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Inviscid incompressible limit for capillary fluids with density-dependent viscosity

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Abstract. The asymptotic limit of the Navier–Stokes–Korteweg system for barotropic capillary fluids with density-dependent viscosities in the low-Mach number and vanishing viscosity regime is established in $\mathbb{R}^d$, with $d = 2, 3$. In the relative energy framework, we prove the convergence of weak solutions of the Navier–Stokes–Korteweg system to the strong solution of the incompressible Euler system. The convergence is obtained through the use of suitable dispersive estimates for an acoustic system altered by the presence of the Korteweg tensor.

Contents

Keywords: Barotropic compressible fluids, Density-dependent viscosity, Navier–Stokes–Korteweg model, Capillary fluids, Inviscid incompressible limit.

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1. Introduction

The aim of this paper is to characterize the asymptotic limit in the low-Mach and vanishing viscosity limit regime of the following (re-scaled) compressible Navier–Stokes–Korteweg system in $(0, T) \times \mathbb{R}^3$:

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{\nabla p(\varrho)}{\varepsilon^2} - 2\nu \operatorname{div}(\varrho \mathbb{D}(\mathbf{u})) - 2\kappa^2 \varrho \nabla \Delta \varrho = 0, \end{cases} \tag{NSK}$$

complemented with far-field behaviour

$$\varrho \to 1, \quad \sqrt{\rho}\mathbf{u} \to 0, \quad \text{as } |x| \to \infty, \tag{1.1}$$

and the initial conditions

$$\varrho(0, \cdot) = \varrho_0, \qquad \varrho \mathbf{u}(0, \cdot) = \varrho_0 \mathbf{u}_0. \tag{1.2}$$

The unknown variables $\varrho = \varrho(t, x)$, $\mathbf{u} = \mathbf{u}(t, x)$ and $p = p(\varrho)$ represent the mass density, the velocity vector and the pressure of the fluid, respectively. This last is given by a standard power law type

$$p(\varrho) = \varrho^\gamma, \qquad \gamma > 1, \tag{1.3}$$

while

$$\mathbb{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla^\top \mathbf{u})$$

is the symmetric part of $\nabla \mathbf{u}$. The Korteweg tensor $2\kappa^2 \varrho \nabla \Delta \varrho$ can be recast in the form $\varrho \nabla \Delta \varrho = \operatorname{div} \mathbb{K}$ where

$$\mathbb{K} = \left(\varrho \operatorname{div}(\nabla \varrho) + \frac{1}{2}|\nabla \varrho|^2 \right) \mathbb{I} - (\nabla \varrho \otimes \nabla \varrho). \tag{1.4}$$

Fluids for which the Korteweg tensor has the form as in (1.4) are usually termed capillary fluids (see, for example, [5] and references therein). The parameter ε represents the Mach number, while ν and κ are the viscosity and capillary coefficients, respectively. The associated energy is given by

$$E(\varrho, \mathbf{u})(\tau) = \int_{\mathbb{R}^3} \left[\frac{\varrho |\mathbf{u}|^2}{2} + H(\varrho) + \kappa^2 |\nabla \varrho|^2 \right](\tau) dx, \tag{1.5}$$

$$H(\varrho) = \frac{1}{\gamma - 1}(\varrho^\gamma - 1 - \gamma(\varrho - 1)), \tag{1.6}$$

where $H(\varrho)$ takes the far-field behaviour into account.

1.1. Singular limit, scope of the present analysis

Formally, in the limit of $\varepsilon \to 0$ and $\nu \to 0$, we obtain the incompressible Euler system in the whole space

$$\begin{cases} \operatorname{div}_x \mathbf{u}^E = 0, \\ \partial_t \mathbf{u}^E + \mathbf{u}^E \cdot \nabla \mathbf{u}^E + \nabla \Pi = 0, \end{cases} \tag{E}$$

with initial conditions

$$\mathbf{u}^E(0, \cdot) = \mathbf{u}_0^E. \tag{1.7}$$

More precisely, in the present analysis we are interested to prove the convergence of the weak solution of (NSK) to the strong solution of (E) in the limit of $\varepsilon \to 0$ and $\nu \to 0$, namely in the low-Mach (incompressible) and vanishing viscosity limit. The key difficulty of our analysis consists in tackling the singular limit in the presence of a highly nonlinear capillarity tensor, a density-dependent viscosity and disturbing effects caused by fast oscillating acoustic waves. The occurrence of the latter is due to the choice of the so-called ill-prepared initial data (see Sect. 2.3).

The convergence will be obtained within the relative energy inequality framework; we refer, for instance, to [18] and references therein for a comprehensive overview of the method. To the best of our knowledge, the present is the first result addressing the inviscid incompressible limit for capillary fluids and could be seen as a continuation of a previous analysis in which a weak-strong uniqueness result (for a fixed viscosity coefficient $\nu > 0$), together with the high-Mach number limit, has been recently obtained; see [15].

To achieve our result, we thus need the following:

- A suitable form of the relative energy inequality that captures structural properties of compressible flow including the capillarity and density-dependent viscosity tensors, in particular, in the vanishing viscosity limit.
- Appropriate dispersive estimates for the decay of the acoustic waves in the low-Mach number limit that take the capillarity effects into account.

Regarding the first point, in [15] the authors derived a relative energy inequality based on an "augmented" version of the system (NSK) in the same spirit of, e.g. [11–13], where an equation for a velocity of the type $\mathbf{v} = 2\nu\nabla \log(\varrho)$ is properly introduced. In particular, defining the "augmented" velocity as $\mathbf{w} = \mathbf{u} + \mathbf{v}$, the "augmented" system reads as follows:

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \\ \partial_t(\varrho \mathbf{w}) + \operatorname{div}(\varrho \mathbf{w} \otimes \mathbf{u}) + \nabla p(\varrho) - 2\nu \operatorname{div}(\varrho \nabla \mathbf{w}) + 2\nu \operatorname{div}(\varrho \nabla \mathbf{v}) - 2\kappa^2 \varrho \nabla \Delta \varrho = 0, \\ \partial_t(\varrho \mathbf{v}) + \operatorname{div}(\varrho \mathbf{v} \otimes \mathbf{u}) + 2\nu \operatorname{div}(\varrho \nabla^t \mathbf{u}) = 0. \end{cases} \tag{NSKa}$$

The use of the "augmented" system is required due to the fact that an H^1 bound for the velocity is no longer available because of the density-dependent viscosity. Consequently, standard application of the Korn's inequality to handle, in particular, the viscous terms in the usual weak-strong uniqueness framework is not possible. The authors in [7] (see also [8]) used this "augmented" version to study, for the first time, the vanishing viscosity limit for a barotropic fluid with density-dependent viscosity. The result has been obtained for a fluid in smooth bounded domain and without the presence of the Korteweg term. However, a recent analysis (see [9]) shows that it is possible to completely avoid the "augmented" version of the system NSK using a suitable form of the relative energy inequality. Consequently, being the proper framework to perform the vanishing viscosity limit, our relative energy inequality will be consistent with the one considered in [9].

On the other hand, the analysis of the incompressible limit requires a suitable control of the acoustic waves which presence is due to (potential) density fluctuations at finite Mach number. Being the problem (NSK) posed on $\mathbb{R}^3$, suitable dispersive estimates yield the decay of the acoustic waves in the low-Mach number limit. Here, we take the presence of the nonlinear capillarity tensor $2\kappa^2\rho\nabla\Delta\rho$ depending on third-order derivatives of the density into account. Upon linearizing system (NSK) around the constant state ($\rho = 1, u = 0$), it is possible to obtain a fourth-order acoustic wave equation for the density fluctuations $\sigma_\varepsilon = \varepsilon^{-1}(\varrho_\varepsilon - 1)$, namely

$$\partial_{tt}^2\sigma_\varepsilon - \frac{1}{\varepsilon^2}\Delta(1 - 2\varepsilon^2\kappa^2\Delta)\sigma_\varepsilon = 0. \tag{1.8}$$

The presence of the Korteweg tensor modifies the dispersion relation governing the propagation of the acoustic waves with respect to the usual dispersive estimates obtained for the classical wave equation, namely in the absence of the capillarity tensor ($\kappa = 0$), and is reminiscent to the acoustic oscillations occurring in quantum fluids [1,2]. Consequently, the adaptation of the techniques developed in [1,2] allows us to derive suitable estimates for the present analysis (see Sect. 4).

1.2. Overview of previous results

Other results in the context of Korteweg fluids concern the low-Mach number limit for quantum fluids (see [1,2,21]) in which the authors study a weak-weak convergence towards the weak solution of the quantum incompressible Navier–Stokes system.

A similar singular limit analysis for the Korteweg type fluids has also been performed in the case of the quasi-neutral limit; see [10,17]. In the framework of strong solutions, we can refer to [19,20,23,24,28,31].

1.3. Organization of the paper

The manuscript is organized as follows: in Sect. 2, we discuss the existence of weak solutions for the Navier–Stokes–Korteweg system (NSK) together with the existence of the strong solution of the Euler system (E). Then, we present our main result.

Section 3 is devoted to the uniform bounds. In Sect. 4, we introduce the acoustic system related to the Navier–Stokes–Korteweg system (NSK) and the dispersive estimates. In Sect. 5, we derive the relative energy inequality suitable for our analysis. In Sect. 6, we study the incompressible and vanishing viscosity limit in the whole space $\mathbb{R}^3$. Section 7 is devoted to some comments and observation that concern the incompressible and vanishing viscosity limit in the 2D case.

1.4. Notation

We denote by $C_c^\infty([0,T)\times\mathbb{R}^3;\mathbb{R}^d)$ the space of periodic smooth functions with values in $\mathbb{R}^d$ with compact support in $[0,T)\times\mathbb{R}^3$ and by $L^p(\mathbb{R}^3)$ the standard Lebesgue spaces. The Sobolev spaces of functions with s distributional derivative in $L^p(\mathbb{R}^3)$ are $W^{s,p}(\mathbb{R}^3)$. In the case, $p=2$, $W^{s,p}(\mathbb{R}^3)=H^s(\mathbb{R}^3)$. The Bochner spaces for time-dependent functions with values in Banach spaces X are denoted by $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$. The space $C(0,T;X_w)$ is the space of continuous functions endowed with the weak topology. By $\mathbf{Q}$ and $\mathbf{P}$, we denote the Helmholtz–Leray projectors on irrotational and divergence-free vector fields, respectively:

$$\mathbf{Q}=\nabla\Delta^{-1}\mathrm{div},\qquad \mathbf{P}=\mathbf{I}-\mathbf{Q}. \tag{1.9}$$

The Helmholtz–Leray projectors are defined as Riesz multipliers and as such bounded linear operators on $W^{s,p}(\mathbb{R}^d)$ for $1<p<\infty$ and $s\in\mathbb{R}$, see, for example, [32, Chapter 7].

2. Background and main result

We introduce the concepts of weak solutions for the Navier–Stokes–Korteweg system (NSK) and of strong solutions for the target Euler system (E). For both systems, solutions are considered in $\mathbb{R}^3$ with given far-field condition (1.1).

2.1. Existence of weak solutions for capillary fluids

Definition 2.1. A triple $(\varrho,\mathbf{u},\mathcal{T})$, with $\varrho\geq 0$, is said to be a weak solution to (NSK) with initial data (1.2) if the following conditions are satisfied:

(i) Integrability conditions:

$$\varrho\in L^\infty(0,T;H^1_{\mathrm{loc}}(\mathbb{R}^3))\cap L^2(0,T;H^2_{\mathrm{loc}}(\mathbb{R}^3)),\qquad \sqrt{\varrho}\mathbf{u}\in L^\infty(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^3)),$$
$$\varrho^{\frac{\gamma}{2}}\in L^\infty(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^3))\cap L^2(0,T;H^1_{\mathrm{loc}}(\mathbb{R}^3)),\qquad \nabla\sqrt{\varrho}\in L^\infty(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^3)),$$
$$\mathcal{T}\in L^2(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^3)),\qquad \varrho\mathbf{u}\in C([0,T);L^{\frac{3}{2}}_{\mathrm{loc,w}}(\mathbb{R}^3)).$$

(ii) Continuity equation: for any $\varphi\in C_c^\infty([0,T]\times\mathbb{R}^3;\mathbb{R})$ and for all $\tau\in[0,T]$,

$$-\int_0^\tau\int_{\mathbb{R}^3}\left(\varrho\partial_t\varphi+\sqrt{\varrho}\sqrt{\varrho}\mathbf{u}\cdot\nabla\varphi\right)dxdt=\int_{\mathbb{R}^3}\varrho(0,\cdot)\varphi(0,\cdot)dx-\int_{\mathbb{R}^3}\varrho(\tau,\cdot)\varphi(\tau,\cdot)dx. \tag{2.1}$$

(iii) Momentum equation: for any fixed $l = 1, 2, 3$, $\phi \in C_c^\infty([0,T] \times \mathbb{R}^3; \mathbb{R}^3)$ and for all $\tau \in [0, T]$,

$$-\int_0^\tau \int_{\mathbb{R}^3} \sqrt{\varrho}\sqrt{\varrho}\mathbf{u} \cdot \partial_t \phi \, dxdt - \int_0^\tau \int_{\mathbb{R}^3} (\sqrt{\varrho}\mathbf{u} \otimes \sqrt{\varrho}\mathbf{u}) : \nabla\phi \, dxdt$$
$$+2\nu \int_0^\tau \int_{\mathbb{R}^3} \sqrt{\varrho}\mathcal{S}(\mathbf{u}) : \nabla\phi \, dxdt - \frac{1}{\varepsilon^2} \int_0^\tau \int_{\mathbb{R}^3} p(\varrho)\mathrm{div}\phi \, dxdt$$
$$-2\kappa^2 \int_0^\tau \int_{\mathbb{R}^3} \left(\nabla\varrho \cdot \nabla(\varrho \mathrm{div}\phi) - \frac{1}{2}|\nabla\varrho|^2 \mathrm{div}\phi + \nabla\varrho \otimes \nabla\varrho : \nabla\phi \right) dxdt$$
$$= \int_{\mathbb{R}^3} (\varrho\mathbf{u} \cdot \phi)(0, \cdot)dx - \int_{\mathbb{R}^3} (\varrho\mathbf{u} \cdot \phi)(\tau, \cdot)dx. \tag{2.2}$$

(iv) Dissipation: for any $\xi \in C_c^\infty([0,T] \times \mathbb{R}^3; \mathbb{R})$ and for all $\tau \in [0, T]$,

$$\int_0^\tau \int_{\mathbb{R}^3} (\sqrt{\varrho}\mathcal{T})\xi \, dxdt = -\int_0^\tau \int_{\mathbb{R}^3} \varrho\mathbf{u} \cdot \nabla\xi \, dxdt - \int_0^\tau \int_{\mathbb{R}^3} 2(\sqrt{\varrho}\mathbf{u} \otimes \nabla\sqrt{\varrho})\xi \, dxdt. \tag{2.3}$$

(v) Energy inequality: for almost all $\tau \in [0, T]$, the energy as defined in (1.5) satisfies the inequality

$$E(\varrho, \mathbf{u})(\tau) + 2\nu \int_0^\tau \int_{\mathbb{R}^3} |\mathcal{S}(\mathbf{u})(\tau, x)|^2 dxdt \le E(\varrho^0, \mathbf{u}^0), \tag{2.4}$$

where $\mathcal{S}$ denotes the symmetric part of the tensor $\mathcal{T}$.

(vi) BD-entropy inequality: there exists $C > 0$ such that for almost all $\tau \in [0, T]$ the Bresch–Desjardins entropy inequality holds

$$\mathcal{B}(\varrho, \mathbf{u})(\tau) + 2\nu \int_0^\tau \int_{\mathbb{R}^3} |\mathcal{A}(\mathbf{u})(\tau, x)|^2 dxdt$$
$$+ \frac{8\nu}{\gamma^2} \int_0^\tau \int_{\mathbb{R}^3} \left| \nabla\varrho^{\frac{\gamma}{2}}(\tau, x) \right|^2 dxdt + 4\nu\kappa^2 \int_0^\tau \int_{\mathbb{R}^3} |\Delta\varrho(\tau, x)|^2 dxdt \le C\mathcal{B}(\varrho^0, \mathbf{u}^0), \tag{2.5}$$

with

$$\mathcal{B}(\varrho, \mathbf{u})(\tau) = \int_{\mathbb{R}^3} \left[\frac{1}{2}|\sqrt{\varrho}u + 2\nu\nabla\sqrt{\varrho}|^2 + \kappa^2|\nabla\varrho|^2 + H(\varrho) \right](\tau)dx. \tag{2.6}$$

and $\mathcal{A}$ denoting the anti-symmetric part of $\mathcal{T}$.

Here, $\mathcal{S}(\mathbf{u})$ and $\mathcal{A}(\mathbf{u})$ are the symmetric and anti-symmetric part of the tensor $\mathcal{T}(\mathbf{u})$, respectively, defined by

$$\sqrt{\varrho}\mathcal{T}(\mathbf{u}) = \nabla(\varrho\mathbf{u}) - 2(\sqrt{\varrho}\mathbf{u}) \otimes \nabla\sqrt{\varrho} \quad \text{in} \quad \mathcal{D}'\left([0, T] \times \mathbb{R}^3\right). \tag{2.7}$$

Remark 2.1. For smooth solutions, the energy inequality (in fact an equality) for the system (NSK) reads

$$E(\varrho, \mathbf{u})(\tau) + 2\nu \int_0^\tau \int_{\mathbb{R}^3} \varrho |\mathbb{D}(\mathbf{u})|^2 dxdt \leq E(\varrho^0, \mathbf{u}^0). \tag{2.8}$$

For weak solutions, we only require the energy inequality (2.4) to hold where the dissipation (2.3) is formulated in distributional sense. Weak solutions are commonly constructed through approximation procedures, and at present, it appears to be unclear whether arbitrary finite energy weak solutions satisfy the inequality (2.8). Indeed, a lack of compactness rules out to conclude that the weak limit in $L^2_{t,x}$ of $\sqrt{\varrho_n}\mathbb{D}(\mathbf{u}_n)$ is given by $\sqrt{\varrho}\mathbb{D}(\mathbf{u})$, see [5, Remark 2.2]. In general, the approximation procedure only yields the information

$$\sqrt{\varrho_n}\mathbb{D}(\mathbf{u}_n) \rightharpoonup \mathcal{S} \text{ in } L^2_{t,x}.$$

The viscous term $\varrho\mathbb{D}(\mathbf{u})$ has hence to be understood as $\sqrt{\varrho}\mathcal{S}$. Similar arguments hold for the BD-entropy inequality.

We postulate the following existence result of global finite energy weak solutions. While to the best of the authors' knowledge this result is not available in the literature for (NSK) considered on $\mathbb{R}^3$ with far field (1.1), the respective result for the problem posed on $\mathbb{T}^3$ is proven in [5].

Theorem 2.1. *Given initial data* $(\varrho^0, \mathbf{u}^0)$ *of finite energy and BD-entropy, then, there exists at least a global weak solution* $(\varrho, \mathbf{u}, \mathcal{T})$ *of* (NSK) *in the sense of Definition 2.1.*

Remark 2.2. It is conceivable that global existence of weak solutions to (NSK) on $\mathbb{R}^3$ with far-field conditions (1.1) can be achieved by combining the result [5] for (NSK) posed on the torus $\mathbb{T}^3$ and the invading domains approach developed in [3] in the case of the quantum Navier–Stokes equations (QNS), see also [16]. Specifically, the approach in [3] consists in the following steps: given initial data with non-trivial far-field conditions, one constructs suitable periodic initial data on growing tori $\mathbb{T}^3_n = \mathbb{R}^3/n\mathbb{Z}^3$ with $n \in \mathbb{N}$. Based on the existence result on periodic domains [4,27], one infers the existence of a sequence of finite energy weak solutions $(\rho_n, u_n)_{n\in\mathbb{N}}$ on $\mathbb{T}^3_n$ that are also solutions to a suitably truncated formulation of the system. Next, a suitable extension of $(\rho_n, u_n)_{n\in\mathbb{N}}$ on $\mathbb{R}^3$ is shown to yield a sequence of approximate solutions on the whole space. By means of the stability properties of these approximate truncated solutions one proves convergence to a truncated solution on the whole space. Finally, the constructed solution is shown to also satisfy the weak formulation of the equations. Note that the approach [5] relies on suitable truncations of the velocity field and the mass density at different scales in the momentum equation, while for QNS [3,27] a truncation of the velocity field turns out to be sufficient. This is linked to the fact that QNS allows for stronger uniform estimates compared to (NSK). Upon taking into account these additional difficulties, we expect that the approach of [3,5] can be adapted to infer global existence of weak solutions to (NSK) on $\mathbb{R}^3$. This is left for future work.

2.2. Local well-posedness for the incompressible Euler equations

We recall here the following classical result (see [25,26], for example) for the target Euler system (E):

Theorem 2.2. *Given* $\mathbf{u}_0^E \in W^{3,2}(\mathbb{R}^3)$ *with* $\operatorname{div}\mathbf{u}_0^E = 0$*, there exist* $T^* > 0$ *and a unique solution to the initial value problem (E) - (1.7) such that for all* $0 < T < T^*$ *it holds,*

$$\mathbf{u}^E \in C^k([0,T), W^{3-k,2}(\mathbb{R}^3;\mathbb{R}^3)),\ \Pi \in C^k([0,T), W^{3-k,2}(\mathbb{R}^3)),\ k = 0,1,2,3$$

$$\|\mathbf{u}^E\|_{W^{k,\infty}(0,T;W^{3-k,2}(\mathbb{R}^3;\mathbb{R}^3))} + \|\Pi\|_{W^{k,\infty}(0,T;W^{3-k,2}(\mathbb{R}^3))} \le c(T, \|\mathbf{u}_0^E\|_{W^{3,2}(\mathbb{R}^3)}). \tag{2.9}$$

2.3. Main result

Our main result rigorously characterizes the aforementioned asymptotic regime of (NSK) in the case of the following set of *ill-prepared initial data*. Specifically,

$$\varrho(0,\cdot) = \varrho_\varepsilon^0 = 1 + \varepsilon\sigma_\varepsilon^0, \quad \sigma_\varepsilon^0 \in L^\infty(\mathbb{R}^3) \cap H^1(\mathbb{R}^3) \text{ uniformly bounded,}$$
$$\sigma_\varepsilon^0 \to s^0 \quad \text{in } H^1(\mathbb{R}^3), \tag{2.10}$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_\varepsilon^0, \quad \sqrt{\varrho_\varepsilon^0}\mathbf{u}_\varepsilon^0 \in L^2(\mathbb{R}^3), \quad \sqrt{\varrho_\varepsilon^0}(\mathbf{u}_\varepsilon^0 - \mathbf{u}^0) \to 0 \quad \text{in} \quad L^2(\mathbb{R}^3). \tag{2.11}$$

as $\varepsilon \to 0$.

Having recalled and collected all the preliminary notions we need in the sequel, we are ready to state our main result.

Theorem 2.3. *Suppose* $\nu \to 0$ *as* $\varepsilon \to 0$*. Assume the initial data* $(\varrho_\varepsilon^0, \mathbf{u}_\varepsilon^0)$ *to be of uniformly bounded finite energy and BD-entropy, i.e. there exists* $C > 0$ *independent of* $\varepsilon, \nu > 0$ *such that*

$$E(\varrho_\varepsilon^0, \mathbf{u}_\varepsilon^0) \le C, \quad B(\varrho_\varepsilon^0, \mathbf{u}_\varepsilon^0) \le C. \tag{2.12}$$

and satisfy (2.10), (2.11)*. Let* $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathcal{T}_\varepsilon)$ *be a weak solution of (NSK) and* $\mathbf{u}^E$ *the unique solution to the initial value problem (E) - (1.7) on* $[0,T) \times \mathbb{R}^3$, $0 < T < T^*$*, with initial datum* $\mathbf{u}_0^E = \mathbf{P}(\mathbf{u}^0) \in W^{3,2}(\mathbb{R}^3)$*. Then, as* $\varepsilon \to 0$*,*

$$\varrho_\varepsilon - 1 \to 0 \ \text{ in } \ L^\infty(0,T; L^2(\mathbb{R}^3) + L^\gamma(\mathbb{R}^3)) \cap L^\infty(0,T; H^s(\mathbb{R}^3)) \tag{2.13}$$

for all $0 \le s < 1$ *and*

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \mathbf{u}^E \ \text{ in } \ L^2(0,T; L^2_{\text{loc}}(\mathbb{R}^3)), \tag{2.14}$$

for any $0 < T < T^*$.

Remark 2.3. Note that in (2.10), we do not require that the initial data for the density are well prepared but allow for the presence of fast oscillating acoustic waves. The quantity s^0 will serve as initial data for the linear acoustic system in Sect. 4 and subsequently in the relative entropy scheme in Section 5. Further, the initial datum for the solution to the target system is given by $\mathbf{u}_0^E = \mathbf{P}(\mathbf{u}^0)$, where $\mathbf{P}$ denotes the Leray projector on divergence-free vector fields defined in (1.9).

Remark 2.4. We also provide the analogue of Theorem 2.3 in the case $d = 2$, the main difference being the weaker dispersive decay of the acoustic waves and the well-posedness properties of the $2D$-Euler equations (E). Posed on $\mathbb{R}^2$, singularity formation in finite time is ruled out and we obtain global solutions. We refer to Sect. 7 for details.

3. Uniform bounds

Owing to (2.4) and (2.6), the weak solutions under consideration satisfy the following uniform bounds. To take the non-trivial far field (1.1) into account, we will repeatedly rely on the following fact: let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ with $\nabla f \in L^2(\mathbb{R}^3)$, then there exists $c \in \mathbb{R}$ such that $f - c \in L^6(\mathbb{R}^3)$ (see [22, Theorem 4.5.9] for a proof).

Lemma 3.1. *Assume that the initial data* $(\varrho_\varepsilon^0, \mathbf{u}_\varepsilon^0)$ *satisfy* (2.12). *Then, the following hold:*

(i) $\varrho_\varepsilon^0 - 1 \in H^1(\mathbb{R}^3)$ *uniformly bounded and*

$$\|\varrho_\varepsilon^0 - 1\|_{L^2(\mathbb{R}^3)} \leq C \begin{cases} \varepsilon^{\frac{4}{6-\gamma}} & 1 < \gamma < 2 \\ \varepsilon & \gamma \geq 2, \end{cases}$$

(ii) $\sqrt{\varrho_\varepsilon^0} - 1 \in L^2(\mathbb{R}^3)$ *uniformly bounded with*

$$\|\sqrt{\varrho_\varepsilon^0} - 1\|_{L^2(\mathbb{R}^3)} \leq C \begin{cases} \varepsilon^{\frac{4}{6-\gamma}} & 1 < \gamma < 2 \\ \varepsilon & \gamma \geq 2, \end{cases}$$

(iii) $\sqrt{\varrho_\varepsilon^0}\mathbf{u}_\varepsilon^0 \in L^2(\mathbb{R}^3)$ *uniformly bounded.*

(iv) the initial momentum satisfies $\varrho_\varepsilon^0\mathbf{u}_\varepsilon^0 \in L^2(\mathbb{R}^3) + L^{\frac{3}{2}}(\mathbb{R}^3)$ *uniformly bounded,*

(v) the initial density fluctuations σ_ε^0 *satisfy* $\varepsilon\sigma_\varepsilon^0 \in H^1(\mathbb{R}^3)$ *uniformly bounded. Provided that* $\gamma \geq 2$, *then also* $\sigma_\varepsilon^0 \in L^2(\mathbb{R}^3)$ *uniformly bounded.*

For $1 < \gamma < 2$, the L^2-norm of the initial density fluctuations may diverge as $\mathcal{O}(\varepsilon^{-\frac{2-\gamma}{6-\gamma}})$. However, such a bound is not required in the sequel.

Remark 3.1. Note that the above uniform bounds are only based on the finite energy assumption and do not use additional information from the viscous dissipation and the BD-entropy inequality.

Proof. The assumptions (2.12) yield that

$$\nabla\varrho_\varepsilon^0 \in L^2(\mathbb{R}^3), \quad H(\varrho_\varepsilon^0) \in L^1(\mathbb{R}^3),$$

uniformly bounded and with H defined as in (1.5) and thus nonnegative. To prove (i), we note that by convexity of the function $s \to s^\gamma - 1 - \gamma(s-1)$ for $\gamma > 1$ it is possible to conclude that

$$\int_{\mathbb{R}^3} \left|\varrho_\varepsilon^0 - 1\right|^2 \mathbf{1}_{\{|\varrho_\varepsilon^0 - 1| \le \frac{1}{2}\}} + \left|\varrho_\varepsilon^0 - 1\right|^\gamma \mathbf{1}_{|\varrho_\varepsilon^0 - 1| > \frac{1}{2}} \mathrm{d}x \le C\varepsilon^2, \tag{3.1}$$

see [29] for details. If $\gamma \ge 2$, this yields

$$\|\varrho_\varepsilon^0 - 1\|_{L^2(\mathbb{R}^3)} \le C\varepsilon$$

and in particular $\varrho_\varepsilon^0 - 1 \in H^1(\mathbb{R}^3)$ uniformly bounded provided that $\gamma \ge 2$. Moreover, for any $\gamma > 1$ it follows from $\nabla\varrho_\varepsilon^0 \in L^2(\mathbb{R}^3)$ that there exists $c \in \mathbb{R}$ such that $\varrho_\varepsilon^0 - c \in L^6(\mathbb{R}^3)$ uniformly bounded. The bound (3.1) then entails that $\varrho_\varepsilon^0 - 1 \in L^6(\mathbb{R}^3)$. For $1 < \gamma < 2$, it follows by interpolation that

$$\|(\varrho_\varepsilon^0 - 1)\mathbf{1}_{|\varrho_\varepsilon^0 - 1| > \frac{1}{2}}\|_{L^2} \le \|(\varrho_\varepsilon^0 - 1)\mathbf{1}_{|\varrho_\varepsilon^0 - 1| > \frac{1}{2}}\|_{L^\gamma}^\theta \|(\varrho_\varepsilon^0 - 1)\mathbf{1}_{|\varrho_\varepsilon^0 - 1| > \frac{1}{2}}\|_{L^6}^{1-\theta} \le C\varepsilon^{\frac{2}{\gamma}\theta}$$

with $\theta = \frac{2\gamma}{6-\gamma}$. Upon observing that

$$\left|\sqrt{\varrho_\varepsilon^0} - 1\right| = \left|(1 + \sqrt{\varrho_\varepsilon^0})^{-1}(\varrho_\varepsilon^0 - 1)\right| \le \left|(\varrho_\varepsilon^0 - 1)\right|,$$

the desired bound (ii) is implied by (i). While (iii) is a direct consequence of (2.12), the statement (iv) follows from $\sqrt{\varrho_\varepsilon^0} - 1 \in H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and $\sqrt{\varrho_\varepsilon^0}\mathbf{u}_\varepsilon^0 \in L^2(\mathbb{R}^3)$ as

$$\varrho_\varepsilon^0\mathbf{u}_\varepsilon^0 = \sqrt{\varrho_\varepsilon^0}\mathbf{u}_\varepsilon^0 + (\sqrt{\varrho_\varepsilon^0} - 1)\sqrt{\varrho_\varepsilon^0}\mathbf{u}_\varepsilon^0 \in L^2(\mathbb{R}^3) + L^{\frac{3}{2}}(\mathbb{R}^3).$$

For the statement (v), it suffices to note that $\varrho_\varepsilon^0 - 1 \in H^1(\mathbb{R}^3)$ uniformly bounded yields $\varepsilon\sigma_\varepsilon^0 \in L^2(\mathbb{R}^3)$ uniformly bounded. Moreover, the L^2-decay rate for $\varrho_\varepsilon^0 - 1$ yields $\sigma_\varepsilon^0 \in L^2(\mathbb{R}^3)$ uniformly bounded provided that $\gamma \ge 2$ and

$$\|\sigma_\varepsilon^0\|_{L^2(\mathbb{R}^3)} \le C\varepsilon^{\frac{\gamma-2}{6-\gamma}}$$

for $1 < \gamma < 2$. □

Along the same lines of Lemma 3.1, one infers the following uniform bounds for weak solutions to (NSK) provided that (2.12) holds:

$$\|\varrho_\varepsilon - 1\|_{L_t^\infty H_x^1} \le C, \quad \|\varrho_\varepsilon - 1\|_{L_t^\infty L_x^2} \le C \begin{cases} \varepsilon^{\frac{4}{6-\gamma}} & 1 < \gamma < 2, \\ \varepsilon & \gamma \ge 2; \end{cases} \tag{3.2}$$

$$\|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L_t^\infty L_x^2} \le C, \quad \|\varrho_\varepsilon\mathbf{u}_\varepsilon\|_{L_t^\infty L_x^2 + L_x^{\frac{3}{2}}} \le C. \tag{3.3}$$

By interpolation, one has that $\rho_\varepsilon - 1$ converges strongly to 0 in $L^\infty(0, T; H^s(\mathbb{R}^3))$ for all $0 \le s < 1$. In particular, for all $2 \le q < 6$ there exists $C > 0$ such that

$$\|\sqrt{\varrho_\varepsilon} - 1\|_{L_t^\infty L_x^q} \le \|\varrho_\varepsilon - 1\|_{L_t^\infty L_x^q} \le C\varepsilon^\beta \tag{3.4}$$

where $\beta = \beta(q) = \frac{4}{5}\left(1 - 3\left(\frac{1}{2} - \frac{1}{q}\right)\right)$ by interpolation. The viscous dissipation and the BD-entropy inequality further allow for the following bounds.

Lemma 3.2. *Assume that* $(\varrho_\varepsilon, u_\varepsilon)$ *is a finite energy weak solution to* (NSK) *with initial data* $(\varrho_\varepsilon^0, u_\varepsilon^0)$ *satisfying* (2.12). *Then, there exits* $C > 0$ *independent of* ε *such that*

$$\nu\|\sqrt{\varrho_\varepsilon} - 1\|_{L_t^\infty H_x^1} \le C, \quad \|\sqrt{\nu}\nabla\varrho_\varepsilon^{\frac{\gamma}{2}}\|_{L_t^2 L_x^2} \le C,$$

$$\|\sqrt{\nu}\Delta\varrho_\varepsilon\|_{L_t^2 L_x^2} \le C, \quad \|\sqrt{\nu}\mathcal{S}(\mathbf{u})\|_{L_t^2 L_x^2} \le C, \quad \|\sqrt{\nu}\mathcal{A}(\mathbf{u})\|_{L_t^2 L_x^2} \le C. \tag{3.5}$$

Proof. The bounds are immediate consequences of the energy and BD-entropy inequalities stated in Definition 2.1. □

4. Acoustic waves

The analysis of the incompressible limit requires a suitable control of the acoustic waves. Indeed, their presence is due to density fluctuations at finite Mach number and may create highly oscillating phenomena. Being the problem (NSK) posed on $\mathbb{R}^d$, suitable dispersive estimates yield the decay of the acoustic waves in the low-Mach number limit. As will be detailed below, it is possible to obtain a fourth-order acoustic wave equation for the *density fluctuations*

$$\sigma_\varepsilon = \varepsilon^{-1}(\varrho_\varepsilon - 1)$$

by a suitable linearization of the system (NSK) around the constant (incompressible) state, namely

$$\partial_{tt}^2\sigma_\varepsilon - \frac{1}{\varepsilon^2}\Delta(1 - 2\varepsilon^2\kappa^2\Delta)\sigma_\varepsilon = 0. \tag{4.1}$$

We notice that in the absence of the capillarity tensor ($\kappa = 0$), we have the usual dispersion relation for the classical wave equation. Here and due to the Korteweg tensor, this dispersion corresponds, at the leading order, to the one already observed in quantum fluids such as the QHD or quantum Navier–Stokes equations, see [1,2]. Denoting the sequence of momenta by $\mathbf{m}_\varepsilon = \varrho_\varepsilon\mathbf{u}_\varepsilon$ and linearizing (NSK) around the constant state ($\varrho = 1, \mathbf{m}_\varepsilon = 0$), one obtains the acoustic system

$$\begin{aligned} &\partial_t\sigma_\varepsilon + \frac{1}{\varepsilon}\operatorname{div}\mathbf{Q}(\mathbf{m}_\varepsilon) = 0, \\ &\partial_t\mathbf{Q}(\mathbf{m}_\varepsilon) + \frac{\gamma}{\varepsilon}\nabla\left(\sigma_\varepsilon - 2\kappa^2\varepsilon^2\Delta\sigma_\varepsilon\right) = \mathbf{Q}(F_\varepsilon), \end{aligned} \tag{4.2}$$

where $\mathbf{Q}$ defined in (1.9) denotes the Leray projector on curl-free vector fields and with

$$F_\varepsilon := -\mathrm{div}\,(\varrho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + 2\nu \mathrm{div}\,(\varrho_\varepsilon \mathbb{D} u_\varepsilon) - \frac{(\gamma-1)}{\varepsilon^2}\nabla H(\varrho_\varepsilon) + \varepsilon^2 G_\varepsilon$$

and where

$$\varepsilon^{-2}\nabla p(\varrho_\varepsilon) = \frac{\gamma}{\varepsilon}\nabla\sigma_\varepsilon + \frac{(\gamma-1)}{\varepsilon^2}\nabla H(\varrho_\varepsilon).$$

In particular, for the dispersive stress tensor div $\mathbb{K}$, as defined in (1.4), we obtain the identity

$$\begin{aligned}\mathrm{div}\,\mathbb{K} &= \mathrm{div}\,\left((1+\varepsilon\sigma_\varepsilon)\mathrm{div}\,(\varepsilon\nabla\sigma_\varepsilon)\mathbb{I} + \frac{1}{2}\left|\varepsilon\nabla\sigma_\varepsilon|^2\right)\mathbb{I}\right) - \mathrm{div}\,(\varepsilon\nabla\sigma_\varepsilon\otimes\varepsilon\nabla\sigma_\varepsilon)\\ &= \varepsilon\nabla\Delta\sigma_\varepsilon + \varepsilon^2 G_\varepsilon\end{aligned}$$

with

$$G_\varepsilon := \sigma_\varepsilon\nabla\Delta\sigma_\varepsilon.$$

While we discarded all terms in div $\mathbb{K}$ being multiplied by ε^2 into F_ε, we keep the leading order linear term $2\varepsilon\kappa^2\nabla\Delta\sigma_\varepsilon$ which, as already mentioned, alters the classical dispersion relation. Consequently, the decay of σ_ε in space-time norms can be shown to be stronger compared to the classical wave equations, see Proposition 4.1 and [21, Remark 4.10]. The method of exploiting altered dispersion relations through accurate Strichartz estimates has also been employed, for example, in the analysis of the quasi-neutral limit for Navier–Stokes–Korteweg system [17].

We sketch the derivation of the respective Strichartz estimates and refer to [1, Section 4] for further details. We symmetrize (4.2) by applying

$$\widetilde{\sigma_\varepsilon} = (1-2\varepsilon^2\kappa^2\Delta)^{\frac{1}{2}}\sigma_\varepsilon, \qquad \widetilde{\mathbf{m}}_\varepsilon = (-\Delta)^{-\frac{1}{2}}\mathrm{div}\,\mathbf{m}_\varepsilon \tag{4.3}$$

to obtain

$$\begin{aligned}\partial_t\widetilde{\sigma_\varepsilon} + \frac{1}{\varepsilon}(-\Delta)^{\frac{1}{2}}(1-2\varepsilon^2\kappa^2\Delta)^{\frac{1}{2}}\widetilde{\mathbf{m}}_\varepsilon &= 0,\\ \partial_t\widetilde{\mathbf{m}}_\varepsilon - \frac{1}{\varepsilon}(-\Delta)^{\frac{1}{2}}(1-2\varepsilon^2\kappa^2\Delta)^{\frac{1}{2}}\widetilde{\sigma_\varepsilon} &= \tilde{F}_\varepsilon,\end{aligned} \tag{4.4}$$

where

$$\tilde{F}_\varepsilon = (-\Delta)^{-\frac{1}{2}}\mathrm{div}\,\mathbf{Q}\,(F_\varepsilon)\,.$$

As described in [1], the system (4.4) can be characterized by means of the linear semi-group operator e^{itH_ε} where H_ε is defined via the Fourier multiplier

$$\phi_\varepsilon(|\xi|) = \frac{|\xi|}{\varepsilon}\sqrt{1+2\varepsilon^2\kappa^2|\xi|^2}. \tag{4.5}$$

The dispersion given by (4.5) mimics wave-like behaviour for low frequencies $|\xi| < \varepsilon^{-1}$, while it is Schrödinger like for high frequencies $|\xi| > \varepsilon^{-1}$. A stationary phase argument then leads to the desired Strichartz (dispersive) estimates [1, Section 4].

In order to recall the dispersive estimates suitable for our analysis, we start saying that a pair of Lebesgue exponents (p, q) is called *"Schrödinger admissible"* if

$$2 \leq p, q \leq \infty \quad \text{and} \quad \frac{2}{p} + \frac{3}{q} = \frac{3}{2}.$$

The following proposition holds.

Proposition 4.1 ([1, Proposition 4.2]). *Fix $\varepsilon > 0$ and $s \in \mathbb{R}$. There exists a constant $C > 0$ independent from T and ε such that for any (p, q)-admissible pair and any $\alpha_0 \in [0, \frac{1}{2}(\frac{1}{2} - \frac{1}{q})]$, the following hold true,*

$$\|e^{itH_\varepsilon} f\|_{L^p(0,T;W^{s-\alpha_0,q}(\mathbb{R}^3))} \leq C\varepsilon^{\alpha_0} \|f\|_{H^s(\mathbb{R}^3)}. \tag{4.6}$$

For the purpose of the inviscid incompressible limit (see Sect. 6), it turns out to be sufficient to consider the acoustic system (4.2) in its homogeneous form

$$\begin{aligned} &\partial_t s_\varepsilon + \frac{1}{\varepsilon} \operatorname{div} \nabla\Phi_\varepsilon = 0, \\ &\partial_t \nabla\Phi_\varepsilon + \frac{\gamma}{\varepsilon} \nabla \left(s_\varepsilon - 2\kappa^2 \varepsilon^2 \Delta s_\varepsilon \right) = 0, \end{aligned} \tag{4.7}$$

with initial data $(s^0, \nabla\Phi_0)$, where we denoted by Φ_ε the acoustic potential, i.e. $\nabla\Phi_\varepsilon = \mathbf{Q}(\mathbf{m}_\varepsilon)$. For the sake of clarity, we choose a distinct notation for the solution s_ε of the linear homogeneous equation (4.7) and the solution σ_ε to (4.2). In particular, we will see in Sect. 6 that the initial data $(s^0, \nabla\Phi_0)$ considered will be given by a properly regularized version of the data $(s^0, \mathbf{Q}(\mathbf{u}_0))$ in (2.11), (2.10).

Remark 4.1. Note that the fourth-order equation (4.1) formally follows from (4.7) by applying the differential operators ∂_t and div to the first and second equation, respectively.

From Proposition 4.1 and combining the transformation (4.3) with the Strichartz estimates of Proposition 4.1, one derives the following bounds for solutions to (4.7). We refer the reader to [1, Proposition 4.5] for details.

Proposition 4.2. *Let $s \in \mathbb{R}$, $(s^0, \mathbf{m}^0)$ be such that $s^0 \in H^s(\mathbb{R}^3)$, $\varepsilon\nabla s^0 \in H^s(\mathbb{R}^3)$ and $\mathbf{Q}(\mathbf{m}^0) \in H^s(\mathbb{R}^3)$ and denote by $(s_\varepsilon, \mathbf{m}_\varepsilon)$ the unique solution to the homogeneous system (4.7) with initial data $(s^0, \mathbf{Q}(\mathbf{m}^0))$. Then, for all $T > 0$, all Schrödinger-admissible pair (p, q) and any $\alpha \in [0, \frac{1}{2}(\frac{1}{2} - \frac{1}{q})]$ the following holds true*

$$\begin{aligned} &\|(s_\varepsilon, \mathbf{Q}(\mathbf{m}_\varepsilon))\|_{L^p(0,T;W^{s,q}(\mathbb{R}^3))} \\ &\quad \leq C\varepsilon^\alpha \left(\|s^0\|_{H^{s+\alpha}(\mathbb{R}^3)} + \|\varepsilon\nabla s^0\|_{H^{s+\alpha}(\mathbb{R}^3)} + \|\mathbf{Q}(\mathbf{m}^0)\|_{H^{s+\alpha}(\mathbb{R}^3)} \right). \end{aligned}$$

Note that $(s_\varepsilon, \mathbf{Q}(\mathbf{m}_\varepsilon))$ can be bounded in terms of $(\widetilde{\sigma_\varepsilon}, \widetilde{\mathbf{m}}_\varepsilon)$ in $L^p(0, T; W^{s,q}(\mathbb{R}^3))$ in view of (4.3), while one has

$$\left\| (\tilde{s}^0_\varepsilon, \tilde{m}^0) \right\|_{H^s(\mathbb{R}^3)} \leq \|s^0\|_{H^s(\mathbb{R}^3)} + \|\varepsilon\nabla s^0\|_{H^s(\mathbb{R}^3)} + \|\mathbf{Q}(\mathbf{m}^0)\|_{H^s(\mathbb{R}^3)}$$

which leads to the above estimate.

5. Relative energy inequality

In the same spirit of [9], we derive the relative energy inequality related to the Navier–Stokes–Korteweg system (NSK). For $\tau \in [0, T]$, we introduce the following relative energy functional

$$\mathcal{E}(\varrho, \mathbf{u} \,|\, r, \mathbf{U})(\tau) = \int_{\mathbb{R}^3} \frac{1}{2} |\sqrt{\varrho}\mathbf{u} - \sqrt{\varrho}\,\mathbf{U}|^2(\tau, \cdot) dx + \kappa^2 \int_{\mathbb{R}^3} |\nabla \varrho - \nabla r|^2(\tau, \cdot) dx \\ + \frac{1}{\varepsilon^2} \int_{\mathbb{R}^3} \left[H(\varrho) - H(r) - H'(r)(\varrho - r) \right] (\tau, \cdot) dx,$$

where $(\varrho, \mathbf{u})$ is a weak solution to (NSK) and $(r, \mathbf{U})$ is a pair of smooth (arbitrary) test functions. In the following, we simply write $\mathcal{E}(\tau)$ in place of $\mathcal{E}(\varrho, \mathbf{u} \,|\, r, \mathbf{U})(\tau)$. First, thanks to the energy inequality (2.4), we infer that

$$\mathcal{E}(s)\Big|_{s=0}^{s=\tau} \le \int_{\mathbb{R}^3} \left(\frac{1}{2} \varrho |\mathbf{U}|^2 - \varrho \mathbf{u} \cdot \mathbf{U} \right) (s, \cdot) dx \Big|_{s=0}^{s=\tau} + \kappa^2 \int_{\mathbb{R}^3} \left(|\nabla r|^2 - 2 \nabla \varrho \cdot \nabla r \right) (s, \cdot) dx \Big|_{s=0}^{s=\tau} \\ - \frac{1}{\varepsilon^2} \int_{\mathbb{R}^3} \left(H(r) + H'(r)(\varrho - r) \right) (s, \cdot) dx \Big|_{s=0}^{s=\tau} \\ - \nu \int_0^\tau \int_{\mathbb{R}^3} |\mathcal{S}(\mathbf{u})|^2 \, dx dt. \tag{5.1}$$

Next, we test the continuity equation by $\frac{1}{2}|\mathbf{U}|^2$ and $2\kappa^2 \Delta r$, and the momentum equation by $\mathbf{U}$, to get

$$\int_0^\tau \int_{\mathbb{R}^3} \frac{1}{2} \varrho \partial_t |\mathbf{U}|^2 \, dx dt + \int_0^\tau \int_{\mathbb{R}^3} \frac{1}{2} \varrho \mathbf{u} \cdot \nabla |\mathbf{U}|^2 \, dx dt = \int_{\mathbb{R}^3} \frac{1}{2} \varrho |\mathbf{U}|^2 (s, \cdot) dx \Big|_{s=0}^{s=\tau},$$

$$-2\kappa^2 \int_0^\tau \int_{\mathbb{R}^3} \varrho \partial_t \Delta r \, dx dt - 2\kappa^2 \int_0^\tau \int_{\mathbb{R}^3} \varrho \mathbf{u} \cdot \nabla \Delta r \, dx dt = 2\kappa^2 \int_{\mathbb{R}^3} \nabla \varrho \cdot \nabla r (s, \cdot) dx \Big|_{s=0}^{s=\tau}$$

and

$$- \int_0^\tau \int_{\mathbb{R}^3} \varrho \mathbf{u} \cdot \partial_t \mathbf{U} \, dx dt - \int_0^\tau \int_{\mathbb{R}^3} (\sqrt{\varrho}\mathbf{u} \otimes \sqrt{\varrho}\mathbf{u}) : \nabla \mathbf{U} \, dx dt - \frac{1}{\varepsilon^2} \int_0^\tau \int_{\mathbb{R}^3} p(\varrho) \mathrm{div}\, \mathbf{U} \, dx dt \\ -2\kappa^2 \int_0^\tau \int_{\mathbb{R}^3} \left(\nabla \varrho \cdot \nabla (\varrho \mathrm{div} \mathbf{U}) - \frac{1}{2} |\nabla \varrho|^2 \mathrm{div} \mathbf{U} + \nabla \varrho \otimes \nabla \varrho : \nabla \mathbf{U} \right) dx dt \\ +2\nu \int_0^\tau \int_{\mathbb{R}^3} \sqrt{\varrho} \mathcal{S}(\mathbf{u}) : \nabla \mathbf{U} dx dt = - \int_{\mathbb{R}^3} (\varrho \mathbf{u} \cdot \mathbf{U})(s, \cdot) dx \Big|_{s=0}^{s=\tau}.$$

Moreover, we have

$$\kappa^2 \int_{\mathbb{R}^3} |\nabla r(s, \cdot)|^2 dx \Big|_{s=0}^{s=\tau} = -2\kappa^2 \int_0^\tau \int_{\mathbb{R}^3} \partial_t r \Delta r dx dt.$$

Then, using the continuity equation, we have

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^3} \left[\partial_t \left(H(r) + H'(r)(\varrho - r)\right)\right] dxdt \\
&= \int_0^T \int_{\mathbb{R}^3} \left[H'(r)\partial_t r + \partial_t(H'(r))(\varrho - r) + H'(r)\partial_t \varrho - H'(r)\partial_t r\right] dxdt \\
&= \int_0^T \int_{\mathbb{R}^3} \left[\partial_t(H'(r))(\varrho - r) - H'(r)\mathrm{div}_x(\varrho \mathbf{u})\right] dxdt \\
&= \int_0^T \int_{\mathbb{R}^3} \left[\partial_t(H'(r))(\varrho - r) + \varrho \mathbf{u} \cdot \nabla_x(H'(r))\right] dxdt.
\end{aligned}
\tag{5.2}
$$

Therefore, from (5.1), we obtain

$$
\begin{aligned}
\mathcal{E}(s)\Big|_{s=0}^{s=\tau} \leq & \int_0^\tau \int_{\mathbb{R}^3} \frac{1}{2}\varrho \partial_t |\mathbf{U}|^2 \, dxdt + \int_0^\tau \int_{\mathbb{R}^3} \frac{1}{2}\varrho \mathbf{u} \cdot \nabla |\mathbf{U}|^2 \, dxdt \\
& - \int_0^\tau \int_{\mathbb{R}^3} \varrho \mathbf{u} \cdot \partial_t \mathbf{U} \, dxdt - \int_0^\tau \int_{\mathbb{R}^3} (\sqrt{\varrho}\mathbf{u} \otimes \sqrt{\varrho}\mathbf{u}) : \nabla \mathbf{U} \, dxdt \\
& + 2\kappa^2 \int_0^\tau \int_{\mathbb{R}^3} (\varrho \partial_t \Delta r + \varrho \mathbf{u} \cdot \nabla \Delta r - \partial_t r \Delta r) dxdt \\
& - 2\kappa^2 \int_0^\tau \int_{\mathbb{R}^3} \left(\nabla \varrho \cdot \nabla(\varrho \mathrm{div}\mathbf{U}) - \frac{1}{2}|\nabla \varrho|^2 \mathrm{div}\mathbf{U} + \nabla \varrho \otimes \nabla \varrho : \nabla \mathbf{U}\right) dxdt \\
& + 2\nu \int_0^\tau \int_{\mathbb{R}^3} \sqrt{\varrho}\mathcal{S}(\mathbf{u}) : \nabla \mathbf{U} dxdt \\
& - \frac{1}{\varepsilon^2} \int_0^\tau \int_{\mathbb{R}^3} \left[\partial_t(H'(r))(\varrho - r) + \varrho \mathbf{u} \cdot \nabla(H'(r)) + p(\varrho)\mathrm{div}\mathbf{U}\right] dxdt \\
& - \nu \int_0^\tau \int_{\mathbb{R}^3} |\mathcal{S}(\mathbf{u})|^2 \, dxdt.
\end{aligned}
\tag{5.3}
$$

Rearranging the relative energy inequality above, we obtain

$$
\begin{aligned}
\mathcal{E}(s)\Big|_{s=0}^{s=\tau} + \nu \int_0^\tau \int_{\mathbb{R}^3} |\mathcal{S}(\mathbf{u})|^2 \, dxdt & \\
\leq & \int_0^\tau \int_{\mathbb{R}^3} \left[\partial_t \mathbf{U} \cdot (\varrho \mathbf{U} - \varrho \mathbf{u}) + (\sqrt{\varrho}\mathbf{u} \cdot \nabla)\mathbf{U} \cdot (\sqrt{\varrho}\mathbf{U} - \sqrt{\varrho}\mathbf{u})\right] dxdt \\
& + 2\kappa^2 \int_0^\tau \int_{\mathbb{R}^3} (\varrho \partial_t \Delta r + \varrho \mathbf{u} \cdot \nabla \Delta r - \partial_t r \Delta r) dxdt \\
& - 2\kappa^2 \int_0^\tau \int_{\mathbb{R}^3} \left(\nabla \varrho \cdot \nabla(\varrho \mathrm{div}\mathbf{U}) - \frac{1}{2}|\nabla \varrho|^2 \mathrm{div}\mathbf{U} + \nabla \varrho \otimes \nabla \varrho : \nabla \mathbf{U}\right) dxdt \\
& + 2\nu \int_0^\tau \int_{\mathbb{R}^3} \sqrt{\varrho}\mathcal{S}(\mathbf{u}) : \nabla \mathbf{U} dxdt \\
& - \frac{1}{\varepsilon^2} \int_0^\tau \int_{\mathbb{R}^3} \left[\partial_t(H'(r))(\varrho - r) + \varrho \mathbf{u} \cdot \nabla(H'(r)) + p(\varrho)\mathrm{div}\mathbf{U}\right] dxdt \\
= & I_1 + ... + I_5.
\end{aligned}
\tag{5.4}
$$

Apart from the presence of the Korteweg terms, relation (5.4) is reminiscent of the one derived in [9] (see relation (4.7)).

6. Inviscid incompressible limit

For a fixed $\delta > 0$, we consider the following ansatz

$$r = 1 + \varepsilon s_{\varepsilon,\delta}, \quad \mathbf{U} = \mathbf{u}^E + \nabla\Phi_{\varepsilon,\delta} \tag{6.1}$$

in the relative energy inequality (5.4). Precisely, $\mathbf{u}^E$ is the solution of the target system (E) with initial datum $\mathbf{u}_0^E = \mathbf{P}(\mathbf{u}_0)$ and $s_{\varepsilon,\delta}$, $\nabla\Phi_{\varepsilon,\delta}$ denote the solution of the acoustic system (4.7) with the regularized initial data

$$s_\delta^0 = s^0 * \eta_\delta, \qquad \nabla\Phi_\delta^0 = \nabla\Phi^0 * \eta_\delta, \qquad \mathbf{Q}(\mathbf{u}^0) = \nabla\Phi^0, \tag{6.2}$$

for $\eta_\delta \in C_c^\infty(\mathbb{R}^3)$ a standard mollifier and $(s^0, \mathbf{u}^0)$ as in (2.10) and (2.11). The above choice is motivated by the fact that we consider the (arbitrary) smooth pair $(r, \mathbf{U})$ as the sum of the incompressible Euler equations and the contribution from acoustic waves. The latter disappear in the low-Mach number limit as the next corollary shows. By means of the Strichartz estimates of Proposition 4.2, we infer the following decay in ε in Strichartz norms.

Corollary 6.1. *Let $s \in \mathbb{R}$, and $(s_\delta^0, \nabla\Phi_\delta^0)$ be as in (6.2). For any Schrödinger-admissible pair (p, q) with $q > 2$ and $\alpha, \delta > 0$ sufficiently small and $T > 0$, there exists $C > 0$ only depending on the constant in (2.12) and $\delta > 0$ such that the unique solution $(s_{\varepsilon,\delta}, \nabla\Phi_{\varepsilon,\delta})$ to the (4.7) satisfies*

$$\|(s_{\varepsilon,\delta}, \nabla\Phi_{\varepsilon,\delta})\|_{L^p(0,T;W^{s,q}(\mathbb{R}^3))} \leq C\varepsilon^\alpha. \tag{6.3}$$

Note that the solution $(s_{\varepsilon,\delta}, \nabla\Phi_{\varepsilon,\delta})$ satisfies the energy conservation

$$\|(s_{\varepsilon,\delta}, \nabla\Phi_{\varepsilon,\delta})\|_{L^\infty(\mathbb{R},L^2(\mathbb{R}^3))} = \|(s_\delta^0, \nabla\Phi_\delta^0)\|_{L^2(\mathbb{R}^3)}, \tag{6.4}$$

and similarly for all $s \in \mathbb{R}$ it holds

$$\|(s_{\varepsilon,\delta}, \nabla\Phi_{\varepsilon,\delta})\|_{L^\infty(\mathbb{R},H^s(\mathbb{R}^3))} = \|(s_\delta^0, \nabla\Phi_\delta^0)\|_{H^s(\mathbb{R}^3)} \tag{6.5}$$

due to the fact that e^{itH_ε} constitutes a unitary semi-group operator.

In the following, we will consider the ansatz (6.1) and we will handle the terms $I_1, \ldots, I_5$ in (5.4) in order to bound the relative entropy functional. We start with the relative energy functional for the initial data, whose required convergence (see (2.11), (2.10)) is crucial in order to prove Theorem 2.3 in terms of Gronwall's type arguments (see Sect. 6.6).

6.1. Initial data

From the assumptions on the initial data $(\varrho_\varepsilon^0, \mathbf{u}_\varepsilon^0)$ in (2.10)-(2.11) and $(r, \mathbf{U})$ in (6.2), respectively, we infer that

$$\mathcal{E}(\varrho_\varepsilon, \mathbf{u}_\varepsilon \,|\, r, \mathbf{U})(0)$$

$$= \int_{\mathbb{R}^3} \frac{1}{2} \left| \sqrt{\varrho_\varepsilon^0} (\mathbf{u}_\varepsilon^0 - \mathbf{u}_0) \right|^2 dx + \kappa^2 \varepsilon^2 \int_{\mathbb{R}^3} \left| \nabla \sigma_\varepsilon^0 - \nabla s_\delta^0 \right|^2 dx$$

$$+ \int_{\mathbb{R}^3} \frac{1}{\varepsilon^2} \left[H\left(1 + \varepsilon \sigma_\varepsilon^0\right) - \varepsilon H'\left(1 + \varepsilon s_\delta^0\right)\left(\sigma_\varepsilon^0 - s_\delta^0\right) - H\left(1 + \varepsilon s_\delta^0\right) \right] dx. \quad (6.6)$$

Now, setting $a = 1 + \varepsilon \sigma_\varepsilon^0$ and $b = 1 + \varepsilon s_\delta^0$, we observe that there exist $\varepsilon_0 > 0$ and $C > 0$ independent of ε such that for all $\varepsilon \in (0, \varepsilon_0)$ it holds

$$H(a) = H(b) + H'(b)(a - b) + \frac{1}{2} H''(\xi)(a - b)^2, \quad \xi \in (a, b),$$

$$\left| H(a) - H'(b)(a - b) - H(b) \right| \leq C\, |a - b|^2 .$$

Consequently, we have

$$\int_{\mathbb{R}^3} \frac{1}{\varepsilon^2} \left[H\left(1 + \varepsilon \sigma_\varepsilon^0\right) - \varepsilon H'\left(1 + \varepsilon s_\delta^0\right)\left(\sigma_\varepsilon^0 - s_\delta^0\right) - H\left(1 + \varepsilon s_\delta^0\right) \right] dx$$

$$\leq C \int_{\mathbb{R}^3} \frac{1}{\varepsilon^2} \left| \varepsilon \left(\sigma_\varepsilon^0 - s_\delta^0\right) \right|^2 dx = C \left\| \sigma_\varepsilon^0 - s_\delta^0 \right\|_{L^2(\mathbb{R}^3)}^2 . \quad (6.7)$$

It follows

$$\mathcal{E}(\varrho_\varepsilon, \mathbf{u}_\varepsilon \,|\, r, \mathbf{U})(0)$$

$$\leq C \left\| \sqrt{\varrho_\varepsilon^0} (\mathbf{u}_\varepsilon^0 - \mathbf{u}_0) \right\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}^2 + C(\kappa^2)\varepsilon^2 \left\| \nabla \sigma_\varepsilon^0 - \nabla s_\delta^0 \right\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}^2 + C \left\| \sigma_\varepsilon^0 - s_\delta^0 \right\|_{L^2(\mathbb{R}^3)}^2 . \quad (6.8)$$

6.2. Convective terms

Next, we provide bounds for the convective terms corresponding to I_1 in (5.4). In the following, we drop the subscript ε in $(\rho_\varepsilon, u_\varepsilon)$ for the sake of a concise notation. Specifically, we have

$$I_1 = \int_0^\tau \int_{\mathbb{R}^3} \left[\partial_t \mathbf{U} \cdot (\varrho \mathbf{U} - \varrho \mathbf{u}) + (\sqrt{\varrho} \mathbf{u} \cdot \nabla) \mathbf{U} \cdot (\sqrt{\varrho} \mathbf{U} - \sqrt{\varrho} \mathbf{u}) \right] dxdt$$

$$= \int_0^\tau \int_{\mathbb{R}^3} (\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U}) \cdot (\varrho \mathbf{U} - \varrho \mathbf{u}) dxdt$$

$$+ \int_0^\tau \int_{\mathbb{R}^3} \nabla \mathbf{U} (\sqrt{\varrho} \mathbf{U} - \sqrt{\varrho} \mathbf{u}) \cdot (\sqrt{\varrho} \mathbf{U} - \sqrt{\varrho} \mathbf{u})\, dxdt = I_1^{(1)} + I_1^{(2)}.$$

As $\nabla U \in L^\infty_{t,x}$, the term $I_1^{(2)}$ is controlled by

$$|I_1^{(2)}| \leq C \int_0^\tau \mathcal{E}(\cdot, t) dt$$

, while $I_1^{(1)}$ can be written as follows

$$I_1^{(1)} = \int_0^\tau \int_{\mathbb{R}^3} (\varrho\mathbf{U} - \varrho\mathbf{u}) \cdot (\partial_t \mathbf{u}^E + \mathbf{u}^E \cdot \nabla \mathbf{u}^E)\, dxdt$$

$$+ \int_0^\tau \int_{\mathbb{R}^3} (\varrho\mathbf{U} - \varrho\mathbf{u}) \cdot \partial_t \nabla \Phi_{\varepsilon,\delta}\, dxdt + \int_0^\tau \int_{\mathbb{R}^3} (\varrho\mathbf{U} - \varrho\mathbf{u}) \otimes \nabla \Phi_{\varepsilon,\delta} : \nabla \mathbf{u}^E\, dxdt$$

$$+ \int_0^\tau \int_{\mathbb{R}^3} (\varrho\mathbf{U} - \varrho\mathbf{u}) \otimes \mathbf{u}^E : \nabla^2 \Phi_{\varepsilon,\delta}\, dxdt + \frac{1}{2} \int_0^\tau \int_{\mathbb{R}^3} (\varrho\mathbf{U} - \varrho\mathbf{u}) \cdot \nabla |\nabla \Phi_{\varepsilon,\delta}|^2\, dxdt. \tag{6.9}$$

Now, we analyse the first term in (6.9). We have

$$\int_0^\tau \int_{\mathbb{R}^3} (\varrho\mathbf{U} - \varrho\mathbf{u}) \cdot (\partial_t \mathbf{u}^E + \mathbf{u}^E \cdot \nabla \mathbf{u}^E)\, dxdt$$
$$= \int_0^\tau \int_{\mathbb{R}^3} \varrho\mathbf{u} \cdot \nabla \Pi\, dxdt - \int_0^\tau \int_{\mathbb{R}^3} \varrho\mathbf{U} \cdot \nabla \Pi\, dxdt.$$

In view of (3.3), there exists $\overline{\mathbf{u}} \in L^\infty(0,T; L^2(\mathbb{R}^3))$ such that $\sqrt{\varrho}\mathbf{u} \rightharpoonup^* \overline{\mathbf{u}}$ in $L_t^\infty L_x^2$. Together with (3.4), it follows that

$$\varrho\mathbf{u} = (\sqrt{\rho} - 1)\sqrt{\rho}u + \sqrt{\rho}u \rightharpoonup^* \overline{\mathbf{u}} \quad \text{weakly-(*) in } L_t^\infty(L_x^2 + L_x^{\frac{3}{2}-}).$$

In particular, one has that

$$\int_0^\tau \int_{\mathbb{R}^3} \varrho\mathbf{u} \cdot \nabla \Pi\, dxdt \to \int_0^\tau \int_{\mathbb{R}^3} \overline{\mathbf{u}} \cdot \nabla \Pi\, dxdt. \tag{6.10}$$

We consider the weak formulation of the continuity equation

$$\int_0^\tau \int_{\mathbb{R}^3} (\varrho \partial_t \varphi + \varrho\mathbf{u} \cdot \nabla \varphi)\, dxdt = \int_{\mathbb{R}^3} \varrho(\tau,\cdot)\varphi(\tau,\cdot) dx - \int_{\mathbb{R}^3} \varrho(0,\cdot)\varphi(0,\cdot) dx$$

and upon relying on the uniform bound (2.9) for Π and approximation of Sobolev functions by smooth functions of compact support we may use Π as a test function. In view of (6.10) and

$$\left[\int_{\mathbb{R}^3} \varrho \Pi\, dx|_0^\tau - \int_0^\tau \int_{\mathbb{R}^3} \varrho \partial_t \Pi\, dxdt \right] \to 0 \quad \text{as} \quad \varepsilon \to 0$$

thanks to Lemma 3.1 and (3.2), we have

$$\int_0^\tau \int_{\mathbb{R}^3} \varrho\mathbf{u} \cdot \nabla \Pi\, dxdt \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Next, we consider

$$\int_0^\tau \int_{\mathbb{R}^3} \varrho\mathbf{U} \cdot \nabla \Pi\, dxdt = \int_0^\tau \int_{\mathbb{R}^3} (\varrho - 1)\mathbf{U} \cdot \nabla \Pi\, dxdt + \int_0^\tau \int_{\mathbb{R}^3} \mathbf{U} \cdot \nabla \Pi\, dxdt.$$

Again thanks to 3.2, it follows

$$\int_0^\tau \int_{\mathbb{R}^3} (\varrho - 1)\mathbf{U} \cdot \nabla \Pi \, dxdt \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

while

$$\int_0^\tau \int_{\mathbb{R}^3} \mathbf{U} \cdot \nabla \Pi \, dxdt = \int_0^\tau \int_{\mathbb{R}^3} \mathbf{u}^E \cdot \nabla \Pi \, dxdt + \int_0^\tau \int_{\mathbb{R}^3} \nabla \Phi_{\varepsilon,\delta} \cdot \nabla \Pi \, dxdt.$$

Performing an integration by parts yields

$$\int_0^\tau \int_{\mathbb{R}^3} \operatorname{div} \mathbf{u}^E \cdot \Pi \, dxdt = 0$$

thanks to the incompressibility condition $\operatorname{div}_x \mathbf{u}^E = 0$. For the other term, using again integration by parts and the acoustic equation (4.7), we have

$$\int_0^\tau \int_{\mathbb{R}^3} \nabla \Phi_{\varepsilon,\delta} \cdot \nabla \Pi \, dxdt = -\int_0^\tau \int_{\mathbb{R}^3} \Delta \Phi_{\varepsilon,\delta} \Pi \, dxdt = \varepsilon \int_0^\tau \int_{\mathbb{R}^3} \partial_t s_{\varepsilon,\delta} \, \Pi \, dxdt$$

$$= \left[\varepsilon \int_{\mathbb{R}^3} s_{\varepsilon,\delta} \, \Pi \, dx |_0^\tau - \varepsilon \int_0^\tau \int_{\mathbb{R}^3} s_{\varepsilon,\delta} \, \partial_t \Pi \, dxdt \right] \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Now, we analyse the second term in (6.9). We have

$$\int_0^\tau \int_{\mathbb{R}^3} (\varrho \mathbf{U} - \varrho \mathbf{u}) \cdot \partial_t \nabla \Phi_{\varepsilon,\delta} \, dxdt = \int_0^\tau \int_{\mathbb{R}^3} \varrho \mathbf{u}^E \cdot \partial_t \nabla \Phi_{\varepsilon,\delta} \, dxdt$$

$$- \int_0^\tau \int_{\mathbb{R}^3} \varrho \mathbf{u} \cdot \partial_t \nabla \Phi_{\varepsilon,\delta} \, dxdt + \frac{1}{2} \int_0^\tau \int_{\mathbb{R}^3} \varrho \partial_t |\nabla \Phi_{\varepsilon,\delta}|^2 \, dxdt$$

$$= \int_0^\tau \int_{\mathbb{R}^3} \varrho \mathbf{u}^E \cdot \partial_t \nabla \Phi_{\varepsilon,\delta} \, dxdt + I^I + I^{II},$$

where

$$I^I = -\int_0^\tau \int_{\mathbb{R}^3} \varrho \mathbf{u} \cdot \partial_t \nabla \Phi_{\varepsilon,\delta} \, dxdt, \quad I^{II} = \frac{1}{2} \int_0^\tau \int_{\mathbb{R}^3} \varrho \partial_t |\nabla \Phi_{\varepsilon,\delta}|^2 \, dxdt, \quad (6.11)$$

The terms I^I and I^{II} will be handled later in combination with other terms. For the remaining one, we have

$$\int_0^\tau \int_{\mathbb{R}^3} \varrho \mathbf{u}^E \cdot \partial_t \nabla \Phi_{\varepsilon,\delta} \, dxdt = \int_0^\tau \int_{\mathbb{R}^3} (\varrho - 1) \mathbf{u}^E \cdot \partial_t \nabla \Phi_{\varepsilon,\delta} \, dxdt$$

$$+ \int_0^\tau \int_{\mathbb{R}^3} \mathbf{u}^E \cdot \partial_t \nabla \Phi_{\varepsilon,\delta} \, dxdt$$

where, thanks to $\operatorname{div}_x \mathbf{u}^E = 0$, the second term

$$\int_0^\tau \int_{\mathbb{R}^3} \operatorname{div}_x \mathbf{u}^E \cdot \partial_t \Phi_{\varepsilon,\delta} \, dxdt = 0.$$

Using the acoustic equation (4.7), the first term is written as follows

$$\int_0^\tau \int_{\mathbb{R}^3} (\varrho - 1)\mathbf{u}^E \cdot \partial_t \nabla \Phi_{\varepsilon,\delta} \, dxdt$$

$$= \left[\int_0^\tau \int_{\mathbb{R}^3} \gamma(\varrho - 1)\mathbf{u}^E \cdot \left(-\frac{1}{\varepsilon} \nabla s_{\varepsilon,\delta} + 2\kappa^2 \varepsilon \Delta s_{\varepsilon,\delta} \right) \, dxdt \right] \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

We conclude analysing the last three terms in (6.9). Owing to Corollary 6.1, we have

$$\begin{aligned} &\left[\int_0^\tau \int_{\mathbb{R}^3} (\varrho \mathbf{U} - \varrho \mathbf{u}) \otimes \nabla \Phi_{\varepsilon,\delta} : \nabla \mathbf{u}^E \, dxdt \right. \\ &\quad + \int_0^\tau \int_{\mathbb{R}^3} (\varrho \mathbf{U} - \varrho \mathbf{u}) \otimes \mathbf{u}^E : \nabla^2 \Phi_{\varepsilon,\delta} \, dxdt \\ &\quad \left. + \frac{1}{2} \int_0^\tau \int_{\mathbb{R}^3} (\varrho \mathbf{U} - \varrho \mathbf{u}) \cdot \nabla |\nabla \Phi_{\varepsilon,\delta}|^2 \, dxdt \right] \to 0 \quad \text{as} \quad \varepsilon \to 0. \end{aligned}$$

In particular, the last term writes

$$\begin{aligned} &\frac{1}{2} \int_0^\tau \int_{\mathbb{R}^3} (\varrho \mathbf{U} - \varrho \mathbf{u}) \cdot \nabla |\nabla \Phi_{\varepsilon,\delta}|^2 \, dxdt \\ &= \frac{1}{2} \int_0^\tau \int_{\mathbb{R}^3} \left(\sqrt{\rho} - 1 \right) \left(\sqrt{\varrho} \mathbf{U} - \sqrt{\varrho} \mathbf{u} \right) \cdot \nabla |\nabla \Phi_{\varepsilon,\delta}|^2 + \left(\sqrt{\varrho} \mathbf{U} - \sqrt{\varrho} \mathbf{u} \right) \cdot \nabla |\nabla \Phi_{\varepsilon,\delta}|^2 \, dxdt. \end{aligned}$$

The former converges to 0 in view of the uniform bounds (3.4) and the latter by virtue of Corollary 6.1.

6.3. Korteweg terms

The Korteweg terms amount to I_2 and I_3 as defined in (5.4). For a fixed $\kappa > 0$ and from (6.1), we have for I_2 that

$$\begin{aligned} I_2 &= 2\kappa^2 \int_0^\tau \int_{\mathbb{R}^3} (\varrho \partial_t \Delta r + \varrho \mathbf{u} \cdot \nabla \Delta r - \partial_t r \Delta r) dxdt \\ &= \int_0^\tau \int_{\mathbb{R}^3} (\varepsilon \varrho \partial_t \Delta s_{\varepsilon,\delta} + \varepsilon (\varrho \mathbf{u}) \cdot \nabla \Delta s_{\varepsilon,\delta} - \varepsilon^2 \partial_t s_{\varepsilon,\delta} \Delta s_{\varepsilon,\delta}) dxdt \to 0 \ \text{ as } \ \varepsilon \to 0 \end{aligned}$$

thanks to the uniform bounds (3.2), (3.3) for $\rho - 1$ and ρu, respectively, as well as the bounds (6.5) and (6.3) for $s_{\varepsilon,\delta}$, $\Phi_{\varepsilon,\delta}$. Upon using (6.1) and $\operatorname{div} \mathbf{u}^E = 0$, we write I_3 as

$$\begin{aligned} I_3 &= -2\kappa^2 \int_0^\tau \int_{\mathbb{R}^3} \left(\nabla \varrho \cdot \nabla (\varrho \operatorname{div} \mathbf{U}) - \frac{1}{2} |\nabla \varrho|^2 \operatorname{div} \mathbf{U} + \nabla \varrho \otimes \nabla \varrho : \nabla \mathbf{U} \right) dxdt. \\ &= -\kappa^2 \int_0^\tau \int_{\mathbb{R}^3} |\nabla \varrho|^2 \Delta \Phi_{\varepsilon,\delta} dxdt - 2\kappa^2 \int_0^\tau \int_{\mathbb{R}^3} \varrho \nabla \varrho \cdot \nabla \Delta \Phi_{\varepsilon,\delta} dxdt \\ &\quad - 2\kappa^2 \int_0^\tau \int_{\mathbb{R}^3} \nabla \varrho \otimes \nabla \varrho : \nabla (\mathbf{u}^E + \nabla \Phi_{\varepsilon,\delta})) dxdt \\ &= I_3^{(1)} + I_3^{(2)} + I_3^{(3)}. \end{aligned}$$

The integral I_3^1 is bounded as

$$\begin{aligned}
&\left|\kappa^2\int_0^\tau\int_{\mathbb{R}^3}\frac{1}{2}|\nabla\varrho|^2\Delta\Phi_{\varepsilon,\delta}dxdt\right| \le \kappa^2\int_0^\tau\int_{\mathbb{R}^3}|\nabla\varrho-\nabla r|^2\left|\Delta\Phi_{\varepsilon,\delta}\right| + |\nabla r|^2\left|\Delta\Phi_{\varepsilon,\delta}\right|dxdt\\
&\le \kappa^2\|\Delta\Phi_{\varepsilon,\delta}\|_{L^\infty([0,T]\times\mathbb{R}^3)}\int_0^\tau\int_{\mathbb{R}^3}|\nabla\varrho-\nabla r|^2dxdt + \|\nabla s_{\varepsilon,\delta}\|^2_{L^2L^4}\|\Delta\Phi_{\varepsilon,\delta}\|_{L^2L^2}\\
&\le C\int_0^\tau\mathcal{E}(\cdot,t)dt + C\kappa^2T^\beta\varepsilon^\alpha
\end{aligned}$$

for some $\alpha,\beta>0$ small from (6.3). Further and due to (6.5), for any $s>\frac{5}{2}$ it holds

$$\|\Delta\Phi_{\varepsilon,\delta}\|_{L^\infty([0,T]\times\mathbb{R}^3)} \le \|\nabla\Phi_{\varepsilon,\delta}\|_{L^\infty H^s} \le C. \tag{6.12}$$

The term I_3^2 obeys the bound

$$\begin{aligned}
&\left|\int_0^\tau\int_{\mathbb{R}^3}\rho\nabla\rho\nabla\Delta\Phi_{\varepsilon,\delta}dxdt\right| = \left|\int_0^\tau\int_{\mathbb{R}^3}(\rho-1)\nabla\rho\nabla\Delta\Phi_{\varepsilon,\delta}+\nabla(\rho-1)\nabla\Delta\Phi_{\varepsilon,\delta}dxdt\right|\\
&\le T^\beta\|\rho-1\|_{L^\infty L^4}\|\nabla\rho\|_{L^\infty L^2}\|\nabla\Delta\Phi_{\varepsilon,\delta}\|_{L^{\frac{8}{3}}L^4} + \left|\int_0^\tau\int_{\mathbb{R}^3}\nabla(\rho-1)\nabla\Delta\Phi_{\varepsilon,\delta}dxdt\right|\\
&\le CT^\beta\varepsilon^\alpha + \left|\int_0^\tau\int_{\mathbb{R}^3}\nabla(\rho-1)\nabla\Delta\Phi_{\varepsilon,\delta}dxdt\right|.
\end{aligned}$$

Note that $\nabla(\rho-1)\in L^\infty(0,T;L^2(\mathbb{R}^3))$ is uniformly bounded and $\rho-1\to 0$ strongly in $L^\infty(0,T;H^s(\mathbb{R}^3))$ for all $0\le s<1$ from (3.2). It follows that $\nabla(\rho-1)\rightharpoonup^* 0$ in $L^\infty(0,T;L^2(\mathbb{R}^3))$ as $\varepsilon\to 0$. Hence,

$$\int_0^\tau\int_{\mathbb{R}^3}\nabla(\rho-1)\nabla\Delta\Phi_{\varepsilon,\delta}dxdt \to 0 \quad \text{as } \varepsilon\to 0.$$

Finally, to bound $I_3^{(3)}$, one observes that

$$\begin{aligned}
|I_3^{(3)}| &\le \kappa^2\int_0^\tau\int_{\mathbb{R}^3}|\nabla\rho|^2|\nabla\mathbf{U}|\,dxdt\\
&\le 2\kappa^2\|\nabla\mathbf{U}\|_{L^\infty([0,T]\times\mathbb{R}^3)}\int_0^\tau\int_{\mathbb{R}^3}|\nabla\rho-\nabla r|^2 + 2\kappa^2\int_0^\tau\int_{\mathbb{R}^3}|\nabla r|^2|\nabla\mathbf{U}|\,dxdt.
\end{aligned}$$

Using that $\mathbf{U}=\mathbf{u}^E+\nabla\Phi_{\varepsilon,\delta}$ is smooth and arguing as for $I_3^{(1)}+I_3^{(2)}$, we conclude that

$$|I_3^{(3)}| \le C\int_0^\tau\mathcal{E}(\cdot,t)dt + C\kappa T^\beta\varepsilon^\alpha$$

for some $\alpha,\beta>0$.

6.4. Viscous terms

We have

$$I_4 = 2\nu\int_0^\tau\int_{\mathbb{R}^3}\sqrt{\varrho}\mathcal{S}(\mathbf{u}):\nabla\mathbf{u}^E dxdt + 2\nu\int_0^\tau\int_{\mathbb{R}^3}\sqrt{\varrho}\mathcal{S}(\mathbf{u}):\nabla\nabla\Phi_{\varepsilon,\delta}\,dxdt = I_4^{(1)}+I_4^{(2)}.$$

Now,

$$
\begin{aligned}
I_4^{(1)} &= 2\nu \int_0^\tau \int_{\mathbb{R}^3} (\sqrt{\varrho}-1)\mathcal{S}(\mathbf{u}) : \nabla \mathbf{u}^E dxdt + 2\nu \int_0^\tau \int_{\mathbb{R}^3} \mathcal{S}(\mathbf{u}) : \nabla \mathbf{u}^E dxdt. \\
&\le C\sqrt{\nu}\left[\int_0^\tau \int_{\mathbb{R}^3} |\sqrt{\varrho}-1|^2 dxdt\right]^{1/2} \sqrt{\nu}\left[\int_0^\tau \int_{\mathbb{R}^3} |\mathcal{S}(\mathbf{u})|^2 dxdt\right]^{1/2} \\
&\quad + C\sqrt{\nu}\left[\int_0^\tau \int_{\mathbb{R}^3} |\mathcal{S}(\mathbf{u})|^2 dxdt\right]^{1/2} \sqrt{\nu} \\
&\le \frac{C}{2}\nu \int_0^\tau \int_{\mathbb{R}^3} |\sqrt{\varrho}-1|^2 dxdt + \frac{C}{2}\nu \int_0^\tau \int_{\mathbb{R}^3} |\mathcal{S}(\mathbf{u})|^2 dxdt \\
&\quad + \frac{C}{2}\nu \int_0^\tau \int_{\mathbb{R}^3} |\mathcal{S}(\mathbf{u})|^2 dxdt + C(\nu).
\end{aligned}
$$

The first term goes to zero as $\varepsilon \to 0$ thanks to (3.2). The viscous terms containing $\mathcal{S}(\mathbf{u})$ can be absorbed on the left-hand side of (5.4) and $C(\nu) \to 0$ as $\nu \to 0$. For the second term, we have

$$
\begin{aligned}
I_4^{(2)} &= 2\nu \int_0^\tau \int_{\mathbb{R}^3} (\sqrt{\varrho}-1)\mathcal{S}(\mathbf{u}) : \nabla\nabla\Phi_{\varepsilon,\delta}\, dxdt + 2\nu \int_0^\tau \int_{\mathbb{R}^3} \mathcal{S}(\mathbf{u}) : \nabla\nabla\Phi_{\varepsilon,\delta}\, dxdt \\
&\le C\nu \|\sqrt{\varrho}-1\|_{L_t^4 L_x^6} \|\mathcal{S}(\mathbf{u})\|_{L_t^2 L_x^2} \|\nabla\nabla\Phi_{\varepsilon,\delta}\|_{L_t^4 L_x^3} \\
&\quad + C\sqrt{\nu}\left[\int_0^\tau \int_{\mathbb{R}^3} |\mathcal{S}(\mathbf{u})|^2 dxdt\right]^{1/2} \sqrt{\nu}\left[\int_0^\tau \int_{\mathbb{R}^3} |\nabla\nabla\Phi_{\varepsilon,\delta}|^2 dxdt\right]^{1/2} \\
&\le C\varepsilon^\alpha \sqrt{\nu} + C\sqrt{\nu}\left[\int_0^\tau \int_{\mathbb{R}^3} |\mathcal{S}(\mathbf{u})|^2 dxdt\right]^{1/2} \\
&\quad \sqrt{\nu}\left[\int_0^\tau \int_{\mathbb{R}^3} |\nabla\nabla\Phi_{\varepsilon,\delta}|^2 dxdt\right]^{1/2}, \quad (0 \le \alpha \le 1/12).
\end{aligned}
$$

Here, we used (3.4), (6.3) and the second term on the right-hand side can be handled similarly as in $I_4^{(1)}$.

6.5. Pressure terms

We recall

$$
I_5 = -\frac{1}{\varepsilon^2}\int_0^\tau \int_{\mathbb{R}^3} \left[\partial_t(H'(r))(\varrho - r) + \varrho\mathbf{u}\cdot\nabla(H'(r)) + p(\varrho)\mathrm{div}\mathbf{U}\right] dxdt.
$$

We first consider the second term in I_5. Using $r = 1 + \varepsilon s_{\varepsilon,\delta}$, we have

$$
-\frac{1}{\varepsilon^2}\int_0^\tau \int_{\mathbb{R}^3} \varrho\mathbf{u}\cdot\nabla(H'(r))dxdt = -\frac{1}{\varepsilon}\int_0^\tau \int_{\mathbb{R}^3} \varrho\mathbf{u}\cdot H''(r)\nabla s_{\varepsilon,\delta} dxdt
$$

$$
= -\int_0^\tau \int_{\mathbb{R}^3} \varrho\mathbf{u}\cdot\nabla s_{\varepsilon,\delta} \frac{H''(1+\varepsilon s_{\varepsilon,\delta}) - H''(1)}{\varepsilon} dxdt - \frac{1}{\varepsilon}\int_0^\tau \int_{\mathbb{R}^3} \varrho\mathbf{u}\cdot H''(1)\nabla s_{\varepsilon,\delta} dxdt,
$$

where $H''(1) = p'(1) = \gamma$. Realizing that

$$\left| \frac{H''(1 + \varepsilon s_{\varepsilon,\delta}) - H''(1)}{\varepsilon} \right| \le C|s_{\varepsilon,\delta}|,$$

we have

$$\left[\int_0^\tau \int_{\mathbb{R}^3} \varrho \mathbf{u} \cdot \nabla s_{\varepsilon,\delta} \frac{H''(1 + \varepsilon s_{\varepsilon,\delta}) - H''(1)}{\varepsilon} dxdt \right] \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

from Corollary 6.1. For the other term, using the acoustic equation (4.7),

$$-\frac{\gamma}{\varepsilon} \int_0^\tau \int_{\mathbb{R}^3} \varrho \mathbf{u} \cdot \nabla s_{\varepsilon,\delta} dxdt = \int_0^\tau \int_{\mathbb{R}^3} \varrho \mathbf{u} \cdot \partial_t \nabla \Phi_{\varepsilon,\delta} \, dxdt - 2\gamma \varepsilon \kappa^2 \int_0^\tau \int_{\mathbb{R}^3} \varrho \mathbf{u} \cdot \nabla \Delta s_{\varepsilon,\delta} \, dxdt$$

where, without loss of generality, we assumed $H''(1) = 1$. The first term cancels with its counterpart I^I in (6.11), while the second converges to 0. Now,

$$\begin{aligned}
&\frac{1}{\varepsilon^2} \int_0^\tau \int_{\mathbb{R}^3} \partial_t (H'(r))(r - \varrho) - p(\varrho) \mathrm{div} \mathbf{U} dxdt \\
&\quad = \int_0^\tau \int_{\mathbb{R}^3} \frac{1 - \varrho}{\varepsilon} H''(r) \partial_t s_{\varepsilon,\delta} \, dxdt + \int_0^\tau \int_{\mathbb{R}^3} s_{\varepsilon,\delta} H''(r) \partial_t s_{\varepsilon,\delta} \, dxdt \\
&\quad \quad - \int_0^\tau \int_{\mathbb{R}^3} \frac{p(\varrho)}{\varepsilon^2} \Delta \Phi_{\varepsilon,\delta} dxdt \\
&= \int_0^\tau \int_{\mathbb{R}^3} \frac{1 - \varrho}{\varepsilon} H''(1) \partial_t s_{\varepsilon,\delta} + \int_0^\tau \int_{\mathbb{R}^3} \frac{1 - \varrho}{\varepsilon} (H''(r) - H''(1)) \partial_t s_{\varepsilon,\delta} dxdt \\
&\qquad + \int_0^\tau \int_{\mathbb{R}^3} s_{\varepsilon,\delta} H''(r) \partial_t s_{\varepsilon,\delta} \, dxdt - \int_0^\tau \int_{\mathbb{R}^3} \frac{p(\varrho)}{\varepsilon^2} \Delta \Phi_{\varepsilon,\delta} \\
&= \int_0^\tau \int_{\mathbb{R}^3} s_{\varepsilon,\delta} H''(r) \partial_t s_{\varepsilon,\delta} \, dxdt + \int_0^\tau \int_{\mathbb{R}^3} \frac{1 - \varrho}{\varepsilon} (H''(r) - H''(1)) \Delta \Phi_{\varepsilon,\delta} dxdt \\
&\qquad - \int_0^\tau \int_{\mathbb{R}^3} \frac{p(\varrho) - p'(1)(\varrho - 1) - p(1)}{\varepsilon^2} \Delta \Phi_{\varepsilon,\delta} dxdt.
\end{aligned}$$

Consequently, for these remaining terms we have

$$\begin{aligned}
&\left[- \int_0^\tau \int_{\mathbb{R}^3} \frac{p(\varrho) - p'(1)(\varrho - 1) - p(1)}{\varepsilon^2} \Delta \Phi_{\varepsilon,\delta} \, dxdt \right. \\
&\left. - \int_0^\tau \int_{\mathbb{R}^3} \frac{1 - \varrho}{\varepsilon} \frac{(H''(r) - H''(1))}{\varepsilon} \Delta \Phi_{\varepsilon,\delta} \, dxdt \right] \to 0 \quad \text{as} \quad \varepsilon \to 0,
\end{aligned}$$

by combining (3.2) for $\gamma \ge 2$ and (3.1) for $1 < \gamma < 2$ and Corollary 6.1. Finally,

$$\begin{aligned}
&\int_0^\tau \int_{\mathbb{R}^3} s_{\varepsilon,\delta} H''(r) \partial_t s_{\varepsilon,\delta} \, dxdt \\
&= \int_0^\tau \int_{\mathbb{R}^3} s_{\varepsilon,\delta} H''(1) \partial_t s_{\varepsilon,\delta} \, dxdt + \int_0^\tau \int_{\mathbb{R}^3} s_{\varepsilon,\delta} (H''(r) - H''(1)) \partial_t s_{\varepsilon,\delta} \, dxdt
\end{aligned}$$

where, using similar arguments as above,

$$\left[\int_0^\tau \int_{\mathbb{R}^3} s_{\varepsilon,\delta}(H''(r) - H''(1))\partial_t s_{\varepsilon,\delta}\, dxdt\right] \to 0 \quad \text{as} \quad \varepsilon \to 0$$

and

$$\int_0^\tau \int_{\mathbb{R}^3} s_{\varepsilon,\delta} H''(1)\partial_t s_{\varepsilon,\delta}\, dxdt = \frac{1}{2}\int_{\mathbb{R}^3} |s_{\varepsilon,\delta}|^2 dx|_0^\tau.$$

Back to I^{II} in (6.11), we have

$$\begin{aligned}&\frac{1}{2}\int_0^\tau \int_{\mathbb{R}^3} \varrho\partial_t |\nabla\Phi_{\varepsilon,\delta}|^2\, dxdt\\ &\quad = \frac{1}{2}\int_0^\tau \int_{\mathbb{R}^3} (\varrho - 1)\partial_t |\nabla\Phi_{\varepsilon,\delta}|^2\, dxdt + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\Phi_{\varepsilon,\delta}|^2\, dx|_0^\tau.\end{aligned}$$

By acoustic energy conservation, we have

$$\frac{1}{2}\int_{\mathbb{R}^3} |s_{\varepsilon,\delta}|^2 dx|_0^\tau + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\Phi_{\varepsilon,\delta}|^2\, dx|_0^\tau = 0$$

and

$$\frac{1}{2}\int_0^\tau \int_{\mathbb{R}^3} (\varrho - 1)\partial_t |\nabla\Phi_{\varepsilon,\delta}|^2\, dxdt =$$

$$\frac{1}{2}\int_0^\tau \int_{\mathbb{R}^3} (\varrho - 1)\nabla\Phi_{\varepsilon,\delta}\cdot\left(-\frac{1}{\varepsilon}\nabla s_{\varepsilon,\delta} + 2\varepsilon\kappa^2\nabla\Delta s_{\varepsilon,\delta}\right) dxdt \to 0 \quad \text{as} \quad \varepsilon \to 0$$

from (3.2) for $\gamma \geq 2$ and (3.1) for $1 < \gamma < 2$ as well as Corollary 6.1.

6.6. Proof of Theorem 2.3

From (5.4) and all the estimates above, we end up with

$$\mathcal{E}(\tau) \leq \mathcal{E}(0) + C\int_0^\tau \mathcal{E}(\cdot, t)dt + \chi(\nu, \varepsilon) \tag{6.13}$$

where $\chi(\nu, \varepsilon) \to 0$ as $(\nu, \varepsilon) \to 0$ (for fixed $\delta > 0$). In virtue of the integral form of the Gronwall's inequality, we have

$$\mathcal{E}(\tau) \leq C(T)\mathcal{E}(0) + \chi(\nu, \varepsilon). \tag{6.14}$$

Consequently, sending $\varepsilon \to 0$ first, and then $\delta \to 0$, thanks to (2.10), (2.11) and (6.8), we have

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \mathcal{E}(\tau) = 0 \tag{6.15}$$

uniformly in $\tau \in (0, T)$. Now, considering $\mathbf{U} = \mathbf{u}^E + \nabla\Phi_{\varepsilon,\delta}$, for any compact set $K \subset \mathbb{R}^3$ we have

$$\|\sqrt{\varrho}(\mathbf{u} - \mathbf{u}^E)\|_{L^2_t L^2_x(K)} \leq \|\sqrt{\varrho}(\mathbf{u} - \mathbf{U})\|_{L^2_t L^2_x(K)} + \|\sqrt{\varrho}\nabla\Phi_{\varepsilon,\delta}\|_{L^p_t L^q_x(K)} \tag{6.16}$$

for $p, q > 2$. The first quantity on the right-hand side of (6.16) goes to zero thanks to (6.15), while

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \|\sqrt{\varrho}\nabla\Phi_{\varepsilon,\delta}\|_{L^p_t L^q_x(K)} = 0 \tag{6.17}$$

thanks to (3.4), (6.3) and (6.5) provided that (p, q) is a Schrödinger-admissible pair. In fact, we have

$$\|\sqrt{\varrho}\nabla\Phi_{\varepsilon,\delta}\|_{L^p_t L^q_x(K)} \leq \|(\sqrt{\varrho}-1)\nabla\Phi_{\varepsilon,\delta}\|_{L^p_t L^q_x(K)} + \|\nabla\Phi_{\varepsilon,\delta}\|_{L^p_t L^q_x(K)} \to 0 \quad \text{as} \quad \varepsilon \to 0$$

for a fixed $\delta > 0$. Consequently, (6.17) holds. Theorem 2.3 is proved.

7. Comments on the 2D case

In this section, we discuss an extension of our main result to the problem posed on $\mathbb{R}^2$. To that end, we only highlight the key steps and modifications needed for its proof. The relevant differences compared to $d = 3$ are given by

- the properties of the target system, namely the $2D$ Euler equations,
- while $\varrho_\epsilon - 1$ enjoys slightly better integrability properties due to Sobolev embedding, the dispersion of the acoustic waves is weaker in $\mathbb{R}^2$ compared to $\mathbb{R}^3$.

To start and concerning the $2D$ incompressible Euler equations, one has global existence of strong solutions as singularity formation in finite time is ruled out by the Beale–Kato–Majda criterion and the conservation of the L^∞-norm of the vorticity, see, for example, [30].

Theorem 7.1. *Given* $\mathbf{u}_0^E \in W^{3,2}(\mathbb{R}^2)$ *with* $\operatorname{div}\mathbf{u}_0 = 0$, *there exists a unique solution*

$$\mathbf{u}^E \in C^k([0,\infty), W^{3-k,2}(\mathbb{R}^2;\mathbb{R}^2)),\ \Pi \in C^k([0,\infty), W^{3-k,2}(\mathbb{R}^2)),\ k = 0, 1, 2, 3$$

to the initial value problem (E) - (1.7) such that for all $0 < T < \infty$ *it holds*

$$\|\mathbf{u}^E\|_{W^{k,\infty}(0,T;W^{3-k,2}(\mathbb{R}^2;\mathbb{R}^2))} + \|\Pi\|_{W^{k,\infty}(0,T;W^{3-k,2}(\mathbb{R}^2))} \leq c(T, \|\mathbf{u}_0^E\|_{W^{3,2}(\mathbb{R}^2)}). \tag{7.1}$$

For the primitive system (NSK) posed on $[0, \infty) \times \mathbb{R}^2$, we postulate existence of global finite energy weak solutions according to Definition 2.1 adapted to $d = 2$. Similarly to the case $d = 3$, we expect that global existence can be inferred in the spirit of Theorem 2.1 by relying on [3,5].

Theorem 7.2. *Given initial data* $(\varrho^0, \mathbf{u}^0)$ *of finite energy and BD-entropy, then, there exists at least a global weak solution* $(\varrho, \mathbf{u}, \mathcal{T})$ *of* (NSK) *posed on* $[0, T) \times \mathbb{R}^2$ *in the sense of Definition 2.1.*

The constructed finite energy weak solutions satisfy the following uniform bounds.

Lemma 7.1. *Let* $(\varrho_\epsilon, \mathbf{u}^0_\varepsilon)$ *be a FEWS to* (NSK) *posed on* $[0,T)\times\mathbb{R}^2$ *with initial data* $(\varrho^0_\varepsilon, \mathbf{u}^0_\varepsilon)$ *satisfying*

$$E(\varrho^0_\varepsilon, \mathbf{u}^0_\varepsilon) \le C, \quad B(\varrho^0_\varepsilon, \mathbf{u}^0_\varepsilon) \le C,$$

for some $C > 0$ *independent of* $\epsilon > 0$. *Then, the following hold:*

(i) $\varrho^0_\varepsilon - 1 \in H^1(\mathbb{R}^2)$ *and* $\varrho_\epsilon - 1 \in L^\infty(0,T;H^1(\mathbb{R}^2))$ *uniformly bounded and*

$$\|\varrho^0_\varepsilon - 1\|_{L^2(\mathbb{R}^2)} \le C\varepsilon, \qquad \|\varrho_\varepsilon - 1\|_{L^\infty(0,T;L^2(\mathbb{R}^2))} \le C\varepsilon,$$

(ii) $\sqrt{\varrho^0_\varepsilon} - 1 \in L^2(\mathbb{R}^2)$, $\sqrt{\varrho_\varepsilon} - 1 \in L^\infty(0,T;L^2(\mathbb{R}^2))$ *uniformly bounded with*

$$\|\sqrt{\varrho^0_\varepsilon} - 1\|_{L^2(\mathbb{R}^2)} \le C\varepsilon, \qquad \|\sqrt{\varrho_\varepsilon} - 1\|_{L^\infty(0,T;L^2(\mathbb{R}^2))} \le C\varepsilon,$$

(iii) $\sqrt{\varrho^0_\varepsilon}\mathbf{u}^0_\varepsilon \in L^2(\mathbb{R}^2)$ *uniformly bounded,*

(iv) the momentum satisfies $\varrho^0_\varepsilon\mathbf{u}^0_\varepsilon \in L^2(\mathbb{R}^2) + L^p(\mathbb{R}^2)$, $\varrho^0_\varepsilon\mathbf{u}^0_\varepsilon \in L^\infty(0,T;L^2(\mathbb{R}^2) + L^p(\mathbb{R}^2))$ *uniformly bounded for all* $1 \le p < 2$,

(v) the density fluctuations σ^0_ε *satisfy* $\sigma^0_\varepsilon \in L^2(\mathbb{R}^2)$, $\sigma_\varepsilon \in L^\infty(0,T;L^2(\mathbb{R}^2))$ *uniformly bounded as well as* $\varepsilon\sigma^0_\varepsilon \in H^1(\mathbb{R}^2)$, $\varepsilon\sigma_\varepsilon \in L^\infty(0,T;H^1(\mathbb{R}^2))$ *uniformly bounded.*

Proof. The only modification in the proof compared to one of Lemma 3.1 consists in the fact that the argument yielding $\varrho_\varepsilon - 1 \in L^6(\mathbb{R}^3)$ from the bound $\nabla\varrho_\epsilon \in L^2(\mathbb{R}^3)$ does not hold for $d = 2$. Instead, we rely on the fact that if f is a measurable function such that $\nabla f \in L^2(\mathbb{R}^2)$ and $\mathrm{supp}(f)$ is of finite Lebesgue measure, then

$$\|f\|_{L^p(\mathbb{R}^2)} \le \|\nabla f\|_{L^2(\mathbb{R}^2)}\mathcal{L}^2\left(\mathrm{supp}(f)\right)^{\frac{1}{p}}. \tag{7.2}$$

For a proof of (7.2), see, for instance, [14, Inequality (3.10)] and also [21, Proof of Lemma 3.1]. From the analogue of (3.1) for $d = 2$ and the Chebyshev inequality, we infer that

$$\mathcal{L}^2(\{|\rho_\varepsilon - 1| > \frac{1}{2}\}) \le \frac{1}{2^\gamma}\int_{\mathbb{R}^2} |\rho_\varepsilon(t) - 1|^\gamma\, 1_{\{|\rho_\varepsilon - 1| > c\}}\mathrm{d}x \le C\varepsilon^2,$$

where $\mathcal{L}^2$ denotes the Lebesgue measure. Consider a smooth cut-off $\chi \in C^\infty_c(\mathbb{R})$ such that $\mathbf{1}_{[3/4,5/4]}(r) \le \chi(r) \le \mathbf{1}_{[1/2,3/2]}(r)$. Applying the inequality (7.2) to $(\varrho_\varepsilon - 1)(1 - \chi(\rho_\varepsilon))$ yields

$$\begin{aligned}\|(\varrho_\varepsilon - 1)(1 - \chi(\rho_\varepsilon))\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))} &\le \|\nabla\varrho_\varepsilon\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))}\mathcal{L}^2\left(\mathrm{supp}(1 - \chi(\rho_\varepsilon))\right)^{\frac{1}{2}} \\ &\le C\varepsilon\|\nabla\varrho_\varepsilon\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))} \le C\varepsilon.\end{aligned}$$

We obtain that

$$\|\varrho_\epsilon - 1\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))} \le C\varepsilon, \quad \|\nabla(\varrho_\epsilon - 1)\|_{L^\infty(0,\infty;L^2(\mathbb{R}^2))} \le C.$$

The proof for the respective bounds of $\varrho^0_\epsilon - 1$ follows verbatim. □

Note that opposite to Lemma 3.1, none of these bounds depends on whether $\gamma < 2$ or $\gamma \geq 2$ for $d = 2$. In particular, by interpolation it follows that

$$\|\rho_\varepsilon\|_{L^\infty(0,T;H^s(\mathbb{R}^2))} \leq C\varepsilon^{1-s}$$

and for all $2 \leq q < \infty$ there exists $C > 0$ such that

$$\begin{aligned} \|\sqrt{\varrho_\varepsilon^0} - 1\|_{L^q} &\leq \|\varrho_\varepsilon^0 - 1\|_{L_x^q} \leq C\varepsilon^{\frac{2}{q}} \\ \|\sqrt{\varrho_\varepsilon} - 1\|_{L_t^\infty L_x^q} &\leq \|\varrho_\varepsilon - 1\|_{L_t^\infty L_x^q} \leq C\varepsilon^{\frac{2}{q}} \end{aligned} \tag{7.3}$$

In addition, the analogue of Lemma 3.2 remains valid for $d = 2$.

Next, we provide the suitable decay of acoustic waves in space-time norms. The dispersion relation (4.5) is non-homogeneous and mimics wave-like behaviour for low and Schrödinger-like behaviour for high frequencies. While it exhibits a regularizing effect for low frequencies for $d > 2$ providing decay of order ε^δ at the expense of a loss or regularity of order δ with $\delta > 0$ arbitrarily small, no such regularizing effect occurs for $d = 2$, see [2, Section 3]. However, separating frequencies above and below the threshold $\frac{1}{\varepsilon}$ and relying on the wave-like estimate for low frequencies combined with an interpolation argument, the following Strichartz estimates are shown to hold in [2, Proposition 3.8].

Definition 7.1. The exponents (q, r) are said to be *θ-admissible* if $2 \leq q, r \leq \infty$, $(q, r, \theta) \neq (2, \infty, 1)$ and

$$\frac{1}{q} + \frac{\theta}{r} = \frac{\theta}{2}.$$

We say that a pair is Schrödinger or wave admissible if $\theta = \frac{d}{2}$ or $\theta = \frac{d-1}{2}$, respectively. Further, we denote $\beta = \beta(r) := \frac{1}{2} - \frac{1}{r}$.

Proposition 7.1 ([2]). *Let $\varepsilon > 0$ and $\theta \in [0, 1)$. Then, for any $\frac{2-\theta}{2}$-admissible pair (q, r) and $s_0 = 3\beta(r)\theta$, it holds*

$$\|e^{itH_\varepsilon} f\|_{L^q(0,T;L^r(\mathbb{R}^2))} \leq C\varepsilon^{\frac{s}{3}}\|f\|_{\dot{H}^s(\mathbb{R}^2)}. \tag{7.4}$$

The estimate allows one to infer decay in Strichartz norms at the cost of arbitrarily small regularity.

Remark 7.1. Being the symbol ϕ_ε non-homogeneous, it does not allow for a separation of scales. Therefore, the ε-dependent estimates cannot be obtained by a simple scaling argument. Further, such estimates also appear in the context of the Gross–Pitaevskii equation. For $\theta = 0$, (7.1) yields Schrödinger-like Strichartz estimates that, however, do not provide decay in ε. In [6, Corollary B.1], the estimate (7.4) is proven for $\theta = 1$ and $d \geq 2$ and low frequencies in the framework of the (GP)-equation. For high frequencies, a Schrödinger-type estimate is obtained. In this regard, (7.4) can be interpreted as a refinement of [6, Corollary B.1].

With these key ingredients at hand and observing that the proof of the relative entropy method can then be adapted in a straightforward way, we have the following main result for the asymptotic limit for $d = 2$. Specifically and analogue to the case $d = 3$, we consider ill-prepared initial data

$$\varrho(0,\cdot) = \varrho_\varepsilon^0 = 1 + \varepsilon\sigma_\varepsilon^0, \quad \sigma_\varepsilon^0 \in L^\infty(\mathbb{R}^2) \cap H^1(\mathbb{R}^2) \text{ uniformly bounded}, \ \sigma_\varepsilon^0 \to s^0 \ \text{ in } H^1(\mathbb{R}^2), \tag{7.5}$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_\varepsilon^0, \quad \sqrt{\varrho_\varepsilon^0}\mathbf{u}_\varepsilon^0 \in L^2(\mathbb{R}^2), \quad \sqrt{\varrho_\varepsilon^0}(\mathbf{u}_\varepsilon^0 - \mathbf{u}^0) \to 0 \ \text{ in } \ L^2(\mathbb{R}^2). \tag{7.6}$$

as $\varepsilon \to 0$.

Theorem 7.3. *Suppose $\nu \to 0$ as $\varepsilon \to 0$. Assume the initial data $(\varrho_\varepsilon^0, \mathbf{u}_\varepsilon^0)$ to be of uniformly bounded finite energy and BD-entropy, i.e. there exists $C > 0$ independent of $\varepsilon, \nu > 0$ such that*

$$E(\varrho_\varepsilon^0, \mathbf{u}_\varepsilon^0) \leq C, \quad B(\varrho_\varepsilon^0, \mathbf{u}_\varepsilon^0) \leq C. \tag{7.7}$$

and satisfy (7.5), (7.6).

Let $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \mathcal{T}_\varepsilon)$ be a weak solution of (NSK) and $\mathbf{u}^E$ the unique solution to the initial value problem (E)–(1.7) on $[0,\infty) \times \mathbb{R}^2$ with initial datum $\mathbf{u}_0^E = \mathbf{P}(\mathbf{u}^0) \in W^{3,2}(\mathbb{R}^3)$. Then, for all $T > 0$ as $\varepsilon \to 0$, it holds

$$\|\varrho_\varepsilon - 1\|_{L^\infty(0,T;H^s(\mathbb{R}^2))} \leq C\varepsilon^{1-s} \tag{7.8}$$

for all $0 < s < 1$ and

$$\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon \to \mathbf{u}^E \ \textit{ in } \ L^2(0,T;L^2_{\mathrm{loc}}(\mathbb{R}^2)). \tag{7.9}$$

We note that different to Theorem 2.3, we may consider arbitrarily large times $T > 0$ as solutions to the limit system are global. The convergence in Orlicz space is implied by the convergence in $L^\infty(0,T;H^s(\mathbb{R}^2))$ due to the Sobolev embeddings for $d = 2$ and any $\gamma > 1$.

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