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Convergence analysis of nonconforming $H(\mathrm{div})$-finite elements for the damped time-harmonic Galbrun's equation

Martin Halla[1]

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Abstract

We consider the damped time-harmonic Galbrun's equation, which is used to model stellar oscillations. We introduce a discontinuous Galerkin finite element method (DGFEM) with $H(\mathrm{div})$-elements, which is nonconforming with respect to the convection operator. We report a convergence analysis, which is based on the frameworks of discrete approximation schemes and T-compatibility. A novelty is that we show how to interpret a DGFEM as a discrete approximation scheme and this approach enables us to apply compact perturbation arguments in a DG-setting, and to circumvent any extra regularity assumptions on the solution. The advantage of the proposed $H(\mathrm{div})$-DGFEM compared to H^1-conforming methods is that we do not require a minimal polynomial order or any special assumptions on the mesh structure. Further, we extend the analysis of the symmetric interior penalty DGFEM to a DGFEM without a penalty term, which considerably improves the smallness assumption on the Mach number to a fairly explicit bound. In addition, the method is robust with respect to the drastic changes of magnitude of the density and sound speed, which occur in stars.

Mathematics Subject Classification 35L05 · 35Q85 · 65N30

1 Introduction

In this article we introduce and analyze a particular finite element method to approximatively solve the damped time-harmonic Galbrun's equation

✉ Martin Halla
martin.halla@kit.edu

[1] Institute for Applied and Numerical Mathematics, Karlsruhe Institute of Technology, Englerstraße 2, Karlsruhe 76131, Baden-Württemberg, Germany

$$-\nabla\left(\rho c_s^2 \operatorname{div}\mathbf{u}\right) + (\operatorname{div}\mathbf{u})\nabla p - \nabla(\nabla p\cdot\mathbf{u}) - \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)^2\mathbf{u} \tag{1a}$$
$$+ (\operatorname{Hess}(p) - \rho\operatorname{Hess}(\phi))\mathbf{u} + \gamma\rho(-i\omega)\mathbf{u} = \mathbf{f} \quad \text{in } \mathcal{O},$$
$$\boldsymbol{\nu}\cdot\mathbf{u} = 0 \quad \text{on } \partial\mathcal{O}, \tag{1b}$$

where the Lagragian displacement vector $\mathbf{u}$ is the only unknown, ρ, p, ϕ, c_s, $\mathbf{b}$, Ω and $\mathbf{f}$ denote density, pressure, gravitational potential, sound speed, background velocity, angular velocity of the frame and sources, $\partial_{\mathbf{b}} := \sum_{l=1}^{3}\mathbf{b}_l\partial_{\mathbf{x}_l}$ denotes the directional derivative in direction $\mathbf{b}$, $\operatorname{Hess}(p)$ the Hessian of p, $\mathcal{O}\subset\mathbb{R}^3$ a bounded domain, and damping is modeled by the term $-i\omega\gamma\rho\mathbf{u}$ with damping coefficient γ. Galbrun's equation was first derived in [1] as a linearization of the nonlinear Euler's equation and serves as a model in aeroacoustics [2] and in an extended form in asterophysics [3]. In the time-domain Galbrun's equation was analyzed in [4]. In the time-harmonic domain a well-posedness analysis in an aeroacoustic setting was reported in [5] through the introduction of an additional transport equation. Different to that, in a stellar context well-posedness results were reported in [6, 7] by exploiting the damping effects in stars. Concerning the numerical approximation of Galbrun's equation it is well known [8] that naive discretizations may yield unreliable results.

To construct stable methods a path is to follow the analysis from the continuous level [6] and to try to mimic the analysis on the discrete level. Here the main tool of [6] was the concept of weak T-coercivity analysis, which is to construct an explicit test function operator to prove Fredholmness with index zero. The crucial point here is that introducing this T operator explicitly its error with respect to discrete versions can be studied. Indeed such a framework was introduced in [9] and applied successfully to various Maxwell eigenvalue problems [10–12] and perfectly matched layer methods [13, 14]. However, it turned out to require too strong assumptions to allow the analysis of discretizations to Galbrun's equation.

As a remedy a version with weaker assumptions was introduced in [15] and successfully applied for the convergence analysis of *divergence stable* $\mathbf{H}^1$-conforming finite element discretizations of Galbrun's equation. Therein the so-called *divergence stability* is ensured by assumptions on the mesh structure and the polynomial order of the method. However, those methods have a significant computational cost, e.g. in three dimensions general meshes are speculated to require a minimal polynomial degree between six and eight [16, 17] and barycentrically refined meshes require a minimal polynomial degree three [18]. Although we note that barycentric refinement produces a lot additional degrees of freedom without a reduction of the element diameters.

In this article we employ $H(\operatorname{div})$-conforming finite elements and treat the non-conformity with respect to the convection operator with a discontinuous Galerkin technique. In particular, we apply a reconstruction operator to lift the jumps and avoid a stabilization term to optimize the assumption on the smallness of the Mach number. The obtained method does not require any special assumption on the meshes and works for all polynomial orders greater equal than one. Most importantly the method is robust with respect to drastic changes of magnitude of the density and sound speed, which occur in stars. Apart from proving the convergence of the method a major contribution of this article is to show how to interpret a DGFEM as a discrete approximation scheme (DAS) [19]—a concept to perform numerical analysis rooted in the 1970 s (see Sect. 3.1 for a definition).

The remainder of this article is structured as follows. In Sect. 2 we introduce the applied $H(\text{div})$-discontinuous Galerkin finite element method. In Sect. 3 we recall the abstract framework from [15] and show that our $H(\text{div})$-DGFE-method constitutes an asymptotically consistent *discrete approximation scheme*. In Sect. 4 we introduce discrete operators $(T_n)_{n\in\mathbb{N}}$, analyze their properties and report our main convergence result in Theorem 19.

2 Formulation of the $H(\text{div})$-DDG-FEM

For two Hilbert spaces $(X, \langle\cdot,\cdot\rangle_X)$, $(Y, \langle\cdot,\cdot\rangle_Y)$ let $L(X, Y)$ be the space of bounded linear operators from X to Y, and set $L(X) := L(X, X)$. For any space X of scalar valued functions let $\mathbf{X} := X^3$. For $A \in L(X)$ let the bounded sesquilinear form $a(\cdot,\cdot)$ be defined by the relation

$$\langle Au, u'\rangle_X = a(u, u') \quad \text{for all } u, u' \in X, \tag{2}$$

and vice-versa for a given bounded sesquilinear form $a(\cdot,\cdot)$ let $A \in L(X)$ be defined by the relation (2). We call $A \in L(X)$ coercive, if $\inf_{u\in X\setminus\{0\}} |\langle Au, u\rangle|/\|u\|_X^2 > 0$. For a bijective operator $T \in L(X)$ we call $A \in L(X)$ weakly left (right) T-coercive, if there exists a compact operator $K \in L(X)$ such that $T^*A + K$ $(AT + K)$ is coercive. Note that historically the notion of left T-coercivity was used, because then the operator T selects a suitable test function: $\langle T^*Au, u\rangle_X = a(u, Tu)$. However, when conducting the stability/compatibility analysis at the discrete level the notion of right weak T-coercivity seems favorable [15], because it avoids the introduction and subsequent treatment of adjoint operators. For expressions $A, B \in \mathbb{R}$ we employ the notation $A \lesssim B$, if there exists a constant $C > 0$ such that $A \le CB$. The constant $C > 0$ may be different at each occurrence and can depend on the domain $\mathcal{O}$, the physical parameters $\rho, c_s, p, , \phi, \gamma, \mathbf{b}, \omega, \Omega$, and on the sequence of Galerkin spaces $(X_n)_{n\in\mathbb{N}}$. However, it will always be independent of the index n and any involved functions ($u, v \in X$, $u_n \in X_n$, etc.) which may appear in the terms A and B.

2.1 Variational formulation

Let $\mathcal{O} \subset \mathbb{R}^3$ be a bounded convex Lipschitz polyhedron, $\omega \in \mathbb{R}\setminus\{0\}$ and $\Omega \in \mathbb{R}^3$. For brevity all function spaces without specified domain are considered on the domain $\mathcal{O}$, e.g. $L^2 = L^2(\mathcal{O})$, etc.. Let $c_s, \rho \in W^{1,\infty}(\mathcal{O}, \mathbb{R})$, $\gamma \in L^\infty(\mathcal{O}, \mathbb{R})$ and constants $\underline{c_s}, \overline{c_s}, \underline{\rho}, \overline{\rho}, \underline{\gamma}, \overline{\gamma} > 0$ be such that $\underline{c_s} \le c_s(\mathbf{x}) \le \overline{c_s}$, $\underline{\rho} \le \rho(\mathbf{x}) \le \overline{\rho}$ and $\underline{\gamma} \le \gamma(\mathbf{x}) \le \overline{\gamma}$ for all $\mathbf{x} \in \mathcal{O}$. In addition let $\mathbf{b} \in W^{1,\infty}(\mathcal{O}, \mathbb{R}^3)$ be compactly supported in $\mathcal{O}$ and $p, \phi \in W^{2,\infty}(\mathcal{O}, \mathbb{R})$. For a scalar function u we consider its gradient to be a column vector $\nabla u := (\partial_{\mathbf{x}_1}u, \partial_{\mathbf{x}_2}u, \partial_{\mathbf{x}_3}u)^\top$, its Hessian to be a matrix $\text{Hess}(u) := (\partial_{\mathbf{x}_n}\partial_{\mathbf{x}_m}u)_{n,m=1,2,3}$, and for a (column) vectorial function $\mathbf{u}$ we consider its gradient to be a matrix $\nabla\mathbf{u} := (\partial_{\mathbf{x}_m}\mathbf{u}_n)_{n,m=1,2,3}$. We abbreviate the L^2- and $\mathbf{L}^2$-

scalar products as $\langle\cdot,\cdot\rangle$. For any space $X \subset L^2$ let $X_* := \{u \in X \colon \langle u, 1\rangle = 0\}$ and $L^2_0 := L^2_*$. For functions $f \in W^{1,\infty}$, $\mathbf{f} \in \mathbf{W}^{1,\infty}$ we denote their Lipschitz constants as C^L_f and $C^L_{\mathbf{f}}$. In addition we introduce the space $\mathbf{H}^1_{\boldsymbol{\nu}0} := \{\mathbf{u} \in \mathbf{H}^1 \colon \boldsymbol{\nu}\cdot\mathbf{u} = 0 \text{ on } \partial\mathcal{O}\}$ and the weighted semi norm $|\mathbf{u}|^2_{\mathbf{H}^1_{c_s^2\rho}} := \|c_s\rho^{1/2}\nabla\mathbf{u}\|^2_{(L^2)^{3\times 3}}$. Let $\partial_{\mathbf{b}} := \mathbf{b}\cdot\nabla = \sum_{l=1}^3 \mathbf{b}_l\partial_{\mathbf{x}_l}$, for which the following integration by parts formula holds:

$$\langle\rho\partial_{\mathbf{b}}\mathbf{u}, \mathbf{u}'\rangle = -\langle\rho\mathbf{u}, \partial_{\mathbf{b}}\mathbf{u}'\rangle,$$

provided that the conservation of mass $\operatorname{div}(\rho\mathbf{b}) = 0$ holds true. We further introduce

$$\begin{aligned} \mathbb{X} &:= \{\mathbf{u} \in \mathbf{L}^2 \colon \operatorname{div}\mathbf{u} \in L^2, \quad \partial_{\mathbf{b}}\mathbf{u} \in \mathbf{L}^2, \quad \boldsymbol{\nu}\cdot\mathbf{u} = 0 \text{ on } \partial\mathcal{O}\},\\ \langle\mathbf{u},\mathbf{u}'\rangle_{\mathbb{X}} &:= \langle\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{u}'\rangle + \langle\partial_{\mathbf{b}}\mathbf{u}, \partial_{\mathbf{b}}\mathbf{u}'\rangle + \langle\mathbf{u},\mathbf{u}'\rangle, \end{aligned}$$

which constitutes a Hilbert space [6, Lemma 2.1]. We recall that $\mathbf{C}^\infty_0$ is dense in $\mathbb{X}$ [15, Theorem 6]. Let

$$\begin{aligned} a(\mathbf{u},\mathbf{u}') := &\langle c_s^2\rho\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{u}'\rangle - \langle\rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}'\rangle\\ &+ \langle\operatorname{div}\mathbf{u}, \nabla p\cdot\mathbf{u}'\rangle + \langle\nabla p\cdot\mathbf{u}, \operatorname{div}\mathbf{u}'\rangle + \langle(\operatorname{Hess}(p) - \rho\operatorname{Hess}(\phi))\mathbf{u},\mathbf{u}'\rangle\\ &- i\omega\langle\gamma\rho\mathbf{u},\mathbf{u}'\rangle \end{aligned} \tag{3}$$

for all $\mathbf{u},\mathbf{u}' \in \mathbb{X}$. Then assuming the conservation of mass $\operatorname{div}(\rho\mathbf{b}) = 0$ the variational formulation of (1) is $a(\mathbf{u},\mathbf{u}') = \langle\mathbf{f},\mathbf{u}'\rangle$ for all $\mathbf{u}' \in \mathbb{X}$ [6, Sect. 2.3].

2.2 *H*(div)-finite elements

Let $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be a sequence of tetrahedral meshes of $\mathcal{O}$, and $\mathcal{F}_n$ ($\mathcal{F}_n^{\text{int}}$) be the collection of the (interior) faces of $\mathcal{T}_n$. For $\tau \in \mathcal{T}_n$ denote $\mathcal{F}_\tau \subset \mathcal{F}_n$ the faces of τ, and $D_\tau := \operatorname{int}\{\bigcup\tilde\tau \colon \tilde\tau \in \mathcal{T}_n, \overline{\tilde\tau}\cap\overline{\tau} \neq \emptyset\}$ the macro element of τ which consists of all elements that share a face with τ. Also for $F \in \mathcal{F}_n$ we define the macro element $\tau_F := \operatorname{int}\{\bigcup\tau \colon F \in \mathcal{F}_\tau, \tau \in \mathcal{T}_n\}$. Further for $\mathcal{U} \in \{\tau, D_\tau, F, \tau_F \colon \tau \in \mathcal{T}_n, F \in \mathcal{F}_n\}$ and a scalar function g we abbreviate $\underline{g}_{\mathcal{U}} := \inf_{\mathbf{x}\in\mathcal{U}} g(\mathbf{x})$ and $\overline{g}_{\mathcal{U}} := \sup_{\mathbf{x}\in\mathcal{U}} g(\mathbf{x})$. For $\tau \in \mathcal{T}_n$ and $F \in \mathcal{F}_n$ let h_τ and h_F respectively be their diameters. Let $\mathfrak{h} \colon \mathcal{F}_n \to \mathbb{R}$ be defined by $\mathfrak{h}|_F := h_F$ for $F \in \mathcal{F}_n$ and $h_n := \max_{\tau\in\mathcal{T}_n} h_\tau$. We assume that $\lim_{n\to\infty} h_n = 0$ and that $(\mathcal{T}_n)_{n\in\mathbb{N}}$ is shape regular, i.e., there exists a constant $C_{\text{sh}} > 0$ such that

$$h_\tau \le C_{\text{sh}} h_F \quad \text{for all } \tau \in \mathcal{T}_n, F \in \mathcal{F}_\tau, n \in \mathbb{N}. \tag{4}$$

Let P_k be the space of scalar polynomials with maximal degree $k \in \mathbb{N}_0$. In this article we consider Brezzi-Douglas-Marini elements [20, Chapter 14.5.1] due to their approximations properties. In principle other $H(\text{div})$-elements such as Raviart-Thomas elements can easily be treated with the framework of this article, although some details need to be treated with care. We consider the polynomial degree $k \in \mathbb{N} = \{1, 2, \ldots\}$

to be uniform and fixed in the entire article. We introduce the finite element spaces

$$\begin{aligned}
\mathbb{X}_n &:= \{\mathbf{u} \in H_0(\mathrm{div})\colon \mathbf{u}|_\tau \in \mathbf{P}_k \text{ for all } \tau \in \mathcal{T}_n\},\\
\mathbb{X}_n^{\mathrm{wbc}} &:= \{\mathbf{u} \in H(\mathrm{div})\colon \mathbf{u}|_\tau \in \mathbf{P}_k \text{ for all } \tau \in \mathcal{T}_n\},\\
Q_n &:= \{u \in L^2_0\colon u|_\tau \in P_{k-1} \text{ for all } \tau \in \mathcal{T}_n\},\\
Q_n^{\mathrm{wbc}} &:= \{u \in L^2\colon u|_\tau \in P_{k-1} \text{ for all } \tau \in \mathcal{T}_n\}.
\end{aligned}$$

We consider the spaces Q_n, Q_n^{wbc} to be equipped with the standard L^2-scalar product, whereas the scalar product on $\mathbb{X}_n$ will be specified later on. For each $\tau \in \mathcal{T}_n$ let $\pi^d_\tau\colon \mathbf{H}^s(\tau) \to \mathbf{P}_k$, $s > 1/2$ and $\pi^l_\tau\colon L^2(\tau) \to P_k$ be the respective standard local interpolation operators, and $\pi^d_n\colon \mathbf{H}^s \to \mathbb{X}_n^{\mathrm{wbc}}$, $s > 1/2$, $\pi^d_n|_\tau := \pi^d_\tau$, $\tau \in \mathcal{T}_n$ and $\pi^l_n\colon L^2 \to Q_n^{\mathrm{wbc}}$, $\pi^l_n|_\tau := \pi^l_\tau$, $\tau \in \mathcal{T}_n$ be their global versions.

To be more specific, π^d_τ and π^l_τ are defined in terms of basis functions and degrees of freedom as described, e.g., in [20, Ch. 5, Ch. 14]. Since the BDM element is only scarcely treated in [20] we refer to [21, p. 62, p. 106-107] for its degrees of freedom and basis functions. Note that π^l_τ is nothing else than the L^2-orthogonal projection onto $P_k(\tau)$.

Note that $\pi^d_n\mathbf{v} \in \mathbb{X}_n$ if $\boldsymbol{\nu}\cdot\mathbf{v} = 0$ on $\partial\mathcal{O}$ and $\pi^l_n v \in Q_n$ if $v \in L^2_0$. We recall the commutation $\operatorname{div}\pi^d_n = \pi^l_n \operatorname{div}$ and the approximation (and boundedness) properties [20, Theorem 11.13]

$$|\mathbf{v} - \pi^d_n\mathbf{v}|_{\mathbf{H}^m(\tau)} \le C_{\mathrm{apr}} h_\tau^{r-m} |\mathbf{v}|_{\mathbf{H}^r(\tau)}, \qquad |v - \pi^l_n v|_{H^m(\tau)} \le C_{\mathrm{apr}} h_\tau^{r-m} |v|_{H^r(\tau)} \tag{5}$$

and an approximation result on faces [20, Remark 12.17]

$$\|\mathbf{v} - \pi^d_n\mathbf{v}\|_{\mathbf{L}^2(\partial\tau)} \le C_{\mathrm{ab}} h_\tau^{r-1/2} |\mathbf{v}|_{\mathbf{H}^r(\tau)} \tag{6}$$

with constants $C_{\mathrm{apr}}, C_{\mathrm{ab}} > 0$ for all $r \in [1, k+1]$, $m \in [0, r]$, $\mathbf{v} \in \mathbf{H}^r(\tau)$, $v \in H^r(\tau)$, $\tau \in \mathcal{T}_n$, $n \in \mathbb{N}$. In addition, we recall the discrete trace inequality [20, Lemma 12.8]

$$\|u\|_{L^2(\partial\tau)} \le C_{\mathrm{dt}} h_\tau^{-1/2} \|u\|_{L^2(\tau)} \tag{7}$$

and the discrete inverse inequality [20, Lemma 12.15]

$$|u|_{H^1(\tau)} \le C_{\mathrm{inv}} h_\tau^{-1} \|u\|_{L^2(\tau)} \tag{8}$$

with constants $C_{\mathrm{dt}}, C_{\mathrm{inv}} > 0$ for all $u \in P_k$, $\tau \in \mathcal{T}_n$, $n \in \mathbb{N}$.

2.3 Distributional discontinuous Galerkin method

Let $\mathbf{H}^1_{\mathrm{pw}}(\mathcal{T}_n) := \{\mathbf{u} \in \mathbf{L}^2\colon \mathbf{u}|_\tau \in \mathbf{H}^1(\tau) \text{ for all } \tau \in \mathcal{T}_n\}$. For $F \in \mathcal{F}_n^{\mathrm{int}}$ we denote the neighboring elements of F as $\tau_1, \tau_2 \in \mathcal{T}_n$. Then for $\mathbf{u} \in \mathbf{H}^1_{\mathrm{pw}}(\mathcal{T}_n)$ and $F \in \mathcal{F}_n^{\mathrm{int}}$ we denote the traces of $\mathbf{u}|_{\tau_1}$ and $\mathbf{u}|_{\tau_2}$ on F as $\mathbf{u}_1$ and $\mathbf{u}_2$ respectively. Thus for $\mathbf{u} \in \mathbf{H}^1_{\mathrm{pw}}(\mathcal{T}_n)$

and $F \in \mathcal{F}_n^{\text{int}}$ we define the following average and jump terms

$$\{\mathbf{u}\} := \frac{1}{2}(\mathbf{u}_1 + \mathbf{u}_2), \qquad [\![\mathbf{u}]\!]_{\mathbf{b}} := (\mathbf{b}\cdot\boldsymbol{\nu}_1)\mathbf{u}_1 + (\mathbf{b}\cdot\boldsymbol{\nu}_2)\mathbf{u}_2 \quad \text{on } F,$$

where $\boldsymbol{\nu}_1, \boldsymbol{\nu}_2$ are the outer unit normal vectors on the elements τ_1, τ_2 adjacent to the face F. In addition we introduce the abbreviations

$$\langle\cdot,\cdot\rangle_{\mathcal{F}_n^{\text{int}}} := \sum_{F\in\mathcal{F}_n^{\text{int}}} \langle\cdot,\cdot\rangle_{\mathbf{L}^2(F)}, \quad \|\cdot\|^2_{\mathcal{F}_n^{\text{int}}} := \langle\cdot,\cdot\rangle_{\mathcal{F}_n^{\text{int}}}.$$

We introduce a lifting operator (see, e.g., [22, Chapter 4.3]) related to the differential operator $\partial_{\mathbf{b}}$. Note that we choose a sign convention for R_n^F as in [23], which is opposite to the one used in [22, Chapter 4.3]. Let $l \in \mathbb{N}$.

$$\mathbf{Q}_n := \{\boldsymbol{\psi}_n \in \mathbf{L}^2 \colon \boldsymbol{\psi}_n|_\tau \in \mathbf{P}_l \text{ for all } \tau \in \mathcal{T}_n\}.$$

Then for $\mathbf{u}_n \in \mathbb{X}_n$ and $F \in \mathcal{F}_n^{\text{int}}$ let $R_n^F\mathbf{u}_n \in \mathbf{Q}_n$ be the solution to

$$\langle R_n^F\mathbf{u}_n, \boldsymbol{\psi}_n\rangle = -\langle[\![\mathbf{u}_n]\!]_{\mathbf{b}}, \{\boldsymbol{\psi}_n\}\rangle_{\mathbf{L}^2(F)} \quad \text{for all } \boldsymbol{\psi}_n \in \mathbf{Q}_n.$$

We observe that $\operatorname{supp}(R_n^F\mathbf{u}_n) = \tau_F$ and it easily follows with (7) that

$$\|R_n^F\mathbf{u}_n\|_{\mathbf{L}^2} \le C_{\text{dt}}\|\mathfrak{h}^{-1/2}[\![\mathbf{u}_n]\!]_{\mathbf{b}}\|_{\mathbf{L}^2(F)} \quad \text{for all } \mathbf{u}_n \in \mathbb{X}_n,\, F \in \mathcal{F}_n^{\text{int}},\, n \in \mathbb{N}. \tag{9}$$

Then we define $R_n := \sum_{F\in\mathcal{F}_n^{\text{int}}} R_n^F$ and the linear operator $D_{\mathbf{b}}^n \colon \mathbb{X}_n \to \mathbf{Q}_n$ by

$$(D_{\mathbf{b}}^n\mathbf{u}_n)|_\tau := \partial_{\mathbf{b}}(\mathbf{u}_n|_\tau) + (R_n\mathbf{u}_n)|_\tau \quad \text{for all } \tau \in \mathcal{T}_n.$$

We remark that for less complicated equations the natural choice for l is $l = k-1$, while $l = k$ might simplify the implementation [22]. However, for the DG method applied in this article it is advisable to choose indeed $l = k$ to exploit the full potential convergence rate (see Theorem 19). Also note that $l \ge 1$ (i.e. $l = 0$ is excluded) is necessary to obtain Lemma 8 and subsequent results. We introduce the following scalar product on $\mathbb{X}_n$

$$\langle\mathbf{u}_n, \mathbf{u}_n'\rangle_{\mathbb{X}_n} := \langle\operatorname{div}\mathbf{u}_n, \operatorname{div}\mathbf{u}_n'\rangle + \langle\mathbf{u}_n, \mathbf{u}_n'\rangle + \langle D_{\mathbf{b}}^n\mathbf{u}_n, D_{\mathbf{b}}^n\mathbf{u}_n'\rangle.$$

Now let

$$\begin{aligned} a_n(\mathbf{u}_n, \mathbf{u}_n') :=\, &\langle c_s^2\rho\operatorname{div}\mathbf{u}_n, \operatorname{div}\mathbf{u}_n'\rangle - \langle\rho(\omega + iD_{\mathbf{b}}^n + i\Omega\times)\mathbf{u}_n, (\omega + iD_{\mathbf{b}}^n + i\Omega\times)\mathbf{u}_n'\rangle \\ &+ \langle\operatorname{div}\mathbf{u}_n, \nabla p\cdot\mathbf{u}_n'\rangle + \langle\nabla p\cdot\mathbf{u}_n, \operatorname{div}\mathbf{u}_n'\rangle \\ &+ \langle(\operatorname{Hess}(p) - \rho\operatorname{Hess}(\phi))\mathbf{u}_n, \mathbf{u}_n'\rangle - i\omega\langle\gamma\rho\mathbf{u}_n, \mathbf{u}_n'\rangle \ \text{ for all } \mathbf{u}_n, \mathbf{u}_n' \in \mathbb{X}_n. \end{aligned} \tag{10}$$

Thus it easily follows that $\sup_{n\in\mathbb{N}}\|A_n\|_{L(\mathbb{X}_n)} < \infty$.

3 Abstract framework

3.1 Discrete approximation schemes and T-compatibility

We remark that in this section we use the symbols $\tilde{A}, \tilde{A}_n$ for generic operators, because the symbols A, A_n are already occupied due to the introduction of the sesquilinear forms $a(\cdot,\cdot)$ and $a_n(\cdot,\cdot)$ in (3) and (10). For a Hilbert space X and $\tilde{A} \in L(X)$ we consider *discrete approximation schemes* (DAS) of $(X, \tilde{A})$ in the following way. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of finite dimensional Hilbert spaces and $\tilde{A}_n \in L(X_n)$. Note that we do not demand that the spaces X_n are subspaces of X. Instead we demand that there exist operators $p_n \in L(X, X_n)$ such that $\lim_{n\to\infty} \|p_n u\|_{X_n} = \|u\|_X$ for each $u \in X$. We then define the following properties of a discrete approximation scheme:

- A sequence $(u_n)_{n\in\mathbb{N}}, u_n \in X_n$ is said to *converge* to $u \in X$, if $\lim_{n\to\infty} \|p_n u - u_n\|_{X_n} = 0$.
- A sequence $(u_n)_{n\in\mathbb{N}}, u_n \in X_n$ is said to be *compact*, if for every subsequence $\mathbb{N}' \subset \mathbb{N}$ there exists a subsubsequence $\mathbb{N}'' \subset \mathbb{N}'$ such that $(u_n)_{n\in\mathbb{N}''}$ converges (to a $u \in X$).
- A sequence $(\tilde{A}_n)_{n\in\mathbb{N}}, \tilde{A}_n \in L(X_n)$ is said to be *asymptotically consistent* or to *approximate* $\tilde{A} \in L(X)$, if $\lim_{n\to\infty} \|\tilde{A}_n p_n u - p_n \tilde{A} u\|_{X_n} = 0$ for each $u \in X$.
- A sequence of operators $(\tilde{A}_n)_{n\in\mathbb{N}}, \tilde{A}_n \in L(X_n)$ is said to be *compact*, if for every bounded sequence $(u_n)_{n\in\mathbb{N}}, u_n \in X_n, \|u_n\|_{X_n} \leq C$ the sequence $(\tilde{A}_n u_n)_{n\in\mathbb{N}}$ is compact.
- A sequence of operators $(\tilde{A}_n)_{n\in\mathbb{N}}, \tilde{A}_n \in L(X_n)$ is said to be *stable*, if there exist constants $C, n_0 > 0$ such that $\tilde{A}_n$ is invertible and $\|\tilde{A}_n^{-1}\|_{L(X_n)} \leq C$ for all $n > n_0$.
- A sequence of operators $(\tilde{A}_n)_{n\in\mathbb{N}}, \tilde{A}_n \in L(X_n)$ is said to be *regular*, if $\|u_n\|_{X_n} \leq C$ and the compactness of $(\tilde{A}_n u_n)_{n\in\mathbb{N}}$ imply the compactness of $(u_n)_{n\in\mathbb{N}}$.

The following theorem will be our tool to prove the regularity of approximations.

Theorem 1 (Theorem 3 of [15]) *Assume the existence of a constant $C > 0$, sequences $(\tilde{A}_n)_{n\in\mathbb{N}}$, $(T_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, $(K_n)_{n\in\mathbb{N}}$ and $B \in L(X)$ which satisfy the following: for each $n \in \mathbb{N}$ one has that $\tilde{A}_n, T_n, B_n, K_n \in L(X_n)$, $\|T_n\|_{L(X_n)}$, $\|T_n^{-1}\|_{L(X_n)}$, $\|B_n\|_{L(X_n)}$, $\|B_n^{-1}\|_{L(X_n)} \leq C$, B is bijective, $(K_n)_{n\in\mathbb{N}}$ is compact,*

$$\lim_{n\to\infty} \|T_n p_n u - p_n T u\|_{X_n} = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|B_n p_n u - p_n B u\|_{X_n} = 0 \; \textit{for each } u \in X,$$

and

$$\tilde{A}_n T_n = B_n + K_n.$$

Then $(\tilde{A}_n)_{n\in\mathbb{N}}$ is regular.

We recall the following lemma, which shows that DAS-regularity (not to be mistaken by smoothness) together with the bijectivity of the continuous problem imply convergence.

Lemma 2 (Lemmas 1 & 2 of [15]) *Let $\tilde{A} \in L(X)$ be bijective and $(X_n, \tilde{A}_n, p_n)_{n\in\mathbb{N}}$ be a discrete approximation scheme of $(X, \tilde{A})$ which is regular and asymptotically consistent. Then $(\tilde{A}_n)_{n\in\mathbb{N}}$ is stable. If u, u_n are the solutions to $\tilde{A}u = f$ and $\tilde{A}_n u_n = f_n \in X_n$, and $\lim_{n\to\infty} \|p_n f - f_n\|_{X_n} = 0$, then $\lim_{n\to\infty} \|p_n u - u_n\|_{X_n} = 0$.*

3.2 Interpretation of the $H(\mathrm{div})$-DDG-FEM as DAS

For $\mathbf{u} \in \mathbb{X}$ let $p_n\mathbf{u} \in \mathbb{X}_n$ be the solution to

$$\langle p_n\mathbf{u}, \mathbf{u}'_n\rangle_{\mathbb{X}_n} = \langle \operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{u}'_n\rangle_{L^2} + \langle \mathbf{u}, \mathbf{u}'_n\rangle_{\mathbf{L}^2} + \langle \partial_{\mathbf{b}}\mathbf{u}, D^n_{\mathbf{b}}\mathbf{u}'_n\rangle_{\mathbf{L}^2} \quad \text{for all } \mathbf{u}'_n \in \mathbb{X}_n.$$

It easily follows that $p_n \in L(\mathbb{X}, \mathbb{X}_n)$ and $\|p_n\|_{L(\mathbb{X},\mathbb{X}_n)} \leq 1$. In addition, there holds the Galerkin orthogonality

$$0 = \langle \operatorname{div}(\mathbf{u} - p_n\mathbf{u}), \operatorname{div}\mathbf{u}'_n\rangle_{L^2} + \langle \mathbf{u} - p_n\mathbf{u}, \mathbf{u}'_n\rangle_{\mathbf{L}^2} + \langle \partial_{\mathbf{b}}\mathbf{u} - D^n_{\mathbf{b}} p_n\mathbf{u}, D^n_{\mathbf{b}}\mathbf{u}'_n\rangle_{\mathbf{L}^2} \quad (11)$$

for all $\mathbf{u}'_n \in \mathbb{X}_n$. In order to analyze p_n further we introduce the distance function $d_n(\mathbf{u}, \mathbf{u}_n)$ between $\mathbf{u} \in \mathbb{X}$ and $\mathbf{u}_n \in \mathbb{X}_n$ as

$$d_n(\mathbf{u}, \mathbf{u}_n) := \sqrt{\|\operatorname{div}\mathbf{u} - \operatorname{div}\mathbf{u}_n\|^2_{L^2} + \|\mathbf{u} - \mathbf{u}_n\|^2_{\mathbf{L}^2} + \|\partial_{\mathbf{b}}\mathbf{u} - D^n_{\mathbf{b}}\mathbf{u}_n\|^2_{\mathbf{L}^2}}.$$

The introduction of $d_n(\cdot,\cdot)$ is necessary, because in general the jump $[\![\mathbf{u}]\!]_{\mathbf{b}}$ is not well-defined for $\mathbf{u} \in \mathbb{X}$ and hence the convenient measure of error $\|\mathbf{u} - \mathbf{u}_n\|_{\mathbb{X}_n}$ is not well-defined without assuming extra regularity of the solution $\mathbf{u}$. It can easily be seen that $d_n(\cdot,\cdot)$ satisfies the triangle inequalities

$$d_n(\mathbf{u}, \mathbf{u}_n) \leq d_n(\tilde{\mathbf{u}}, \mathbf{u}_n) + \|\mathbf{u} - \tilde{\mathbf{u}}\|_{\mathbb{X}}, \qquad d_n(\mathbf{u}, \mathbf{u}_n) \leq d_n(\mathbf{u}, \tilde{\mathbf{u}}_n) + \|\mathbf{u}_n - \tilde{\mathbf{u}}_n\|_{\mathbb{X}_n}$$

for all $\mathbf{u}, \tilde{\mathbf{u}} \in \mathbb{X}$, $\mathbf{u}_n \tilde{\mathbf{u}}_n \in \mathbb{X}_n$.

Lemma 3 *For each $\boldsymbol{u} \in \boldsymbol{H}^1_{\boldsymbol{\nu}0}$ one has $d_n(\boldsymbol{u}, p_n\boldsymbol{u}) \leq d_n(\boldsymbol{u}, \pi^d_n\boldsymbol{u})$.*

Proof We compute by means of (11) that

$$\begin{aligned} d_n(\mathbf{u}, p_n\mathbf{u})^2 &= \|\operatorname{div}(\mathbf{u} - p_n\mathbf{u})\|^2_{L^2} + \|\mathbf{u} - p_n\mathbf{u}\|^2_{\mathbf{L}^2} + \|\partial_{\mathbf{b}}\mathbf{u} - D^n_{\mathbf{b}} p_n\mathbf{u}\|^2_{\mathbf{L}^2} \\ &= \langle \operatorname{div}(\mathbf{u} - p_n\mathbf{u}), \operatorname{div}(\mathbf{u} - \pi^d_n\mathbf{u})\rangle_{L^2} + \langle \mathbf{u} - p_n\mathbf{u}, \mathbf{u} - \pi^d_n\mathbf{u}\rangle_{\mathbf{L}^2} \\ &\quad + \langle \partial_{\mathbf{b}}\mathbf{u} - D^n_{\mathbf{b}} p_n\mathbf{u}, \partial_{\mathbf{b}}\mathbf{u} - D^n_{\mathbf{b}}\pi^d_n\mathbf{u}\rangle_{\mathbf{L}^2} \\ &\leq d_n(\mathbf{u}, p_n\mathbf{u}) d_n(\mathbf{u}, \pi^d_n\mathbf{u}) \end{aligned}$$

which proves the claim. □

Lemma 4 *For each $\boldsymbol{u} \in \boldsymbol{H}^1_{\boldsymbol{\nu}0} \cap \boldsymbol{H}^{1+s}$, $s > 0$, one has $d_n(\boldsymbol{u}, \pi^d_n\boldsymbol{u}) \lesssim h_n^{\min\{s,k\}}\|\boldsymbol{u}\|_{\boldsymbol{H}^{1+s}}$.*

Proof The claim follows from the definition of $d_n(\cdot,\cdot)$ that

$$\|\partial_{\mathbf{b}}\mathbf{u}-D^n_{\mathbf{b}}\pi^d_n\mathbf{u}\|_{\mathbf{L}^2(\tau)}\le\|\partial_{\mathbf{b}}\mathbf{u}-\partial_{\mathbf{b}}\pi^d_n\mathbf{u}\|_{\mathbf{L}^2(\tau)}+\|R_n\pi^d_n\mathbf{u}\|_{\mathbf{L}^2(\tau)},$$

(5), (6), (9) and $[\![\mathbf{u}]\!]_{\mathbf{b}}=0$. □

Lemma 5 *For each $\boldsymbol{u}\in\mathbb{X}$ we have that* $\lim_{n\to\infty}d_n(\boldsymbol{u},p_n\boldsymbol{u})=0$.

Proof Since $\mathbf{C}^\infty_0$ is dense in $\mathbb{X}$ [15, Theorem 6] for each $\epsilon>0$ we can choose $\tilde{\mathbf{u}}\in\mathbf{C}^\infty_0$ such that $\|\mathbf{u}-\tilde{\mathbf{u}}\|_{\mathbb{X}}<\epsilon$. Then we estimate

$$\begin{aligned}d_n(\mathbf{u},p_n\mathbf{u})&\le d_n(\mathbf{u},p_n\tilde{\mathbf{u}})+\|p_n(\mathbf{u}-\tilde{\mathbf{u}})\|_{\mathbb{X}_n}\\&\le d_n(\tilde{\mathbf{u}},p_n\tilde{\mathbf{u}})+\|p_n(\mathbf{u}-\tilde{\mathbf{u}})\|_{\mathbb{X}_n}+\|\mathbf{u}-\tilde{\mathbf{u}}\|_{\mathbb{X}}\\&\le d_n(\tilde{\mathbf{u}},p_n\tilde{\mathbf{u}})+2\epsilon.\end{aligned}$$

Thus it follows with Lemmas 3 and 4 that $\limsup_{n\to\infty}d_n(\mathbf{u},p_n\mathbf{u})\le2\epsilon$. Since $\epsilon>0$ was arbitrary the claim follows. □

Lemma 6 *For each $\boldsymbol{u}\in\boldsymbol{H}^1_{\boldsymbol{\nu}0}$ we have that* $\lim_{n\to\infty}d_n(\boldsymbol{u},\pi^d_n\boldsymbol{u})=0$.

Proof In principle we proceed as in the proof of Lemma 5. However, to construct a suitable smooth approximation $\tilde{\mathbf{u}}$ which respects the boundary condition $\boldsymbol{\nu}\cdot\tilde{\mathbf{u}}=0$ we need to introduce some technical details. Let $\mathcal{U}_1\subset\mathcal{U}_2\subset\mathcal{O}$ be such that $\operatorname{supp}\mathbf{b}\subset\mathcal{U}_1$, $\operatorname{dist}(\operatorname{supp}\mathbf{b},\partial\mathcal{U}_1)>0$, $\operatorname{dist}(\mathcal{U}_1,\partial\mathcal{U}_2)>0$ and $\operatorname{dist}(\mathcal{U}_2,\partial\mathcal{O})>0$. Note that such $\mathcal{U}_1,\mathcal{U}_2$ exist, because by assumption $\operatorname{dist}(\partial\mathcal{O},\operatorname{supp}\mathbf{b})>0$. Then let $\chi\in C^\infty$ be such that $\chi=0$ on $\partial\mathcal{O}$ and $\chi=1$ in $\mathcal{U}_2$. For $\mathbf{u}\in\mathbf{H}^1_{\boldsymbol{\nu}0}$ let $\mathbf{u}_1:=\chi\mathbf{u}$ and $\mathbf{u}_2:=(1-\chi)\mathbf{u}$. Let $\epsilon>0$ be given. Since $\mathbf{u}_1\in\mathbf{H}^1_0$ and $\mathbf{C}^\infty_0$ is dense in $\mathbf{H}^1_0$ we can find $\tilde{\mathbf{u}}_1\in\mathbf{C}^\infty_0$ such that $\|\mathbf{u}_1-\tilde{\mathbf{u}}_1\|_{\mathbf{H}^1}<\epsilon$. Then we have the estimates

$$d_n(\mathbf{u},\pi^d_n\mathbf{u})\le d_n(\mathbf{u}_1,\pi^d_n\mathbf{u}_1)+d_n(\mathbf{u}_2,\pi^d_n\mathbf{u}_2)$$

and

$$\begin{aligned}d_n(\mathbf{u}_1,\pi^d_n\mathbf{u}_1)&\le d_n(\tilde{\mathbf{u}}_1,\pi^d_n\tilde{\mathbf{u}}_1)+\|\mathbf{u}_1-\tilde{\mathbf{u}}_1\|_{\mathbb{X}}+\|\pi^d_n(\mathbf{u}_1-\tilde{\mathbf{u}}_1)\|_{\mathbb{X}_n}\\&\lesssim d_n(\tilde{\mathbf{u}}_1,\pi^d_n\tilde{\mathbf{u}}_1)+(1+\sup_{m\in\mathbb{N}}\|\pi^d_m\|_{L(\mathbf{H}^1_{\boldsymbol{\nu}0},\mathbb{X}_m)})\|\mathbf{u}_1-\tilde{\mathbf{u}}_1\|_{\mathbf{H}^1}\\&\lesssim h_n\|\tilde{\mathbf{u}}_1\|_{\mathbf{H}^2}+\epsilon.\end{aligned}$$

Thus $\limsup_{n\in\mathbb{N}}d_n(\mathbf{u}_1,\pi^d_n\mathbf{u}_1)\lesssim\epsilon$ and hence $\lim_{n\in\mathbb{N}}d_n(\mathbf{u}_1,\pi^d_n\mathbf{u}_1)=0$. For $\mathbf{u}_2$ we use the smoothing operators $\mathcal{K}^d_{\epsilon,0},\mathcal{K}^b_{\epsilon,0}$ defined in [24, (4.1)] which satisfy the subsequently used commutation property $\nabla\cdot\mathcal{K}^d_{\epsilon,0}\mathbf{g}=\mathcal{K}^b_{\epsilon,0}\nabla\cdot\mathbf{g}$ [24, Lem. 4.3(iii)], set $\tilde{\mathbf{u}}_2:=\mathcal{K}^d_{\epsilon,0}\mathbf{u}_2$ and estimate

$$d_n(\mathbf{u}_2,\pi^d_n\mathbf{u}_2)\le d_n(\tilde{\mathbf{u}}_2,\pi^d_n\tilde{\mathbf{u}}_2)+\|\mathbf{u}_2-\tilde{\mathbf{u}}_2\|_{\mathbb{X}}+\|\pi^d_n(\mathbf{u}_2-\tilde{\mathbf{u}}_2)\|_{\mathbb{X}_n}.$$

In addition we compute that

$$\|\pi_n^d(\mathbf{u}_2-\tilde{\mathbf{u}}_2)\|^2_{\mathbb{X}_n} = \|\pi_n^d(\mathbf{u}_2-\tilde{\mathbf{u}}_2)\|^2_{\mathbf{L}^2} + \|\operatorname{div}\pi_n^d(\mathbf{u}_2-\tilde{\mathbf{u}}_2)\|^2_{\mathbf{L}^2} + \sum_{\tau\in\mathcal{T}_n}\|D^n_{\mathbf{b}}\pi_n^d(\mathbf{u}_2-\tilde{\mathbf{u}}_2)\|^2_{\mathbf{L}^2(\tau)}.$$

Let $q\in(2,6)$. We use the bound $\|\pi_n^d\mathbf{v}\|_{\mathbf{L}^2}\lesssim\|\mathbf{v}\|_{\mathbf{L}^q}+\|\operatorname{div}\mathbf{v}\|_{L^2}$, $\mathbf{v}\in H(\operatorname{div})\cap\mathbf{L}^q$ [20, Chapter 17.2]. It follows that

$$\begin{aligned}\|\pi_n^d(\mathbf{u}_2-\tilde{\mathbf{u}}_2)\|^2_{\mathbf{L}^2} &+ \|\operatorname{div}\pi_n^d(\mathbf{u}_2-\tilde{\mathbf{u}}_2)\|^2_{\mathbf{L}^2}\\ &= \|\pi_n^d(\mathbf{u}_2-\tilde{\mathbf{u}}_2)\|^2_{\mathbf{L}^2} + \|\pi_n^l\operatorname{div}(\mathbf{u}_2-\tilde{\mathbf{u}}_2)\|^2_{\mathbf{L}^2}\\ &\lesssim \|\mathbf{u}_2-\tilde{\mathbf{u}}_2\|^2_{\mathbf{L}^q} + \|\operatorname{div}\mathbf{u}_2-\operatorname{div}\tilde{\mathbf{u}}_2\|^2_{\mathbf{L}^2}\\ &= \|(1-\mathcal{K}^d_{\epsilon,0})\mathbf{u}_2\|^2_{\mathbf{L}^q} + \|(1-\mathcal{K}^b_{\epsilon,0})\operatorname{div}\mathbf{u}_2\|^2_{\mathbf{L}^2}.\end{aligned}$$

In addition, there exists an $\epsilon_0>0$ such that $\tilde{\mathbf{u}}_2|_{\mathcal{U}_1}=(\mathcal{K}_{\epsilon,0}\mathbf{u}_2)|_{\mathcal{U}_1}=0$ for all $\epsilon\in(0,\epsilon_0)$, which we assume henceforth. There also exists an index $n_0>0$ such that $D_\tau\subset\mathcal{U}_1$ for all $\tau\in\mathcal{T}_n$ with $\tau\cap\operatorname{supp}\mathbf{b}\neq\emptyset$ for all $n>n_0$, which we assume henceforth. Thus

$$\sum_{\tau\in\mathcal{T}_n}\|D^n_{\mathbf{b}}\pi_n^d(\mathbf{u}_2-\tilde{\mathbf{u}}_2)\|^2_{\mathbf{L}^2(\tau)} \le \sum_{\tau\in\mathcal{T}_n}\|\mathbf{b}\|^2_{\mathbf{L}^\infty(D_\tau)}\|\mathbf{u}_2-\tilde{\mathbf{u}}_2\|^2_{\mathbf{H}^1(D_\tau)}=0.$$

Further we note that $\|\mathbf{u}_2-\tilde{\mathbf{u}}_2\|_{\mathbb{X}}=\|\mathbf{u}_2-\tilde{\mathbf{u}}_2\|_{H(\operatorname{div})}$, because $\operatorname{supp}\mathbf{b}\cap\operatorname{supp}(\mathbf{u}_2-\tilde{\mathbf{u}}_2)=\emptyset$. Altogether we obtain that

$$d_n(\mathbf{u}_2,\pi_n^d\mathbf{u}_2)\le h_n\|\tilde{\mathbf{u}}_2\|_{\mathbf{H}^2}+\|(1-\mathcal{K}^d_{\epsilon,0})\mathbf{u}_2\|^2_{\mathbf{L}^q}+\|(1-\mathcal{K}^b_{\epsilon,0})\operatorname{div}\mathbf{u}_2\|^2_{\mathbf{L}^2}$$

and hence

$$\limsup_{n\in\mathbb{N}} d_n(\mathbf{u}_2,\pi_n^d\mathbf{u}_2)\lesssim\|(1-\mathcal{K}^d_{\epsilon,0})\mathbf{u}_2\|^2_{\mathbf{L}^q}+\|(1-\mathcal{K}^b_{\epsilon,0})\operatorname{div}\mathbf{u}_2\|^2_{\mathbf{L}^2}.$$

Due to the continuous Sobolev embedding $\mathbf{H}^1\hookrightarrow\mathbf{L}^q$ it holds that $\mathbf{u}_2\in\mathbf{L}^q$, and the right-hand side of the former inequality tends to zero for $\epsilon\to 0$. Thus $\lim_{n\in\mathbb{N}} d_n(\mathbf{u}_2,\pi_n^d\mathbf{u}_2)=0$ and the proof is finished. □

Lemma 7 *For each* $\boldsymbol{u}\in\mathbb{X}$ *we have that* $\lim_{n\to\infty}\|p_n\boldsymbol{u}\|_{\mathbb{X}_n}=\|\boldsymbol{u}\|_{\mathbb{X}}$.

Proof We compute

$$\begin{aligned}\|p_n\mathbf{u}\|^2_{\mathbb{X}_n} &= \langle p_n\mathbf{u},p_n\mathbf{u}\rangle_{\mathbb{X}_n} = \langle\operatorname{div}\mathbf{u},\operatorname{div}p_n\mathbf{u}\rangle_{L^2}+\langle\mathbf{u},p_n\mathbf{u}\rangle_{\mathbf{L}^2}+\langle\partial_{\mathbf{b}}\mathbf{u},D^n_{\mathbf{b}}p_n\mathbf{u}\rangle_{\mathbf{L}^2}\\ &= \|\mathbf{u}\|^2_{\mathbb{X}}+\langle\operatorname{div}\mathbf{u},\operatorname{div}(p_n\mathbf{u}-\mathbf{u})\rangle_{L^2}+\langle\mathbf{u},p_n\mathbf{u}-\mathbf{u}\rangle_{\mathbf{L}^2}\\ &\quad+\langle\partial_{\mathbf{b}}\mathbf{u},D^n_{\mathbf{b}}p_n\mathbf{u}-\partial_{\mathbf{b}}\mathbf{u}\rangle_{\mathbf{L}^2}.\end{aligned}$$

Since

$$
\begin{aligned}
&|\langle \operatorname{div} \mathbf{u}, \operatorname{div}(p_n\mathbf{u} - \mathbf{u})\rangle_{L^2} + \langle \mathbf{u}, p_n\mathbf{u} - \mathbf{u}\rangle_{\mathbf{L}^2} + \langle \partial_{\mathbf{b}}\mathbf{u}, D^n_{\mathbf{b}} p_n\mathbf{u} - \partial_{\mathbf{b}}\mathbf{u}\rangle_{\mathbf{L}^2}| \\
&\quad \le \|\mathbf{u}\|_{\mathbb{X}} d_n(\mathbf{u}, p_n\mathbf{u})
\end{aligned}
$$

the claim follows from Lemma 5. □

Thus $(\mathbb{X}_n, A_n, p_n)_{n\in\mathbb{N}}$ forms a discrete approximation scheme of $(\mathbb{X}, A)$. Before we establish the asymptotic consistency of this scheme we need to state a weak compactness result.

Lemma 8 *Let $(\boldsymbol{u}_n)_{n\in\mathbb{N}}$, $\boldsymbol{u}_n \in \mathbb{X}_n$ satisfy $\sup_{n\in\mathbb{N}} \|\boldsymbol{u}_n\|_{\mathbb{X}_n} < \infty$. Then there exist $\boldsymbol{u} \in \mathbb{X}$ and a subsequence $\mathbb{N}' \subset \mathbb{N}$ such that $\boldsymbol{u}_n \xrightarrow{L^2} \boldsymbol{u}$, $\operatorname{div} \boldsymbol{u}_n \xrightarrow{L^2} \operatorname{div} \boldsymbol{u}$ and $D^n_{\boldsymbol{b}}\boldsymbol{u}_n \xrightarrow{L^2} \partial_{\boldsymbol{b}}\boldsymbol{u}$.*

Proof Since $\mathbf{u}_n$, $D^n_{\mathbf{b}}\mathbf{u}_n$ and $\operatorname{div} \mathbf{u}_n$ are bounded sequences in $\mathbf{L}^2$ and L^2 respectively, there exist $\mathbf{u}, \mathbf{g} \in \mathbf{L}^2$, $q \in L^2$ and a subsequence $\mathbb{N}' \subset \mathbb{N}$ such that $\mathbf{u}_n \xrightarrow{\mathbf{L}^2} \mathbf{u}$, $D^n_{\mathbf{b}}\mathbf{u}_n \xrightarrow{\mathbf{L}^2} \mathbf{g}$ and $\operatorname{div} \mathbf{u}_n \xrightarrow{L^2} q$. It remains to show that $\mathbf{g} = \partial_{\mathbf{b}}\mathbf{u}$ and $q = \operatorname{div} \mathbf{u}$. To this end we work with a distributional technique. For conforming differential operators (here the divergence operator) this technique is quite standard and for a reconstructed differential operator it can be found, e.g., in the proof of [23, Theorem 5.2]. Let $\boldsymbol{\psi} \in \mathbf{C}^\infty_0$ and $\psi \in C^\infty_0$. Then

$$
\langle q, \psi\rangle = \lim_{n\in\mathbb{N}'} \langle \operatorname{div} \mathbf{u}_n, \psi\rangle = \lim_{n\in\mathbb{N}'} -\langle \mathbf{u}_n, \nabla\psi\rangle = -\langle \mathbf{u}, \nabla\psi\rangle
$$

and hence $q = \operatorname{div} \mathbf{u}$. Let $\boldsymbol{\psi}_n$ be the lowest order standard $\mathbf{H}^1$-interpolant of $\boldsymbol{\psi}$ on the mesh $\mathcal{T}_n$. We compute

$$
\begin{aligned}
\langle D^n_{\mathbf{b}}\mathbf{u}_n, \boldsymbol{\psi}\rangle &= \langle D^n_{\mathbf{b}}\mathbf{u}_n, \boldsymbol{\psi} - \boldsymbol{\psi}_n\rangle + \langle D^n_{\mathbf{b}}\mathbf{u}_n, \boldsymbol{\psi}_n\rangle \\
&= \langle D^n_{\mathbf{b}}\mathbf{u}_n, \boldsymbol{\psi} - \boldsymbol{\psi}_n\rangle + \sum_{\tau\in\mathcal{T}_n} \langle \partial_{\mathbf{b}}\mathbf{u}_n, \boldsymbol{\psi}_n\rangle_{\mathbf{L}^2(\tau)} - \langle [\![\mathbf{u}_n]\!]_{\mathbf{b}}, \{\boldsymbol{\psi}_n\}\rangle_{\mathcal{F}^{\mathrm{int}}_n} \\
&= \langle D^n_{\mathbf{b}}\mathbf{u}_n, \boldsymbol{\psi} - \boldsymbol{\psi}_n\rangle + \sum_{\tau\in\mathcal{T}_n} \langle \partial_{\mathbf{b}}\mathbf{u}_n, \boldsymbol{\psi}_n\rangle_{\mathbf{L}^2(\tau)} - \langle (\boldsymbol{\nu}\cdot\mathbf{b})\mathbf{u}_n, \boldsymbol{\psi}_n\rangle_{\mathbf{L}^2(\partial\tau)} \\
&= \langle D^n_{\mathbf{b}}\mathbf{u}_n, \boldsymbol{\psi} - \boldsymbol{\psi}_n\rangle - \langle \mathbf{u}_n, (\partial_{\mathbf{b}} + \operatorname{div}(\mathbf{b}))\boldsymbol{\psi}_n\rangle \\
&= -\langle \mathbf{u}_n, (\partial_{\mathbf{b}} + \operatorname{div}(\mathbf{b}))\boldsymbol{\psi}\rangle + \langle D^n_{\mathbf{b}}\mathbf{u}_n, \boldsymbol{\psi} - \boldsymbol{\psi}_n\rangle \\
&\quad + \langle \mathbf{u}_n, (\partial_{\mathbf{b}} + \operatorname{div}(\mathbf{b}))(\boldsymbol{\psi} - \boldsymbol{\psi}_n)\rangle.
\end{aligned}
$$

Since $\|\boldsymbol{\psi} - \boldsymbol{\psi}_n\|_{\mathbf{H}^1} \lesssim h_n\|\boldsymbol{\psi}\|_{\mathbf{H}^2}$ and $\|\mathbf{u}_n\|_{\mathbb{X}_n} \lesssim 1$ it follows that

$$
\langle \mathbf{g}, \boldsymbol{\psi}\rangle = \lim_{n\to\infty} \langle D^n_{\mathbf{b}}\mathbf{u}_n, \boldsymbol{\psi}\rangle = \lim_{n\to\infty} -\langle \mathbf{u}_n, (\partial_{\mathbf{b}} + \operatorname{div}(\mathbf{b}))\boldsymbol{\psi}\rangle = -\langle \mathbf{u}, (\partial_{\mathbf{b}} + \operatorname{div}(\mathbf{b}))\boldsymbol{\psi}\rangle
$$

and hence $\partial_{\mathbf{b}}\mathbf{u} = \mathbf{g}$. □

Theorem 9 *For each $\boldsymbol{u} \in \mathbb{X}$ we have that $\lim_{n\to\infty} \|A_n p_n\boldsymbol{u} - p_n A\boldsymbol{u}\|_{\mathbb{X}_n} = 0$.*

Proof Let $\mathbf{u} \in \mathbb{X}$ and $(\mathbf{u}_n)_{n\in\mathbb{N}}$, $\mathbf{u}_n \in \mathbb{X}_n$, $\|\mathbf{u}_n\|_{\mathbb{X}_n} = 1$ be such that

$$\|A_n p_n \mathbf{u} - p_n A\mathbf{u}\|_{\mathbb{X}_n} \leq |\langle A_n p_n \mathbf{u} - p_n A\mathbf{u}, \mathbf{u}_n\rangle_{\mathbb{X}_n}| + 1/n.$$

For each arbitrary subsequence $\mathbb{N}' \subset \mathbb{N}$ we consider $(\mathbf{u}_n)_{n\in\mathbb{N}'}$ and apply Lemma 8, which yields a subsequence of $\mathbb{N}'$ and an element in $\mathbb{X}$ which we name $\mathbb{N}'' \subset \mathbb{N}'$ and $\mathbf{u}' \in \mathbb{X}$ which satisfy the weak convergence properties stated in Lemma 8. Then it follows with the definition of p_n that

$$\begin{aligned}
\lim_{n\in\mathbb{N}''} \langle p_n A\mathbf{u}, \mathbf{u}_n\rangle_{\mathbb{X}_n} &= \lim_{n\in\mathbb{N}''} \big(\langle \operatorname{div} A\mathbf{u}, \operatorname{div} \mathbf{u}_n\rangle + \langle A\mathbf{u}, \mathbf{u}_n\rangle + \langle \partial_{\mathbf{b}} A\mathbf{u}, D^n_{\mathbf{b}}\mathbf{u}_n\rangle\big)\\
&= \langle \operatorname{div} A\mathbf{u}, \operatorname{div} \mathbf{u}'\rangle + \langle A\mathbf{u}, \mathbf{u}'\rangle + \langle \partial_{\mathbf{b}} A\mathbf{u}, \partial_{\mathbf{b}}\mathbf{u}'\rangle\\
&= \langle A\mathbf{u}, \mathbf{u}'\rangle_{\mathbb{X}} = a(\mathbf{u}, \mathbf{u}').
\end{aligned}$$

We further compute that

$$\begin{aligned}
&\langle A_n p_n \mathbf{u}, \mathbf{u}_n\rangle_{\mathbb{X}_n} = a_n(p_n\mathbf{u}, \mathbf{u}_n)\\
&\quad= \langle c_s^2\rho \operatorname{div} p_n\mathbf{u}, \operatorname{div} \mathbf{u}_n\rangle - \langle \rho(\omega + iD^n_{\mathbf{b}} + i\Omega\times)p_n\mathbf{u}, (\omega + iD^n_{\mathbf{b}} + i\Omega\times)\mathbf{u}_n\rangle\\
&\qquad+ \langle \operatorname{div} p_n\mathbf{u}, \nabla p \cdot \mathbf{u}_n\rangle + \langle \nabla p \cdot p_n\mathbf{u}, \operatorname{div} \mathbf{u}_n\rangle + \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi)) p_n\mathbf{u}, \mathbf{u}_n\rangle\\
&\qquad- i\omega\langle \gamma\rho p_n\mathbf{u}, \mathbf{u}_n\rangle\\
&\quad= \left.\begin{aligned}&\langle c_s^2\rho \operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{u}_n\rangle - \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}, (\omega + iD^n_{\mathbf{b}} + i\Omega\times)\mathbf{u}_n\rangle\\
&+\langle \operatorname{div} \mathbf{u}, \nabla p \cdot \mathbf{u}_n\rangle + \langle \nabla p \cdot \mathbf{u}, \operatorname{div} \mathbf{u}_n\rangle\\
&+\langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\mathbf{u}, \mathbf{u}_n\rangle - i\omega\langle \gamma\rho\mathbf{u}, \mathbf{u}_n\rangle\end{aligned}\right\} \quad (12)\\
&\qquad\left.\begin{aligned}&+ \langle c_s^2\rho \operatorname{div}(p_n\mathbf{u} - \mathbf{u}), \operatorname{div} \mathbf{u}_n\rangle\\
&-\langle \rho((\omega + i\Omega\times)(p_n\mathbf{u} - \mathbf{u}) + iD^n_{\mathbf{b}} p_n\mathbf{u} - i\partial_{\mathbf{b}}\mathbf{u}), (\omega + iD^n_{\mathbf{b}} + i\Omega\times)\mathbf{u}_n\rangle\\
&+ \langle \operatorname{div}(p_n\mathbf{u} - \mathbf{u}), \nabla p \cdot \mathbf{u}_n\rangle + \langle \nabla p \cdot (p_n\mathbf{u} - \mathbf{u}), \operatorname{div} \mathbf{u}_n\rangle\\
&+ \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))(p_n\mathbf{u} - \mathbf{u}), \mathbf{u}_n\rangle \quad - i\omega\langle \gamma\rho(p_n\mathbf{u} - \mathbf{u}), \mathbf{u}_n\rangle.\end{aligned}\right\} \quad (13)
\end{aligned}$$

It holds that $\lim_{n\in\mathbb{N}''}(12) = a(\mathbf{u}, \mathbf{u}')$. Further we estimate that $|(13)| \lesssim d_n(\mathbf{u}, p_n\mathbf{u})$ and hence $\lim_{n\in\mathbb{N}''}(13) = 0$ due to Lemma 5. Thus altogether we obtain that $\lim_{n\in\mathbb{N}''} \|A_n p_n\mathbf{u} - p_n A\mathbf{u}\|_{\mathbb{X}_n} = 0$ and hence $\lim_{n\to\infty} \|A_n p_n\mathbf{u} - p_n A\mathbf{u}\|_{\mathbb{X}_n} = 0$, which finishes the proof. ◻

4 Convergence analysis

4.1 Weak right T-coercivity

First we recall how the well-posedness of the continuous problem (1) is established [6]. That is the injectivity of (1) [6, Lemma 3.7], which follows in a straightforward fashion combined with the weak T-coercivity of (1). Actually for the latter we use in the current article *right* T-coercivity instead of *left* T-coercivity as in [6], and we also choose a slightly different construction of T compared to [6]. The reasons for these changes are to be aligned with the forthcoming discrete analysis. To construct T we first derive a topological decomposition of $\mathbb{X}$. Thus we introduce

$$\mathbf{q} := c_s^{-2}\rho^{-1}\nabla p \tag{14}$$

as in [6, (3.3)] and for $\mathbf{u} \in H_0(\mathrm{div})$ we seek a solution $v \in H^2$ to

$$(\mathrm{div} + \mathbf{q}\cdot)\nabla v = (\mathrm{div} + \mathbf{q}\cdot)\mathbf{u} \qquad \text{in } \mathcal{O}, \tag{15a}$$
$$\boldsymbol{\nu} \cdot \nabla v = 0 \qquad \text{on } \partial\mathcal{O}. \tag{15b}$$

Note at this point that we only demand $\mathbf{u} \in H_0(\mathrm{div}) \subset \mathbb{X}$ and the reason why we emphasize this is that the discrete spaces satisfy $\mathbb{X}_n \subset H_0(\mathrm{div})$, but $\mathbb{X}_n \not\subset \mathbb{X}$ (in general). To start with we consider (15) as variational problem in H^1. If a solution $v \in H^1$ to (15) exists, then it follows with convenient regularity theory [25, Theorem 2.17] that the map $\mathbf{u} \to v$ is in $L(H_0(\mathrm{div}), H^2)$. Although the sesquilinear form associated to the left hand-side of (15) is only weakly coercive and we cannot guarantee the injectivity of the associated operator. As a remedy we consider the problem on H^2_* and introduce in addition to the low order perturbation $\mathbf{q}\cdot$ another perturbation through an operator M. In addition we replace $\mathbf{q}\cdot$ by $P_{L^2_0}\mathbf{q}\cdot$ (with $P_{L^2_0}$ being the orthogonal projection from L^2 to L^2_0) to enable a suitable perturbation analysis. Thus let $H^2_{*,\mathrm{Neu}} := \{\phi \in H^2_* \colon \boldsymbol{\nu} \cdot \nabla\phi = 0\}$ and

$$M := \sum_{l=1}^{L} \psi_l \langle \mathrm{div}\,\cdot, \mathrm{div}\,\nabla\phi_l \rangle,$$

where the number $L \in \mathbb{N}_0$ is the dimension of the kernel space of $(\mathrm{div} + P_{L^2_0}\mathbf{q}\cdot)\nabla \in L(H^2_{*,\mathrm{Neu}}, L^2_0)$, $\phi_l \in H^2_{*,\mathrm{Neu}}$, $l = 1, \ldots, L$, is an orthonormal basis with respect to the $H^2_{*,\mathrm{Neu}}$-equivalent inner product $\langle \Delta\cdot, \Delta\cdot \rangle$ of the kernel space, and $\psi_l \in L^2_0$, $l = 1, \ldots, L$, is an orthonormal basis of the L^2_0-orthogonal complement of $(\mathrm{div} + P_{L^2_0}\mathbf{q}\cdot)\nabla H^2_{*,\mathrm{Neu}}$. Then we consider the problem: find $v \in H^2_{*,\mathrm{Neu}}$ satisfying

$$(\mathrm{div} + P_{L^2_0}\mathbf{q}\cdot + M)\nabla v = (\mathrm{div} + P_{L^2_0}\mathbf{q}\cdot + M)\mathbf{u} \qquad \text{in } \mathcal{O}, \tag{16a}$$
$$\boldsymbol{\nu} \cdot \nabla v = 0 \qquad \text{on } \partial\mathcal{O}, \tag{16b}$$

instead of (15). Thus for $\mathbf{u} \in \mathbb{X}$ we set

$$\mathbf{v} := P_V\mathbf{u} := \nabla v, \tag{17}$$

$\mathbf{w} := \mathbf{u} - \mathbf{v}$ and

$$T\mathbf{u} := \mathbf{v} - \mathbf{w}, \tag{18}$$

where v is the solution to (16). Note that at this point we introduced the projection operator P_V, but whenever the context is clear we will use the shorthand notation

$\mathbf{v} = P_V\mathbf{u}$, $\mathbf{w} = \mathbf{u} - P_V\mathbf{u}$ (which suppresses the dependence on $\mathbf{u}$). It follows from its construction that $T \in L(\mathbb{X})$ and $TT = \mathrm{Id}_{\mathbb{X}}$. It follows from the definition of $\mathbf{w}$ and (16) that $(\operatorname{div} + P_{L^2_0}\mathbf{q}\cdot + M)\mathbf{w} = 0$, which we rearrange as $(\operatorname{div} + \mathbf{q}\cdot)\mathbf{w} = (\mathrm{Id}_{L^2} - P_{L^2_0})(\mathbf{q}\cdot\mathbf{w}) - M\mathbf{w}$ for the forthcoming analysis. Since the operators $\mathrm{Id}_{L^2} - P_{L^2_0}$ and M are compact the proof of [6, Theorem 3.11] needs only to be adapted slightly to obtain that A is weakly right T-coercive. We do not give more details at this point, because the proof will be contained in the proof of Theorem 18.

4.2 Construction and properties of T_n

Next we introduce a discrete variant T_n of T and analyze its properties.

4.2.1 Definition of T_n

For $\mathbf{u} \in H_0(\operatorname{div})$ let $\tilde{v} \in H^2_*$ be the solution to

$$(\operatorname{div} + P_{L^2_0}\mathbf{q}\cdot + M)\nabla\tilde{v} = (\operatorname{div} + \pi^l_n P_{L^2_0}\mathbf{q}\cdot + M)\mathbf{u}, \tag{19a}$$

$$\boldsymbol{\nu}\cdot\nabla\tilde{v} = 0. \tag{19b}$$

Note that in comparison to (16) in the right hand-side of (19) the term $P_{L^2_0}\mathbf{q}\cdot$ is changed to $\pi^l_n P_{L^2_0}\mathbf{q}\cdot$ and therefore we use a new symbol $\tilde{v}$ to denote the solution to (19). Subsequently for $\mathbf{u} \in H_0(\operatorname{div})$ and $\tilde{v}$ being the solution to (19) let

$$\tilde{\mathbf{v}} := P_{\tilde{V}_n}\mathbf{u} := \nabla\tilde{v} \quad \text{and} \quad \mathbf{v}_n := P_{V_n}\mathbf{u} := \pi^d_n\nabla\tilde{v}. \tag{20}$$

For $\mathbf{u}_n \in \mathbb{X}_n$ let $\mathbf{w}_n := \mathbf{u}_n - \mathbf{v}_n$ and set

$$T_n\mathbf{u}_n := \mathbf{v}_n - \mathbf{w}_n. \tag{21}$$

Again, let us mention that when possible we avoid to use the explicit operators $P_{\tilde{V}_n}$, P_{V_n} and use the abbreviations $\mathbf{v}_n$, $\mathbf{w}_n$ instead.

4.2.2 Boundedness of T_n

Lemma 10 *There exists a constant $C > 0$ such that $\|P_{V_n}\|_{L(\mathbb{X}_n)} \le C$ for all $n \in \mathbb{N}$.*

Proof Let $\mathbf{u}_n \in \mathbb{X}_n$ be given and $\tilde{v}$ be the solution to (19). First we note that $\|\tilde{v}\|_{H^2} \lesssim \|\mathbf{u}_n\|_{\mathbb{X}_n}$. Since $\nabla\tilde{v} \in \mathbf{H}^1$ the function $\pi^d_n\nabla\tilde{v}$ is well defined and the uniform boundedness of P_{V_n} follows from $\operatorname{div}\pi^d_n\nabla\tilde{v} = \pi^l_n \operatorname{div}\nabla\tilde{v}$, the point-wise convergence of π^d_n, π^l_n, (5), (6) and $[\![\nabla\tilde{v}]\!]_{\mathbf{b}} = 0$. □

Lemma 11 *There exists a constant $C > 0$ such that $\|T_n\|_{L(\mathbb{X}_n)} \le C$ for all $n \in \mathbb{N}$.*

Proof Follows from Lemma 10 and the definition (21) of T_n. □

4.2.3 Stability of T_n

Lemma 12 *Let* $O_n := P_{V_n}P_{V_n} - P_{V_n}$. *Then,* $\lim_{n\to\infty} \|O_n\|_{L(\mathbb{X}_n)} = 0$.

Proof Let $\mathbf{u}_n \in \mathbb{X}_n$ and $\tilde{v}_1$ be the solution to (19). Then $P_{V_n}\mathbf{u}_n = \pi_n^d \nabla\tilde{v}_1$. Let $\tilde{v}_2$ be the solution to (19) with $\mathbf{u}_n$ being replaced by $\pi_n^d\nabla\tilde{v}_1$ in the right-hand side. We compute

$$\begin{aligned}(\operatorname{div} + P_{L_0^2}\mathbf{q}\cdot + M)\nabla\tilde{v}_2 &= (\operatorname{div} + \pi_n^l P_{L_0^2}\mathbf{q}\cdot + M)\pi_n^d\nabla\tilde{v}_1 \\ &= \pi_n^l(\operatorname{div} + P_{L_0^2}\mathbf{q}\cdot + M)\nabla\tilde{v}_1 + M\pi_n^d\nabla\tilde{v}_1 - \pi_n^l M\nabla\tilde{v}_1 \\ &= \pi_n^l(\operatorname{div} + P_{L_0^2}\mathbf{q}\cdot + M)\mathbf{u}_n + M(\pi_n^d - \mathrm{Id}_{\mathbb{X}})\nabla\tilde{v}_1 \\ &\quad + (\mathrm{Id}_{L_0^2} - \pi_n^l)M\nabla\tilde{v}_1 \\ &= (\operatorname{div} + \pi_n^l P_{L_0^2}\mathbf{q}\cdot + M)\mathbf{u}_n + \tilde{O}_n\mathbf{u}_n\end{aligned} \tag{22}$$

with $\tilde{O}_n\mathbf{u}_n := M(\pi_n^d - \mathrm{Id}_{\mathbb{X}})\nabla\tilde{v}_1 + (\mathrm{Id}_{L_0^2} - \pi_n^l)M\nabla\tilde{v}_1 + (\pi_n^l - \mathrm{Id}_{L_0^2})M\mathbf{u}_n$. Since M is a compact operator which maps into L_0^2 and $\mathrm{Id}_{L_0^2} - \pi_n^l$ converges point-wise to zero on L_0^2, it follows that $(\mathrm{Id}_{L_0^2} - \pi_n^l)M$ tends to zero in the operator norm. For the remaining first term in $\tilde{O}_n$ we estimate

$$\begin{aligned}\|M(\pi_n^d - \mathrm{Id}_{\mathbb{X}})P_{\tilde{V}_n}\|_{L(\mathbb{X}_n, L_0^2)} &\lesssim \|\operatorname{div}(\pi_n^d - \mathrm{Id}_{\mathbb{X}})P_{\tilde{V}_n}\|_{L(\mathbb{X}_n, L_0^2)} \\ &= \|(\pi_n^l - \mathrm{Id}_{L_0^2})\operatorname{div} P_{\tilde{V}_n}\|_{L(\mathbb{X}_n, L_0^2)}.\end{aligned}$$

We compute

$$\begin{aligned}\operatorname{div}\nabla\tilde{v}_1 &= (\operatorname{div} + \pi_n^l P_{L_0^2}\mathbf{q}\cdot + M)\mathbf{u}_n - (P_{L_0^2}\mathbf{q}\cdot + M)\nabla\tilde{v}_1, \\ \pi_n^l \operatorname{div}\nabla\tilde{v}_1 &= (\operatorname{div} + \pi_n^l P_{L_0^2}\mathbf{q}\cdot + M)\mathbf{u}_n - \pi_n^l(P_{L_0^2}\mathbf{q}\cdot + M)\nabla\tilde{v}_1 + (\pi_n^l - \mathrm{Id})M\mathbf{u}_n.\end{aligned}$$

Hence

$$\begin{aligned}\|M(\pi_n^d - \mathrm{Id}_{\mathbb{X}})P_{\tilde{V}_n}\|_{L(\mathbb{X}_n, L_0^2)} &\lesssim \|(P_{L_0^2} - \pi_n^l)(P_{L_0^2}\mathbf{q}\cdot + M)P_{\tilde{V}_n}\|_{L(H_0(\operatorname{div}), L_0^2)} \\ &\quad + \|(\pi_n^l - \mathrm{Id})M\|_{L(H_0(\operatorname{div}), L_0^2)}.\end{aligned}$$

The former right-hand side tends to zero due to the previously used arguments and

$$\|(P_{L_0^2} - \pi_n^l)(P_{L_0^2}\mathbf{q}\cdot P_{\tilde{V}_n})\|_{L(H_0(\operatorname{div}), L_0^2)} \lesssim h_n\|\mathbf{q}\cdot P_{\tilde{V}_n}\|_{L(H_0(\operatorname{div}), H^1)} \lesssim h_n.$$

Now it follows from $(\operatorname{div} + P_{L_0^2}\mathbf{q}\cdot + M)\nabla(\tilde{v}_2 - \tilde{v}_1) = \tilde{O}_n\mathbf{u}_n$ that

$$\|(P_{V_n}P_{V_n} - P_{V_n})\mathbf{u}_n\|_{\mathbb{X}_n} \lesssim \|\nabla\tilde{v}_2 - \nabla\tilde{v}_1\|_{\mathbf{H}^1} \lesssim \|\tilde{O}_n\|_{L(\mathbb{X}_n, L_0^2)}\|\mathbf{u}_n\|_{\mathbb{X}_n}$$

and the claim is proven. □

Lemma 13 *There exist constants $n_0, C > 0$ such that T_n is invertible and $\|T_n^{-1}\|_{L(\mathbb{X}_n)} \leq C$ for all $n > n_0$.*

Proof It follows from Lemma 12 and the definition of T_n that $T_n T_n = 4P_{V_n}P_{V_n} - 4P_{V_n} + \mathrm{Id}_{\mathbb{X}_n} = \mathrm{Id}_{\mathbb{X}_n} + 4O_n$. Since $\|O_n\|_{L(\mathbb{X}_n)}$ tends to zero there exists an index $n_0 > 0$ such that $\|O_n\|_{L(\mathbb{X}_n)} < 1/8$ for all $n > n_0$ and hence $\|(\mathrm{Id}_{\mathbb{X}_n} + 4O_n)^{-1}\|_{L(\mathbb{X}_n)} \leq 2$ for $n > n_0$. It follows that $T_n^{-1} = (\mathrm{Id}_{\mathbb{X}_n} + 4O_n)^{-1} T_n$ and thus $\|T_n^{-1}\|_{L(\mathbb{X}_n)} \leq 2\|T_n\|_{L(\mathbb{X}_n)}$, $n > n_0$. The claim follows now from Lemma 11. □

4.2.4 Asymptotic consistency of T_n

Lemma 14 *One has that* $\lim_{n\to\infty} \|(P_{V_n} p_n - p_n P_V)\boldsymbol{u}\|_{\mathbb{X}_n} = 0$ *for each* $\boldsymbol{u} \in \mathbb{X}$.

Proof Recall that P_V, $P_{\tilde{V}_n}$ and P_{V_n} are defined in (17) and (20) respectively. We estimate

$$\begin{aligned}
&\|(P_{V_n} p_n - p_n P_V)\mathbf{u}\|_{\mathbb{X}_n} \\
&\quad \leq d_n(P_V\mathbf{u}, p_n P_V\mathbf{u}) + d_n(P_V\mathbf{u}, P_{V_n} p_n\mathbf{u}) \\
&\quad = d_n(P_V\mathbf{u}, p_n P_V\mathbf{u}) + d_n(P_V\mathbf{u}, \pi_n^d P_{\tilde{V}_n} p_n\mathbf{u}) \\
&\quad \leq d_n(P_V\mathbf{u}, p_n P_V\mathbf{u}) + d_n(P_V\mathbf{u}, \pi_n^d P_{\tilde{V}_n}\mathbf{u}) + \|\pi_n^d P_{\tilde{V}_n}(\mathbf{u} - p_n\mathbf{u})\|_{\mathbb{X}_n} \\
&\quad \lesssim d_n(P_V\mathbf{u}, p_n P_V\mathbf{u}) + d_n(P_V\mathbf{u}, \pi_n^d P_V\mathbf{u}) + \|P_V\mathbf{u} - P_{\tilde{V}_n}\mathbf{u}\|_{\mathbb{X}} + \|\mathbf{u} - p_n\mathbf{u}\|_{H(\mathrm{div})} \\
&\quad \lesssim d_n(P_V\mathbf{u}, p_n P_V\mathbf{u}) + d_n(P_V\mathbf{u}, \pi_n^d P_V\mathbf{u}) + \|(P_{L^2_0} - \pi_n^l)(P_{L^2_0}\mathbf{q}\cdot\mathbf{u})\|_{L^2} \\
&\qquad + d_n(\mathbf{u}, p_n\mathbf{u}).
\end{aligned}$$

The claim follows now from Lemmas 5 and 6 and the point-wise convergence of π_n^l. □

Lemma 15 *One has that* $\lim_{n\to\infty} \|(T_n p_n - p_n T)\boldsymbol{u}\|_{L(\mathbb{X},\mathbb{X}_n)} = 0$ *for each* $\boldsymbol{u} \in \mathbb{X}$.

Proof The assertion follows from Lemma 14 and the definitions (18), (21) of T, T_n. □

4.3 Discrete weak T_n-coercivity

Let us collect some properties, to emphasize the essential ingredients for the proof of the forthcoming Theorem 18. We recall the identity [6, (3.7)]

$$\langle c_s^2\rho \operatorname{div}\mathbf{v}, \operatorname{div}\mathbf{v}\rangle = |\mathbf{v}|^2_{\mathbf{H}^1_{c_s^2\rho}} + \langle K_G\mathbf{v}, \mathbf{v}\rangle_{\mathbf{V}} \tag{23}$$

with a compact operator $K_G \in L(\mathbf{V})$ for all $\mathbf{v} \in \mathbf{V}$, where

$$\mathbf{V} := \{\nabla v \colon v \in H^2_{*,\mathrm{Neu}}\}, \quad \|\cdot\|_{\mathbf{V}} := |\cdot|_{\mathbf{H}^1_{c_s^2\rho}}.$$

Note that actually (23) is in [6] formulated only for a subspace of $\mathbf{V}$. However, the proof to extend this result to $\mathbf{V}$ requires no changes at all. The purpose of the next lemma is to work out an estimate in weighted norms, which is essential to obtain the robustness with respect to $\underline{\rho}/\overline{\rho}$ and $\underline{c_s}/\overline{c_s}$.

Lemma 16 *We have that*

$$\|\rho^{1/2} D_{\boldsymbol{b}}^n \pi_n^d \boldsymbol{v}\|_{\boldsymbol{L}^2}^2 \leq (C_\pi^\#)^2 (1 + h_n^2 \tilde{C}_\pi) \|c_s^{-1}\boldsymbol{b}\|_{\boldsymbol{L}^\infty}^2 |\boldsymbol{v}|_{\boldsymbol{H}^1_{c_s^2\rho}}^2$$

with constants $\tilde{C}_\pi > 0$,

$$(C_\pi^\#)^2 := 2\big((C_{\mathrm{ab}} C_{\mathrm{sh}} C_{\mathrm{dt}})^2 + \sup_{n\in\mathbb{N}} \sup_{\tau\in\mathcal{T}_n} \|\pi_n^d\|_{L(\boldsymbol{H}^1_*(\tau))}^2\big), \qquad \|\cdot\|_{\boldsymbol{H}^1_*(\tau)} := |\cdot|_{\boldsymbol{H}^1(\tau)}$$

for all $\boldsymbol{v} \in \boldsymbol{H}^1_{\boldsymbol{\nu}0}$, $n \in \mathbb{N}$.

Proof For each $\tau \in \mathcal{T}_n$ we estimate

$$\begin{aligned}
\|\rho^{1/2} \partial_{\mathbf{b}} \pi_n^d \mathbf{v}\|_{\mathbf{L}^2(\tau)}^2 &\leq \|c_s^{-1}\mathbf{b}\|_{\mathbf{L}^\infty}^2 \overline{c_s}_\tau^2 \overline{\rho}_\tau |\pi_n^d \mathbf{v}|_{\mathbf{H}^1(\tau)}^2 \\
&\leq \|c_s^{-1}\mathbf{b}\|_{\mathbf{L}^\infty}^2 \overline{c_s}_\tau^2 \overline{\rho}_\tau \|\pi_n^d\|_{L(\mathbf{H}^1_*(\tau))}^2 |\mathbf{v}|_{\mathbf{H}^1(\tau)}^2 \\
&\leq \|c_s^{-1}\mathbf{b}\|_{\mathbf{L}^\infty}^2 \|\pi_n^d\|_{L(\mathbf{H}^1_*(\tau))}^2 \Big(1 + h_n^2 \frac{1}{\underline{c_s}^2 \underline{\rho}} (C^L_{c_s\rho^{1/2}})^2\Big)^2 |\mathbf{v}|_{\mathbf{H}^1_{c_s^2\rho}(\tau)}^2,
\end{aligned}$$

where we have used that

$$\overline{c_s}_\tau \overline{\rho}_\tau^{1/2} |v|_{\mathbf{H}^1(\tau)} = \|\frac{\overline{c_s}_\tau \overline{\rho}_\tau^{1/2}}{c_s \rho^{1/2}} \nabla v\|_{\mathbf{L}^2_{c_s^2\rho}(\tau)} \leq \frac{\overline{c_s}_\tau \overline{\rho}_\tau^{1/2}}{\underline{c_s}_\tau \underline{\rho}_\tau^{1/2}} |v|_{H^1_{c_s^2\rho}(\tau)} \leq (1 + \frac{h_n C^L_{c_s\rho^{1/2}}}{\underline{c_s}_\tau \underline{\rho}_\tau^{1/2}}) |v|_{H^1_{c_s^2\rho}(\tau)}$$

for a scalar function v. We further compute

$$\begin{aligned}
\|\rho^{1/2} R_n \pi_n^d \mathbf{v}\|_{\mathbf{L}^2(\tau)} = \|\rho^{1/2} \sum_{F\in\mathcal{F}_\tau} R_n^F \pi_n^d \mathbf{v}\|_{\mathbf{L}^2(\tau)} &\leq C_{\mathrm{dt}} \overline{\rho}_\tau \|\mathfrak{h}^{-1/2} [\![\pi_n^d \mathbf{v}]\!]_{\mathbf{b}}\|_{\mathbf{L}^2(\partial\tau)} \\
&= C_{\mathrm{dt}} \overline{\rho}_\tau \|\mathfrak{h}^{-1/2} [\![\pi_n^d \mathbf{v} - \mathbf{v}]\!]_{\mathbf{b}}\|_{\mathbf{L}^2(\partial\tau)}.
\end{aligned}$$

Hence we deduce by means of (6) that

$$\begin{aligned}
C_{\mathrm{dt}}^2 \sum_{\tau\in\mathcal{T}_n} \overline{\rho}_\tau \|\mathfrak{h}^{-1/2} [\![\pi_n^d \mathbf{v} - \mathbf{v}]\!]_{\mathbf{b}}\|_{\mathbf{L}^2(\partial\tau)}^2 &\leq C_{\mathrm{dt}}^2 \sum_{\tau\in\mathcal{T}_n} \overline{\rho}_\tau \sum_{F\in\mathcal{F}_\tau} \|\mathfrak{h}^{-1/2} [\![\pi_n^d \mathbf{v} - \mathbf{v}]\!]_{\mathbf{b}}\|_{\mathbf{L}^2(F)}^2 \\
&\leq C_{\mathrm{dt}}^2 \sum_{\tau\in\mathcal{T}_n} \overline{\rho}_\tau \sum_{F\in\mathcal{F}_\tau} \Big(\frac{1}{2} \sum_{j=1}^2 \|\mathfrak{h}^{-1/2} (\boldsymbol{\nu}\cdot\mathbf{b})((\pi_n^d \mathbf{v})_j - \mathbf{v})\|_{\mathbf{L}^2(F)}\Big)^2 \\
&\leq \frac{C_{\mathrm{dt}}^2}{2} \sum_{\tau\in\mathcal{T}_n} \overline{\rho}_\tau \sum_{F\in\mathcal{F}_\tau} \sum_{j=1}^2 \|\mathfrak{h}^{-1/2} (\boldsymbol{\nu}\cdot\mathbf{b})((\pi_n^d \mathbf{v})_j - \mathbf{v})\|_{\mathbf{L}^2(F)}^2
\end{aligned}$$

$$\leq \|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty} C^2_{\mathrm{dt}} \sum_{\tau\in\mathcal{T}_n} \overline{c_s}_\tau^{-2}\overline{\rho}_{\mathcal{O}_\tau} \|\mathfrak{h}^{-1/2}((\pi_n^d\mathbf{v})|_\tau - \mathbf{v})\|^2_{\mathbf{L}^2(\partial\tau)}$$

$$\leq \|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty} C^2_{\mathrm{ab}} C^2_{\mathrm{sh}} C^2_{\mathrm{dt}} \sum_{\tau\in\mathcal{T}_n} \overline{c_s}_\tau^{-2}\overline{\rho}_{\mathcal{O}_\tau} |\mathbf{v}|^2_{\mathbf{H}^1(\tau)}$$

$$\leq \|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty} C^2_{\mathrm{ab}} C^2_{\mathrm{sh}} C^2_{\mathrm{dt}} \Big(1 + (Ch_n)^2 \frac{1}{\underline{c_s}^2\underline{\rho}} (C^L_{c_s\rho^{1/2}})^2\Big)^2 |\mathbf{v}|^2_{\mathbf{H}^1_{c_s^2\rho}}$$

with a constant $C > 0$ that only depends on C_{sh}. Thus the claim follows. □

It further follows from (22) that for $\mathbf{u}_n \in \mathbb{X}_n$ (with $\mathbf{u}_n = \mathbf{v}_n + \mathbf{w}_n$ as defined in Sect. 4.2.1)

$$(\operatorname{div} + \pi_n^l P_{L^2_0}\mathbf{q}\cdot)\mathbf{w}_n = -M\mathbf{w}_n - \tilde{O}_n\mathbf{u}_n, \tag{24}$$

where we recall that $M \in L(H_0(\mathrm{div}), L^2_0)$ is compact and $\|\tilde{O}_n\|_{L(\mathbb{X}_n, L^2_0)}$ tends to zero. The next lemma shows that operators such as M lead to compact sequences of operators in the sense of discrete approximation schemes.

Lemma 17 *Let $K_n^{EP_V}$, $K_n^{K_G}$, $K_n^M \in L(\mathbb{X}_n)$ be defined by*

$$\begin{aligned}
\langle K_n^{EP_V}\boldsymbol{u}_n, \boldsymbol{u}'_n\rangle_{\mathbb{X}_n} &:= \langle P_{V_n}\boldsymbol{u}_n, P_{V_n}\boldsymbol{u}'_n\rangle_{\boldsymbol{L}^2},\\
\langle K_n^{\mathrm{mean}}\boldsymbol{u}_n, \boldsymbol{u}'_n\rangle_{\mathbb{X}_n} &:= \langle \mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}_n), \mathrm{mean}(\boldsymbol{q}\cdot\boldsymbol{w}'_n)\rangle_{L^2},\\
\langle K_n^{K_G}\boldsymbol{u}_n, \boldsymbol{u}'_n\rangle_{\mathbb{X}_n} &:= \langle K_G P_{\tilde{V}_n}\boldsymbol{u}_n, K_G P_{\tilde{V}_n}\boldsymbol{u}'_n\rangle_{V},\\
\langle K_n^{M}\boldsymbol{u}_n, \boldsymbol{u}'_n\rangle_{\mathbb{X}_n} &:= \langle M\boldsymbol{u}_n, M\boldsymbol{u}'_n\rangle_{L^2},
\end{aligned}$$

for all $\boldsymbol{u}_n, \boldsymbol{u}'_n \in \mathbb{X}_n$. Then $(K_n^{EP_V})_{n\in\mathbb{N}}$, $(K_n^{K_G})_{n\in\mathbb{N}}$, $(K_n^M)_{n\in\mathbb{N}}$ are compact in the sense of discrete approximation schemes.

Proof Let $(\mathbf{u}_n)_{n\in\mathbb{N}}$, $\mathbf{u}_n \in \mathbb{X}_n$ be a given bounded sequence $\|\mathbf{u}_n\|_{\mathbb{X}_n} \leq 1$ for each $n \in \mathbb{N}$. Let an arbitrary subsequence $\mathbb{N}' \subset \mathbb{N}$ be given. Recall the compact Sobolev embedding $E_{\mathbf{H}^1_{\boldsymbol{\nu}0},\mathbf{L}^2} \in L(\mathbf{H}^1_{\boldsymbol{\nu}0}, \mathbf{L}^2)$ and that $P_{V_n} = \pi_n^d P_{\tilde{V}_n}$, $P_{\tilde{V}_n} \in L(H_0(\mathrm{div}), \mathbf{H}^1_{\boldsymbol{\nu}0})$, $n \in \mathbb{N}$ are uniformly bounded. Then there exists a $\mathbf{z} \in \mathbf{L}^2$ and a subsequence $\mathbb{N}'' \subset \mathbb{N}'$ such that $\lim_{n\in\mathbb{N}''} \|\mathbf{z} - P_{\tilde{V}_n}\mathbf{u}_n\|_{\mathbf{L}^2} = 0$. It further follows that

$$\begin{aligned}
\|\mathbf{z} - P_{V_n}\mathbf{u}_n\|_{\mathbf{L}^2} = \|\mathbf{z} - \pi_n^d P_{\tilde{V}_n}\mathbf{u}_n\|_{\mathbf{L}^2} &\leq \|\mathbf{z} - P_{\tilde{V}_n}\mathbf{u}_n\|_{\mathbf{L}^2} + \|(1-\pi_n^d)P_{\tilde{V}_n}\mathbf{u}_n\|_{\mathbf{L}^2}\\
&\lesssim \|\mathbf{z} - P_{\tilde{V}_n}\mathbf{u}_n\|_{\mathbf{L}^2} + h_n\|P_{\tilde{V}_n}\mathbf{u}_n\|_{\mathbf{H}^1} \xrightarrow{n\in\mathbb{N}''} 0.
\end{aligned}$$

We want to show that $\lim_{n\in\mathbb{N}'''} \|p_n P_V^*\mathbf{z} - K_n^{EP_V}\mathbf{u}_n\|_{\mathbb{X}_n} = 0$ for a subsequence $\mathbb{N}''' \subset \mathbb{N}''$. To this end let $\mathbf{u}'_n \in \mathbb{X}_n$, $\|\mathbf{u}'_n\|_{\mathbb{X}_n} = 1$, $n \in \mathbb{N}''$ be such that $\|p_n P_V^*\mathbf{z} - K_n^{EP_V}\mathbf{u}_n\|_{\mathbb{X}_n} \leq |\langle p_n P_V^*\mathbf{z} - K_n^{EP_V}\mathbf{u}_n, \mathbf{u}'_n\rangle_{\mathbb{X}_n}| + 1/n$. By means of Lemma 8 we choose $\mathbb{N}''' \subset \mathbb{N}''$ and $\mathbf{u} \in \mathbb{X}$ such that $\mathbf{u}'_n$ converges weakly to $\mathbf{u}$ in the sense of

Lemma 8. We compute

$$\begin{aligned}\langle p_n P_V^* \mathbf{z}, \mathbf{u}_n' \rangle_{\mathbb{X}_n} &= \langle \operatorname{div} P_V^* \mathbf{z}, \operatorname{div} \mathbf{u}_n' \rangle + \langle P_V^* \mathbf{z}, \mathbf{u}_n' \rangle + \langle \partial_{\mathbf{b}} P_V^* \mathbf{z}, D_{\mathbf{b}}^n \mathbf{u}_n' \rangle \\ &\xrightarrow{n\in\mathbb{N}'''} \langle \operatorname{div} P_V^* \mathbf{z}, \operatorname{div} \mathbf{u} \rangle + \langle P_V^* \mathbf{z}, \mathbf{u} \rangle + \langle \partial_{\mathbf{b}} P_V^* \mathbf{z}, \partial_{\mathbf{b}} \mathbf{u} \rangle = \langle P_V^* \mathbf{z}, \mathbf{u} \rangle_{\mathbb{X}} = \langle \mathbf{z}, P_V \mathbf{u} \rangle\end{aligned}$$

and

$$\begin{aligned}\langle K_n^{EP_V} \mathbf{u}_n, \mathbf{u}_n' \rangle_{\mathbb{X}_n} &= \langle P_{V_n} \mathbf{u}_n, P_{V_n} \mathbf{u}_n' \rangle_{\mathbf{L}^2} \\ &= \langle P_{V_n} \mathbf{u}_n - \mathbf{z}, P_{V_n} \mathbf{u}_n' \rangle_{\mathbf{L}^2} + \langle \mathbf{z}, \pi_n^d P_{\tilde{V}_n} \mathbf{u}_n' \rangle_{\mathbf{L}^2} \\ &= \langle P_{V_n} \mathbf{u}_n - \mathbf{z}, P_{V_n} \mathbf{u}_n' \rangle_{\mathbf{L}^2} + \langle \mathbf{z}, \pi_n^d P_V \mathbf{u}_n' \rangle_{\mathbf{L}^2} + \langle \mathbf{z}, \pi_n^d (P_{\tilde{V}_n} - P_V) \mathbf{u}_n' \rangle_{\mathbf{L}^2} \\ &= \langle P_{V_n} \mathbf{u}_n - \mathbf{z}, P_{V_n} \mathbf{u}_n' \rangle_{\mathbf{L}^2} + \langle \mathbf{z}, P_V \mathbf{u}_n' \rangle_{\mathbf{L}^2} + \langle \mathbf{z}, (\pi_n^d - 1) P_V \mathbf{u}_n' \rangle_{\mathbf{L}^2} \\ &\quad + \langle \mathbf{z}, \pi_n^d (P_{\tilde{V}_n} - P_V) \mathbf{u}_n' \rangle_{\mathbf{L}^2} \\ &= \langle P_{V_n} \mathbf{u}_n - \mathbf{z}, P_{V_n} \mathbf{u}_n' \rangle_{\mathbf{L}^2} + \langle \mathbf{z}, P_V \mathbf{u}_n' \rangle_{\mathbf{L}^2} + \langle \mathbf{z}, (\pi_n^d - 1) P_V \mathbf{u}_n' \rangle_{\mathbf{L}^2} \\ &\quad + \langle \mathbf{z}, (\pi_n^d - 1)(P_{\tilde{V}_n} - P_V) \mathbf{u}_n' \rangle_{\mathbf{L}^2} + \langle \mathbf{z}, (P_{\tilde{V}_n} - P_V) \mathbf{u}_n' \rangle_{\mathbf{L}^2}.\end{aligned}$$

As previously, we estimate

$$\begin{aligned}\|(1-\pi_n^d) P_V \mathbf{u}_n'\|_{\mathbf{L}^2} + \|(1-\pi_n^d)(P_{\tilde{V}_n} - P_V) \mathbf{u}_n'\|_{\mathbf{L}^2} &\lesssim h_n (\|P_V \mathbf{u}_n'\|_{\mathbf{H}^1} + \|P_{\tilde{V}_n} \mathbf{u}_n'\|_{\mathbf{H}^1}) \\ &\lesssim h_n.\end{aligned}$$

In addition, we can write $(P_{\tilde{V}_n} - P_V)\mathbf{u}_n' = S(\pi_n^l - P_{L_0^2})(\mathbf{q} \cdot \mathbf{u}_n')$ with

$$S := \nabla\big((\operatorname{div} + P_{L_0^2} \mathbf{q} \cdot + M)\nabla\big)^{-1} \in L(L_0^2, \mathbf{L}^2)$$

and hence

$$\begin{aligned}\langle \mathbf{z}, (P_{\tilde{V}_n} - P_V) \mathbf{u}_n' \rangle_{\mathbf{L}^2} &= \langle \mathbf{z}, S(\pi_n^l - P_{L_0^2})(\mathbf{q} \cdot \mathbf{u}_n') \rangle_{\mathbf{L}^2} \\ &= \langle (\pi_n^l - P_{L_0^2}) S^* \mathbf{z}, P_{L_0^2}(\mathbf{q} \cdot \mathbf{u}_n') \rangle_{L_0^2} \xrightarrow{n\in\mathbb{N}'''} 0,\end{aligned}$$

where we used that π_n^l is an orthogonal projection which converges point-wise. Since

$$\langle \mathbf{z}, P_V \mathbf{u}_n' \rangle_{\mathbf{L}^2} = \langle P_V^* \mathbf{z}, \mathbf{u}_n' \rangle_{H_0(\operatorname{div})} \xrightarrow{n\in\mathbb{N}'''} \langle P_V^* \mathbf{z}, \mathbf{u} \rangle_{H_0(\operatorname{div})} = \langle \mathbf{z}, P_V \mathbf{u} \rangle_{\mathbf{L}^2}$$

the claim for $(K_n^{EP_V})_{n\in\mathbb{N}}$ follows. The proofs for $(K_n^{\mathrm{mean}})_{n\in\mathbb{N}}$, $(K_n^{K_G})_{n\in\mathbb{N}}$ and $(K_n^M)_{n\in\mathbb{N}}$ can be derived by a very same technique. □

In order to formulate Theorem 18 we introduce some additional quantities. For a selfadjoint matrix m let $\lambda_-(m)$ be its smallest eigenvalue. We introduce the matrix function $\underline{\underline{m}} := -\rho^{-1}\,\mathrm{Hess}(p) + \mathrm{Hess}(\phi)$. Further let

$$C_{\underline{\underline{m}}} := \max\left\{0, \sup_{x\in\mathcal{O}} \frac{-\lambda_-(\underline{\underline{m}}(x))}{\gamma(x)}\right\} \text{ and } \theta := \arctan(C_{\underline{\underline{m}}}/|\omega|) \in [0,\pi/2), \quad \text{for } \omega \neq 0.$$

Remark 1 Let us compare the assumption on the smallness of $\|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty}$ in Thm. 18 with the one of the $\mathbf{H}^1$-conforming FEM [15, Thm. 23, (3.19)]: $\frac{1}{(C_\pi^\#)^2}\frac{1}{1+C^2_{\underline{\underline{m}}}/|\omega|^2}$ vs. $\beta^2_{\mathrm{disc}}\frac{\underline{\underline{c_s}}^2\underline{\underline{\rho}}}{\overline{\overline{c_s}}^2\overline{\overline{\rho}}}\frac{1}{1+C_M^2/\omega^2}$ (where the constant C_M from [15] equals $C_{\underline{\underline{m}}}$). Both include the part $\frac{1}{1+C^2_{\underline{\underline{m}}}/|\omega|^2}$, which originates from the analysis at the continuous level. The h-independent terms $\frac{1}{(C_\pi^\#)^2}$ and (the squared discrete inf-sup constant of the divergence) β^2_{disc} are artefacts of the respective finite element discretizations. Of course they reduce the admissible range of $\|c_s^{-1}\mathbf{b}\|^2_{\mathbf{L}^\infty}$ compared to the continuous level, but this can be considered to be an acceptable restriction. On the other hand, the factor $\frac{\underline{\underline{c_s}}^2\underline{\underline{\rho}}}{\overline{\overline{c_s}}^2\overline{\overline{\rho}}}$ which we avoided with the FEM of the present article can be of magnitude 10^{22} for realistic parameters!

Theorem 18 *If* $\|c_s^{-1}\boldsymbol{b}\|^2_{\boldsymbol{L}^\infty} < \frac{1}{(C_\pi^\#)^2}\frac{1}{1+C^2_{\underline{\underline{m}}}/|\omega|^2}$, *then* $A_nT_n = B_n + K_n$ *with* $(B_n \in L(\mathbb{X}_n))_{n\in\mathbb{N}}$ *is uniformly bounded and stable,* $(K_n \in L(\mathbb{X}_n))_{n\in\mathbb{N}}$ *is compact, and there exists a bijective operator* $B \in L(\mathbb{X})$ *such that* $\lim_{n\to\infty}\|B_np_n\boldsymbol{u} - p_nB\boldsymbol{u}\|_{\mathbb{X}_n} = 0$ *for each* $\boldsymbol{u} \in \mathbb{X}$.

Proof We will use that

$$\begin{aligned}
a_n(\mathbf{u}_n,\mathbf{u}'_n) = {} & \langle c_s^2\rho\big(\operatorname{div}\mathbf{u}_n + \pi_n^l P_{L^2_0}(\mathbf{q}\cdot\mathbf{u}_n)\big), \operatorname{div}\mathbf{u}'_n + \pi_n^l P_{L^2_0}(\mathbf{q}\cdot\mathbf{u}'_n)\rangle \\
& - \langle c_s^2\rho\pi_n^l P_{L^2_0}(\mathbf{q}\cdot\mathbf{u}_n), \pi_n^l P_{L^2_0}(\mathbf{q}\cdot\mathbf{u}'_n)\rangle - \langle\rho(i\omega\gamma + \underline{\underline{m}})\mathbf{u}_n, \mathbf{u}'_n\rangle \\
& + \langle c_s^2\rho \operatorname{div}\mathbf{u}_n, (I-\pi_n^l)P_{L^2_0}(\mathbf{q}\cdot\mathbf{u}'_n) + \mathrm{mean}(\mathbf{q}\cdot\mathbf{u}'_n)\rangle \\
& + \langle c_s^2\rho\big((I-\pi_n^l)P_{L^2_0}(\mathbf{q}\cdot\mathbf{u}_n) + \mathrm{mean}(\mathbf{q}\cdot\mathbf{u}_n)\big), \operatorname{div}\mathbf{u}'_n\rangle \\
& - \langle\rho(\omega + iD^n_{\mathbf{b}} + i\Omega\times)\mathbf{u}_n, (\omega + iD^n_{\mathbf{b}} + i\Omega\times)\mathbf{u}'_n\rangle.
\end{aligned}$$

Step 1. (definition of B_n and K_n). Let $K_G \in L(\mathbf{V})$ be the compact operator from (23). Let

$$\begin{aligned}
\langle\tilde{B}_n\mathbf{u}_n,\mathbf{u}'_n\rangle_{\mathbb{X}_n} := {} & \\
& \langle c_s^2\rho \operatorname{div}\mathbf{v}_n, \operatorname{div}\mathbf{v}'_n\rangle - \langle\rho iD^n_{\mathbf{b}}\mathbf{v}_n, iD^n_{\mathbf{b}}\mathbf{v}'_n\rangle \\
& + \langle c_s^2\rho\pi_n^l P_{L^2_0}(\mathbf{q}\cdot\mathbf{w}_n), \pi_n^l P_{L^2_0}(\mathbf{q}\cdot\mathbf{w}'_n)\rangle && \text{(25a)} \\
& - \langle\rho iD^n_{\mathbf{b}}\mathbf{v}_n, (\omega + iD^n_{\mathbf{b}} + i\Omega\times)\mathbf{w}'_n\rangle \\
& + \langle\rho(\omega + iD^n_{\mathbf{b}} + i\Omega\times)\mathbf{w}_n, iD^n_{\mathbf{b}}\mathbf{v}'_n\rangle && \text{(25b)} \\
& + \langle\rho(\omega + iD^n_{\mathbf{b}} + i\Omega\times)\mathbf{w}_n, (\omega + iD^n_{\mathbf{b}} + i\Omega\times)\mathbf{w}'_n\rangle
\end{aligned}$$

$$
\begin{aligned}
&+ \langle \rho(i\gamma + \underline{\underline{m}})\mathbf{w}_n, \mathbf{w}_n' \rangle && \text{(25c)}\\
&+ \langle \mathbf{v}_n, \mathbf{v}_n' \rangle + C_1 \langle K_G P_{\tilde{V}_n} \mathbf{u}_n, K_G P_{\tilde{V}_n} \mathbf{u}_n' \rangle_{\mathbf{V}} + \langle M\mathbf{w}_n, M\mathbf{w}_n' \rangle \\
&+ \langle \tilde{O}_n \mathbf{u}_n, \tilde{O}_n \mathbf{u}_n' \rangle && \text{(25d)}
\end{aligned}
$$

and

$$
\begin{aligned}
\langle \tilde{K}_n \mathbf{u}_n, \mathbf{u}_n' \rangle_{\mathbb{X}_n} :=& \\
& C_2(\langle \mathbf{v}_n, \mathbf{v}_n' \rangle + \langle K_G P_{\tilde{V}_n} \mathbf{u}_n, K_G P_{\tilde{V}_n} \mathbf{u}_n' \rangle_{\mathbf{V}} + \langle \tilde{O}_n \mathbf{u}_n, \tilde{O}_n \mathbf{u}_n' \rangle && \text{(26a)}\\
&+ \langle M\mathbf{w}_n, M\mathbf{w}_n' \rangle + \langle \operatorname{mean}(\mathbf{q}\cdot\mathbf{w}_n), \operatorname{mean}(\mathbf{q}\cdot\mathbf{w}_n') \rangle) && \text{(26b)}\\
&+ \langle c_s^2 \rho \mathbf{q}\cdot\mathbf{v}_n, \operatorname{div}\mathbf{v}_n' \rangle + \langle c_s^2 \rho \operatorname{div}\mathbf{v}_n, \mathbf{q}\cdot\mathbf{v}_n' \rangle \\
&- \langle \rho(\omega + i\Omega\times)\mathbf{v}_n, (\omega + i\Omega\times)\mathbf{v}_n' \rangle && \text{(26c)}\\
&- \langle \rho(\omega + i\Omega\times)\mathbf{v}_n, i D_{\mathbf{b}}^n \mathbf{v}_n' \rangle - \langle \rho i D_{\mathbf{b}}^n \mathbf{v}_n, (\omega + i\Omega\times)\mathbf{v}_n' \rangle \\
&- i\omega \langle \gamma\rho \mathbf{v}_n, \mathbf{v}_n' \rangle && \text{(26d)}\\
&- \langle \rho \underline{\underline{m}} \mathbf{v}_n, \mathbf{v}_n' \rangle && \text{(26e)}\\
&- \langle \rho \underline{\underline{m}} \mathbf{v}_n, \mathbf{w}_n' \rangle - i\omega \langle \gamma\rho \mathbf{v}_n, \mathbf{w}_n' \rangle \\
&- \langle c_s^2 \rho \pi_n^l P_{L_0^2}(\mathbf{q}\cdot\mathbf{v}_n), \pi_n^l P_{L_0^2}(\mathbf{q}\cdot\mathbf{w}_n') \rangle && \text{(26f)}\\
&- \langle \rho(\omega + i\Omega\times)\mathbf{v}_n, (\omega + i D_{\mathbf{b}}^n + i\Omega\times)\mathbf{w}_n' \rangle && \text{(26g)}\\
&- \langle c_s^2 \rho (\operatorname{div} + \pi_n^l P_{L_0^2}\mathbf{q}\cdot)\mathbf{v}_n, M\mathbf{w}_n' + \tilde{O}_n \mathbf{u}_n' \rangle && \text{(26h)}\\
&+ \langle c_s^2 \rho \big((\operatorname{Id} - \pi_n^l) P_{L_0^2}(\mathbf{q}\cdot\mathbf{v}_n) + \operatorname{mean}(\mathbf{q}\cdot\mathbf{v}_n)\big), \operatorname{div}\mathbf{w}_n' \rangle && \text{(26i)}\\
&+ \langle c_s^2 \rho \operatorname{div}\mathbf{v}_n, (\operatorname{Id} - \pi_n^l) P_{L_0^2}(\mathbf{q}\cdot\mathbf{w}_n') + \operatorname{mean}(\mathbf{q}\cdot\mathbf{v}_n') \rangle && \text{(26j)}\\
&+ \langle \rho \underline{\underline{m}} \mathbf{w}_n, \mathbf{v}_n' \rangle + i\omega \langle \gamma\rho \mathbf{w}_n, \mathbf{v}_n' \rangle \\
&+ \langle c_s^2 \rho \pi_n^l P_{L_0^2}(\mathbf{q}\cdot\mathbf{w}_n), \pi_n^l P_{L_0^2}(\mathbf{q}\cdot\mathbf{v}_n') \rangle && \text{(26k)}\\
&+ \langle \rho(\omega + i D_{\mathbf{b}}^n + i\Omega\times)\mathbf{w}_n, (\omega + i\Omega\times)\mathbf{v}_n' \rangle && \text{(26l)}\\
&+ \langle c_s^2 \rho (M\mathbf{w}_n + \tilde{O}_n \mathbf{u}_n), (\operatorname{div} + \pi_n^l P_{L_0^2}\mathbf{q}\cdot)\mathbf{v}_n' \rangle && \text{(26m)}\\
&- \langle c_s^2 \rho \big((\operatorname{Id} - \pi_n^l) P_{L_0^2}(\mathbf{q}\cdot\mathbf{w}_n) + \operatorname{mean}(\mathbf{q}\cdot\mathbf{w}_n)\big), \operatorname{div}\mathbf{v}_n' \rangle && \text{(26n)}\\
&- \langle c_s^2 \rho \operatorname{div}\mathbf{w}_n, (\operatorname{Id} - \pi_n^l) P_{L_0^2}(\mathbf{q}\cdot\mathbf{v}_n') + \operatorname{mean}(\mathbf{q}\cdot\mathbf{v}_n') \rangle && \text{(26o)}\\
&- \langle c_s^2 \rho (M\mathbf{w}_n + \tilde{O}_n \mathbf{u}_n), M\mathbf{w}_n' + \tilde{O}_n \mathbf{u}_n' \rangle && \text{(26p)}\\
&- \langle c_s^2 \rho \big((\operatorname{Id} - \pi_n^l) P_{L_0^2}(\mathbf{q}\cdot\mathbf{w}_n) + \operatorname{mean}(\mathbf{q}\cdot\mathbf{w}_n)\big), \operatorname{div}\mathbf{w}_n' \rangle && \text{(26q)}\\
&- \langle c_s^2 \rho \operatorname{div}\mathbf{w}_n, (\operatorname{Id} - \pi_n^l) P_{L_0^2}(\mathbf{q}\cdot\mathbf{w}_n') + \operatorname{mean}(\mathbf{q}\cdot\mathbf{w}_n') \rangle && \text{(26r)}
\end{aligned}
$$

and

$$
\begin{aligned}
\langle K_n \mathbf{u}_n, \mathbf{u}_n' \rangle_{\mathbb{X}_n} :=& \\
& - C_2 \langle \mathbf{v}_n, \mathbf{v}_n' \rangle - (C_1 + C_2) \langle K_G P_{\tilde{V}_n} \mathbf{u}_n, K_G P_{\tilde{V}_n} \mathbf{u}_n' \rangle_{\mathbf{V}} && \text{(27a)}
\end{aligned}
$$

$$
\begin{aligned}
&- (1 + C_2)\langle M\mathbf{w}_n, M\mathbf{w}'_n\rangle - C_2\langle \mathrm{mean}(\mathbf{q}\cdot\mathbf{w}_n), \mathrm{mean}(\mathbf{q}\cdot\mathbf{w}'_n)\rangle \\
&- (1 + C_2)\langle \tilde{O}_n\mathbf{u}_n, \tilde{O}_n\mathbf{u}'_n\rangle
\end{aligned} \tag{27b}
$$

for all $\mathbf{u}_n, \mathbf{u}'_n \in \mathbb{X}_n$, where the constants $C_1, C_2 > 0$ will be specified later on. Then with $B_n := \tilde{B}_n + \tilde{K}_n$ it holds that $A_nT_n = B_n + K_n$. We discuss the details of this decomposition in the following. First note that $\langle A_nT_n\mathbf{u}_n, \mathbf{u}'_n\rangle_{\mathbb{X}_n} = a_n(T_n\mathbf{u}_n, \mathbf{u}'_n) = a_n(\mathbf{v}_n - \mathbf{w}_n, \mathbf{v}'_n + \mathbf{w}'_n)$. Second note that the operators $\tilde{K}_n$ and K_n contain only terms which are intuitively compact, i.e., they are sums of operators tending to zero, having finite dimensional range, involving the compact operator K_G or a compact L^2-embedding ($\mathbf{v}_n \hookrightarrow\hookrightarrow \mathbf{L}^2$).

Nevertheless, proving that each term in $\tilde{K}_n$ does indeed yield a *compact sequence of operators in the sense of discrete approximation schemes* seems tedious. Instead, we only prove this compactness for K_n and absorb $\tilde{K}_n$ into the coercive part B_n. To ensure the coercivity of B_n we choose the constants C_1, C_2 sufficiently large.

On the other hand, the compactness of $(K_n)_{n\in\mathbb{N}}$ is ensured by Lemma 17 and $\lim_{n\to\infty}\|\tilde{O}_n\|_{L(\mathbb{X}_n,L^2)} = 0$. Note that we added the line (25d) into the definition of $\tilde{B}_n$ for stability reasons and together with (26a), (26b) they cancel out with (27a), (27b). Also note that in (26) we grouped together all terms of the kinds $(\mathbf{v}_n, \mathbf{v}'_n)$, $(\mathbf{v}_n, \mathbf{w}'_n)$, $(\mathbf{w}_n, \mathbf{v}'_n)$, $(\mathbf{w}_n, \mathbf{w}'_n)$ respectively and we included blank lines to emphasize those different blocks. At several places we applied (24). The uniform boundedness of B_n, $n \in \mathbb{N}$ follows straightforwardly.

Step 2. (coercivity of $\tilde{B}_n$). The commutation property $\operatorname{div}\pi_n^d = \pi_n^l \operatorname{div}$ will enable us to adapt (23) to $\mathbf{v}_n$ in an apt way. To this end we compute

$$
\begin{aligned}
\operatorname{div}\mathbf{v}_n &= \operatorname{div}\pi_n^d\nabla\tilde{v} = \pi_n^l\Delta\tilde{v} = \pi_n^l\Big(-(P_{L^2_0}\mathbf{q}\cdot +M)\nabla\tilde{v} + (\operatorname{div} + \pi_n^l P_{L^2_0}\mathbf{q}\cdot +M)\mathbf{u}_n\Big) \\
&= -(P_{L^2_0}\mathbf{q}\cdot +M)\nabla\tilde{v} + (\operatorname{div} + \pi_n^l P_{L^2_0}\mathbf{q}\cdot +M)\mathbf{u}_n \\
&\quad + (\mathrm{Id} - \pi_n^l)(P_{L^2_0}\mathbf{q}\cdot +M)\nabla\tilde{v} + (\pi_n^l - \mathrm{Id})M\mathbf{u}_n \\
&= \Delta\tilde{v} + (\mathrm{Id} - \pi_n^l)(P_{L^2_0}\mathbf{q}\cdot +M)P_{\tilde{V}_n}\mathbf{u}_n + (\pi_n^l - \mathrm{Id})M\mathbf{u}_n \\
&=: \Delta\tilde{v} + \hat{O}_n\mathbf{u}_n.
\end{aligned}
$$

By the same technique as used previously in the proof of Lemma 12 it follows that $\lim_{n\to\infty}\|\hat{O}_n\|_{L(\mathbb{X}_n,L^2_0)} = 0$. Thus with $\langle \check{O}_n\mathbf{u}_n, \mathbf{u}'_n\rangle_{\mathbb{X}_n} := \langle c_s^2\rho\operatorname{div}\mathbf{v}_n, \hat{O}_n\mathbf{u}'_n\rangle + \langle c_s^2\rho\hat{O}_n\mathbf{u}_n, \operatorname{div}\mathbf{v}'_n\rangle + \langle c_s^2\rho\hat{O}_n\mathbf{u}_n, \hat{O}_n\mathbf{u}'_n\rangle$ it holds that $\lim_{n\to\infty}\|\check{O}_n\|_{\mathbb{X}_n} = 0$ and

$$
\langle c_s^2\rho\operatorname{div}\mathbf{v}_n, \operatorname{div}\mathbf{v}_n\rangle = \langle c_s^2\rho\Delta\tilde{v}, \Delta\tilde{v}\rangle + \langle \check{O}_n\mathbf{u}_n, \mathbf{u}_n\rangle_{\mathbb{X}_n}. \tag{28}
$$

Thus (28) and (23) yield that

$$
\langle c_s^2\rho\operatorname{div}\mathbf{v}_n, \operatorname{div}\mathbf{v}_n\rangle = |\nabla\tilde{v}|^2_{\mathbf{H}^1_{c_s^2\rho}} + \langle K_G P_{\tilde{V}_n}\mathbf{u}_n, P_{\tilde{V}_n}\mathbf{u}_n\rangle_{\mathbf{V}} + \langle \check{O}_n\mathbf{u}_n, \mathbf{u}_n\rangle_{\mathbb{X}_n}. \tag{29}
$$

Due to the smallness assumption on the Mach number there exist $\epsilon \in (0, 1)$, $\tau \in (0, \pi/2 - \theta)$ and $n_0 > 0$ such that

$$C_{\theta,\tau,\epsilon,n_0} := 1 - (C_\pi^\#)^2 (1 + \sup_{n>n_0} h_n^2 \tilde{C}_\pi) \|c_s^{-1}\mathbf{b}\|_{\mathbf{L}^\infty}^2 \big(1 + \tan^2(\theta+\tau)(1-\epsilon)^{-1}\big) - 2\epsilon$$

is positive. Henceforth we assume that $n > n_0$. Now we estimate by means of a weighted Young's inequality and the definition of θ that

$$\begin{aligned}
&\frac{1}{\cos(\theta+\tau)} \operatorname{Re}\Big(e^{-i(\theta+\tau)\operatorname{sgn}\omega} \langle \tilde{B}_n \mathbf{u}_n, \mathbf{u}_n\rangle_{\mathbb{X}_n}\Big) \\
&\quad = \|c_s \rho^{1/2} \operatorname{div} \mathbf{v}_n\|_{L^2}^2 - \|\rho^{1/2} D_{\mathbf{b}}^n \mathbf{v}_n\|_{\mathbf{L}^2}^2 + \|\mathbf{v}_n\|_{\mathbf{L}^2}^2 + C_1 \|K_G P_{\tilde{V}_n} \mathbf{u}_n\|_{\mathbf{V}}^2 + \|M\mathbf{w}_n\|_{L^2}^2 \\
&\qquad + \|\tilde{O}_n \mathbf{u}_n\|_{L^2}^2 + \|c_s \rho^{1/2} \pi_n^l P_{L_0^2}(\mathbf{q}\cdot\mathbf{w}_n)\|_{\mathbf{L}^2}^2 + \|\rho^{1/2}(\omega + iD_{\mathbf{b}}^n + i\Omega\times)\mathbf{w}_n\|_{\mathbf{L}^2}^2 \\
&\qquad + \langle \rho \underline{\underline{m}} \mathbf{w}_n, \mathbf{w}_n\rangle_{\mathbf{L}^2} + 2\tan(\theta+\tau)\operatorname{sgn}\omega \operatorname{Im}(\langle \rho(\omega + iD_{\mathbf{b}}^n + i\Omega\times)\mathbf{w}_n, iD_{\mathbf{b}}^n \mathbf{v}_n\rangle) \\
&\qquad - |\omega| \tan(\theta+\tau) \|(\gamma\rho)^{1/2}\mathbf{w}_n\|_{\mathbf{L}^2}^2 \\
&\quad \geq \|c_s \rho^{1/2} \operatorname{div} \mathbf{v}_n\|_{L^2}^2 - \big(1 + \tan^2(\theta+\tau)(1-\epsilon)^{-1}\big) \|\rho^{1/2} D_{\mathbf{b}}^n \mathbf{v}_n\|_{\mathbf{L}^2}^2 + \|\mathbf{v}_n\|_{\mathbf{L}^2}^2 \\
&\qquad + C_1 \|K_G P_{\tilde{V}_n} \mathbf{u}_n\|_{\mathbf{V}}^2 + \|M\mathbf{w}_n\|_{L^2}^2 + \|\tilde{O}_n \mathbf{u}_n\|_{L^2}^2 + \|c_s \rho^{1/2} \pi_n^l P_{L_0^2}(\mathbf{q}\cdot\mathbf{w}_n)\|_{\mathbf{L}^2}^2 \\
&\qquad + \epsilon \|\rho^{1/2}(\omega + iD_{\mathbf{b}}^n + i\Omega\times)\mathbf{w}_n\|_{\mathbf{L}^2}^2 + |\omega|\big(\tan(\theta+\tau) - \tan\theta\big)\|(\gamma\rho)^{1/2}\mathbf{w}_n\|_{\mathbf{L}^2}^2.
\end{aligned}$$

Thence Lemma 16 and (29) yield that

$$\begin{aligned}
&\|c_s \rho^{1/2} \operatorname{div} \mathbf{v}_n\|_{L^2}^2 - \big(1 + \tan^2(\theta+\tau)(1-\epsilon)^{-1}\big) \|\rho^{1/2} D_{\mathbf{b}}^n \mathbf{v}_n\|_{\mathbf{L}^2}^2 + \|\mathbf{v}_n\|_{\mathbf{L}^2}^2 \\
&\qquad + C_1 \|K_G P_{\tilde{V}_n} \mathbf{u}_n\|_{\mathbf{V}}^2 \\
&\quad \geq \epsilon\big(\|c_s \rho^{1/2} \operatorname{div} \mathbf{v}_n\|_{L^2}^2 + \|\rho^{1/2} D_{\mathbf{b}}^n \mathbf{v}_n\|_{\mathbf{L}^2}^2\big) + C_{\theta,\tau,\epsilon,n_0} |\nabla \tilde{v}|_{\mathbf{H}^1_{c_s^2\rho}}^2 + \|\mathbf{v}_n\|_{\mathbf{L}^2}^2 \\
&\qquad + \Big(C_1 - \frac{1}{4\delta}\Big) \|K_G P_{\tilde{V}_n} \mathbf{u}_n\|_{\mathbf{V}}^2 - \big(\delta \sup_{m\in\mathbb{N}} \|P_{\tilde{V}_m}\|_{L(\mathbb{X}_m,\mathbf{V})}^2 + \|\check{O}_n\|_{L(\mathbb{X}_n)}\big) \|\mathbf{u}_n\|_{\mathbb{X}_n}^2 \\
&\quad \geq \epsilon \min\{\underline{c_s}^2 \underline{\rho}, \underline{\rho}, 1\} \|\mathbf{v}_n\|_{\mathbb{X}_n}^2 + \Big(C_1 - \frac{1}{4\delta}\Big) \|K_G P_{\tilde{V}_n} \mathbf{u}_n\|_{\mathbf{V}}^2 \\
&\qquad - \big(\delta \sup_{m\in\mathbb{N}} \|P_{\tilde{V}_m}\|_{L(\mathbb{X}_m,\mathbf{V})}^2 + \|\check{O}_n\|_{L(\mathbb{X}_n)}\big) \|\mathbf{u}_n\|_{\mathbb{X}_n}^2.
\end{aligned}$$

Further (24) yields that

$$4\big(\|M\mathbf{w}_n\|_{L^2}^2 + \|\tilde{O}_n \mathbf{u}_n\|_{L^2}^2 + \|\pi_n^l P_{L_0^2}(\mathbf{q}\cdot\mathbf{w}_n)\|_{\mathbf{L}^2}^2\big) \geq \|\operatorname{div} \mathbf{w}_n\|_{L^2}^2$$

and hence

$$\begin{aligned}
&\|M\mathbf{w}_n\|_{L^2}^2 + \|\tilde{O}_n \mathbf{u}_n\|_{L^2}^2 + \|c_s \rho^{1/2} \pi_n^l P_{L_0^2}(\mathbf{q}\cdot\mathbf{w}_n)\|_{\mathbf{L}^2}^2 + \epsilon \|\rho^{1/2}(\omega + iD_{\mathbf{b}}^n + i\Omega\times)\mathbf{w}_n\|_{\mathbf{L}^2}^2 \\
&\qquad + |\omega|\big(\tan(\theta+\tau) - \tan\theta\big)\|(\gamma\rho)^{1/2}\mathbf{w}_n\|_{\mathbf{L}^2}^2 \gtrsim \|\mathbf{w}_n\|_{\mathbb{X}_n}^2.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{\cos(\theta+\tau)} \operatorname{Re}\left(e^{-i(\theta+\tau)\operatorname{sgn}\omega}\langle \tilde{B}_n \mathbf{u}_n, \mathbf{u}_n\rangle_{\mathbb{X}_n}\right)\\
&\quad\geq C_{\tilde{B}}\|\mathbf{u}_n\|^2_{\mathbb{X}_n} + \left(C_1 - \frac{1}{4\delta}\right)\|K_G P_{\tilde{V}_n}\mathbf{u}_n\|^2_{\mathbf{V}}\\
&\qquad - \big(\delta \sup_{m\in\mathbb{N}} \|P_{\tilde{V}_m}\|^2_{L(\mathbb{X}_m,\mathbf{V})} + \|\check{O}_n\|_{L(\mathbb{X}_n)}\big)\|\mathbf{u}_n\|^2_{\mathbb{X}_n}
\end{aligned}$$

with a constant $C_{\tilde{B}} > 0$ independent of δ, C_1, $n > n_0$. Hence we can choose $\delta > 0$ and $n_1 > n_0$ such that

$$\begin{aligned}
\frac{1}{\cos(\theta+\tau)} \operatorname{Re}\left(e^{-i(\theta+\tau)\operatorname{sgn}\omega}\langle \tilde{B}_n \mathbf{u}_n, \mathbf{u}_n\rangle_{\mathbb{X}_n}\right) &\geq \frac{C_{\tilde{B}}}{2}\|\mathbf{u}_n\|^2_{\mathbb{X}_n}\\
&\quad + \left(C_1 - \frac{1}{4\delta}\right)\|K_G P_{\tilde{V}_n}\mathbf{u}_n\|^2_{\mathbf{V}}
\end{aligned}$$

for all $n > n_1$. Now we choose $C_1 > 1/(4\delta)$ to obtain that

$$\frac{1}{\cos(\theta+\tau)} \operatorname{Re}\left(e^{-i(\theta+\tau)\operatorname{sgn}\omega}\langle \tilde{B}_n \mathbf{u}_n, \mathbf{u}_n\rangle_{\mathbb{X}_n}\right) \geq \frac{C_{\tilde{B}}}{2}\|\mathbf{u}_n\|^2_{\mathbb{X}_n}$$

for all $n > n_1$.

Step 3. (coercivity of B_n). To start with, we estimate the first term in (26i) and note that (26j), (26n), (26o), (26q), (26r) can be estimated similarly. To this end we compute that

$$\begin{aligned}
|\langle c_s^2\rho(\operatorname{Id}-\pi_n^l)P_{L^2_0}(\mathbf{q}\cdot\mathbf{w}_n), \operatorname{div}\mathbf{v}_n'\rangle| &= |\langle P_{L^2_0}(\mathbf{q}\cdot\mathbf{w}_n), (\operatorname{Id}-\pi_n^l)(c_s^2\rho\operatorname{div}\mathbf{v}_n')\rangle|\\
&\leq \|\mathbf{q}\|_{\mathbf{L}^\infty}\|\mathbf{w}_n\|_{\mathbf{L}^2}\|(\operatorname{Id}-\pi_n^l)(c_s^2\rho\operatorname{div}\mathbf{v}_n')\|_{L^2}.
\end{aligned}$$

Then we apply a discrete commutator technique [26] and estimate

$$\begin{aligned}
\|(\operatorname{Id}-\pi_n^l)(c_s^2\rho\operatorname{div}\mathbf{v}_n')\|^2_{L^2} &= \sum_{\tau\in\mathcal{T}_n}\|(\operatorname{Id}-\pi_n^l)(c_s^2\rho\operatorname{div}\mathbf{v}_n')\|^2_{L^2(\tau)}\\
&= \sum_{\tau\in\mathcal{T}_n}\|(\operatorname{Id}-\pi_n^l)((c_s^2\rho - c_\tau)\operatorname{div}\mathbf{v}_n')\|^2_{L^2(\tau)}\\
&\leq \sum_{\tau\in\mathcal{T}_n}\|(c_s^2\rho - c_\tau)\operatorname{div}\mathbf{v}_n'\|^2_{L^2(\tau)}\\
&\leq (C^L_{c_s^2\rho})^2 h_n^2 \sum_{\tau\in\mathcal{T}_n}\|\operatorname{div}\mathbf{v}_n'\|^2_{L^2(\tau)} = (C^L_{c_s^2\rho})^2 h_n^2\|\operatorname{div}\mathbf{v}_n'\|^2_{L^2}
\end{aligned}$$

with suitably chosen constants c_τ, $\tau\in\mathcal{T}_n$. Let

$$|\mathbf{u}_n|^2_{Y_n} := \|\mathbf{v}_n\|^2_{\mathbf{L}^2} + \|K_G P_{\tilde{V}_n}\mathbf{u}_n\|^2_{\mathbf{V}} + \|\tilde{O}_n\mathbf{u}_n\|^2_{L^2} + \|M\mathbf{w}_n\|^2_{L^2} + \|\operatorname{mean}(\mathbf{q}\cdot\mathbf{w}_n)\|^2_{L^2}.$$

We estimate

$$\frac{1}{\cos(\theta+\tau)} \operatorname{Re}\left(e^{-i(\theta+\tau)\operatorname{sgn}\omega} \langle \tilde{K}_n \mathbf{u}_n, \mathbf{u}_n \rangle_{\mathbb{X}_n}\right) \\ \geq C_2 |\mathbf{u}_n|^2_{Y_n} - h_n C_{Y,1} \|\mathbf{u}_n\|^2_{\mathbb{X}_n} - C_{Y,2} \|\mathbf{u}_n\|_{\mathbb{X}_n} |\mathbf{u}_n|_{Y_n}$$

with constants $C_{Y,1}, C_{Y,2} > 0$. Thus

$$\frac{1}{\cos(\theta+\tau)} \operatorname{Re}\left(e^{-i(\theta+\tau)\operatorname{sgn}\omega} \langle B_n \mathbf{u}_n, \mathbf{u}_n \rangle_{\mathbb{X}_n}\right) \\ \geq \frac{C_{\tilde{B}}}{2} \|\mathbf{u}_n\|^2_{\mathbb{X}_n} + C_2 |\mathbf{u}_n|^2_{Y_n} - h_n C_{Y,1} \|\mathbf{u}_n\|^2_{\mathbb{X}_n} - C_{Y,2} \|\mathbf{u}_n\|_{\mathbb{X}_n} |\mathbf{u}_n|_{Y_n} \\ \geq (C_{\tilde{B}}/4 - h_n C_{Y,1}) \|\mathbf{u}_n\|^2_{\mathbb{X}_n} + (C_2 - C^2_{Y,2}/C^2_{\tilde{B}}) |\mathbf{u}_n|^2_{Y_n}.$$

Now we choose $C_2 > C^2_{Y,2}/C^2_{\tilde{B}}$ and obtain the uniform stability of B_n for large enough index n.

Step 4. (asymptotic consistency of B_n). Similarly to the discrete setting, it holds that $AT = B + K$ with

$$\begin{aligned} \langle B\mathbf{u}, \mathbf{u}' \rangle_{\mathbb{X}} := \\ & \langle c_s^2 \rho \operatorname{div} \mathbf{v}, \operatorname{div} \mathbf{v}' \rangle - \langle \rho i \partial_{\mathbf{b}} \mathbf{v}, i \partial_{\mathbf{b}} \mathbf{v}' \rangle + \langle c_s^2 \rho P_{L_0^2}(\mathbf{q} \cdot \mathbf{w}), P_{L_0^2}(\mathbf{q} \cdot \mathbf{w}') \rangle \\ & - \langle \rho i \partial_{\mathbf{b}} \mathbf{v}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}' \rangle + \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}, i\partial_{\mathbf{b}} \mathbf{v}' \rangle \\ & + \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}' \rangle + \langle \rho(i\gamma + \underline{\underline{m}})\mathbf{w}, \mathbf{w}' \rangle \\ & + \langle \mathbf{v}, \mathbf{v}' \rangle + C_1 \langle K_G \mathbf{v}, K_G \mathbf{v}' \rangle_{\mathbf{V}} + \langle M\mathbf{w}, M\mathbf{w}' \rangle \\ & + C_2(\langle \mathbf{v}, \mathbf{v}' \rangle + \langle K_G \mathbf{v}, K_G \mathbf{v}' \rangle_{\mathbf{V}} + \langle M\mathbf{w}, M\mathbf{w}' \rangle + \langle \operatorname{mean}(\mathbf{q} \cdot \mathbf{w}), \operatorname{mean}(\mathbf{q} \cdot \mathbf{w}') \rangle) \\ & + \langle c_s^2 \rho \mathbf{q} \cdot \mathbf{v}, \operatorname{div} \mathbf{v}' \rangle + \langle c_s^2 \rho \operatorname{div} \mathbf{v}, \mathbf{q} \cdot \mathbf{v}' \rangle - \langle \rho(\omega + i\Omega\times)\mathbf{v}, (\omega + i\Omega\times)\mathbf{v}' \rangle \\ & - \langle \rho(\omega + i\Omega\times)\mathbf{v}, i\partial_{\mathbf{b}} \mathbf{v}' \rangle - \langle \rho i \partial_{\mathbf{b}} \mathbf{v}, (\omega + i\Omega\times)\mathbf{v}' \rangle - i\omega \langle \gamma \rho \mathbf{v}, \mathbf{v}' \rangle - \langle \rho \underline{\underline{m}} \mathbf{v}, \mathbf{v}' \rangle \\ & - \langle \rho \underline{\underline{m}} \mathbf{v}, \mathbf{w}' \rangle - i\omega \langle \gamma \rho \mathbf{v}, \mathbf{w}' \rangle - \langle c_s^2 \rho P_{L_0^2}(\mathbf{q} \cdot \mathbf{v}), P_{L_0^2}(\mathbf{q} \cdot \mathbf{w}') \rangle \\ & - \langle \rho(\omega + i\Omega\times)\mathbf{v}, (\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}' \rangle - \langle c_s^2 \rho(\operatorname{div} + P_{L_0^2} \mathbf{q}\cdot)\mathbf{v}, M\mathbf{w}' \rangle \\ & + \langle c_s^2 \rho \operatorname{mean}(\mathbf{q} \cdot \mathbf{v}), \operatorname{div} \mathbf{w}' \rangle + \langle c_s^2 \rho \operatorname{div} \mathbf{v}, \operatorname{mean}(\mathbf{q} \cdot \mathbf{w}') \rangle \\ & + \langle \rho \underline{\underline{m}} \mathbf{w}, \mathbf{v}' \rangle + i\omega \langle \gamma \rho \mathbf{w}, \mathbf{v}' \rangle + \langle c_s^2 \rho P_{L_0^2}(\mathbf{q} \cdot \mathbf{w}), P_{L_0^2}(\mathbf{q} \cdot \mathbf{v}') \rangle \\ & + \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}, (\omega + i\Omega\times)\mathbf{v}' \rangle + \langle c_s^2 \rho M\mathbf{w}, (\operatorname{div} + P_{L_0^2} \mathbf{q}\cdot)\mathbf{v}' \rangle \\ & - \langle c_s^2 \rho \operatorname{div} \mathbf{w}, \operatorname{mean}(\mathbf{q} \cdot \mathbf{v}') \rangle - \langle c_s^2 \rho \operatorname{div} \mathbf{w}, \operatorname{mean}(\mathbf{q} \cdot \mathbf{v}') \rangle \\ & - \langle c_s^2 \rho \operatorname{mean}(\mathbf{q} \cdot \mathbf{w}), \operatorname{div} \mathbf{w}' \rangle - \langle c_s^2 \rho \operatorname{div} \mathbf{w}, \operatorname{mean}(\mathbf{q} \cdot \mathbf{w}') \rangle - \langle c_s^2 \rho M\mathbf{w}, M\mathbf{w}' \rangle \end{aligned}$$

and

$$\langle K\mathbf{u}, \mathbf{u}' \rangle_{\mathbb{X}} := -C_2 \langle \mathbf{v}, \mathbf{v}' \rangle - (C_1 + C_2) \langle K_G \mathbf{v}, K_G \mathbf{v}' \rangle_{\mathbf{V}} - (1 + C_2) \langle M\mathbf{w}, M\mathbf{w}' \rangle \\ - C_2 \langle \operatorname{mean}(\mathbf{q} \cdot \mathbf{w}), \operatorname{mean}(\mathbf{q} \cdot \mathbf{w}') \rangle$$

for all $\mathbf{u}, \mathbf{u}' \in \mathbb{X}$. In addition, the coercivity of B follows along the same lines of the respective proof for B_n. To prove the asymptotic consistency of B, B_n we first show the asymptotic consistency of K, K_n. Thus let $\mathbf{u} \in \mathbb{X}$ be given. We need to show that $\lim_{n\to\infty} \|p_n K\mathbf{u} - K_n p_n \mathbf{u}\|_{\mathbb{X}_n} = 0$. Let $\mathbf{u}'_n \in \mathbb{X}_n$, $\|\mathbf{u}'_n\|_{\mathbb{X}_n} = 1$, $n \in \mathbb{N}$ be such that $\|p_n K\mathbf{u} - K_n p_n \mathbf{u}\|_{\mathbb{X}_n} \leq |\langle p_n K\mathbf{u} - K_n p_n \mathbf{u}, \mathbf{u}'_n\rangle_{\mathbb{X}_n}| + 1/n$. Let $\mathbb{N}' \subset \mathbb{N}$ be an arbitrary subsequence. Due to Lemma 8 there exist $\mathbb{N}'' \subset \mathbb{N}'$ and $\mathbf{u}' \in \mathbb{X}$ such that $\mathbf{u}'_n \xrightarrow{\mathbf{L}^2} \mathbf{u}'$, $\operatorname{div} \mathbf{u}'_n \xrightarrow{L^2} \operatorname{div} \mathbf{u}'$, $D^n_{\mathbf{b}} \mathbf{u}'_n \xrightarrow{\mathbf{L}^2} \partial_{\mathbf{b}} \mathbf{u}'$, and we conveniently compute

$$\begin{aligned}\langle p_n K\mathbf{u}, \mathbf{u}'_n\rangle_{\mathbb{X}_n} &= \langle \operatorname{div} K\mathbf{u}, \operatorname{div} \mathbf{u}'_n\rangle + \langle K\mathbf{u}, \mathbf{u}'_n\rangle + \langle \partial_{\mathbf{b}} K\mathbf{u}, D^n_{\mathbf{b}} \mathbf{u}'_n\rangle \\ &\xrightarrow{n\in\mathbb{N}''} \langle \operatorname{div} K\mathbf{u}, \operatorname{div} \mathbf{u}'\rangle + \langle K\mathbf{u}, \mathbf{u}'\rangle + \langle \partial_{\mathbf{b}} B\mathbf{u}, \partial_{\mathbf{b}} \mathbf{u}'\rangle = \langle K\mathbf{u}, \mathbf{u}'\rangle_{\mathbb{X}}.\end{aligned}$$

On the other hand we estimate

$$\begin{aligned}|\langle \mathbf{v}, \mathbf{v}'_n\rangle - \langle P_{V_n} p_n \mathbf{u}, \mathbf{v}'_n\rangle| &= |\langle P_V \mathbf{u} - \pi^d_n P_{\tilde{V}_n} p_n \mathbf{u}, \mathbf{v}'_n\rangle| \\ &\lesssim |\langle P_V \mathbf{u} - \pi^d_n P_{\tilde{V}_n} \mathbf{u}, \mathbf{v}'_n\rangle| + d_n(\mathbf{u}, p_n \mathbf{u}) \\ &\lesssim |\langle P_V \mathbf{u} - \pi^d_n P_V \mathbf{u}, \mathbf{v}'_n\rangle| + d_n(\mathbf{u}, p_n \mathbf{u}) \\ &\quad + \|(P_{L^2_0} - \pi^l_n)(\mathbf{q}\cdot\mathbf{u})\|_{L^2} \\ &\lesssim h_n \|P_V \mathbf{u}\|_{\mathbf{H}^1} + d_n(\mathbf{u}, p_n \mathbf{u}) + \|(P_{L^2_0} - \pi^l_n)(\mathbf{q}\cdot\mathbf{u})\|_{L^2}\end{aligned}$$

and

$$\begin{aligned}&|\langle K_G \mathbf{v}, K_G P_{\tilde{V}_n} \mathbf{u}'_n\rangle_{\mathbf{V}} - \langle K_G P_{\tilde{V}_n} p_n \mathbf{u}, K_G P_{\tilde{V}_n} \mathbf{u}'_n\rangle_{\mathbf{V}}| \\ &\quad = |\langle K_G(P_V \mathbf{u} - P_{\tilde{V}_n} p_n \mathbf{u}), K_G P_{\tilde{V}_n} \mathbf{u}'_n\rangle_{\mathbf{V}}| \\ &\quad \lesssim |\langle K_G(P_V \mathbf{u} - P_{\tilde{V}_n} \mathbf{u}), K_G P_{\tilde{V}_n} \mathbf{u}'_n\rangle_{\mathbf{V}}| + d_n(\mathbf{u}, p_n \mathbf{u}) \\ &\quad \lesssim \|(P_{L^2_0} - \pi^l_n)(\mathbf{q}\cdot\mathbf{u})\|_{L^2} + d_n(\mathbf{u}, p_n \mathbf{u})\end{aligned}$$

and

$$\begin{aligned}&|\langle M\mathbf{w}, M\mathbf{w}'_n\rangle_{\mathbf{V}} - \langle M\mathbf{w}_n(p_n \mathbf{u}), M\mathbf{w}'_n\rangle| \\ &\quad = |\langle M(\mathbf{w} - \mathbf{w}_n(p_n \mathbf{u})), M\mathbf{w}'_n\rangle| \\ &\quad = |\langle M(\mathbf{u} - P_V \mathbf{u} - (p_n \mathbf{u} - P_{V_n} p_n \mathbf{u})), M\mathbf{w}'_n\rangle| \\ &\quad \lesssim \|\mathbf{u} - P_V \mathbf{u} - (p_n \mathbf{u} - P_{V_n} p_n \mathbf{u})\|_{H(\operatorname{div})} \\ &\quad \lesssim \|P_V \mathbf{u} - P_{V_n} p_n \mathbf{u}\|_{H(\operatorname{div})} + d_n(\mathbf{u}, p_n \mathbf{u}) \\ &\quad \lesssim \|p_n P_V \mathbf{u} - P_{V_n} p_n \mathbf{u}\|_{H(\operatorname{div})} + d_n(\mathbf{u}, p_n \mathbf{u}) + d_n(P_V \mathbf{u}, p_n P_V \mathbf{u})\end{aligned}$$

and

$$\begin{aligned}&|\langle \operatorname{mean}(\mathbf{q}\cdot\mathbf{w}), \operatorname{mean}(\mathbf{q}\cdot\mathbf{w}'_n)\rangle_{\mathbf{V}} - \langle \operatorname{mean}(\mathbf{q}\cdot\mathbf{w}_n(p_n \mathbf{u})), \operatorname{mean}(\mathbf{q}\cdot\mathbf{w}'_n)\rangle| \\ &\quad \lesssim \|\mathbf{u} - P_V \mathbf{u} - (p_n \mathbf{u} - P_{V_n} p_n \mathbf{u})\|_{H(\operatorname{div})}\end{aligned}$$

$$\lesssim \|p_n P_V \mathbf{u} - P_{V_n} p_n \mathbf{u}\|_{H(\mathrm{div})} + d_n(\mathbf{u}, p_n \mathbf{u}) + d_n(P_V \mathbf{u}, p_n P_V \mathbf{u}).$$

Thus

$$\lim_{n\to\infty} |\langle K_n p_n \mathbf{u}, \mathbf{u}'_n\rangle_{\mathbb{X}_n} + C_2\langle \mathbf{v}, \mathbf{v}'_n\rangle + (C_1 + C_2)\langle K_G \mathbf{v}, K_G P_{\tilde{V}_n} \mathbf{u}'_n\rangle_{\mathbf{V}} \\ + (1 + C_2)\langle M\mathbf{w}, M\mathbf{w}'_n\rangle + C_2\langle \mathrm{mean}(\mathbf{q}\cdot\mathbf{w}), \mathrm{mean}(\mathbf{q}\cdot\mathbf{w}'_n)\rangle| = 0.$$

We further use the operator $S := \nabla\big((\mathrm{div} + P_{L^2_0}\mathbf{q}\cdot + M)\nabla\big)^{-1} \in L(L^2_0, \mathbf{V})$ (where we changed the space $\mathbf{L}^2$ to $\mathbf{V}$ compared to the proof of Lemma 17) and compute

$$\begin{aligned}
\langle K_G \mathbf{v}, K_G P_{\tilde{V}_n}\mathbf{u}'_n\rangle_{\mathbf{V}} &= \langle K_G^* K_G \mathbf{v}, P_{\tilde{V}_n}\mathbf{u}'_n\rangle_{\mathbf{V}}\\
&= \langle K_G^* K_G \mathbf{v}, P_V \mathbf{u}'_n\rangle_{\mathbf{V}} + \langle K_G^* K_G \mathbf{v}, S(\pi^l_n - P_{L^2_0})(\mathbf{q}\cdot\mathbf{u}'_n)\rangle_{\mathbf{V}}\\
&= \langle P_V^* K_G^* K_G \mathbf{v}, \mathbf{u}'_n\rangle_{H_0(\mathrm{div})} + \langle(\pi^l_n - P_{L^2_0})S^* K_G^* K_G \mathbf{v}, \mathbf{q}\cdot\mathbf{u}'_n\rangle_{L^2}\\
&\xrightarrow{n\in\mathbb{N}''} \langle P_V^* K_G^* K_G \mathbf{v}, \mathbf{u}'\rangle_{H_0(\mathrm{div})} = \langle K_G \mathbf{v}, K_G \mathbf{v}'\rangle_{\mathbf{V}}
\end{aligned}$$

and hence

$$\begin{aligned}
\lim_{n\in\mathbb{N}''} \Big(&C_2\langle \mathbf{v}, \mathbf{v}'_n\rangle + (C_1 + C_2)\langle K_G\mathbf{v}, K_G P_{\tilde{V}_n}\mathbf{u}'_n\rangle_{\mathbf{V}} + (1 + C_2)\langle M\mathbf{w}, M\mathbf{w}'_n\rangle\\
&+ C_2\langle \mathrm{mean}(\mathbf{q}\cdot\mathbf{w}), \mathrm{mean}(\mathbf{q}\cdot\mathbf{w}'_n)\rangle\Big)\\
&= C_2\langle \mathbf{v}, \mathbf{v}'\rangle + (C_1 + C_2)\langle K_G\mathbf{v}, K_G\mathbf{v}'\rangle_{\mathbf{V}} + (1 + C_2)\langle M\mathbf{w}, M\mathbf{w}'\rangle\\
&+ C_2\langle \mathrm{mean}(\mathbf{q}\cdot\mathbf{w}), \mathrm{mean}(\mathbf{q}\cdot\mathbf{w}')\rangle\\
&= \langle K\mathbf{u}, \mathbf{u}'\rangle_{\mathbb{X}}.
\end{aligned}$$

Thus we obtained that $\lim_{n\to\infty}\|p_n K\mathbf{u} - K_n p_n \mathbf{u}\|_{\mathbb{X}_n} = 0$ for each $\mathbf{u} \in \mathbb{X}$. It remains to recall $B_n = A_n T_n - K_n$, $B = AT - K$ and to estimate

$$\begin{aligned}
&\|(p_n B - B_n p_n)\mathbf{u}\|_{\mathbb{X}_n}\\
&\le \|(p_n K - K_n p_n)\mathbf{u}\|_{\mathbb{X}_n} + \|(p_n AT - A_n T_n p_n)\mathbf{u}\|_{\mathbb{X}_n}\\
&\le \|(p_n K - K_n p_n)\mathbf{u}\|_{\mathbb{X}_n} + \|(p_n A - A_n p_n)T\mathbf{u}\|_{\mathbb{X}_n}\\
&\quad + \|A_n\|_{L(\mathbb{X}_n)}\|(p_n T - T_n p_n)\mathbf{u}\|_{\mathbb{X}_n}.
\end{aligned}$$

Thus $\lim_{n\to\infty}\|(p_n B - B_n p_n)\mathbf{u}\|_{\mathbb{X}_n} = 0$ follows from the just proven asymptotic consistency of K_n, K, Theorem 9, Lemma 15 and from the uniform boundedness of $(A_n)_{n\in\mathbb{N}}$. □

4.4 Convergence results

Theorem 19 *Let $\boldsymbol{f} \in \boldsymbol{L}^2$ and $\boldsymbol{u} \in \mathbb{X}$ be the solution to $a(\boldsymbol{u}, \boldsymbol{u}') = \langle \boldsymbol{f}, \boldsymbol{u}'\rangle$ for all $\boldsymbol{u}' \in \mathbb{X}$. Then, there exists an $n_0 > 0$ such that for all $n > n_0$ the solution $\boldsymbol{u}_n \in \mathbb{X}_n$ to $a_n(\boldsymbol{u}_n, \boldsymbol{u}'_n) = \langle \boldsymbol{f}, \boldsymbol{u}'_n\rangle$ for all $\boldsymbol{u}'_n \in \mathbb{X}_n$ exists and $\lim_{n\to\infty} d_n(\boldsymbol{u}, \boldsymbol{u}_n) = 0$. In addition,* *if* $\boldsymbol{u} \in \boldsymbol{H}^{1+s}$, $\operatorname{div} \boldsymbol{u} \in \boldsymbol{H}^1$, $\rho \in W^{1+s,\infty}$ *and* $\boldsymbol{b} \in \boldsymbol{W}^{1+s,\infty}$ *with* $s > 0$, *then* $d_n(\boldsymbol{u}, \boldsymbol{u}_n) \lesssim h_n^{\min(1+s,k)} + h_n^{\min(s,l)}$.

Proof Due to Theorems 9 and 18 we can apply Theorem 1. Since A is injective and hence bijective Lemma 2 yields that $(A_n)_{n\in\mathbb{N}}$ is stable. Let $\mathbf{g} \in \mathbb{X}$ be such that $\langle \mathbf{g}, \mathbf{u}' \rangle_{\mathbb{X}} = \langle \mathbf{f}, \mathbf{u}' \rangle_{\mathbf{L}^2}$ for all $\mathbf{u}' \in \mathbb{X}$ and $\mathbf{g}_n \in \mathbb{X}_n$ be such that $\langle \mathbf{g}_n, \mathbf{u}_n' \rangle_{\mathbb{X}_n} = \langle \mathbf{f}, \mathbf{u}_n' \rangle_{\mathbf{L}^2}$ for all $\mathbf{u}_n' \in \mathbb{X}_n$. To obtain that $\lim_{n\to\infty} d_n(\mathbf{u}, \mathbf{u}_n) = 0$ it remains to show that $\lim_{n\to\infty} \|p_n\mathbf{g} - \mathbf{g}_n\|_{\mathbb{X}_n} = 0$. We proceed conveniently and choose $\mathbf{u}_n' \in \mathbb{X}_n$, $\|\mathbf{u}_n'\|_{\mathbb{X}_n} = 1$, $n \in \mathbb{N}$ such that $\|p_n\mathbf{g} - \mathbf{g}_n\|_{\mathbb{X}_n} \leq |\langle p_n\mathbf{g} - \mathbf{g}_n, \mathbf{u}_n' \rangle_{\mathbb{X}_n}| + 1/n$. For an arbitrary subsequence $\mathbb{N}' \subset \mathbb{N}$ we choose $\mathbf{u}' \in \mathbb{X}$ and $\mathbb{N}'' \subset \mathbb{N}'$ as in Lemma 8 and obtain that

$$\begin{aligned}\langle p_n\mathbf{g} - \mathbf{g}_n, \mathbf{u}_n' \rangle_{\mathbb{X}_n} &= \langle \operatorname{div} \mathbf{g}, \operatorname{div} \mathbf{u}_n' \rangle + \langle \mathbf{g}, \mathbf{u}_n' \rangle + \langle \partial_{\mathbf{b}}\mathbf{g}, D_{\mathbf{b}}^n \mathbf{u}_n' \rangle - \langle \mathbf{f}, \mathbf{u}_n' \rangle \\ &\xrightarrow{n\in\mathbb{N}''} \langle \mathbf{g}, \mathbf{u}' \rangle_{\mathbb{X}} - \langle \mathbf{f}, \mathbf{u}' \rangle = 0,\end{aligned}$$

from which it follows that $\lim_{n\to\infty} \|p_n\mathbf{g} - \mathbf{g}_n\|_{\mathbb{X}_n} = 0$, and hence $\lim_{n\to\infty} d_n(\mathbf{u}, \mathbf{u}_n) = 0$. To obtain the convergence rate we first estimate

$$d_n(\mathbf{u}, \mathbf{u}_n) \leq d_n(\mathbf{u}, p_n\mathbf{u}) + \|p_n\mathbf{u} - \mathbf{u}_n\|_{\mathbb{X}_n} \lesssim d_n(\mathbf{u}, p_n\mathbf{u}) + \|A_n(p_n\mathbf{u} - \mathbf{u}_n)\|_{\mathbb{X}_n}$$

and further compute that

$$\begin{aligned}\|A_n(p_n\mathbf{u} - \mathbf{u}_n)\|_{\mathbb{X}_n} &= \sup_{\|\mathbf{u}_n'\|_{\mathbb{X}_n}=1} |a_n(p_n\mathbf{u} - \mathbf{u}_n, \mathbf{u}_n')| = O\big(d_n(\mathbf{u}, p_n\mathbf{u}), n \to \infty\big) \\ &+ \sup_{\|\mathbf{u}_n'\|_{\mathbb{X}_n}=1} \Big|\langle c_s^2\rho \operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{u}_n' \rangle - \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}, (\omega + iD_{\mathbf{b}}^n + i\Omega\times)\mathbf{u}_n' \rangle \\ &+ \langle \operatorname{div} \mathbf{u}, \nabla p \cdot \mathbf{u}_n' \rangle + \langle \nabla p \cdot \mathbf{u}, \operatorname{div} \mathbf{u}_n' \rangle + \langle (\operatorname{Hess}(p) - \rho \operatorname{Hess}(\phi))\mathbf{u}, \mathbf{u}_n' \rangle \\ &- i\omega\langle \gamma\rho\mathbf{u}, \mathbf{u}_n' \rangle - \langle \mathbf{f}, \mathbf{u}_n' \rangle_{\mathbf{L}^2}\Big|.\end{aligned}$$

The next step is to integrate by parts

$$\langle c_s^2\rho \operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{u}_n' \rangle = -\langle \nabla(c_s^2\rho \operatorname{div} \mathbf{u}), \mathbf{u}_n' \rangle, \quad \langle \nabla p \cdot \mathbf{u}, \operatorname{div} \mathbf{u}_n' \rangle = -\langle \nabla(\nabla p \cdot \mathbf{u}), \mathbf{u}_n' \rangle,$$

where the left arguments in the $\mathbf{L}^2$-scalar products of the former right-hand sides are in $\mathbf{L}^2$ due to the assumptions $\mathbf{u} \in \mathbf{H}^1$, $\operatorname{div} \mathbf{u} \in H^1$, $c_s, \rho \in W^{1,\infty}$ and $p \in W^{2,\infty}$. Note that due these assumptions and because $\mathbf{u}$ solves the PDE with $\mathbf{f} \in \mathbf{L}^2$ we also have $\partial_{\mathbf{b}}\partial_{\mathbf{b}}\mathbf{u} \in \mathbf{L}^2$. The integration by parts of the operator $D_{\mathbf{b}}^n$ is performed similarly as in the proof of Lemma 8. Thus let $\boldsymbol{\psi}_n \in \mathbf{Q}_n$ be a suitable $\mathbf{H}^1$ projection of $\rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}$, e.g. $\boldsymbol{\psi}_n = \mathcal{J}_n(\rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u})$ with $\mathcal{J}_n$ as in [24, (6.4)]. We compute

$$\langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}, D_{\mathbf{b}}^n\mathbf{u}_n' \rangle = \langle \boldsymbol{\psi}_n, D_{\mathbf{b}}^n\mathbf{u}_n' \rangle + \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u} - \boldsymbol{\psi}_n, D_{\mathbf{b}}^n\mathbf{u}_n' \rangle$$

and

$$\langle \boldsymbol{\psi}_n, D_{\mathbf{b}}^n\mathbf{u}_n' \rangle = \sum_{\tau\in\mathcal{T}_n} \langle \boldsymbol{\psi}_n, \partial_{\mathbf{b}}\mathbf{u}_n' + R_n\mathbf{u}_n' \rangle_{\mathbf{L}^2(\tau)}$$

$$
\begin{aligned}
&= \sum_{\tau\in\mathcal{T}_n} \langle \boldsymbol{\psi}_n, \partial_{\mathbf{b}} \mathbf{u}_n' \rangle_{\mathbf{L}^2(\tau)} - \langle \{\boldsymbol{\psi}_n\}, [\![\mathbf{u}_n']\!]_{\mathbf{b}} \rangle_{\mathcal{F}_n^{\text{int}}} \\
&= \sum_{\tau\in\mathcal{T}_n} \langle \boldsymbol{\psi}_n, \partial_{\mathbf{b}} \mathbf{u}_n' \rangle_{\mathbf{L}^2(\tau)} - \langle \boldsymbol{\psi}_n, (\boldsymbol{\nu}\cdot\mathbf{b})\mathbf{u}_n' \rangle_{\mathbf{L}^2(\partial\tau)} \\
&= -\langle (\partial_{\mathbf{b}} + \operatorname{div}(\mathbf{b}))\boldsymbol{\psi}_n, \mathbf{u}_n' \rangle \\
&= -\langle (\partial_{\mathbf{b}} + \operatorname{div}(\mathbf{b}))\rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}, \mathbf{u}_n' \rangle \\
&\quad + \langle (\partial_{\mathbf{b}} + \operatorname{div}(\mathbf{b}))\big(\boldsymbol{\psi}_n - \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}\big), \mathbf{u}_n' \rangle \\
&= -\langle \rho\partial_{\mathbf{b}}(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}, \mathbf{u}_n' \rangle - \langle \operatorname{div}(\rho\mathbf{b})(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}, \mathbf{u}_n' \rangle \\
&\quad + \langle (\partial_{\mathbf{b}} + \operatorname{div}(\mathbf{b}))\big(\boldsymbol{\psi}_n - \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}\big), \mathbf{u}_n' \rangle.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sup_{\|\mathbf{u}_n'\|_{\mathbb{X}_n}=1} \Big| &\langle c_s^2 \rho \operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{u}_n' \rangle - \langle \rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}, (\omega + iD_{\mathbf{b}}^n + i\Omega\times)\mathbf{u}_n' \rangle \\
&+ \langle \operatorname{div}\mathbf{u}, \nabla p \cdot \mathbf{u}_n' \rangle + \langle \nabla p\cdot\mathbf{u}, \operatorname{div}\mathbf{u}_n' \rangle + \langle (\operatorname{Hess}(p) - \rho\operatorname{Hess}(\phi))\mathbf{u}, \mathbf{u}_n' \rangle \\
&- i\omega\langle \gamma\rho\mathbf{u}, \mathbf{u}_n' \rangle - \langle \mathbf{f}, \mathbf{u}_n' \rangle_{\mathbf{L}^2} \Big| \\
&\lesssim \|\rho(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u} - \boldsymbol{\psi}_n\|_{\mathbf{H}^1} \lesssim h_n^{\min(s,l)}
\end{aligned}
$$

due to the properties of $\mathcal{J}_n$ and (5). Since we can estimate $d_n(\mathbf{u}, p_n\mathbf{u})$ with Lemma 4 the claim follows. □

Remark 2 The unusual regularity assumptions and convergence rate $h_n^{\min(1+s,k)} + h_n^{\min(s,l)}$ of Theorem 19 deserve some discussion. First we note that for a right-hand side $\mathbf{f} \in \mathbf{L}^2$ it follows that $-\nabla(c_s^2\rho\operatorname{div}\mathbf{u}) + \rho\partial_{\mathbf{b}}\partial_{\mathbf{b}}\mathbf{u} \in \mathbf{L}^2$, which neither allows us to deduce that $\nabla(c_s^2\rho\operatorname{div}\mathbf{u}) \in \mathbf{L}^2$ nor that $\rho\partial_{\mathbf{b}}\partial_{\mathbf{b}}\mathbf{u} \in \mathbf{L}^2$. This is the reason why we explicitly need to assume that $\mathbf{u} \in \mathbf{H}^1$ and $\operatorname{div}\mathbf{u} \in H^1$.

If the polynomial degree l of the lifting operator R_n is chosen as $l = k$, and if $k \leq s$, then we obtain the convenient rate h_n^k. However, if the solution and the parameters have only a maximal regularity $s < \infty$ and $k \geq 1 + s$, then we obtain only the rate h_n^s, which is one power less then the conveniently expected rate. The reason for this unusual result is that we employ a DG method without a ($\mathbf{b}$-jump) stabilization term, which results in a weaker norm.

Remark 3 In principle the previous analysis of this article can also be applied to other DG variants such as the symmetric interior penalty method, where we note that the sign of the coefficient of the penalty term $\langle \mathfrak{h}^{-1}[\![\mathbf{u}_n]\!]_{\mathbf{b}}, [\![\mathbf{u}_n']\!]_{\mathbf{b}} \rangle_{\mathcal{F}_n^{\text{int}}}$ in the sesquilinear form needs to be negative (or have a suitable complex sign). Of course, the norm $\|\cdot\|_{\mathbb{X}_n}$ and everything related would need to be adapted. Further note that it follows along the lines of the proof of [27, Theorem 8] that there exists a constant $C > 0$ such that

$$
\begin{aligned}
\|(\mathfrak{h}\rho)^{1/2}\{\partial_{\mathbf{b}}\mathbf{u}_n\}\|_{\mathcal{F}_n^{\text{int}}} &\leq C(\|\rho^{1/2}\partial_{\mathbf{b}}\mathbf{u}_n\|_{\mathbf{L}^2} + \|\rho^{1/2}\mathbf{u}_n\|_{\mathbf{L}^2}), \\
\|(\mathfrak{h}\rho)^{1/2}\{(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}_n\}\|_{\mathcal{F}_n^{\text{int}}} &\leq C(\|\rho^{1/2}(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{u}_n\|_{\mathbf{L}^2} + \|\rho^{1/2}\mathbf{u}_n\|_{\mathbf{L}^2})
\end{aligned}
$$

for all $\mathbf{u}_n \in \mathbb{X}_n$, $n \in \mathbb{N}$, where $\partial_{\mathbf{b}}\mathbf{u}_n$ is interpreted piece-wise with respect to the mesh $\mathcal{T}_n$. Note the inconvenient term $\|\rho^{1/2}\mathbf{u}_n\|_{\mathbf{L}^2}$ in the former right hand-sides. Then it follows that for a large enough penalty parameter $\alpha > 0$ the following coercivity estimate holds

$$\begin{aligned}
&\Big|\|\rho^{1/2}(i\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}_n\|^2_{\mathbf{L}^2} + \alpha\|\mathfrak{h}^{-1/2}[\![\mathbf{w}_n]\!]_{\mathbf{b}}\|^2_{\mathcal{F}_n^{\text{int}}} + i\omega\|(\rho\gamma)^{1/2}\mathbf{w}_n\|^2_{\mathbf{L}^2} \\
&\quad - \langle\rho\{(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}_n\}, [\![\mathbf{w}_n]\!]_{\mathbf{b}}\rangle_{\mathcal{F}_n^{\text{int}}} - \langle[\![\mathbf{w}_n]\!]_{\mathbf{b}}, \rho\{(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}_n\}\rangle_{\mathcal{F}_n^{\text{int}}}\Big| \\
&\quad \gtrsim \|(\omega + i\partial_{\mathbf{b}} + i\Omega\times)\mathbf{w}_n\|^2_{\mathbf{L}^2} + \|\mathfrak{h}^{-1/2}[\![\mathbf{w}_n]\!]_{\mathbf{b}}\|^2_{\mathcal{F}_n^{\text{int}}} + \|(\rho\gamma)^{1/2}\mathbf{w}_n\|^2_{\mathbf{L}^2}.
\end{aligned} \tag{30}$$

Now the crux is that in the proof of Theorem 18 we multiply with $e^{-i(\theta+\tau)\,\mathrm{sgn}\,\omega}$ and hence the real part of the coefficient of $\|(\rho\gamma)^{1/2}\mathbf{u}_n\|^2_{\mathbf{L}^2}$ depends on θ and will become small for large θ. Thus α will depend on θ as well and has to be chosen sufficiently large to guarantee a coercivity estimate of kind (30). Until now this produces no severe drawbacks. However, when repeating the respective estimates in the proof of Theorem 18 the additional term $-\alpha\|\mathfrak{h}^{-1/2}[\![\pi_n^d\tilde{\mathbf{v}}]\!]_{\mathbf{b}}\|^2_{\mathcal{F}_n^{\text{int}}}$ needs to be estimated by $|\nabla\tilde{\mathbf{v}}|^2_{\mathbf{H}^1_{c_s^\rho}}$, which leads to a more restrictive assumption on the smallness of the Mach number. We conclude that the penalty parameter α needs to be balanced in a nontrivial way to guarantee a coercivity estimate for $\mathbf{w}_n$ while avoiding an unnecessarily restrictive assumption on the smallness of the Mach number.

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