IFAC Journal of Systems and Control 36 (2026) 100402

Contents lists available at ScienceDirect

IFAC Journal of Systems and Control

journal homepage: www.elsevier.com/locate/ifacsc

Full length article

On the numerical calculation of transfer functions of linear time-invariant partial differential equations

Jan M. Schaßberger [a,*], Gudrun Thäter [b], Lutz Gröll [a]

[a] Karlsruhe Institute of Technology, Hermann-von-Helmholtz-Platz 1, Eggenstein-Leopoldshafen, 76344, Germany
[b] Karlsruhe Institute of Technology, Kaiserstraße 12, Karlsruhe, 76131, Germany

ARTICLE INFO

Article history:
Received 28 May 2025
Received in revised form 23 January 2026
Accepted 27 February 2026
Available online 7 March 2026

Keywords:
Partial differential equation
Transfer function
Model approximation

ABSTRACT

Linear time-invariant (LTI) partial differential equations (PDEs) with one time and one spatial coordinate are a particularly important system class, since they are, for instance, obtained by linearizing nonlinear PDE models about an equilibrium solution. Since this system class still poses challenges for control engineering purposes, a change to an input-output description in the form of transfer function models is advantageous. However, it is usually impossible to derive a closed-form transfer function for linear PDEs directly. For this reason, we present a new approach to non-parametric transfer functions of LTI PDEs, in which either a reformulated Cauchy-like problem or a boundary value problem is to be solved numerically. The addressable system class is extended to PDEs with inhomogeneous boundary conditions and PDEs involving mixed partial derivatives. In addition, a possible reduction of the state vector's order for certain PDEs is discussed. Two examples are used to demonstrate the application and accuracy of the approach.

1. Introduction

Transfer functions are an indispensable system representation of the input-output (I/O-) behavior in the Laplace domain. They take an important role in control engineering analysis and design of linear time-invariant (LTI) ordinary differential equation (ODE) systems, i.e., for purely linear systems and linear approximations. For LTI ODEs, transfer functions are derived by applying the Laplace transform to the system model followed by solving a system of algebraic equation for the desired I/O-relation. To use this approach for distributed-parameter systems, either an ODE approximation model is derived directly or the ODE is obtained via discretization of a partial differential equation (PDE) model. Thus, starting from a nonlinear distributed-parameter system, these approaches can be summarized by

Path 1: Linearization of an ODE approximation model (Karlström & Breitholtz, 1992),
Path 2: Linearization of a discretized PDE model (Aalto, 2008),
Path 3: Discretization of a linearized PDE model (Lopes dos Santos et al., 2010),

as shown in Fig. 1. However, all three approaches lead to different transfer functions. Even more importantly, they deviate to an unknown extend from the PDE's actual one, which can be obtained by applying the Laplace transform directly to the PDE model (Curtain & Morris, 2009) shown as Path 4 in Fig. 1.

A prerequisite for the efficient application of Path 4 are models described by LTI PDEs with one time and one spatial coordinate. In this case, the Laplace transform yields a complex-valued differential equation w.r.t. the spatial coordinate. With the PDE's boundary conditions, it typically forms a boundary value problem (BVP) that must be solved in closed form to obtain the system's transfer function (Curtain & Morris, 2009). Unfortunately, this is only possible for comparatively simple systems with typically constant coefficients, which is a major limitation especially for the application to PDE models of industrial processes. These are usually nonlinear, so that linearization about a (time-independent) equilibrium yields LTI PDEs with spatially-dependent coefficients. Hence, Path 4 in Fig. 1 is seldom applicable.

To overcome this issue, we introduce a new broadly applicable approach for the numerical calculation of transfer functions of LTI PDEs. It is based on the idea of solving the Laplace domain differential equation using numerical methods as shown by Path 5 in Fig. 1. This leads to the PDE's actual transfer function in a non-parametric form. The latter can either be used directly for non-parametric analysis or design methods, like loop-shaping, or as a basis for determining a parametric approximation. Having demonstrated the advantages of the new approach over the use of finite-dimensional approximations using an example in Schaßberger et al. (2025), the present work focuses

* Corresponding author.
E-mail addresses: jan.schassberger@kit.edu (J.M. Schaßberger), gudrun.thaeter@kit.edu (G. Thäter), lutz.groell@kit.edu (L. Gröll).

https://doi.org/10.1016/j.ifacsc.2026.100402

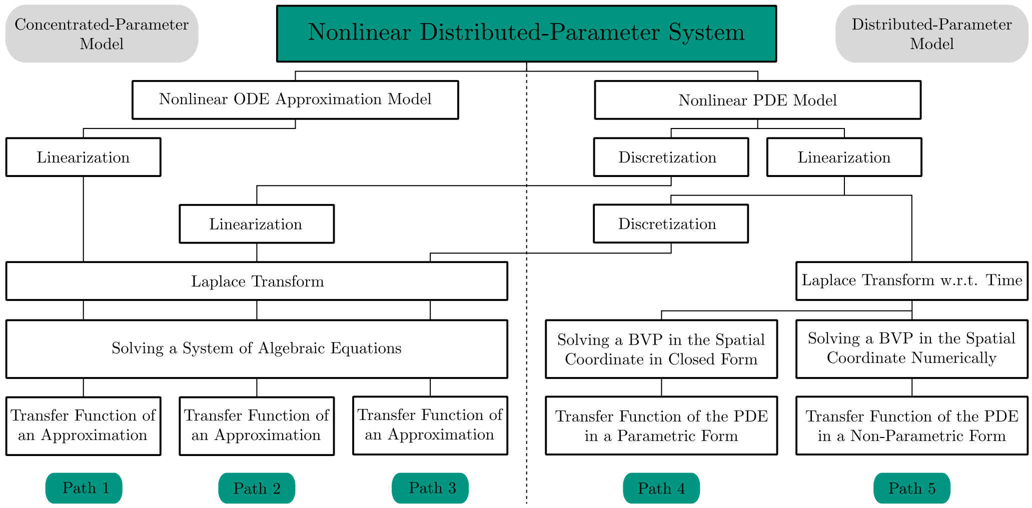


Fig. 1. Approaches for the calculation of transfer functions of nonlinear distributed-parameter systems.

on introducing the approach for general causal mixed-boundary control systems with pointwise distributed actuation and measurement. Moreover, we present a second numerical calculation procedure, which allows to extend the addressable system class significantly.

In the following, we first discuss the considered solution concept, introduce the notation and recall the properties of the Laplace transform in Section 2. Afterwards, the calculation approach is presented in Section 3 and the extension to a broader class of PDEs is addressed in Section 4. Finally, the application to two examples is demonstrated and a comparison with their closed-form solutions is performed in Section 5.

2. Preliminaries

We introduce the notation used in the present work in Section 2.1 and discuss the role of the solution concept in Section 2.2. Afterwards, the properties of the Laplace transform in the context of PDEs and the definition of the transfer function are recalled in Sections 2.3 and 2.4, respectively.

2.1. Notation

$\mathbb{R}^+ = \{c \in \mathbb{R}: c \geq 0\}$ denotes the closed positive real axis. We define $\mathbb{R}^+_{0-}$ as $\mathbb{R}^+$ with a neighborhood. The spatial domain is $\bar{\mathcal{Z}} = \{z \in \mathbb{R} : 0 \leq z \leq 1\}$ and $\mathcal{Z}$ denotes its subset where no boundary conditions apply. Moreover, we have $\mathbb{C}^+_\alpha = \{s \in \mathbb{C}: \operatorname{Re} s > \alpha\}$ and $\overline{\mathbb{C}^+_\alpha} = \mathbb{C}^+_\alpha \cup (\alpha + \mathrm{i}\mathbb{R}) \cup \{\infty\}$. With $PC(\mathbb{R};\mathbb{R}^n)$, $C^\infty(\mathbb{R};\mathbb{R}^n)$ and $L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$, we denote the piecewise continuous, the smooth and the locally integrable functions, respectively. The step function is denoted by $1(t)$. With the notation $x_{\square,i}(t)$, we refer to the i-th element of the vector $x_\square(t)$, by $x_{\square,ij}(t)$ to the element in the i-th row and the j-th column of the matrix $X_\square(t)$. The abbreviation $\partial^n_\square x(z,t) := \frac{\partial^n}{\partial \square^n} x(z,t)$ with $n \in \mathbb{N}$ refers to the n-th partial derivative of $x(z,t)$ w.r.t. $\square \in \{z, t\}$. Analogously, we use for mixed derivatives $\partial^n_z \partial^m_t x(z,t) := \frac{\partial^{n+m}}{\partial^n z \partial^m t} x(z,t)$ with $n, m \in \mathbb{N}$. The spatial derivatives of a function $x(z)$, are denoted by the prime notation $x'(z) := \frac{d}{dz} x(z)$.

2.2. Solution concepts

The choice of the concept is essential for the interpretation of the PDE's solution, its derivatives and the calculus used. For this reason, a selection of concepts is briefly discussed in the following. When considering *classical* solutions, all partial derivatives occurring in the PDE exist always and everywhere in the classical sense (Ebert & Reissig, 2018), which requires sufficient smoothness. This concept, however, is typically too restrictive for many control engineering purposes as discontinuities in the input signals are usually to be allowed. Although a sufficiently smooth input signal can be obtained by smoothing the input with a low pass filter (Deutscher, 2012), the latter must then be taken into account in controller design, which usually leads to a loss of performance. Moreover, finding or proving the existence of classical solutions can be particularly difficult for models of real world applications. A further concept are *mild* solutions, where the solution has sufficient smoothness almost everywhere to fulfill the PDE in the classical sense. It is predominantly used for finite- or infinite-dimensional ordinary differential equations, their relevance in the context of PDEs is explained by the fact that certain PDEs can be written as abstract differential equations and vice versa (Curtain & Zwart, 2020). For many PDEs, however, this requires a reformulation of the actual equation, e.g. for boundary control systems or equation with time derivatives with an order greater than one. Such a reformulation must be found on the one hand and may be accompanied by additional restrictions, e.g. on the smoothness of the input signals, on the other (Curtain & Zwart, 2020). Furthermore, advanced mathematical concepts are needed like the Pettis-integral for the impulse response (Curtain & Zwart, 2020). A more elegant way to deal with insufficient smoothness, even discontinuities, is offered by the concepts of *weak* and *distributional* solutions due to a different interpretation of a function's derivative using test functions. The biggest difference between the latter two concepts is that the distributional solution fulfills the PDE pointwise in a distributional sense, while the weak solution only fulfills the variational formulation, i.e, the PDE in an integral sense (Ebert & Reissig, 2018). In addition, the latter usually also requires a certain regularity of the solution and its derivatives. For the considerations in the present work, the concept of distributional solutions is the most appropriate one, since

- a transfer function corresponds to the Laplace transform of a system's impulse response.
- it guarantees the interchangeability of mixed partial derivatives.
- it allows to define in-domain point-measurement.

There are essentially two closely related concepts to deal with distributional solutions, namely the Mikusiński operator calculus and the Laplace transform. The former is more powerful, since it avoids certain assumptions required in the Laplace transform and is applicable to functions that are not exponentially bounded (Buschman, 1996). However, the full potential of the Mikusiński operator calculus is often not needed, since, for example, hyperexponential growth of solutions is questionable in practical applications. Hence, we focus on the well-established Laplace transform, which can be extended from the classical to the distributional calculus.

2.3. Laplace transform

In the addressed class of LTI PDEs, vector-valued functions with one (input or output signals) or two independent variables (state signals) occur. To determine the Laplace transform of the latter, one might consider the application of a multivariable transform as e.g. in Ditkin et al. (2017) or Debnath (2016). However, in context of transfer functions of PDEs, the Laplace transform is applied only w.r.t. time to all signals (Curtain & Morris, 2009), whereas the spatial coordinate is treated as a constant. In the present section, we introduce the unilateral Laplace transform w.r.t. time and restrict ourselves to causal real-valued signals, which are most important for technical applications. To highlight the characteristic features of the transformation in the distributional sense, we first briefly recall the definition in the classical calculus for functions on $\mathbb{R}^+$.

Definition 1 (*Laplace Transform in the Classical Calculus (Hinrichsen & Pritchard, 2005)*)**.** Suppose $f \in L^1_{\mathrm{loc}}(\mathbb{R}^+;\mathbb{R})$ and $f_\alpha \in L^1(\mathbb{R}^+;\mathbb{R})$, $t \mapsto \mathrm{e}^{-\alpha t}f(t) =: f_\alpha(t)$ for some $\alpha \in \mathbb{R}$. Moreover, let $s = \sigma + \mathrm{i}\omega$ with $\sigma, \omega \in \mathbb{R}$. Then, the unilateral Laplace transform of f is defined on $\overline{\mathbb{C}^+_\alpha}$ by

$$\hat{f}(s) := \mathfrak{L}\{f\}(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\mathrm{d}t, \quad \mathrm{Re}\, s \geq \alpha. \tag{1}$$

The Laplace transform $\hat{f}$ is continuous on $\overline{\mathbb{C}^+_\alpha}$, analytic on $\mathbb{C}^+_\alpha$ and bounded.

The set of Laplace-transformable functions on $\overline{\mathbb{C}^+_\alpha}$ is denoted by

$$\mathcal{L}_\alpha(\mathbb{R}^+;\mathbb{R}) := \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^+;\mathbb{R}), f_\alpha \in L^1(\mathbb{R}^+;\mathbb{R})\}. \tag{2}$$

In order to apply the Laplace transform to differential equations, one also needs to define it for a function's derivatives. Assume that n is the degree of the highest derivative of $f \in \mathcal{L}(\mathbb{R}^+;\mathbb{R}^n)$ occurring in the equations, $f \in C^n((0,\infty);\mathbb{R})$ and $f^{(n)} \in \mathcal{L}_\alpha(\mathbb{R}^+;\mathbb{R})$. Then, it holds

$$\mathfrak{L}\left\{f^{(n)}\right\}(s) = s^n\hat{f}(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0^+) \tag{3}$$

and (Doetsch, 1950)

$$f^{(n)} \in \mathcal{L}_\alpha(\mathbb{R}^+;\mathbb{R}) \;\Rightarrow\; f^{(i)} \in \mathcal{L}_\alpha(\mathbb{R}^+;\mathbb{R}),\ i = 1,\ldots,n-1.$$

Before we introduce the Laplace transform for distributions, we recall the essentials needed in the following. Distributions are the generalization of the locally integrable functions and are defined as continuous linear functionals on the set of test functions ϕ. For the Schwartz distributions, ϕ are the infinitely differentiable functions $C^\infty_c(\mathbb{R};\mathbb{R})$ with compact support equipped with a topology, commonly denoted as $\mathcal{D}(\mathbb{R};\mathbb{R}) := C^\infty_c(\mathbb{R};\mathbb{R})$. Hence, a Schwartz distribution f is defined by

$$\langle f,\phi\rangle = \begin{cases} \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x, & \text{if } f \in L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}), \\ \mathrm{value}(\phi), & \text{depending on the definition} \\ & \text{of } f,\ \text{if } f \notin L^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}). \end{cases} \tag{4}$$

The set of all such distributions forms the dual space of $\mathcal{D}(\mathbb{R};\mathbb{R})$ denoted by

$$\mathcal{D}'(\mathbb{R};\mathbb{R}) = \Big\{ \langle f,\cdot\rangle : \mathcal{D}(\mathbb{R};\mathbb{R}) \to \mathbb{R} \mid \langle f,\cdot\rangle \text{ is linear and continuous} \Big\}. \tag{5}$$

However, the Laplace transform can only be successfully applied to a certain subset, namely the tempered distributions $S'(\mathbb{R};\mathbb{C}) \subset \mathcal{D}'(\mathbb{R};\mathbb{C})$. The adjustment of the image into $\mathbb{C}$ becomes necessary due to the complex-valued kernel of the Laplace transform, even if only real-valued functions are addressed. The tempered distributions are the continuous linear functionals of slow growth on the Schwartz space $\mathcal{S}(\mathbb{R};\mathbb{C})$. The latter is the vector space of $C^\infty(\mathbb{R};\mathbb{C})$-functions that, together with all their derivatives, decrease faster than any power of $1/|\tau|$ as $|\tau| \to \infty$, so that for any $k, n \in \mathbb{N}$ and $\tau \in \mathbb{R}$ it holds (Beffa, 2024)

$$\lim_{|\tau|\to\infty} \left|\tau^k D^n \phi(\tau)\right| = 0. \tag{6}$$

Since our focus is on causal real-valued signals, the class of distribution needs to be restricted to those with support in $\mathbb{R}^+_{0-}$, which is $\mathbb{R}^+$ with a neighborhood of $[0,\infty)$. This ensures the existence of the limit from the left of zero. Thus, we can define the unilateral Laplace transform as follows.

Definition 2 (*Laplace Transform in the Distributional Calculus (Beffa, 2024)*)**.** Let $f \in \mathcal{D}'(\mathbb{R}^+_{0-};\mathbb{R})$ and γ be a smooth function with left-bounded support and $\gamma(t) = 1$ in $\mathbb{R}^+_{0-}$, so that $f_\alpha \in \mathcal{S}'(\mathbb{R}^+_{0-};\mathbb{R})$ and $\gamma(t)\mathrm{e}^{-(s-\alpha)t} \in \mathcal{S}(\mathbb{R}^+_{0-};\mathbb{C})$ for some $\alpha \in \mathbb{R}$. Then, the Laplace transform of f is defined by[1]

$$\hat{f}(s) := \mathfrak{L}\{f(t)\} = \langle f_\alpha(t), \gamma(t)\mathrm{e}^{-(s-\alpha)t}\rangle, \quad \mathrm{Re}\, s \geq \alpha, \tag{7}$$

or commonly abbreviated by[2]

$$\hat{f}(s) := \mathfrak{L}\{f(t)\} = \langle f(t), \mathrm{e}^{-st}\rangle, \quad \mathrm{Re}\, s \geq \alpha. \tag{8}$$

Analogously to the classical calculus, one can define the set of Laplace-transformable distributions

$$\mathcal{L}'_\alpha(\mathbb{R}^+_{0-};\mathbb{R}) = \Big\{ f \in \mathcal{D}'(\mathbb{R}^+_{0-};\mathbb{R}), \alpha \in \mathbb{R}, f_\alpha \in \mathcal{S}'(\mathbb{R}^+_{0-};\mathbb{R}) \Big\}. \tag{9}$$

The definition of the distributional derivative of $f \in \mathcal{L}'_\alpha(\mathbb{R}^+_{0-};\mathbb{R})$ implies $f^{(n)} \in \mathcal{L}'_\alpha(\mathbb{R}^+_{0-};\mathbb{R})$ for all $n \in \mathbb{N}$ (Chirilă et al., 2021). Thus, the Laplace transform of a distribution's derivative is (Bahar, 1969)

$$\mathfrak{L}\{f^n(t)\} = s^n\hat{f}(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0^-). \tag{10}$$

When comparing Definitions 1 and 2, it becomes immediately clear that every function that is Laplace-transformable in the classical sense is also Laplace-transformable in the distributional sense. However, the latter has the major advantage that no additional assumptions have to be made regarding the smoothness of the solution in the context of derivatives, as distributions are infinitely differentiable (Beffa, 2024). As a consequence, the order of partial derivatives can be changed arbitrarily. This is in contrast to the classical calculus, where interchanging partial

[1] The definition (7) may appear unnecessarily complicated compared to (8). However, it prevents the test function in (7) from growing arbitrarily on the left side of the complex plane. Moreover, the condition $\gamma(t) = 1$ does not change the value of the integral (4) defining the distribution.

[2] This is clearly not a pair of a tempered distribution and a test function in the Schwartz space. However, this abbreviation is still reasonable, when restricting the distribution's domain to a compact subset of $\mathbb{R}^+_{0-}$, which is always possible for technical systems. Then (8) can be read as a distribution with compact support in $\mathcal{E}'$ and an indefinitely differentiable function in $\mathcal{E}$.

derivatives requires additional assumptions given in the well-known theorems of Schwartz, Clairaut or Young. The reader may note an additional distinction concerning the initial values. In case of the classical calculus, the right-sided initial values need to be considered, whereas the left-sided ones are needed in case of distributions. This can lead to discrepancies between the assumed and actual initial values, as discussed in Bahar (1969). To keep the presentation simple, we use the classical notation instead of the bracket notation also in the distributional sense up to small modifications in the initial values and the lower limit of integration w.r.t. time.

Let now $f\colon \bar{\mathcal{Z}} \times \mathbb{R}^+_{0-} \to \mathbb{R}$ with $f(z,\cdot) \in \mathcal{L}'_\alpha(\mathbb{R}^+_{0-};\mathbb{R})$. In the context of PDEs, it is assumed that limits, derivatives (Schiff, 1999) and integrals (McCollum & Brown, 1965) w.r.t. the spatial coordinate pass through the Laplace transform. Hence, the transform of the function's spatial derivative $\partial_z f(z,t)$ is given by

$$\mathfrak{L}\{\partial_z f(z,t)\} := \int_{0-}^{\infty} \partial_z f(z,t)\mathrm{e}^{-st}\,\mathrm{d}t = \partial_z \hat{f}(z,s)$$

and analogously that of the m-th spatial derivative by

$$\mathfrak{L}\{\partial_z^m f(z,t)\} = \partial_z^m \hat{f}(z,s). \tag{11}$$

The integral of the function f w.r.t. some subset of the spatial domain $Z \subset \bar{\mathcal{Z}}$ becomes

$$\mathfrak{L}\left\{\int_Z f(z,t)\,\mathrm{d}z\right\} := \int_Z \hat{f}(z,s)\,\mathrm{d}z, \tag{12}$$

whenever the integral exists. By combining (10) and (11), the Laplace transform of mixed derivatives of the form $\partial_z^m\partial_t^n f(z,t)$ with $m,n \in \mathbb{N}$ is

$$\mathfrak{L}\{\partial_z^m\partial_t^n x(z,t)\} = \partial_z^m \mathfrak{L}\{\partial_t^n x(z,t)\}. \tag{13}$$

With the interchangeability property, it follows

$$\mathfrak{L}\{\partial_t^n\partial_z^m x(z,t)\} = \partial_z^m \mathfrak{L}\{\partial_t^n x(z,t)\}. \tag{14}$$

The given formulas allow the transformation of a LTI PDE, its boundary conditions and output equations to the Laplace domain, from which an input-output description in form of a transfer function can be derived.

2.4. Transfer functions

In engineering science, the transfer function $\hat{g}(s)$ of a LTI system with one input $u(t) \in \mathbb{R}$ and one output $y(t) \in \mathbb{R}$ is defined as $\hat{g}(s) = \hat{y}(s)/\hat{u}(s)$ for zero initial conditions. More precisely, the necessary analytical continuation of the quotient defines the transfer function, since the quotient would otherwise only be valid in the common domain in which the input's and output's Laplace transforms are analytical. For vectorial[3] inputs $u(t) \in \mathbb{R}^m$ and outputs $y(t) \in \mathbb{R}^p$, the role of the quotient changes to a complex-valued mapping $\hat{G}\colon \mathbb{C} \to \mathbb{C}^{p\times m}$, i.e.,

$$\hat{y}(s) = \hat{G}(s)\hat{u}(s). \tag{15}$$

In the context of the distributional calculus, it is more common to define the transfer function as the analytical continuation of the Laplace transform of the impulse response. Both definitions are equivalent for finite- and infinite-dimensional state space systems (Zwart, 2004). However, infinite-dimensional systems have irrational transfer functions, where the analytical continuation over the entire complex plane must, if possible, be defined with greater care, e.g. the branch cuts has to be specified. The specifics that arise in the context of irrational transfer functions are discussed, for example, in Smirnov (1964). It is therefore advantageous to define the transfer function of infinite-dimensional systems without the need for an analytical continuation. Such a definition is given in Zwart (2004) under the name transmission function, which is called in the present contribution transfer function defined via exponential signals.

Definition 3 (*Transfer Function Defined Via Exponential Signals (Transmission Function (Zwart, 2004))*). Let $u(t) = u_0\mathrm{e}^{s_0t}\cdot 1(t)$ and suppose there exists an initial value function such that $y(t) = y_0\mathrm{e}^{s_0t}\cdot 1(t)$ for all $t \geq 0$. Then, we call $\hat{G}(s_0)$ the transfer function at s_0, if for all $u_0 \in \mathbb{C}^m$, $y_0 \in \mathbb{C}^p$ can be written as

$$y_0 = \hat{G}(s_0)u_0. \tag{16}$$

Hence, $\hat{G}$ exists at all points $s_0 \in \mathbb{C}$ that are neither isolated nor non-isolated singularities of $\hat{G}$ except for the removable ones. Although Definition 3 avoids the analytical continuation, it is hardly suitable for the actual calculation of the transfer function as shown in Appendix A. This is due to the need to specify the initial values and the pointwise character of the definition. Furthermore, theorems based on this definition, such as in Curtain and Zwart (2020), are rarely rigorously applicable to models of technical systems, because their prerequisites are difficult to verify. Therefore, we prefer the approaches based on the Laplace transform, since these are better suited for calculations and lead to structurally identical differential equations defining the transfer function (Curtain & Morris, 2009). The issues related to the analytical continuation are overcome by initially considering $\mathbb{C}$ as the domain of $\hat{G}$. This means that the singular points are not explicitely exluded in the numerical calculation, as they are not known a priori. Since their location becomes visible from the numerically calculated $\hat{G}$, the domain of $\hat{G}$ can be defined a posteriori.

3. Introduction of the new numerical approach

The application of the Laplace transform to a system of LTI PDEs and its boundary conditions generally leads to an ordinary boundary value problem in the spatial coordinate. The latter relates the Laplace transforms of the inputs and outputs of the system, so that by solving the boundary value problem in closed form, the I/O-transfer function can be derived. Unfortunately, this is only possible for comparatively simple systems. In the present contribution, we therefore focus on the numerical calculation of the transfer function in a non-parametric form by performing the following steps:

1. Introduction of the considered class of PDEs,
2. Application of the Laplace transform,
3. Reformulation into first order differential equations,
4. Numerical calculation of the transfer function via the
    4.1. Transition matrix approach in Section 3.4.1,
    4.2. ODE boundary value approach in Section 3.4.2.

Afterwards, a comparison of both approaches is performed in Section 3.5. For the moment, we restrict the class of PDEs and discuss its extension in Section 4.

3.1. Considered restricted class of PDEs

In the following, the well-posedness of the LTI PDE systems in the sense of the distributional calculus is assumed. The

[3] The Laplace transform is to be interpreted as a Bochner integral. Its main features are the Banach space-valued functions (here $\mathbb{R}^m$ and $\mathbb{R}^p$), a common multiplier e^{-st} and a joint region of convergence.

inputs[4] $u \in \mathcal{L}'_\alpha(\mathbb{R}^+_{0^-};\mathbb{R}^m)$ are considered to act on the boundary (index b) and/or pointwise distributed (index d). The latter is a special type of distributed input in the form of a product of a purely time-dependent signal and a spatially-dependent shape function. Also the output signals $y \in \mathcal{L}'_\alpha(\mathbb{R}^+_{0^-};\mathbb{R}^p)$ are split in a similar way, so that we have

$$u(t) = \begin{bmatrix} u_\mathrm{b}(t) \\ u_\mathrm{d}(t) \end{bmatrix} \in \mathbb{R}^{m_\mathrm{b}+m_\mathrm{d}}, \quad y(t) = \begin{bmatrix} y_\mathrm{b}(t) \\ y_\mathrm{d}(t) \end{bmatrix} \in \mathbb{R}^{p_\mathrm{b}+p_\mathrm{d}}. \tag{17}$$

Let $x\colon \bar{\mathcal{Z}} \times \mathbb{R}^+_{0^-} \to \mathbb{R}^n$ with $x(z,\cdot) \in \mathcal{L}'_\alpha(\mathbb{R}^+_{0^-};\mathbb{R}^n)$ be the state vector of the PDE. It is characterized by the vectors of its highest order partial derivatives $\nu_t \in \{\nu \in \mathbb{N}^n \setminus \{0^n\}\colon \nu_i \geq 0\}$, $\nu_z \in \{\nu \in \mathbb{N}^n\colon \nu_i > 0\}$ and their maximal elements $\overline{\nu}_t = \max_i\{\nu_{t,i}\}$ and $\overline{\nu}_z = \max_i\{\nu_{z,i}\}$. Moreover, the system of PDE is assumed to not involve mixed partial derivatives, so that it can be written as

$$\sum_{i=1}^{\overline{\nu}_t} A_{ti}(z)\partial_t^i x(z,t) + \sum_{j=0}^{\overline{\nu}_z} A_{zj}(z)\partial_z^j x(z,t) = B_\mathrm{d}(z)u_\mathrm{d}(t), \quad z \in \mathcal{Z},\ t > 0^-, \tag{18a}$$

where $A_{ti}, A_{zi} \in PC(\bar{\mathcal{Z}};\mathbb{R}^{n\times n})$ and $B_\mathrm{d} \in PC(\bar{\mathcal{Z}};\mathbb{R}^{n\times m_\mathrm{d}})$. The initial values of (18a) are

$$x(z,0^-) = x_0(z), \quad z \in \bar{\mathcal{Z}}, \tag{18b}$$

with $x_0 \in PC(\bar{\mathcal{Z}};\mathbb{R}^n)$. In general, the system has boundary conditions at both ends of the spatial domain

$$\begin{aligned} &f_0(x(0,t), \partial_z x(0,t), \ldots, u_\mathrm{b}(t)) = 0, \quad t > 0^-, \\ &f_1(x(1,t), \partial_z x(1,t), \ldots, u_\mathrm{b}(t)) = 0, \quad t > 0^-, \end{aligned} \tag{18c}$$

where f_0, f_1 are linear functions of the state vector, its derivatives and the boundary input vector with constant coefficients. We assume that the spatial derivative of the highest order of all entries x_i of the states vector occurring in f_0, f_1 are smaller than ν_{zi}.

The system description is completed by the boundary outputs

$$y_\mathrm{b}(t) = C_{\mathrm{b}0}x(0,t) + C_{\mathrm{b}1}x(1,t), \tag{19a}$$

where $C_{\mathrm{b}0}, C_{\mathrm{b}1} \in \mathbb{R}^{p_\mathrm{b}\times n}$ are constant matrices, and by the pointwise distributed outputs

$$y_\mathrm{d}(t) = \int_{\bar{\mathcal{Z}}} C_\mathrm{d}(z)x(z,t)\,\mathrm{d}z, \tag{19b}$$

with $C_\mathrm{d} \in \mathcal{D}'(\bar{\mathcal{Z}};\mathbb{R}^{p_\mathrm{d}\times n})$. Note that $C_\mathrm{d} \in \mathcal{D}'$ is used to allow Dirac functions as shape functions for modeling in-domain point measurements.

3.2. Application of the Laplace transform

With the assumptions in the previous section, the Laplace transform according to Definition 2 applied to (18a) and (18b) yields

$$\sum_{i=1}^{\overline{\nu}_t} A_{ti}(z)\Big(s^i\hat{x}(z,s) - \sum_{\ell=1}^{i} s^{i-\ell}x_0^{(\ell-1)}(z)\Big) + \sum_{j=0}^{\overline{\nu}_z} A_{zj}(z)\partial_z^j\hat{x}(z,s) = B_\mathrm{d}(z)\hat{u}_\mathrm{d}(s), \quad z \in \mathcal{Z}, \tag{20a}$$

where $\hat{x}\colon \bar{\mathcal{Z}} \times \overline{\mathbb{C}^+_\alpha} \to \mathbb{C}^n$ and $\hat{u}_\mathrm{d}\colon \overline{\mathbb{C}^+_\alpha} \to \mathbb{C}^{m_\mathrm{d}}$. Since the complex variable s is a parameter in (20a), the partial derivative ∂_z^j becomes an ordinary derivative so that (20a) is a system of complex-valued ordinary differential equations. For this reason, in the ongoing text, the partial derivative w.r.t. z is replaced by the prime sign, i.e., $\partial_z\hat{x}(z,s) = \hat{x}'(z,s)$. Using this notation, the Laplace transform of (18c) is

$$\begin{aligned} &f_0(\hat{x}(0,s), \hat{x}'(0,s), \ldots, \hat{u}_\mathrm{b}(s)) = 0, \\ &f_1(\hat{x}(1,s), \hat{x}'(1,s), \ldots, \hat{u}_\mathrm{b}(s)) = 0, \end{aligned} \tag{20b}$$

where $\hat{u}_\mathrm{b}\colon \overline{\mathbb{C}^+_\alpha} \to \mathbb{C}^{m_\mathrm{b}}$. Analogously, the output Eqs. (19) in the Laplace domain are

$$\hat{y}_\mathrm{b}(s) = C_{\mathrm{b}0}\hat{x}(0,s) + C_{\mathrm{b}1}\hat{x}(1,s), \tag{21a}$$

$$\hat{y}_\mathrm{d}(s) = \int_{\bar{\mathcal{Z}}} C_\mathrm{d}(z)\hat{x}(z,s)\,\mathrm{d}z, \tag{21b}$$

where $\hat{y}_\mathrm{b}\colon \overline{\mathbb{C}^+_\alpha} \to \mathbb{C}^{m_\mathrm{b}}$ and $\hat{y}_\mathrm{d}\colon \overline{\mathbb{C}^+_\alpha} \to \mathbb{C}^{m_\mathrm{d}}$.

Since we aim for the system's transfer function, the initial values are considered to be zero, i.e., $x_0(z) = 0_n$ for $z \in \bar{\mathcal{Z}}$. If the initial values are of interest, they can be treated as the distributed inputs.

3.3. Reformulation into first order differential equations

The Laplace transform of (18a), i.e., (20a), is a vector differential equation of order greater or equal than one, which can be reformulated as a system of first order differential equations by introducing a new state vector $\hat{\tilde{x}}\colon \bar{\mathcal{Z}} \times \overline{\mathbb{C}^+_\alpha} \to \mathbb{C}^{\tilde{\nu}_z}$

$$\hat{\tilde{x}}(z,s) = \Big[\ \hat{x}_1(z,s),\ \hat{x}'_1(z,s),\ \ldots,\ \hat{x}_1^{(\nu_{z1}-1)}(z,s),\ \hat{x}_2(z,s),\ \ldots,\ \hat{x}_n^{(\nu_n-1)}(z,s)\ \Big]^\top \tag{22}$$

with $\tilde{\nu}_z = \sum_{i=1}^n \nu_{z,i}$. Thus, (20a) becomes

$$\tilde{E}(z)\hat{\tilde{x}}'(z,s) = \tilde{A}(z,s)\hat{\tilde{x}}(z,s) + \tilde{B}_\mathrm{d}(z)\hat{u}_\mathrm{d}(s), \quad z \in \mathcal{Z}, \tag{23a}$$

where $\tilde{E} \in PC(\bar{\mathcal{Z}};\mathbb{R}^{\tilde{\nu}_z\times\tilde{\nu}_z})$, $\tilde{A} \in PC(\bar{\mathcal{Z}} \times \overline{\mathbb{C}^+_\alpha};\mathbb{C}^{\tilde{\nu}_z\times\tilde{\nu}_z})$ and $\tilde{B}_\mathrm{d} \in PC(\bar{\mathcal{Z}};\mathbb{R}^{\tilde{\nu}_z\times m_\mathrm{d}})$. The boundary conditions (18c) written with the extended state vector are

$$\begin{aligned} &\tilde{f}_0(\hat{\tilde{x}}(0,s), \hat{u}_\mathrm{b}(s)) = 0, \\ &\tilde{f}_1(\hat{\tilde{x}}(1,s), \hat{u}_\mathrm{b}(s)) = 0, \end{aligned} \tag{23b}$$

and the output equations

$$\begin{aligned} \hat{y}_\mathrm{b}(s) &= \tilde{C}_{\mathrm{b}0}\hat{\tilde{x}}(0,s) + \tilde{C}_{\mathrm{b}1}\hat{\tilde{x}}(1,s), \\ \hat{y}_\mathrm{d}(s) &= \int_{\bar{\mathcal{Z}}} \tilde{C}_\mathrm{d}(z)\hat{\tilde{x}}(z,s)\,\mathrm{d}z, \end{aligned} \tag{24}$$

where $\tilde{C}_{\mathrm{b}0}, \tilde{C}_{\mathrm{b}1} \in \mathbb{R}^{p_\mathrm{b}\times\tilde{\nu}_z}$ and $\tilde{C}_\mathrm{d} \in \mathcal{D}'(\bar{\mathcal{Z}};\mathbb{R}^{p_d\times\tilde{\nu}_z})$. At this point, it should be noted that in contrast to the standard state space extension of ODE systems in control theory, no general form of the block structure can be given. For example, if the highest order spatial derivatives of several state variables appear in the same equation, the extension yields a non-diagonal $\tilde{E}$. Thus, the extension cannot be formalized but needs to be found via direct calculation. An example is given in Appendix B.

3.4. Numerical calculation

The boundary value problem (23) can in general not be solved in closed form. For this reason, we introduce two numerical approaches in the present section. The choice of the approach depends on the properties of $\tilde{E}$ as shown in Table 1. Two approaches are proposed, the so-called transition matrix approach (Approach 1) and the ODE boundary value approach (Approach 2). In the former, the boundary value problem in the Laplace domain (23) is reformulated in a complex-valued ODE Cauchy-like problem, whereas the latter addresses it directly using numerical methods.

[4] In the present work all signals representing the influence of the environment on the system are denoted as inputs (Hinrichsen & Pritchard, 2005), i.e., both controlled inputs and uncontrolled ones like disturbances.

Table 1
Choice of the numerical approach.

Property		Approach
1	$\lvert \det(\tilde{E}(z))\rvert \geq \varepsilon > 0 \;\; \forall z \in \bar{\mathcal{Z}}$	1 & 2
2	$\exists z_0 \in \bar{\mathcal{Z}}: \det(\tilde{E}(z_0)) = 0$	2[a]

[a] Generalized transition matrices (Berger & Ilchmann, 2013) allow to treat systems with property 2 in a similar way as in approach 1, which, however, is not pursued in the present work.

3.4.1. Transition matrix approach

If (23a) has the Property 1 as in Table 1, by left multiplication with the inverse of $\tilde{E}(z)$, one obtains an identity matrix on the left side together with modified matrices on the right side of (23a). Hence, we start with

$$\hat{\tilde{x}}'(z,s) = \tilde{A}(z,s)\hat{\tilde{x}}(z,s) + \tilde{B}_{\mathrm{d}}(z)\hat{u}_{\mathrm{d}}(s), \quad z \in \mathcal{Z}. \tag{25}$$

The solution $\hat{\tilde{x}}(z,s)$ of (25) can formally be written with the state transition matrix $\hat{\Phi}: \bar{\mathcal{Z}} \times \bar{\mathcal{Z}} \times \overline{\mathbb{C}_\alpha^+} \to \mathbb{C}^{\tilde{\nu}_z \times \tilde{\nu}_z}$

$$\hat{\tilde{x}}(z,s) = \hat{\Phi}(z,0,s)\hat{\tilde{x}}(0,s) + \int_0^z \hat{\Phi}(z,\zeta,s)\tilde{B}_{\mathrm{d}}(\zeta)\hat{u}_{\mathrm{d}}(s)\,\mathrm{d}\zeta, \quad z \in \bar{\mathcal{Z}}, \tag{26}$$

which is the solution of the matrix-valued differential equation

$$\begin{aligned} \hat{\Phi}'(z,0,s) &= \tilde{A}(z,s)\hat{\Phi}(z,0,s), \qquad z \in \bar{\mathcal{Z}} \setminus \{0\}, \\ \hat{\Phi}(0,0,s) &= I_{\tilde{\nu}_z}. \end{aligned} \tag{27}$$

In order to use (26) for the calculation of $\hat{G}$, it is necessary to express $\hat{\tilde{x}}(0,s)$ as a function of the boundary inputs. To this end, $\hat{\tilde{x}}(1,s)$ in the second equation of (23b) is expressed using (26) for $z = 1$ leading to

$$\hat{\tilde{x}}(1,s) = \hat{\Phi}(1,0,s)\hat{\tilde{x}}(0,s) + \int_0^1 \hat{\Phi}(1,\zeta,s)\tilde{B}_{\mathrm{d}}(\zeta)\,\mathrm{d}\zeta\;\hat{u}_{\mathrm{d}}(s), \quad z \in \bar{\mathcal{Z}}. \tag{28}$$

Since (23b) and (28) are linear equations in $\hat{\tilde{x}}(0,s)$, $\hat{u}_{\mathrm{b}}(s)$ and $\hat{u}_{\mathrm{d}}(s)$ with constant coefficients, we can write

$$\hat{\tilde{x}}(0,s) = \hat{B}_{\mathrm{b}}(s)\hat{u}_{\mathrm{b}}(s) + \hat{B}_{\mathrm{d}}(s)\hat{u}_{\mathrm{d}}(s), \tag{29}$$

with $\hat{B}_{\mathrm{b}}(s) \in \mathbb{C}^{\tilde{\nu}_z \times m_{\mathrm{b}}}$ and $\hat{B}_{\mathrm{d}}(s) \in \mathbb{C}^{\tilde{\nu}_z \times m_{\mathrm{d}}}$. Inserting (29) in (26) leads to

$$\hat{\tilde{x}}(z,s) = \hat{\Phi}(z,0,s)\hat{B}_{\mathrm{b}}(s)\hat{u}_{\mathrm{b}}(s) + \left(\hat{\Phi}(z,0,s)\hat{B}_{\mathrm{d}}(s) + \int_0^z \hat{\Phi}(z,\zeta,s)\tilde{B}_{\mathrm{d}}(\zeta)\,\mathrm{d}\zeta \right)\hat{u}_{\mathrm{d}}(s), \quad z \in \bar{\mathcal{Z}}. \tag{30}$$

While $\hat{\Phi}(z,0,s)$ in (30) can directly be determined numerically from (27), for $\hat{\Phi}(z,\zeta,s)$ the transition matrix relation

$$\hat{\Phi}(z,\zeta,s) = \hat{\Phi}(z,0,s)\hat{\Phi}^{-1}(\zeta,0,s), \tag{31}$$

can be applied. It is obvious that calculating the inverse of $\hat{\Phi}(\zeta,0,s)$ in (31) can be avoided by solving the system

$$\hat{\Phi}^\top(\zeta,0,s)\hat{\Phi}^\top(z,\zeta,s) = \hat{\Phi}^\top(z,0,s). \tag{32}$$

With $\hat{\Phi}(z,\zeta,s)$ calculated from (32), the integral term in (30) can be determined. In a final step, (30) is inserted in the output Eqs. (24) such that after reordering the transfer function matrix $\hat{G}: \mathbb{C} \to \mathbb{C}^{p \times m}$ follows

$$\begin{bmatrix} \hat{y}_{\mathrm{b}}(s) \\ \hat{y}_{\mathrm{d}}(s) \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{G}_{\mathrm{bb}}(s) & \hat{G}_{\mathrm{bd}}(s) \\ \hat{G}_{\mathrm{db}}(s) & \hat{G}_{\mathrm{dd}}(s) \end{bmatrix}}_{\hat{G}(s)} \begin{bmatrix} \hat{u}_{\mathrm{b}}(s) \\ \hat{u}_{\mathrm{d}}(s) \end{bmatrix}. \tag{33}$$

The matrix entries in (33) are

$$\hat{G}_{\mathrm{bb}}(s) = \left(\tilde{C}_{\mathrm{b0}} + \tilde{C}_{\mathrm{b1}}\hat{\Phi}(1,0,s)\right)\hat{B}_{\mathrm{b}}(s), \tag{34a}$$

$$\hat{G}_{\mathrm{bd}}(s) = \left(\tilde{C}_{\mathrm{b0}} + \tilde{C}_{\mathrm{b1}}\hat{\Phi}(1,0,s)\right)\hat{B}_{\mathrm{d}}(s) + \tilde{C}_{\mathrm{b1}}\int_{\bar{\mathcal{Z}}} \hat{\Phi}(1,\zeta,s)\tilde{B}_{\mathrm{d}}(z)\,\mathrm{d}\zeta, \tag{34b}$$

$$\hat{G}_{\mathrm{db}}(s) = \int_{\bar{\mathcal{Z}}} \tilde{C}_{\mathrm{d}}(z)\hat{\Phi}(z,0,s)\hat{B}_{\mathrm{b}}(s)\,\mathrm{d}z, \tag{34c}$$

$$\hat{G}_{\mathrm{dd}}(s) = \int_{\bar{\mathcal{Z}}} \tilde{C}_{\mathrm{d}}(z)\left(\hat{\Phi}(z,0,s)\hat{B}_{\mathrm{d}}(s) + \int_0^z \hat{\Phi}(z,\zeta,s)\tilde{B}_{\mathrm{d}}(\zeta)\,\mathrm{d}\zeta \right)\mathrm{d}z, \tag{34d}$$

with $\hat{G}_{\mathrm{bb}}: \mathbb{C} \to \mathbb{C}^{p_{\mathrm{b}} \times m_{\mathrm{b}}}$, $\hat{G}_{\mathrm{db}}: \mathbb{C} \to \mathbb{C}^{p_{\mathrm{d}} \times m_{\mathrm{b}}}$, $\hat{G}_{\mathrm{bd}}: \mathbb{C} \to \mathbb{C}^{p_{\mathrm{b}} \times m_{\mathrm{d}}}$ and $\hat{G}_{\mathrm{dd}}: \mathbb{C} \to \mathbb{C}^{p_{\mathrm{d}} \times m_{\mathrm{d}}}$.

Let us summarize the **calculation procedure**

1. Chose $\Omega \subset \mathbb{C}$ and discretize the set in points $s_0 \in \Omega$,
2. Solve (27) for $s = s_0$ numerically to obtain $\hat{\Phi}(\cdot,0,s_0)$ with a suitable ODE solver,
3. The integrals in (34b)–(34d) are calculated via some standard numerical quadrature, where the necessary points $\hat{\Phi}(z_i,\zeta_j,s_0)$ are determine in a subroutine solving the equation system (32),
4. Combining all determined terms yields $\hat{G}(s_0)$ via (34),
5. Repeat steps 2–4 for all $s_0 \in \Omega$.

While in practice distributed inputs often occur, distributed measurements are rare. Thus, the double integral in (34d) simplifies to a single integral in case of in-domain point measurements. This calculation approach is demonstrated in Section 5.1.2 and Section 5.2.2.

3.4.2. ODE boundary value approach

While the transfer function matrix is derived in the transition matrix approach via an I/O-relation of the form of (15), in the ODE boundary value approach it is obtained from the Laplace transform of the impulse response. As common, the latter is defined as the matrix of the system responses g_{ij} observed at the i-th output for the excitation of j-th input with a Dirac function while the other inputs as well as the initial value are zero. Since the distributional Laplace transform of the Dirac function is one, the j-th input's Laplace transform is

$$\hat{u}(s)_j = \begin{bmatrix} \hat{u}_{\mathrm{b}}(s)_j \\ \hat{u}_{\mathrm{d}}(s)_j \end{bmatrix} = e_j^{m_{\mathrm{b}}+m_{\mathrm{d}}}, \tag{35}$$

where $e_j^{m_{\mathrm{b}}+m_{\mathrm{d}}}$ denote the canonical basis vectors of the $(m_{\mathrm{b}}+m_{\mathrm{d}})$-dimensional space $\mathbb{C}^{m_{\mathrm{b}}+m_{\mathrm{d}}}$. Hence, $\hat{u}_{\mathrm{b}}(s)$ and $\hat{u}_{\mathrm{d}}(s)$ are to be replaced in (23) by

$$\begin{aligned} \hat{u}_{\mathrm{b}}(s)_j &= \begin{cases} e_j^{m_{\mathrm{b}}}, & j \leq m_{\mathrm{b}} \\ 0_{m_{\mathrm{d}}}, & \text{else}, \end{cases} \\ \hat{u}_{\mathrm{d}}(s)_j &= \begin{cases} 0_{m_{\mathrm{d}}}, & j \leq m_{\mathrm{b}} \\ e_{j-m_{\mathrm{b}}+1}^{m_{\mathrm{d}}}, & \text{else}. \end{cases} \end{aligned} \tag{36}$$

Eq. (23) modified in this way represents an ODE boundary value problem, which can be solved numerically for a certain e_j. With the calculated $\hat{\tilde{x}}(z,s)_j$, one obtains from (24)

$$\begin{aligned} \hat{g}_{\mathrm{b}}(s)_j &= C_{\mathrm{b0}}\hat{\tilde{x}}(0,s)_j + C_{\mathrm{b1}}\hat{\tilde{x}}(1,s)_j, \\ \hat{g}_{\mathrm{d}}(s)_j &= \int_{\bar{\mathcal{Z}}} C_{\mathrm{d}}(z)\hat{\tilde{x}}(z,s)_j\,\mathrm{d}z, \end{aligned} \tag{37}$$

such that

$$\hat{g}(s)_j := \begin{bmatrix} \hat{g}_b(s)_j \\ \hat{g}_d(s)_j \end{bmatrix}. \quad (38)$$

This calculation needs to be carried out for all inputs j to determine $\hat{G}: \mathbb{C} \to \mathbb{C}^{p\times m}$ by

$$\hat{G}(s) = \begin{bmatrix} \hat{g}(s)_1 & \dots & \hat{g}(s)_{m_b+m_d} \end{bmatrix}. \quad (39)$$

Let us summarize the **calculation procedure**

1. Choose $\Omega \subset \mathbb{C}$ and discretize the set in points $s_0 \in \Omega$,
2. Solve (23) for $s = s_0$ and $\hat{u}_b(s_0)_j$, $\hat{u}_d(s_0)_j$, as in (35), with a suitable BVP solver to obtain $\hat{x}(\cdot, s_0)$,
3. Calculate the integral in (37) via some standard numerical quadrature,
4. Combine all determined terms to obtain $\hat{g}(s_0)_j$ via (38),
5. Repeat steps 2–4 for all $j \in [1, m_b + m_d]$ to obtain $\hat{G}(s_0)$ via (39),
6. Repeat steps 2–5 for all $s_0 \in \Omega$.

The application of this approach is shown in Section 5.1.3.

3.5. Comparison of the approaches

In the following, a brief comparison of the most important properties of the approaches is given. The transition matrix approach

+ is computationally advantageous (Cauchy problem), since it is typically faster and better-suited for complex-valued s_0 with large imaginary part.
+ yields the whole $\hat{G}(s_0)$ at once.
+ allows to express $\hat{G}(s)$ via $\hat{\Phi}(z, 0, s)$.
– is not applicable for singular $\tilde{E}$.
– may fail for certain s_0 (see Section 5.1.4).
– requires the calculation of $\hat{\Phi}$, whose number of entries scales quadratically with the number of PDE states.

However, the ODE boundary value approach

+ is broadly applicable.
+ requires only the numerical calculation of a vector with the size of the PDE's state vector for each $\hat{u}(s)_j$ as in (35).
– only allows to define $\hat{G}(s)$ implicitly as the solution of an ODE BVP.
– necessitates j-times repeated calculation for all $\hat{u}(s)_j$ to obtain $\hat{G}(s_0)$ as in (39).
– requires BVP solvers, which are typically less robust.

Both approaches have shown sufficient accuracy for all examples investigated by the authors.

4. Extension of the class of PDEs

The class of PDEs that can be addressed by the proposed numerical calculation approaches is much larger than discussed in Section 3. The extension to systems with inhomogeneous boundary conditions is discussed in Section 4.1. The application to PDEs involving state variables without spatial derivatives is presented in Section 4.2 and mixed partial derivatives are considered in Section 4.3.

4.1. Inhomogeneous boundary conditions

The assumption of homogeneous boundary conditions (18c) is due to the goal of obtaining a transfer functions in the common form of (15), where the inputs are allowed to be arbitrary functions. Otherwise, a transfer function with an additional input occurs, caused by the inhomogeneous part. However, this additional input is a fixed function instead of an arbitrary one. In order to avoid this situation, input and state transformations can be applied to the PDE to eliminate the inhomogeneous part of the boundary conditions. Hence, this is a preliminary step before the application of the calculation approaches.

4.2. State variables without spatial derivatives

Assume that no equation in(18) depends on a partial spatial derivative of the component x_j, i.e., $\nu_{z,j} = 0$. In other words, the j-th columns of A_{zi} are zero columns for all $i \geq 1$, i.e., not in A_{z0}. Hence, by shifting the j-th element to the end of the state vector, the system (23) reads with $\hat{\bar{x}}: \bar{\mathcal{Z}} \times \overline{\mathbb{C}_\alpha^+} \to \mathbb{C}^{\tilde{\nu}_z-1}$ and $\hat{\xi}: \bar{\mathcal{Z}} \times \overline{\mathbb{C}_\alpha^+} \to \mathbb{C}$

$$\tilde{E}(z)\begin{bmatrix} \hat{\bar{x}}'(z,s) \\ \hat{\xi}'(z,s) \end{bmatrix} = \tilde{A}(z,s)\begin{bmatrix} \hat{\bar{x}}(z,s) \\ \hat{\xi}(z,s) \end{bmatrix} + \tilde{B}_d(z)\hat{u}_d(s), \quad z \in \mathcal{Z}, \quad (40)$$

where the matrices are

$$\tilde{E}(z) = \begin{bmatrix} \tilde{E}_{\bar{x}\bar{x}}(z) & 0^{\tilde{\nu}_z-1} \\ \tilde{e}_{\xi\bar{x}}^\top(z) & 0 \end{bmatrix}, \qquad \tilde{B}_d = \begin{bmatrix} \tilde{B}_{d\bar{x}}(z) \\ \tilde{b}_{d\xi}^\top(z) \end{bmatrix},$$
$$\tilde{A}(z,s) = \begin{bmatrix} \tilde{A}_{\bar{x}\bar{x}}(z,s) & \tilde{a}_{\bar{x}\xi}(z,s) \\ \tilde{a}_{\xi\bar{x}}^\top(z,s) & \tilde{a}_{\xi\xi}(z,s) \end{bmatrix}. \quad (41)$$

If $\left|\tilde{a}_{\xi\xi}(z,s)\right| \geq \epsilon > 0$ for almost every $(z,s) \in \bar{\mathcal{Z}} \times \overline{\mathbb{C}_\alpha^+}$, the last row of (40) can be rearranged w.r.t. $\hat{\xi}(z,s)$

$$\hat{\xi}(z,s) = \Big(\tilde{e}_{\xi\bar{x}}^\top(z)\hat{\bar{x}}'(z,s) - \tilde{a}_{\xi\bar{x}}^\top(z,s)\hat{\bar{x}}(z,s) - \tilde{b}_{d\xi}^\top(z)\hat{u}_d(s) \Big) / \tilde{a}_{\xi\xi}(z,s), \qquad z \in \mathcal{Z}. \quad (42)$$

When inserting (42) in the remaining equation of (40)

$$\tilde{E}_{\bar{x}\bar{x}}(z)\hat{\bar{x}}'(z,t) = \tilde{A}_{\bar{x}\bar{x}}(z,s)\hat{\bar{x}}(z,t) + \tilde{a}_{\bar{x}\xi}(z,s)\hat{\xi}(z,s) + \tilde{B}_{d\bar{x}}(z)\hat{u}_d(s), \qquad z \in \mathcal{Z},$$

a system with the order $\tilde{\nu}_z - 1$ is obtained. An example for this system class is shown in Section 5.2.

4.3. Mixed partial derivatives

Due to the interchangeability of the partial derivatives in the distributional calculus, any LTI PDE involving mixed partial derivatives can be written in the compact form

$$\sum_{i=0}^{\bar{\nu}}\sum_{j=0}^{i} A_{ij}(z)\partial_t^i\partial_z^j x(z,t) = B_d(z)u_d(t), \; z \in \mathcal{Z}, \; t > 0^-, \quad (43)$$

with suitable boundary conditions and initial value, where $\bar{\nu} = \max\{\bar{\nu}_t, \bar{\nu}_z\}$, $A_{ij} \in PC(\bar{\mathcal{Z}}; \mathbb{R}^{n\times n})$ and $B_d \in PC(\bar{\mathcal{Z}}; \mathbb{R}^{n\times m_d})$. Let $x: \bar{\mathcal{Z}} \times \mathbb{R}_{0-}^+ \to \mathbb{R}^n$ with $x(z,\cdot) \in \mathcal{L}'_\alpha(\mathbb{R}_{0-}^+; \mathbb{R}^n)$, $u_d \in \mathcal{L}'_\alpha(\mathbb{R}_{0-}^+; \mathbb{R}^m)$ and $x_0(z) = 0$ for all $z \in \bar{\mathcal{Z}}$, then the Laplace-transformed system reads

$$\sum_{i=0}^{\bar{\nu}}\sum_{j=0}^{i} s^i A_{ij}(z)\partial_z^j\hat{x}(z,s) = B_d(z)\hat{u}_d(s), \quad z \in \mathcal{Z}, \quad (44)$$

where $\hat{x}: \bar{\mathcal{Z}} \times \overline{\mathbb{C}_\alpha^+} \to \mathbb{C}^n$ and $\hat{u}: \overline{\mathbb{C}_\alpha^+} \to \mathbb{C}^m$. By introducing an extended state vector $\hat{\bar{x}}: \bar{\mathcal{Z}} \times \overline{\mathbb{C}_\alpha^+} \to \mathbb{C}^{\tilde{\nu}_z}$, (44) can be rewritten as system of first order equations

$$\tilde{E}(z,s)\hat{\bar{x}}'(z,s) = \tilde{A}(z,s)\hat{\bar{x}}(z,s) + \tilde{B}_d\hat{u}_d(s), \quad z \in \mathcal{Z}. \quad (45)$$

In contrast to (23a), the matrix $\tilde{E}$ might have a s-dependence. However, as the complex frequency only appears as a parameter, the numerical approach can be chosen according to the properties of $\tilde{E}$ as specified in Table 1.

5. Application

We begin with the heat equation as an example of the system class introduced in Section 3 and apply both numerical calculation approaches in Section 5.1. Afterwards, a transport PDE with an additional equation with zero characteristic speed is considered in Section 5.2, which belongs to the extended system class from Section 4.2. The parameters used in the simulations are summarized in Table 2.

5.1. Heat equation with Dirichlet boundary conditions

The one-dimensional heat equation describes the temperature distribution $x\colon \bar{\mathcal{Z}} \times \mathbb{R}^+_{0^-} \to \mathbb{R}$ in a long rod

$$\partial_t x(z,t) = a^2 \partial_z^2 x(z,t), \qquad z \in \mathcal{Z},\ t > 0^-, \tag{46a}$$
$$x(z,0^-) = x_0(z), \qquad z \in \bar{\mathcal{Z}}, \tag{46b}$$
$$x(0,t) = 0, \qquad t > 0^-, \tag{46c}$$
$$x(1,t) = u(t), \qquad t > 0^-, \tag{46d}$$
$$y(t) = x(z_0,t), \tag{46e}$$

which is a boundary control system with point measurement. From (46), one reads $\nu_t = 1$, $\nu_z = 2$ and $C_d(z) = \delta(z - z_0)$. Assume that $u, x(z,\cdot), y \in \mathcal{L}'_\alpha(\mathbb{R}^+_{0^-};\mathbb{R})$, $x_0 \in PC(\bar{\mathcal{Z}};\mathbb{R})$ and $a \in \mathbb{R}^+$, $z_0 \in \mathcal{Z}$. The application of the Laplace transform to (46) leads to a 2nd order BVP w.r.t. the spatial coordinate similar to (20)

$$\hat{x}''(z,s) = \frac{s}{a^2}\hat{x}(z,s) + \frac{1}{a^2}x_0(z), \qquad z \in \mathcal{Z}, \tag{47a}$$
$$\hat{x}(0,s) = 0, \tag{47b}$$
$$\hat{x}(1,s) = \hat{u}(s), \tag{47c}$$
$$\hat{y}(s) = \hat{x}(z_0,s) \tag{47d}$$

with $\hat{x}\colon \bar{\mathcal{Z}} \times \overline{\mathbb{C}^+_\alpha} \to \mathbb{C}$ and $\hat{u}, \hat{y}\colon \overline{\mathbb{C}^+_\alpha} \to \mathbb{C}$. In the following, we derive the closed-form transfer function in Section 5.1.1, apply the transition matrix approach in Section 5.1.2 and the ODE boundary value approach in Section 5.1.3 and thus consider $x_0(z) \equiv 0$ for all $z \in \bar{\mathcal{Z}}$. A comparison is performed in Section 5.1.4.

5.1.1. Closed-form solution

The general solution of (47a) is

$$\hat{x}(z,s) = c_1 \sinh\left(\tfrac{\sqrt{s}}{a}z\right) + c_2 \cosh\left(\tfrac{\sqrt{s}}{a}z\right), \quad z \in \bar{\mathcal{Z}}, \tag{48}$$

so that from the boundary conditions (47b) and (47c) it follows $c_1 = \hat{u}(s)/\sinh(\sqrt{s}/a)$ and $c_2 = 0$. Then, with (47d) the transfer function as defined in (15) is

$$\hat{g}(s) := \frac{\hat{y}(s)}{\hat{u}(s)} = \sinh\left(\tfrac{\sqrt{s}}{a}z_0\right)\sinh\left(\tfrac{\sqrt{s}}{a}\right)^{-1}. \tag{49}$$

5.1.2. Numerical solution via the transition matrix approach

Since $\nu_z = 2$ in (46), (47) is reformulated in a first order system with $\hat{\tilde{x}}\colon \bar{\mathcal{Z}} \times \overline{\mathbb{C}^+_\alpha} \to \mathbb{C}^2$ of the form $\hat{\tilde{x}}(z,s) = \begin{bmatrix}\hat{x}(z,s) & \hat{x}'(z,s)\end{bmatrix}^\top$

$$\hat{\tilde{x}}'(z,s) = \begin{bmatrix}0 & 1\\ s/a^2 & 0\end{bmatrix}\hat{\tilde{x}}(z,s), \qquad z \in \mathcal{Z}, \tag{50a}$$
$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\hat{\tilde{x}}(0,s) = \begin{bmatrix}0\\0\end{bmatrix}, \tag{50b}$$
$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\hat{\tilde{x}}(1,s) = \begin{bmatrix}1\\0\end{bmatrix}\hat{u}(s), \tag{50c}$$
$$\hat{y}(s) = \begin{bmatrix}1 & 0\end{bmatrix}\hat{\tilde{x}}(z_0,s). \tag{50d}$$

Table 2
Parameters used in the simulations in Section 5.

(a) Heat equation (46).

Parameter	Value
a	$4.8 \cdot 10^{-5}$
z_0	0.8

(b) Transport PDE (57).

Parameter	Value
λ	1
a_{12}	0.2
a_{21}	0.2
a_{22}	0.3

This ODE boundary value problem corresponds to (23) with $\tilde{E}(z) = I_2$ and has the solution

$$\hat{\tilde{x}}(z,s) = \hat{\Phi}(z,0,s)\hat{\tilde{x}}(0,s), \quad z \in \bar{\mathcal{Z}}, \tag{51}$$

where $\hat{\Phi}\colon \bar{\mathcal{Z}} \times \bar{\mathcal{Z}} \times \overline{\mathbb{C}^+_\alpha} \to \mathbb{C}^2$ is defined by

$$\hat{\Phi}'(z,0,s) = \begin{bmatrix}0 & 1\\ s/a^2 & 0\end{bmatrix}\hat{\Phi}(z,0,s), \quad z \in \bar{\mathcal{Z}} \setminus \{0\}, \qquad \hat{\Phi}(0,0,s) = I_2. \tag{52}$$

The boundary conditions (50b) and (50c) specify only the first component $\hat{\tilde{x}}_1(0,s) = 0$ of the vector directly. From (50c), the second component can be obtained with (51)

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\hat{\Phi}(1,0,s)\hat{\tilde{x}}(0,s) = \begin{bmatrix}1\\0\end{bmatrix}\hat{u}(s). \tag{53}$$

Let $\hat{\Phi}(z,0,s) = [\hat{\phi}_{ij}(z,0,s)]$, then (53) reads

$$\hat{\phi}_{11}(1,0,s)\hat{\tilde{x}}_1(0,s) + \hat{\phi}_{12}(1,0,s)\hat{\tilde{x}}_2(0,s) = \hat{u}(s), \tag{54}$$

leading to

$$\hat{\tilde{x}}_2(0,s) = \hat{\phi}_{12}^{-1}(1,0,s)\hat{u}(s) \tag{55}$$

and

$$\hat{y}(s) = \begin{bmatrix}1 & 0\end{bmatrix}\hat{\tilde{x}}(z_0,s), \tag{56a}$$
$$= \begin{bmatrix}1 & 0\end{bmatrix}\hat{\Phi}(z_0,0,s)\hat{\tilde{x}}(0,s), \tag{56b}$$
$$= \underbrace{\hat{\phi}_{12}(z_0,0,s)\hat{\phi}_{12}^{-1}(1,0,s)}_{\hat{g}(s)}\hat{u}(s). \tag{56c}$$

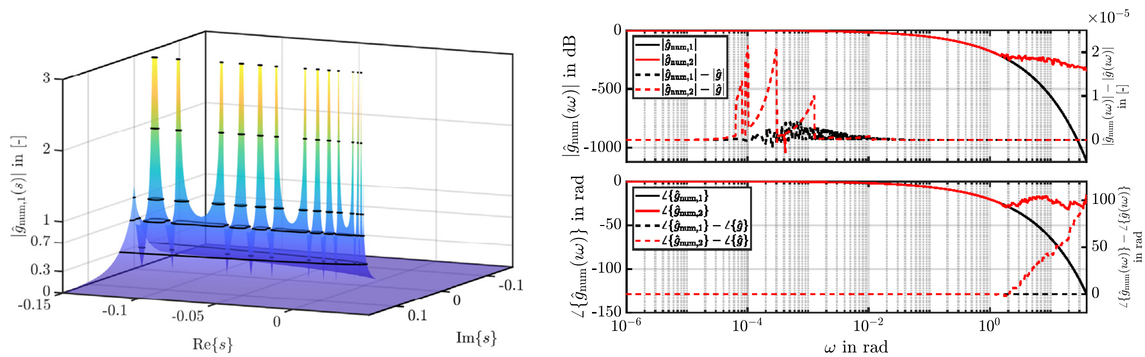
For the example under consideration, the calculation procedure in Section 3.4.1 becomes

1. Discretize $\Omega \subset \mathbb{C}$,
2. Solve (52) for $s = s_0$ numerically,
3. This step is not necessary, since (46e) is an in-domain point relation rather than an integral one,
4. Calculate $\hat{g}(s_0)$ according to (56c),
5. Repeat steps 2–4 for all $s_0 \in \Omega$.

5.1.3. Numerical solution via the ODE boundary value approach

As described in Section 3.4.2, we choose $\hat{u}(s) = 1$ and obtain the first order boundary value problem

$$\hat{\tilde{x}}'(z,s) = \begin{bmatrix}0 & 1\\ s/a^2 & 0\end{bmatrix}\hat{\tilde{x}}(z,s), \qquad z \in \mathcal{Z}, \tag{57a}$$
$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\hat{\tilde{x}}(0,s) = \begin{bmatrix}0\\0\end{bmatrix}, \tag{57b}$$



(a) Magnitude of $\hat{g}_{\text{num}}(s)$.

(b) Bode plot of $\hat{g}_{\text{num}}(s)$ and its difference to $\hat{g}(s)$.

Fig. 2. Illustrations of the numerical transfer function of the heat equation (46) determined with the transition matrix approach $\hat{g}_{\text{num},1}$ and the ODE boundary value approach $\hat{g}_{\text{num},2}$. The closed-form transfer function is denoted by $\hat{g}$. For the numerical calculation, Matlab's ODE45 and BVP5C are used, both with RelTol $= 10^{-6}$ and AbsTol $= 10^{-8}$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \hat{\hat{x}}(1,s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{57c}$$

From the numerically determined solution $\hat{\hat{x}}(\cdot, s_0)$ of Eq. (57), $\hat{g}(s_0)$ is calculated via

$$\hat{g}(s_0) := \hat{y}(s_0) = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{\hat{x}}(z_0, s_0). \tag{58}$$

Hence, the calculation procedure from Section 3.4.2 becomes

1. Choose $\Omega \subset \mathbb{C}$ and discretize the set in points $s_0 \in \Omega$,
2. Solve (57) for $s = s_0$ with a suitable BVP solver,
3. This step is not necessary, since (46e) is an in-domain point relation rather than an integral one,
4. Determine $\hat{g}(s_0)$ via (58),
5. This steps is not necessary, since (46) has only one input,
6. Repeat steps 2–5 for all $s_0 \in \Omega$.

5.1.4. Comparison

The numerical calculation procedures lead to the transfer function of (46) in a non-parametric form, which are illustrated as magnitude plot over the complex plane in Fig. 2(a) and in a Bode diagram in Fig. 2(b). From the deviation shown as black dashed line, it can be concluded that the transition matrix approach achieves good overall agreement with the closed-form solution over the whole frequency range. The ODE boundary value approach yields larger deviations for $\omega > 1$ illustrated by the red dashed line. However, these are still insignificant from a practical point of view.

Invisible in the diagrams in Fig. 2 is that the transition matrix approach fails at $s = 0$, which corresponds to a removable singularity (Curtain & Morris, 2009) caused by

$$\hat{\phi}_{12}(z,0,s) = \frac{\alpha}{\sqrt{s}} \sinh(\sqrt{s}z/\alpha), \quad z \in \bar{\mathcal{Z}}. \tag{59}$$

However, $\hat{g}(0)$ can be approximated by the values in its neighborhood or using the ODE boundary value approach, which yields the correct value $\hat{g}(0) = 0.8$. The overall calculation time, however, exceeds that of the transition matrix approach significantly.

5.2. Hyperbolic PDE with a state variable without spatial derivative

Consider the example from de Andrade et al. (2022) with $x\colon \bar{\mathcal{Z}} \times \mathbb{R}_{0^-}^+ \to \mathbb{R}^2$, $x(z,\cdot) \in \mathcal{L}'_\alpha(\mathbb{R}_{0^-}^+;\mathbb{R}^2)$ and $u, y \in \mathcal{L}'_\alpha(\mathbb{R}_{0^-}^+;\mathbb{R})$[5]

$$\partial_t x(z,t) = \Lambda \partial_z x(z,t) + A x(z,t), \qquad z \in \mathcal{Z},\ t > 0^-, \tag{60a}$$

$$x(z,0^-) = x_0(z), \qquad z \in \bar{\mathcal{Z}}, \tag{60b}$$

$$x(0,t) = bu(t), \qquad t > 0^-, \tag{60c}$$

$$y(t) = c^\top x(1,t), \tag{60d}$$

where $\lambda, a_{ij} > 0$ and the matrices are as follows

$$\Lambda = \begin{bmatrix} -\lambda & 0 \\ 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad b = \begin{bmatrix} 1 & 0 \end{bmatrix}^\top, \qquad c^\top = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{61}$$

The system (60) is characterized by $\nu_t = [1\ \ 1]^\top$, $\nu_z = [1\ \ 0]^\top$ and thus belongs to the model class described in Section 4.2. Applying the Laplace transform and rearranging the equation leads with $x_0(z) \equiv 0_2$ for all $z \in \bar{\mathcal{Z}}$ to

$$\hat{x}_1'(z,s) = -\tfrac{s}{\lambda}\hat{x}_1(z,s) + \tfrac{a_{12}}{\lambda}\hat{x}_2(z,s), \qquad z \in \mathcal{Z}, \tag{62a}$$

$$0 = a_{21}\hat{x}_1(z,s) + (a_{22} - s)\hat{x}_2(z,s), \quad z \in \bar{\mathcal{Z}}. \tag{62b}$$

By solving (62b) for $\hat{x}_2(z,s)$ and plugging it into (62a) as described in Section 4.2, one obtains

$$\hat{x}_1'(z,s) = \tfrac{1}{\lambda}\left(\tfrac{a_{12}a_{21}}{s - a_{22}} - s\right)\hat{x}_1(z,s), \quad z \in \mathcal{Z} \tag{63a}$$

$$\hat{x}_1(0,s) = \hat{u}(s), \tag{63b}$$

$$\hat{y}(s) = \hat{x}_1(1,s). \tag{63c}$$

In the following, we derive the closed form solution in Section 5.2.1 and apply the transition matrix approach in Section 5.2.2. A comparison is performed in Section 5.2.3.

5.2.1. Closed-form solution

The closed-form transfer function of (63) can be written with the exponential function

$$\hat{y}(s) = \exp\left(\tfrac{1}{\lambda}\left(\tfrac{a_{12}a_{21}}{s-a_{22}} - s\right)\right)\hat{u}(s). \tag{64}$$

5.2.2. Transition matrix approach

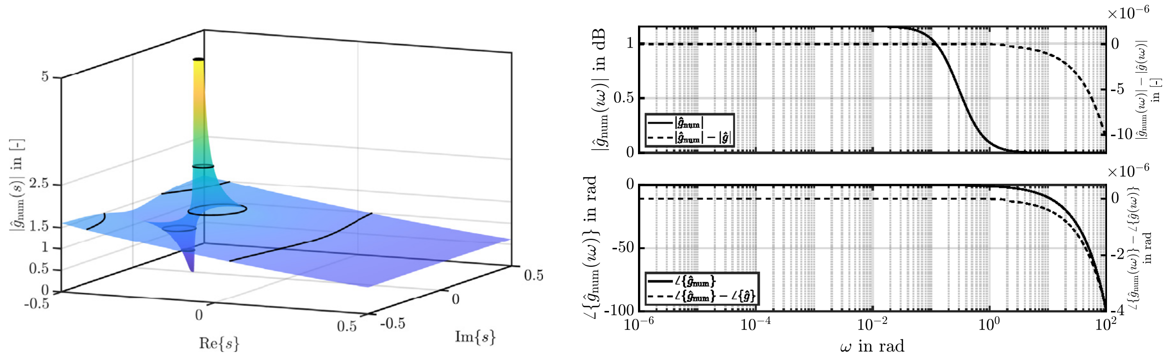
The transition matrix approach can be applied to (63) as described in Section 3.4.1. The differential equation for $\hat{\phi}\colon \bar{\mathcal{Z}} \times \bar{\mathcal{Z}} \times \mathbb{C}_\alpha^+ \to \mathbb{C}$ reads

$$\hat{\phi}'(z,0,s) = \tfrac{1}{\lambda}\left(\tfrac{a_{12}a_{21}}{s-a_{22}} - s\right)\hat{\phi}(z,0,s), \quad z \in \mathcal{Z}, \qquad \hat{\phi}(0,0,s) = 1 \tag{65}$$

and the transfer function $\hat{g}(s)$ of (60) follows from

$$\hat{y}(s) = \underbrace{\hat{\phi}(1,0,s)}_{\hat{g}(s)}\hat{u}(s). \tag{66}$$

[5] The second equation in (60a) does not have a boundary condition so that (60a) is defined on $\bar{\mathcal{Z}}$ for $x_2(z,t)$, but on $\mathcal{Z}$ for $x_1(z,t)$. This is the reason for the different domains in (62).



(a) Magnitude of $\hat{g}_{\text{num}}(s)$.

(b) Bode plot of $\hat{g}_{\text{num}}(s)$ and its difference to $\hat{g}(s)$.

Fig. 3. Two illustrations of the numerical transfer function $\hat{g}_{\text{num}}$ of the hyperbolic PDE with Dirichlet boundary conditions (60) determined with the transition matrix approach. The closed-form transfer function is denoted by $\hat{g}$. Matlab's ODE45 is used for the numerical calculation with RelTol $= 10^{-6}$ and AbsTol $= 10^{-8}$.

For the present example, the calculation procedure in Section 3.4.1 becomes

1. Discretize $\Omega \subset \mathbb{C}$,
2. Solve (65) for $s = s_0$ numerically,
3. This step is not necessary, since (60d) is a boundary output,
4. Calculate $\hat{g}(s_0)$ according to (66),
5. Repeat steps 2–4 for all $s_0 \in \Omega$.

5.2.3. Comparison

Carrying out the numerical calculations lead to the transfer function shown in Fig. 3. In Fig. 3(b), an excellent agreement between the closed-form and the numerically calculated transfer function can be observed over the whole frequency range.

6. Conclusion

The new numerical approach allows to calculate the transfer function of LTI PDEs with one time and one spatial coordinate in a non-parametric form. It overcomes both the need for an approximation step as in Paths 1–3 and the limitations w.r.t. the system class of Path 4 in Fig. 1. Two approaches for solving the complex-valued ordinary boundary problem in the Laplace domain are presented. The first one is based on a reformulation into a Cauchy-like problem, whereas the second one solves the BVP directly. The determined non-parametric transfer function can, for instance, be visualized in a Bode plot or as a magnitude plot over the complex plane. This allows the application of non-parametric analysis or design methods to the PDE's actual transfer function. For example, the Bode plot can be used in loop shaping control design, whereas from the magnitude plot the location of zeros and singularities follow. Alternatively, a parametric approximation of the calculated non-parametric transfer function can serve as a starting point for modern model-based frequency-domain design methods. The accuracy of both calculation approaches has proven to be sufficient for practical applications.

CRediT authorship contribution statement

Jan M. Schaßberger: Writing – review & editing, Writing – original draft, Visualization, Software. **Gudrun Thäter:** Writing – review & editing, Supervision, Formal analysis. **Lutz Gröll:** Writing – review & editing, Supervision, Methodology, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Transfer functions defined via exponential signals

In Section 2.4, one of the transfer function's definitions is via the complex-valued pointwise gain to the input $u_0 e^{s_0 t} \cdot 1(t)$ for a suitable initial value, which is not necessarily zero. The concept shall be briefly explained via two examples.

A.1. Integrator ODE

Consider the integrator ODE with $u, x, y \in \mathcal{L}'(\mathbb{R}^+_{0-}; \mathbb{C})$

$$\begin{aligned} \dot{x}(t) &= u(t), \qquad t > 0^-, \\ x(0^-) &= x_0, \\ y(t) &= x(t), \end{aligned} \tag{A.1}$$

with an initial value $x_0 \in \mathbb{R}$ that is generally different from zero. Applying the Laplace transform to (A.1) leads for $u(t) = u_0 e^{s_0 t} \cdot 1(t)$ to

$$\hat{x}(s) = \frac{u_0}{s(s - s_0)} + \frac{x_0}{s}, \tag{A.2a}$$

$$\hat{y}(s) = \hat{x}(s). \tag{A.2b}$$

Combining (A.2a) and (A.2b), one obtains

$$\hat{y}(s) = -\frac{u_0/s_0}{s} + \frac{u_0/s_0}{s - s_0} + \frac{x_0}{s}. \tag{A.3}$$

For $x_0 = u_0/s_0$, the output of the system is a purely exponential signal

$$y(t) = \frac{1}{s_0} u_0 e^{s_0 t} \cdot 1(t), \tag{A.4}$$

yielding $\hat{g}(s_0) = 1/s_0$, which is the value of the transfer function of (A.1) $\hat{g}(s) = 1/s$ evaluated at $s = s_0 \neq 0$.

A.2. Transport PDE

Consider the transport equation with $x\colon \bar{\mathcal{Z}} \times \mathbb{R}^+_{0-} \to \mathbb{C}$ and $u, x(z,\cdot), y \in \mathcal{L}'(\mathbb{R}^+_{0-}; \mathbb{C})$

$$\partial_t x(z,t) = -\lambda \partial_z x(z,t), \qquad z \in \mathcal{Z},\ t > 0^-, \tag{A.5}$$

$$
\begin{aligned}
x(z,0^-) &= x_0(z), && z \in \bar{\mathcal{Z}},\\
x(0,t) &= u(t), && t > 0^-,\\
y(t) &= x(1,t),
\end{aligned}
$$

with the initial value function $x_0(z) \in PC(\bar{\mathcal{Z}})$. When applying the Laplace transform to (A.5) for $u(t) = u_0 e^{s_0 t} \cdot 1(t)$, one obtains

$$\hat{x}'(z,s) = -\frac{s}{\lambda}\hat{x}(z,s) + \frac{1}{\lambda}x_0(z), \qquad z \in \mathcal{Z}, \tag{A.6a}$$

$$\hat{x}(0,s) = \frac{u_0}{s-s_0}, \tag{A.6b}$$

$$\hat{y}(s) = \hat{x}(1,s), \tag{A.6c}$$

with $\hat{x}\colon \bar{\mathcal{Z}} \times \overline{\mathbb{C}_\alpha^+} \to \mathbb{C}$ and $\hat{y}\colon \overline{\mathbb{C}_\alpha^+} \to \mathbb{C}$. The general solution of (A.6a) is

$$\hat{x}(z,s) = c(s)e^{-sz/\lambda} + \frac{1}{\lambda}\int_0^z e^{s(\zeta - z)/\lambda}x_0(\zeta)\,d\zeta. \tag{A.7}$$

Combining (A.6b), (A.6c) and (A.7), the Laplace transform of the system output can be expressed by

$$\hat{y}(s) = \frac{u_0}{s-s_0}e^{-s/\lambda} + \frac{1}{\lambda}\int_0^1 e^{s(\zeta-1)/\lambda}x_0(\zeta)\,d\zeta. \tag{A.8}$$

Now suppose $x_0(z) = u_0 e^{-s_0 z/\lambda}$, so that (A.8) reads

$$\hat{y}(s) = \frac{u_0}{s-s_0}e^{-s/\lambda} + \frac{u_0}{\lambda}\int_0^1 e^{((s-s_0)\zeta - s)/\lambda}\,d\zeta, \tag{A.9}$$

which leads to

$$\hat{y}(s) = \frac{u_0}{s-s_0}e^{-s/\lambda} + \left[\frac{u_0}{s-s_0}e^{((s-s_0)\zeta - s)/\lambda}\right]_0^1 \tag{A.10}$$

so that $\hat{y}$ becomes

$$
\begin{aligned}
\hat{y}(s) &= \frac{u_0}{s-s_0}e^{-s/\lambda} + \frac{u_0}{s-s_0}\left(e^{-s_0/\lambda} - e^{-s/\lambda}\right),\\
&= \frac{u_0}{s-s_0}e^{-s_0/\lambda}.
\end{aligned} \tag{A.11}
$$

Hence, we obtain a purely exponential output

$$y(t) = e^{-s_0/\lambda}u_0 e^{s_0 t} \cdot 1(t), \tag{A.12}$$

yielding $\hat{g}(s_0) = e^{-s_0/\lambda}$, which is the value of the well-known transfer function of (A.5) $\hat{g}(s) = e^{-s/\lambda}$ at $s = s_0$.

Appendix B. Reformulation into a system of first order equations

Let $x\colon \bar{\mathcal{Z}} \times \mathbb{R}_{0^-}^+ \to \mathbb{R}^2$ with $x(z,\cdot) \in \mathcal{L}'(\mathbb{R}_{0^-}^+;\mathbb{R}^2)$ denote the state vector and $u \in \mathcal{L}'(\mathbb{R}_{0^-}^+;\mathbb{R}^2)$ the input vector of a PDE system consisting of a reaction-advection and a reaction-advection-diffusion equation

$$
\begin{aligned}
&A_{t1}\partial_t x(z,t) + A_{z2}\partial_z^2 x(z,t) + A_{z1}\partial_z x(z,t)\\
&\qquad + A_{z0}x(z,t) = Bu(t), \qquad z \in \mathcal{Z},\ t > 0^-,
\end{aligned} \tag{B.1}
$$

where

$$
\begin{aligned}
A_{t1} &= \mathrm{diag}(1,\ 1), \quad & A_{z2} &= \mathrm{diag}(0,\ a), \quad & B &= [b_{ij}],\\
& & A_{z1} &= [\lambda_{ij}],\\
& & A_{z0} &= [a_{ij}].
\end{aligned}
$$

With $\hat{x}\colon \bar{\mathcal{Z}} \times \overline{\mathbb{C}_\alpha^+} \to \mathbb{C}^2$, $\hat{u}\colon \overline{\mathbb{C}_\alpha^+} \to \mathbb{C}^2$ and zero initial value, (B.1) reads in the Laplace domain

$$
\begin{aligned}
&sA_{t1}\hat{x}(z,s) + A_{z2}\hat{x}''(z,s) + A_{z1}\hat{x}'(z,s)\\
&\qquad + A_{z0}\hat{x}(z,s) = B\hat{u}(s), \qquad z \in \mathcal{Z}.
\end{aligned} \tag{B.2}
$$

With the extended state vector $\hat{\xi}\colon \bar{\mathcal{Z}} \times \overline{\mathbb{C}_\alpha^+} \to \mathbb{C}^3$

$$\hat{\xi}_1(z,s) := \hat{x}_1(z,s), \quad \hat{\xi}_2(z,s) := \hat{x}_2(z,s), \quad \hat{\xi}_3(z,s) := \hat{x}_2'(z,s),$$

(B.2) can be written as a first order system

$$\tilde{E}\hat{\xi}'(z,s) = \tilde{A}(s)\hat{\xi}(z,s) + \tilde{B}\hat{u}(s), \qquad z \in \mathcal{Z},$$

where

$$
\tilde{E} = \begin{bmatrix} \lambda_{11} & 0 & 0\\ 0 & 1 & 0\\ \lambda_{21} & 0 & -a \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} b_{11} & b_{12}\\ 0 & 0\\ b_{21} & b_{22} \end{bmatrix},
$$

$$
\tilde{A}(s) = \begin{bmatrix} -(a_{11}+s) & -a_{12} & -\lambda_{12}\\ 0 & 0 & 1\\ -a_{21} & -(a_{22}+s) & -\lambda_{22} \end{bmatrix}.
$$

Data availability

No data was used for the research described in the article.

References

Aalto, H. (2008). Transfer Functions for Natural Gas Pipeline Systems. *IFAC Proceedings Volumes*, *41*(2), 889–894.

Bahar, L. (1969). The Laplace Transform of the Derivative of a Function with Finite Jumps. *"Journal of the Franklin Institute"*, *288*(4), 275–289.

Beffa, F. (2024). *Weakly Nonlinear Systems*. Springer.

Berger, T., & Ilchmann, A. (2013). On the Standard Canonical Form of Time-Varying Linear DAEs. *Quart. Appl. Math.*, *71*, 69–87.

Buschman, R. (1996). *Integral Transformations, Operational Calculus, and Generalized Functions*. Springer.

Chirilă, A., Marin, M., & Öchsner, A. (2021). *Distribution Theory Applied to Differential Equations*. Springer.

Curtain, R., & Morris, K. (2009). Transfer Functions of Distributed Parameter Systems: A Tutorial. *Automatica*, *45*(5), 1101–1116.

Curtain, R., & Zwart, H. (2020). Introduction to Infinite-Dimensional Systems Theory. In *texts in applied mathematics*: *vol. 71*, Springer.

de Andrade, G., Vazquez, R., Karafyllis, I., & Krstic, M. (2022). Backstepping Control of a Hyperbolic PDE System with Zero Characteristic Speed. *IFAC-PapersOnLine*, *55*(26), 168–173.

Debnath, L. (2016). The Double Laplace Transforms and Their Properties with Applications to Functional, Integral and Partial Differential Equations. *Int. J. Appl. Comput. Math*, *2*, 223–241.

Deutscher, J. (2012). *Zustandsregelung verteiltparametrischer Systeme [State-Space Control of Distributed Parameter Systems]*. Springer.

Ditkin, V., Prudnikov, A., & Wishart, D. (2017). *Operational Calculus in Two Variables and Its Applications*. Dover Publications Inc.

Doetsch, G. (1950). Handbuch der Laplace-Transformation [Handbook of the Laplace Transform]. In *Lehrbücher und Monographien aus dem Gebiete der Exakten Wissenschaften – Mathematische Reihe*: *vol. 14*, Birkhäuser.

Ebert, M., & Reissig, M. (2018). *Methods for Partial Differential Equations*. Springer.

Hinrichsen, D., & Pritchard, A. (2005). Mathematical Systems Theory I. In *texts in applied mathematics*: *vol. 48*, Springer.

Karlström, A., & Breitholtz, C. (1992). Experimental Validation of a Packed Bed Distillation Model. *Chemical Engineering & Technology*, *15*(6), 406–416.

McCollum, P., & Brown, B. (1965). *Laplace Transform Tables and Theorems*. Holt, Rinehart and Winston.

Lopes dos Santos, P., Azevedo-Perdicoúlis, T., Ramos, J., Jank, G., Martins de Carvalho, J., & Milhinhos, J. (2010). A Lumped Transfer Function Model for High Pressure Gas Pipelines. In *49th IEEE conference on decision and control* (pp. 3487–3492).

Schaßberger, J., Hagenmeyer, V., & Gröll, L. (2025). A New Numerical Approach to Transfer Functions of LTI-PDEs. *IFAC-PapersOnLine*, *59*(1), 97–102.

Schiff, J. (1999). The Laplace Transform: Theory and Applications. In *undergraduate texts in mathematics*, Springer.

Smirnov, V. (1964). *A Course of Higher Mathematics*: *vol. 3*, Pergamon, Part 2.

Zwart, H. (2004). Transfer Functions for Infinite-Dimensional Systems. *Systems & Control Letters*, *52*, 247–255.