\|$ as $M \to \infty$ uniformly for $\xi \in B_R$, because we can bound the sum by a geometric series. Therefore, by Lemma 5.15, $\prod_{m=0}^{M-1} \frac{\|A^{(k)}(\lambda^{-km}\xi)\|}{\|A(0)^k\|} \lesssim e^{\|\xi\|} \lesssim 1$ as $M \to \infty$ uniformly for $\xi \in B_R$.
- We have $\|A^{(r)}(\lambda^{-kM}\xi)\| \lesssim 1$ as $M \to \infty$ uniformly for $r \in \{0, \cdots, k-1\}$, $\xi \in B_R$.

Putting both of these facts together, we have

$$\frac{\|A^{(N)}(\xi)\|}{\|A(0)^k\|^M} \leq \|A^{(r)}(\lambda^{-kM}\xi)\| \prod_{m=0}^{M-1} \frac{\|A^{(k)}(\lambda^{-km}\xi)\|}{\|A(0)^k\|} \lesssim 1$$

as $N \to \infty$ uniformly for $\xi \in B_R$.

Finally, we have $M = \frac{N}{k} - \frac{r}{k}$ which means

$$\|A^{(N)}(\xi)\| \lesssim \|A(0)^k\|^M = \|A(0)^k\|^{\frac{N}{k}} \|A(0)^k\|^{-\frac{r}{k}} \lesssim \|A(0)^k\|^{\frac{N}{k}}$$

as $N \to \infty$ uniformly for $\xi \in B_R$. This concludes the proof of *(i)*.

(ii) For the proof of *(ii)*, we distinguish two cases:

- Assume $A(0)^k = 0$ for some $k \in \mathbb{N}$: then $\rho(A(0)) = 0$. This means we want to prove $\|A^{(N)}(\xi)\|$ decays faster than any exponential function. As in the first part, for all $N \in \mathbb{N}$, we can write $N = kM + r$ with $M \in \mathbb{N}$, $r \in \{0, \cdots, k-1\}$. Then we have

$$\frac{1}{N}\log\|A^{(N)}(\xi)\| = \frac{1}{N}\log\|A^{(r)}(\lambda^{-kM}\xi)\| + \frac{1}{N}\sum_{m=0}^{M-1}\log\|A^{(k)}(\lambda^{-km}\xi)\|$$
$$\lesssim -\frac{1}{N}\sum_{m=0}^{M-1} m \lesssim -N$$

  This implies that $\frac{1}{N}\log\|A^{(N)}(\xi)\| \to -\infty$ as $N \to \infty$, uniformly for $\xi \in B_R$, therefore $\|A^{(N)}(\xi)\| \lessapprox 0 = \rho(A(0))$ as $N \to \infty$ uniformly for $\xi \in B_R$.

- Now assume $A(0)$ is not nilpotent, so we can apply part *(i)* for every $k \in \mathbb{N}$. Define a sequence $(a_n)_{n\in\mathbb{N}}$ by

$$a_N := \sup_{\xi \in B_R} \log\|A^{(N)}(\xi)\|$$

  By the submultiplicativity of the matrix norm, this sequence is subadditive, so Fekete's Lemma tells us that $\lim_{N\to\infty} \frac{a_N}{N} = \inf_{N\in\mathbb{N}} \frac{a_N}{N}$. Let $\chi := \lim_{N\to\infty} \frac{a_N}{N} = \lim_{N\to\infty} \frac{1}{N} \log \sup_{\xi\in B_R} \|A^{(N)}(\xi)\|$. Then, by part *(i)*, for all $k \in \mathbb{N}$ we have

$$\chi = \lim_{M\to\infty} \frac{a_{kM}}{kM} \leq \frac{1}{k}\log\|A(0)^k\|$$

  By taking the limit as $k \to \infty$, we obtain $\chi \leq \lim_{k\to\infty} \frac{1}{k}\log\|A(0)^k\| = \log \rho(A(0))$, which concludes the proof.

□

Lemma 5.17. *Let $A : \mathbb{R}^d \to \mathbb{C}^{\ell\times\ell}$ be a matrix function satisfying Assumption 5.1, and assume there exists an eigenvector v be an eigenvector of $A(0)^*$ whose eigenvalue $\overline{\mu}$ satisfies $|\overline{\mu}|^k = \|A(0)^k\| > 0$ for some $k \in \mathbb{N}$. We denote the conjugate of $\overline{\mu}$ by μ.*

Define $w^{(N)}(\xi) := \overline{\mu}^{-N} A^{(N)*}(\xi)v$. Then:

(i) *The limit* $w(\xi) := \lim_{N\to\infty} w^{(N)}(\xi)$ *exists for all* ξ.

(ii) $\|w(\xi) - w^{(N)}(\xi)\| = O(\lambda^{-N}\|\xi\|)$

(iii) $\|w(\xi) - v\| = O(\|\xi\|)$

(iv) $\|w(\xi)\| = 1 + O(\|\xi\|)$

(v) $\langle A^{(N)}(\xi)w(\xi), v\rangle = \mu^N(1 + O(\|\xi\|))$

(vi) $\|A^{(N)}(\xi)w(\xi)\| \asymp |\mu|^N$

as $\xi \to 0$, $N \to \infty$.

Proof. By Lemma 5.16, we have $\|A^{(N)}(\xi)\| \lesssim \|A(0)^k\|^{\frac{N}{k}} = |\mu|^N$ as $N \to \infty$: therefore $\|\overline{\mu}^{-N-1}A^{(N)*}(\xi)\| \lesssim 1$ as $\xi \to 0, N \to \infty$.

Then the differences $w^{(N+1)}(\xi) - w^{(N)}(\xi)$ are bounded by

$$\begin{aligned}\|w^{(N+1)}(\xi) - w^{(N)}(\xi)\| &= \|\overline{\mu}^{-N-1}A^{(N+1)*}(\xi)v - \overline{\mu}^{-N}A^{(N)*}(\xi)v\| \\ &\le \|\overline{\mu}^{-N-1}A^{(N)*}(\xi)\|\|A(\lambda^{-N}\xi)^*v - \overline{\mu}v\| \\ &\le \|\overline{\mu}^{-N-1}A^{(N)*}(\xi)\|\|A(\lambda^{-N}\xi)^* - A(0)^*\|\|v\| \\ &= O(\lambda^{-N}\|\xi\|)\end{aligned}$$

as $\xi \to 0, N \to \infty$.

We can write $w^{(N)}(\xi) = v + \sum_{n=1}^{N}\left(w^{(n)}(\xi) - w^{(n-1)}(\xi)\right)$: as $N \to \infty$, the latter sum is bounded by a telescoping series with exponent λ, therefore the limit $w(\xi)$ exists and satisfies *(i)* and *(iii)*. Furthermore, we have

$$w(\xi) - w^{(N)}(\xi) = \sum_{n=N+1}^{\infty} w^{(n)}(\xi) - w^{(n-1)}(\xi) = O(\lambda^{-N}\|\xi\|)$$

as $N \to \infty, \xi \to 0$, where again we are bounding the series by a geometric series. Statement *(iv)* follows directly from *(iii)*.

For *(v)*, we have

$$\begin{aligned}\langle A^{(N)}(\xi)w(\xi), v\rangle &= \langle w(\xi), \overline{\mu}^N\overline{\mu}^{-N}A^{(N)}(\xi)^*v\rangle \\ &= \mu^N\langle w(\xi), w^{(N)}(\xi)\rangle \\ &= \mu^N(\|w(\xi)\|^2 + \langle w(\xi), w^{(N)}(\xi) - w(\xi)\rangle) \\ &= \mu^N(1 + O(\|\xi\|))\end{aligned}$$

as $\xi \to 0, N \to \infty$, using *(ii)* and *(iv)*.

Now we only have to prove *(vi)*. From Lemma 5.16 we obtain $\|A^{(N)}(\xi)\| \lesssim |\mu|^N$, which is the upper bound. For the lower bound, we use *(v)*:

$$\begin{aligned}\|A^{(N)}(\xi)w(\xi)\| &\geq |\langle A^{(N)}(\xi)w(\xi), v\rangle| \\ &= |\mu|^N(1 + O(\|\xi\|))\end{aligned}$$ □

Example 5.18. If $A(\xi)$ is the normalized Fourier matrix of a primitive substitution rule, the vector given by $v_j = m_{\mathbb{R}^d}(\tau_j)$ is an eigenvector of $A(0)^*$ with eigenvalue $1 = \|A(0)\|$. Then one can show that $w(\xi)$ is given by $w(\xi)_j = \widehat{\mathbb{1}_{\tau_j}}(\xi)$ (up to a constant factor), as this is the vector function that satisfies $w(\xi) = A(\xi)^* w(\lambda^{-1}\xi)$. Then $w(\xi)$ can be understood as the cocycle analogue of an eigenvector, and the proof of Lemma 5.17 is analogous to the power method for finding the top eigenvector.

5.3.2 Defining the subspaces

From now on, we let A be a matrix-valued function satisfying Assumption 5.1. As before, we define $A^{(N)}(\xi) = A(\lambda^{-N+1}\xi) \cdots A(\xi)$. Let χ_j, E_j, P_j be as in Section 5.1 and define $m_j := \dim E_j$.

We will want to pick a basis of $\mathbb{C}^\ell$ of the following form.

Definition 5.19. A *descending Jordan basis* for a matrix $A \in \mathbb{C}^{\ell \times \ell}$ is a basis $v_1, \cdots, v_\ell$ with the following properties:

- A is in lower Jordan normal form with respect to this basis.
- The diagonal entries μ_j with respect to the basis satisfy $|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_\ell|$.

Example 5.20. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{pmatrix}.$$

Then the standard basis is a descending Jordan basis for A.

Every square matrix over the complex numbers has a Jordan normal form, so in particular it admits a descending Jordan basis. Furthermore, given a descending Jordan basis, we can choose an appropriate inner product on $\mathbb{C}^\ell$ which makes this basis orthonormal. **For the rest of this section, we fix a descending Jordan basis $v_1, \cdots, v_\ell$ for $A(0)$ and an inner product on $\mathbb{C}^\ell$ such that the basis is orthonormal with respect to the inner product.** Let $\mu_1, \cdots, \mu_\ell$ be corresponding diagonal entries of $A(0)$.

By the definition of a descending Jordan basis, the first m_1 vectors span E_1, the next m_2 span E_2, and so on. For $j \in \{0, \cdots, l+1\}$, define $m_{(j)} = m_1 + \cdots + m_j$ as the sum of the first j multiplicities for $j = 0, \cdots, l+1$, setting $m_{(0)} := 0$. As the basis is orthonormal, $A(0)^*$ is in upper Jordan normal form with respect to this basis, and the diagonal entries are given by $\overline{\mu}_j$.

We will characterize the Lyapunov spaces of the cocycle by using exterior algebra. Recall that, for $A \in \mathbb{C}^{\ell\times\ell}$, $A^{\wedge q}$ is the *q-th exterior power* of A, which is the linear operator on $\mathbb{C}^{\ell\wedge q}$ which satisfies

$$A^{\wedge q}(w_1 \wedge \cdots \wedge w_q) = Aw_1 \wedge \cdots \wedge Aw_q$$

for all $w_1, \cdots, w_q \in \mathbb{C}^\ell$.

In our case, for all $q \in [\ell]$, we can consider the matrix function $A^{\wedge q}$ defined by $A^{\wedge q}(\xi) = A(\xi)^{\wedge q}$, and its associated cocycle $A^{(N)\wedge q}(\xi) = A(\lambda^{-N+1}\xi)^{\wedge q} \cdots A(\xi)^{\wedge q}$.

Then we can define:

Definition 5.21. Let $j \in [\ell]$. We define

$$\begin{aligned} \alpha_j &= v_1 \wedge \cdots \wedge v_j \\ \mu_{(j)} &= \mu_1\mu_2 \cdots \mu_j \\ \beta_j^{(N)}(\xi) &= \overline{\mu_{(j)}}^{-N} A^{(N)*\wedge j}\alpha_j \\ \beta_j(\xi) &= \lim_{N\to\infty} \beta_j^{(N)}(\xi) \end{aligned}$$

whenever the latter limit exists. Here, by $A^{(N)*\wedge j}$ we mean the product $(A^{(N)\wedge j}(\xi))^* = A^{\wedge j}(\xi)^* \cdots A^{\wedge j}(\lambda^{-N+1}\xi)^*$.

We want to use Lemma 5.17 to prove that, for appropriate choices of q, the limits $\beta_q(\xi)$ exist. We will do this by choosing $q = m_{(j)}$ for $j = 1, \cdots, l+1$: then α_q will be an eigenvector satisfying the assumptions of Lemma 5.17.

Lemma 5.22. *Let $j \in [l]$ and $q := m_{(j)}$. Then the following properties hold:*

(i) The matrix function $A^{\wedge q}$ satisfies Assumption 5.1: that is, we have the bound $\|A(\xi)^{\wedge q} - A(0)^{\wedge q}\| = O(\|\xi\|)$ as $\xi \to 0$.

(ii) $A(0)^{\wedge q*}$ has the eigenvalue $\overline{\mu}_{(j)}$ with the eigenvector α_q.

(iii) There exists $k \in \mathbb{N}$ such that $\|A(0)^{\wedge q}\|^k = |\mu_{(q)}|^k$.

Proof. Part *(i)* follows from Lemma 5.14, using the fact that A satisfies Assumption 5.1.

For the rest, we note that the structure of $A(0)^{\wedge q*}$ is described by Lemma 5.13, as the vectors $v_1, \cdots, v_q$ put $A(0)^*$ in upper Jordan normal form by assumption: this means the $A(0)^{\wedge q*}$ is in upper triangular form, where the diagonal entries are products of the eigenvalues of $A(0)^*$ with multiplicity, and the off-diagonal entries are are associated to vectors in the same Jordan block of $A(0)^*$.

By the way we have chosen q, the vectors $v_1, \cdots, v_q$ are the generalized eigenvectors of $A(0)^*$ associated to the exponents $\chi_1, \cdots, \chi_j$, so that $|\mu_t| < e^{\chi_j}$ for all $t > q$. In particular, if $s \leq q$ and $t > q$, v_s and v_t correspond to different Jordan blocks in $A(0)^*$. Then, using Lemma 5.13, the transformation matrix of $A(0)^{\wedge q*}$ can be written as

$$\begin{pmatrix} \mu_{(q)} & \mathbf{0}^\top \\ \mathbf{0} & M \end{pmatrix}$$

where $\mathbf{0}$ denotes the column vector of zeros of appropriate size, and M is a matrix with spectral norm strictly less than $|\mu_{(q)}|$.

Therefore, for any sufficiently large $k \in \mathbb{N}$, we have $\|M\|^k < |\mu_{(q)}|^k$ and therefore $\|(A(0)^{\wedge q*})^k\| = |\mu_{(q)}|^k$, which concludes the proof of *(iii)*. □

Example 5.23. We illustrate the phenomena described in Lemma 5.22 for the concrete example A from Example 5.20. This matrix has, has $\mu_1 = 1, \mu_2 = 1, \mu_3 = \frac{1}{2}, \mu_4 = \frac{1}{2}$, so it has the Lyapunov exponents $\chi_1 = 0$ and $\chi_2 = \log \frac{1}{2}$, with multiplicities $m_1 = 2$ and $m_2 = 2$. Taking the exterior power of A^* with $q = m_{(1)} = 2$, and representing it with respect to the basis $\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$, we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

From this, we see it can be written in the block form given in Lemma 5.22. One can compute that the operator norm of $(A^{\wedge 2*})^3$ is equal to 1.

Theorem 5.24. *Let $j \in [l+1]$, set $q := m_{(j)}$. Then the following properties hold:*

(i) The limit $\beta_q(\xi) := \lim_{N\to\infty} \beta_q^{(N)}(\xi)$ exists and is an q-blade or zero.

(ii) $\|\beta_q^{(N)}(\xi) - \beta_q(\xi)\| = O(\lambda^{-N}\|\xi\|)$ as $N \to \infty$, $\xi \to 0$.

(iii) $\|\beta_q(\xi) - \alpha_q\| = O(\|\xi\|)$ as $N \to \infty, \xi \to 0$.

(iv) $\|\beta_q(\xi)\| = 1 + O(\|\xi\|)$ as $N \to \infty$, $\xi \to 0$.

(v) $\langle A^{(N)\wedge q}(\xi)\beta_q(\xi), \alpha_q\rangle = |\mu_{(q)}|^N(1 + O(\|\xi\|))$ as $N \to \infty$, $\xi \to 0$.

(vi) $\|A^{(N)\wedge q}(\xi)\beta_q(\xi)\| \asymp |\mu_{(q)}|^N$ as $N \to \infty$, $\xi \to 0$.

as $\xi \to 0, N \to \infty$.

Proof. By Lemma 5.22, the matrix function $A^{\wedge q}$ satisfies all the assumptions of Lemma 5.17, with respect to the vector α_q: from this, almost every statement directly by taking $v = \alpha_q$. The only thing we have not proved yet is that $\beta_q(\xi)$ is a q-blade or zero, but this follows from the fact that $\beta_q^{(N)}(\xi)$ is a blade for all N and Lemma 5.11. □

Definition 5.25. For $j \in [l+1]$, define the subspaces

$$Y_j(\xi) := \left(\operatorname{span} \beta_{m_{(j)}}(\xi)\right)^{\perp},$$

and set $Y_0(\xi) = \mathbb{C}^\ell$. Define

$$E_j(\xi) := Y_{j-1}(\xi) \cap Y_j(\xi)^{\perp}$$

for $j \in [l+1]$, and let $P_j(\xi)$ be the orthogonal projection onto $E_j(\xi)$.

5.3.3 Proving the inequalities

Our goal will be to show that the projections $P_j(\xi)$ we just defined satisfy the inequalities from Theorem 5.3. In particular, this will mean that the subspaces $Y_j(\xi)$ are the *Lyapunov subspaces* of linear cocycle of the sequence $A(\lambda^{-N+1}\xi)$ (see [13] for a definition).

First, we make the following crucial observation:

Lemma 5.26. *Let $j \in [l+1]$. Then*

$$A(\xi)Y_j(\xi) \subseteq Y_j(\lambda^{-1}\xi)$$

Proof. Set $q := m_{(j)}$ and let $x \in Y_j(\xi)$. By definition, this means that for all $u \in \operatorname{span} \beta_q(\xi)$, we have $\langle x, u \rangle = 0$. Furthermore, by definition of the β_q, we know that $A(\xi)^* \beta_q(\lambda^{-1}\xi)$ and $\beta_q(\xi)$ are linearly dependent, so $\operatorname{span} A(\xi)^*\beta_q(\lambda^{-1}\xi) = \operatorname{span} \beta_q(\xi)$, Now let $u' \in \operatorname{span} \beta_q(\lambda^{-1}\xi)$: then $A(\xi)^* u' \in \operatorname{span} A(\xi)^*\beta_q(\lambda^{-1}\xi) = \operatorname{span} \beta_q(\xi)$, which implies $\langle A(\xi)x, u' \rangle = \langle x, A(\xi)^* u' \rangle = 0$: and thus $A(\xi)x \in \operatorname{span} \beta_q(\xi)^{\perp}$, which is what we wanted to show. □

Using this, we are ready to prove the bounds in Theorem 5.3.

Lemma 5.27. *Let $j, k \in [l+1]$, such that $j < k$. Then*

$$\|P_j A^{(N)}(\xi) P_k(\xi) x\| \lesssim \lambda^{-N} \|\xi\| \|P_k(\xi)x\|$$

as $N \to \infty, \xi \to 0$ uniformly for $x \in \mathbb{C}^\ell$.

Proof. Let $y = P_k(\xi)x$. Without loss of generality we can assume that $y \neq 0$, as otherwise we have 0 on both sides of the inequality. Also let $q := m_{(j)}$.

By definition of the Lyapunov subspaces, we have $A^{(N)}(\xi)Y_k(\xi) \subset Y_k(\lambda^{-N}\xi)$, therefore

$$A^{(N)}y = (I - P_{(j)}(\xi))A^{(N)}(\xi)y$$

Define $P_{(j)} := P_1 + \cdots + P_j$ and $P_{(j)}(\xi) := P_1(\xi) + \cdots + P_j(\xi)$ for $j \in [l+1]$. By Theorem 5.24, we have $\|P_m - P_m(\xi)\| = O(\|\xi\|)$ for all $m \in [l]$, which implies $\|P_{(m)} - P_{(m)}(\xi)\| = O(\|\xi\|)$, therefore $\|P_k(I - P_{(j)}(\xi))\| = O(\|\xi\|)$ as $\xi \to 0$. Therefore

$$\begin{aligned} \frac{\|P_j A^{(N)}(\xi)y\|}{\|A^{(N)}(\xi)y\|} &= \frac{\|P_j(I - P_{(j)}(\lambda^{-N}\xi))A^{(N)}(\xi)y\|}{\|A^{(N)}(\xi)y\|} \\ &\leq \|P_j(I - P_{(j)}(\lambda^{-N}\xi))\| \\ &= O(\lambda^{-N}\|\xi\|) \end{aligned}$$

as $N \to \infty, \xi, x \to 0$. □

Lemma 5.28. *There exists a radius $R > 0$ such that the following properties hold.*

(i) If $x \in Y_j(\xi)$, $\|A^{(N)}(\xi)x\| \lessapprox e^{-\chi_j N}\|x\|$ as $N \to \infty$, uniformly for x and $\xi \in B_R$.

(ii) If $x \in Y_j(\xi)^\perp$, $\|A^{(N)}(\xi)x\| \gtrapprox e^{-\chi_{j-1} N}\|x\|$ as $N \to \infty$, uniformly for x and $\xi \in B_R$.

(iii) If $j \in [l]$ and E_{j+1} has a basis of eigenvectors of $A(0)$, then (i) holds with $\lesssim$ instead of $\lessapprox$.

(iv) If E_j has a basis of eigenvectors of $A(0)$, then (ii) holds with $\gtrsim$ instead of $\gtrapprox$.

Proof. In the subsequent we write $q := m_{(j)}$. Note that, using the definition of the cocycle and Lemma 5.16, we have $\|A^{(N)\wedge q}(\xi)\| \lessapprox e^{(\chi_1 + \cdots + \chi_j)N}$ as $N \to \infty$ uniformly for $\xi \in B_R$: we will use this fact multiple times in the proof.

(i) Let $x \in Y_j(\xi)$. Then, by Lemma 5.27, we have

$$\|A^{(N)}(\xi)x\| \lesssim \|(1 - P_1 - \cdots - P_j)A^{(N)}(\xi)x\|$$

as $N \to \infty$ uniformly for all small enough ξ. Furthermore we have

$$|\langle A^{(N)\wedge q}\beta_q(\xi), \alpha_q\rangle| \asymp |\mu_{(q)}|^N \asymp \|A^{(N)\wedge q}\beta_q(\xi)\|$$

as $N \to \infty, \xi \to 0$. Therefore, if we let $\pi_q^{(N)}(\xi)$ be the orthogonal projection onto span $A^{(N)\wedge q}\beta_q(\xi)$ and use Lemma 5.12, we can conclude that there exist some radius $R > 0$, some number $N_0 \in \mathbb{N}$ and some constant $0 < C < 1$ such that

$\|P_1+\cdots+P_j-\pi_q^{(N)}(\xi)\| \le C$ for all $\xi \in B_R$ and N large enough. Therefore, there exists some radius $R > 0$ such that

$$\|A^{(N)}(\xi)x\| \lesssim \|(1-P_1-\cdots-P_j)A^{(N)}(\xi)x\| \lesssim \|(1-\pi_q^{(N)}(\xi))A^{(N)}(\xi)x\|$$

as $N \to \infty$ uniformly for x and $\xi \in B_R$. The last term is, by definition, orthogonal to $A^{(N)\wedge q}\beta_q(\xi)$, therefore its norm can be estimated using Lemma 5.16 to obtain

$$\|(1-\pi_q^{(N)}(\xi))A^{(N)}(\xi)x\| = \frac{\|A^{(N)\wedge(q+1)}(\beta_q(\xi)\wedge x)\|}{\|A^{(N)\wedge q}\beta_q(\xi)\|} \lessapprox e^{-\chi_j N}\|x\|$$

as $N \to \infty$ uniformly for x and $\xi \in B_R$.

(ii) Let $x \in Y_j(\xi)^\perp$: that is, $x \in \operatorname{span}\beta_q(\xi)$. This means there exists a unique $(q-1)$-blade $\gamma(\xi,x)$ orthogonal to x such that $\beta_q(\xi) = \gamma(\xi,x)\wedge x$. Therefore, there exists a radius $R > 0$ such that

$$\|A^{(N)}(\xi)x\| \ge \frac{\|A^{(N)\wedge q}(\xi)\beta_q(\xi)\|}{\|A^{(N)}(\xi)\gamma(\xi,x)\|} \gtrapprox e^{-\chi_{j-1}N}\|x\|$$

as $N \to \infty$ uniformly for x and $\xi \in B_R$. Again, we are using Lemma 5.16 in the last step.

(iii) Assume E_{j+1} has a basis of eigenvectors of $A(0)$. Then, applying Lemma 5.13 to the basis $v_1, \cdots, v_\ell$, we deduce that the top eigenspaces of $A(0)^{\wedge(q+1)}$ also have a basis of eigenvectors, therefore $\|A(0)^{\wedge(q+1)}\| = \rho(A(0)^{\wedge(q+1)})$. Then, by Lemma 5.16, $\|A^{(N)\wedge(q+1)}(\xi)\| \lesssim \rho(A(0)^{\wedge(q+1)})^N$ as $N \to \infty$ uniformly for $\xi \in B_R$. Then the same argument as in *(i)* using $\lesssim$ instead of $\lessapprox$.

(iv) Same as in *(iii)*. □

Lemma 5.29. *Let $j, k \in [l+1]$ with $j < k$.*
Then

$$\|P_k A^{(N)}(\xi)P_j(\xi)x\| \lessapprox \max(e^{\chi_k-\chi_j}, \lambda^{-1})^N \|A^{(N)}(\xi)P_j(\xi)x\|$$

as $N \to \infty$ uniformly for $x \in \mathbb{C}^\ell$ and $\xi \in B_R$.

Sketch. The following argument is inspired by Dubiner [22]. The idea is as follows: if $x \in Y_j(0)$, multiplying it by $A(0)$ will amplify its component in $E_j(0)$ relative to the norm of x, so $\frac{A(0)^N x}{\|A(0)^N x\|}$ will "converge to E_j".

Now, if $x \in Y_j(\xi)$, multiplying it by $A(\xi)$ will also tend to amplify the component in $E_j(\xi)$, but there is also an error of order $O(\xi\|x\|)$ relative to the $\xi = 0$ case: in particular, $A(\xi)x$ may fail to be in $E_j(\lambda^{-n}x)$ even if $x \in E_j(\xi)$. We will need to control this error, which accounts for the worse bound in the lemma, when compared to Lemma 5.27. □

Proof. Let $y = P_j(\xi)x$: without loss of generality we can assume $y \neq 0$. Also let

$$\begin{aligned} C_N &:= \frac{\|P_k A^{(N)}(\xi)y\|}{\|A^{(N)}(\xi)y\|} \\ q_N &:= \frac{\|A^{(N-1)}(\xi)y\|}{\|A^{(N)}(\xi)y\|} e^{\chi_j} \\ \mu &:= e^{\chi_k - \chi_j} \end{aligned}$$

Also let $L_1 > 0$ be such that $e^{-\chi_k}\|A(\lambda^{-N}\xi) - A(0)\| \leq L_1 \lambda^{-N}$ for all $\xi \in B_R$.

We claim that the following inequality holds:

$$C_N \leq \left(\prod_{k=1}^{N} q_k\right) \mu^N C_0 + L_1 \sum_{k=1}^{N} \left(\prod_{n=N-k}^{N} q_n\right) \lambda^{-N+k} \mu^{k-1} \tag{5.8}$$

Assuming that this inequality holds, we can prove the lemma. To see this, note that we have $\prod_{n=1}^{N} q_n \approx 1$ as $N \to \infty$ uniformly for $\xi \in B_R, x \in \mathbb{C}^\ell$: therefore, there exists an increasing function $\epsilon : \mathbb{N} \to \mathbb{R}_{\geq 0}$ such that $\epsilon(N)^{-1} \leq \prod_{n=1}^{N} q_N \leq \epsilon(N)$ for all $N \in \mathbb{N}$, $x \in \mathbb{C}^\ell$ and $\xi \in B_R$, and $\limsup_{N\to\infty} \frac{1}{N} \log \epsilon(N) \leq 0$. Then we have

$$\begin{aligned} C_N &\leq \left(\prod_{k=1}^{N} q_k\right) \mu^N C_0 + L_1 \sum_{k=1}^{N} \left(\prod_{n=N-k}^{N} q_n\right) \lambda^{-N+k} \mu^{k-1} \\ &\leq \mu^N \epsilon(N) C_0 + \sum_{k=1}^{N} \epsilon(N)\epsilon(N-k)\lambda^{-N+k}\mu^{k-1} \\ &\leq \max(\mu, \lambda^{-1})^N (\epsilon(N)C_0 + N\epsilon(N)^2 \mu^{-1} L_1) \\ &\lessapprox \max(\mu, \lambda^{-1})^N \end{aligned}$$

as $N \to \infty$ uniformly for $x \in \mathbb{C}^\ell, \xi \in B_R$, which is the statement of the lemma. Therefore, as long as we can prove (5.8), we are done.

In order to prove (5.8), note that C_N satisfies the following recursive inequality for $N \in \mathbb{N}$.

$$\begin{aligned} C_N = \frac{\|P_k A^{(N)}(\xi)y\|}{\|A^{(N)}y\|} &\leq q_N e^{-\chi_j} \frac{\|P_k A^{(N)}(\xi)y\|}{\|A^{(N-1)}y\|} \\ &\leq q_N e^{-\chi_j} \frac{\|P_k A(0) A^{(N-1)}(\xi)y\|}{\|A^{(N-1)}y\|} \\ &\quad + q_N e^{-\chi_j} \frac{\|P_k(A(\lambda^{-N+1}\xi) - A(0))A^{(N-1)}(\xi)y\|}{\|A^{(N-1)}y\|} \\ &\leq q_N \mu C_{N-1} + L_1 q_N \lambda^{-N+1} \end{aligned}$$

Using this, we can prove (5.8) by induction. For $N = 0$, the inequality reads $C_0 \leq C_0$, so nothing is to be done. For higher N we have:

$$\begin{aligned}
C_N &\leq q_N \mu C_{N-1} + L_1 q_N \lambda^{-N+1} \\
&\leq q_N \mu \left(\left(\prod_{n=1}^{N-1} q_n \right) \mu^{N-1} C_0 + \sum_{k=1}^{N-1} L_1 \left(\prod_{n=N-k}^{N-1} q_n \right) \lambda^{-N+k+1} \mu^{k-2} \right) + L_1 q_N \lambda^{-N+1} \\
&= \left(\left(\prod_{k=1}^{N} q_k \right) \mu^N C_0 + L_1 \sum_{k=1}^{N-1} \left(\prod_{n=N-k}^{N} q_n \right) \lambda^{-N+k+1} \mu^{k-2} \right) + L_1 q_N \lambda^{-N+1} \\
&= \left(\prod_{k=1}^{N} q_k \right) \mu^N C_0 + L_1 \sum_{k=0}^{N-1} \left(\prod_{n=N-k}^{N} q_n \right) \lambda^{-N+k+1} \mu^{k-2} \\
&= \left(\prod_{k=1}^{N} q_k \right) \mu^N C_0 + L_1 \sum_{k=1}^{N} \left(\prod_{n=N-k}^{N} q_n \right) \lambda^{-N+k} \mu^{k-1}
\end{aligned}$$

Which finishes the proof by induction of Inequality (5.8), and hence the lemma. □

Now we are ready to prove Theorem 5.3.

Proof of Theorem 5.3. Let A be a matrix function satisfying Assumption 5.1. Note that it suffices to prove that there exists some norm on $\mathbb{C}^\ell$ such that the inequalities in Theorem 5.3 hold: as all norms on $\mathbb{C}^\ell$ are equivalent, this means the inequalities will hold for any norm.

Pick a descending Jordan basis $v_1, \cdots, v_\ell$ of $A(0)$ and pick an inner product of $\mathbb{C}^\ell$ which makes this basis orthonormal. Then we can define the projections $P_j(\xi)$ as in Definition 5.25, using this inner product, and all the lemmas we have proven so far hold.

Then, using the norm induced by this inner product, we check the inequalities.

- Inequalities (5.3), (5.7) and (5.6) follow from Lemma 5.28.
- Inequality (5.5) follows from Lemma 5.29.
- Inequality (5.4) follows from Lemma 5.27.

This concludes the proof. □

6 Main Argument

In this section, we use the tools we have developed to study the diffraction of substitution rules around the origin and provide sufficient conditions for their hyperuniformity in terms of their substitution matrix.

6.1 Statement

Let $\mathcal{S}$ be a primitive, stone, FLC substitution rule on $\mathbb{R}^d$ with ℓ colors. Let $\lambda > 1$ be its scaling constant and M_{full} be its full substitution matrix as defined in Chapter 4 (now we call it *full* substitution matrix to distinguish it from the spherical substitution matrix M_{sph} we will define in Chapter 7). Recall that M_{full} is a primitive matrix with Perron–Frobenius eigenvalue λ^d. For each $\mu \in \operatorname{Spec} M_{\text{full}}$, denote the corresponding generalized eigenspace by E_μ.

For every $w \in \mathbb{C}^\ell$, let $\hat{\eta}_w$ be the diffraction measure of the substitution rule $\mathcal{S}$ with weights w, and define the following constants:

$$\begin{aligned}
\beta_\parallel(w) &:= d - \max\left\{\log_\lambda|\mu| \;\middle|\; \mu \in \operatorname{Spec} M_{\text{full}} \setminus \{\lambda^d\},\ w \notin E_\mu^\perp\right\},\\
\beta_\perp(w) &:= d + 1 - \max\left\{\log_\lambda|\mu| \;\middle|\; \mu \in \operatorname{Spec} M_{\text{full}} \setminus \{\lambda^d\},\ w \in E_\mu^\perp\right\},\\
\beta(w) &:= \min\left(\beta_\perp(w), \beta_\parallel(w)\right).
\end{aligned}$$

Note these expressions make sense even if $0 \in \operatorname{Spec} M_{\text{full}}$, in which case $\log 0 = -\infty$: we never need to subtract two infinities. We also define $\max\emptyset = -\infty$.

Recall that we defined the asymptotic notation $\lessapprox$, such that $f(r) \lessapprox r^\alpha$ as $r \to 0$ if and only if for every $\varepsilon > 0$ we have $f(r) \lesssim r^{\alpha-\epsilon}$ (see Section 2.4 for details).

The most general theorem on the decay of $\hat{\eta}_w$ around the origin we will prove is the following:

Theorem 6.1. *Let $w \in \mathbb{C}^\ell$, and let $\hat{\eta}_w$ be the diffraction measure of $\mathcal{S}$ with weights w. Then the diffraction satisfies*

$$\hat{\eta}_w(B_r) \lessapprox r^{2\beta(w)}$$

as $r \to 0$.

Remark 6.2. Recall that, as we defined in Remark 4.21, it is possible to *recenter* the substitution rule $\mathcal{S}$ in a way that displaces the canonical prototiles but still results in the same tilings. As Theorem 6.1 only depends on the substitution matrix M_{full}, and not on the particular displacements, the bounds we obtain stay stable under recentering. In particular, if Theorem 6.1 proves that $\mathcal{S}$ is (Class I) hyperuniform for weights w, any of its recentered substitution rules will also be (Class I) hyperuniform for weights w.

From Theorem 6.1, we get the following criterion for hyperuniformity.

Corollary 6.3. *Assume $d \in \{1, 2\}$. Then, for every $w \in \mathbb{C}^\ell$, the following holds: if $|\mu| < \lambda^{\frac{d}{2}}$ for all $\mu \in \operatorname{Spec} M_{\text{full}} \setminus \{\lambda^d\}$ such that $w \notin E_\mu^\perp$, then $\mathcal{S}$ is hyperuniform for weights w.*

Proof. By definition, $\mathbf{\Phi}$ is hyperuniform for weights w if and only if $\hat{\eta}_w(B_r) = o(r^d)$ as $r \to 0$, where $\hat{\eta}_w$ is the diffraction measure of $\langle \mathbf{\Phi}, w \rangle$. Then we have:

- For all $\mu \in \operatorname{Spec} M_{\text{full}} \setminus \{\lambda^d\}$ with $w \notin E_\mu^\perp$, we have $|\mu| < \lambda^{\frac{d}{2}}$ by assumption. Therefore $\beta_\parallel(w) > d - \frac{d}{2} = \frac{d}{2}$.
- For all $\mu \in \operatorname{Spec} M_{\text{full}} \setminus \{\lambda^d\}$ with $w \in E_\mu^\perp$, we have $|\mu| < \lambda^d$, as λ^d is the PF eigenvalue of the substitution matrix. Therefore $\beta_\perp(w) > d + 1 - d = 1$.

By Theorem 6.1, we have $\hat{\eta}_w(B_r) \lessapprox r^{2\beta(w)}$ as $r \to 0$. But for $d = 1, 2$, we have $\beta(w) = \min(\beta_\perp(w), \beta_\parallel(w)) > \min(\frac{d}{2}, 1) \geq \frac{d}{2}$, so $\hat{\eta}_w(B_r) = o(r^d)$ as $r \to 0$. This means $\mathcal{S}$ is hyperuniform for weights w. ☐

6.2 Proof of Theorem 6.1

Let $\boldsymbol{A}$ be the normalized Fourier matrix of the substitution rule $\mathcal{S}$. Let $\chi_1 > \cdots > \chi_l > \chi_{l+1}$ be the Lyapunov exponents of $\boldsymbol{A}(0)$ and $E_1, \cdots, E_{l+1}$ their corresponding spaces of generalized eigenvectors with projections $P_1, \cdots, P_{l+1}$, as defined in Section 5.1. Also let $R > 0$ be small enough so that the conclusions of Theorem 5.3 hold for the normalized Fourier matrix $\boldsymbol{A}$: let $P_1(\xi), \cdots, P_{l+1}(\xi)$ be the projections from the theorem. By Theorem 4.53, the diffraction matrix $\hat{\boldsymbol{H}}$ of $\mathcal{S}$ admits a self-similar density $(\boldsymbol{h}, \nu)$ on $B_R^\times$.

Fix $w \in \mathbb{C}^\ell$, and define the two following subsets of $[l+1]$:

$$\mathcal{J}_\parallel(w) = \{j \in [l+1] \mid w \notin E_j^\perp\}$$
$$\mathcal{J}_\perp(w) = \{j \in [l+1] \mid w \in E_j^\perp\}$$

Then the constants from Theorem 6.1 can be restated as follows:

Lemma 6.4. *The constants $\beta_\|$ and $\beta_\perp$ from Theorem 6.1 are given by*

- $\beta_\|(w) = \min\{-\frac{\chi_j}{\log\lambda} \mid j \in \mathcal{J}_\|(w) \setminus \{1\}\}$
- $\beta_\perp(w) = 1 + \min\{-\frac{\chi_j}{\log\lambda} \mid j \in \mathcal{J}_\perp(w) \setminus \{1\}\}$

Proof. Recall that $\beta_\|(w)$ is defined as

$$d - \max\left\{\log_\lambda|\mu| \;\middle|\; \mu \in \operatorname{Spec} M_{\text{full}} \setminus \{\lambda^d\},\ w \notin E_\mu^\perp\right\}$$

Using the fact that $\boldsymbol{A}(0) = \lambda^{-d} M_{\text{full}}$, we see that $\mu \in \operatorname{Spec} M_{\text{full}}$ satisfies $\log\frac{|\mu|}{\lambda^d} = \chi_j$ if and only if generalized eigenspace E_μ is contained in E_j. Therefore $w \in E_j^\perp$ holds if and only if $w \in E_\mu^\perp$ for every $\mu \in \operatorname{Spec} M_{\text{full}}$ such that $\log\frac{|\mu|}{\lambda^d} = \chi_j$. If this is the case, we have:

$$-\frac{\chi_j}{\log\lambda} = -\frac{\log|\mu| - d\log\lambda}{\log\lambda} = d - \log_\lambda|\mu|.$$

Therefore the definitions of $\beta_\|(w)$ and $\beta_\perp(w)$ coincide with the equations in the statement of the lemma. This concludes the proof. □

The proof of Theorem 6.1 hinges on the following lemma.

Lemma 6.5. *For all $j \in [l+1] \setminus \{1\}$ and $k \in \mathcal{J}_\|(w)$, we have*

$$\|P_k \boldsymbol{A}^{(N)}(\lambda^{-1}\xi) P_j(\xi) x\| \lessapprox \lambda^{-\beta(w)N} \|P_j(\xi)x\|$$

as $N \to \infty$ uniformly for $\xi \in B_R$, $x \in \mathbb{C}^\ell$.

Proof. Depending on j and k, we will find α, γ such that $\alpha + \gamma \geq \beta(w)$ and the following inequalities hold:

$$\begin{aligned} \|\boldsymbol{A}^{(N)}(\lambda^{-1}\xi) P_j(\xi) x\| &\lessapprox \lambda^{-\alpha N} \|P_j(\xi) x\| \\ \|P_k \boldsymbol{A}^{(N)}(\lambda^{-1}\xi) P_j(\xi) x\| &\lessapprox \lambda^{-\gamma N} \|\boldsymbol{A}^{(N)}(\lambda^{-1}\xi) P_j(\xi) x\| \end{aligned} \tag{6.1}$$

as $N \to \infty$ uniformly for $\xi \in B_R$, $x \in \mathbb{C}^\ell$. If we achieve this, we have

$$\|P_k \boldsymbol{A}^{(N)}(\xi) P_j(\xi) x\| = \frac{\|P_k \boldsymbol{A}^{(N)}(\lambda^{-1}\xi) P_j(\xi) x\|}{\|\boldsymbol{A}^{(N)}(\lambda^{-1}\xi) P_j(\xi) x\|} \|\boldsymbol{A}^{(N)}(\lambda^{-1}\xi) P_j(\xi) x\| \lessapprox \lambda^{-(\alpha+\gamma)N}$$

as $N \to \infty$ uniformly for $\xi \in B_R$, $x \in \mathbb{C}^\ell$, which concludes the proof of the lemma. (If the denominator of the fraction is 0 the left hand side is 0 as well, so the inequality still holds.)

In order to find α, γ, we need to consider different cases. We will use Theorem 5.3 for all of these, using the characterization of $\beta_\|$ and $\beta_\perp$ from Lemma 6.4.

- **Case** $j \notin \mathcal{J}_\perp(w)$**:** In this case, Equation (5.3) from Theorem 5.3 reads

$$\|\boldsymbol{A}^{(N)}(\lambda^{-1}\xi)P_j(\xi)x\| \lessapprox e^{\chi_j N}\|P_j(\xi)x\|$$

  Furthermore, we have

$$\|P_k\boldsymbol{A}^{(N)}(\lambda^{-1}\xi)P_j(\xi)x\| \lessapprox \|\boldsymbol{A}^{(N)}(\lambda^{-1}\xi)P_j(\xi)x\|$$

  Then, using the fact that $e^{\chi_j N} = \lambda^{\frac{\chi_j}{\log\lambda}N}$, we see that Inequalities (6.1) hold with $\alpha = -\frac{\chi_j}{\log\lambda}$ and $\gamma = 0$. As $\alpha \geq \beta_\parallel(w) \geq \beta(w)$, we have $\alpha + \gamma \geq \beta(w)$.

- **Case** $j \in \mathcal{J}_\perp(w), j > k$**:** In this case, Equations (5.3) and (5.4) from Theorem 5.3 read:

$$\|\boldsymbol{A}^{(N)}(\lambda^{-1}\xi)P_j(\xi)x\| \lessapprox e^{\chi_j N}\|P_j(\xi)x\|$$
$$\|P_k\boldsymbol{A}^{(N)}(\lambda^{-1}\xi)P_j(\xi)x\| \lesssim \lambda^{-N}\|\boldsymbol{A}^{(N)}(\lambda^{-1}\xi)P_j(\xi)x\|$$

  Therefore, Inequalities (6.1) hold if we set $\alpha = -\frac{\chi_j}{\log\lambda}$ and $\gamma = 1$. In this case $\alpha + \gamma = 1 - \frac{\chi_j}{\log\lambda} \geq \beta_\perp \geq \beta$.

- **Case** $j \in \mathcal{J}_\perp(w), j < k$**:** In this case, Equations (5.3) and (5.5) from Theorem 5.3 read:

$$\|\boldsymbol{A}^{(N)}(\lambda^{-1}\xi)P_j(\xi)x\| \lessapprox e^{\chi_j N}\|P_j(\xi)x\|$$
$$\|P_k\boldsymbol{A}^{(N)}(\lambda^{-1}\xi)P_j(\xi)x\| \lessapprox \max(e^{\chi_k-\chi_j}, \lambda^{-1})^N\|\boldsymbol{A}^{(N)}(\lambda^{-1}\xi)P_j(\xi)x\|$$

  Therefore, Inequalities (6.1) hold if we set $\alpha = -\frac{\chi_j}{\log\lambda}$ and $\gamma = \min(\frac{\chi_j-\chi_k}{\log\lambda}, 1)$. In this case we have either $\alpha+\gamma = -\frac{\chi_j}{\log\lambda} + 1 \geq \beta_\perp$ or $\alpha+\gamma = -\chi_j + \frac{\chi_j-\chi_k}{\log\lambda} = -\frac{\chi_k}{\log\lambda} \geq \beta_\parallel$. In both cases we have $\alpha + \gamma \geq \beta$.

Then, for every j, k as in the statement of the lemma, we have found α, γ such that Inequalities (6.1) hold and $\alpha + \gamma \geq \beta(w)$. This concludes the proof. □

In Lemma 6.5, we exclude the case $j = 1$ This is justified by the following lemma.

Lemma 6.6. *For ν-almost every $\xi \in B_R$, we have $P_1(\xi)\boldsymbol{h}(\xi)P_1(\xi) = 0$.*

Proof. The first Lyapunov exponent of $\boldsymbol{A}(0)$, $\chi_1 = 0$, has multiplicity $m_1 = 1$. Therefore, by Inequality (5.6) of Theorem 5.3, we have

$$\|\boldsymbol{A}^{(N)}(\xi)P_1(\xi)x\| \gtrsim \|P_1(\xi)x\|$$

as $N \to \infty$ uniformly for all $\xi \in B_R, x \in \mathbb{C}^\ell$. This means there exist some $c > 0$ and

$N_0 \in \mathbb{N}$ such that

$$\|\boldsymbol{A}^{(N)}(\xi)P_1(\xi)x\| \geq c\|P_1(\xi)x\|$$

for all $\xi \in B_R$, $N \geq N_0$.

As $\boldsymbol{h}(\xi)$ is positive semidefinite for all ξ, we can find $f_1(\xi), \cdots, f_\ell(\xi)$ such that $\boldsymbol{h}(\xi) = \sum_{t=1}^{\ell} f_t(\xi)f_t(\xi)^*$. Therefore

$$\begin{aligned}
\|\boldsymbol{h}(\lambda^{-N}\xi)\| &= \|\boldsymbol{A}^{(N)}(\lambda^{-1}\xi)\boldsymbol{h}(\xi)\boldsymbol{A}^{(N)*}(\lambda^{-1}\xi)\| \\
&\geq \|\boldsymbol{A}^{(N)}(\lambda^{-1}\xi)P_1(\xi)\boldsymbol{h}(\xi)P_1(\xi)\boldsymbol{A}^{(N)*}(\lambda^{-1}\xi)\| \\
&= \sum_{t=1}^{\ell}\|\boldsymbol{A}^{(N)}(\lambda^{-1}\xi)P_1(\xi)f_t(\xi)\|^2 \\
&\geq c\sum_{t=1}^{\ell}\|P_1(\xi)f_t(\xi)\|^2 \\
&\geq c\|P_1(\xi)\boldsymbol{h}(\xi)P_1(\xi)\|
\end{aligned}$$

for all $\xi \in B_R$, $N \geq N_0$.

On the other hand, by Lemma 4.58, we have $\boldsymbol{h}(\lambda^{-N}\xi) \to 0$ as $N \to \infty$, so the inequality we just proved implies $\|P_1(\xi)\boldsymbol{h}(\xi)P_1(\xi)\| = 0$, as we wanted to show. □

Proof of Theorem 6.1. By Lemma 4.57, it suffices to show

$$\langle \boldsymbol{h}(\lambda^{-N}\xi)\, w, w\rangle \;\lessapprox\; \lambda^{-2\beta(w)N}\,\|\boldsymbol{h}(\xi)\| \qquad \text{as } N \to \infty, \tag{6.2}$$

uniformly for $\xi \in B_R$.

By Lemma 4.50, $\boldsymbol{h}(\xi)$ is positive semidefinite for ν-almost every ξ. Hence for ν-almost every ξ there exist orthogonal vectors $f_1(\xi), \ldots, f_\ell(\xi)$ (possibly some zero) with

$$\boldsymbol{h}(\xi) = \sum_{t=1}^{\ell} f_t(\xi)f_t(\xi)^*.$$

Therefore

$$\langle \boldsymbol{h}(\lambda^{-N}\xi)\, w, w\rangle = \sum_{t=1}^{\ell}|\langle \boldsymbol{A}^{(N)}(\xi)f_t(\xi),\, w\rangle|^2,$$

and it is enough to prove

$$|\langle \boldsymbol{A}^{(N)}(\xi)f_t(\xi),\, w\rangle| \;\lessapprox\; \lambda^{-\beta(w)N}\,\|f_t(\xi)\|,$$

as $N \to \infty$, uniformly for $\xi \in B_R$ and for all $t \in [\ell]$.

We can decompose this further using the projections $P_j(\xi)$ and P_j from Theorem 5.3.

We have $\sum_{j=1}^{l+1} P_j = \sum_{j=1}^{l+1} P_j(\xi) = I$, so we can bound

$$\begin{aligned} \left|\langle \boldsymbol{A}^{(N)}(\xi) f_t(\xi), w\rangle\right| &= \left|\left\langle \Big(\sum_{k=1}^{l+1} P_k\Big) \boldsymbol{A}^{(N)}(\lambda^{-1}\xi) \Big(\sum_{j=1}^{l+1} P_j(\xi) f_t(\xi)\Big), w \right\rangle\right| \\ &= \left|\sum_{j,k=1}^{l+1} \langle \boldsymbol{A}^{(N)}(\lambda^{-1}\xi) P_j(\xi) f_t(\xi), P_k^* w\rangle\right| \\ &\leq \sum_{j,k=1}^{l+1} \left|\langle \boldsymbol{A}^{(N)}(\lambda^{-1}\xi) P_j(\xi) f_t(\xi), P_k^* w\rangle\right|. \end{aligned}$$

Therefore, in order to prove the theorem, it suffices to prove

$$|\langle \boldsymbol{A}^{(N)}(\lambda^{-1}\xi) P_j(\xi) f_t(\xi), P_k^* w\rangle| \lessapprox \lambda^{-\beta(w)N} \|P_j(\xi) f_t(\xi)\| \|P_k^* w\| \tag{$\star$}$$

as $N \to \infty$ uniformly for all $\xi \in B_R$, $t \in [\ell]$ and $j, k \in [l+1]$. If we are able to prove this inequality, we are done.

It suffices to prove ($\star$) for all ξ, t, j, k such that $P_j(\xi) f_t(\xi) \neq 0$ and $P_k^* w \neq 0$, as otherwise both sides of the asymptotic inequality are 0. But:

- By Lemma 6.6, we have $P_1(\xi) f_t(\xi) = 0$.
- If $k \notin \mathcal{J}_{\|}(w)$, we have $k \in \mathcal{J}_{\perp}(w)$. By definition, this means $w \perp E_k$: as P_k is a projection with image E_k, this means $P_k^* w = 0$

This means that we only need prove the inequality ($\star$) for the $j, k \in [l+1]$ such that $j \neq 1$ and $k \in \mathcal{J}_{\|}(w)$. These are precisely the indices we considered in Lemma 6.5: therefore, Lemma 6.5 proves the ($\star$) uniformly for all $\xi \in B_R$, $t \in [\ell]$ and $j, k \in [l+1]$. This completes the proof. □

7 Examples

In this last chapter, we will apply the results of Chapter 6 to a wide variety of substitution rules.

7.1 One-dimensional substitution rules with two colors

First, we turn our attention to the simplest nontrivial substitution rules possible: one-dimensional substitution rules with two colors. In this case, the substitution matrix only has one non-PF eigenvalue, so Theorem 6.1 specializes to the following:

Corollary 7.1. *Let $\mathcal{S}$ be a primitive, stone, FLC substitution rule on $\mathbb{R}$ with $\ell = 2$. Let λ be its scaling constant, μ_2 be the smallest eigenvalue of the full substitution matrix M_{full}, and v_2 be the corresponding right eigenvector. For all $w \in \mathbb{C}^2$, let $\hat{\eta}_w$ be the diffraction of $\mathcal{S}$ with weights w. Then we have*

$$\hat{\eta}_w(B_r) \lessapprox r^{2\beta(w)}$$

as $r \to 0$, where

$$\beta(w) := \begin{cases} 2 - \log_\lambda|\mu_2| & \text{if } w \perp v_2 \\ 1 - \log_\lambda|\mu_2| & \text{otherwise} \end{cases}$$

Furthermore, the inequality holds with $\lesssim$ unless $\mu_2 = 0$, in which case we have $\hat{\eta}_w(B_r) \lessapprox 0$ as $r \to 0$.

In the generic case where w is not orthogonal to the second eigenvector, this is the bound originally conjectured by Oğuz, Socolar, Steinhardt, and Torquato [47] and has been checked for several examples by the same authors and also by Baake and Grimm [6], as we will see in the examples below.

Proof of Corollary 7.1. The only part of Corollary 7.1 that does not follow directly from Theorem 6.1 is the claim that we can replace $\lessapprox$ with $\lesssim$. In order to see this, we repeat the outline of the proof of Theorem 6.1, which simplifies significantly. In particular, M_{full} is diagonalizable, as it has two distinct eigenvalues.

Let $R > 0$ be a radius such that the conclusions of Theorem 5.3 hold for the normalized Fourier matrix $\boldsymbol{A}(\xi)$ of $\boldsymbol{\mathcal{S}}$. The eigenvalues of $\boldsymbol{A}(0) = \lambda^{-1} M_{\mathrm{full}}$ are 1 and $\lambda^{-1}\mu_2$, hence the Lyapunov exponents are $\chi_1 = 0$ and $\chi_2 = \log_\lambda|\mu_2| - 1$.

Let $\boldsymbol{h}$ be a self-similar density of $\hat{\boldsymbol{H}}|_{B_R^\times}$ with base ν. As $\boldsymbol{h}(\xi)$ is positive semidefinite, it is diagonalizable everywhere: furthermore, by Lemma 6.6, we have $P_1(\xi)\boldsymbol{h}(\xi)P_1(\xi)^* = 0$ for ν-almost every $\xi \in B_R^\times$.

Therefore, $\boldsymbol{h}(\xi)$ has rank at most 1 for ν-almost every $\xi \in B_R^\times$, so there exists a vector-valued function $\boldsymbol{f} : B_R^\times \to \mathbb{C}^2$ with $P_2(\xi)\boldsymbol{f}(\xi) = \boldsymbol{f}(\xi)$ such that $\boldsymbol{h}(\xi) = \boldsymbol{f}(\xi)\boldsymbol{f}(\xi)^*$.

Now we consider two cases:

- Assume $w \not\perp v_2$: then $\beta(w) = 1 - \log_\lambda|\mu_2|$, so $\lambda^{-\beta(w)} = e^{\chi_2}$. Furthermore, we have $P_2\boldsymbol{f}(\xi) = \boldsymbol{f}(\xi)$ for ν-almost every $\xi \in B_R^\times$. Then we can compute:

$$\begin{aligned}
\langle \boldsymbol{h}(\lambda^{-N}\xi)w, w\rangle &= |\langle \boldsymbol{f}(\lambda^{-N}\xi), w\rangle|^2 \\
&= |\langle \boldsymbol{A}^{(N)}(\lambda^{-1}\xi)P_2\boldsymbol{f}(\xi), w\rangle|^2 \\
&\le \|\boldsymbol{A}^{(N)}(\lambda^{-1}\xi)P_2\boldsymbol{f}(\xi)\|^2\|w\|^2 \\
&\lesssim e^{2\chi_2 N} \\
&= \lambda^{-2\beta(w)N}
\end{aligned}$$

  as $N \to \infty$ uniformly for $\xi \in B_R^\times$. In the last inequality we use Theorem 5.3, where we have $\lesssim$ instead of $\lessapprox$ because $\boldsymbol{A}(0)$ is diagonalizable.

- Assume $w \perp v_2$: then $\beta(w) = 2 - \log_\lambda|\mu_2|$, so $\lambda^{-\beta(w)} = \lambda^{-1}e^{\chi_2}$.

  As w is orthogonal to v_2, we have $P_2^* w = 0$, so $P_1^* w = w$.

$$\begin{aligned}
\langle \boldsymbol{h}(\lambda^{-N}\xi)w, w\rangle &= |\langle \boldsymbol{f}(\lambda^{-N}\xi), w\rangle|^2 \\
&= |\langle \boldsymbol{A}^{(N)}(\lambda^{-1}\xi)P_2\boldsymbol{f}(\xi), P_1^* w\rangle|^2 \\
&\le \|P_1\boldsymbol{A}^{(N)}(\lambda^{-1}\xi)P_2\boldsymbol{f}(\xi)\|^2\|w\|^2 \\
&\lesssim \lambda^{-1}e^{2\chi_2 N} \\
&= \lambda^{-2\beta(w)N}
\end{aligned}$$

  as $N \to \infty$ uniformly for $\xi \in B_R^\times$.

By Theorem 4.57, this is enough to prove $\hat{\eta}_w(B_r) \lesssim r^{2\beta(w)}$ as $r \to 0$. □

In what follows, we only consider examples where the canonical prototiles are intervals: every stone substitution rule with this property is face-to-face, hence FLC.

Remark 7.2. So far we have considered only *geometric* substitution rules, which act on multisets of points in $\mathbb{R}^d$. In the one-dimensional case, the are roughly equivalent to *symbolic* substitution rules, i.e. maps $\varrho_0 : \mathcal{A} \to \mathcal{A}^*$ where $\mathcal{A}$ is a finite alphabet and $\mathcal{A}^*$ is the set of finite words over $\mathcal{A}$. Indeed, every primitive symbolic substitution rule ϱ_0 can be turned into a primitive geometric substitution rule unique up to recentering, and every geometric substitution rule with interval prototiles is obtained in this way. See Baake and Grimm [5] for details on this correspondence. For each example we consider below, we explicitly give the geometric substitution rule, and also write the corresponding symbolic substitution for the convenience of experienced readers.

Fibonacci substitution rule

The most famous example of a one-dimensional substitution rule is the *Fibonacci substitution rule*, given symbolically by $1 \mapsto 12, 2 \mapsto 1$.

Example 7.3. The *(geometric) Fibonacci substitution rule* on $\mathbb{R}$ is a substitution rule $\mathcal{S}_{Fib} = (\lambda, \mathbf{\Delta})$ with 2 colors, where $\lambda = \tau = \frac{1+\sqrt{5}}{2}$ and the displacement matrix $\mathbf{\Delta} \in \mathcal{P}_{fin}(\mathbb{R})^{2\times 2}$ is given by

$$\mathbf{\Delta} = \begin{pmatrix} \{0\} & \{0\} \\ \{1\} & \emptyset \end{pmatrix}.$$

Figure 7.1: A tiling obtained from the Fibonacci substitution rule.

This is a primitive, stone, FLC substitution rule, with canonical prototiles $\tau_1 = [0, 1]$ and $\tau_2 = [0, \tau^{-1}]$. Its substitution matrix is given by

$$M_{\text{full}} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

with eigenvalues $\mu_1 = \tau$ and $\mu_2 = -\tau^{-1}$: the eigenvector to μ_2 is $(\tau^{-2}, -\tau^{-1})^\top$, so w is orthogonal to v_2 if and only if w is proportional to the PF eigenvector $v_{PF} = (\tau^{-1}, \tau^{-2})^\top$. Then $\beta(w) = 1 - \log_\tau \tau^{-1} = 2$ if w is not proportional to v_{PF}, and $\beta(w) = 2 - \log_\tau \tau^{-1} = 3$ if it is.

Therefore, using Corollary 7.1, we obtain the following.

Corollary 7.4. *Let $w \in \mathbb{C}^2$ and $\hat{\eta}_w$ be the diffraction of $\mathcal{S}_{Fib}$ with weights w. Then we have*

$$\hat{\eta}_w(B_r) \lesssim r^4$$

as $r \to 0$, so $\mathcal{S}_{Fib}$ is Class I hyperuniform for weights w. Furthermore, if w is proportional to v_{PF}, then

$$\hat{\eta}_w(B_r) \lesssim r^6$$

as $r \to 0$.

The generic bound was already known [6, 7, 47]: it can also be proven using the model set description of the Fibonacci tilings. The better bound for PF weights has not been noticed before, to the best of our knowledge.

Thue–Morse substitution rule

Example 7.5. The *Thue–Morse substitution rule* on $\mathbb{R}$ is a substitution rule $\mathcal{S}_{TM} = (\lambda, \boldsymbol{\Delta})$, where $\lambda = 2$ and the displacement matrix $\boldsymbol{\Delta} \in \mathcal{P}_{fin}(\mathbb{R})^{2\times 2}$ is given by

$$\boldsymbol{\Delta} = \begin{pmatrix} \{0\} & \{1\} \\ \{1\} & \{0\} \end{pmatrix}$$

This is a primitive, stone, FLC substitution rule with canonical prototiles $\tau_1 = \tau_2 = [0, 1]$.

Its substitution matrix is given by

$$M_{\text{full}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Here the second eigenvalue is $\mu_2 = 0$, so we obtain the following.

Corollary 7.6. *For any weights w, $\hat{\eta}_w(B_r) \lessapprox 0$ as $r \to 0$. In particular, $\mathcal{S}_{TM}$ is hyperuniform of Class I for all weights w.*

In fact, in this case, one obtains periodic solutions for $w = (1, 1)^\top$. For $w = (1, -1)^\top$ (the *balanced case*) Baake and Grimm [3] derived sharper asymptotics.

A periodic two-color substitution

Example 7.7 ([5, Ex. 4.2])**.** Consider the substitution rule $\mathcal{S}$ on $\mathbb{R}$ with scaling constant $\lambda = 3$ and displacement matrix

$$\boldsymbol{\Delta} = \begin{pmatrix} \{0, 2\} & \{1\} \\ \{1\} & \{0, 2\} \end{pmatrix}$$

In this case, the canonical prototiles are $\tau_1 = \tau_2 = [0, 1]$, and the second eigenvalue is $\mu_2 = 1$. Therefore, Corollary 7.1 would give the bound $\hat{\eta}_w(B_r) \lesssim r^2$. However, it turns out that the self-similar sets of this rule are all periodic, hence we have $\hat{\eta}_w(B_r) = 0$ for all w and all $r < 1$. This is an example where the bound from Corollary 7.1 is not optimal.

An antihyperuniform example

Now we consider the substitution rule with symbolic form $1 \mapsto 11112, 2 \mapsto 12222$.

Example 7.8. Define the substitution rule $\mathcal{S}$ on $\mathbb{R}$ with $\lambda = 5$ and

$$\mathbf{\Delta} = \begin{pmatrix} \{0,1,2,3\} & \{0\} \\ \{4\} & \{1,2,3,4\} \end{pmatrix}$$

This substitution rule is primitive and stone with canonical prototiles $\tau_1 = \tau_2 = [0, 1]$. The substitution matrix is

$$M_{\text{full}} = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$$

with smallest eigenvalue $\mu_2 = 3$ and corresponding eigenvector $v_2 = (1, -1)^\top$.

Corollary 7.9. *Let $\mathcal{S}$ be as above and $w \in \mathbb{C}^2$.*

(i) If $w = (1,1)^\top$ (that is, if one gives all tiles the same weight), we have $\hat{\eta}_w(B_r) \lesssim r^{2(2-\log_5 3)}$. In particular, because $2(2 - \log_5 3) = \sim 2.6$, $\mathcal{S}$ is Class I hyperuniform for this choice of weights.

(ii) If w is not a multiple of $(1,1)^\top$, then $\hat{\eta}_w(B_r) \lesssim r^{2(1-\log_5 3)}$.

Note that $2(1-\log_5 3) = \sim 0.6 < 1$: therefore, part (ii) does not prove hyperuniformity for $w \in (1,1)^\top$. In fact, with more careful analysis, one can show that, in fact, $\hat{\eta}_w(B_r) \asymp r^{0.6}$, so that $\mathcal{S}$ is *antihyperuniform* for these weights. This is peculiar for the following reason: if $\mathbf{\Lambda} = (\Lambda_1, \Lambda_2) \in \Omega$, after recentering if necessary, the sets Λ_1 and $\Lambda_1 \cup \Lambda_2$ are MLD equivalent, but one of them is hyperuniform and the other one is not. This means that hyperuniformity is not preserved under MLD equivalence, contradicting standing conjectures in the field. See [31] for a more detailed discussion of this phenomenon.

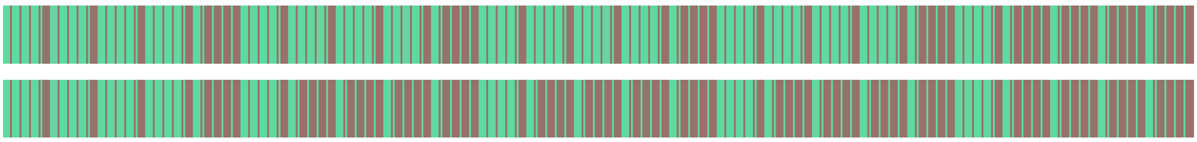


Figure 7.2: Two tilings constructed from the rule in Example 7.8, with one color in green and the other in red. One can see that tilings constructed with this rule tend to have large patches with mostly one color, followed by large patches with mostly the other color. This suggests the tilings may not be hyperuniform.

7.2 Symmetric substitution rules

In two or more dimensions, most substitution tilings are defined via symmetry: there often exists some group of rigid motions which is compatible with the substitution rule. In this case, we will be able to use Theorem 6.1 to prove that a large class of symmetric substitution tilings are hyperuniform.

In this section, we introduce a formalism to describe symmetric substitution rules, use it to define several examples, and discuss their hyperuniformity and degree of uniformity.

7.2.1 Symmetric substitution systems

First, we define symmetric substitution systems, and show how one can efficiently construct them and check their basic properties.

Definition 7.10. Let $G < O(d)$ be a finite subgroup equipped with an action on $[\ell]$, in addition to its usual action on $\mathbb{R}^d$. We say $\mathcal{S}$ is a *G-symmetric substitution rule* if the displacement matrix $\mathbf{\Delta} = (\Delta_{j,k})_{j,k\in[\ell]}$ satisfies

$$\Delta_{g\cdot j, g\cdot k} = g \cdot \Delta_{j,k}$$

for all $g \in G$, $j, k \in [\ell]$.

If $\mathcal{S}$ is a G-symmetric substitution rule, it suffices to define the displacements for a set of representatives of the orbits of G on $[\ell]$. Furthermore, in order to check whether it is a stone substitution rule, it suffices to check it on this set of representatives.

Lemma 7.11. *Let $\mathcal{S}$ be a primitive, G-symmetric substitution rule with canonical prototiles $\tau_1, \cdots, \tau_\ell$. Then, for all $g \in G$, we have*

$$g \cdot \tau_j = \tau_{g\cdot j}$$

Proof. Let $\mathcal{J}_G \subset [\ell]$ be a set of representatives of the orbits of G, and, for $j \in [\ell]$, define new tiles

$$\tau'_j = \bigcup_{g\in G} g \cdot \tau_{g^{-1}\cdot j}.$$

Then the tiles $\tau'_1, \cdots, \tau'_\ell$ are compact and satisfy the equation

$$D_\lambda \tau'_j = \bigcup_{i\in[\ell]} \bigcup_{x\in\Delta_{j,i}} T_x \tau'_i.$$

By Theorem 4.16, the canonical prototiles of $\mathcal{S}$ are unique, hence $\tau_j = \tau'_j$ for all $j \in [\ell]$. As the prototiles τ'_j satisfy $\tau'_j = g \cdot \tau'_{g^{-1}\cdot j}$, we are done. □

Lemma 7.12. *Let $\ell \in \mathbb{N}$, and let $G < O(d)$ be a finite subgroup equipped with an action on $[\ell]$. Let $\mathcal{J}_G \subset [\ell]$ be a set of representatives of the orbits of G, and let $(\varrho_j)_{j\in\mathcal{J}_G}$ be a collection of finite ℓ-color subsets of $\mathbb{R}^d$. Then there exists a unique G-symmetric substitution rule $\mathcal{S}$ such that $\varrho(\boldsymbol{o}_j) = \varrho_j$ for all $j \in \mathcal{J}_G$.*

Lemma 7.13. *Let $\mathcal{S}$ be a G-symmetric primitive substitution rule with ℓ colors and $\tau_1, \cdots, \tau_\ell$ be compact subsets of $\mathbb{R}^d$. Then the following are equivalent:*

(i) $\mathcal{S}$ is a stone substitution with canonical prototiles $\tau_1, \cdots, \tau_\ell$.

(ii) The following conditions hold:

  (i) For all $j \in [\ell]$, $g \in G$, $\tau_{g\cdot j} = g \cdot \tau_j$

  (ii) For some set of representatives $\mathcal{J}_G \subset [\ell]$ of the action of G and all $j_0 \in \mathcal{J}_G$ and $k \in [\ell]$, $\varrho(\boldsymbol{o}_{j_0})$ tiles $D_\lambda \tau_j$ with prototiles $\tau_1, \cdots, \tau_\ell$.

Proof. The implication (i) ⇒ (ii) is clear by the definition of a G-symmetric substitution rule and Lemma 7.11.

So assume (ii) holds: we want to show $\varrho(\boldsymbol{o}_j)$ tiles $D_\lambda \tau_j$ with prototiles $\tau_1, \cdots, \tau_\ell$ for all $j \in [\ell]$. So let $\mathcal{J}_G$ be a set of representatives of the orbits of G on $[\ell]$. Then, for every $j \in [\ell]$, there exists $j_0 \in \mathcal{J}_G$ and $g \in G$ such that $j = g \cdot j_0$. Then

$$\varrho(\boldsymbol{o}_j) = \varrho(\boldsymbol{o}_{g\cdot j_0}) = g \cdot \varrho(\boldsymbol{o}_{j_0})$$

because of the symmetry of $\mathcal{S}$. By assumption, $\varrho(\boldsymbol{o}_{j_0})$ tiles $D_\lambda \tau_{j_0}$ with prototiles $\tau_1, \cdots, \tau_\ell$: therefore, $g \cdot \varrho(\boldsymbol{o}_{j_0}) = \varrho(\boldsymbol{o}_j)$ tiles $g \cdot D_\lambda \tau_{j_0} = D_\lambda \tau_j$ with the same prototiles. This concludes the proof. □

7.2.2 The spherical substitution matrix

Let $\mathcal{S}$ be a substitution rule with ℓ colors. If $\mathcal{S}$ is G-symmetric for some finite group G, this has consequences for the structure of the substitution matrix M_{full}, which we will study in the rest of this section. These will help us apply Theorem 6.1 to prove hyperuniformity of symmetric substitution systems.

Let $\mathcal{S}$ be a G-symmetric substitution rule with ℓ colors. Then the action of G on $[\ell]$ induces a unitary action of G on $\mathbb{C}^\ell$, by permuting the standard basis vectors: that is, $g \cdot e_j = e_{g\cdot j}$ for $j \in [\ell]$. The substitution and Fourier matrices of $\mathcal{S}$ interact with this action as follows:

Lemma 7.14.

(i) The full substitution matrix M_{full} of $\mathcal{S}$ satisfies $g \cdot M_{\text{full}} = M_{\text{full}} \cdot g$ for all $g \in G$.

(ii) The normalized Fourier matrix $\boldsymbol{A}(\xi)$ *satisfies* $g \cdot \boldsymbol{A}(\xi) = \boldsymbol{A}(g \cdot \xi) \cdot g$ *for all* $g \in G$.

Proof. By definition of a G-symmetric substitution rule, we have

$$\begin{aligned}(g \cdot \boldsymbol{A}(\xi))_{jk} &= \boldsymbol{A}(\xi)_{g \cdot j,k} \\ &= \lambda^{-d} \sum_{x \in \Delta_{g \cdot j,k}} e^{2\pi i \langle x, \xi \rangle} \\ &= \lambda^{-d} \sum_{x \in \Delta_{j,g^{-1} \cdot k}} e^{2\pi i \langle g^{-1} x, \xi \rangle} \\ &= \boldsymbol{A}(\xi)_{j,g^{-1} \cdot k} \\ &= (\boldsymbol{A}(g \cdot \xi) \cdot g)_{jk}\end{aligned}$$

for all $j, k \in [\ell]$. This shows (ii). As $M_{\text{full}} = \boldsymbol{A}(0)$, (i) immediately follows. □

As M_{full} commutes with the action of G, the generalized eigenspaces of M_{full} are G-invariant. Furthermore, as G acts on $\mathbb{C}^\ell$ by permutations, its action is unitary. Then the following holds.

Lemma 7.15. *Let* $V \subset \mathbb{C}^\ell$ *be a subspace which is both* G*-invariant and* M_{full}*-invariant. Then, for every* $\mu \in \operatorname{Spec} M_{\text{full}}$ *with generalized eigenspace* E_μ*, exactly one of the following holds:*

(i) $\mu \in \operatorname{Spec} M_{\text{full}}|_V$ *and* $E_\mu \cap V \neq 0$

(ii) $\mu \notin \operatorname{Spec} M_{\text{full}}|_V$ *and* $E_\mu \perp V$

Therefore, we can get information about the spectral theory of M_{full} by decomposing the substitution matrix into certain G-invariant subspaces. In particular, let V_{sph} be the space of G-invariant vectors in $\mathbb{C}^\ell$. This is also M_{full}-invariant, so we can define the following:

Definition 7.16. Let $\mathcal{J}_G$ be a set of representatives of the orbits of G on $[\ell]$, and let $i_1 < \cdots < i_{\ell_0}$ be the elements of $\mathcal{J}_G$. Then the *spherical substitution matrix* M_{sph} is the $\ell_0 \times \ell_0$ matrix defined by

$$(M_{\text{sph}})_{jk} = \sum_{g \in G} \#\Delta_{i_j, g \cdot i_k}$$

for $j, k \in [\ell_0]$.

The spherical substitution matrix M_{sph} is the transformation matrix of the restriction of the substitution operator M_{full} to the subspace V_{sph}. Therefore we obtain the following:

Lemma 7.17. *Let* $\lambda \in \operatorname{Spec} M_{\text{full}}$ *and* E_λ *be its eigenspace. Then exactly one of the following holds:*

(i) $\lambda \in \operatorname{Spec} M_{\text{sph}}$ *and* λ *has a* G*-invariant eigenvector.*

(ii) $\lambda \notin \operatorname{Spec} M_{\text{sph}}$ *and every eigenvector of* λ *is orthogonal to* V_{sph}*.*

Using this lemma, we can apply Theorem 6.1 to get a sufficient condition for hyperuniformity for G-symmetric substitution systems with G-invariant eigenvectors, as follows:

Corollary 7.18. *Let* $\mathcal{S}$ *be a* G*-symmetric, primitive, FLC substitution rule on* $\mathbb{R}^2$*. Then the following holds: if* $|\mu| < \lambda$ *for all* $\mu \in \operatorname{Spec} M_{\text{sph}} \setminus \{\lambda^2\}$ *and* $w \in \mathbb{C}^\ell$ *is a* G*-invariant vector, then* $\mathcal{S}$ *is hyperuniform for weights* w*.*

Remark 7.19. The spherical matrix corresponds to the trivial representation of G. One can obtain similar matrices by looking at other irreducible representations of G: this gives a decomposition of M_{full} into blocks corresponding to the irreducible components of the action of G. Alternatively, one can write M_{full} as a matrix with entries in the *group ring* $\mathbb{C}[G]$, providing a compact description of the substitution matrix (this requires some care if the action is not free). This has been used by Sadun [49] (see also [12]) to study the cohomology of substitution spaces. We will also use this later in specific examples.

7.2.3 Examples

In this section, we give examples of symmetric substitution systems in dimension 2, and discuss their hyperuniformity and spectral properties.

For every example we consider, the symmetry group G will either be the cyclic group of rotations or the dihedral group.

- $G = C_n$, where the generator R acts on $\mathbb{R}^2$ by rotation by $\frac{2\pi}{n}$ radians counterclockwise. For $i \in [n]$, we set $g_i := R^{i-1}$, so we have $G = \{g_1, g_2, \cdots, g_n\}$.
- $G = D_n$, the dihedral group of order $2n$, generated by the rotation R by $\frac{2\pi}{n}$ radians counterclockwise, and the reflection S in the x-axis. For $i \in [n]$, we set $g_i := R^{i-1}$ and $g_{n+i} := R^{i-1}S$: this gives us an ordering $G = \{g_1, g_2 \cdots, g_{2n}\}$.

Remark 7.20 (A picture is worth a thousand words)**.** For these examples, explicitly writing down the displacement matrix $\boldsymbol{\Delta}$ would often be cumbersome and not very enlightening. As is common in the literature, we will instead define the substitution rule by showing a picture of the patches defined by $\varrho(\boldsymbol{o}_j)$ for all $j \in \mathcal{J}_G$, where $\mathcal{J}_G$ is a set of representatives of the orbits of G on $[\ell]$: as long as one marks each tile in a way that makes its color clear, this is enough to determine $\varrho(\boldsymbol{o}_j)$ for all $j \in \mathcal{J}_G$, and hence the substitution rule $\mathcal{S}$ (see Lemma 7.13). The way G operates on the colors will also be apparent from the pictures.

Chair substitution rule

Example 7.21. We define the *chair substitution rule* $\mathcal{S}$ on $\mathbb{R}^2$ as the C_4-symmetric rule given by Figure 7.4. We check that this is a primitive, stone, FLC substitution rule.

- From the picture, it is apparent that the support of $\varrho(\boldsymbol{o}_1)$ is exactly $D_2\tau_1$: therefore $\mathcal{S}$ is stone with scaling constant $\lambda = 2$ and the pictured prototiles $\tau_1, \tau_2, \tau_3, \tau_4$.

- After applying the substitution rule twice, all four orientations of the original prototile appear, hence $\mathcal{S}$ is primitive.

- Figure 7.5 shows a portion of a chair tiling, i.e. a tiling obtained from a legal multiset of $\mathcal{S}$. From the figure, we see that the tiling is not edge-to-edge, but it is *sibling edge-to-edge* [32]: the intersection between two tiles is either empty, a vertex, a side of both tiles, or one half of a side of one tile. This implies that it is FLC, as there are only finitely many ways to surround each tile satisfying these rules.

There is only one prototile up to rotation. Therefore, the spherical substitution matrix is $M_{\text{sph}} = (4)$.

Figure 7.3: The four prototiles of the chair substitution rule, with the origin marked in black. The rotational symmetry is apparent.

Corollary 7.22. *The chair rule is hyperuniform for constant weights.*

The full substitution matrix of the chair substitution rule is given by

$$M_{\text{full}} = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \end{pmatrix}:$$

the first row can be read from Figure 7.4, and the other rows follow from the C_4-

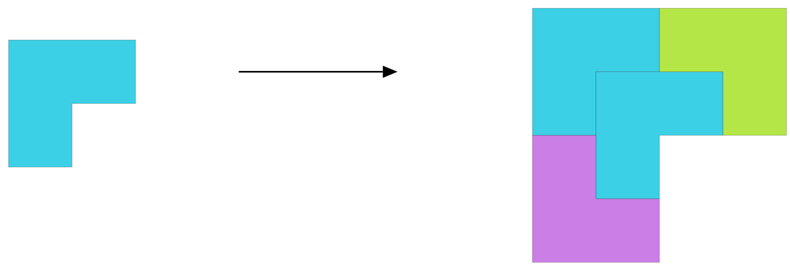


Figure 7.4: The prototile τ_1 and the patch of $\varrho(\boldsymbol{o}_1)$.

symmetry. It has eigenvalues $4, 2, 2, 0$, with eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ i \\ -1 \\ -i \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ -i \\ -1 \\ i \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

Then Theorem 6.1 gives us the following bounds.

Corollary 7.23. *Let $\mathcal{S}$ be the chair substitution rule and $w \in \mathbb{C}^4$. Then*

(i) If $w \in \operatorname{span}\{v_1, v_4\}$, then $\hat{\eta}_w(B_r) \lessapprox r^4$ as $r \to 0$. In particular, $\mathcal{S}$ is hyperuniform of Class I with these weights.

(ii) If $w \notin \operatorname{span}\{v_1, v_4\}$, then $\hat{\eta}_w(B_r) \lessapprox r^2$ as $r \to 0$.

Proof. If $w \in \operatorname{span}\{v_1, v_4\}$, w is orthogonal to the eigenspaces of the eigenvalue 2, therefore $\beta_\perp(w) = 3 - \log_2 2 = 2$ and $\beta_\|(w) = 2 - \log_2 0 = \infty$. Then $\beta(w) = 2$, so Theorem 6.1 implies $\hat{\eta}_w(B_r) \lessapprox r^4$.

Otherwise, we have $\beta(w) = \beta_\|(w) = 2 - \log_2 2 = 1$, so $\hat{\eta}_w(B_r) \lessapprox r^2$. □

Part *(ii)* does not prove hyperuniformity for these weights: if the bound were sharp, this would mean that the chair substitution rule is not hyperuniform for these weights.

In the literature, one often considers a variant of the chair substitution rule, the *block substitution rule* [4], which has the same substitution matrix but where the prototiles are all squares: as the substitution matrix is the same, the bounds from Corollary 7.23 still hold.

For the block substitution rule, self-similar sets are *limit-periodic*, and Baake and Grimm [4] computed the diffraction measure explicitly. In this case, we can prove that

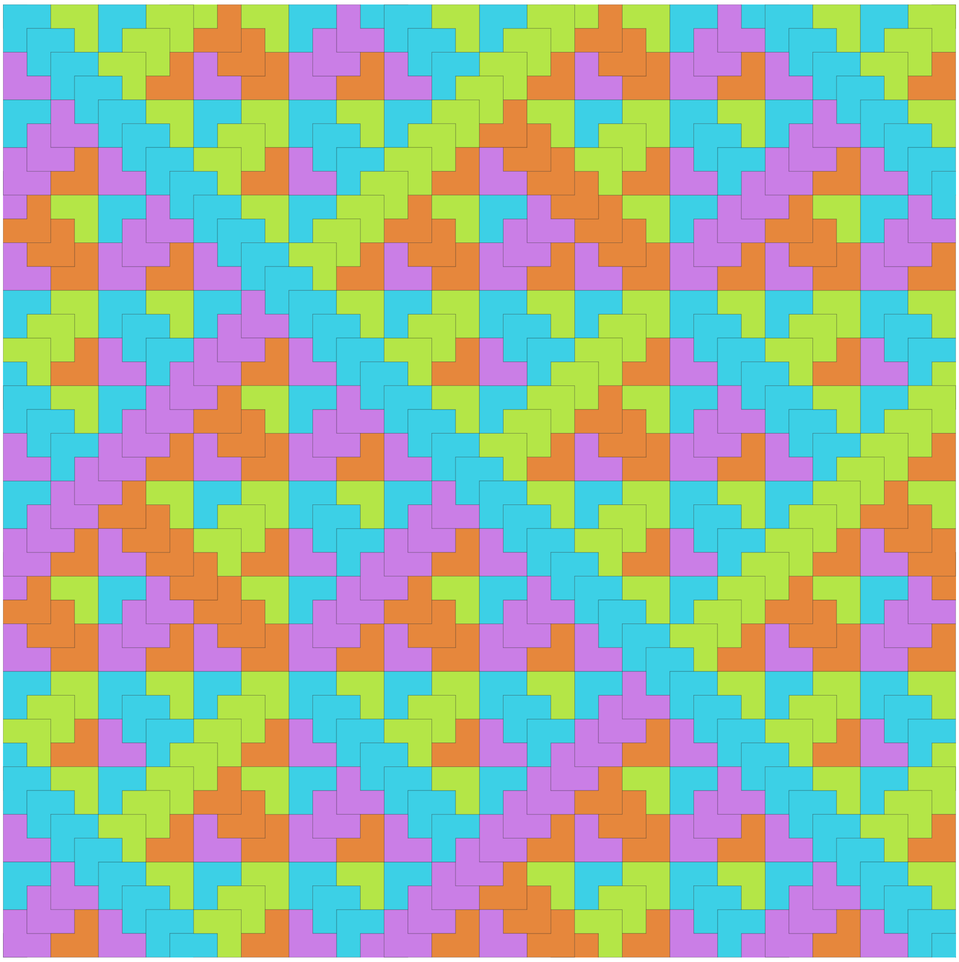


Figure 7.5: A chair tiling.

there are choices of weights for which the block substitution rule is not hyperuniform.

Theorem 7.24. *Let $\mathcal{S}_{block}$ be the block substitution rule and $w = (1,0,0,0)^\top$. Then $\hat{\eta}_w(B_r) \gtrsim r^2$ as $r \to 0$.*

Proof. By Theorem 4.57, it suffices to find a subset $A \subset B_r$ which is not a null set and such that $\hat{\eta}_w(D_{2^{-n}}A) \gtrsim 4^{-n}$: as $\mathcal{S}_{block}$ is pure point diffractive, it suffices to do this for a single point.

In fact, Baake and Grimm [4, Equation (25)] showed that

$$\hat{\eta}_w(2^{-n}(1,0)^\top) = \left| \frac{1}{4^n} \frac{1}{1-\epsilon_n} \right|^2 \asymp 4^{-n}$$

as $n \to \infty$, where $\epsilon_n = e^{-2\pi i 2^{-n}}$. This completes the proof. ☐

Similar results should hold for other choices of weights not in $\operatorname{span}\{v_1, v_4\}$. This suggests that the same is true for the chair substitution rule.

Penrose substitution rule

Example 7.25. The *Penrose substitution rule* $\mathcal{S}_{Pen}$, is a D_{10}-symmetric substitution rule defined as depicted in Figure 7.6. It is primitive, stone and FLC, with scaling constant $\lambda = \tau = \frac{1+\sqrt{5}}{2}$.

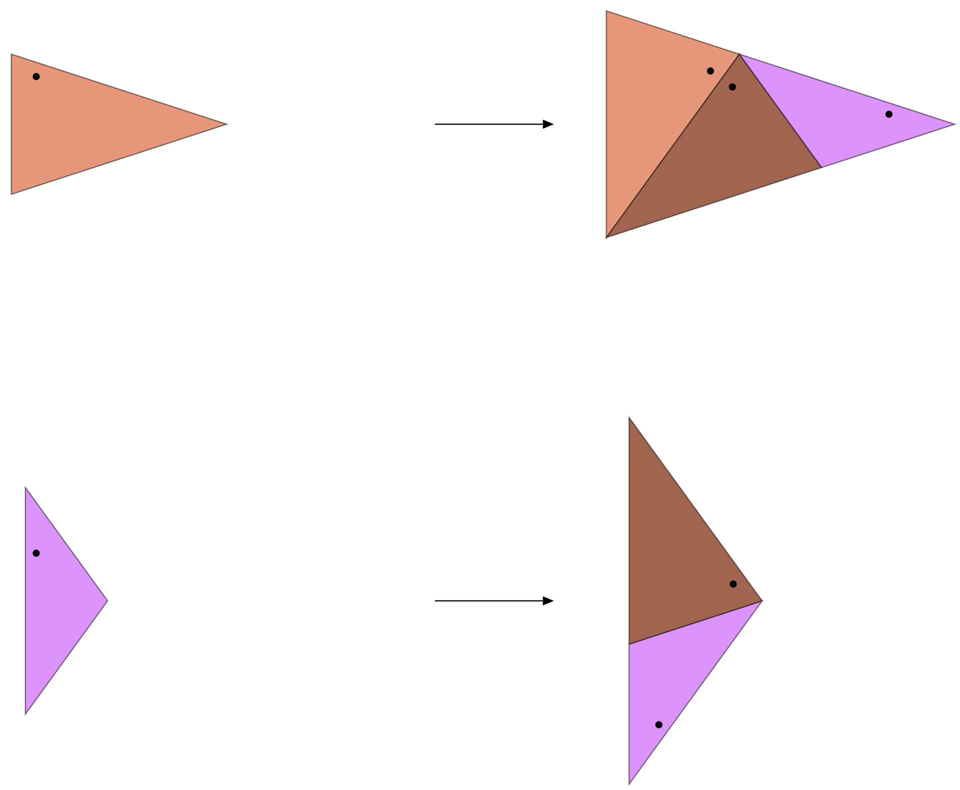


Figure 7.6: The prototiles τ_1 and τ_{21} on the left, together with the supports of $\varrho(\boldsymbol{o}_1)$ and $\varrho(\boldsymbol{o}_{21})$ on the right. On the right, the origin of each tile is marked. We picture the tiles $\tau_1, \cdots, \tau_{10}$ in pink, $\tau_{11}, \cdots, \tau_{20}$ in brown, and the tiles $\tau_{21}, \cdots, \tau_{30}$ in light blue, note that the triangles have an axis of reflection symmetry, so the colors are needed to distinguish them.

Here, $M_{\mathrm{sph}} = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ with non-PF eigenvalue $-\tau^{-2}$, so the hypothesis of Corollary 7.18 holds for symmetric weights. Therefore:

Theorem 7.26. *The Penrose rule is hyperuniform for any symmetric choice of weights, such as* $w = (1, 1, \cdots)^{\top}$.

In order to write exact bounds for the diffraction around the origin, we need the full substitution matrix. This is a 40×40 matrix, but, as mentioned in Remark 7.19, we

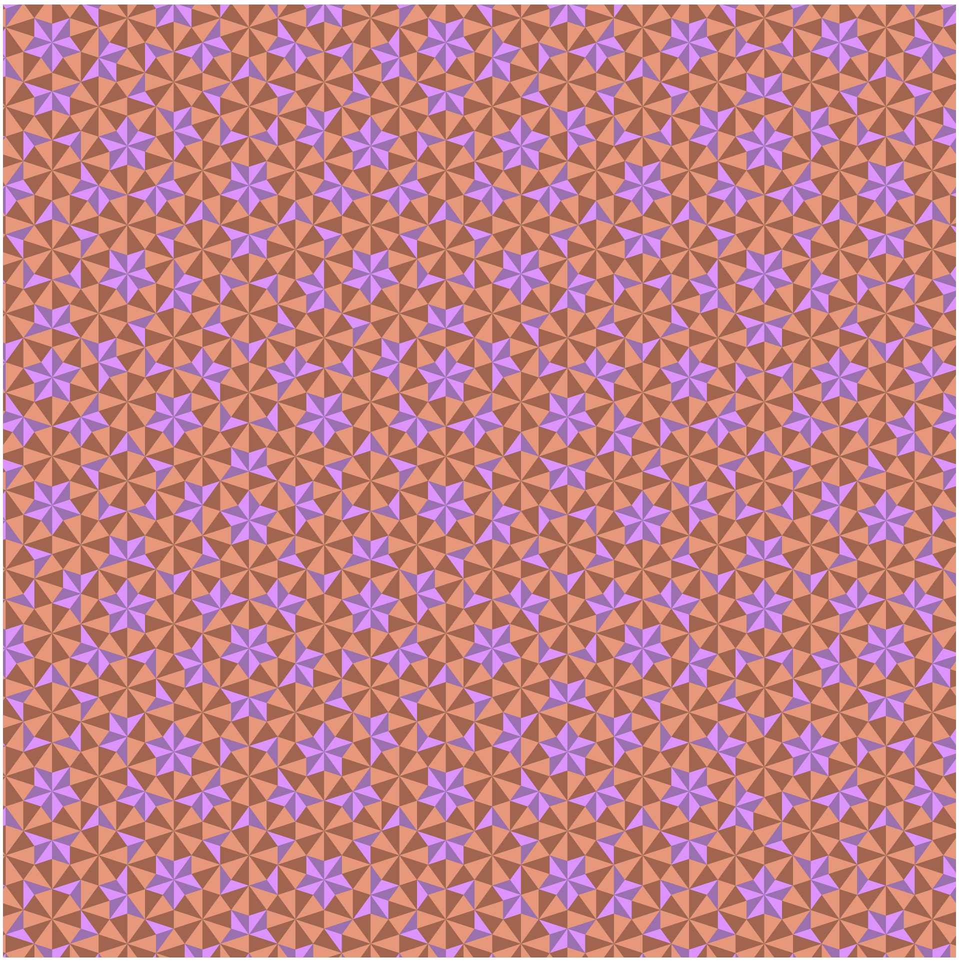


Figure 7.7: Example Penrose tiling. As before, the tiles τ_{11} to τ_{20} and τ_{31} to τ_{40} are depicted in darker colors to distinguish them.

can write it in a compact way using symmetry.

Theorem 7.27. *For symmetric weights w, $\hat{\eta}_w(B_r) \lessapprox r^\alpha$ as $r \to 0$ for the Penrose rule, where $\alpha = 2(3 - \log_\varphi(\mu_2)) = \sim 2.85$.*

Proof. In particular, as D_{10} acts freely on $\mathbb{C}^{40}$, we can assume it acts diagonally on $\mathbb{C}^{20} \oplus \mathbb{C}^{20}$. For $g \in D_{10}$, let $Z(g) \in \mathbb{C}^{20\times 20}$ be the unique G-equivariant matrix such that $Z(g)e_1 = e_{g\cdot 1}$. Then, by inspecting Figure 7.6, we see that the full substitution matrix is given by

$$M_{\text{full}} = \begin{pmatrix} Z(R^7) + Z(R^6 S) & Z(R^7) \\ Z(R^3 S) & Z(R^4) \end{pmatrix}$$

Using a computer algebra system, one can compute that this matrix is diagonalizable and its second largest eigenvalue is $\mu_2 = \sim 2.13$. □

This bound is probably not sharp: numerical experiments and heuristic calculations [25, 33] suggest that the best exponent is $\hat{\eta}_w(B_r) \lessapprox r^8$, at least for constant weights.

Proving this rigorously would require a more detailed analysis of the diffraction of the Penrose rule, such as using the representation of Penrose tilings as model sets; see Baake and Grimm [5, Example 7.11].

Remark 7.28. Note that the map $Z : G \to \mathbb{C}^{20\times 20}$ used above is *not* the action of G on $\mathbb{C}^{20}$: as G is non-abelian, the matrices representing the action on G are not G-equivariant. Instead, if we identify $\mathbb{C}^{20}$ with the group ring $\mathbb{C}[D_{10}]$ and let G act by left multiplication, then $Z(g)$ is the matrix representing *right* multiplication by g.

Danzer's 7-fold tiling

Example 7.29. Danzer's 7-fold tiling is a D_7-symmetric substitution rule with scaling constant $\lambda = \sqrt{1 + \frac{\sin(\frac{2\pi}{7})}{\frac{\pi}{7}}} = {\sim}2.8$: the displacements and canonical prototiles are shown in Figure 7.8, and an example tiling is shown in Figure 7.9. One can deduce from the pictures that the substitution rule is primitive, stone and FLC.

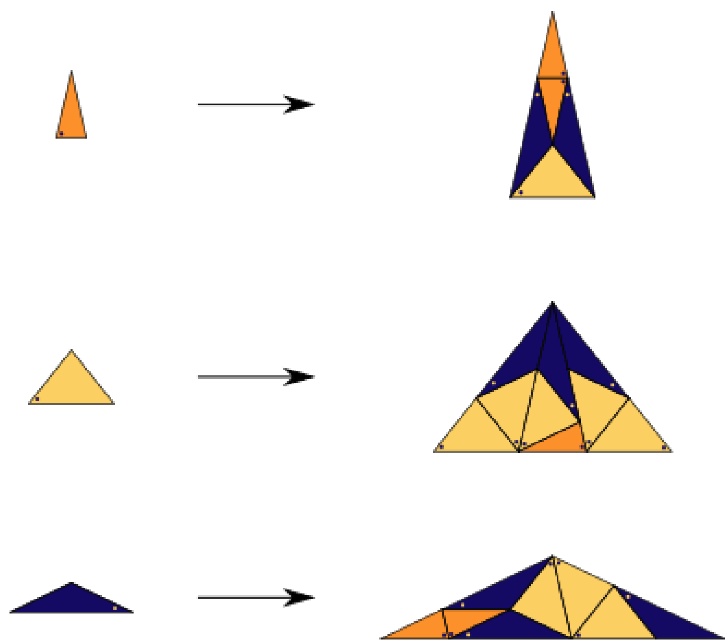


Figure 7.8: Danzer's 7-fold substitution rule. Figure by Frettlöh, Harriss, and Gähler [24] licensed under CC BY-NC-SA 2.0.

Note that λ is not a Pisot number, hence its dynamical spectrum has no eigenvalues and its diffraction has no pure point component [26, 54]. Thanks to our approach using self-similar densities, we are able to estimate its diffraction around the origin exactly the same as for the previous examples: we do not even need to figure out whether the diffraction is singular continuous or absolutely continuous.

Figure 7.9: A Danzer tiling. Figure by Frettlöh, Harriss, and Gähler [24] licensed under CC BY-NC-SA 2.0.

Theorem 7.30. *Danzer's 7-fold rule is hyperuniform for any symmetric choice of weights.*

Proof. The spherical substitution matrix $M_{\mathrm{sph}} = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 3 \end{pmatrix}$ has non-PF eigenvalues approximately ~ 2.1 and ~ 0.06, both smaller than λ. □

Theorem 7.31. *The diffraction of Danzer's 7-fold rule has $\hat{\eta}_w(B_r) \lessapprox r^\alpha$ with $\alpha = \sim 2.6$. for any symmetric choice of weights w.*

Proof. Similarly to the Penrose case, D_7 acts freely on the colors of Danzer's 7-fold rule, so, after ordering the colors appropriately, we can write

$$M_{\mathrm{full}} = \begin{pmatrix} Z(S)+Z(R^7S) & Z(R^4) & Z(R^3)+Z(R^{10}S) \\ I & I+Z(S)+R^3+Z(R^{10})Z(S)+Z(R^4) & Z(R^3)Z(S)+Z(R^{10})+Z(S) \\ Z(R^4)Z(S)+Z(R^{10}) & Z(R^4)+Z(R^5)+Z(R^8)Z(S) & Z(S)+Z(R^6)+Z(R^8) \end{pmatrix},$$

where r and s are the 14×14 permutation matrices representing the actions of R and S on $\mathbb{C}^{28}$, and I is the 28×28 identity matrix.

One can compute using numerical methods that the second largest eigenvalue is approximately $\mu_2 = \sim 4.3$. Thus

$$\beta_{\|}(w) = 2 - \log_\lambda(\sim 2.1) = \sim 2.6$$
$$\beta_{\perp}(w) \geq 3 - \log_\lambda(\sim 4.3) = \sim 1.6$$

which means giving $2\beta(w) = \sim 3.1$. □

Other examples

The *Tilings encyclopedia* [24] is a large compendium of interesting tilings including many coming from symmetric substitution rules. As long as they are primitive, stone and FLC, one can apply Theorem 6.1 to try to prove hyperuniformity for many of these tilings: in order to do this, we only need to find their spherical matrix. All the information we need can be obtained from the pictures in the encyclopedia.

Theorem 7.32. *The following substitution rules are hyperuniform for any symmetric choice of weights:*

(i) Godréche–Lançon–Billard (modified as in [11, Fig. 3]).

(ii) CAP

(iii) Watanabe Ito Soma 12-fold

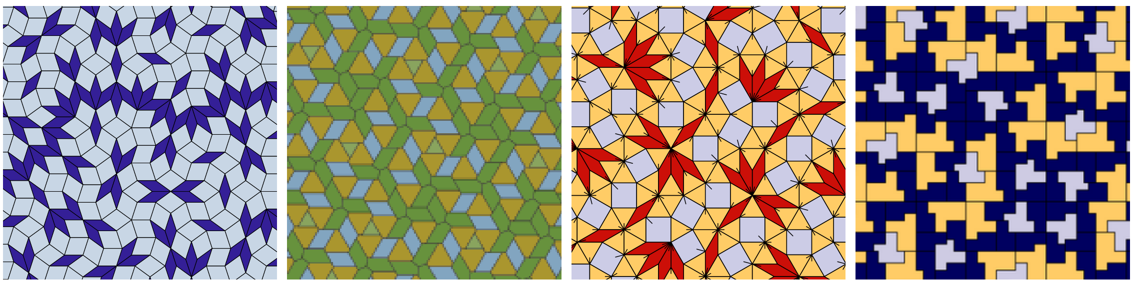
(iv) Ammann A3

Proof. Note that, for rules (i), (ii) and (iii), the prototiles depicted in the encyclopedia are not the canonical prototiles. However, we still know the rules are stone, because they define self-similar Delone subsets. For the Godréche–Lançon–Billard, the canonical prototiles have fractal boundary and have been computed by Godrèche and Lançon [30]. The advantage of the polygonal prototiles depicted in the encyclopedia is that they prove the substitution rules are FLC, as they define edge-to-edge tilings.

In order to prove the tilings are hyperuniform, we need to compute their spherical substitution matrix, and compare largest non-PF eigenvalue μ_2 with the scaling constant λ. We do this in Table 7.1. If the scaling constant is not written in the encyclopedia, we can compute it as $\lambda = \sqrt{\lambda_{PF}}$, where λ_{PF} is the PF eigenvalue. For all of the example, we see that $|\mu_2| < \lambda$, which means they are hyperuniform for any choice of symmetric weights by Corollary 7.18. □

Rule	λ	M_{sph}	μ_2
Godréche–Lançon–Billard (modified)	$\frac{1}{2}(5+\sqrt{5}) = {\sim}3.6$	$\left(\begin{smallmatrix} 10 & 5 \\ 5 & 5 \end{smallmatrix}\right)$	$\frac{1}{2}(15-5\sqrt{5}) = {\sim}1.9$
CAP	$\frac{3+\sqrt{5}}{2}$	$\begin{pmatrix} 3 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 3 & 0 & 2 & 3 \end{pmatrix}$	1
Watanabe Ito Soma 12-fold	$2+\sqrt{3}$	$\begin{pmatrix} 7 & 8 & 16 \\ 2 & 3 & 6 \\ 2 & 2 & 4 \end{pmatrix}$	$7-4\sqrt{3}$
Ammann A3	$\frac{3+\sqrt{5}}{2}$	$\begin{pmatrix} 1 & 3 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	1

Table 7.1: For each rule in Theorem 7.32, we list its scaling constant, spherical substitution matrix, and largest non-PF eigenvalue. In all of these cases, we have $|\mu_2| < \lambda$, so the substitution rule is hyperuniform for symmetric weights.



(a) A GLB tiling **(b)** A CAP tiling **(c)** A WIS tiling **(d)** An Ammann tiling

Figure 7.10: The tilings obtained from the substitution rules considered in Theorem 7.32. Figures (a), (b), (c) and (d), by Frettlöh, Harriss, and Gähler [24] licensed under CC BY-NC-SA 2.0.

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