



Visibility in Hypercubes

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Abstract

A subset M of vertices in a graph G is a mutual-visibility set if any two vertices u and v in M “see” each other in G , that is, there exists a shortest u, v -path in G that contains no elements of M as internal vertices. The mutual-visibility number $\mu(G)$ of a graph G is the largest size of a mutual-visibility set in G . Let $n \in \mathbb{N}$ and Q_n be an n -dimensional hypercube. Cicerone, Di Fonso, Di Stefano, Navarra, and Piselli showed that $2^n / \sqrt{n} \leq \mu(Q_n) \leq 2^{n-1}$. In this paper, we prove that $\mu(Q_n) > 0.186 \cdot 2^n$ and thus establish that $\mu(Q_n) = \Theta(2^n)$. We also consider the chromatic mutual-visibility number, $\chi_\mu(G)$, defined as the smallest number of colors used on vertices of G , such that every color class is a mutual-visibility set in G . Klavžar, Kuziak, Valenzuela-Tripodoro, and Yero asked whether $\chi_\mu(Q_n) = O(1)$. We answer their question in the negative, namely, we show that $\chi_\mu(Q_n)$ is a growing function of n . Moreover, we show that $\chi_\mu(Q_n) = O(\log \log n)$. Finally, we study the so-called total mutual-visibility number of graphs and give asymptotically tight bounds on this parameter for hypercubes.

Keywords Visibility · Mutual-visibility · Hypercubes · Daisies

1 Introduction

Let G be a simple graph and $M \subseteq V(G)$. For any two vertices $u, v \in V(G)$, we say that u and v are M -visible if there exists a shortest u, v -path in G that contains no vertices from $M \setminus \{u, v\}$. Note that any two adjacent vertices are M -visible.

Definition 1.1 Let M be a subset of vertices in a graph G . We say that M is a mutual-visibility set if any two vertices $u, v \in M$ are M -visible. The largest size of a mutual-

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visibility set in G is denoted by $\mu(G)$ and called the mutual-visibility number of G .

The systematic investigation of mutual-visibility in graphs was pioneered by Di Stefano [12], and has garnered extensive attention and subsequent research [4, 5, 7–11, 19, 20, etc.] in recent years. The exact value of $\mu(G)$ is only known for very few graph classes, e.g., cliques, complete bipartite graphs, trees, cycles, grids [12], butterflies [7], Cartesian products of paths and cycles [19], Cartesian products of cliques, and direct products of cliques [10]. It has been shown in the initial work by Di Stefano [12] that determining $\mu(G)$ for any given graph G is NP-complete. By considering the mutual-visibility set consisting of all neighbors of the maximum degree vertex, we can find a large mutual-visibility set in any dense graph. As the mutual-visibility problem was initially motivated by establishing efficient and confidential communication in networks, the research on $\mu(G)$ mainly focuses on sparse and highly connected graphs, such as product graphs and hypercube-like graphs.

The primary objective of this paper is to explore the mutual-visibility phenomenon in hypercubes. For $n \in \mathbb{N}$, the n -dimensional hypercube Q_n is a graph on the vertex set $2^{[n]}$, where two vertices $A, B \in 2^{[n]}$ form an edge in Q_n if and only if their symmetric difference $A \Delta B := (A \setminus B) \cup (B \setminus A)$ has size 1. Cicerone, Di Fonso, Di Stefano, Navarra, and Piselli [7] initially showed that $2^n / \sqrt{n} \leq \mu(Q_n) \leq 2^{n-1}$. The upper bound was later improved by Korže and Vesel [20] to $\mu(Q_n) \leq \frac{59}{128} \cdot 2^n$ for $n \geq 7$. Bodnár [2] recently obtained $\mu(Q_n) \leq \frac{1916879}{4718592} \cdot 2^n$ for sufficiently large n using flag algebras. Here, we prove that $\mu(Q_n)$ is at least a constant fraction of 2^n and hence determine $\mu(Q_n)$ up to a multiplicative constant.

Theorem 1.2 *For every $n \in \mathbb{N}$, we have $\mu(Q_n) > \frac{14}{75} \cdot 2^n$.*

We derive Theorem 1.2 by establishing a connection between mutual-visibility sets in Q_n and a special class of hypergraphs called daisies. Let $r, s, t \in \mathbb{N}$ with $\min\{r, s\} \geq t$. Following the notation of Bollobás, Leader, and Malvenuto [3], an (r, s, t) -daisy $\mathcal{D}_r(s, t)$ is a collection of all r -element sets T satisfying $A \subseteq T \subseteq B$, for some given $A \subseteq B$ with $|A| = r - t$ and $|B| = r + s - t$. The Turán number $\text{ex}(n, \mathcal{D}_r(s, t))$ is the largest size of a set family $\mathcal{F}_r \subseteq \binom{[n]}{r}$ containing no (r, s, t) -daisy. By constructing dense mutual-visibility sets in Q_n via daisy-free families (see Section 2.1), we obtain the following lower bound on $\mu(Q_n)$.

Theorem 1.3 *Let $n, d \in \mathbb{N}$ with $n \geq d \geq 3$. Then*

$$\mu(Q_n) \geq \frac{1}{d} \left(\sum_{r=d}^{n-d} \text{ex}(n, \mathcal{D}_r(2d, d)) + 2 \sum_{r=0}^{d-1} \binom{n}{r} \right).$$

Klavžar, Kuziak, Valenzuela-Tripodoro, and Yero [18] recently introduced a coloring version of the mutual-visibility problem.

Definition 1.4 Let G be a graph. The smallest number of colors in a vertex-coloring of G such that each color class is a mutual-visibility set is called the chromatic mutual-visibility number of G and denoted by $\chi_\mu(G)$.

Equivalently, $\chi_\mu(G)$ is the smallest integer such that $V(G)$ can be partitioned into $\chi_\mu(G)$ mutual-visibility sets. It is clear that $\chi_\mu(G) \geq |V(G)|/\mu(G)$. Naturally, one might ask whether $\chi_\mu(G) = O(|V(G)|/\mu(G))$ holds in general. Knowing that $\mu(Q_n) = \Omega(2^n)$ from an earlier arXiv version of this paper, Klavžar et al. [18] raised the following question:

Is there an absolute constant $C > 0$, such that $\chi_\mu(Q_n) \leq C$ holds for all $n \in \mathbb{N}$?

We answer their question in the negative with the following result. In particular, this shows that $\chi_\mu(G)$ can be arbitrarily far from the trivial lower bound $|V(G)|/\mu(G)$.

Theorem 1.5 *For any $q > 0$ there exists some $n_0 > 0$, such that $\chi_\mu(Q_n) > q$ for all $n \geq n_0$. On the other hand, $\chi_\mu(Q_n) = O(\log \log n)$.*

We also study a natural variant of mutual-visibility introduced by Cicerone, Di Stefano, Klavžar, and Yero [10].

Definition 1.6 Let G be a graph and $M \subseteq V(G)$. We say that M is a total mutual-visibility set if any two vertices $u, v \in V(G)$ are M -visible. The largest size of a total mutual-visibility set in G is denoted by $\mu_t(G)$ and called the total mutual-visibility number of G .

Since a total mutual-visibility set is always a mutual-visibility set, we have $\mu_t(G) \leq \mu(G)$. The exact value of $\mu_t(G)$ is also known for rather few graph classes, see [4–8, 10, 21, 27]. Furthermore, determining $\mu_t(G)$ for any given graph G , as demonstrated in [8], is also NP-complete. We obtain some tight bounds on $\mu_t(Q_n)$, improving the previous result $\mu_t(Q_n) \geq 2^{n-2}/(n^2 - n)$ by Bujtás, Klavžar, and Tian [6].

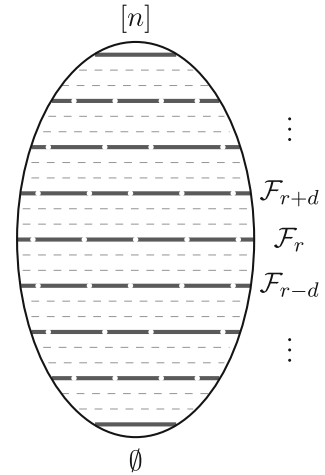
Theorem 1.7 *For every $n \in \mathbb{N}$, we have $2^{n-1}/n \leq \mu_t(Q_n) \leq 2^n/n$. If $n = 2^m - 1$ with $m \in \mathbb{N}$, then $2^n/(n + 1) \leq \mu_t(Q_n) \leq 2^n/n$.*

The remainder of this paper is organized as follows. In Section 2.1, we exhibit our construction of a large mutual-visibility set in Q_n and prove the key lemma. Section 2.2 is dedicated to proving Theorem 1.3. We derive Theorem 1.2 from Theorem 1.3 in Section 2.3. In Section 3 we establish Theorem 1.5. Finally, we prove Theorem 1.7 in Section 4. Section 5 contains concluding remarks and open questions.

2 Mutual-Visibility in Q_n

Given $n \in \mathbb{N}$ and $r \in \{0, 1, \dots, n\}$, the r^{th} layer \mathcal{L}_r of the hypercube Q_n is $\binom{[n]}{r}$, i.e., the family of all r -subsets of $[n] = \{1, \dots, n\}$. We say that the layer \mathcal{L}_r is between the layers $\mathcal{L}_{r'}$ and $\mathcal{L}_{r''}$ if $r' < r < r''$ or $r'' < r < r'$. We denote by $\text{dist}(A, B)$ the distance in Q_n between vertices A and B . Cicerone et al. [7, Theorem 1] observed that for any r , $\mathcal{L}_r \cup \mathcal{L}_{r+d}$ with $d \geq 3$ is a mutual-visibility set in Q_n , giving a lower bound $\mu(Q_n) \geq \Omega(2^n/\sqrt{n})$. Here we construct a larger mutual-visibility set.

Fig. 1 Construction of a dense mutual-visibility set.



2.1 Construction of a Mutual-Visibility Set

Let $d \geq 3$. Let \mathcal{F}_r be a $\mathcal{D}_r(2d, d)$ -free subfamily of \mathcal{L}_r for each $r \in \{d, \dots, n - d\}$ and let $\mathcal{F}_r = \mathcal{L}_r$ for $r \in \{0, \dots, d - 1\} \cup \{n - d + 1, \dots, n\}$. Let

$$M(\lambda) = \bigcup_{r \equiv \lambda \pmod{d}} \mathcal{F}_r.$$

We say that the layers of Q_n containing vertices of $M(\lambda)$ are *selected layers*. See Figure 1 for an illustration of $M(\lambda)$.

Lemma 2.1 *The set $M = M(\lambda)$ is a mutual-visibility set in Q_n for any $\lambda \in \mathbb{Z}$.*

Proof of Lemma 2.1 For any two vertices $A, B \in M$ we shall verify that they are M -visible. Cicerone et al. [7, Theorem 1] observed the following.

Claim 1. For any two vertices $A, B \in M \cap (\mathcal{L} \cup \mathcal{L}')$, where \mathcal{L} and \mathcal{L}' are two consecutive selected layers, there exists a shortest A, B -path in Q_n that is internally disjoint from $\mathcal{L} \cup \mathcal{L}'$ and only goes through the layers between \mathcal{L} and \mathcal{L}' .

A proof of Claim 1 is included in the appendix for completeness. This implies that any two vertices of M that are either from the same layer or from some consecutive selected layers “see” each other. For any other two vertices of M , we shall argue that there exists a shortest path between them going through the “holes” in the selected layers in between, see Figure 2.

Claim 2. For any $A, B \in 2^{[n]}$, where $|A| \leq r - d$ and $|B| \geq r + d$, there exists $C \in \mathcal{L}_r \setminus \mathcal{F}_r$ such that $A \cap B \subseteq C \subseteq A \cup B$.

Indeed, since $|A \cap B| \leq |A| \leq r - d$ and $|A \cup B| \geq |B| \geq r + d$, there are some $A' \in \mathcal{L}_{r-d}$ and $B' \in \mathcal{L}_{r+d}$ with $A \cap B \subseteq A' \subseteq B' \subseteq A \cup B$. Since \mathcal{F}_r is $\mathcal{D}_r(2d, d)$ -free, for any $A' \in \mathcal{L}_{r-d}$ and $B' \in \mathcal{L}_{r+d}$ with $A' \subseteq B'$, the family \mathcal{F}_r omits some

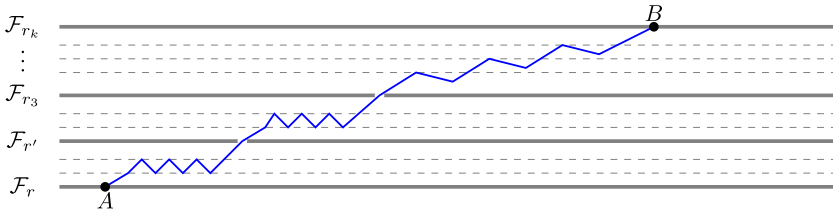


Fig. 2 A shortest A, B -path going through “holes” in the layers

$C \in \mathcal{L}_r$ satisfying $A' \subseteq C \subseteq B'$. In particular, $C \in \mathcal{L}_r \setminus \mathcal{F}_r$ and $A \cap B \subseteq C \subseteq A \cup B$. This proves Claim 2.

Claim 3. For any two vertices A, B of Q_n and any C such that $A \cap B \subseteq C \subseteq A \cup B$, there is a shortest A, B -path in Q_n containing C .

Note that $\text{dist}(A, B) = |A \Delta B|$. Let C be an arbitrary set with $A \cap B \subseteq C \subseteq A \cup B$. Then $\text{dist}(A, C) = |A \Delta C|$ and $\text{dist}(C, B) = |C \Delta B|$. Thus there is an A, B -walk W through C of length $|A \Delta C| + |C \Delta B|$. We have that $|A \Delta C| + |C \Delta B| = (|A \setminus C| + |C \setminus B|) + (|B \setminus C| + |C \setminus A|) = |A \setminus B| + |B \setminus A| = |A \Delta B|$. Since the length of W is $\text{dist}(A, B)$, W is a shortest A, B -path. This completes the proof of Claim 3.

Now, consider $A \in \mathcal{F}_r$ and $B \in \mathcal{F}_{r'}$ for $r' \geq r$. Recall that $r' \equiv r \pmod{d}$. We shall prove by induction on $r' - r$, that there is a shortest A, B -path in Q_n , whose internal vertices are not in M and are contained in layers between \mathcal{L}_r and $\mathcal{L}_{r'}$. For $r' - r = 0$ or $r' - r = d$, such a path exists by Claim 1. If $r' - r \geq 2d$, let $C \in \mathcal{L}_{r+d} \setminus \mathcal{F}_{r+d}$ such that $A \cap B \subseteq C \subseteq A \cup B$. Such a C exists by Claim 2.

By induction, there is a shortest (in Q_n) A, C -path that is internally disjoint from M and goes through the layers between \mathcal{L}_r and \mathcal{L}_{r+d} . Similarly there is a shortest (in Q_n) C, B -path that is internally disjoint from M and goes through the layers between \mathcal{L}_{r+d} and $\mathcal{L}_{r'}$. The union P of these two paths is an A, B -path whose internal vertices are not in M . Moreover, by Claim 3 there is a shortest A, B -path P' in Q_n that contains C . We see that the length of the A, C -subpath of P is at most the length of the A, C -subpath of P' . Similarly, the C, B -subpath of P has length at most the length of C, B -subpath of P' . Hence P is not longer than P' , i.e., P is a desired shortest A, B -path. \square

2.2 Proof of Theorem 1.3

Proof of Theorem 1.3 By Lemma 2.1 we see that Q_n contains pairwise disjoint mutual-visibility sets $M(0), M(1), \dots, M(d-1)$ whose total size is $\sum_{r=d}^{n-d} \text{ex}(n, \mathcal{D}_r(2d, d)) + 2 \sum_{r=0}^{d-1} \binom{n}{r}$. Thus some $M(i)$ has size at least $\frac{1}{d} \left(\sum_{r=d}^{n-d} \text{ex}(n, \mathcal{D}_r(2d, d)) + 2 \sum_{r=0}^{d-1} \binom{n}{r} \right)$. \square

2.3 Proof of Theorem 1.2

Let $r, s, t \in \mathbb{N}$ with $\min\{r, s\} \geq t$. The Turán density of (r, s, t) -daisies is defined as

$$\pi(\mathcal{D}_r(s, t)) := \lim_{n \rightarrow \infty} \frac{\text{ex}(n, \mathcal{D}_r(s, t))}{\binom{n}{r}}.$$

We shall use the following statement from the proof of Theorem 5 in the paper by Ellis, Ivan, and Leader [13]. Let $q \in \mathbb{N}$ be a fixed prime power. Then

$$\lim_{r \rightarrow \infty} \pi(n, \mathcal{D}_r(q + 2, 2)) \geq \prod_{k=1}^{\infty} (1 - q^{-k}). \tag{2.1}$$

We also need the following monotonicity statement mentioned in the literature, see, e.g., Keevash [17] and Sidorenko [24] and we include its proof in the appendix.

Lemma 2.2 *Let r, s, t be positive integers with $\min\{r, s\} \geq t$. Then for all $n \geq r$*

$$\frac{\text{ex}(n, \mathcal{D}_r(s, t))}{\binom{n}{r}} \geq \frac{\text{ex}(n + 1, \mathcal{D}_r(s, t))}{\binom{n+1}{r}} \quad \text{and} \quad \pi(\mathcal{D}_r(s, t)) \geq \pi(\mathcal{D}_{r+1}(s, t)).$$

Proof of Theorem 1.2 Since the stated lower bound is trivial when $n \leq 3$, without loss of generality we assume $n > 3$. We shall derive Theorem 1.2 from Theorem 1.3 with $d = 3$. For that we need to find the lower bounds on $\text{ex}(n, \mathcal{D}_r(6, 3))$. Since every $\mathcal{D}_r(6, 3)$ contains a copy of $\mathcal{D}_r(5, 2)$, we have $\text{ex}(n, \mathcal{D}_r(6, 3)) \geq \text{ex}(n, \mathcal{D}_r(5, 2))$. Applying (2.1) with $q = 3$ gives

$$\lim_{r \rightarrow \infty} \pi(\mathcal{D}_r(5, 2)) \geq \prod_{k=1}^{\infty} (1 - 3^{-k}) > 0.56.$$

Accordingly, for $n \geq r \geq 2$ Lemma 2.2 implies that $\pi(\mathcal{D}_r(5, 2)) > 0.56$ and $\text{ex}(n, \mathcal{D}_r(5, 2)) > 0.56 \binom{n}{r}$. Now by Theorem 1.3 with $d = 3$, we obtain

$$\begin{aligned} \mu(Q_n) &\geq \frac{1}{3} \left(\sum_{r=3}^{n-3} \text{ex}(n, \mathcal{D}_r(6, 3)) + 2 \sum_{r=0}^2 \binom{n}{r} \right) \\ &\geq \frac{1}{3} \left(\sum_{r=3}^{n-3} \text{ex}(n, \mathcal{D}_r(5, 2)) + 2 \sum_{r=0}^2 \binom{n}{r} \right) > \frac{0.56}{3} \cdot 2^n. \end{aligned}$$

□

3 Mutual-Visibility Coloring of Q_n

Proof of Theorem 1.5 Fix any positive integer q . We first show that there exists some $n_0 > 0$, such that $\chi_\mu(Q_n) > q$ holds for all $n \geq n_0$. Given integers $s \geq r \geq 1$, the q -color hypergraph Ramsey number $R_r(s; q)$ is defined as the smallest integer N such that any q -coloring of $\binom{[N]}{r}$ contains a monochromatic copy of $\binom{[s]}{r}$. Since q is fixed in the beginning, we simply write $R_r(s) = R_r(s; q)$. Now let $n_0 = q \cdot (R_2 \circ R_3 \circ \dots \circ R_{2q}(2q))$. Let $n \geq n_0$ and fix an arbitrary q -coloring of $V(Q_n)$. Following an approach from [1], we iteratively apply the hypergraph Ramsey theorem and see that there is a sequence of sets $X_{2q} \subseteq X_{2q-1} \subseteq \dots \subseteq X_2 \subseteq X_1 \subseteq [n]$, such that $\binom{X_r}{r}$ is monochromatic for each $r \in [2q]$ and $|X_{2q}| = 2q$. Let Q be the subgraph of Q_n induced by $2^{X_{2q}}$, in particular, Q is a copy of Q_{2q} . We have that every layer of Q is monochromatic. Since Q contains $2q + 1$ layers and there are only q colors, by the pigeonhole principle at least three layers of Q receive the same color. Let $\mathcal{L}_i \cap V(Q)$, $\mathcal{L}_j \cap V(Q)$, and $\mathcal{L}_k \cap V(Q)$ be the three layers of Q contained in the same color class that we denote by M , where $i < j < k$. Consider two vertices $A \in \mathcal{L}_i \cap V(Q)$ and $B \in \mathcal{L}_k \cap V(Q)$ with $A \subseteq B$. Observe that every shortest A, B -path in Q_n goes through some vertex $C \in \mathcal{L}_j$ satisfying $A \subseteq C \subseteq B$. As Q is a copy of a hypercube, all such vertices C are contained in $\mathcal{L}_j \cap V(Q)$. Namely, every shortest A - B path must go through the layer $\mathcal{L}_j \cap V(Q)$, so M is not a mutual-visibility set. This implies that $\chi_\mu(Q_n) > q$.

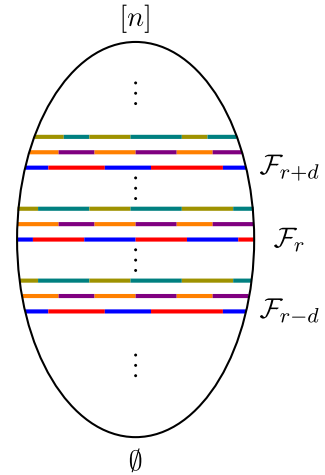
Now we shall prove the upper bound on $\chi_\mu(Q_n)$. All logarithms below are base 2. Assume that n is sufficiently large and let $d = \lceil \log \log n \rceil \geq 3$. Consider a layer \mathcal{L}_r and let $\lambda \in \{0, 1, \dots, d - 1\}$ such that $r \equiv \lambda \pmod{d}$. If $d \leq r \leq n - d$, we color \mathcal{L}_r uniformly at random in two colors $(\lambda, 1)$ and $(\lambda, 2)$. If $r \leq d - 1$ or $r \geq n - d + 1$ we color \mathcal{L}_r simply with one color $(\lambda, 1)$. See Figure 3 for an illustration.

We claim that with positive probability every color class is a mutual-visibility set. By Lemma 2.1, it is sufficient to prove that for each r with $d \leq r \leq n - d$, with positive probability every color class in \mathcal{L}_r is $D_r(2d, d)$ -free. Fix such an r and let $\mathcal{J} = \mathcal{J}_r(d)$ be the collection of all copies of daisies $D_r(2d, d)$ in the layer \mathcal{L}_r .

For any $J \in \mathcal{J}$, let $\mathcal{E}(J)$ be the event that J is monochromatic. Let $p = p(J)$ be the probability that $\mathcal{E}(J)$ occurs. Since $|J| = \binom{2d}{d}$,

$$p = 2^{-|J|+1} = 2^{-\binom{2d}{d}+1}.$$

Fig. 3 Each color class is a mutual-visibility set.



Let $g = g(J) = |\{J' \in \mathcal{J} \setminus \{J\} : J' \cap J \neq \emptyset\}|$. We have that

$$g \leq \binom{2d}{d} \binom{r}{d} \binom{n-r}{d} \leq \binom{2d}{d} \binom{n}{2d} < \left(\frac{en}{d}\right)^{2d}.$$

Indeed, each member of J belongs to $\binom{r}{d} \binom{n-r}{d}$ distinct copies of $D_r(2d, d)$. Thus $\mathcal{E}(J)$ is mutually independent of all but at most g other events $\mathcal{E}(J')$, $J' \in \mathcal{J}$. Now since

$$e \cdot p \cdot (g + 1) \leq 2^{1 - \binom{2d}{d}} n^{2d} < 2^{-(\log n)^2 / \sqrt{100 \log \log n} + 2 \log n \log \log n} < 1,$$

by Lovász Local Lemma [14] (see also [26, Theorem 1.5]), with positive probability there exists no monochromatic copy of $D_r(2d, d)$ in \mathcal{L}_r . □

Remark. Although the proof above shows that $\chi_\mu(Q_n)$ grows with n , the dependence of the lower bound on n is quite unsatisfactory. In fact, it is even difficult to express the lower bound on $\chi_\mu(Q_n)$ in terms of n , since for $\chi_\mu(Q_n) > q$ we require n to be a composition of q -color hypergraph Ramsey numbers.

4 Total Mutual-Visibility in Q_n

In this section it is helpful to regard the hypercube Q_n as a graph on the vertex set $\{0, 1\}^n$. For $x, y \in \{0, 1\}^n$, the *Hamming distance* between x and y , corresponding to the distance in the graph Q_n , denoted $\text{dist}(x, y)$, is the number of coordinates where x and y differ. For a set $M \subseteq \{0, 1\}^n$, we say that M *avoids Hamming distance* d if for any two vertices $x, y \in M$, $\text{dist}(x, y) \neq d$. Bujtás et al. [6] showed the following characterization of a total mutual-visibility set in Q_n .

Lemma 4.1 ([6, Theorem 6]) *$M \subseteq \{0, 1\}^n$ is a total mutual-visibility set in Q_n if and only if M avoids the Hamming distance 2.*

By taking a random subset of $\{0, 1\}^n$ and applying the deletion method to the pairs of vertices with Hamming distance 2, Bujtás et al. [6] proved the lower bound $\mu_t(Q_n) \geq 2^{n-2}/(n^2-n)$ for all $n \geq 2$. The same bound (with a slightly better constant) can also be derived from a greedy algorithm. On the other hand, our improvement on the lower bound comes from a coding theory perspective.

A subset M of $\{0, 1\}^n$ is a *single error correcting code*, if the Hamming distance between any two elements of M is at least 3. Let for $x \in \{0, 1\}^n$, $\mathcal{B}(x, 1)$ be the set of all elements from $\{0, 1\}^n$ at Hamming distance at most 1 from x , i.e., a unit ball with center x . Note that $|\mathcal{B}(x, 1)| = n + 1$ and if M is a single error correcting code, then for any distinct $x, y \in M$, $\mathcal{B}(x, 1) \cap \mathcal{B}(y, 1) = \emptyset$. Thus, the size of M is at most $2^n/(n + 1)$. In particular, if $|M| = 2^n/(n + 1)$, then M is called a *perfect single error correcting code*. Hamming [16] showed that perfect codes exist for certain values of n .

Lemma 4.2 (Hamming [16]) *Let $n = 2^m - 1$ for some integer $m \geq 2$. Then there exists a perfect single error correcting code $M \subseteq \{0, 1\}^n$.*

The *weight* of an element $x \in \{0, 1\}^n$ is defined to be the number of 1’s in x . Given $w \in [0, n]$, let $A(n, 4, w)$ denote the largest size of a set M of elements from $\{0, 1\}^n$ with weight w and pairwise Hamming distance at least 4. Employing a simple combinatorial argument, Graham and Sloane [15] proved the following lower bound on $A(n, 4, w)$. We include the proof here for completeness and later discussion.

Lemma 4.3 ([15, Theorem 1]) *Let $n \in \mathbb{N}$ and $w \in [0, n]$. Then $A(n, 4, w) \geq \binom{n}{w}/n$.*

Proof For $\lambda \in [n]$ we define $\mathcal{C}_{w,\lambda}$ to be the set of $x \in \{0, 1\}^n$ with weight w and $\|x\| \equiv \lambda \pmod{n}$, where $\|x\|$ denotes the sum of non-zero entries’ positions in x . For example, $\|(0, 1, 1)\| = 2 + 3 = 5$. Because all elements in $\mathcal{C}_{w,\lambda}$ have the same weight, the pairwise Hamming distance in $\mathcal{C}_{w,\lambda}$ is even. If $x, y \in \mathcal{C}_{w,\lambda}$ and $\text{dist}(x, y) = 2$, then $\|x\| \not\equiv \|y\| \pmod{n}$, a contradiction. Namely, the pairwise Hamming distance in $\mathcal{C}_{w,\lambda}$ is at least 4. Furthermore, $|\bigcup_{\lambda \in [n]} \mathcal{C}_{w,\lambda}| = \binom{n}{w}$, since each element from $\{0, 1\}^n$ of weight w is in $\mathcal{C}_{w,\lambda}$, for some λ . Let \mathcal{C}_w be the largest among all $\mathcal{C}_{w,\lambda}$. By averaging we have $|\mathcal{C}_w| \geq \binom{n}{w}/n$. □

4.1 Proof of Theorem 1.7

Proof of Theorem 1.7 We start by showing the stated upper bound on $\mu_t(Q_n)$. Let $M \subseteq \{0, 1\}^n$ be a maximum total mutual-visibility set in Q_n . By Lemma 4.1 we know that M avoids the Hamming distance 2. We define a bipartite graph G on the vertex set $M \cup (\{0, 1\}^n \setminus M)$, such that $x \in M$ and $y \in \{0, 1\}^n \setminus M$ are adjacent if and only if $\text{dist}(x, y) = 2$. We shall count the number of edges in G . First, for every $x \in M$, there are $\binom{n}{2}$ vertices in $\{0, 1\}^n$ with Hamming distance 2 from x , that is, the number of ways to flip two coordinates of x . Since $M \subseteq \{0, 1\}^n$ avoids the Hamming distance 2, all such vertices are in $\{0, 1\}^n \setminus M$, namely,

$$e(G) = \sum_{x \in M} d_G(x) = |M| \binom{n}{2}. \tag{4.1}$$

On the other hand, we claim that $d_G(y) \leq \lfloor \frac{n}{2} \rfloor$ holds for every $y \in \{0, 1\}^n \setminus M$. Suppose there exists some $y \in \{0, 1\}^n \setminus M$ with $d_G(y) > \lfloor \frac{n}{2} \rfloor$. Each neighbor of y is obtained by flipping two coordinates of y . Thus, by the pigeonhole principle, there must exist two neighbors $u, v \in M$ of y and distinct $i, j, k \in [n]$, such that u is obtained by flipping the i -th and k -th coordinates of y , and v is obtained by flipping the j -th and k -th coordinates of y . But then the Hamming distance between u and v is exactly 2, a contradiction. Therefore,

$$e(G) = \sum_{y \in \{0,1\}^n \setminus M} d_G(y) \leq \frac{(2^n - |M|)n}{2}.$$

Together with (4.1), we deduce the upper bound $\mu_t(Q_n) = |M| \leq \frac{2^n}{n}$.

Now we prove the lower bounds on $\mu_t(Q_n)$.

Case 1: $n = 2^m - 1$ with $m \in \mathbb{N}$.

If $m = 1$, then Q_n is isomorphic to K_2 . In this case, the whole vertex set of Q_n is a total mutual-visibility set, hence, $\mu_t(Q_n) \geq 2^n/(n + 1)$. If $m \geq 2$, then Lemma 4.2 implies that there exists a subset $M \subseteq \{0, 1\}^n$ with pairwise Hamming distance at least 3 and $|M| = 2^n/(n + 1)$. By Lemma 4.1, such an M is a total mutual-visibility set in Q_n , thus, $\mu_t(Q_n) \geq 2^n/(n + 1)$.

Case 2: n is an arbitrary natural number.

For each $w \in [0, n]$, consider a set C_w of elements from $\{0, 1\}^n$ with weight w and pairwise Hamming distance at least 4, such that $|C_w| \geq \binom{n}{w}/n$. Such a set exists by Lemma 4.3. Furthermore, we define

$$\mathcal{A} := \bigcup_{\substack{w \in [0, n] \\ w \equiv \{0, 1\} \pmod{4}}} C_w \quad \text{and} \quad \mathcal{B} := \bigcup_{\substack{w \in [0, n] \\ w \equiv \{2, 3\} \pmod{4}}} C_w.$$

It is easy to see that \mathcal{A} avoids the Hamming distance 2. Indeed, for any two elements $x, y \in \{0, 1\}^n$ with weight $w(x) \leq w(y)$, if $\text{dist}(x, y) = 2$, then either $w(x) = w(y)$ or $w(x) - w(y) = 2$. If $w(x) = w(y)$, then x and y must belong to the same C_w , meaning that $\text{dist}(x, y) \geq 4$. If $w(x) - w(y) = 2$, then $w(x) \in \{0, 1\} \pmod{4}$ and $w(y) \in \{0, 1\} \pmod{4}$ can not hold simultaneously, a contradiction.

By symmetry, \mathcal{B} also avoids the Hamming distance 2. Then, Lemma 4.1 ensures that both \mathcal{A} and \mathcal{B} are total mutual-visibility sets in Q_n . Furthermore, since $|\mathcal{A} \cup \mathcal{B}| = \sum_{w=0}^n |C_w| \geq 2^n/n$, we have $\mu_t(Q_n) \geq \max\{|\mathcal{A}|, |\mathcal{B}|\} \geq 2^{n-1}/n$. □

5 Concluding Remarks

In this paper we establish tight bounds (up to a multiplicative constant) on $\mu(Q_n)$ and $\mu_t(Q_n)$. Moreover, our proofs provide explicit constructions of nearly optimal mutual-visibility and total mutual-visibility sets in hypercubes.

A natural generalisation of the hypercube Q_n is the so-called *Hamming graph* $H(n, q)$ with $n, q \in \mathbb{N}$, which is defined as a graph on the vertex set $[q]^n$, where two

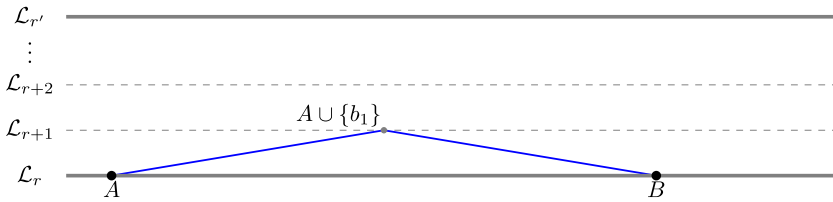


Fig. 4 A shortest A, B -path when $|A| = |B|$ and $k = 1$

vertices $x, y \in [q]^n$ form an edge if and only if their Hamming distance equals 1. Bujtás et al. [6] proved non-trivial bounds for $\mu_t(H(n, q))$. It might be possible to improve their lower bound using certain probabilistic techniques and non-binary error correcting codes, see, e.g., Lindström [22] and Schönheim [23]. On the other hand, the parameter $\mu(H(n, q))$ for $n > 2$ has not been studied to the best of our knowledge and might be interesting to investigate.

For the coloring version of the mutual-visibility problem, we have proved that $\omega(1) = \chi_\mu(Q_n) = O(\log \log n)$. It would be interesting to determine the order of magnitude of $\chi_\mu(Q_n)$. In addition, Q_n is the only known class of graphs with $\chi_\mu(Q_n) \gg |V(Q_n)|/\mu(Q_n)$. Are there other classes of graphs that exhibit the same behavior?

Lastly, we note that our proofs in Section 4 can be adapted to settle the coloring version of the total mutual-visibility problem for hypercubes. Let $\chi_\mu^{\text{total}}(G)$ be the smallest number of colors needed to color $V(G)$ such that each color class is a total mutual-visibility set in G . In the proof of Lemma 4.3 we have actually partitioned every layer of Q_n into n total mutual-visibility sets. Combining this with the lower bound proof of Theorem 1.7, one can show that $\chi_\mu^{\text{total}}(Q_n) \leq 2n$. On the other hand, the upper bound on $\mu_t(Q_n)$ implies that $\chi_\mu^{\text{total}}(Q_n) \geq n$. The function $\chi_\mu^{\text{total}}(G)$ has not been studied elsewhere and could be of independent interest.

Appendix

Claim 1 *Let $n \in \mathbb{N}$ and $r, r' \in \{0, \dots, n\}$ with $r + 3 \leq r'$. For any distinct vertices $A, B \in \mathcal{L}_r \cup \mathcal{L}_{r'}$, there exists a shortest A, B -path in Q_n that is internally disjoint from $\mathcal{L}_r \cup \mathcal{L}_{r'}$ and only goes through the layers between \mathcal{L}_r and $\mathcal{L}_{r'}$.*

Proof of Claim 1 Fix arbitrary $A, B \in \mathcal{L}_r \cup \mathcal{L}_{r'}$, $A \neq B$.

Case 1: $|A| = |B|$.

Without loss of generality assume $A, B \in \mathcal{L}_r$, the case of $A, B \in \mathcal{L}_{r'}$ follows by a similar argument. From $|A| = |B|$ we can deduce that $|A \setminus B| = |B \setminus A|$. Let $A \setminus B = \{a_1, \dots, a_k\}$ and $B \setminus A = \{b_1, \dots, b_k\}$ for some $k \geq 1$. If $k = 1$, then $(A, A \cup \{b_1\}, B)$ is a desired shortest A, B -path, see Figure 4.

If $k \geq 2$, we first take $(A, A \cup \{b_1\})$. Then, we iteratively add b_{i+1} and delete a_i for $i \in [k - 1]$ till we reach the vertex $A \cup \{b_1, \dots, b_k\} \setminus \{a_1, \dots, a_{k-1}\}$. Finally, we delete a_k and reach B , completing the construction of a desired shortest A, B -path. See Figure 5 for an illustration.

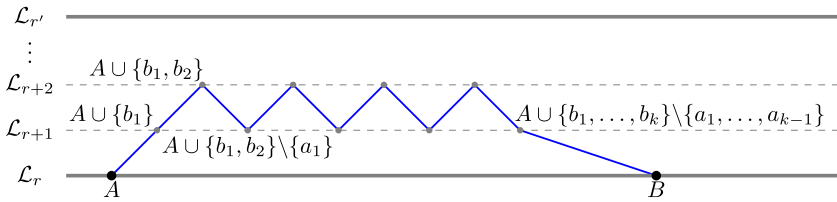


Fig. 5 A shortest A, B -path when $|A| = |B|$ and $k \geq 2$

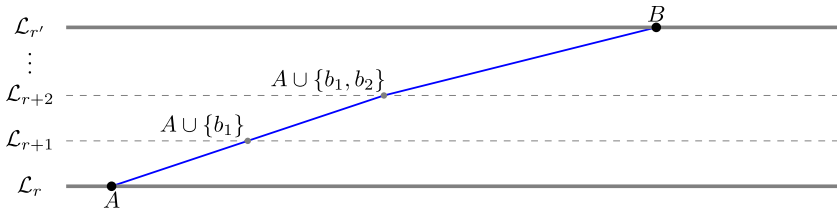


Fig. 6 A shortest A, B -path when $|A| \neq |B|$ and $A \setminus B = \emptyset$

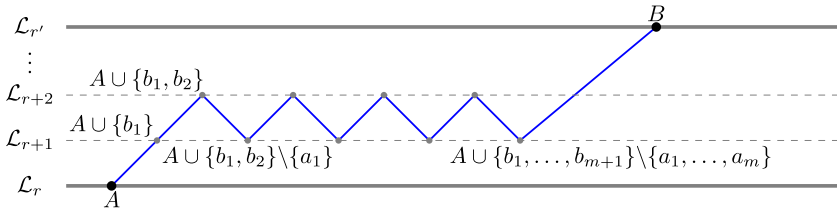


Fig. 7 A shortest A, B -path when $|A| \neq |B|$ and $A \setminus B \neq \emptyset$

Case 2: $|A| \neq |B|$.

Assume that $A \in \mathcal{L}_r$ and $B \in \mathcal{L}_{r+r'}$. It follows that $|B \setminus A| - |A \setminus B| = r' - r$. Let $B \setminus A = \{b_1, \dots, b_k\}$ for some $k \geq r' - r$. If $A \setminus B = \emptyset$, then we obtain the desired A - B path by simply adding elements of $B \setminus A$ to A one at a time, see Figure 6.

If $A \setminus B \neq \emptyset$, let $A \setminus B = \{a_1, \dots, a_m\}$ with $m = k - (r' - r)$. We first take $(A, A \cup \{b_1\})$, then we iteratively add b_{i+1} and delete a_i for $i \in [m]$ till we reach the vertex $A \cup \{b_1, \dots, b_{m+1}\} \setminus \{a_1, \dots, a_m\}$, afterwards we add b_{m+2}, \dots, b_k one at a time and eventually reach B . This constructs the shortest A, B -path, see Figure 7 for an illustration. □

Proof of Lemma 2.2 Let $\mathcal{F} \subseteq \binom{[n+1]}{r}$ be a largest $\mathcal{D}_r(s, t)$ -free family, namely, $|\mathcal{F}| = \text{ex}(n + 1, \mathcal{D}_r(s, t))$. For every $A \in \binom{[n+1]}{n}$, we define $\mathcal{F}_A := \mathcal{F} \cap \binom{A}{r}$. Since \mathcal{F}_A is a $\mathcal{D}_r(s, t)$ -free family with ground set of size n , we have $|\mathcal{F}_A| \leq \text{ex}(n, \mathcal{D}_r(s, t))$. On the other hand, since every member of \mathcal{F} is contained in $\binom{[n+1]}{n-r} = n + 1 - r$ different \mathcal{F}_A 's, we can bound $\text{ex}(n + 1, \mathcal{D}_r(s, t))$ as follows

$$\text{ex}(n + 1, \mathcal{D}_r(s, t)) = |\mathcal{F}| = \frac{\sum_{A \in \binom{[n+1]}{n}} |\mathcal{F}_A|}{n + 1 - r} \leq \frac{(n + 1)\text{ex}(n, \mathcal{D}_r(s, t))}{n + 1 - r}.$$

Consequently, we have

$$\frac{\text{ex}(n, \mathcal{D}_r(s, t))}{\binom{n}{r}} \geq \frac{(n + 1 - r)\text{ex}(n + 1, \mathcal{D}_r(s, t))}{(n + 1)\binom{n}{r}} = \frac{\text{ex}(n + 1, \mathcal{D}_r(s, t))}{\binom{n+1}{r}}.$$

To show the second monotonicity, let $\mathcal{F} \subseteq \binom{[n+1]}{r+1}$ be a largest $\mathcal{D}_{r+1}(s, t)$ -free family. For every $x \in [n + 1]$, we define $\mathcal{F}_x := \{A \setminus \{x\} : A \in \mathcal{F} \text{ with } x \in A\}$. It is a simple observation that \mathcal{F}_x is a $\mathcal{D}_r(s, t)$ -free family with ground set of size n , so we have $|\mathcal{F}_x| \leq \text{ex}(n, \mathcal{D}_r(s, t))$. Moreover, since every member of \mathcal{F} contributes one member to $r + 1$ different \mathcal{F}_x 's, we obtain the following bound

$$\text{ex}(n + 1, \mathcal{D}_{r+1}(s, t)) = |\mathcal{F}| = \frac{\sum_{x \in [n+1]} |\mathcal{F}_x|}{r + 1} \leq \frac{(n + 1)\text{ex}(n, \mathcal{D}_r(s, t))}{r + 1}.$$

Accordingly,

$$\frac{\text{ex}(n, \mathcal{D}_r(s, t))}{\binom{n}{r}} \geq \frac{(r + 1)\text{ex}(n + 1, \mathcal{D}_{r+1}(s, t))}{(n + 1)\binom{n}{r}} = \frac{\text{ex}(n + 1, \mathcal{D}_{r+1}(s, t))}{\binom{n+1}{r+1}}.$$

Taking the limit for $n \rightarrow \infty$, it holds that $\pi(D_r(s, t)) \geq \pi(\mathcal{D}_{r+1}(s, t))$. □

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Declarations

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