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A classification of 5-dimensional manifolds, homogeneous souls of codimension two and non-diffeomorphic pairs

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Abstract
Let $T(\gamma)$ be the total space of the canonical line bundle γ over $\mathbb{C}\mathrm{P}^1$ and r an integer, which is divisible by an odd prime. We prove that $L_r^3 \times T(\gamma)$ admits an infinite sequence of metrics of nonnegative sectional curvature with pairwise non-homeomorphic souls, where L_r^3 is a 3-dimensional lens space with fundamental group of order r. Furthermore, we classify a class of non-simply connected 5-manifolds up to diffeomorphism and use this result to give first examples of manifolds N, which admit two complete metrics of nonnegative sectional curvature with souls S and S' of codimension two such that S and S' are diffeomorphic whereas the pairs (N, S) and (N, S') are not diffeomorphic. These results give solutions to two problems posed by Igor Belegradek, Slawomir Kwasik and Reinhard Schultz.

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1 Introduction

In [15] we found among the set of total spaces of principal S^1-fibre bundles over $S^2 \times S^2$ infinite sequences of simply and tangentially homotopy equivalent but pairwise non-homeomorphic manifolds. This topological result was a main step in the discovery of the first examples of manifolds, which admit infinitely many complete metrics of nonnegative sectional curvature and pairwise non-homeomorphic souls of codimension three. In this work a further topological study of these manifolds leads to new phenomena in the field of nonnegative sectional curvature, which partially involve well studied objects such as the three dimensional lens spaces and the tautological bundle over $\mathbb{C}\mathrm{P}^1$.

In 1972 J. Cheeger and D. Gromoll [6] proved that any complete open Riemannian manifold with nonnegative sectional curvature is diffeomorphic to the total space of the normal bundle of a totally geodesic and totally convex submanifold, called a *soul*.

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This fundamental structural result on nonnegatively curved complete Riemannian manifolds led to further questions and results concerning the existence of infinitely large families of souls of a specific codimension or moduli spaces of complete metrics of nonnegative sectional curvature.

For example I. Belegradek [4] constructed the first examples of manifolds, which admit infinite sequences of nonnegatively curved metrics with pairwise non-homeomorphic souls of codimension at least five. In [12] V. Kapovitch, A. Petrunin and W. Tuschmann found infinite sequences of nonnegatively curved metrics with souls of codimension at least eleven and with upper bounds for the curvature and the diameter. Furthermore, they proved that a certain moduli space of complete metrics of nonnegative sectional curvature on a 22-dimensional manifold has infinitely many connected components.

Let $L^{a,b}$ be the total space of the principal S^1-fibre bundle over $S^2 \times S^2$ given by the first Chern class $ax + by$, where x, y are the standard generators of $H^2(S^2; \mathbb{Z})$ of the first and the second factor of the base respectively. And furthermore, we denote the set $\{L^{a,b} | (0, 0) \neq (a, b) \in \mathbb{Z}^2\}$ by $\mathcal{L}$.

These manifolds form a subclass of the class of lens space bundles over S^2: Let L^3_a be the classical 3-dimensional lens space with fundamental group isomorphic to $\mathbb{Z}/a$, equipped with the induced metric from S^3. The manifold $L^{a,b}$ is diffeomorphic to the total space of the L^3_a-bundle over S^2 with clutching function

$$S^1 \to \mathrm{Isom}(L^3_a), \;\; z \mapsto (x \mapsto z^b x).$$

Theorem 1 *Let a, b be integers such that their greatest common divisor is divisible by an odd prime. The total space of a complex line bundle over $L^{a,b}$ with primitive first Chern class admits an infinite sequence of complete metrics of nonnegative sectional curvature with pairwise non-homeomorphic souls.*

This theorem provides the first examples of manifolds, which possess infinitely many metrics of nonnegative curvature with pairwise non-homeomorphic souls of codimension two and therefore we obtain solutions to [5, Problem 4.8 (i)].

Let $T(\gamma)$ be the total space of the canonical line bundle γ over $\mathbb{C}\mathrm{P}^1$. If we consider L^3_r as the total space of the principal S^1-fibre bundle over S^2 with first Chern class rx then we obtain the following theorem as a special case of Theorem 1.

Theorem 2 *If r is divisible by an odd prime then $L^3_r \times T(\gamma)$ admits an infinite sequence of complete metrics of nonnegative sectional curvature with pairwise non-homeomorphic souls.*

Let N be a smooth manifold and $\mathfrak{R}_{sec\geq 0}(N)$ be the set of smooth and complete metrics on N of nonnegative sectional curvature. We can equip $\mathfrak{R}_{sec\geq 0}(N)$ with the topology of smooth convergence on compact subsets and we denote this topological space by $\mathfrak{R}^c_{sec\geq 0}(N)$. The diffeomorphism group of N acts on $\mathfrak{R}^c_{sec\geq 0}(N)$ via pullback. The orbit space under this action equipped with the quotient topology is called the *moduli space* of complete metrics on N of nonnegative sectional curvature and it is denoted by $\mathfrak{M}^c_{sec\geq 0}(N)$.

The normal bundles of the souls that appear in Theorem 2 have nontrivial rational Euler class and thus, we may conclude by a result of V. Kapovitch, A. Petrunin and

W. Tuschmann [12, Lem. 6.1] that the souls of metrics, which lie in the same path component of $\mathfrak{M}^c_{sec\geq 0}(L^3_r \times T(\gamma))$ have to be diffeomorphic. This fact and Theorem 2 imply the following

Theorem 3 *If r is divisible by an odd prime then $\mathfrak{M}^c_{sec\geq 0}(L^3_r \times T(\gamma))$ has infinitely many components.*

Let the topology on $\mathfrak{R}_{sec\geq 0}(N)$ be the one induced by uniform smooth convergence. We call the resulting topological space $\mathfrak{R}^u_{sec\geq 0}(N)$ and the corresponding moduli space is denoted by $\mathfrak{M}^u_{sec\geq 0}(N)$. There is the obvious map

$$\mathfrak{M}^u_{sec\geq 0}(N) \to \mathfrak{M}^c_{sec\geq 0}(N).$$

We immediately realize that Theorem 3 implies the existence of infinitely many connected components of $\mathfrak{M}^u_{sec\geq 0}(L^3_r \times T(\gamma))$ if r is not a power of two.

A further chapter of this article is devoted to a classification theorem for a class of smooth closed non-simply connected 5-dimensional manifolds.

Theorem 4 *Let r be odd, greater than one and not divisible by three. Furthermore, let N and N' be oriented smooth closed spin 5-dimensional manifolds with cyclic fundamental group of order r and which are homotopy equivalent to manifolds in $\mathcal{L}$. Then N and N' are oriented diffeomorphic if and only if there exists an orientation preserving homotopy equivalence $h : N' \to N$ such that*

(i) $\rho(h_*(g), N) = \rho(g, N')$ *for all* $g \in \pi_1(N') \setminus \{0\}$ *and*
(ii) $\Delta(N) \sim \Delta(N')$,

where the ρ-invariant is defined in [1, p. 589], $\Delta(\cdot)$ is the Reidemeister torsion as defined in [13, p. 405] and $\sim$ denotes that there exist isomorphisms between the fundamental groups of N, N' and $\mathbb{Z}/r$ s.t. the Reidemeister torsions correspond to the same element in the Whitehead group $Wh(\mathbb{Z}/r)$.

We apply Theorem 4 to a classification of non-simply connected manifolds in $\mathcal{L}$.

Theorem 5 *Let r be as in Theorem 4 and $L^{a,b}$, $L^{a',b'} \in \mathcal{L}$ with $\pi_1(L^{a,b}) \cong \pi_1(L^{a',b'}) \cong \mathbb{Z}/r$. Furthermore, let (m, n), (m', n') be pairs of integers such that $m\frac{b}{r} + n\frac{a}{r} = 1 = m'\frac{b'}{r} + n'\frac{a'}{r}$. Then $L^{a,b}$ and $L^{a',b'}$ are diffeomorphic if and only if there exist $\epsilon, \epsilon', \delta \in \{\pm 1\}$ and $k, k' \in \mathbb{Z}/r$ such that*

$$ab = \delta a'b',$$

$$\left(\epsilon m + k\frac{a}{r}\right)\left(\epsilon n - k\frac{b}{r}\right) \equiv \delta\left(\epsilon' m' + k'\frac{a'}{r}\right)\left(\epsilon' n' - k'\frac{b'}{r}\right) \bmod r,$$

$$\frac{b}{r}\left(\epsilon m + k\frac{a}{r}\right) - \frac{a}{r}\left(\epsilon n - k\frac{b}{r}\right) \equiv \left(\frac{b'}{r}\left(\epsilon' m' + k'\frac{a'}{r}\right) - \frac{a'}{r}\left(\epsilon n' - k'\frac{b'}{r}\right)\right) \bmod r.$$

As the ρ-invariant is als a topological invariant [16, Ch. 14B], Theorem 4 and 5 are also homeomorphism classifications.

We use this explicit classification result and the surgery theory, which lies behind it to prove Theorem 6 stated below.

Let x be an element of $H^2(L^{a,b};\mathbb{Z})$ and $N_x^{a,b}$ be the total space of the complex line bundle over $L^{a,b}$ with first Chern class x.

Theorem 6 *Let r, b be integers, where r is as in Theorem 4.*

(i) If b is not zero but divisible by r and there is a unit s in $\mathbb{Z}/r$ such that $s^2=-1$ then there exists a primitive element $y \in H^2(L^{r,br};\mathbb{Z})$ such that $N_y^{r,br}$ has the following property: $N_y^{r,br}$ admits complete metrics g and g' of nonnegative sectional curvature with souls S, S' respectively such that S and S' are diffeomorphic whereas the pairs $(N_y^{r,br}, S)$ and $(N_y^{r,br}, S')$ aren't diffeomorphic.

(ii) Let $x \in H^2(L^{r,br};\mathbb{Z})$ be primitive and g, g' be complete metrics of $N_x^{r,br}$ of nonnegative sectional curvature with souls S and S' respectively such that $S \in \mathcal{L}$. If $\mathbb{Z}/r$ does not contain a non-trivial unit s with $s^3=1$ and b is zero or not divisible by r then $(N_x^{r,br}, S)$ and $(N_x^{r,br}, S')$ are diffeomorphic if and only if S and S' are diffeomorphic.

If $r=5$ then 2 is a unit in $\mathbb{Z}/5$ with $2^2=-1$ thus, Theorem 6 (i) yields solutions to Problem 4.9 in [5]. The statement of Theorem 6 (i) also holds if we require that there exists a non-trivial unit s in $\mathbb{Z}/r$ with $s^2=1$. It is worth mentioning that Theorem 6 (ii) especially holds for complete metrics on $L_r^3 \times T(\gamma)$ of nonnegative sectional curvature. Any real line bundle over a soul $L^{a,b}$ has to be trivial if the order of its fundamental group is odd. It is worth mentioning that for such manifolds two is the smallest possible codimension where non-diffeomorphic pairs as described in Theorem 6 (i) may occur. Furthermore, Theorem 6 (i) gives another way to get non-path-connected components of the moduli space $\mathfrak{M}^u_{sec\geq 0}(\cdot)$.

The main purpose of the previous theorem is to highlight some new geometric topololgical phenomena. The specific choice of manifolds keeps, to the author's belief, the technical part of the proof as simple as possible.

The structure of this work and some remarks:

Souls of codimension two. We apply results by K. Grove and W. Ziller [9] to prove the existence of complete metrics of nonnegative sectional curvature on total spaces of complex line bundles over $L^{a,b}$ given by a primitive integral cohomology class of degree two. Furthermore, we apply techniques from surgery theory to prove Theorem 1.

Topological Classification. We prove Theorem 4 and classify the non-simply connected total spaces of principal S^1-fibre bundles over $S^2 \times S^2$ with finite fundamental group of odd order not divisible by three up to diffeomorphism and prove Theorem 5.

By using similar methods as we use for proving Theorem 4 one can generalize this differential topological classification result to a classification of smooth closed 5-manifolds with π_1 finite cyclic of odd order not divisible by three, π_2 torsion-free, where π_1 acts trivially on π_2. Examples of such manifolds are total spaces of principal S^1-fibre bundles over simply connected smooth closed 4-manifolds with π_2 torsion-free and $\mathrm{rk}(\pi_2) \geq 1$, e.g. $\mathbb{C}\mathrm{P}^2 \# \pm \mathbb{C}\mathrm{P}^2$ or lens space bundles over S^2 with fibre a 3-dimensional lens space.

If we choose π_2 to be trivial then we are in the realm of 5-dimensional fake lens spaces, which have been classified up to homeomorphism by C.T.C. Wall [16, Part 3, 14 E]. Thus, the classification results here may be viewed as a special extension of Wall's results, which are also complementing results that were obtained by Duan-Liang [7] and Hambelton-Su [10], who studied S^1-fibre bundles over 4-manifolds with trivial fundamental group and fundamental group $\mathbb{Z}/2$ resp. Furthermore, this classification could also have interesting applications in contact topology [8] or might play a role in Sasakian geometry [3].

2 Souls of codimension two

Lemma 7 *The total space of any complex line bundle over $L^{a,b} \in \mathcal{L}$ admits a complete metric of nonnegative sectional curvature with soul $L^{a,b}$.*

Proof We know by definition that $L^{a,b}$ is the total space of a principal S^1-fibre bundle over $S^2 \times S^2$, where $\Pi_{a,b} : L^{a,b} \twoheadrightarrow S^2 \times S^2$ is its projection map. The Gysin sequence of that fibre bundle implies that $\Pi^*_{a,b} : H^2(S^2 \times S^2; \mathbb{Z}) \to H^2(L^{a,b}; \mathbb{Z})$ is surjective, where $\Pi^*_{a,b}$ is the map on cohomolgy groups induced by $\Pi_{a,b}$. Thus, any principal S^1-fibre bundle over $L^{a,b}$ is the pullback of a principal S^1-fibre bundle over $S^2 \times S^2$. Hence, the total space of a principal S^1-fibre bundle over $L^{a,b}$ is (canonically) diffeomorphic to a principal T^2-bundle over $S^2 \times S^2$. From [9, Thm. 4.5] it follows that the total space of any principal S^1-fibre bundle over $L^{a,b}$ admits a complete metric of nonnegative sectional curvature, which is invariant under the T^2-action. Let α be a complex line bundle over $L^{a,b}$, $T(\alpha)$ its total space and let α' be the corresponding principal S^1-bundle over $L^{a,b}$. Furthermore, let ϕ be the principal T^2-bundle over $S^2 \times S^2$ with total space $T(\phi)$, where the inclusion of the first S^1-factor of T^2 into T^2 induces a S^1-fibre bundle, which is α'. Thus, $T(\alpha)$ and $T(\phi)$ are (canonically) diffeomorphic. The total space $T(\alpha)$ can be obtained in the following way:

$$T(\alpha) = \mathbb{C} \overset{S^1 \times \{1\}}{\times} T(\phi),$$

where we glue $\mathbb{C}$ fibrewise to the first S^1-factor of the T^2-fibres of $T(\phi)$ according to the standard S^1-action on $\mathbb{C}$. Thus, if we choose the standard metric on $\mathbb{C}$ and the T^2-invariant metric on $T(\phi)$ from [9] then it follows from O'Neill's formula for the sectional curvature for Riemannian submersions that $T(\alpha)$ admits a complete metric of nonnegative sectional curvature, where the soul is the zero-section of α. □

Let a, b be as in Theorem 1 and r be their greatest common divisor. From [15, Prop. 16] we know that there are infinitely many pairwise non-homeomorphic total spaces of S^1-bundles over $S^2 \times S^2$, which lie in the same tangential homotopy type of $L^{a,b}$. By Corollary 19 we may choose the homotopy equivalences between these manifolds such that their normal invariants are trivial. Following the idea of the proof of [5, Thm. 14.1] we show the following: If we choose a complex line bundle over $L^{a,b}$ with primitive first Chern class then all the total spaces of the pullback bundles with respect to homotopy equivalences with trivial normal invariant are diffeomorphic. Thus, by [15, Thm. 1.1.] and the contruction of the metric above Theorem 1 follows.

We give a sketch of the proof of [5, Thm. 14.1] applied to our situation:

Let α be a complex line bundle over $L^{a,b}$. By $D(\alpha)$ we denote the total space of the associated disc bundle, by $\mathbf{P}$ the disc bundle projection and by $S(\alpha)$ the boundary of $D(\alpha)$. If $f : N \to L^{a,b}$ is a homotopy equivalence then f induces a homotopy equivalence $f(\alpha) : D(f^*\alpha) \to D(\alpha)$. The normal invariants of f and $f(\alpha)$ are related by the following formula:

$$\mathfrak{q}(f(\alpha)) = \mathbf{P}^*\mathfrak{q}(f), \tag{1}$$

where $\mathfrak{q}(\cdot)$ denotes the normal invariant. This is for example proved in [5, Lem. 5.9]. Hence, if f has trivial normal invariant then the same holds for $f(\alpha)$.

A *simple homotopy structure* on $D(\alpha)$ is a simple homotopy equivalence $f : N \to D(\alpha)$, where N is a smooth manifold. Another simple homotopy structure $f' : N' \to D(\alpha)$ and f are said to be equivalent if there exists a diffeomorphism $d : N' \to N$ such that $f \circ d$ and f' are homotopic. The set of equivalence classes of simple homotopy structures on $D(\alpha)$ is called the *simple structure set* of $D(\alpha)$ and it is denoted by $S^s(D(\alpha))$ and the class, which is represented by $id : D(\alpha) \to D(\alpha)$ is called the *base point* of $S^s(D(\alpha))$. In Wall's surgery exact sequence associated to $S^s(D(\alpha))$ there is the map

$$\Delta : L^s_8\big(\pi_1(D(\alpha)), \pi_1(S(\alpha))\big) \to S^s(D(\alpha))$$

which has the property that the elements of the image of Δ are represented by simple homotopy equivalences with trivial normal invariant and vice versa any homotopy equivalence with trivial normal invariant represents an element of the image of Δ. Thus, if $L^s_8(\pi_1(D(\alpha)), \pi_1(S(\alpha)))$ is trivial then $f(\alpha)$ would represent the base point of $S^s(D(\alpha))$, which means that $D(\alpha)$ and $D(f^*\alpha)$ would be diffeomorphic and hence, the same would be true for their interiors. If $i_* : \pi_1(S(\alpha)) \to \pi_1(D(\alpha))$ is an isomorphism then it would follow from Wall's π-π-Theorem that $L^s_8(\pi_1(D(\alpha)), \pi_1(S(\alpha)))$ is trivial.

Lemma 8 *Let α be a complex line bundle over $L^{a,b}$ with primitive first Chern class then $i_* : \pi_1(S(\alpha)) \to \pi_1(D(\alpha))$ is an isomorphism.*

Proof The second stage of the Postnikov tower of $L^{a,b}$ is $L^\infty_r \times \mathbb{C}\mathrm{P}^\infty$ [15, Lem. 2.4], where L^∞_r is the infinite dimensional lens space with fundamental group isomorphic to $\mathbb{Z}/r$. $H^2(L^\infty_r \times \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}) \cong H^2(L^\infty_r; \mathbb{Z}) \oplus H^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$, which is isomorphic to $\mathbb{Z}/r \oplus \mathbb{Z}$. Let z be the standard generator of $H^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$ and v_1 a generator of $H^1(L^\infty_r; \mathbb{Z}/r)$. Let $\big(H^1\big(L^{a,b}; \mathbb{Z}/r\big)\big)^*$ be the set of units of $H^1\big(L^{a,b}; \mathbb{Z}/r\big)$ and $\mathrm{Pri}(H^2(L^{a,b}; \mathbb{Z}))$ the set of elements in $H^2(L^{a,b}; \mathbb{Z})$, which generate a $\mathbb{Z}$-summand. By [15, Lem. 2.9] there is a 1–1 correspondence between

$$\Big(H^1\big(L^{a,b}; \mathbb{Z}/r\big)\Big)^* \oplus \mathrm{Pri}(H^2(L^{a,b}; \mathbb{Z}))$$

and the set of homotopy classes of maps g from $L^{a,b}$ to $L_r^\infty \times \mathbb{CP}^\infty$, which induce isomorphisms on π_1 and π_2. The correspondence is given by

$$g^*(v_1) \text{ and } g^*(z).$$

Thus, a complex line bundle over $L^{a,b}$ with primitive first Chern class $\bar{z}$ is the pullback of a complex line bundle over $L_r^\infty \times \mathbb{CP}^\infty$ under a map g with $g^*(z) = \bar{z}$. The homotopy exact sequence associated to the principal S^1-fibre bundle over $L_r^\infty \times \mathbb{CP}^\infty$ with first Chern class z starts as follows:

$$\pi_2(\mathbb{CP}^\infty) \xrightarrow{\partial} \pi_1(S^1) \to \pi_1(S_z) \to \pi_1(L_r^\infty) \to 0,$$

where S_z is the associated sphere bundle. The boundary map ∂ is an isomorphism since it is induced by the Euler class of the bundle. Hence, $\pi_1(S_z) \to \pi_1(L_r^\infty)$ is an isomorphism. But by using the 2-smoothing g the analogues statement transfers to the S^1-fibre bundle over $L^{a,b}$ given by the first Chern class $\bar{z}$. □

3 Topological classification

Let $L^{a,b} \in \mathcal{L}$. Before we give a proof of Theorem 4 we gather some differential topological properties of $L^{a,b}$ ([15, Sect. 2]):

(i) $\pi_1(L^{a,b}) \cong \mathbb{Z}/\gcd(a, b)$ and $\pi_2(L^{a,b}) \cong \mathbb{Z}$.
(ii) The tangent bundle of $L^{a,b}$ is stably trivial, which implies that all its stable characteristic classes are trivial.
(iii) The Reidemeister torsion as defined in [13, p. 405] is trivial.

How surgery gets involved in the classification of manifolds of the type we consider was partially explained in section 2 of [15]. There we found a connection between a bordism classification and a homotopy classification:

Let $\pi_1(L^{a,b}) \cong \mathbb{Z}/r$. The second Postnikov stage of $L^{a,b}$ is $L_r^\infty \times \mathbb{CP}^\infty =: B_r$ ([15, Lemma 10]). The bordism group of our interest is $\Omega_5^{Spin}(B_r)$, which is defined to be the set

$$\big\{(M, f) |\, M \text{ a closed smooth 5-dimensional spin manifold}, \quad f : M \to B_r\big\}$$

modulo an equivalence relation, which is given as follows: $(M, g) \sim (N, g')$ if there exists a 6-dimensional smooth spin manifold with boundary equal to the disjoint union of M and N and a map $F : W \to B_r$, which restricted to the boundary components is the map f, f' respectively.

The Postnikov decomposition of $L^{a,b}$ yields maps $f : L^{a,b} \to B_r$, which induce isomorphisms on the first and second homotopy groups. We call such maps *normal 2-smoothings*. Let $f : L^{a,b} \to B_r$ and $f' : L^{a',b'} \to B_r$ be normal 2-smoothings then $\big(L^{a,b}, f\big)$, $\big(L^{a',b'}, f'\big)$ define elements in $\Omega_5^{Spin}(B_r)$. Assume that $\big(L^{a,b}, f\big)$

and $\left(L^{a',b'}, f'\right)$ represent the same element in $\Omega_5^{Spin}(B_r)$ then there exists a bordism (W, F) between them as described above and we call (W, F) a *normal (co)bordism*. Furthermore, we may assume that F is a 3-equivalence [11, Prop. 4].

In the following sections we describe elements of relevant groups that are central in surgery theory, and for enhanced readability, we denote $L^{a,b}$ by N_0 and $L^{a',b'}$ by N_1.

Let $L_6^{s,\tau}(\mathbb{Z}/r, S)$ be the set consisting of stable equivalence classes of weakly based non-singular (-1)-quadratic forms over the ring $\Lambda := \mathbb{Z}[\mathbb{Z}/r]$. In this context, stabilization refers to taking the orthogonal sum with the (-1)-hyperbolic forms. A "weakly based" form means that there is a choice of an equivalence class of bases for the underlying free Λ-module, where two bases are equivalent if the change-of-basis matrix has trivial Whitehead torsion. The S in the notation $L_6^{s,\tau}(\mathbb{Z}/r, S)$ refers to the choice of a so-called form parameter, as introduced in [2], which will be discussed in little more detail later.

It is easy to show that $L_6^{s,\tau}(\mathbb{Z}/r, S)$ forms a group, with the group structure given by taking the orthogonal sum of the quadratic forms. The simple Wall group $L_6^s(\mathbb{Z}/r, S)$ and $L_6^{s,\tau}(\mathbb{Z}/r, S)$ are related to each other by the following exact sequence:

$$0 \to L_6^s(\mathbb{Z}/r, S) \xrightarrow{i} L_6^{s,\tau}(\mathbb{Z}/r, S) \xrightarrow{\tau} \mathrm{Wh}(\mathbb{Z}/r),$$

where i denotes the canonical inclusion, and the map τ sends stable equivalence classes of weakly based non-singular (-1)-quadratic forms over Λ to the Whitehead torsion of the matrix representation of the adjoint of these quadratic forms, with respect to the chosen weak equivalence class of the basis.

Now, let us associate to (W, F) an element in $L_6^{s,\tau}(\mathbb{Z}/r, S(N_0 \times I))$, where $S(N_0 \times I)$ is explained below. To understand this, we recall Wall's definition of a quadratic form on an even-dimensional compact manifold. We equip W with a base point and orient it at this point. Wall defines a skew-Hermitian form λ on the group of regular homotopy classes of immersions of 3-dimensional spheres into W, which roughly speaking is given by transversal double point intersections, where the two branches of the intersection are joined at the base point. This form takes values in Λ, and it is called the "equivariant intersection form" associated with W. Similarly, Wall assigns to each immersion u an element $\mu(u) \in \frac{\Lambda}{\langle a+\bar{a}\rangle}$, which is given by the self-intersection of the immersion. When we compose μ with the quotient map onto $\frac{\Lambda}{\langle a+\bar{a},1\rangle}$, we call the result $\tilde{\mu}$.

By [11, Prop. 4], we may assume that F is a 3-equivalence, and we identify $\pi_1(W)$ with $\mathbb{Z}/r$. For simplicity, we denote N by N_0 and M by N_1. Since (W, N_i) is 2-connected, $\pi_3(W, N_i)$ and $H_3(W, N_i; \Lambda)$ are isomorphic under the relative Hurewicz homomorphism. By Poincaré duality, it follows that $H_3(W, N_i; \Lambda)$ is the only possibly non-vanishing homology group of the pair (W, N_i), and by [16, Lemma 2.3.], $H_3(W, N_i; \Lambda)$ is a stably free Λ-module with a preferred equivalence class of s-basis (for a definition of s-basis, see [13, p.369]). If we take the connected sum of W with $\#_k(S^3 \times S^3)$ for some sufficiently large $k \in \mathbb{N}$, we may assume that $H_3(W, N_i; \Lambda)$ is a free Λ-module.

The intersection form $\lambda : H_3(W, N_0; \Lambda) \times H_3(W, N_1; \Lambda) \to \Lambda$ is unimodular, which follows from Poincaré duality. Furthermore, [16, Theorem 2.1.] tells us that this form is even simple if $H_3(W, N_0; \Lambda)$ and $H_3(W, N_1; \Lambda)$ are equipped with preferred bases.

Let $K\pi_3(W)$ be $\ker(F_* : \pi_3(W) \to \pi_3(B_r))$, $K\pi_3(N_i)$ be $\ker(f_{i_*} : \pi_3(N_i) \to \pi_3(B_r))$, and $\mathrm{Im}K\pi_3(N_i)$ be the image of $K\pi_3(N_i)$ under the homomorphism induced by the inclusion $N_i \hookrightarrow W$. We claim that $\mathrm{Im}K\pi_3(N_0) = \mathrm{Im}K\pi_3(N_1)$.

To prove this, suppose there exists an element $x \in \mathrm{Im}K\pi_3(N_0)$ that does not lie in $\mathrm{Im}K\pi_3(N_1)$. Then, by the homotopy exact sequence associated with (W, N_1), x represents a non-trivial element in $\pi_3(W, N_1)$. Since λ is non-degenerate and $\pi_3(W) \to \pi_3(W, N_0)$ is surjective, there exists a $y \in \pi_3(W)$ such that $\lambda(x, y) \neq 0$. However, since x is trivial in $\pi_3(W, N_0)$, we must have $\lambda(x, y) = 0$, which leads to a contradiction. By interchanging the roles of N_0 and N_1, we conclude that $\mathrm{Im}K\pi_3(N_0) = \mathrm{Im}K\pi_3(N_1)$.

The map $\pi_3(W) \to \pi_3(W, N_i)$ induces an isomorphism

$$\frac{K\pi_3(W)}{\mathrm{Im}K\pi_3(N_i)} \xrightarrow{\approx} \pi_3(W, N_i),$$

which can be seen using the following diagram:

$$\begin{array}{ccccccc} & & \pi_4(B, W) & & & & \\ & & \downarrow & \searrow & & & \\ \pi_3(N_i) & \to & \pi_3(W) & \to & \pi_3(W, N_i) & \to & 0 \\ & \searrow & \downarrow & & & & \\ & & \pi_3(B) & & & & \\ & & \downarrow & & & & \\ & & 0, & & & & \end{array}$$

where the diagonal maps are surjective.

We now explain what S stands for: By $S(W)$, we denote the subgroup of Λ that projects onto the image of μ restricted to $\mathrm{Im}K\pi_3(N_0)$. Since $\mathrm{Im}K\pi_3(N_0) = \mathrm{Im}K\pi_3(N_1)$, $S(W)$ is equal to $S(N_0 \times I) = S(N_1 \times I)$, and we simply denote this subgroup as S.

If we equip $\frac{K\pi_3(W)}{\mathrm{Im}K\pi_3(N_0)}$ with the basis induced by the preferred basis on $H_3(W, N_0; \Lambda)$, then the map $\pi_3(W) \to \pi_3(W, N_i)$, coming from the inclusion $W \hookrightarrow (W, N_i)$, induces the form

$$\bar{\lambda} : \frac{\pi_3(W)}{\mathrm{Im}K\pi_3(N_0)} \times \frac{\pi_3(W)}{\mathrm{Im}K\pi_3(N_0)} \to \Lambda,$$

where $\bar{\lambda}$ is the induced skew-Hermitian form on the quotient.

This is a unimodular skew-Hermitian form, and $(\bar{\lambda}, \tilde{\mu})$ represents an element $[(\bar{\lambda}, \tilde{\mu})] =: \Theta(W, F)$ in $L_6^{s,\tau}(\mathbb{Z}/r, S(N_0 \times I))$. This element does not depend on the choice of a normal bordism (W, F).

Finally, for every $\Theta \in L_6^{s,\tau}(\mathbb{Z}/r, S)$, there exists a 3-equivalence (M, f') and a bordism (W', F') between (N_0, f) and (M, f') such that $\Theta(W', F') = \Theta$. Moreover, (M, f') is, up to s-cobordism, completely determined by Θ. This result was proved in [16, Theorem 5.8] for the case of simple homotopy equivalences into a finite Poincaré complex, and the proof clearly extends to our situation, leading to the following lemma.

Lemma 9 *Up to normal s-cobordism, there is a 1–1 correspondence between elements* $\theta \in L_6^{s,\tau}(\mathbb{Z}/r, S)$ *and the bordism classes of normal 2-smoothings* $(N; g)$.

As above, Wall's proof for simple homotopy equivalences extends literally to our situation on 2-smoothings, technically explained in details in [11], leading to the following theorem.

Theorem 10 (W, F) *is bordant relative boundary to an s-cobordism if and only if* $\Theta(W, F) = 0$.

Now, we have a closer look on the so-called *multisignature*. The multisignature extends the notion of the ordinary signature of a quadratic form over the integers in a certain sense. The most general definition of the multisignature, which one can find in [17] or [16, p. 174] is applicable to unimodular forms over group rings of finite groups with either the standard or a non-standard involution. Since we deal with forms over group rings with trivial orientation since all manifolds are orientable we give a definition of the multisignature, which is equivalent to the general definition restricted to the orientable case [16, p. 175]:

Let $\mathcal{M}$ be a free Λ-module and $\lambda : \mathcal{M} \times \mathcal{M} \to \Lambda$ a non-degenerate skew-hermitian form. Furthermore, we denote by p_c the map from Λ to $\mathbb{Z}$, which sends $a_0 + \sum_{\alpha \in \mathbb{Z}/r \setminus \{0\}} a_\alpha$ to a_0. The composition $p_c \circ \lambda$ is a skew-hermitian $\mathbb{Z}$-valued non-degenerate form. We extend $p_c \circ \lambda$ to $\mathcal{M}^{\mathbb{C}} := \mathcal{M} \otimes_{\mathbb{Z}} \mathbb{C}$ in the obvious way, which yields a skew-hermitian unimodular $\mathbb{C}$-valued form $\lambda_{\mathbb{C}}$, i.e. $\lambda_{\mathbb{C}}(x, y) = -\overline{\lambda_{\mathbb{C}}(y, x)}$ for all $x, y \in \mathcal{M}^{\mathbb{C}}$.

It is clear that $\mathcal{M}^{\mathbb{C}}$ inherits a $\mathbb{Z}/r$-action from $\mathcal{M}$ and we easily realize that $\lambda_{\mathbb{C}}(x\alpha, y\alpha) = \lambda_{\mathbb{C}}(x, y)$, for all $x, y \in \mathcal{M}^{\mathbb{C}}$ and $\alpha \in \mathbb{Z}/r$. Now we choose a positive definite $\mathbb{Z}/r$-invariant hermitian form $\langle \cdot, \cdot \rangle_{\mathbb{Z}/r}$ on $\mathcal{M}^{\mathbb{C}}$: Let $(\tilde{z}_1, ..., \tilde{z}_l)$ be a complex basis of $\mathcal{M}^{\mathbb{C}}$ and let $\langle \cdot, \cdot \rangle$ be the standard hermitian form on $\mathcal{M}^{\mathbb{C}}$, i.e. $\left\langle \tilde{z}_i, \tilde{z}_j \right\rangle = \delta_{ij}$. We define the following $\mathbb{Z}/r$-invariant hermitian product: $\langle \cdot, \cdot \rangle_{\mathbb{Z}/r} := \sum_{\alpha \in \mathbb{Z}/r} \langle \alpha(\cdot), \alpha(\cdot) \rangle$. There is the following linear map A of $\mathcal{M}^{\mathbb{C}}$ to itself defined by $\lambda_{\mathbb{C}}(x, y) = \langle x, Ay \rangle$ for all $x, y \in \mathcal{M}^{\mathbb{C}}$. It follows that A is a skew-hermitian $\mathbb{Z}/r$-equivariant automorphism of $\mathcal{M}^{\mathbb{C}}$ and that the eigenvalues of A are purely imaginary and nonzero. Thus, $\mathcal{M}^{\mathbb{C}} = \mathcal{M}^{\mathbb{C}}_+ \oplus \mathcal{M}^{\mathbb{C}}_-$, where $\mathcal{M}^{\mathbb{C}}_\pm$ is the sum of the eigenspaces corresponding to positive multiples of $\pm i$. Since the eigenspaces $\mathcal{M}^{\mathbb{C}}_+$ and $\mathcal{M}^{\mathbb{C}}_-$ are $\mathbb{Z}/r$-invariant they define complex $\mathbb{Z}/r$-representations. We denote the characters of these $\mathbb{Z}/r$-representations by r_+ and r_- respectively.

Definition 11 (i) The multisignature of λ, which we denote by $MS(\lambda)$ is the element of the complex representation ring $RU(\mathbb{Z}/r)$ given by

$$r_+ - r_-.$$

(ii) Let g be an element of $\mathbb{Z}/r$, then the multisignature of g w.r.t. λ is the difference of the characters of $r_+(g)$ and $r_-(g)$, i.e. $MS(g,\lambda) := r_+(g) - r_-(g)$.

The multisignature is well defined. This follows from the following three facts:

(1) The characters depend continuously on the inner product.
(2) The space of all $\mathbb{Z}/r$-invariant hermitian products on $\mathcal{M}^{\mathbb{C}}$ is connected.
(3) The characters of a compact group are discrete.

Let r be odd. If $\mathbb{Z}/r$ is the trivial group one knows that the signature induces an isomorphism between $L^s_{4m}(\{1\})$ and $8\mathbb{Z}$. It is also true that the so called Arf-invariant gives an isomorphism between $L^s_{4m+2}(\{1\})$ and $\mathbb{Z}/2$. The multisignature is an important tool for distinguishing elements in $L^s_{2k}(\mathbb{Z}/r)$:

Theorem 12 *([16, Thm. 13A.4.])There is a decomposition*

$$L^s_{2k}(\mathbb{Z}/r) = L^s_{2k}(\{1\}) \oplus \widetilde{L}^s_{2k}(\mathbb{Z}/r),$$

where the multisignature maps $\widetilde{L}^s_{2k}(\pi)$ injectively to the characters (real or imaginary as appropriate) trivial on 1 and divisible by 4.

In the following we study the surgery obstruction group $L^{s,\tau}_6(\mathbb{Z}/r, S)$, and for further details we refer to Section 3.4 in [14]. First we have a closer look on $L^s_6(\mathbb{Z}/r)$:

As already indicated in Theorem 12 the functorial character of the Wall L-groups yields a decomposition of $L^s_6(\mathbb{Z}/r)$, i.e. $L^s_6(\mathbb{Z}/r) = L^s_6(\{1\}) \oplus \widetilde{L}^s_6(\mathbb{Z}/r)$. Let $[(\Lambda^d, \lambda, \mu)] =: E$ be an element in $L^s_6(\mathbb{Z}/r)$. The first coordinate of E with respect to the above decomposition is detected by the Arf-invariant in the following sense:

To an element $[(\Lambda^d, \lambda, \mu)]$ in $L^s_6(\mathbb{Z}/r)$ one can assign an element in $L^s_6(\{1\})$: As we identify Λ with $\mathbb{Z}^b$ (for some $b \in \mathbb{N}$) in a canonical way we regard Λ^d as a $\mathbb{Z}$-module. Let $\epsilon : \Lambda \to \mathbb{Z}$ be the augmentation map, which is a ring homomorphism and let $\tilde{\epsilon}$ be the projection of $\frac{\Lambda}{\{2a|a\in\Lambda\}}$ to $\frac{\epsilon(\Lambda)}{\epsilon(\{2a|a\in\Lambda\})} \cong \mathbb{Z}/2$.

We compose λ with ϵ and compose the quadratic refinement μ with $\tilde{\epsilon}$ then we obtain an integral non-degenerate skew-hermitian quadratic form, which represents an element e in $L^s_6(\{1\})$. We define $\mathrm{Arf}(E)$ to be the *classical Arf-invariant* of e. The second coordinate of E with respect to the decomposition above is according to Theorem 11 detected by the multisignature.

The difference between $L^s_6(\mathbb{Z}/r)$ and $L^s_6(\mathbb{Z}/r, S)$ comes from the different choices of form parameters. In order to understand $L^s_6(\mathbb{Z}/r, S)$ we do the same considerations as for $L^s_6(\mathbb{Z}/r)$. We observe that elements in $L^s_6(\mathbb{Z}/r, S)$ are uniquely determined by the multisignature since the Arf-invariant of elements in $L^s_6(\{1\}, S)$ is trivial.

If the multisignature of $\left(\bar{\lambda} : \Lambda^d \times \Lambda^d \to \Lambda, \bar{\mu}\right)$ is zero, then this implies that there is a basis $\mathcal{B}$ of Λ^d such that $(\bar{\lambda}, \bar{\mu})$ represents the zero-element in $L^s_6(\mathbb{Z}/r, S)$ and hence, in $L^{s,\tau}_6(\mathbb{Z}/r, S)$ if one chooses Λ^d with the basis $\mathcal{B}$. Let $(\bar{\lambda}, \bar{\mu})$ be the equivariant intersection form and the self-intersection associated to (W, F), which represents the surgery obstruction $\Theta(W, F)$ ([15, Sect. 2]). The notations $(\bar{\lambda}, \bar{\mu})$ and $(\bar{\lambda}, \bar{\mu})_{\mathcal{B}}$ shall indicate that Λ^d is equipped with a preferred basis and the basis $\mathcal{B}$ respectively.

By abuse of notation a canonically based (-1)-hyperbolic form over (Λ, S) is again denoted by $H^n_{-1}(\Lambda)$ (for some $n \in \mathbb{N}$). If $(\bar{\lambda}, \bar{\mu})_{\mathcal{B}}$ represents the zero element in

$L_6^s(\mathbb{Z}/r, S)$, then there is a $l \in \mathbb{N}$ such that $H_{-1}^l(\Lambda) \oplus (\bar{\lambda}, \bar{\mu})_{\mathcal{B}}$ is isomorphic to $H_{-1}^{l'}(\Lambda)$ for some $l' \in \mathbb{N}$, where the isomorphism is simple. Let B be the matrix of base change with respect to the bases in $H_{-1}^l(\Lambda) \oplus (\bar{\lambda}, \bar{\mu})$ and $H_{-1}^l(\Lambda) \oplus (\bar{\lambda}, \bar{\mu})_{\mathcal{B}}$. The element in $\mathrm{Wh}(\mathbb{Z}/r)$ that is represented by B is denoted by $\tau(B)$ and we define $\mathrm{MS}(\Theta(W, F))$ to be $\mathrm{MS}(\bar{\lambda}, \bar{\mu})$ and $\mathrm{MS}(\alpha, \Theta(W, F))$ to be $\mathrm{MS}(\alpha, \bar{\lambda}, \bar{\mu})$. These considerations imply

Proposition 13 $\Theta(W, F) = 0$ *if and only if* $\mathrm{MS}(\Theta(W, F))$ *and* $\tau(B)$ *are trivial.*

The proof of Theorem 3 in [11] implies the following

Theorem 14 *W is relative boundary bordant to an h-cobordism $\left(W_h; L^{a,b}, L^{a',b'}\right)$ if and only if the multisignature of the equivariant intersection pairing $\bar{\lambda}$ on $\frac{K\pi_3(W)}{\mathrm{Im}K\pi_3(L^{a,b})}$ is trivial. In this case the vanishing of the algebraic torsion $\tau(B)$ is equivalent to the vanishing of the Whitehead torsion of the inclusion $L^{a,b} \hookrightarrow W_h$.*

Before we begin with the proof of Theorem 4 we explain what the *equivariant $\mathbb{Z}/r$-signature* of a 6-dimensional bordism W, on which $\mathbb{Z}/r$ operates smoothly and in an orientation preserving way is. For more details we refer to [1].

We know that there is the following skew-hermitian form called the intersection form of W:

$$\lambda \;\; H^3(W, \partial W; \mathbb{Z}) \times H^3(W, \partial W; \mathbb{Z}) \to \mathbb{Z}.$$

This form is $\mathbb{Z}/r$-invariant, i.e. $\lambda(gu, gv) = \lambda(u, v)$. This follows from the assumption that $\mathbb{Z}/r$ acts on W in an orientation preserving way.

The radical $\mathrm{rad}(\lambda)$ of λ equals $\mathrm{Ker}(i^* : H^3(W, \partial W; \mathbb{Z}) \to H^3(W, \mathbb{Z}))$. We easily see that the form

$$\bar{\lambda} \;\; \frac{H^3(W, \partial W; \mathbb{Z})}{\mathrm{rad}(\lambda)} \times \frac{H^3(W, \partial W; \mathbb{Z})}{\mathrm{rad}(\lambda)} \to \mathbb{Z},$$

given by

$$\bar{\lambda}([u], [v]) := \lambda(u, v)$$

is a non-degenerate $\mathbb{Z}/r$-invariant skew-hermitian form.

In the following we denote $\frac{H^3(W,\partial W;\mathbb{Z})}{\mathrm{rad}(\lambda)}$ by $\hat{H}(W)$. Tensoring $\hat{H}(W)$ with $\mathbb{C}$ over $\mathbb{Z}$ yields a complex vector space, which we denote by $\hat{H}(W)^{\mathbb{C}}$. We extend $\bar{\lambda}$ to $\hat{H}(W)^{\mathbb{C}}$ in the following way:

$$\bar{\lambda}_{\mathbb{C}} \;\; \hat{H}(W)^{\mathbb{C}} \times \hat{H}(W)^{\mathbb{C}} \to \mathbb{C}, \;\; (x \otimes z_1, y \otimes z_2) \mapsto \bar{\lambda}(x, y) \cdot (z_1 \cdot z_2).$$

This complex valued quadratic form is a skew-hermitian $\mathbb{Z}/r$-invariant unimodular form.

As in the definition of the multisignature we obtain unitary $\mathbb{Z}/r$-representations r_i, r_{-i}, which lead us to

Definition 15 The $\mathbb{Z}/r$-signature $\text{sign}(\mathbb{Z}/r, W)$ of W is defined as

$$r_i - r_{-i} \in RU(\mathbb{Z}/r)$$

Let $g \in \mathbb{Z}/r$ then

$$\text{sign}(g, W) := r_i(g) - r_{-i}(g).$$

Proof of Theorem 4 We make use of the following proposition, which summarizes the results of Prop 2.5 and Prop. 2.6 in [15].

Proposition 16 *Let r be as in Theorem 4 and N, M be closed smooth oriented spin 5-manifolds with vanishing first Pontrjagin class and f and g normal 2-smoothings from N and M to B_r respectively. Then the following statements are equivalent:*

(i) (N, f) *and* (M, g) *represent the same element in* $\Omega_5^{Spin}(B_r)$.
(ii) $f_*[N] = g_*[M] \in H_5(B_r; \mathbb{Z})$.
(iii) *N and M are homotopy equivalent.*

To apply this proposition we have to show that the manifolds under consideration in Theorem 4 have vanishing first Pontrjagin class. Let N be such a manifold. Then N is homotopy equivalent to a manifold in $\mathcal{L}$ and by Lemma 2.1 in [15] the fundamental group operates trivially on the homotopy groups of its universal covering space, which is $S^2 \times S^3$. Let $p_1(S^2 \times S^3)$ and $p_1(N)$ be the first Pontrijagin class of $S^2 \times S^3$ and N resp. and $\pi^* : H^4(N; \mathbb{Z}) \to H^4(S^2 \times S^3; \mathbb{Z})$ be the induced map of the universal covering $\pi : S^2 \times S^3 \to N$ on the fourth intergral cohomology groups of the respective manifolds. Since we know that $p_1(S^2 \times S^3) = \pi^*(p_1(N))$ and that under these conditions $\pi^* : H^4(N; \mathbb{Z}) \to H^4(S^2 \times S^3; \mathbb{Z})$ is injective we can conclude from the fact that $S^2 \times S^3$ has vanishing first Pontrjagin class that this also holds for N.

Let us now assume that there exists a normal bordism (W, F) between normal 2-smoothings (N, f) and (N', f') and we may choose (W, F) such that F is a 3-equivalence (see [11, Prop. 4]). We identify $\pi_1(W)$ with $\mathbb{Z}/r$ and denote $\mathbb{Z}[\mathbb{Z}/r]$ by Λ. In the following we relate the multisignature of $\Theta(W, F)$ to the $\mathbb{Z}/r$-equivariant signature of $\widetilde{W}$. We equip W and $\widetilde{W}$ with basepoints b and $\tilde{b}$ respectively such that $\tilde{b}$ is a lift of b under the universal covering map. We want to study the unimodular skew-hermitian form

$$\bar{\lambda} : \frac{K\pi_3(W)}{\text{Im}K\pi_3(N)} \times \frac{K\pi_3(W)}{\text{Im}K\pi_3(N)} \to \Lambda$$

which comes from the intersection pairing defined on $K\pi_3(W)$.

Since $\pi_3(B_r) = 0$ we observe that $K\pi_3(W) := \text{Ker}(F_\star : \pi_3(W) \to \pi_3(B_r)) = \pi_3(W)$.

There is the following composition of maps

$$\pi_3(\widetilde{W}) \overset{\mathcal{H}}{\to} H_3(\widetilde{W}; \mathbb{Z}) \overset{(\cap[\widetilde{W}, \partial\widetilde{W}])^{-1}}{\to} H^3(\widetilde{W}, \partial\widetilde{W}; \mathbb{Z}) \to$$

$$\rightarrow \underbrace{\frac{H^3(\widetilde{W}, \partial \widetilde{W}; \mathbb{Z})}{\mathrm{Ker}(i^* : H^3(\widetilde{W}, \partial \widetilde{W}; \mathbb{Z}) \rightarrow H^3(\widetilde{W}; \mathbb{Z}))}}_{=:\hat{H}^3(\widetilde{W})}$$

which we call Φ, where $(\cap[\widetilde{W}, \partial \widetilde{W}])^{-1}$ is the inverse of the Poincaré-Lefschetz isomorphism. The map Φ is $\mathbb{Z}/r$-equivariant and since $\widetilde{W}$ is 1-connected we know that the Hurewicz map $\mathcal{H}$ is surjective. We have seen that the $\mathbb{Z}/r$-signature for $\widetilde{W}$ is defined for the non-singular pairing $\bar{\mathcal{I}} : \hat{H}^3(\widetilde{W}) \times \hat{H}^3(\widetilde{W}) \rightarrow \mathbb{Z}$ which comes from the cup-product-pairing on $H^3(\widetilde{W}, \partial \widetilde{W}; \mathbb{Z})$. We denote the intersection pairing on $K\pi_3(W) = \pi_3(W)$ by λ. Let us recall how λ was defined. For a detailed exposition of the following we refer to [16, Ch. 5].

Elements of $\pi_3(W)$ are regular homotopy classes of immersions of S^3 into W. For $\gamma_0, \gamma_1 \in \pi_3(W)$ we find representatives (i_0, w_0) and (i_1, w_1) such that the images of these maps intersect transversally only in finitely many points. We denote the set of intersection points by D. From γ_0 and γ_1 we obtain a well defined element in Λ, namely $\lambda(\gamma_0, \gamma_1) := \sum_{d \in D} \epsilon(d)\alpha(d)$, where $\epsilon(d) \in \{\pm 1\}$ and $\alpha(d) \in \mathbb{Z}/r$ ([16, p. 45]). There is a unique lift $\widetilde{i_j}$ of i_j to $\widetilde{W}$ determined by $\tilde{b}$. Let $\lambda_{\mathbb{Z}}\big([\widetilde{i_0}], [\widetilde{i_1}]\big)$ be the $\mathbb{Z}$-valued algebraic intersection number of the homology classes $[\widetilde{i_0}]$ and $[\widetilde{i_1}]$ then we may identify $\lambda(\gamma_0, \gamma_1)$ with $\sum_{\alpha \in \mathbb{Z}/r} \lambda_{\mathbb{Z}}([\widetilde{i_0}], [l_{\alpha^{-1}} \circ \widetilde{i_1}])\alpha$, where $l_{\alpha^{-1}}$ denotes the left multiplication by α^{-1}. This means that

$$p_c \circ \lambda(i_0, i_1) = \lambda_{\mathbb{Z}}([\widetilde{i_0}], [\widetilde{i_1}]).$$

But $\lambda_{\mathbb{Z}}([\widetilde{i_0}], [\widetilde{i_1}]) = [\widetilde{i_0}]^* \cup [\widetilde{i_1}]^* \in H^6(\widetilde{W}, \partial \widetilde{W}; \mathbb{Z})$, where $[\widetilde{i_j}]^*$ denotes the Poincaré-Lefschetz dual of $[\widetilde{i_j}]$. All in all we obtain

$$p_c \circ \lambda = \bar{\mathcal{I}}(\Phi(\cdot), \Phi(\cdot)).$$

Thus, the radicals of $p_c \circ \lambda$, λ and $\bar{\mathcal{J}}(\Phi(\cdot), \Phi(\cdot))$ coincide and are equal to $\mathrm{Ker}(\Phi)$, which is $\mathrm{Im} K\pi_3(N)$. From the definitions of the multisignature and the equivariant signature we conclude:

Computing the multisignature of $\bar{\lambda}$ is the same as computing the $\mathbb{Z}/r$-signature of $\widetilde{W}$.

Thus,

$$MS(\alpha, \Theta(W, F)) = \mathrm{sign}(\alpha, \widetilde{W}), \quad \forall \alpha \in \mathbb{Z}/r. \tag{2}$$

Let $\alpha \in \mathbb{Z}/r \setminus \{0\}$ then from (2) and Novikov's additivity formula for the equivariant signature it follows that

$$MS(\alpha, \Theta(W, F)) = \rho(\alpha, N) - \rho(\alpha, N').$$

Since $\widetilde{W}$ is 6-dimensional it follows that $\operatorname{sign}(\widetilde{W})$ is trivial and thus,

$$MS(\alpha, \Theta(W, F)) = 0, \quad \forall \alpha \in \mathbb{Z}/r$$

if and only if

$$\rho(\beta, N') = \rho\big((f_*^{-1} \circ f'_*)(\beta), N\big), \forall \beta \in \pi_1(N') \setminus \{0\}. \tag{3}$$

If the ρ-invariant condition holds then we know from Theorem 14 that W is normally bordant to an h-cobordism.

Since N and N' are homotopy equivalent to manifolds in $\mathcal{L}$, having cyclic fundamental groups, $\pi_1(N)$ and $\pi_1(N')$ operate trivially on the rational cohomology ring of the universal covering spaces $\widetilde{N}$ and $\widetilde{N'}$ respectively. These facts imply that the Reidemeister torsions $\Delta(N)$ and $\Delta(N')$ according to [13, p. 405] are defined. Since by assumption $\Delta(N) \sim \Delta(N')$ [13, Thm. 12.8] imply that the Whitehead torsion of the inclusion of one boundary component into this h-cobordism is trivial. By applying the s-cobordism theorem N and N' are diffeomorphic.

The theorem follows by the following considerations: If there is a homotopy equivalence $h : N' \to N$ and $f : N \to B_r$ is a normal 2-smoothing then $f' := f \circ h : N' \to B_r$ is a normal 2-smoothing of N'. Condition (i) in Theorem 4 is equivalent to condition (3). And by the considerations above N and N' are diffeomorphic if conditions (i) and (ii) of Theorem 4 hold. □

Our next aim is to classify the non-simply connected manifolds in $\mathcal{L}$, where the order of the fundamental group is coprime to 6. The orientation convention for the manifolds in $\mathcal{L}$ is the following: Since these manifolds are total spaces of principal S^1-fibre bundles over $S^2 \times S^2$ we orient them by orienting the base and the fibre in the standard way.

Proof of Theorem 5

Theorem 17 *([15, Thm. 8]) Let r be as in Theorem 4 and $L^{a,b}$, $L^{a',b'} \in \mathcal{L}$ with $\pi_1(L^{a,b}) \cong \pi_1(L^{a',b'}) \cong \mathbb{Z}/r$. Furthermore, let (m, n), (m', n') be pairs of integers such that $m\frac{b}{r} + n\frac{a}{r} = 1 = m'\frac{b'}{r} + n'\frac{a'}{r}$. Then $L^{a,b}$ and $L^{a',b'}$ are oriented homotopy equivalent if and only if there exist $s, s' \in (\mathbb{Z}/r)^*$, $\epsilon, \epsilon' \in \{\pm 1\}$ and $k, k' \in \mathbb{Z}/r$ such that*

$$s^3 \frac{ab}{r^2} \equiv s'^3 \frac{a'b'}{r^2} \bmod r,$$

$$s\left(\epsilon m + k\frac{a}{r}\right)\left(\epsilon n - k\frac{b}{r}\right) \equiv s'\left(\epsilon' m' + k'\frac{a'}{r}\right)\left(\epsilon' n' - k'\frac{b'}{r}\right) \bmod r,$$

$$s^2\left(\frac{b}{r}\left(\epsilon m + k\frac{a}{r}\right) - \frac{a}{r}\left(\epsilon n - k\frac{b}{r}\right)\right) \equiv s'^2\left(\frac{b'}{r}\left(\epsilon' m' + k'\frac{a'}{r}\right) - \frac{a'}{r}\left(\epsilon n' - k'\frac{b'}{r}\right)\right) \bmod r.$$

If we change the orientation of one of the manifolds in Theorem 5 and Theorem 17 then we have to insert a minus sign on the corresponding side of the congruences. Since it is important for later discussions we explain where the parameters ϵ, k and s come from:

Let us fix a choice of tuples (m, n) as in Theorem 17. By [15, Lem. 15] we can explicitly give a 1–1 correspondence between the set of homotopy classes of normal 2-smoothings of $L^{a,b}$ and the set of triples

$$\left\{(\epsilon, k, s) | \epsilon \in \{\pm 1\}, k \in \mathbb{Z}/r, s \in (\mathbb{Z}/r)^*\right\}$$

by sending a normal 2-smoothing $g : L^{a,b} \to B_r$ to

$$g^*(v_1) = s\alpha, \tag{4}$$

$$g^*(z) = \epsilon\left(m\Pi^*_{a,b}(x) - n\Pi^*_{a,b}(y)\right) + k\left(\frac{a}{r}\Pi^*_{a,b}(x) + \frac{b}{r}\Pi^*_{a,b}(y)\right), \tag{5}$$

where $\Pi_{a,b}$ is the projection map of the associated S^1-fibre bundle of $L^{a,b}$ and v_1 and z are generators of $H^1(L_r^\infty; \mathbb{Z}/r)$ and $H^2(\mathbb{CP}^\infty; \mathbb{Z})$ respectively.

We know that the universal covering space of $L^{a,b}$ is $L^{\frac{a}{r}\frac{b}{r}}$, and that the deck transformation on $D^{\frac{a}{r}\frac{b}{r}}$ by $\pi_1(L^{a,b})$ is given by fibrewise rotation by angles corresponding to the r'th roots of unity. This perspective yields a canonical identification of $\pi_1(L^{a,b})$ with $\mathbb{Z}/r$. The disc bundle $D^{\frac{a}{r}\frac{b}{r}}$ associated to the S^1-fibre bundle structure with the $\mathbb{Z}/r$-action canonically extended serves as a convinient choice of a bordism. Furthermore, this $\mathbb{Z}/r$-bordism has trivial equivariant signature since on the one hand the $\mathbb{Z}/r$-action is homotopically trivial, as it sits in an S^1-action and on the other hand the dimension of the bordism is not divisible by 4, which means that the ordinary signature is trivial. The fixed point set is just $S^2 \times S^2$ and the normal bundle of the fixed point set is isomorphic to the 2-dimensional real vector bundle given by the Euler class $\frac{a}{r}x + \frac{b}{r}y$. Let g be a non-trivial element of $\mathbb{Z}/r$ and θ_g the rotation angle between 0 and π of the action by g then the ρ-invariant associated to the action of g is defined to be the evaluation of certain characteristic polynomials depending on the Chern-, Pontrjagin classes of the normal bundle $\mathcal{N}$ of $S^2 \times S^2$ and the Pontrjagin classes of $S^2 \times S^2$, on the fundamental class of $S^2 \times S^2$:

$$\left(i \tan \frac{\theta_\beta}{2}\right)^{-1} \sum_{j=0} \mathcal{L}_j\left(S^2 \times S^2\right) \sum_r \mathcal{M}_r^{\theta_\beta}(\mathcal{N}_{\theta_\beta}) \left[S^2 \times S^2\right]_\pm ,$$

where $\left[S^2 \times S^2\right]_\pm$ is ± 1 times the standard fundamental class of $S^2 \times S^2$. The sign in $\left[S^2 \times S^2\right]_\pm$ depends on how β operates on $L^{\frac{a}{r},\frac{b}{r}}$ and we conclude that

$$\rho\left(-g, L^{\frac{a}{r},\frac{b}{r}}\right) = -\rho\left(g, L^{\frac{a}{r},\frac{b}{r}}\right). \tag{6}$$

The formula for the ρ-invariant is the following:

$$\rho : \pi_1\big(L^{a,b}\big) \setminus \{0\} \to \mathbb{C}, \quad g \mapsto -i \frac{\cos\left(\frac{\theta_g}{2}\right)}{2r^2 \sin^3\left(\frac{\theta_g}{2}\right)} ab \tag{7}$$

([14, p. 88]), which is clearly injective if $ab \neq 0$. If the ρ-invariant of $L^{a,b}$ and $L^{a',b'}$ shall coincide then the non-triviality of (7) implies that $|ab| = |a'b'|$. And (6) together with the injectivity of (7) implies that $s' = \delta s$ if $ab = \delta a'b'$, where $\delta \in \{\pm 1\}$. Hence, the conditions of Theorem 4 are fulfilled for $L^{a,b}$ and $L^{a',b'}$ if and only if there exists $m, m', n, n', k, k', \epsilon, \epsilon'$ and $\delta \in \{\pm 1\}$ as stated in Theorem 17 such that

$$\frac{ab}{r^2} = \delta \frac{a'b'}{r^2},$$

$$\left(\epsilon m + k\frac{a}{r}\right)\left(\epsilon n - k\frac{b}{r}\right) \equiv \delta \left(\epsilon' m' + k'\frac{a'}{r}\right)\left(\epsilon' n' - k'\frac{b'}{r}\right) \bmod r,$$

$$\begin{aligned} \frac{b}{r}\left(\epsilon m + k\frac{a}{r}\right) - \frac{a}{r}\left(\epsilon n - k\frac{b}{r}\right) \equiv \underbrace{\delta^2}_{=1} \Bigg(\frac{b'}{r}\left(\epsilon' m' + k'\frac{a'}{r}\right) \\ -\frac{a'}{r}\left(\epsilon n' - k'\frac{b'}{r}\right)\Bigg) \bmod r. \end{aligned}$$

Proof of Theorem 6 (i) The idea of the proof of Theorem 6 (i) is the following: We prove for $L^{r,qr}$ that there is a self-homotopy equivalence h with trivial normal invariant and a primitive element y of $H^2(L^{r,qr}; \mathbb{Z})$ such that there is no self-diffeomorphism ϕ such that

$$h^*(y) = \pm \phi^*(y). \tag{8}$$

Since h has trivial normal invariant the total spaces of the complex line bundles with first Chern class y and $h^*(y)$ respectively are diffeomorphic (proof of Theorem 1). But if there exists a self-diffeomorphism of the total space, which sends the zero-section of the one bundle to the zero-section of the other then ϕ would preserve the normal bundle structure, i.e. formula (8) would hold, which can not be the case.

Let (W, F) be a normal bordism between $(L^{r,qr}, f_0)$ and $(L^{r,qr}, f_1)$, where $f_0, f_1 : L^{r,qr} \to B_r$ are normal 2-smoothings. We may assume that F is a 3-equivalence. As we have already seen the surgery obstruction $\Theta(W, F)$ is determined by its multisignature and may be canonically identified with an element in $L^s(\mathbb{Z}/r, S)$. As already mentioned to any element Θ' of $L^{s,\tau}(\mathbb{Z}/r, S)$ there exists a normal bordism (W', F') between $(L^{r,qr}, f_0)$ and (N, f'), where f' is a normal 2-smoothing and (N, f') is unique up to normal s-cobordism. Let us take the element Θ_s of $L^s(\mathbb{Z}/r, S)$, which has the same multisignature (up to isomorphism) as $\Theta(W, F)$.

Then Wall's surgery implies: There exists

(i) a bordism $(W_s; L^{r,qr}, N_s)$,
(ii) a map $F_s : W_s \to L^{r,qr} \times I$, where $F_s|_{L^{r,qr}}$ is the identity and $F_s|_{N_s}$ is a simple homotopy equivalence and
(iii) a stable normal framing Φ_s of $F_s^* \nu_{L^{r,qr}} \oplus \tau_{W_s}$, where $\nu_{L^{r,qr}}$ is the stable normal bundle of $L^{r,qr}$.

$F_s|_{N_s}$ is unique up to normal s-cobordism. And thus, Θ_s maps to an element in $S^s(L^{r,qr})$, the simple structure set of $L^{r,qr}$. Let $F_0 : L^{r,qr} \times I \to B_r$ be a homotopy of f_0 then $F_0 \circ F_s : W_s \to B_r$ is normal bordism between $(L^{r,qr}, f_0)$ and $(N_s, F_0 \circ F_s|_{N_s})$. These considerations together with Lemma 9 yield the following

Lemma 18 *(N_1, f_1) and $(N_s, F_0 \circ F_s|_{N_s})$ differ by a normal s-cobordism thus, the map f_1 can be obtained from f_0 by precomposing with a simple self-homotopy equivalence with trivial normal invariant.*

Let us have a look on Theorem 17 in the case where $L^{a,b} = L^{a',b'} = L^{r,qr}$ and we choose $m = 0$ and $n = 1$. Recall that the further parameters, which are used in Theorem 17 determine the images of generators of $H^1(L_r^\infty; \mathbb{Z}/r)$ and $H^2(\mathbb{C}P^\infty; \mathbb{Z})$ under the map, which is induced by a normal 2-smoothing ((4),(5)) and if we choose the opposite orientation we have to insert a minus to the corresponding sides of the congruences. We claim that if q is not zero but divisible by r then the congruences in Theorem 17 hold for: $s, s' = 1$, $\epsilon = 1$, $\epsilon' = -1$, $k = -1$ and $k' = s$: The first equation is trivially fulfilled since $q \equiv 0 \bmod r$. The other two congruences are:

$$sk\epsilon \equiv s'k'\epsilon' \bmod r, \tag{9}$$
$$s^2\epsilon \equiv s'^2\epsilon' \bmod r. \tag{10}$$

We realize that these congruences hold for the above choice of parameters. Hence, by Lemma 18 there is a self-homotopy equivalence h with trivial normal invariant such that

$$h^*\left(m\Pi^*_{r,rq}(x) - n\Pi^*_{r,rq}(y) - \Pi^*_{r,rq}(x) + q\Pi^*_{r,rq}(y)\right)$$

equals

$$-\left(m\Pi^*_{r,rq}(x) - n\Pi^*_{r,rq}(y)) + s(\Pi^*_{r,rq}(x) + q\Pi^*_{r,rq}(y)\right)$$

which obviously is not ± 1 times its preimage. But if there was a self-diffeomorphism ϕ fulfilling the same cohomological property as h then we could construct a normal s-cobordism:

Glue $W_1 := \left(L^{r,qr} \times I_1; L^{r,qr} \times \{0\}, L^{r,qr} \times \{1\}\right)$ and $W_2 := \left(L^{r,qr} \times I_2; L^{r,qr} \times \{0\}, L^{r,qr} \times \{1\}\right)$ together by identifying $L^{r,qr} \times \{1\}$ with $L^{r,qr} \times \{0\}$ via ϕ. We call the result $W_1 \cup_\phi W_2$ and by D we denote the canonical diffeomorphism from $W_1 \cup_\phi W_2$ to $L^{r,qr} \times I$. Let $f : L^{r,qr} \to B_r$ be the normal 2-smoothing, which corresponds to the choice of the parameters s, ϵ and k ((3),(4)) and $F : L^{r,qr} \times I \to B_r$ be a homotopy of f. Then $(W_1 \cup_\phi W_2, F \circ D)$ is a normal s-cobordism between $(L^{r,qr}, f)$ and $\left(L^{r,qr}, F \circ D|_{L^{r,qr}\times\{1\}}\right)$, where $F \circ D|_{L^{r,qr}\times\{1\}}$ is the normal 2-smoothing from $L^{r,qr}$ to B_r, which corresponds to the choice of the parameters s', ϵ' and k'. But since $r^2 q \neq 0$ the ρ-invariant is injective and since $(W_1 \cup_\phi W_2, F \circ \delta)$ is a normal s-cobordism the equations in Theorem 17 should hold for $s = \pm s'$ and ϵ', k' as above (proof of Theorem 17). Equation (9) implies that $s = k' = \pm k = \pm 1$ but $s^2 = -1$, which is a contradiction.

If there is a non-trivial unit s in $\mathbb{Z}/r$ with $s^2 = 1$ then we choose the parameters as follows:s, $s' = 1, \epsilon = 1 = \epsilon', k = 1$ and $k' = s$. They fulfill the congruences (9) and (10) and the statement of Theorem 6 (i) follows from the considerations above. □

Proof of Theorem 6 (ii) If $(N_x^{r,qr}, S)$ and $(N_x^{r,qr}, S')$ are diffeomorphic then trivially S, S' are diffeomorphic.

From [5, Cor. 5.12] it follows that there is a homotopy equivalence $h : S \to S'$, which pulls back the normal bundle.

If q is zero then we realize that since the ρ-invariant is zero any isomorphism on the second cohomology group induced by a self-homotopy equivalence can be induced by a self-diffeomorphism. This follows from the proof of Theorem 17 and Lemma 18, where the simple homotopy equivalence is a diffeomorphism.

If q is not divisible by three and there is no non-trivial unit s in $\mathbb{Z}/r$ such that $s^3 = 1$ then the first equation in Theorem 17 implies that $s' = \pm s$ depending on the chosen orientations. But the above considerations imply that any set of parameters with $s' = \pm s$ fulfilling the equations in Theorem 17 also fulfills the equations in Theorem 5. Hence, again any isomorphism on the second cohomology group induced by a self-homotopy equivalence can be induced by a self-diffeomorphism. □

Corollary 19 *Let r be as in Theorem 1. All the manifolds in $\mathcal{L}$, which are homotopy equivalent to $L^{a,b}$ with $\pi_1(L^{a,b}) \cong \mathbb{Z}/r$ lie in the orbit of the basepoint of the simple structure set of $L^{a,b}$ under the action of $L_6^s(\mathbb{Z}/r)$. Thus, the homotopy equivalences between manifolds in $\mathcal{L}$ may be chosen in such a way that they have trivial normal invariant.*

Proof If $L^{a,b}$ and $L^{a',b'}$ are homotopy equivalent then there exists a normal bordism (W, F) between normal 2-smoothings of $L^{a,b}$ and $L^{a',b'}$. The considerations before Lemma 9 and Lemma 18 imply that there exists a simple homotopy equivalence between $L^{a,b}$ and $L^{a',b'}$. □

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