Published for SISSA by Springer

Received: October 27, 2025
Accepted: January 24, 2026
Published: March 12, 2026

Integral of the double-emission eikonal function for two massive emitters at an arbitrary angle

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Abstract: We present a semi-analytic calculation of the integrated double-emission eikonal function of two massive emitters whose momenta are at an arbitrary angle to each other. This result is needed for extending the nested soft-collinear subtraction scheme [*Eur. Phys. J. C* **77** (2017) 248] to processes with massive partons.

Keywords: Higher-Order Perturbative Calculations, Quark Masses

ArXiv ePrint: 2510.04778

JHEP03(2026)117

https://doi.org/10.1007/JHEP03(2026)117

Contents

1 Introduction

Subtraction schemes for higher-order QCD computations are essential for improving the reliability of theoretical predictions in the context of collider physics. For next-to-next-to-leading order (NNLO) computations, significant progress in developing such schemes for *massless* partons occurred in recent years [1–15]. To extend these results to *massive* partons, further scheme-dependent calculations are required. In this paper, we describe a computation of the integral of the double-emission eikonal function for two massive emitters whose momenta are at an arbitrary angle to each other. This integral is an important ingredient needed for making the nested soft-collinear subtraction scheme applicable to processes with heavy quarks. Calculations of a similar integral in the context of two different slicing schemes have been reported earlier in refs. [16, 17].

In principle, one can attempt to integrate the eikonal function for two massive emitters analytically,[1] but this approach rapidly becomes unnecessarily complicated. Because of this,

[1] In fact, this is what was done in ref. [17] for a particular scheme choice. We also note that the analytic calculation for the back-to-back kinematics in the context of the nested subtraction scheme was performed in ref. [18]. Earlier, a numerical computation of a similar quantity was performed in ref. [19].

in the current paper we pursue a *semi-analytic* approach where we extract all $1/\varepsilon$ divergences of the integrated eikonal function analytically, and construct a representation for the finite remainder that can be computed numerically without a regulator.

Our approach to this problem is based on the observation that soft and collinear singularities of the eikonal function can be easily subtracted. This observation was used in the calculation of the N-jettiness soft function at NNLO QCD described in ref. [20]. It was also used very recently in the computation of the integral of the eikonal function of massless and massive emitters with momenta at arbitrary angles to each other [21]. In this paper, we largely follow the approach of ref. [21], although there are important differences between the massive-massless and the massive-massive cases. These differences can be summarized by noticing that the massive-massless case has stronger infra-red singularities but simpler integrals, whereas in the massive-massive case the situation is reversed.

The rest of the paper is organized as follows. In section 2 we introduce the eikonal function, explain our conventions, and define the integral of the eikonal function that is needed in the context of the nested soft-collinear subtraction scheme [1]. In section 3, the integral of the single-emission eikonal function, and the so-called iterative contribution to the double-emission eikonal function are discussed. In section 4 we explain how the integral of the non-iterative piece of the double-emission eikonal function is computed. We describe in detail the subtraction of infra-red singularities, and explain how to express the different contributions that arise along the way through easier-to-compute phase-space integrals. We explain how to extract the infra-red and collinear singularities from such integrals, and construct finite remainders that are computed numerically. In section 5, we discuss the results including the $1/\varepsilon$ terms, the implementation of the finite remainders in a numerical code, and multiple checks that have been performed to ensure their correctness. We conclude in section 6. Useful technical details including definitions of integrals, and aspects of their calculation can be found in appendices.

2 Conventions

We follow ref. [21] and study a generic partonic process

$$0 \to h_1(p_1) + \cdots + h_n(p_n) + H_{n+1}(p_{n+1}) + \cdots + H_N(p_N) + f_1(k_1) + f_2(k_2), \tag{2.1}$$

where h_i and H_i are massless and massive partons, respectively, and $f_{1,2}$ are two massless, potentially unresolved partons which can be either two gluons or a $q\bar{q}$ pair. We consider the double-soft limit, $k_1, k_2 \to 0$, with all other momenta in eq. (2.1) fixed. In this limit, the amplitude squared of the process in eq. (2.1) factorizes. It becomes [22]:

- if $f_{1,2}$ are gluons,

$$\begin{aligned}\lim_{k_1,k_2\to 0} |\mathcal{M}^{gg}(\{p\},k_1,k_2)|^2 \approx g_{s,b}^4 \Bigg\{ &\frac{1}{2}\sum_{i,j,k,l}^{N} \mathcal{S}_{ij}(k_1)\mathcal{S}_{kl}(k_2)|\mathcal{M}^{\{(ij),(kl)\}}(\{p\})|^2 \\ &- C_A \sum_{i,j}^{N} \mathcal{S}_{ij}(k_1,k_2)|\mathcal{M}^{(ij)}(\{p\})|^2 \Bigg\},\end{aligned} \tag{2.2}$$

- if $f_1 = q$ and $f_2 = \bar{q}$,

$$\lim_{k_1,k_2\to 0} |\mathcal{M}^{q\bar{q}}(\{p\},k_1,k_2)|^2 \approx g_{s,b}^4\, T_R \sum_{i,j}^{N} \mathcal{I}_{ij}(k_1,k_2)|\mathcal{M}^{(ij)}(\{p\})|^2 \,. \tag{2.3}$$

Quantities that appear in the above equations include two Casimir operators of the SU(3) group, $C_A = 3, T_R = 1/2$, the bare strong coupling constant $g_{s,b}$, as well as the color-correlated matrix elements of the process without two soft partons

$$|\mathcal{M}^{\{(ij),(kl)\}}(\{p\})|^2 = \langle \mathcal{M}(\{p\})|\{\boldsymbol{T}_i\cdot\boldsymbol{T}_j,\boldsymbol{T}_k\cdot\boldsymbol{T}_l\}|\mathcal{M}(\{p\})\rangle \,, \tag{2.4}$$

$$|\mathcal{M}^{\{(ij)\}}(\{p\})|^2 = \langle \mathcal{M}(\{p\})|\boldsymbol{T}_i\cdot\boldsymbol{T}_j|\mathcal{M}(\{p\})\rangle \,. \tag{2.5}$$

The quantities $\boldsymbol{T}_i$ are the color-charge operators [23], and $\{\ldots,\ldots\}$ denotes an anti-commutator. Sums in eqs. (2.2), (2.3) run over all pairs of hard color-charged emitters.

In eq. (2.2), the term containing the product of two single-eikonal factors

$$\mathcal{S}_{ij}(k) = \frac{(p_i\cdot p_j)}{(p_i\cdot k)(p_j\cdot k)} \,, \tag{2.6}$$

is the *Abelian* contribution. We note that $S_{ij}(k)$ also appears in the single-emission eikonal function relevant for computations at next-to-leading order.

The *non-Abelian* term, proportional to the color factor C_A, is more complicated. The eikonal function $\mathcal{S}_{ij}(k_1,k_2)$ reads

$$\mathcal{S}_{ij}(k_1,k_2) = \mathcal{S}_{ij}^0(k_1,k_2) + \left[m_i^2\mathcal{S}_{ij}^m(k_1,k_2) + m_j^2\mathcal{S}_{ji}^m(k_1,k_2)\right] , \tag{2.7}$$

where quantities that appear inside the square brackets, explicitly depend on the masses of the two emitters, $m_{i,j}$. In addition to this explicit dependence, both functions $\mathcal{S}_{ij}^0(k_1,k_2)$ and $\mathcal{S}_{ij}^m(k_1,k_2)$ *implicitly* depend on these masses, since the momenta of hard emitters are on-shell, $p_{i,j}^2 = m_{i,j}^2$.

The first term in eq. (2.7), $\mathcal{S}_{ij}^0(k_1,k_2)$ is the same for massless and massive emitters [22]. It reads

$$\begin{aligned}
\mathcal{S}_{ij}^0(k_1,k_2) = {} & \frac{(1-\varepsilon)}{(k_1\cdot k_2)^2}\frac{[(p_i\cdot k_1)(p_j\cdot k_2)+i\leftrightarrow j]}{(p_i\cdot k_{12})(p_j\cdot k_{12})} \\
& - \frac{(p_i\cdot p_j)^2}{2(p_i\cdot k_1)(p_j\cdot k_2)(p_i\cdot k_2)(p_j\cdot k_1)}\left[2-\frac{[(p_i\cdot k_1)(p_j\cdot k_2)+i\leftrightarrow j]}{(p_i\cdot k_{12})(p_j\cdot k_{12})}\right] \\
& + \frac{(p_i\cdot p_j)}{2(k_1\cdot k_2)}\left[\frac{2}{(p_i\cdot k_1)(p_j\cdot k_2)}+\frac{2}{(p_j\cdot k_1)(p_i\cdot k_2)}-\frac{1}{(p_i\cdot k_{12})(p_j\cdot k_{12})}\right. \\
& \left.\times\left(4+\frac{[(p_i\cdot k_1)(p_j\cdot k_2)+i\leftrightarrow j]^2}{(p_i\cdot k_1)(p_j\cdot k_2)(p_i\cdot k_2)(p_j\cdot k_1)}\right)\right] ,
\end{aligned} \tag{2.8}$$

where we have used the abbreviation $k_{12} = k_1 + k_2$. The other two contributions in eq. (2.7) are only relevant for the massive emitters. The function $\mathcal{S}_{ij}^m(k_1, k_2)$ is given by [24][2]

$$\begin{aligned}\mathcal{S}_{ij}^m(k_1,k_2) = & \frac{(p_i\cdot p_j)(p_j\cdot k_{12})}{2(p_i\cdot k_1)(p_j\cdot k_2)(p_i\cdot k_2)(p_j\cdot k_1)(p_i\cdot k_{12})} \\ & - \frac{1}{2(k_1\cdot k_2)(p_i\cdot k_{12})(p_j\cdot k_{12})}\left(\frac{(p_j\cdot k_1)^2}{(p_i\cdot k_1)(p_j\cdot k_2)} + \frac{(p_j\cdot k_2)^2}{(p_i\cdot k_2)(p_j\cdot k_1)}\right).\end{aligned} \tag{2.9}$$

In the quark-antiquark case, the eikonal function $\mathcal{I}_{ij}(k_1, k_2)$ reads

$$\mathcal{I}_{ij}(k_1,k_2) = \frac{[(p_i\cdot k_1)\,(p_j\cdot k_2) + i \leftrightarrow j] - (p_i\cdot p_j)\,(k_1\cdot k_2)}{(k_1\cdot k_2)^2\,(p_i\cdot k_{12})\,(p_j\cdot k_{12})}\,, \tag{2.10}$$

and there is no difference between massive and massless emitters.

It is convenient to make use of the color conservation

$$\sum_{i=1}^{N} \boldsymbol{T}_i |\mathcal{M}(\{p\})\rangle = 0, \tag{2.11}$$

and the symmetry of functions $S_{ij} = S_{ji}$ and $I_{ij} = I_{ji}$ to write

$$\sum_{i,j}^{N} \mathcal{S}_{ij}(k_1,k_2)|\mathcal{M}^{(ij)}(\{p\})|^2 = \sum_{i<j}^{N} \widetilde{\mathcal{S}}_{ij}(k_1,k_2)|\mathcal{M}^{(ij)}(\{p\})|^2, \tag{2.12}$$

$$\sum_{i,j}^{N} \mathcal{I}_{ij}(k_1,k_2)|\mathcal{M}^{(ij)}(\{p\})|^2 = \sum_{i<j}^{N} \widetilde{\mathcal{I}}_{ij}(k_1,k_2)|\mathcal{M}^{(ij)}(\{p\})|^2, \tag{2.13}$$

where

$$\widetilde{\mathcal{S}}_{ij} = 2\mathcal{S}_{ij} - \mathcal{S}_{ii} - \mathcal{S}_{jj}, \tag{2.14}$$

$$\widetilde{\mathcal{I}}_{ij} = 2\mathcal{I}_{ij} - \mathcal{I}_{ii} - \mathcal{I}_{jj}. \tag{2.15}$$

To compute the required double-soft contributions, we have to integrate the corresponding eikonal functions $\widetilde{S}_{ij}$ and $\widetilde{I}_{ij}$ over the phase space of two unresolved partons with momenta $k_{1,2}$. Working within the nested soft-collinear subtraction scheme [1], we have to fix the reference frame, and restrict energies of unresolved partons by introducing an upper cut-off $E_{\max}$. Furthermore, energies of unresolved partons must be ordered. We call the parton with the larger (smaller) energy $\mathfrak{m}(\mathfrak{n})$, and refer to their momenta as $k_{\mathfrak{m},\mathfrak{n}}$, instead of $k_{1,2}$, which describe momenta without energy ordering.

We define the required double-emission phase-space integrals as [1]

$$\mathbb{S}\left[\mathcal{S}_{ij}\mathcal{S}_{kl}\right] = \int [\mathrm{d}k_{\mathfrak{m}}][\mathrm{d}k_{\mathfrak{n}}]\theta\left(E_{\max} - k_{\mathfrak{m}}^0\right)\theta\left(k_{\mathfrak{m}}^0 - k_{\mathfrak{n}}^0\right)\mathcal{S}_{ij}(k_{\mathfrak{m}})\mathcal{S}_{kl}(k_{\mathfrak{n}}), \tag{2.16}$$

$$\mathbb{S}\left[\Xi_{ij}\right] = \int [\mathrm{d}k_{\mathfrak{m}}][\mathrm{d}k_{\mathfrak{n}}]\theta\left(E_{\max} - k_{\mathfrak{m}}^0\right)\theta\left(k_{\mathfrak{m}}^0 - k_{\mathfrak{n}}^0\right)\Xi_{ij}\left(k_{\mathfrak{m}}, k_{\mathfrak{n}}\right), \tag{2.17}$$

[2] A different expression for S_{ij}^m is found in ref. [5]. However, both expressions give the same result after summing over i, j in eqs. (2.2), (2.3) thanks to colour conservation.

where the eikonal function Ξ_{ij} is either $\widetilde{\mathcal{S}}_{ij}$ (for gg emission) or $\widetilde{\mathcal{I}}_{ij}$ (for $q\bar{q}$ emission), and

$$[\mathrm{d}k] = \frac{\mathrm{d}^{d-1}k}{2k^0(2\pi)^{d-1}}, \tag{2.18}$$

is the phase-space element. We note that $d = 4 - 2\varepsilon$ is the space-time dimension.[3]

We use the homogeneity of the eikonal functions to extract the dependence of the result on $E_{\max}$; we explained how to do that in ref. [21]. Without repeating this discussion here, we simply quote the result for the correlated part

$$\mathcal{S}\,[\Xi_{ij}] = -\frac{1}{4\varepsilon E_{\max}^{4\varepsilon}} \int [\mathrm{d}l_{\mathfrak{m}}][\mathrm{d}l_{\mathfrak{n}}]\delta\left(1 - l_{\mathfrak{m}} \cdot P\right)\theta\left(l_{\mathfrak{m}} \cdot P - l_{\mathfrak{n}} \cdot P\right)\Xi_{ij}\left(l_{\mathfrak{m}}, l_{\mathfrak{n}}\right). \tag{2.19}$$

The auxiliary four-vector P in the above equation reads $P = (1, \vec{0})$. An identical formula applies to the product of two single eikonal functions $\mathcal{S}[S_{ij}S_{kl}]$.

Our goal is to compute the required soft integrals for two massive emitters. As we already noted, the masses of partons i and j are $m_{i,j}$, respectively. The squares of their four-momenta $p_{i,j}$ are then $p_{i,j}^2 = m_{i,j}^2$. We can choose a reference frame to integrate the eikonal function. The integral in eq. (2.19) is boost-invariant, but a particular choice $P = (1, \vec{0})$ defines the laboratory frame where both heavy partons i and j move with different velocities. In the lab frame, the momenta $p_{i,j}$ are characterized by energies or, equivalently, velocities $\beta_{i,j}$ and their directions. We write

$$p_{i,j} = m_{i,j}\gamma_{i,j}\left(1, \beta_{i,j}\,\vec{\kappa}_{i,j}\right), \tag{2.20}$$

where $\gamma_{i,j} = 1/\sqrt{1-\beta_{i,j}^2}$ and $\vec{\kappa}_{i,j}$ are unit vectors.

However, as we discussed in ref. [21], it is beneficial to work in a different frame where one of the heavy partons is at rest. In this frame

$$P = \gamma_t\left(1, v_t\vec{n}_t\right), \quad p_i = m_i(1, \vec{0}), \quad p_j = E_j\left(1, v_{ij}\vec{n}_j\right), \tag{2.21}$$

with $\gamma_t = \gamma_i$, $v_t = \beta_i$ and $\vec{n}_t = -\vec{\kappa}_i$. Furthermore, v_{ij} is the relative velocity of partons i and j, and E_j and $\vec{n}_j$ are the energy and the direction of a parton j in the rest frame of i.

The relation between these and the laboratory-frame quantities is easy to establish. We find

$$\begin{aligned} E_j &= \frac{p_i p_j}{m_i} = m_j\gamma_i\gamma_j\left(1 - \beta_i\beta_j\cos\theta_{ij}\right), \\ v_{ij} &= \sqrt{1 - \frac{(1-\beta_i^2)(1-\beta_j^2)}{(1-\beta_i\beta_j\cos\theta_{ij})^2}}, \\ \vec{n}_t\cdot\vec{n}_j &= \frac{1}{v_{ij}}\left(1 - \frac{1-\beta_i^2}{1-\beta_i\beta_j\cos\theta_{ij}}\right), \end{aligned} \tag{2.22}$$

where $\cos\theta_{ij} = \vec{\kappa}_i\cdot\vec{\kappa}_j$. As we will see, the integrated eikonal function depends on $\vec{n}_t\cdot\vec{n}_j$, v_{ij}, and v_t, and all these quantities can be expressed in terms of the parameters in the laboratory frame using eq. (2.22).

[3] We use dimensional regularization to regulate soft and collinear divergences throughout this paper.

3 Single emission and iterations

To compute the iterated part of the double-real emission function, we need to integrate the product of two single-emission eikonal functions. The integral reads

$$\mathcal{S}[S_{ij}S_{km}] = -\frac{1}{4\varepsilon E_{\rm max}^{4\varepsilon}} \int [{\rm d}l_{\mathfrak{m}}][{\rm d}l_{\mathfrak{n}}]\delta(1 - l_{\mathfrak{m}}\cdot P)\theta(l_{\mathfrak{m}}\cdot P - l_{\mathfrak{n}}\cdot P)\ S_{ij}(l_{\mathfrak{m}})\ S_{km}(l_{\mathfrak{n}}). \tag{3.1}$$

The gluon momenta $l_{\mathfrak{m},\mathfrak{n}}$ are defined as $l_{\mathfrak{m},\mathfrak{n}} = l^0_{\mathfrak{m},\mathfrak{n}}(1, \vec{n}_{\mathfrak{m},\mathfrak{n}})$, with $\vec{n}^2_{\mathfrak{m},\mathfrak{n}} = 1$. We calculate the integral in eq. (3.1) directly in the laboratory frame, i.e. $P = (1, \vec{0})$. Writing $l^0_{\mathfrak{n}} = \omega\ l^0_{\mathfrak{m}}$ and integrating over ω, we find

$$\mathcal{S}[S_{ij}S_{km}] = \frac{N_\varepsilon^2}{8\varepsilon^2 E_{\rm max}^{4\varepsilon}} \left\langle \frac{\rho_{ij}}{\rho_{i\mathfrak{m}}\rho_{j\mathfrak{m}}} \right\rangle_{\mathfrak{m}} \left\langle \frac{\rho_{km}}{\rho_{k\mathfrak{n}}\rho_{m\mathfrak{n}}} \right\rangle_{\mathfrak{n}}, \tag{3.2}$$

where $\rho_{xy} = 1 - \vec{\beta}_x\cdot\vec{\beta}_y$, with $\vec{\beta}_{\mathfrak{m},\mathfrak{n}} = \vec{n}_{\mathfrak{m},\mathfrak{n}}$. Furthermore, we use

$$N_\varepsilon = \frac{\Omega^{(d-1)}}{2(2\pi)^{d-1}}, \tag{3.3}$$

and

$$\Big\langle \dots \Big\rangle_x = \int \frac{{\rm d}\Omega_x^{(d-1)}}{\Omega^{(d-1)}} \cdots, \tag{3.4}$$

is the integral over directions of the parton x.

Using integrals defined in appendix B, we write eq. (3.2) as

$$\mathcal{S}[S_{ij}S_{km}] = \frac{N_\varepsilon^2}{8\varepsilon^2 E_{\rm max}^{4\varepsilon}}\ I_{11}^{(2)}[\rho_{ii}, \rho_{jj}, \rho_{ij}]\ I_{11}^{(2)}[\rho_{kk}, \rho_{mm}, \rho_{km}]. \tag{3.5}$$

Integrals $I_{11}^{(2)}$ cannot be computed in closed form for arbitrary d and require an expansion in ε. The depth of the required expansion depends on whether lines i, j, k, m are massive or massless. If all lines are massive, the integrals are finite; hence, each of them needs to be expanded to $\mathcal{O}(\varepsilon^2)$.[4] If, on the other hand, there are one or more massless partons, additional singularity is generated by the angular integration, and the angular integral for massive lines through $\mathcal{O}(\varepsilon^3)$ is needed.

We note in passing that these very deep expansions may not be necessary. Indeed, for the purpose of subtractions, the iterative piece will have to be combined with an iteration of Catani's $I^{(1)}$ operator [23, 25], appearing in the virtual corrections; for the massless case, this mechanism was explained in ref. [9]. Similarly, also for the massive case it should lead to a cancellation of the highest $1/\varepsilon^2$ singularities, without destroying the factorized form as in eq. (3.5). Because of this, the required depth of the ε-expansion will be reduced.

[4]The result for the angular integral $I_{11}^{(2)}(\rho_{ii}, \rho_{jj}, \rho_{ij})$ expanded through $\mathcal{O}(\varepsilon^2)$ can be found in the ancillary file to this paper.

4 Non-iterative double-real emission contribution

We consider the integral of the correlated contribution to the double-real eikonal function, defined in eq. (2.19). We will compute it in the rest frame of the parton i. Using the expression for P and other momenta in the rest frame of i given in section 2, and integrating over the energy of the gluon $\mathfrak{m}$, we find

$$\mathcal{S}[\Xi_{ij}] = \frac{\mathcal{N}_A}{\varepsilon} \left\langle \int_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}} \rho_{t\mathfrak{m}}^{4\varepsilon} \theta(\rho_{t\mathfrak{m}} - \omega\rho_{t\mathfrak{n}}) \left[\omega^2 \Xi_{ij}(\mathfrak{m},\mathfrak{n})\right] \right\rangle_{\mathfrak{m}\mathfrak{n}}, \tag{4.1}$$

where $\rho_{tx} = 1 - v_t \vec{n}_t \cdot \vec{n}_x$, $x = \mathfrak{m},\mathfrak{n}$,

$$\mathcal{N}_A = -\frac{\mathcal{N}_\varepsilon^2}{4} \left(\frac{E_{\max}}{\gamma_t}\right)^{-4\varepsilon}, \tag{4.2}$$

and the momenta of gluons $\mathfrak{m}$ and $\mathfrak{n}$ to be used in eq. (4.1) are $l_{\mathfrak{m}} = (1, \vec{n}_{\mathfrak{m}})$ and $l_{\mathfrak{n}} = \omega(1, \vec{n}_{\mathfrak{n}})$.

Eq. (4.1) is a convenient starting point for the computation. We will focus on the calculation of the gluon eikonal function since it is more general than the quark one. Hence, we identify Ξ_{ij} with $\widetilde{S}_{ij}$, cf. eq. (2.14); we will refer to $\mathcal{S}[\widetilde{S}_{ij}]$ as G_{ij}. The integral in eq. (4.1) cannot be computed numerically right away because of divergences. For two massive emitters, these divergences appear in just two cases — i) when the gluon $\mathfrak{n}$ becomes soft, $\omega \to 0$, and ii) when the gluons $\mathfrak{m}$ and $\mathfrak{n}$ become collinear to each other. We will iteratively subtract these singularities from the integrand in eq. (4.1) to construct a finite quantity that can be calculated numerically. However, we stress that because of the overall $1/\varepsilon$ factor in eq. (4.1), it is highly non-trivial to compute all $1/\varepsilon$ poles analytically even if all sources of divergences have been removed from the integrand in that equation.

We start with the soft subtraction and write

$$G_{ij} = \mathcal{S}_\omega[G_{ij}] + \overline{\mathcal{S}}_\omega[G_{ij}]. \tag{4.3}$$

The first term corresponds to the strongly-ordered limit of G_{ij},

$$\mathcal{S}_\omega[G_{ij}] = \frac{\mathcal{N}_A}{\varepsilon} \left\langle \int_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}} \rho_{t\mathfrak{m}}^{4\varepsilon}\, \theta(\rho_{t\mathfrak{m}} - \omega\rho_{t\mathfrak{n}})\, S_\omega \left[\omega^2 \widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right] \right\rangle_{\mathfrak{m}\mathfrak{n}}, \tag{4.4}$$

where the operator S_ω extracts the leading $\mathcal{O}(1/\omega^2)$ singularity from the eikonal function $\widetilde{S}_{ij}$. We will discuss the computation of this quantity in the next section.

The second term

$$\overline{\mathcal{S}}_\omega[G_{ij}] = \frac{\mathcal{N}_A}{\varepsilon} \left\langle \int_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}} \rho_{t\mathfrak{m}}^{4\varepsilon} \theta(\rho_{t\mathfrak{m}} - \omega\rho_{t\mathfrak{n}}) \overline{S}_\omega \left[\omega^2 \widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right] \right\rangle_{\mathfrak{m}\mathfrak{n}}, \tag{4.5}$$

involves the operator $\overline{S}_\omega = 1 - S_\omega$. Hence, it does not possess a soft singularity since it is explicitly subtracted. It remains to isolate and remove the collinear $\mathfrak{m}||\mathfrak{n}$ singularity from it. While such a subtraction is straightforward, we need to do it in such a way, that all divergent contributions can be computed analytically.

To this end, we found it convenient to write $\overline{\mathcal{S}}_\omega[G_{ij}]$ as the sum of two terms

$$\overline{\mathcal{S}}_\omega[G_{ij}] = \overline{\mathcal{S}}_\omega[G_{ij}^{(0)}] + \overline{\mathcal{S}}_\omega[\Delta G_{ij}]. \tag{4.6}$$

The first term reads

$$\overline{\mathcal{S}}_\omega[G_{ij}^{(0)}] = \frac{\mathcal{N}_A}{\varepsilon}\left\langle \int_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}}\,\theta(1-\omega)\,\overline{S}_\omega\left[\omega^2 \widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_{\mathfrak{m}\mathfrak{n}}. \tag{4.7}$$

It corresponds to the soft-subtracted integral in case when the laboratory frame and the rest frame of the parton i coincide. The function $\widetilde{S}_{ij}$ in this case depends on a single direction, which makes the integration much simpler.

The second quantity $\overline{\mathcal{S}}_\omega[\Delta G_{ij}]$ reads

$$\overline{\mathcal{S}}_\omega[\Delta G_{ij}] = \frac{\mathcal{N}_A}{\varepsilon}\left\langle \int_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}}\,f(\omega,\rho_{t\mathfrak{m}},\rho_{t\mathfrak{n}})\,\overline{S}_\omega\left[\omega^2 \widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_{\mathfrak{m}\mathfrak{n}}, \tag{4.8}$$

where

$$f(\omega,\rho_{t\mathfrak{m}},\rho_{t\mathfrak{n}}) = \rho_{t\mathfrak{m}}^{4\varepsilon}\theta(\rho_{t\mathfrak{m}}-\omega\rho_{t\mathfrak{n}}) - \theta(1-\omega). \tag{4.9}$$

The integral in eq. (4.8) is divergent because of the collinear $\mathfrak{m}||\mathfrak{n}$ singularity of the integrand. However, as we will see shortly this divergence is softer than the divergence of the quantity $\overline{\mathcal{S}}_\omega[G_{ij}]$.

To subtract the $\mathfrak{m}||\mathfrak{n}$ divergence, we write $\overline{\mathcal{S}}_\omega[\Delta G_{ij}]$ as the sum of two terms

$$\overline{\mathcal{S}}_\omega[\Delta G_{ij}] = \overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\mathfrak{m}||\mathfrak{n}} + \overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\mathrm{fin}}, \tag{4.10}$$

where

$$\begin{aligned}
\overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\mathfrak{m}||\mathfrak{n}} &= \frac{\mathcal{N}_A}{\varepsilon}\left\langle \int_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}}\,C_{\mathfrak{m}\mathfrak{n}}\,f(\omega,\rho_{t\mathfrak{m}},\rho_{t\mathfrak{n}})\,\overline{S}_\omega\,\left[\omega^2 \widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_{\mathfrak{m}\mathfrak{n}}, \\
\overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\mathrm{fin}} &= \frac{\mathcal{N}_A}{\varepsilon}\left\langle \int_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}}\,\overline{C}_{\mathfrak{m}\mathfrak{n}}\,f(\omega,\rho_{t\mathfrak{m}},\rho_{t\mathfrak{n}})\,\overline{S}_\omega\,\left[\omega^2 \widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_{\mathfrak{m}\mathfrak{n}}.
\end{aligned} \tag{4.11}$$

The operator $\overline{C}_{\mathfrak{m}\mathfrak{n}}$ is defined as $\overline{C}_{\mathfrak{m}\mathfrak{n}} = 1 - C_{\mathfrak{m}\mathfrak{n}}$. The operator $C_{\mathfrak{m}\mathfrak{n}}$ extracts the non-integrable part of the $\mathfrak{m}||\mathfrak{n}$ collinear limit of the integrand, but it does not act on the angular phase space. Since

$$C_{\mathfrak{m}\mathfrak{n}} f(\omega,\rho_{t\mathfrak{m}},\rho_{t\mathfrak{n}}) = (\rho_{t\mathfrak{m}}^{4\varepsilon}-1)\,\theta(1-\omega)\,C_{\mathfrak{m}\mathfrak{n}}, \tag{4.12}$$

we find

$$\overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\mathfrak{m}||\mathfrak{n}} = \frac{\mathcal{N}_A}{\varepsilon}\left\langle \int_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}}(\rho_{t\mathfrak{m}}^{4\varepsilon}-1)\,\theta(1-\omega)\,C_{\mathfrak{m}\mathfrak{n}}\,\overline{S}_\omega\left[\omega^2 \widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_{\mathfrak{m}\mathfrak{n}}. \tag{4.13}$$

To simplify $\overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\mathrm{fin}}$, we split $\overline{C}_{\mathfrak{m}\mathfrak{n}} f(\omega,\rho_{t\mathfrak{m}},\rho_{t\mathfrak{n}})$ into two terms

$$\overline{C}_{\mathfrak{m}\mathfrak{n}}\left[\rho_{t\mathfrak{m}}^{4\varepsilon}\theta(\rho_{t\mathfrak{m}}-\omega\rho_{t\mathfrak{n}}) - \theta(1-\omega)\right] = \overline{C}_a + \overline{C}_b, \tag{4.14}$$

using the following decomposition $\rho_{t\mathfrak{m}}^{4\varepsilon} = 1 + (\rho_{t\mathfrak{m}}^{4\varepsilon}-1)$. The two terms are defined as follows

$$\begin{aligned}
\overline{C}_a &= \overline{C}_{\mathfrak{m}\mathfrak{n}}\left[\theta(\rho_{t\mathfrak{m}}-\omega\rho_{t\mathfrak{n}}) - \theta(1-\omega)\right] = \theta(\rho_{t\mathfrak{m}}-\omega\rho_{t\mathfrak{n}}) - \theta(1-\omega), \\
\overline{C}_b &= \overline{C}_{\mathfrak{m}\mathfrak{n}}\left[(\rho_{t\mathfrak{m}}^{4\varepsilon}-1)\theta(\rho_{t\mathfrak{m}}-\omega\rho_{t\mathfrak{n}})\right] = (\rho_{t\mathfrak{m}}^{4\varepsilon}-1)\left[\theta(\rho_{t\mathfrak{m}}-\omega\rho_{t\mathfrak{n}}) - \theta(1-\omega)C_{\mathfrak{m}\mathfrak{n}}\right].
\end{aligned} \tag{4.15}$$

We then write

$$\overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\mathrm{fin}} = \overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\mathrm{fin,a}} + \overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\mathrm{fin,b}}, \tag{4.16}$$

where

$$
\begin{aligned}
\overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\rm fin,a} &= \frac{\mathcal{N}_A}{\varepsilon}\left\langle \int_0^\infty \frac{{\rm d}\omega}{\omega^{1+2\varepsilon}}\left(\theta(\rho_{\mathfrak{tm}}-\omega\rho_{\mathfrak{tn}})-\theta(1-\omega)\right)\overline{S}_\omega\left[\omega^2\widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_{\mathfrak{mn}}, \\
\overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\rm fin,b} &= \frac{\mathcal{N}_A}{\varepsilon}\left\langle \int_0^\infty \frac{{\rm d}\omega}{\omega^{1+2\varepsilon}}\overline{C}_b\overline{S}_\omega\left[\omega^2\widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_{\mathfrak{mn}}.
\end{aligned}
\tag{4.17}
$$

The quantity $\overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\rm fin,b}$ does not require further manipulations since its integrand is $\mathcal{O}(\varepsilon)$, cf. eq. (4.15); hence, we compute it numerically.

The quantity $\overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\rm fin,a}$ is also finite, but its integrand is not suppressed by ε. Therefore, because of the $1/\varepsilon$ prefactor, it needs to be expanded to linear order in ε. To facilitate such an expansion, we make use of the following identity[5]

$$
\overline{S}_\omega\left[\omega^2\widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\Bigg|_{\substack{\omega\to\frac{1}{\omega}\\ \mathfrak{m}\leftrightarrow\mathfrak{n}}} - \overline{S}_\omega\left[\omega^2\widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right] = \Delta_{ij}, \tag{4.18}
$$

where

$$
\Delta_{ij} = \frac{(\rho_{\mathfrak{m}j}-\rho_{\mathfrak{n}j})(\rho_{\mathfrak{mn}}-\rho_{\mathfrak{m}j}-\rho_{\mathfrak{n}j})}{\rho_{\mathfrak{mn}}\rho_{\mathfrak{m}j}\rho_{\mathfrak{n}j}} + \left(1-v_{ij}^2\right)\frac{(\rho_{\mathfrak{m}j}-\rho_{\mathfrak{n}j})(\rho_{\mathfrak{m}j}+\rho_{\mathfrak{n}j}-\rho_{\mathfrak{mn}})}{\rho_{\mathfrak{mn}}\rho_{\mathfrak{m}j}^2\rho_{\mathfrak{n}j}^2}. \tag{4.19}
$$

It follows that

$$
\overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\rm fin,a} = \overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\rm fin,a_1} + \overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\rm fin,a_2}, \tag{4.20}
$$

where

$$
\overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\rm fin,a_1} = \frac{\mathcal{N}_A}{2\varepsilon}\left\langle \int_0^\infty \frac{{\rm d}\omega}{\omega^{1-2\varepsilon}}(\theta(\omega\rho_{\mathfrak{tn}}-\rho_{\mathfrak{tm}})-\theta(\omega-1))\Delta_{ij}\right\rangle_{\mathfrak{mn}}, \tag{4.21}
$$

and

$$
\overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\rm fin,a_2} = \frac{\mathcal{N}_A}{\varepsilon}\left\langle \int_0^\infty \frac{{\rm d}\omega}{\omega}\,\frac{\omega^{-2\varepsilon}-\omega^{2\varepsilon}}{2}\times[\theta(\rho_{\mathfrak{tm}}-\omega\rho_{\mathfrak{tn}})-\theta(1-\omega)]\,\overline{S}_\omega\left[\omega^2\widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_{\mathfrak{mn}}. \tag{4.22}
$$

The integrand of the latter quantity is $\mathcal{O}(\varepsilon)$, so that it can be computed numerically without further ado.

Putting everything together, we write G_{ij} as

$$
G_{ij} = \mathcal{S}_\omega[G_{ij}] + \overline{\mathcal{S}}_\omega[G_{ij}^{(0)}] + \overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\mathfrak{m}||\mathfrak{n}} + \overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\rm fin,a_1} + \overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\rm num}, \tag{4.23}
$$

where four first terms on the right-hand side require further work, and the last term can be computed numerically by taking the $\varepsilon \to 0$ limit. It reads

$$
\begin{aligned}
\overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\rm num} &= \overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\rm fin,a_2} + \overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\rm fin,b} \\
&= 4\mathcal{N}_A\left\langle \int_0^\infty \frac{{\rm d}\omega}{\omega}\ln\rho_{\mathfrak{tm}}\left[\theta(\rho_{\mathfrak{tm}}-\omega\rho_{\mathfrak{tn}})-\theta(1-\omega)C_{\mathfrak{mn}}\right]\overline{S}_\omega\left[\omega^2\widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_{\mathfrak{mn}} \\
&\quad - 2\mathcal{N}_A\left\langle \int_0^\infty \frac{{\rm d}\omega}{\omega}\ln\omega\left[\theta(\rho_{\mathfrak{tm}}-\omega\rho_{\mathfrak{tn}})-\theta(1-\omega)\right]\overline{S}_\omega\left[\omega^2\widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_{\mathfrak{mn}}.
\end{aligned}
\tag{4.24}
$$

[5] We note that in the case of a soft $q\bar{q}$ pair, Δ_{ij} vanishes.

JHEP03(2026)117

We note that the four terms that require additional work have different degrees of divergence, that we illustrate below

$$\mathcal{S}_\omega[G_{ij}] \sim \mathcal{O}(\varepsilon^{-3}), \qquad \overline{\mathcal{S}}_\omega[G_{ij}^{(0)}] \sim \mathcal{O}(\varepsilon^{-2}),$$
$$\overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\mathfrak{m}||\mathfrak{n}} \sim \mathcal{O}(\varepsilon^{-1}), \quad \overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\mathrm{fin,a_1}} \sim \mathcal{O}(\varepsilon^{-1}).$$

Hence, the most challenging quantity to deal with is the strongly-ordered contribution $\mathcal{S}_\omega[G_{ij}]$. We discuss it in the next section.

4.1 The integral of the strongly-ordered eikonal function $\mathcal{S}_\omega[G_{ij}]$

In this section, we discuss the integration of the strongly-ordered eikonal function $S_\omega[G_{ij}]$,

$$\mathcal{S}_\omega[G_{ij}] = \frac{\mathcal{N}_A}{\varepsilon}\left\langle \int_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}} \rho_{t\mathfrak{m}}^{4\varepsilon} \theta(\rho_{t\mathfrak{m}} - \omega\rho_{t\mathfrak{n}}) S_\omega\left[\omega^2 \widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_{\mathfrak{m}\mathfrak{n}}. \tag{4.25}$$

For the two-gluon case, we find

$$\begin{aligned} S_\omega\left[\omega^2 \widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right] &= \frac{2}{\rho_{\mathfrak{m}j}\rho_{\mathfrak{m}\mathfrak{n}}} + \frac{2}{\rho_{\mathfrak{m}\mathfrak{n}}\rho_{\mathfrak{n}j}} - \frac{2}{\rho_{\mathfrak{m}j}\rho_{\mathfrak{n}j}} - \frac{1}{\rho_{\mathfrak{m}\mathfrak{n}}} + \frac{1}{\rho_{\mathfrak{n}j}} - \frac{\rho_{\mathfrak{m}j}}{\rho_{\mathfrak{m}\mathfrak{n}}\rho_{\mathfrak{n}j}} \\ &\quad + (1-v_{ij}^2)\left(\frac{1}{\rho_{\mathfrak{m}j}^2\rho_{\mathfrak{n}j}} - \frac{1}{\rho_{\mathfrak{m}j}^2\rho_{\mathfrak{m}\mathfrak{n}}} - \frac{1}{\rho_{\mathfrak{m}j}\rho_{\mathfrak{m}\mathfrak{n}}\rho_{\mathfrak{n}j}}\right). \end{aligned} \tag{4.26}$$

The integration over ω is straightforward

$$\int_0^\infty \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}} \rho_{t\mathfrak{m}}^{4\varepsilon} \theta(\rho_{t\mathfrak{m}} - \omega\rho_{t\mathfrak{n}}) = -\frac{1}{2\varepsilon} \rho_{t\mathfrak{m}}^{2\varepsilon}\rho_{t\mathfrak{n}}^{2\varepsilon}, \tag{4.27}$$

and we obtain

$$\begin{aligned} \mathcal{S}_\omega[G_{ij}] = -\frac{\mathcal{N}_A}{2\varepsilon^2}\Bigg\langle \rho_{t\mathfrak{m}}^{2\varepsilon}\rho_{t\mathfrak{n}}^{2\varepsilon}\Bigg[&\frac{2}{\rho_{\mathfrak{m}j}\rho_{\mathfrak{m}\mathfrak{n}}} + \frac{2}{\rho_{\mathfrak{m}\mathfrak{n}}\rho_{\mathfrak{n}j}} - \frac{2}{\rho_{\mathfrak{m}j}\rho_{\mathfrak{n}j}} - \frac{1}{\rho_{\mathfrak{m}\mathfrak{n}}} + \frac{1}{\rho_{\mathfrak{n}j}} - \frac{\rho_{\mathfrak{m}j}}{\rho_{\mathfrak{m}\mathfrak{n}}\rho_{\mathfrak{n}j}} \\ &+(1-v_{ij}^2)\left(\frac{1}{\rho_{\mathfrak{m}j}^2\rho_{\mathfrak{n}j}} - \frac{1}{\rho_{\mathfrak{m}j}^2\rho_{\mathfrak{m}\mathfrak{n}}} - \frac{1}{\rho_{\mathfrak{m}j}\rho_{\mathfrak{m}\mathfrak{n}}\rho_{\mathfrak{n}j}}\right)\Bigg]\Bigg\rangle_{\mathfrak{m}\mathfrak{n}}. \end{aligned} \tag{4.28}$$

This expression involves terms with and without $\rho_{\mathfrak{m}\mathfrak{n}}$, and we find it convenient to separate them. Hence, we write

$$\mathcal{S}_\omega[G_{ij}] = \mathcal{S}_\omega[G_{ij}]_{\mathrm{F}} + \mathcal{S}_\omega[G_{ij}]_{\mathrm{NF}}. \tag{4.29}$$

The first term does not contain $1/\rho_{\mathfrak{m}\mathfrak{n}}$; it reads

$$\mathcal{S}_\omega[G_{ij}]_{\mathrm{F}} = -\frac{\mathcal{N}_A}{2\varepsilon^2}\left\langle \frac{\rho_{t\mathfrak{n}}^{2\varepsilon}}{\rho_{\mathfrak{n}j}}\right\rangle_{\mathfrak{n}} \left\langle \rho_{t\mathfrak{m}}^{2\varepsilon}\left[1 - \frac{2}{\rho_{\mathfrak{m}j}} + \frac{1-v_{ij}^2}{\rho_{\mathfrak{m}j}^2}\right]\right\rangle_{\mathfrak{m}}. \tag{4.30}$$

The angular integrations are actually finite, so that $\mathcal{S}_\omega[G_{ij}]_{\mathrm{F}} \sim \varepsilon^{-2}$. Using integrals defined in the appendix, we derive

$$S_\omega[G_{ij}]_{\mathrm{F}} = -\frac{\mathcal{N}_A}{2\varepsilon^2}\, I_{-2\varepsilon,1}^{(2)}\left[I_{-2\varepsilon}^{(1)}[\rho_{tt}] - 2I_{-2\varepsilon,1}^{(2)} + (1-v_{ij}^2) I_{-2\varepsilon,2}^{(2)}\right]. \tag{4.31}$$

JHEP03(2026)117

We note that unless their arguments are shown explicitly, all integrals in the above formula should be interpreted as $I[\rho_{tt}, \rho_{jj}, \rho_{tj}]$.[6]

The remaining, *non-factorizable*, part is more complicated. It reads

$$\mathcal{S}_\omega[G_{ij}]_{\rm NF} = -\frac{\mathcal{N}_A}{2\varepsilon^2}\left\langle \frac{\rho_{t\mathfrak{m}}^{2\varepsilon}\rho_{t\mathfrak{n}}^{2\varepsilon}}{\rho_{\mathfrak{m}\mathfrak{n}}}\left[\frac{2}{\rho_{\mathfrak{m}j}} + \frac{2}{\rho_{\mathfrak{n}j}} - 1 - \frac{\rho_{\mathfrak{m}j}}{\rho_{\mathfrak{n}j}} - (1-v_{ij}^2)\left(\frac{1}{\rho_{\mathfrak{m}j}^2} + \frac{1}{\rho_{\mathfrak{m}j}\rho_{\mathfrak{n}j}}\right)\right]\right\rangle_{\mathfrak{m}\mathfrak{n}}. \tag{4.32}$$

To simplify eq. (4.32), we use an opportunity to rename $\mathfrak{m} \leftrightarrow \mathfrak{n}$, rewriting it in the following way

$$\mathcal{S}_\omega[G_{ij}]_{\rm NF} = -\frac{\mathcal{N}_A}{2\varepsilon^2}\left\langle \frac{\rho_{t\mathfrak{m}}^{2\varepsilon}\rho_{t\mathfrak{n}}^{2\varepsilon}}{\rho_{\mathfrak{m}\mathfrak{n}}}\left[\frac{4-\rho_{\mathfrak{n}j}}{\rho_{\mathfrak{m}j}} - 1 - (1-v_{ij}^2)\left(\frac{1}{\rho_{\mathfrak{m}j}^2} + \frac{1}{\rho_{\mathfrak{m}j}\rho_{\mathfrak{n}j}}\right)\right]\right\rangle_{\mathfrak{m}\mathfrak{n}}. \tag{4.33}$$

Furthermore, we find it convenient to write

$$\rho_{t\mathfrak{m}}^{2\varepsilon}\rho_{t\mathfrak{n}}^{2\varepsilon} = \frac{\rho_{t\mathfrak{m}}^{4\varepsilon} + \rho_{t\mathfrak{n}}^{4\varepsilon}}{2} - \frac{\left(\rho_{t\mathfrak{m}}^{2\varepsilon} - \rho_{t\mathfrak{n}}^{2\varepsilon}\right)^2}{2}. \tag{4.34}$$

The last term on the right-hand side in eq. (4.34) provides a finite contribution to $S_\omega[G_{ij}]_{\rm NF}$, which can be computed numerically right away. Hence, we define

$$\mathcal{S}_\omega[G_{ij}]_{\rm NF}^{\rm num} = \frac{\mathcal{N}_A}{4\varepsilon^2}\left\langle \frac{\left(\rho_{t\mathfrak{m}}^{2\varepsilon} - \rho_{t\mathfrak{n}}^{2\varepsilon}\right)^2}{\rho_{\mathfrak{m}\mathfrak{n}}}\left[\frac{4-\rho_{\mathfrak{n}j}}{\rho_{\mathfrak{m}j}} - 1 - (1-v_{ij}^2)\left(\frac{1}{\rho_{\mathfrak{m}j}^2} + \frac{1}{\rho_{\mathfrak{m}j}\rho_{\mathfrak{n}j}}\right)\right]\right\rangle_{\mathfrak{m}\mathfrak{n}}. \tag{4.35}$$

The remaining contribution reads

$$\mathcal{S}_\omega[G_{ij}]_{\rm NF}^{\rm div} = -\frac{\mathcal{N}_A}{4\varepsilon^2}\left\langle \frac{\rho_{t\mathfrak{m}}^{4\varepsilon} + \rho_{t\mathfrak{n}}^{4\varepsilon}}{\rho_{\mathfrak{m}\mathfrak{n}}}\left[\frac{4-\rho_{\mathfrak{n}j}}{\rho_{\mathfrak{m}j}} - 1 - (1-v_{ij}^2)\left(\frac{1}{\rho_{\mathfrak{m}j}^2} + \frac{1}{\rho_{\mathfrak{m}j}\rho_{\mathfrak{n}j}}\right)\right]\right\rangle_{\mathfrak{m}\mathfrak{n}}. \tag{4.36}$$

Performing further $\mathfrak{m} \leftrightarrow \mathfrak{n}$ redefinition, we remove $\rho_{t\mathfrak{n}}^{4\varepsilon}$ from the integrand and obtain

$$\mathcal{S}_\omega[G_{ij}]_{\rm NF}^{\rm div} = -\frac{\mathcal{N}_A}{4\varepsilon^2}\left\langle \frac{\rho_{t\mathfrak{m}}^{4\varepsilon}}{\rho_{\mathfrak{m}\mathfrak{n}}}\left[\frac{4-\rho_{\mathfrak{n}j}}{\rho_{\mathfrak{m}j}} + \frac{4-\rho_{\mathfrak{m}j}}{\rho_{\mathfrak{n}j}} - 2 - (1-v_{ij}^2)\left(\frac{1}{\rho_{\mathfrak{m}j}^2} + \frac{1}{\rho_{\mathfrak{n}j}^2} + \frac{2}{\rho_{\mathfrak{m}j}\rho_{\mathfrak{n}j}}\right)\right]\right\rangle_{\mathfrak{m}\mathfrak{n}}. \tag{4.37}$$

To write the above equation in a form convenient for further analysis, we introduce the following integral family (see appendix B.1 for more details)

$$I^{(2)}_{a_1,a_2,a_3,a_4+b_4\varepsilon} = \left\langle \frac{1}{\rho_{\mathfrak{m}\mathfrak{n}}^{a_1}\rho_{\mathfrak{m}j}^{a_2}\rho_{\mathfrak{n}j}^{a_3}\rho_{t\mathfrak{m}}^{a_4+b_4\varepsilon}}\right\rangle_{\mathfrak{m}\mathfrak{n}}, \tag{4.38}$$

and use it to write

$$\begin{aligned} S_\omega[G_{ij}]_{\rm NF}^{\rm div} = -\frac{\mathcal{N}_A}{4\varepsilon^2}\Big\{ & 4I^{(2)}_{1,1,0,-4\varepsilon} + 4I^{(2)}_{1,0,1,-4\varepsilon} - I^{(2)}_{1,1,-1,-4\varepsilon} - I^{(2)}_{1,-1,1,-4\varepsilon} - 2I^{(2)}_{1,0,0,-4\varepsilon} \\ & - (1-v_{ij}^2)\left[I^{(2)}_{1,2,0,-4\varepsilon} + I^{(2)}_{1,0,2,-4\varepsilon} + 2I^{(2)}_{1,1,1,-4\varepsilon}\right]\Big\}. \end{aligned} \tag{4.39}$$

[6]To avoid confusion, we note that in this section we always work in the rest frame of the parton i. This implies that $\rho_{tt} = 1 - v_t^2$, $\rho_{jj} = 1 - v_{ij}^2, \rho_{tj} = 1 - v_{ij} v_t \vec{n}_t \cdot \vec{n}_j$. The relation of various quantities in the above equations to their counterparts in the laboratory frame is given in eq. (2.22).

JHEP03(2026)117

Finally, the strongly-ordered contribution reads

$$\mathcal{S}_\omega[G_{ij}] = \mathcal{S}_\omega[G_{ij}]_{\rm F} + \mathcal{S}_\omega[G_{ij}]^{\rm div}_{\rm NF} + \mathcal{S}_\omega[G_{ij}]^{\rm num}_{\rm NF}, \tag{4.40}$$

where the relevant contributions are given in eqs. (4.31,4.35,4.39).

4.2 The soft-subtracted, double-collinear contirbution

As the next step, we analyse the soft-subtracted, double-collinear contribution. It reads

$$\overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\mathfrak{m}||\mathfrak{n}} = \frac{\mathcal{N}_A}{\varepsilon}\left\langle \int_0^\infty \frac{{\rm d}\omega}{\omega^{1+2\varepsilon}}(\rho_{t\mathfrak{m}}^{4\varepsilon}-1)\theta(1-\omega)C_{\mathfrak{m}\mathfrak{n}}\overline{\mathcal{S}}_\omega\left[\omega^2\widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_{\mathfrak{m}\mathfrak{n}}. \tag{4.41}$$

This term possesses a $1/\varepsilon$ singularity at most, because $\rho_{t\mathfrak{m}}^{4\varepsilon} - 1 \sim \mathcal{O}(\varepsilon)$. To isolate it, we compute the $\mathfrak{m}||\mathfrak{n}$ limit of the soft-subtracted eikonal function and find

$$C_{\mathfrak{m}\mathfrak{n}}\,\overline{S}_\omega\left[\omega^2\widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right] = \frac{4(\vec{n}_j\cdot\vec{r})^2\omega^2(1-\varepsilon)v_{ij}^2}{(\omega+1)^4\rho_{\mathfrak{m}j}^2\rho_{\mathfrak{m}\mathfrak{n}}} + \frac{4\omega\left[(\rho_{\mathfrak{m}j}-2)\,\rho_{\mathfrak{m}j}+1-v_{ij}^2\right]}{(\omega+1)^2\rho_{\mathfrak{m}j}^2\rho_{\mathfrak{m}\mathfrak{n}}}, \tag{4.42}$$

where $\vec{r}$ is the unit vector defined by the following equation

$$\vec{n}_\mathfrak{n} = \cos\theta_{\mathfrak{m}\mathfrak{n}}\vec{n}_\mathfrak{m} + \sin\theta_{\mathfrak{m}\mathfrak{n}}\vec{r}, \quad \vec{r}\cdot\vec{n}_\mathfrak{m} = 0. \tag{4.43}$$

We note that by setting $v_{ij}\to 1$ in eq. (4.42), we reproduce the soft-subtracted double-collinear contribution for the massive-massless case, discussed in ref. [21].

To proceed with the calculation of $\overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\mathfrak{m}||\mathfrak{n}}$, we integrate over directions of a parton $\mathfrak{n}$ in eq. (4.41), starting with the integration over $\vec{r}$. Using

$$r^i r^j \to \frac{\delta^{ij} - \vec{n}^i_\mathfrak{m}\vec{n}^j_\mathfrak{m}}{d-2}, \tag{4.44}$$

we obtain

$$\left\langle C_{\mathfrak{m}\mathfrak{n}}\overline{S}_\omega\left[\omega^2\widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_r = -2\frac{\omega(2+3\omega+2\omega^2)}{(1+\omega)^4}\left[\frac{(2-\rho_{\mathfrak{m}j})}{\rho_{\mathfrak{m}j}\rho_{\mathfrak{m}\mathfrak{n}}} - \frac{1-v_{ij}^2}{\rho_{\mathfrak{m}j}^2\rho_{\mathfrak{m}\mathfrak{n}}}\right]. \tag{4.45}$$

Using this result in eq. (4.41), we find

$$\begin{aligned}\overline{\mathcal{S}}_\omega[\Delta G_{ij}]_{\mathfrak{m}||\mathfrak{n}} = &-\frac{2\mathcal{N}_A}{\varepsilon}\int_0^\infty \frac{{\rm d}\omega}{\omega^{1+2\varepsilon}}\theta(1-\omega)\frac{\omega(2+3\omega+2\omega^2)}{(1+\omega)^4} \\ &\times\left\langle(\rho_{t\mathfrak{m}}^{4\varepsilon}-1)\left(\frac{(2-\rho_{\mathfrak{m}j})}{\rho_{\mathfrak{m}j}\rho_{\mathfrak{m}\mathfrak{n}}} - \frac{1-v_{ij}^2}{\rho_{\mathfrak{m}j}^2\rho_{\mathfrak{m}\mathfrak{n}}}\right)\right\rangle_{\mathfrak{m}\mathfrak{n}}.\end{aligned} \tag{4.46}$$

We integrate over the energy ω, and obtain

$$\begin{aligned}\gamma_\omega = \int_0^1 \frac{{\rm d}\omega}{\omega^{1+2\varepsilon}}\frac{\omega(2+3\omega+2\omega^2)}{(1+\omega)^4} = \frac{11}{12} + \varepsilon\left(\frac{1}{12}+\frac{11}{3}\ln(2)\right) \\ + \varepsilon^2\left(\frac{1}{3}+\frac{11}{18}\pi^2\right) + \varepsilon^3\left(\frac{4}{3}\ln(2)+11\zeta_3\right) + \mathcal{O}\left(\varepsilon^4\right).\end{aligned} \tag{4.47}$$

The remaining integration over directions of $\mathfrak{n}$ is straightforward. Putting everything together, we find

$$\overline{S}_\omega[\Delta G_{ij}]_{\mathfrak{m}||\mathfrak{n}} = \frac{(1-2\varepsilon)\mathcal{N}_A\gamma_\omega}{\varepsilon^2}\left\langle(\rho_{t\mathfrak{m}}^{4\varepsilon}-1)\left[\frac{(2-\rho_{\mathfrak{m}j})}{\rho_{\mathfrak{m}j}} - \frac{1-v_{ij}^2}{\rho_{\mathfrak{m}j}^2}\right]\right\rangle_{\mathfrak{m}}. \tag{4.48}$$

It follows from eq. (4.48), that $\overline{S}_\omega[\Delta G_{ij}]_{\mathfrak{m}||\mathfrak{n}}$ has a $1/\varepsilon$ singularity, since the integrand is $\mathcal{O}(\varepsilon)$. Finally, we express eq. (4.48) through integrals described in the appendix and find

$$\begin{aligned}\overline{S}_\omega[\Delta G_{ij}]_{\mathfrak{m}||\mathfrak{n}} = \frac{1-2\varepsilon}{\varepsilon^2}\mathcal{N}_A\gamma_\omega\Big\{&2I^{(2)}_{-4\varepsilon,1} - 2I^{(1)}_1[\rho_{jj}] - I^{(1)}_{-4\varepsilon}[\rho_{jj}] + 1\\ &- (1-v_{ij}^2)\left[I^{(2)}_{-4\varepsilon,2} - I^{(1)}_2[\rho_{jj}]\right]\Big\}.\end{aligned} \tag{4.49}$$

4.3 The quantity $\overline{S}_\omega[\Delta G_{ij}]_{\mathrm{fin,a_1}}$

In this section, we compute the quantity $\overline{S}_\omega[\Delta G_{ij}]_{\mathrm{fin,a_1}}$ defined in eq. (4.21). In that equation, one can integrate over the gluon energy ω. This gives

$$\overline{S}_\omega[\Delta G_{ij}]_{\mathrm{fin,a_1}} = -\frac{\mathcal{N}_A}{4\varepsilon^2}\left\langle\left[\left(\frac{\rho_{t\mathfrak{m}}}{\rho_{t\mathfrak{n}}}\right)^{2\varepsilon} - 1\right]\Delta_{ij}\right\rangle_{\mathfrak{m}\mathfrak{n}}. \tag{4.50}$$

The function Δ_{ij} defined in eq. (4.19) is anti-symmetric w.r.t. $\mathfrak{m}\leftrightarrow\mathfrak{n}$ permutations. Using this, we find

$$\overline{S}_\omega[\Delta G_{ij}]_{\mathrm{fin,a_1}} = \frac{\mathcal{N}_A}{8\varepsilon^2}\left\langle\left[\left(\frac{\rho_{t\mathfrak{n}}}{\rho_{t\mathfrak{m}}}\right)^{2\varepsilon} - \left(\frac{\rho_{t\mathfrak{m}}}{\rho_{t\mathfrak{n}}}\right)^{2\varepsilon}\right]\Delta_{ij}\right\rangle_{\mathfrak{m}\mathfrak{n}}. \tag{4.51}$$

Since

$$\left(\frac{\rho_{t\mathfrak{n}}}{\rho_{t\mathfrak{m}}}\right)^{2\varepsilon} - \left(\frac{\rho_{t\mathfrak{m}}}{\rho_{t\mathfrak{n}}}\right)^{2\varepsilon} = 4\varepsilon\ln\frac{\rho_{t\mathfrak{n}}}{\rho_{t\mathfrak{m}}} + \mathcal{O}(\varepsilon^3), \tag{4.52}$$

and $\mathcal{O}(\varepsilon^3)$ contribution is irrelevant for computing $\overline{S}_\omega[\Delta G_{ij}]_{\mathrm{fin,a_1}}$ through $\mathcal{O}(\varepsilon^0)$, it follows that the following equations hold

$$\overline{S}_\omega[\Delta G_{ij}]_{\mathrm{fin,a_1}} = \frac{\mathcal{N}_A}{8\varepsilon}\left\langle 4\ln\frac{\rho_{t\mathfrak{n}}}{\rho_{t\mathfrak{m}}}\,\Delta_{ij} + \mathcal{O}(\varepsilon^2)\right\rangle_{\mathfrak{m}\mathfrak{n}} = \frac{\mathcal{N}_A}{4\varepsilon^2}\left\langle\left[\rho_{t\mathfrak{m}}^{-2\varepsilon} - \rho_{t\mathfrak{m}}^{2\varepsilon}\right]\Delta_{ij}\right\rangle_{\mathfrak{m}\mathfrak{n}} + \mathcal{O}(\varepsilon). \tag{4.53}$$

Using the explicit expression for Δ_{ij}, it is straightforward to write the above expression in terms of integrals described in the appendix. We find

$$\begin{aligned}\overline{S}_\omega[\Delta G_{ij}]_{\mathrm{fin,a_1}} = \frac{\mathcal{N}_A}{4\varepsilon^2}\Big\{&\Big[I^{(2)}_{0,0,1,2\varepsilon} - I^{(2)}_{0,1,0,2\varepsilon} - I^{(2)}_{1,-1,1,2\varepsilon} + I^{(2)}_{1,1,-1,2\varepsilon}\\ &+ (1-v_{ij}^2)I^{(2)}_{0,2,1,2\varepsilon} - I^{(2)}_{0,1,2,2\varepsilon} - I^{(2)}_{1,2,0,2\varepsilon} + I^{(2)}_{1,0,2,2\varepsilon}\Big] - (2\varepsilon\to-2\varepsilon)\Big\},\end{aligned} \tag{4.54}$$

where the change in the sign of ε only applies to the last index of integrals $I^{(2)}_{a,b,c,d,2\varepsilon}$.

4.4 Calculation of $\overline{\mathcal{S}}_\omega[G_{ij}^{(0)}]$

The last quantity we require is $\overline{\mathcal{S}}_\omega[G_{ij}^{(0)}]$; it is defined in eq. (4.7). We remind the reader that this quantity can be understood as the integral of the eikonal function in the case where the rest frame of the parton i and the laboratory frame coincide, and where the parton j moves with the velocity v_{ij}. To compute $\overline{\mathcal{S}}_\omega[G_{ij}^{(0)}]$, we write it as an integral over the energy ω

$$\overline{\mathcal{S}}_\omega[G_{ij}^{(0)}] = \frac{\mathcal{N}_A}{\varepsilon}\int_0^1 \mathrm{d}\omega\, \mathcal{A}_{ij}(\omega). \tag{4.55}$$

To obtain the function $\mathcal{A}_{ij}(\omega)$, we integrate over the angles of partons $\mathfrak{m}$ and $\mathfrak{n}$ at fixed ω. The function $\mathcal{A}_{ij}(\omega)$, defined as

$$\mathcal{A}_{ij}(\omega) = \left\langle \omega^{-1-2\varepsilon}\overline{\mathcal{S}}_\omega \left[\omega^2 \widetilde{S}_{ij}(\mathfrak{m},\mathfrak{n})\right]\right\rangle_{\mathfrak{m}\mathfrak{n}}, \tag{4.56}$$

can be written as an integral over the gluon momenta with fixed energies

$$\mathcal{A}_{ij}(\omega) = \int \frac{[\mathrm{d}l_\mathfrak{m}][\mathrm{d}l_\mathfrak{n}]}{\mathcal{N}_\varepsilon^2}\,\delta(1-l_\mathfrak{m}\cdot P)\,\delta(\omega - l_\mathfrak{n}\cdot P)\,\omega^{-2}\overline{\mathcal{S}}_\omega\left[\omega^2\widetilde{S}_{ij}(l_\mathfrak{m},l_\mathfrak{n})\right]. \tag{4.57}$$

The quantity $\mathcal{N}_\varepsilon$ is defined in eq. (3.3), and $P = (1,\vec{0})$. The function $\mathcal{A}_{ij}(\omega)$ can be computed using reverse unitarity [26] in a straightforward way. Using the integration-by-parts technology [27, 28], we express $\mathcal{A}_{ij}(\omega)$ in terms of eight master integrals

$$\begin{aligned}
&J_1 = \langle 1\rangle, && J_2 = \left\langle \frac{1}{D_2}\right\rangle, && J_3 = \left\langle \frac{1}{D_3}\right\rangle, && J_4 = \left\langle \frac{1}{D_2D_3}\right\rangle,\\
&J_5 = \left\langle \frac{1}{D_1D_4}\right\rangle, && J_6 = \left\langle \frac{1}{D_4^2}\right\rangle, && J_7 = \left\langle \frac{1}{D_4}\right\rangle, && J_8 = \left\langle \frac{1}{D_2D_4}\right\rangle.
\end{aligned} \tag{4.58}$$

The four inverse propagators $D_{1,\dots,4}$ read

$$D_1 = l_\mathfrak{m}\cdot l_\mathfrak{n},\; D_2 = l_\mathfrak{m}\cdot p_j,\; D_3 = l_\mathfrak{n}\cdot p_j,\; D_4 = l_\mathfrak{m}\cdot p_j + l_\mathfrak{n}\cdot p_j, \tag{4.59}$$

and the integration measure is defined according to the following equation

$$\left\langle X\right\rangle = \int \frac{[\mathrm{d}l_\mathfrak{m}][\mathrm{d}l_\mathfrak{n}]}{\mathcal{N}_\varepsilon^2}\delta(1-l_\mathfrak{m}\cdot P)\delta(\omega - l_\mathfrak{n}\cdot P)\, X. \tag{4.60}$$

The four-momentum of the parton j is $p_j = (1,\vec{v}_{ij})$, and $\overline{\mathcal{S}}_\omega[G_{ij}^{(0)}]$ depends on the absolute value of the relative velocity v_{ij}. It turns out to be convenient to write the result in terms of the following variable

$$\eta_{ij} = \frac{1-v_{ij}}{1+v_{ij}}. \tag{4.61}$$

We use the differential equations in v_{ij} to compute the required integrals; the boundary conditions are easily obtained by computing the relevant integrals at $v_{ij}=0$. The calculation is described in appendix C. Using the results for the master integrals, we find

$$\begin{aligned}
\overline{\mathcal{S}}_\omega[G_{ij}^{(0)}] = \frac{\mathcal{N}_A}{\varepsilon^2}\Bigg[&-\frac{11}{6} - \frac{11\ln(\eta_{ij})}{12v_{ij}} + \varepsilon\Bigg(\frac{\ln^2(\eta_{ij})}{2} + \frac{83}{9} - \frac{22\ln(2)}{3} + \frac{1}{v_{ij}}\Bigg[\frac{\ln^3(\eta_{ij})}{24}\\
&+\frac{2\ln^2(\eta_{ij})}{3} - \frac{8}{3}\mathrm{Li}_2(1-\eta_{ij}) - \frac{11}{3}\ln(2)\ln(\eta_{ij}) + \frac{131\ln(\eta_{ij})}{18}\Bigg] + \frac{1}{v_{ij}^2}\Bigg[-\frac{\ln^3(\eta_{ij})}{24}\\
&-\frac{\ln^2(\eta_{ij})}{4} - \frac{\pi^2}{12}\ln(\eta_{ij}) + \mathrm{Li}_3(\eta_{ij}) - \frac{1}{2}\ln(\eta_{ij})\mathrm{Li}_2(\eta_{ij}) - \zeta_3\Bigg]\Bigg) + \mathcal{O}(\varepsilon^2)\Bigg].
\end{aligned} \tag{4.62}$$

We do not show here the $\mathcal{O}(\varepsilon^2)$ term in the square brackets, required to obtain $\overline{\mathcal{S}}_\omega[G_{ij}^{(0)}]$ through $\mathcal{O}(\varepsilon^0)$. The corresponding result can be found in the ancillary file provided with this submission.

5 The final results: analysis, implementation and checks

The final result for the integral of the double-emission eikonal function for two soft gluons is obtained by combining different terms displayed in eq. (4.23). The strongly-order contribution $\mathcal{S}_\omega[G_{ij}]$ is further split into analytic and numerical pieces as shown in eq. (4.40). A similar computation has been performed for the soft $q\bar{q}$ pair; for brevity, we do not discuss it and proceed directly to the results.

5.1 Analytic expressions of the poles

In the preceding section we have isolated all divergent contributions needed for the computation of the integrated eikonal function. Hence, we are in the position to combine them, and obtain the divergent part of the integral in an analytic form.

Many integrals required for this computation are obtained by solving the differential equations discussed in appendices; the results are naturally expressed in terms of generalized polylogarithms (GPLs) [29, 30]. Unfortunately, arguments of these GPLs are very complicated, especially because multiple square roots of kinematic variables appear in the course of simplifying systems of differential equations. This leads to significant problems when a numerical evaluation of such an analytic result is attempted.

To devise a representation of the analytic result suitable for fast numerical evaluation, we follow the same procedures as in ref. [21]. To this end, we employ the symbol technique [31, 32] and use it to express all GPLs up to weight four through logarithms, classical polylogarithms Li_n $(n = 2, 3, 4)$ and an additional function $\mathrm{Li}_{2,2}$, defined as

$$\mathrm{Li}_{2,2}(z_1, z_2) = \sum_{i>j>0}^{\infty} \frac{z_1^i z_2^j}{i^2 j^2} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{z_1^i}{(i+j)^2}\frac{(z_1 z_2)^j}{j^2}. \tag{5.1}$$

Since the problem of finding suitable candidate functions to express linear combinations of GPL's in terms of polylogs and $\mathrm{Li}_{2,2}$ is equivalent to an optimization problem, we used the program `Gurobi` [33] to improve the efficiency of this procedure. In the process of manipulating GPLs, we heavily relied on the package `PolyLogTools` [34–36] and used the library `GiNaC` [37, 38] to evaluate GPLs numerically. Finally, we make sure that all functions that appear in the final result are real-valued in the physical region.

Below we showcase the divergent parts of the integrated double-emission eikonal functions for both gluon and quark cases. We write

$$\begin{aligned}
\frac{\mathcal{S}[\widetilde{S}_{ij}]}{\mathcal{N}_A} = \frac{\overline{\mathcal{S}}_\omega[G_{ij}^{(0)}]}{\mathcal{N}_A} &- \frac{1}{\varepsilon^3}\left[1 + \frac{\ln(\eta_{ij})}{v_{ij}}\right] + \frac{1}{\varepsilon^2}\left\{5 - 2\ln\left(1-\beta_i^2\right) + \frac{\ln(\eta_i)}{\beta_i} + \frac{\ln(\eta_j)}{\beta_j}\right. \\
&\left. + \frac{q_3}{v_{ij}} + \frac{1}{v_{ij}^2}\frac{\ln^2(\eta_{ij})}{4}\right\} + \frac{1}{\varepsilon}\left\{-\frac{\ln(\eta_i)\ln(\eta_{ij})}{2\beta_i v_{ij}} - \frac{\ln(\eta_{ij})\ln(\eta_j)}{2v_{ij}\beta_j}\right. \\
&\left. + \frac{q_2}{\beta_i} + \frac{q_7}{v_{ij}} + \frac{q_5}{v_{ij}^2} + \frac{q_1}{\beta_j} + q_4\right\} + \mathcal{O}(\varepsilon^0),
\end{aligned} \tag{5.2}$$

$$\frac{\mathcal{S}[\widetilde{\mathcal{I}}_{ij}]}{\mathcal{N}_A} = \frac{1}{\varepsilon^2}\left[-\frac{\ln(\eta_{ij})}{6v_{ij}} - \frac{1}{3}\right] + \frac{1}{\varepsilon}\left\{\frac{37}{18} - \frac{2}{3}\ln\left(1-\beta_i^2\right) + \frac{\ln(\eta_i)}{3\beta_i} + \frac{\ln(\eta_j)}{3\beta_j} + \frac{q_6}{v_{ij}}\right\} + \mathcal{O}(\varepsilon^0), \tag{5.3}$$

where η_{ij} is defined in eq. (4.61) and

$$\eta_i = \frac{1-\beta_i}{1+\beta_i}, \quad \eta_j = \frac{1-\beta_j}{1+\beta_j}. \tag{5.4}$$

The divergent part of the quantity $\overline{\mathcal{S}}_\omega[G_{ij}^{(0)}]$ can be found in eq. (4.62). The results shown in eqs. (5.2,5.3) contain quantities $q_{1,\ldots,7}$. They read

$$\begin{aligned}
q_1 =\,& 4\mathrm{Li}_2\left(1-\frac{\eta_i\eta_{ij}}{z^2}\right) - \mathrm{Li}_2\left(1-\frac{\eta_i\eta_{ij}}{z}\right) - \mathrm{Li}_2\left(1-\frac{\eta_i}{z}\right) - \mathrm{Li}_2\left(1-\frac{\eta_{ij}}{z}\right) + \mathrm{Li}_2(1-z)\\
&-\frac{3}{2}\ln^2(\eta_i) - 4\ln(2)\ln(\eta_i) + 4\ln(\eta_i)\ln(\eta_i+1) - \frac{1}{2}\ln(\eta_i)\ln(\eta_{ij}) + 4\ln(\eta_i+1)\ln(\eta_{ij})\\
&+\ln(z)\ln(\eta_i) - 8\ln(z)\ln(\eta_i+1) + \frac{1}{2}\ln^2(\eta_{ij}) - 4\ln(2)\ln(\eta_{ij}) - 3\ln(z)\ln(\eta_{ij}) + \frac{7\ln^2(z)}{2}\\
&+8\ln(2)\ln(z) - \frac{19\ln(z)}{3} + \frac{19\ln(\eta_i)}{6} + \frac{19\ln(\eta_{ij})}{6},\\
q_2 =\,& \mathrm{Li}_2\left(1-\frac{\eta_i\eta_{ij}}{z}\right) - 4\mathrm{Li}_2(1-\eta_i) + \mathrm{Li}_2\left(1-\frac{\eta_i}{z}\right) - \mathrm{Li}_2\left(1-\frac{\eta_{ij}}{z}\right) + \mathrm{Li}_2(1-z)\\
&+\frac{3}{2}\ln^2(\eta_i) + 4\ln(2)\ln(\eta_i) - 4\ln(\eta_i)\ln(\eta_i+1) + \frac{1}{2}\ln(\eta_i)\ln(\eta_{ij}) + \frac{\ln^2(z)}{2}\\
&-\ln(z)\ln(\eta_i) - \frac{19\ln(\eta_i)}{6},\\
q_3 =\,& 2\mathrm{Li}_2\left(1-\frac{\eta_i\eta_{ij}}{z}\right) - 2\mathrm{Li}_2\left(1-\frac{\eta_i}{z}\right) + 2\mathrm{Li}_2\left(1-\frac{\eta_{ij}}{z}\right) + 2\mathrm{Li}_2(1-z) - 3\mathrm{Li}_2(1-\eta_{ij})\\
&+2\ln(\eta_i+1)\ln(\eta_{ij}) + \frac{1}{4}\ln^2(\eta_{ij}) - 2\ln(2)\ln(\eta_{ij}) - 2\ln(z)\ln(\eta_{ij}) + \ln^2(z)\\
&+2\ln(\eta_{ij}),\\
q_4 =\,& -3\ln^2(\eta_i) - 8\ln^2(\eta_i+1) - 8\ln(2)\ln(\eta_i) + 8\ln(\eta_i)\ln(\eta_i+1) + 16\ln(2)\ln(\eta_i+1)\\
&-\ln(\eta_i)\ln(\eta_{ij}) + 2\ln(z)\ln(\eta_i) - \frac{1}{4}\ln^2(\eta_{ij}) + 2\ln(z)\ln(\eta_{ij}) - 2\ln^2(z) - 8\ln^2(2)\\
&+\frac{38\ln(2)}{3} - \frac{38}{3}\ln(\eta_i+1) + \frac{19\ln(\eta_i)}{3} - \frac{16}{3},\\
q_5 =\,& \ln(\eta_{ij})\mathrm{Li}_2\left(1-\frac{\eta_i}{z}\right) - \ln(\eta_{ij})\mathrm{Li}_2\left(1-\frac{\eta_i\eta_{ij}}{z}\right) + \ln(\eta_{ij})\mathrm{Li}_2(1-\eta_{ij}) - \mathrm{Li}_2(1-z)\ln(\eta_{ij})\\
&-\ln(\eta_{ij})\mathrm{Li}_2\left(1-\frac{\eta_{ij}}{z}\right) - \frac{1}{4}\ln^3(\eta_{ij}) + \ln(2)\ln^2(\eta_{ij}) + \ln(z)\ln^2(\eta_{ij}) - \frac{1}{2}\ln^2(z)\ln(\eta_{ij})\\
&-\ln(\eta_i+1)\ln^2(\eta_{ij}) - \frac{3}{4}\ln^2(\eta_{ij}),\\
q_6 =\,& \frac{2}{3}\mathrm{Li}_2\left(1-\frac{\eta_i\eta_{ij}}{z}\right) - \frac{2}{3}\mathrm{Li}_2\left(1-\frac{\eta_i}{z}\right) - \frac{2\mathrm{Li}_2(1-\eta_{ij})}{3} + \frac{2}{3}\mathrm{Li}_2\left(1-\frac{\eta_{ij}}{z}\right) + \frac{2\mathrm{Li}_2(1-z)}{3}\\
&+\frac{1}{6}\ln^2(\eta_{ij}) - \frac{4}{3}\ln(2)\ln(\eta_{ij}) - \frac{2}{3}\ln(z)\ln(\eta_{ij}) + \frac{2}{3}\ln(\eta_i+1)\ln(\eta_{ij})\\
&+\frac{\ln^2(z)}{3} + \frac{37\ln(\eta_{ij})}{36},
\end{aligned}$$

$$
\begin{aligned}
q_7 = {}& 6\mathrm{Li}_3\left(\frac{\eta_i-z}{\eta_i\,(1-\eta_{ij})}\right)-6\mathrm{Li}_3\left(\frac{(\eta_i-z)\,\eta_{ij}}{z\,(1-\eta_{ij})}\right)-4\mathrm{Li}_3\left(-\frac{z\,(\eta_i-z)}{\eta_i\,(\eta_{ij}-z)}\right)\\
&+4\mathrm{Li}_3\left(-\frac{(\eta_i-z)\,\eta_{ij}}{z\,(\eta_{ij}-z)}\right)+2\mathrm{Li}_3\left(-\frac{\eta_i\eta_{ij}-z}{z}\right)-4\mathrm{Li}_3\left(-\frac{\eta_i\eta_{ij}-z}{(1-z)z}\right)\\
&+6\mathrm{Li}_3\left(-\frac{\eta_i\eta_{ij}-z}{z\,(1-\eta_{ij})}\right)-6\mathrm{Li}_3\left(-\frac{\eta_i\eta_{ij}-z}{\eta_i\,(1-\eta_{ij})}\right)-2\mathrm{Li}_3\left(\frac{\eta_i\eta_{ij}-z}{\eta_i\eta_{ij}}\right)\\
&+4\mathrm{Li}_3\left(-\frac{z\,(\eta_i\eta_{ij}-z)}{(1-z)\eta_i\eta_{ij}}\right)-4\mathrm{Li}_3\left(\frac{\eta_i\eta_{ij}-z}{\eta_{ij}-z}\right)+4\mathrm{Li}_3\left(\frac{\eta_i\eta_{ij}-z}{\eta_i\,(\eta_{ij}-z)}\right)\\
&+4\mathrm{Li}_3\left(\frac{\eta_i-z}{1-z}\right)-2\mathrm{Li}_3\left(-\frac{\eta_i-z}{z}\right)+2\mathrm{Li}_3\left(\frac{\eta_i-z}{\eta_i}\right)-4\mathrm{Li}_3\left(\frac{\eta_i-z}{(1-z)\eta_i}\right)\\
&-3\mathrm{Li}_3\,(1-\eta_{ij})+3\mathrm{Li}_3\left(-\frac{1-\eta_{ij}}{\eta_{ij}}\right)+6\mathrm{Li}_3\left(\frac{1-\eta_{ij}}{1-z}\right)-6\mathrm{Li}_3\left(\frac{z\,(1-\eta_{ij})}{(1-z)\eta_{ij}}\right)\\
&-6\mathrm{Li}_3\left(-\frac{1-\eta_{ij}}{\eta_{ij}-z}\right)+6\mathrm{Li}_3\left(-\frac{z\,(1-\eta_{ij})}{\eta_{ij}-z}\right)+2\mathrm{Li}_3\left(-\frac{\eta_{ij}-z}{z}\right)-2\mathrm{Li}_3\left(\frac{\eta_{ij}-z}{\eta_{ij}}\right)\\
&+2\mathrm{Li}_3(1-z)-2\mathrm{Li}_3\left(-\frac{1-z}{z}\right)+4\ln(\eta_i)\,\mathrm{Li}_2(1-z)-8\ln(\eta_i+1)\,\mathrm{Li}_2(1-z)\\
&+8\ln(2)\mathrm{Li}_2(1-z)+8\ln(\eta_{ij})\,\mathrm{Li}_2\left(-\frac{1-\eta_i}{2\eta_i}\right)+8\ln(\eta_{ij})\,\mathrm{Li}_2\left(\frac{1-\eta_i}{\eta_i+1}\right)\\
&-7\ln(\eta_i)\,\mathrm{Li}_2\left(1-\frac{\eta_i}{z}\right)+8\ln(\eta_i+1)\,\mathrm{Li}_2\left(1-\frac{\eta_i}{z}\right)-8\ln(2)\mathrm{Li}_2\left(1-\frac{\eta_i}{z}\right)\\
&-3\ln(z)\mathrm{Li}_2\left(\frac{\eta_i-z}{\eta_i\,(1-\eta_{ij})}\right)+3\ln(\eta_i)\,\mathrm{Li}_2\left(\frac{\eta_i-z}{\eta_i\,(1-\eta_{ij})}\right)+6\ln(\eta_{ij})\,\mathrm{Li}_2\left(\frac{\eta_i-z}{\eta_i\,(1-\eta_{ij})}\right)\\
&-9\ln(\eta_i)\,\mathrm{Li}_2\,(1-\eta_{ij})+12\ln(\eta_i+1)\,\mathrm{Li}_2\,(1-\eta_{ij})-12\ln(2)\mathrm{Li}_2\,(1-\eta_{ij})\\
&+5\ln(\eta_i)\,\mathrm{Li}_2\left(\frac{1-\eta_{ij}}{1-z}\right)+6\ln(z)\mathrm{Li}_2\left(\frac{z\,(1-\eta_{ij})}{(1-z)\eta_{ij}}\right)-\ln(\eta_i)\,\mathrm{Li}_2\left(\frac{z\,(1-\eta_{ij})}{(1-z)\eta_{ij}}\right)\\
&-6\ln(\eta_{ij})\,\mathrm{Li}_2\left(\frac{z\,(1-\eta_{ij})}{(1-z)\eta_{ij}}\right)-3\ln(z)\mathrm{Li}_2\left(\frac{(\eta_i-z)\,\eta_{ij}}{z\,(1-\eta_{ij})}\right)+3\ln(\eta_i)\,\mathrm{Li}_2\left(\frac{(\eta_i-z)\,\eta_{ij}}{z\,(1-\eta_{ij})}\right)\\
&+4\ln(z)\mathrm{Li}_2\left(-\frac{z\,(\eta_i-z)}{\eta_i\,(\eta_{ij}-z)}\right)-5\ln(\eta_{ij})\,\mathrm{Li}_2\left(-\frac{z\,(\eta_i-z)}{\eta_i\,(\eta_{ij}-z)}\right)-6\ln(z)\mathrm{Li}_2\left(-\frac{1-\eta_{ij}}{\eta_{ij}-z}\right)\\
&+5\ln(\eta_i)\,\mathrm{Li}_2\left(-\frac{1-\eta_{ij}}{\eta_{ij}-z}\right)-\ln(\eta_i)\,\mathrm{Li}_2\left(-\frac{z\,(1-\eta_{ij})}{\eta_{ij}-z}\right)+4\ln(z)\mathrm{Li}_2\left(-\frac{(\eta_i-z)\,\eta_{ij}}{z\,(\eta_{ij}-z)}\right)\\
&+\ln(\eta_{ij})\,\mathrm{Li}_2\left(-\frac{(\eta_i-z)\,\eta_{ij}}{z\,(\eta_{ij}-z)}\right)+4\ln(\eta_i)\,\mathrm{Li}_2\left(1-\frac{\eta_{ij}}{z}\right)-8\ln(\eta_i+1)\,\mathrm{Li}_2\left(1-\frac{\eta_{ij}}{z}\right)\\
&+8\ln(2)\mathrm{Li}_2\left(1-\frac{\eta_{ij}}{z}\right)+7\ln(\eta_i)\,\mathrm{Li}_2\left(\frac{\eta_i-z}{\eta_i\eta_{ij}-z}\right)+\ln(\eta_i)\,\mathrm{Li}_2\left(\frac{(\eta_i-z)\,\eta_{ij}}{\eta_i\eta_{ij}-z}\right)\\
&-4\ln(\eta_i)\,\mathrm{Li}_2\left(\frac{(1-z)\,(\eta_i-z)\,\eta_{ij}}{(\eta_{ij}-z)\,(\eta_i\eta_{ij}-z)}\right)-4\ln(\eta_i)\,\mathrm{Li}_2\left(\frac{(\eta_i-z)\,(\eta_{ij}-z)}{(1-z)\,(\eta_i\eta_{ij}-z)}\right)\\
&+7\ln(\eta_i)\,\mathrm{Li}_2\left(1-\frac{\eta_i\eta_{ij}}{z}\right)-8\ln(\eta_i+1)\,\mathrm{Li}_2\left(1-\frac{\eta_i\eta_{ij}}{z}\right)+8\ln(2)\mathrm{Li}_2\left(1-\frac{\eta_i\eta_{ij}}{z}\right)
\end{aligned}
$$

JHEP03(2026)117

$$
\begin{aligned}
&-4\ln(z)\mathrm{Li}_2\left(-\frac{\eta_i\eta_{ij}-z}{(1-z)z}\right)-\ln(\eta_{ij})\,\mathrm{Li}_2\left(-\frac{\eta_i\eta_{ij}-z}{(1-z)z}\right)+3\ln(z)\mathrm{Li}_2\left(-\frac{\eta_i\eta_{ij}-z}{z\,(1-\eta_{ij})}\right)\\
&-3\ln(\eta_i)\,\mathrm{Li}_2\left(-\frac{\eta_i\eta_{ij}-z}{z\,(1-\eta_{ij})}\right)+3\ln(z)\mathrm{Li}_2\left(-\frac{\eta_i\eta_{ij}-z}{\eta_i\,(1-\eta_{ij})}\right)-3\ln(\eta_i)\,\mathrm{Li}_2\left(-\frac{\eta_i\eta_{ij}-z}{\eta_i\,(1-\eta_{ij})}\right)\\
&-4\ln(z)\mathrm{Li}_2\left(-\frac{z\,(\eta_i\eta_{ij}-z)}{(1-z)\eta_i\eta_{ij}}\right)+5\ln(\eta_{ij})\,\mathrm{Li}_2\left(-\frac{z\,(\eta_i\eta_{ij}-z)}{(1-z)\eta_i\eta_{ij}}\right)+\frac{7}{2}\ln(\eta_i)\ln^2(\eta_{ij})\\
&-\ln(\eta_i+1)\ln^2(\eta_{ij})+6\ln^2(\eta_i)\ln(\eta_{ij})+8\ln(2)\ln(\eta_i)\ln(\eta_{ij})\\
&-8\ln(\eta_i)\ln(\eta_i+1)\ln(\eta_{ij})-3\ln^2(\eta_{ij})\ln(z-\eta_i\eta_{ij})-8\ln(z)\ln(\eta_i)\ln(\eta_{ij})\\
&+8\ln(z)\ln(\eta_i+1)\ln(\eta_{ij})+3\ln(z)\ln(\eta_{ij})\ln(z-\eta_i\eta_{ij})+2\ln^2(z)\ln(\eta_i)\\
&-4\ln^2(z)\ln(\eta_i+1)-\frac{1}{12}\ln^3(\eta_{ij})+3\ln(1-\eta_{ij})\ln^2(\eta_{ij})+\ln(2)\ln^2(\eta_{ij})\\
&-8\ln(2)\ln(z)\ln(\eta_{ij})-3\ln(z)\ln(1-\eta_{ij})\ln(\eta_{ij})+4\ln(2)\ln^2(z)\\
&-\frac{7\mathrm{Li}_2(1-z)}{3}+\frac{7}{3}\mathrm{Li}_2\left(1-\frac{\eta_i}{z}\right)+\frac{2\mathrm{Li}_2(1-\eta_{ij})}{3}-\frac{7}{3}\mathrm{Li}_2\left(1-\frac{\eta_{ij}}{z}\right)-\frac{7}{3}\mathrm{Li}_2\left(1-\frac{\eta_i\eta_{ij}}{z}\right)\\
&-\ln^2(\eta_{ij})+\frac{13}{3}\ln(2)\ln(\eta_{ij})+\frac{7}{3}\ln(z)\ln(\eta_{ij})-\frac{7\ln^2(z)}{6}+\ln(\eta_i)\ln(\eta_{ij})\\
&-\frac{13}{3}\ln(\eta_i+1)\ln(\eta_{ij})-\frac{23\ln(\eta_{ij})}{6},
\end{aligned}
\tag{5.5}
$$

where $z = (\eta_i\,\eta_j\,\eta_{ij})^{1/2}$.

5.2 Implementation, checks and numerical results

We combine the results described in the previous sections into a computer code where the analytic and numerical parts of the calculation are put together. This code can be obtained from a git-repository using the following command

```
git clone https://github.com/apik/SSmm.git
```

The code contains routines for an efficient numerical evaluation of the polylogarithms Li_n, and the $\mathrm{Li}_{2,2}$ function, required for the analytical part of the result. These routines are constructed following methods described in ref. [39]. Finite remainder functions are integrated using the Vegas Monte-Carlo method [40] as implemented in the GSL library [41].

The time required to achieve a per mille relative precision for a single kinematic point is approximately 1s on a standard single-core CPU. It is limited by numerical integration of the finite remainder. The integration precision is quite comparable in the whole integration range. These results look encouraging. We believe that the efficiency and the speed of the numerical evaluation should not present any problem when used in the context of the subtraction methods for the double-real emission contributions.

We continue with the discussion of the checks of the calculation. As we explained in section 2, integrals of the eikonal functions depend on four parameters — $E_{\max}$, velocities of partons i and j that we refer to as $\beta_{i,j}$, and the angle θ_{ij} between them. Apart from the dependence on $E_{\max}$, which is rather trivial, the dependence of the final results on the other three parameters is complex. We note that for the discussion in this section, we always

JHEP03(2026)117

$N_{i,j}$	N_θ	Gluons(MC)	Gluons(DE)	Quarks(MC)	Quarks(DE)
2	2	$7.9436(20) \cdot 10^{-4}$	$7.9453708 \cdot 10^{-4}$	$-1.8834(5) \cdot 10^{-6}$	$-1.8822142 \cdot 10^{-6}$
2	9	$1.447303(9) \cdot 10^{-2}$	$1.4473147 \cdot 10^{-2}$	$-3.6194(8) \cdot 10^{-5}$	$-3.6179578 \cdot 10^{-5}$
2	16	$3.574673(24) \cdot 10^{-2}$	$3.5746842 \cdot 10^{-2}$	$-9.6734(18) \cdot 10^{-5}$	$-9.6709404 \cdot 10^{-5}$
2	23	$4.918967(27) \cdot 10^{-2}$	$4.9189925 \cdot 10^{-2}$	$-1.39541(23) \cdot 10^{-4}$	$-1.3953750 \cdot 10^{-4}$
9	2	$2.16337(10) \cdot 10^{-2}$	$2.1633179 \cdot 10^{-2}$	$1.6820(8) \cdot 10^{-4}$	$1.6824617 \cdot 10^{-4}$
9	9	$3.67465(11) \cdot 10^{-1}$	$3.6747979 \cdot 10^{-1}$	$1.9800(15) \cdot 10^{-3}$	$1.9813388 \cdot 10^{-3}$
9	16	$8.16731(25) \cdot 10^{-1}$	$8.1673711 \cdot 10^{-1}$	$1.228(4) \cdot 10^{-3}$	$1.2358643 \cdot 10^{-3}$
9	23	$1.053284(32)$	1.0532756	$-1.106(5) \cdot 10^{-3}$	$-1.1036785 \cdot 10^{-3}$
16	2	$1.53621(8) \cdot 10^{-1}$	$1.5363543 \cdot 10^{-1}$	$4.4056(13) \cdot 10^{-3}$	$4.4075784 \cdot 10^{-3}$
16	9	$2.10969(12)$	2.1097537	$4.8217(20) \cdot 10^{-2}$	$4.8238002 \cdot 10^{-2}$
16	16	$3.57157(33)$	3.5712515	$4.554(5) \cdot 10^{-2}$	$4.5557310 \cdot 10^{-2}$
16	23	$3.9824(5)$	3.9818443	$2.278(9) \cdot 10^{-2}$	$2.2664599 \cdot 10^{-2}$
23	2	$3.55465(15)$	3.5546016	$2.10520(26) \cdot 10^{-1}$	$2.1051877 \cdot 10^{-1}$
23	9	$2.01624(27) \cdot 10^{1}$	$2.0161751 \cdot 10^{1}$	$1.1123(5)$	1.1119777
23	16	$1.9866(6) \cdot 10^{1}$	$1.9860241 \cdot 10^{1}$	$9.204(12) \cdot 10^{-1}$	$9.2047726 \cdot 10^{-1}$
23	23	$1.8636(9) \cdot 10^{1}$	$1.8630407 \cdot 10^{1}$	$7.514(15) \cdot 10^{-1}$	$7.5231867 \cdot 10^{-1}$

Table 1. Comparison of the finite $\mathcal{O}(\varepsilon^0)$ parts of the results for the integral of the double-emission eikonal for gluon and quark cases for $\beta_i = \beta_j$. Results obtained with the numerical code (MC) are compared with the calculation based on solving dedicated differential equations (DE) for master integrals in the $\beta_i = \beta_j$ case.

show the results for the function

$$\mathcal{O}_\Xi(\beta_i, \beta_j, \cos\theta_{ij}) = -\frac{4E_{\max}^{4\varepsilon}}{N_\varepsilon^2} \mathcal{S}[\Xi_{ij}]. \tag{5.6}$$

We also note that the numerical code in the git-repository outputs the results for this function as well.

For the discussion of the numerical results, it is useful to consider a grid of benchmark points. We parametrize them by a triplet of integer numbers $\{N_i, N_j, N_\theta\}$, which define $\beta_{i,j}$ and $\cos\theta_{ij}$ according to the following equation

$$\beta_i = \frac{N_i}{25}, \quad \beta_j = \frac{N_j}{25}, \quad \cos\theta_{ij} = \cos\left(\frac{N_\theta \pi}{25}\right), \quad N_x = 1, \ldots, 24. \tag{5.7}$$

A very useful feature of the integrated eikonal functions is their regularity in various kinematic limits, e.g. $\beta_{i,j} \to 0$ or $\beta_i \to \beta_j$. We have used the latter limit to perform extensive checks of the results. Since this limit is regular, we directly obtain the $\beta_i = \beta_j$ numerical values for the function $\mathcal{O}_\Xi$ from the computer code described at the beginning of this section. They are shown in table 1 for several benchmark points.

Reference (DE) values in that table are obtained from a high-precision numerical solution of the system of differential equations for the master integrals using the package `DiffExp` [42]. To explain this further, we note that if $\beta_i = \beta_j = \beta$, the integrated eikonal function depends

JHEP03(2026)117

N_i	N_j	N_θ	MC (gluons)	SecDec(gluons)	MC (quarks)	SecDec(quarks)
2	9	16	0.3477025(18)	0.346(4)	$-1.5041(21)\cdot 10^{-4}$	$-1.0(9)\cdot 10^{-4}$
9	16	23	2.40990(8)	2.397(32)	$1.0100(14)\cdot 10^{-2}$	$1.023(34)\cdot 10^{-2}$
16	23	9	9.6427(3)	9.46(28)	$4.5287(6)\cdot 10^{1}$	$4.54(2)\cdot 10^{1}$

Table 2. Comparison of the finite parts for the gluon and quark integrated double-emission eikonal functions (MC) with the results of direct numerical integration(SecDec).

on two variables only (β and θ_{ij}), so that methods similar to the ones employed in our previous paper [21] apply. We then perform the IBP reduction for the equal-velocity case, and construct a system of differential equations for all required master integrals. Starting with simple boundary conditions at $\beta = 0$, where the dependence on θ_{ij} disappears, we fix the value of the angle to the desired value and numerically solve the system of equations with respect to a single variable β. We compare the results of solving the differential equations, and the results of the numerical integration in table 1. For all considered kinematic points excellent agreement is observed. We note that an exact agreement is found for the $1/\varepsilon$ poles that we do not show in table 1 for the sake of brevity. Finally, we note that we also compared the results of our calculation with the analytic results of ref. [18], that correspond to a particular case of $\beta_i = \beta_j$ and $\theta_{ij} = \pi$, and found excellent agreement.

To check the obtained results for two *different* velocities, $\beta_i \neq \beta_j$, we performed direct numerical integration of the eikonal functions in the laboratory frame using the sector-decomposition approach as implemented in the `pySecDec` package [43]. Since the structure of singularities for $\beta_{i,j} \neq 1$ is simpler than in the case with massless emitters, it is straightforward to subtract the collinear $\mathfrak{m}||\mathfrak{n}$ singularity, and then extract the soft singularity if needed.

To prepare a suitable input for `pySecDec`, we write the function $\mathcal{O}_\Xi$ in the following way

$$\mathcal{O}_\Xi = \frac{1}{\varepsilon}\int_0^1 \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}} \left(\Big\langle (1 - C_{\mathfrak{mn}})\omega^2 \Xi_{ij}(\omega, \vec{n}_\mathfrak{m}, \vec{n}_\mathfrak{n}) \Big\rangle_{\mathfrak{mn}} + \Big\langle C_{\mathfrak{mn}}\omega^2 \Xi_{ij}(\omega, \vec{n}_\mathfrak{m}, \vec{n}_\mathfrak{n}) \Big\rangle_{\mathfrak{mn}} \right), \quad (5.8)$$

where the eikonal factors for gluons and quarks $\Xi_{ij} = \{\widetilde{\mathcal{S}}_{ij}, \widetilde{\mathcal{I}}_{ij}\}$ need to be integrated over directions of partons $\mathfrak{m}$ and $\mathfrak{n}$, and the energy of the parton $\mathfrak{n}$.

The action of the $C_{\mathfrak{mn}}$ operator in eq. (5.8) makes the first term finite in the $\mathfrak{m}||\mathfrak{n}$ limit, and simplifies the second one, rendering it amenable to analytic integration. In particular, as we have seen earlier when discussing the $C_{\mathfrak{mn}}$ limit, integrations over ω and the directions of partons $\mathfrak{m}$ and $\mathfrak{n}$ factorize. We find

$$\int_0^1 \frac{\mathrm{d}\omega}{\omega^{1+2\varepsilon}} \left(\Big\langle C_{\mathfrak{mn}}\omega^2 \Xi_{ij}(\omega, \vec{n}_\mathfrak{m}, \vec{n}_\mathfrak{n}) \Big\rangle_{\mathfrak{m},\mathfrak{n}} \right) = W_\Xi \left(\rho_{ii} I_2^{(1)}[\rho_{ii}] + \rho_{jj} I_2^{(1)}[\rho_{jj}] - 2\rho_{ij} I_{1,1}^{(2)}[\rho_{ii},\rho_{jj},\rho_{ij}] \right), \quad (5.9)$$

with angular integrals provided in appendices A and B. The functions W_Ξ are obtained by integrating over ω; they are

$$W_{\widetilde{\mathcal{S}}} = -\frac{1-2\varepsilon}{12\varepsilon^2}\left(6 - \varepsilon(11 - \varepsilon + 4\varepsilon^2) - 2\varepsilon^2(11 + 4\varepsilon^2)(\psi_{1/2-\varepsilon} - \psi_{-\varepsilon})\right), \quad (5.10)$$

$$W_{\widetilde{\mathcal{I}}} = \frac{1-2\varepsilon}{6\varepsilon(1-\varepsilon)}\left(1 - 2\varepsilon(1-\varepsilon) + \varepsilon(2 - \varepsilon(3 - 4\varepsilon)(\psi_{1/2-\varepsilon} - \psi_{-\varepsilon})\right). \quad (5.11)$$

β_i	β_j	$\cos\theta_{ij}$	ε^{-3}	ε^{-2}	ε^{-1}	ε^{0}
0.08	0.36	-0.4257	$5.7483627 \cdot 10^{-2}$	$3.3156281 \cdot 10^{-2}$	$1.7083994 \cdot 10^{-1}$	$3.477025(18) \cdot 10^{-1}$
0.36	0.64	-0.9686	$3.9188310 \cdot 10^{-1}$	$2.9983506 \cdot 10^{-1}$	1.1508376	2.40990(8)
0.64	0.92	0.4258	$7.0543274 \cdot 10^{-1}$	1.3901524	4.1595510	9.6427(3)

Table 3. Benchmark results for the function $\mathcal{O}_{GG}$, c.f eq. (5.6).

β_i	β_j	$\cos\theta_{ij}$	ε^{-2}	ε^{-1}	ε^{0}
0.08	0.36	-0.4257	$1.9161209 \cdot 10^{-2}$	$2.4858662 \cdot 10^{-6}$	$-1.5041(21) \cdot 10^{-4}$
0.36	0.64	-0.9686	$1.3062770 \cdot 10^{-1}$	$1.3410916 \cdot 10^{-2}$	$1.0100(14) \cdot 10^{-2}$
0.64	0.92	0.4258	$2.3514424 \cdot 10^{-1}$	$2.9111167 \cdot 10^{-1}$	$4.5287(6) \cdot 10^{1}$

Table 4. Benchmark results for the function $\mathcal{O}_{QQ}$, cf. eq. (5.6).

The ψ-functions in the above equation are the standard logarithmic derivatives of Γ-functions.

To integrate terms in the integrand in eq. (5.8) with the $\mathfrak{m}||\mathfrak{n}$ divergences subtracted, i.e. the first term on the r.h.s. of eq. (5.8) without the $1/\varepsilon$ prefactor, we write them separately for gg and $q\bar{q}$ cases in the following way

$$I^{\rm sd}_{GG} = \int_0^1 {\rm d}\omega\; \omega^{-1-2\varepsilon} \left\langle \mathcal{F}^{(0)}_{GG}(\omega, \vec{n}_{\mathfrak{m}}, \vec{n}_{\mathfrak{n}}) + \varepsilon \mathcal{F}^{(1)}_{GG}(\omega, \vec{n}_{\mathfrak{m}}, \vec{n}_{\mathfrak{n}}) \right\rangle_{\mathfrak{m},\mathfrak{n}}, \tag{5.12}$$

$$I^{\rm sd}_{QQ} = \int_0^1 {\rm d}\omega\; \omega^{-2\varepsilon} \left\langle \mathcal{F}^{(0)}_{QQ}(\omega, \vec{n}_{\mathfrak{m}}, \vec{n}_{\mathfrak{n}}) \right\rangle_{\mathfrak{m},\mathfrak{n}}. \tag{5.13}$$

The functions $\mathcal{F}_{GG,QQ}$ are finite in the $\omega \to 0$ limit and are independent of the regularisation parameter ε. Angular integrations over the directions of the parton momenta have to be performed in $(d-1)$-dimensions. We treat these integrations, as well as integrations over ω using the sector decomposition [43].[7]

The results of this evaluation for several benchmark points are shown in table 2, where the finite parts of the integrated double-emission eikonal functions for three kinematic points are given. We observe a rather satisfactory agreement between the two independent calculations of the integral of the double-emission eikonal function.

We would like to conclude this section by showing the results for the integrated double-emission eikonal functions for a few benchmark points. These results, shown in tables 3,4 can be used to verify that the numerical code from the git-repository is interpreted and used properly, including the ε-dependent normalization prefactor.

6 Conclusion

In this paper, we computed the integrals of the double-emission eikonal functions for two massive emitters whose momenta are at an arbitrary angle to each other. This result is one of a few ingredients required for extending the nested soft-collinear subtraction scheme [1] to processes with massive partons.

[7] We note that functions $\mathcal{F}_{GG,QQ}$ are non-singular, so it is possible and, in fact, beneficial to keep them *implicit* during the sector-decomposition process.

JHEP03(2026)117

Our computation is based on the approach already described in refs. [20, 21] where local subtractions were used to remove potential singularities from the eikonal functions. The subtracted terms are integrated analytically, whereas finite remainders, suitable for numerical integration, are computed in four space-time dimensions.

We emphasize that all $1/\varepsilon$ poles that appear in the integral of the double-soft eikonal function are calculated analytically. The analytic results are expressed in terms of standard logarithmic and polylogarithmic functions up to weight four, as well as the function $\mathrm{Li}_{2,2}$. This simple representation enables fast and efficient evaluation of the analytic part of the result.

The numerical code that combines the analytic and the numerical parts of the calculation can be obtained from a git-repository. We have checked it for various kinematic cases, coming to the conclusion that, with very moderate runtimes, it provides a per mille relative precision for generic velocities and scattering angles.

Acknowledgments

This research was supported in part by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under grant no. 396021762 - TRR 257.

A Massless and single-massive angular integrals

In this appendix, we define some of the angular integrals that are used for calculations in section 4. These integrals contain either massless propagators, or a massive and a massless propagator.

We fix the normalization of angular integrals in such a way that the angular volume is equal to one,

$$\left\langle 1 \right\rangle_{\mathfrak{m}} = \left\langle 1 \right\rangle_{\mathfrak{mn}} = 1. \tag{A.1}$$

We begin by introducing the following angular integrals,

$$\left\langle \frac{1}{\rho_{\mathfrak{m}x}^{n}} \right\rangle_{\mathfrak{m}} = I_n^{(1)}\left[\rho_{xx}\right], \quad \rho_{xx} \neq 0, \tag{A.2}$$

$$\left\langle \frac{1}{\rho_{\mathfrak{m}x}^{a}\rho_{\mathfrak{m}y}^{b}} \right\rangle_{\mathfrak{m}} = I_{a,b}^{(0)}\left[\rho_{xy}\right], \quad \rho_{xx} = \rho_{yy} = 0, \tag{A.3}$$

These integrals can be computed in a closed form, in terms of hypergeometric functions. The results read

$$I_n^{(1)}\left[\rho_{11}\right] = \left(1+\sqrt{1-\rho_{11}}\right)^{-n} {}_2F_1\left(n, 1-\varepsilon, 2-2\varepsilon; \frac{2\sqrt{1-\rho_{11}}}{1+\sqrt{1-\rho_{11}}}\right), \tag{A.4}$$

$$I_{a,b}^{(0)}\left[\rho_{12}\right] = \frac{\Gamma\left(2-2\varepsilon\right)\Gamma\left(1-\varepsilon-a\right)\Gamma\left(1-\varepsilon-b\right)}{2^{a+b}\Gamma^2\left(1-\varepsilon\right)\Gamma\left(2-2\varepsilon-a-b\right)} {}_2F_1\left(a, b, 1-\varepsilon; 1-\frac{\rho_{12}}{2}\right). \tag{A.5}$$

B Double-massive angular integrals

We consider angular integrals with two massive propagators defined as

$$\left\langle \frac{1}{\rho_{\mathfrak{m}x}^{c_1}\rho_{\mathfrak{m}y}^{c_2}} \right\rangle_{\mathfrak{m}} = I_{c_1,c_2}^{(2)}\left[\rho_{xx}, \rho_{yy}, \rho_{xy}\right], \quad \rho_{xx} \neq 0, \rho_{yy} \neq 0, \rho_{xy} \neq 0. \tag{B.1}$$

JHEP03(2026)117

They can be mapped onto loop integrals by using the reverse unitarity [26]

$$I^{(2)}_{c_1,c_2} = \frac{\mathcal{N}_\varepsilon}{(2\pi)^{d-1}} \int \mathrm{d}^d k \frac{\delta^+(k^2)\,\delta(1-k\cdot P)}{(k\cdot l_1)^{c_1}\,(k\cdot l_2)^{c_2}}, \tag{B.2}$$

where the scalar products among P, l_1, l_2 read

$$P\cdot l_{1,2} = 1, \quad P^2 = 1, \quad l_1^2 = \rho_{xx} = x, \quad l_2^2 = \rho_{yy} = y, \quad l_1\cdot l_2 = \rho_{xy} = w. \tag{B.3}$$

The powers of the propagators c_i may depend linearly on ε. We write $c_i = a_i + b_i\varepsilon$ and assume that a_i and b_i are integers.

We can reduce the number of integrals with two massive propagators that need to be computed, by using the integration-by-parts (IBP) technology [27, 28]. Because c_i depends on ε, one has to keep powers of propagators symbolic when solving the IBP identities. We note that since the IBP identities relate integrals whose propagator powers differ by integers, integrals with different b_i's cannot be connected by IBP relations.

We use `Kira 3` [44] to perform the IBP reduction with symbolic propagator powers. We find that all integrals $I^{(2)}_{c_1,c_2}$, with $c_i = a_i + \varepsilon b_i$, $i = 1,2$, can be expressed through four master integrals

$$\boldsymbol{f} = \left(I^{(2)}_{b_1\varepsilon,b_2\varepsilon}, \quad I^{(2)}_{1+b_1\varepsilon,b_2\varepsilon}, \quad I^{(2)}_{b_1\varepsilon,1+b_2\varepsilon}, \quad I^{(2)}_{1+b_1\varepsilon,1+b_2\varepsilon} \right)^{\mathrm{T}}. \tag{B.4}$$

To compute them, we define a new integral basis

$$\boldsymbol{g} = \frac{1}{1-2\varepsilon} \left(g_1, \quad g_2, \quad g_3, \quad g_4 \right)^{\mathrm{T}}, \tag{B.5}$$

where

$$\begin{aligned} g_1 &= [1-(b_1+b_2+2)\,\varepsilon]\, I^{(2)}_{b_1\varepsilon,b_2\varepsilon} + b_1\varepsilon I^{(2)}_{1+b_1\varepsilon,b_2\varepsilon} + b_2\varepsilon I^{(2)}_{b_1\varepsilon,1+b_2\varepsilon}, \\ g_2 &= \varepsilon\sqrt{1-x}\, I^{(2)}_{1+b_1\varepsilon,b_2\varepsilon}, \\ g_3 &= \varepsilon\sqrt{1-y}\, I^{(2)}_{b_1\varepsilon,1+b_2\varepsilon}, \\ g_4 &= \varepsilon\sqrt{w^2-xy}\, I^{(2)}_{1+b_1\varepsilon,1+b_2\varepsilon}. \end{aligned} \tag{B.6}$$

The integrals $g_{1,\ldots,4}$ satisfy the differential equations of the form

$$\mathrm{d}\boldsymbol{g} = \varepsilon\,(\mathrm{d}\mathbb{A})\,\boldsymbol{g}, \tag{B.7}$$

where the matrix $\mathbb{A}$ reads

$$\mathbb{A} = \sum_{n_1=0,n_2=0}^{n_1+n_2\le 2} b_1^{n_1} b_2^{n_2} \mathbb{A}_{n_1 n_2}, \tag{B.8}$$

with

$$\mathbb{A}_{00} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{L_7}{2} & L_3 - L_1 & 0 & 0 \\ -\frac{L_8}{2} & 0 & L_4 - L_2 & 0 \\ -\frac{L_9}{2} & -L_{11} & -L_{12} & L_6 - L_5 \end{pmatrix},$$
$$\mathbb{A}_{10} = \begin{pmatrix} -\frac{L_1}{2} & -L_7 & 0 & 0 \\ 0 & -\frac{L_1}{2} & 0 & 0 \\ 0 & \frac{L_{10}}{2} & -\frac{L_2}{2} + L_4 - \frac{L_6}{2} & \frac{L_{12}}{2} \\ 0 & -\frac{L_{11}}{2} & -\frac{L_{12}}{2} & \frac{L_2}{2} - L_5 + \frac{L_6}{2} \end{pmatrix},$$
$$\mathbb{A}_{01} = \begin{pmatrix} -\frac{L_2}{2} & 0 & -L_8 & 0 \\ 0 & -\frac{L_1}{2} + L_3 - \frac{L_6}{2} & \frac{L_{10}}{2} & \frac{L_{11}}{2} \\ 0 & 0 & -\frac{L_2}{2} & 0 \\ 0 & -\frac{L_{11}}{2} & -\frac{L_{12}}{2} & \frac{L_1}{2} - L_5 + \frac{L_6}{2} \end{pmatrix}, \tag{B.9}$$
$$\mathbb{A}_{11} = \begin{pmatrix} 0 & -\frac{L_7}{2} & -\frac{L_8}{2} & \frac{L_9}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbb{A}_{20} = \begin{pmatrix} 0 & -\frac{L_7}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbb{A}_{02} = \begin{pmatrix} 0 & 0 & -\frac{L_8}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the above equations, we have used the following short-hand notations

$$\begin{aligned}
L_1 &= \log(x), & L_2 &= \log(y), \\
L_3 &= \log(1-x), & L_4 &= \log(1-y), \\
L_5 &= \log\left(w^2 - xy\right), & L_6 &= \log\left((1-w)^2 - (1-x)(1-y)\right), \\
L_7 &= \log\left(\frac{1-\sqrt{1-x}}{1+\sqrt{1-x}}\right), & L_8 &= \log\left(\frac{1-\sqrt{1-y}}{1+\sqrt{1-y}}\right), \\
L_9 &= \log\left(\frac{w-\sqrt{w^2-xy}}{w+\sqrt{w^2-xy}}\right), & L_{10} &= \log\left(\frac{1-w-\sqrt{1-x}\sqrt{1-y}}{1-w+\sqrt{1-x}\sqrt{1-y}}\right), \\
L_{11} &= \log\left(\frac{w-x-\sqrt{1-x}\sqrt{w^2-xy}}{w-x+\sqrt{1-x}\sqrt{w^2-xy}}\right), & L_{12} &= \log\left(\frac{w-y-\sqrt{1-y}\sqrt{w^2-xy}}{w-y+\sqrt{1-y}\sqrt{w^2-xy}}\right).
\end{aligned} \tag{B.10}$$

To solve the differential equation (B.7), we use the boundary condition

$$\boldsymbol{g}(x=1, y=1, w=1) = (1,0,0,0)^{\mathrm{T}}, \tag{B.11}$$

and rationalize the square roots in the differential equation, by changing variables

$$x = \frac{4\eta_1}{(\eta_1+1)^2}, \; y = \frac{4\eta_2}{(\eta_2+1)^2}, \; w = \frac{2\left(\eta_1\eta_2 + z^2\right)}{(\eta_1+1)(\eta_2+1)\,z}. \tag{B.12}$$

The resulting differential equations are then simple enough, and the solution can be straightforwardly obtained in terms of GPLs.

B.1 Two-loop double-massive integrals

In section 4.1 we introduced the following integral family

$$I^{(2)}_{a_1,a_2,a_3,a_4+b_4\varepsilon} = \left\langle \frac{1}{\rho_{\mathfrak{mn}}^{a_1}\rho_{\mathfrak{m}y}^{a_2}\rho_{\mathfrak{n}y}^{a_3}\rho_{\mathfrak{m}x}^{a_4+b_4\varepsilon}} \right\rangle_{\mathfrak{mn}}, \tag{B.13}$$

and used such integrals to write the strongly-ordered contribution to the eikonal function $\mathcal{S}_\omega[G_{ij}]$. To compute these integrals we map these integrals onto loop integrals using reverse unitarity

$$\left[\frac{\mathcal{N}_\varepsilon}{(2\pi)^{d-1}}\right]^2 \int \mathrm{d}^d k_1 \mathrm{d}^d k_2 \frac{\delta^+(k_1^2)\,\delta(1-k_1\cdot P)\,\delta^+(k_2^2)\,\delta(1-k_2\cdot P)}{(k_1\cdot k_2)^{a_1}\,(k_1\cdot l_2)^{a_2}\,(k_2\cdot l_2)^{a_3}\,(k_1\cdot l_1)^{a_4+b_4\varepsilon}}, \tag{B.14}$$

and perform the IBP reduction with `Kira 3` and identify twelve master integrals

$$\begin{aligned}
&f_1 = I^{(2)}_{0,0,0,b_4\varepsilon}, && f_2 = I^{(2)}_{0,0,0,1+b_4\varepsilon}, && f_3 = I^{(2)}_{0,1,0,b_4\varepsilon}, && f_4 = I^{(2)}_{0,1,0,1+b_4\varepsilon},\\
&f_5 = I^{(2)}_{0,0,1,b_4\varepsilon}, && f_6 = I^{(2)}_{0,0,1,1+b_4\varepsilon}, && f_7 = I^{(2)}_{0,1,1,b_4\varepsilon}, && f_8 = I^{(2)}_{0,1,1,1+b_4\varepsilon},\\
&f_9 = I^{(2)}_{1,0,1,b_4\varepsilon}, && f_{10} = I^{(2)}_{1,-1,1,b_4\varepsilon}, && f_{11} = I^{(2)}_{1,0,1,1+b_4\varepsilon}, && f_{12} = I^{(2)}_{1,-1,1,1+b_4\varepsilon}.
\end{aligned} \tag{B.15}$$

We then construct differential equations for these master integrals by differentiating them with respect to $x = \rho_{xx}$, $y = \rho_{yy}$ and $w = \rho_{xy}$, and using the reduction to express the result in terms of $f_{1,2,\ldots,12}$. It is also possible to construct a *canonical* basis for these integrals that we will again refer to as $\boldsymbol{g}$. The relation between $\boldsymbol{g}$ and $\boldsymbol{f}$ reads

$$\boldsymbol{g} = \mathrm{T}\boldsymbol{f}, \tag{B.16}$$

where

$$\mathrm{T} = \begin{pmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & 0 \\ T_{31} & 0 & T_{33} \end{pmatrix}, \tag{B.17}$$

with

$$T_{11} = \begin{pmatrix} \frac{(b_4+2)\varepsilon-1}{2\varepsilon-1} & \frac{b_4\varepsilon}{1-2\varepsilon} & 0 & 0 \\ 0 & \frac{\sqrt{1-x}\varepsilon}{1-2\varepsilon} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{1-y}\varepsilon}{1-2\varepsilon} & 0 \\ 0 & 0 & 0 & \frac{\varepsilon\sqrt{w^2-xy}}{1-2\varepsilon} \end{pmatrix}, \quad T_{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{(b_4+2)(b_4\varepsilon-1)}{2b_4(1-2\varepsilon)} & \frac{(b_4+2)\varepsilon}{2(\varepsilon-1)} & 0 & 0 \end{pmatrix},$$
$$T_{33} = \begin{pmatrix} 0 & 0 & \frac{\varepsilon^2\sqrt{w^2-xy}}{(1-2\varepsilon)^2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{1-x}\varepsilon^2}{(1-2\varepsilon)^2} \\ \frac{(b_4+2)\sqrt{1-y}\varepsilon^2}{b_4(1-2\varepsilon)^2} & 0 & 0 & 0 \\ -\frac{2(b_4+2)\varepsilon^2}{b_4(1-2\varepsilon)^2} & \frac{(b_4+2)\varepsilon((b_4+2)\varepsilon-1)}{b_4(1-2\varepsilon)^2} & 0 & -\frac{(b_4+2)\varepsilon^2}{(1-2\varepsilon)^2} \end{pmatrix}, \tag{B.18}$$

and

$$T_{22} = \sqrt{1-y}\,\frac{\varepsilon}{1-2\varepsilon}T_{11}. \tag{B.19}$$

The corresponding matrix $\mathbb{A}$ in the $d\log$ form of differential equation $\mathrm{d}\boldsymbol{g} = \varepsilon \mathrm{d}(\mathbb{A})\, \boldsymbol{g}$ reads

$$\mathbb{A} = \sum_{n=-1}^{2} b_4^n \mathbb{A}_n, \tag{B.20}$$

with

$$\mathbb{A}_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -L_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{L_8}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2L_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{A}_2 = \begin{pmatrix} 0 & -\frac{L_7}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{L_7}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{L_7}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{L_7}{2} & 0 & 0 \end{pmatrix}, \tag{B.21}$$

and

$$\mathbb{A}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{L_7}{2} & L_3 - L_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{L_8}{2} & 0 & L_4 - L_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{L_9}{2} & -L_{11} & -L_{12} & L_6 - L_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{L_8}{2} & 0 & 0 & 0 & L_4 - L_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{L_8}{2} & 0 & 0 & -\frac{L_7}{2} & -L_1 - L_2 + L_3 + L_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{L_8}{2} & 0 & -\frac{L_8}{2} & 0 & 2L_4 - 2L_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{L_8}{2} & -\frac{L_9}{2} & -L_{11} & -L_{12} & -L_2 + L_4 - L_5 + L_6 & 0 & 0 & 0 & 0 \\ \frac{L_9}{4} & \frac{L_{11}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & L_1 + L_2 - 3L_5 + 2L_6 & -L_{11} & -L_{12} & \frac{L_9}{2} \\ \frac{L_7}{4} & \frac{L_1}{2} - \frac{L_2}{2} - L_3 + \frac{L_6}{2} & 0 & 0 & 0 & -L_8 & 0 & 0 & L_{11} & L_3 - L_1 & L_{10} & \frac{L_7}{2} \\ \frac{L_8}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & L_{12} & L_{10} & L_4 - L_2 & \frac{L_8}{2} \\ 0 & 0 & 0 & 0 & L_8 & -2L_{10} & 0 & 0 & -2L_9 & 2L_7 & 2L_8 & -L_1 - L_2 + L_6 \end{pmatrix}, \tag{B.22}$$

$$\mathbb{A}_1 = \begin{pmatrix} -\frac{L_1}{2} & -L_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{L_1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{L_{10}}{2} & -\frac{L_2}{2} + L_4 - \frac{L_6}{2} & \frac{L_{12}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{L_{11}}{2} & -\frac{L_{12}}{2} & \frac{L_2}{2} - L_5 + \frac{L_6}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{L_1}{2} & -L_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{L_1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{L_{10}}{2} & -\frac{L_2}{2} + L_4 - \frac{L_6}{2} & \frac{L_{12}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{L_{11}}{2} & -\frac{L_{12}}{2} & \frac{L_2}{2} - L_5 + \frac{L_6}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{L_2}{2} - L_5 + \frac{L_6}{2} & -\frac{L_{11}}{2} & -\frac{L_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{L_1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{L_{12}}{2} & \frac{L_{10}}{2} & -\frac{L_2}{2} + L_4 - \frac{L_6}{2} & 0 \\ 0 & \frac{L_7}{2} & 0 & 0 & 0 & -L_{10} & 0 & 0 & -L_9 & 2L_7 & L_8 & -\frac{L_1}{2} \end{pmatrix}. \tag{B.23}$$

JHEP03(2026)117

The logarithms $L_{1,\dots,12}$ that appear in the above equations are defined in eq. (B.10). The boundary conditions are easily fixed by using

$$g_n(x=1, y=1, w=1) = \delta_{1n}. \tag{B.24}$$

Using the variable transformation as in eq. (B.12), we remove square roots from the system of equations and solve it in terms of GPLs.

C Integrals for $\overline{\mathcal{S}}_\omega[G_{ij}^{(0)}]$

It remains to discuss the calculation of integrals that are needed for the term $\overline{\mathcal{S}}_\omega[G_{ij}^{(0)}]$, cf. section 4.4. The required eight master integrals are displayed in eq. (4.58). We use them to define a new basis

$$\begin{aligned}
\widetilde{J}_1 &= \frac{J_1}{\omega}, & \widetilde{J}_2 &= \frac{v_{ij} J_2 \varepsilon}{\omega(2\varepsilon-1)}, & \widetilde{J}_3 &= \frac{v_{ij} J_3 \varepsilon}{(2\varepsilon-1)}, \\
\widetilde{J}_4 &= \frac{v_{ij}^2 J_4 \varepsilon^2}{(2\varepsilon-1)^2}, & \widetilde{J}_5 &= \frac{v_{ij} J_5 (\omega+1)\varepsilon^2}{(2\varepsilon-1)^2}, & \widetilde{J}_6 &= \frac{v_{ij}^2 J_6 \varepsilon}{(2\varepsilon-1)^2},
\end{aligned}$$
$$\widetilde{J}_7 = \frac{v_{ij} J_7 (1-4\varepsilon)\varepsilon}{(\omega-1)(2\varepsilon-1)^2} + \frac{v_{ij} J_6 (\omega+1)\varepsilon}{(\omega-1)(2\varepsilon-1)^2} + \frac{2 v_{ij} J_5 (\omega+1)\varepsilon^2}{(\omega-1)(2\varepsilon-1)^2}, \qquad \widetilde{J}_8 = \frac{v_{ij}^2 J_8 \varepsilon^2}{(2\varepsilon-1)^2}. \tag{C.1}$$

These integrals satisfy the following canonical differential equations

$$d\widetilde{\boldsymbol{J}} = \varepsilon (d\mathbb{A}) \widetilde{\boldsymbol{J}}, \tag{C.2}$$

where

$$\mathbb{A} = \begin{pmatrix} \mathbb{A}_{11} & 0 \\ \mathbb{A}_{21} & \mathbb{A}_{22} \end{pmatrix}, \tag{C.3}$$

with

$$\begin{aligned}
\mathbb{A}_{11} &= \begin{pmatrix}
-2L_4 & 0 & 0 & 0 \\
\frac{L_3}{2} - \frac{L_1}{2} & -L_1 + 2L_2 - L_3 - 2L_4 & 0 & 0 \\
\frac{L_3}{2} - \frac{L_1}{2} & 0 & -L_1 + 2L_2 - L_3 - 2L_4 & 0 \\
0 & \frac{L_3}{2} - \frac{L_1}{2} & \frac{L_3}{2} - \frac{L_1}{2} & -2L_1 + 4L_2 - 2L_3 - 2L_4
\end{pmatrix}, \\
\mathbb{A}_{21} &= \begin{pmatrix}
\frac{L_1}{4} - \frac{L_3}{4} & 0 & 0 & 0 \\
-\frac{L_1}{4} - \frac{L_3}{4} - \frac{L_5}{2} + \frac{L_6}{4} + \frac{L_7}{4} & 0 & 0 & 0 \\
-\frac{L_1}{4} + \frac{L_3}{4} + \frac{L_6}{4} - \frac{L_7}{4} & 0 & 0 & 0 \\
0 & \frac{L_3}{2} - \frac{L_1}{2} & 0 & 0
\end{pmatrix}, \\
\mathbb{A}_{22} &= \begin{pmatrix}
-L_1 + 2L_2 - L_3 - 2L_5 & L_1 - L_3 & L_4 & 0 \\
L_1 - L_3 - L_6 + L_7 & -L_1 + 4L_2 - L_3 - 2L_5 - L_6 - L_7 & L_7 - L_6 & 0 \\
L_1 + L_3 + 2L_5 - L_6 - L_7 & -L_1 + L_3 - L_6 + L_7 & 2L_2 - 2L_4 - L_6 - L_7 & 0 \\
L_1 - L_3 & L_4 & L_1 - L_3 & -2L_1 + 4L_2 - 2L_3 - 2L_4
\end{pmatrix}.
\end{aligned} \tag{C.4}$$

JHEP03(2026)117

The logarithms L_i in this case are given by

$$
\begin{aligned}
L_1 &= \log(1 - v_{ij}), & L_5 &= \log(1 + \omega), \\
L_2 &= \log(v_{ij}), & L_6 &= \log(1 + \omega + v_{ij} - \omega v_{ij}), \\
L_3 &= \log(1 + v_{ij}), & L_7 &= \log(1 + \omega - v_{ij} + \omega v_{ij}). \\
L_4 &= \log(\omega), & &
\end{aligned}
$$

The boundary conditions are easily fixed by using

$$\widetilde{J}_n(v_{ij} = 0) = \omega^{-2\varepsilon}\delta_{1n}. \tag{C.5}$$

Finding the solution of the system of differential equations in terms of GPLs is straightforward.

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has code included as electronic supplementary material.

Supplementary material and data. The following files are provided as supplementary material/data and are accessible through the article's webpage.

- `G0partGG`
- `G0partQQ`
- `I11`
- `README.txt`
- `SSggGPL`
- `SSqqGPL`

All the relevant information on how to use the supplementary material is provided in the README.txt file.

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