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On Kato's square root property for the generalized Stokes operator

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A R T I C L E I N F O

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We establish the Kato square root property for the generalized Stokes operator on $\mathbb{R}^d$ with bounded measurable coefficients. More precisely, we identify the domain of the square root of $Au := -\operatorname{div}(\mu\nabla u) + \nabla\phi$, $\operatorname{div}(u) = 0$, with the space of divergence-free H^1-vector fields and further prove the estimates $\|A^{1/2}u\|_{\mathrm{L}^2} \simeq \|\nabla u\|_{\mathrm{L}^2}$. As an application we show that $A^{1/2}$ depends holomorphically on the coefficients μ. Besides the boundedness and measurability as well as an ellipticity condition on μ, there are no requirements on the coefficients.

1. Introduction

At the beginning of the 1960s, Tosio Kato asked, whether, given two Hilbert spaces $V \subseteq H$ with V being dense in H, the domain of the square root of the maximal accretive operator L in H associated to a closed and sectorial sesquilinear form $\mathfrak{a}\colon V \times V \to \mathbb{C}$ always coincides with the form domain, i.e., whether $\mathcal{D}(L^{1/2}) = V$. In the subsequent

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years it became clear by counterexamples of Lions [35] and McIntosh [36] that such a result does not hold in this generality. Lions already specified the question to elliptic operators $L = -\operatorname{div}(\mu\nabla\cdot)$ with bounded measurable coefficients which became known as the Kato square root problem. It was eventually resolved in the whole space to the affirmative by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian [10] in 2002.

In the last two decades there has been a surge of interesting results that revealed, for instance, a deep connection of the square root property to boundary value problems [1,3,6,8,11,12,32], operator-adapted function spaces [8,12,31], maximal regularity and quasilinear PDE [5,7] and many more [4,17,18,23,29,30]. Since the restriction from densely defined, closed and sectorial sesquilinear forms to elliptic operators in divergence form is quite drastic, there has always been the question for which other operators this square root property holds.

For elliptic operators the validity of the square root property was extended to situations including rough boundary geometries and mixed boundary conditions [15,19,24] as well as to operators on submanifolds [38]. Using the theory of Muckenhoupt weights, it was extended to elliptic operators with degenerate coefficients [20–22]. It was further established for Schrödinger operators [16] and parabolic operators [9,40].

The purpose of this paper is to enrich the class of operators sharing the square root property by a nonlocal operator that naturally arises in the theory of fluid mechanics. More precisely, we consider the generalized Stokes operator in the whole space, which is formally given by

$$Au \coloneqq -\operatorname{div}(\mu\nabla u) + \nabla\phi, \quad \operatorname{div}(u) = 0 \tag{1.1}$$

and assume that the coefficients μ are bounded and measurable, and satisfy a Gårding inequality. The sesquilinear form to which A is associated is given by

$$\mathfrak{a} : \mathrm{H}^1_\sigma(\mathbb{R}^d) \times \mathrm{H}^1_\sigma(\mathbb{R}^d) \to \mathbb{C}, \quad (u, v) \mapsto \sum_{\alpha,\beta,i,j=1}^{d} \int_{\mathbb{R}^d} \mu^{ij}_{\alpha\beta} \partial_\beta u_j \overline{\partial_\alpha v_i} \, \mathrm{d}x, \tag{1.2}$$

where $\mathrm{H}^1_\sigma(\mathbb{R}^d) \coloneqq \{u \in \mathrm{H}^1(\mathbb{R}^d; \mathbb{C}^d) : \operatorname{div}(u) = 0\}$. Thus, the square root property for the generalized Stokes operator asks for the identification of $\mathcal{D}(A^{1/2})$ with $\mathrm{H}^1_\sigma(\mathbb{R}^d)$.

Even though the sesquilinear form in (1.2) has the same form as for elliptic operators in divergence form, it has the crucial difference expressed in the fact that the form domain is only given by $\mathrm{H}^1_\sigma(\mathbb{R}^d)$. This results in the appearance of the pressure gradient in (1.1) and, in stark contrast to elliptic operators, has the effect that A is *nonlocal*. The locality of elliptic operators is a key-property for the derivation of one of the most important tools used in the proof of the square root property commonly known as *off-diagonal estimates*. For the resolvent of the generalized Stokes operator such off-diagonal estimates are not known. Recently, the second author established a nonlocal version of such estimates with polynomial decay [43] but, so far, the order of decay is too small and thus insufficient for our purposes. We will improve the order of decay of these estimates by an iterative procedure and thereby attain a sufficient order of decay. Afterwards we will show how to adapt the proof of the square root property for elliptic operators to these new nonlocal estimates.

Let us state the main result of this article in detail. We start with our assumptions on the coefficients.

Assumption 1.1. The coefficients $\mu = (\mu^{ij}_{\alpha\beta})^d_{\alpha,\beta,i,j=1}$ with $\mu^{ij}_{\alpha\beta} \in \mathrm{L}^\infty(\mathbb{R}^d;\mathbb{C})$ for every $1 \leq \alpha, \beta, i, j \leq d$ satisfy for some $\mu_\bullet, \mu^\bullet > 0$ the inequalities

$$\operatorname{Re} \sum_{\alpha,\beta,i,j=1}^{d} \int_{\mathbb{R}^d} \mu^{ij}_{\alpha\beta} \partial_\beta u_j \overline{\partial_\alpha u_i} \, \mathrm{d}x \geq \mu_\bullet \|\nabla u\|^2_{\mathrm{L}^2} \qquad (u \in \mathrm{H}^1(\mathbb{R}^d;\mathbb{C}^d))$$

and

$$\operatorname{esssup}_{x\in\Omega} \|\mu(x)\|_{\mathcal{L}(\mathbb{C}^{d\times d})} \leq \mu^\bullet.$$

The generalized Stokes operator A is realized on $\mathrm{L}^2_\sigma(\mathbb{R}^d)$ via the sesquilinear form in (1.2). Thus, A is given by $Au := f$, where $u \in \mathcal{D}(A)$ and f are associated via

$$\mathcal{D}(A) := \Big\{ u \in \mathrm{H}^1_\sigma(\mathbb{R}^d) : \exists f \in \mathrm{L}^2_\sigma(\mathbb{R}^d) \text{ such that } \\ \forall v \in \mathrm{H}^1_\sigma(\mathbb{R}^d) \text{ it holds } \mathfrak{a}(u,v) = \int_{\mathbb{R}^d} f \cdot \overline{v} \, \mathrm{d}x \Big\}.$$

The main result of this article concerns a characterization of the domain of the square root of A as the space of divergence-free H^1-vector fields. We stress, that neither symmetry nor regularity of the coefficients is assumed.

Theorem 1.2. *Let μ satisfy Assumption 1.1. Then A has the square root property, i.e., we have that $\mathcal{D}(A^{1/2}) = \mathrm{H}^1_\sigma(\mathbb{R}^d)$ and*

$$C^{-1}\|\nabla u\|_{\mathrm{L}^2} \leq \|A^{1/2}u\|_{\mathrm{L}^2} \leq C\|\nabla u\|_{\mathrm{L}^2} \qquad (u \in \mathrm{H}^1_\sigma(\mathbb{R}^d))$$

where $C > 0$ only depends on d, $\mu_\bullet$ and $\mu^\bullet$.

A particular application that Kato had in mind was the Lipschitz dependency of the square roots $A^{1/2}$ with respect to the coefficients μ measured in the L^∞-topology, which is proven via holomorphy. To formulate such properties and statements let us emphasize the coefficients in the notation of A by writing A_μ for the generalized Stokes operator with coefficients μ.

Theorem 1.3. *The set* $\mathcal{O} := \{\mu\colon \mu$ *satisfies Assumption 1.1 for some* $\mu_\bullet, \mu^\bullet > 0\}$ *is open in the* L^∞*-topology and the map*

$$\mathcal{O} \to \mathcal{L}(\mathrm{H}^1_\sigma(\mathbb{R}^d), \mathrm{L}^2_\sigma(\mathbb{R}^d)), \quad \mu \mapsto A_\mu^{1/2}$$

is holomorphic, i.e., for every $\mu \in \mathcal{O}$ *and* $M \in \mathrm{L}^\infty(\mathbb{R}^d; \mathcal{L}(\mathbb{C}^{d\times d}))$ *there exists* $r > 0$ *such that the map*

$$\{z \in \mathbb{C} : |z| < r\} \to \mathcal{L}(\mathrm{H}^1_\sigma(\mathbb{R}^d), \mathrm{L}^2_\sigma(\mathbb{R}^d)), \quad z \mapsto A_{\mu+zM}^{1/2}$$

is holomorphic.

This kind of holomorphy of $\mu \mapsto A_\mu^{1/2}$ implies the above-mentioned local Lipschitz property for small perturbations which reads as follows.

Corollary 1.4. *Let* $\mu_1 \in \mathcal{O}$*. Then there exists* $\delta > 0$ *and a constant* $C > 0$ *depending only on* $(\mu_1)_\bullet, (\mu_1)^\bullet, d$ *and* δ *such that for all* $\mu_2 \in \mathcal{O}$ *with* $\|\mu_2 - \mu_1\|_{\mathrm{L}^\infty(\mathbb{R}^d;\mathcal{L}(\mathbb{C}^{d\times d}))} < \delta$ *we have*

$$\|A_{\mu_2}^{1/2} - A_{\mu_1}^{1/2}\|_{\mathcal{L}(\mathrm{H}^1_\sigma, \mathrm{L}^2_\sigma)} \leq C\|\mu_2 - \mu_1\|_{\mathrm{L}^\infty(\mathbb{R}^d;\mathcal{L}(\mathbb{C}^{d\times d}))}.$$

Let us mention that the generalized Stokes operator arises in the theory of fluid mechanics as the linearization of non-Newtonian fluids, see, e.g., [41, Sec. 12.1]. In this context, symmetric coefficients are of particular interest and the square root property can be established by *Kato's second representation theorem* [34] using pure abstract reasoning. However, already for elliptic operators in divergence form, there is no known proof that establishes the Lipschitz estimate in Corollary 1.4 for symmetric coefficients by remaining in the class of operators with symmetric coefficients. All existing proofs, such as the one presented here, use in a crucial way the square root property for operators with nonsymmetric coefficients. Thus, this Lipschitz estimate can be regarded as nontrivial even if one is only interested in symmetric coefficients.

The outline of this paper is as follows. In Section 2 we collect elementary L^2-estimates on the resolvent of the generalized Stokes operator and then establish the above-mentioned nonlocal off-diagonal estimates of polynomial order. Afterwards we begin to adapt the proof of the square root property for elliptic operators as it was presented in [27]. In Section 3 we shortly recapitulate the reduction to a square function estimate which is then established in Sections 4 and 5. Section 4 presents how the classical principal part approximation can be adapted to these nonlocal off-diagonal estimates of polynomial order, leading — as in the elliptic situation — to a reduction to a Carleson measure estimate. This is established in Section 5 where a $T(b)$-type test function has to be constructed. Compared to elliptic operators there will be an additional constraint in this construction as it has to cope with two ingredients at the same time — divergence-freeness and locality. Here, a suitable Bogovskiĭ correction will be crucial. In the final Section 6 we establish Theorem 1.3 and Corollary 1.4.

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Notation

Throughout this article, the dimension is denoted by d and satisfies $d \geq 2$. We will write $\mathbb{N}$ for the set of all positive integers and $\mathbb{N}_0$ for the set of all non-negative integers. The norms of $\mathbb{C}$ and $\mathbb{C}^d$ will be denoted by $|\cdot|$, all other norms will be labeled accordingly. We will use the euclidean norm on $\mathbb{C}^d$ and $\mathbb{C}^{d\times d}$. The open ball centered in $x \in \mathbb{R}^d$ and with radius $r > 0$ is denoted by $B(x, r)$. Given an open ball B and $\ell \in \mathbb{N}_0$, the ℓ-th dyadic annulus around B is defined by

$$C_\ell(B) \coloneqq B(x, 2^{\ell+1}r) \setminus \overline{B(x, 2^\ell r)} \quad \text{if} \quad \ell \geq 1 \quad \text{and} \quad C_0(B) \coloneqq 2B.$$

Given a cube $Q \subseteq \mathbb{R}^d$ we denote its sidelength by $\ell(Q)$. As for balls, we define

$$C_\ell(Q) \coloneqq 2^{\ell+1}Q \setminus \overline{2^\ell Q} \quad \text{if} \quad \ell \geq 1 \quad \text{and} \quad C_0(Q) \coloneqq 2Q.$$

In estimates it will be convenient to write $\alpha \lesssim \beta$ or $\alpha \gtrsim \beta$ if there exists $C > 0$, depending only on parameters not at stake, such that $\alpha \leq C\beta$ or $C\alpha \geq \beta$. We will write $\alpha \simeq \beta$ if $\alpha \lesssim \beta$ and $\alpha \gtrsim \beta$. In some situations it will be more handy to keep the notation $\alpha \leq C\beta$ or $C\alpha \geq \beta$. In this case C is generic.

The characteristic function of a set $A \subseteq \mathbb{R}^d$ is denoted by $\mathbf{1}_A$. If A is measurable we denote its Lebesgue measure by $|A|$. In the case $0 < |A| < \infty$, we define the mean value of $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ as

$$(f)_A \coloneqq \fint_A f \,\mathrm{d}x \coloneqq \frac{1}{|A|} \int_A f \,\mathrm{d}x.$$

Important function spaces of divergence-free vector fields are given by

$$\begin{aligned} \mathrm{L}^2_\sigma(\mathbb{R}^d) &\coloneqq \{f \in \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^d) \colon \operatorname{div}(f) = 0 \text{ in sense of distributions}\} \\ \mathrm{H}^1_\sigma(\mathbb{R}^d) &\coloneqq \{f \in \mathrm{H}^1(\mathbb{R}^d; \mathbb{C}^d) \colon \operatorname{div}(f) = 0\}. \end{aligned}$$

Recall that

$$\mathrm{C}^\infty_{c,\sigma}(\mathbb{R}^d) \coloneqq \{\varphi \in \mathrm{C}^\infty_c(\mathbb{R}^d; \mathbb{C}^d) \colon \operatorname{div}(\varphi) = 0\}$$

is dense in $\mathrm{L}^2_\sigma(\mathbb{R}^d)$ as well as in $\mathrm{H}^1_\sigma(\mathbb{R}^d)$, see Lemma II.2.5.4 and Lemma II.2.5.5 in [42]. The space of all bounded and antilinear functionals from $\mathrm{H}^1_\sigma(\mathbb{R}^d) \to \mathbb{C}$ is denoted by

$\mathrm{H}^{-1}_{\sigma}(\mathbb{R}^d)$. Given $u \in \mathrm{H}^1(\mathbb{R}^d;\mathbb{C}^d)$, we regard its gradient as the matrix given as the transpose of the Jacobian of u, i.e., $\nabla u = (\partial_\alpha u_i)_{\alpha,i=1}^d$.

For $\alpha \in (0,1)$ the fractional Sobolev space with differentiability α will be denoted by $\mathrm{H}^\alpha(\mathbb{R}^d)$. We write its norm in integral form

$$\|f\|_{\mathrm{H}^\alpha} := \|f\|_{\mathrm{L}^2} + \|f\|_{\dot{\mathrm{H}}^\alpha} \quad \text{where} \quad \|f\|_{\dot{\mathrm{H}}^\alpha} := \Bigg(\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2\alpha}} \,\mathrm{d}x\,\mathrm{d}y \Bigg)^{\frac{1}{2}}.$$

For $f \in \mathrm{H}^\alpha(\mathbb{R}^d)$ we will need to calculate a counterpart of the homogeneous $\dot{\mathrm{H}}^\alpha$-seminorm on a measurable set $A \subseteq \mathbb{R}^d$ which we write as

$$\|f\|_{\dot{\mathrm{H}}^\alpha(A)} := \Bigg(\iint\limits_{A\times A} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2\alpha}} \,\mathrm{d}x\,\mathrm{d}y \Bigg)^{\frac{1}{2}}.$$

2. L^2-theory of the generalized Stokes resolvent

This section is devoted to properties of the generalized Stokes operator. It contains elementary properties that directly follow from classical form theory. The main result of this section is Proposition 2.5 which can be seen as nonlocal off-diagonal estimates for the generalized Stokes resolvent. We start by introducing an extension of the generalized Stokes operator in $\mathrm{H}^{-1}_{\sigma}(\mathbb{R}^d)$.

Definition 2.1. The *weak (generalized) Stokes operator* $\mathcal{A} : \mathrm{H}^1_\sigma(\mathbb{R}^d) \subseteq \mathrm{H}^{-1}_\sigma(\mathbb{R}^d) \to \mathrm{H}^{-1}_\sigma(\mathbb{R}^d)$ associated to the sesquilinear form $\mathfrak{a}$ is defined by

$$\langle \mathcal{A}u, v\rangle_{\mathrm{H}^{-1}_\sigma,\mathrm{H}^1_\sigma} := \mathfrak{a}(u,v) \qquad (u,v \in \mathrm{H}^1_\sigma(\mathbb{R}^d)).$$

Observe that $u \in \mathcal{D}(A)$ if and only if $\mathcal{A}u \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$. Hence, $\mathcal{A}$ is an extension of A. For later reasoning it will be important to deduce a representation of $\mathcal{A}$ as a product of the operators div, μ and ∇. For fixed x, the coefficients may be interpreted as a linear operator from $\mathbb{C}^{d\times d}$ to $\mathbb{C}^{d\times d}$ via

$$\mu(x)G := \Big(\sum_{j,\beta=1}^d \mu^{ij}_{\alpha\beta}(x) G_{\beta j} \Big)_{\alpha,i=1}^d \qquad (G \in \mathbb{C}^{d\times d}).$$

If one defines the dot-product of two matrices $G, H \in \mathbb{C}^{d\times d}$ as $G\cdot H := \sum_{\alpha,i=1}^d G_{\alpha i} H_{\alpha i}$, then the sesquilinear form defined in (1.1) can be written as

$$\mathfrak{a}(u,v) = \int\limits_{\mathbb{R}^d} \mu\nabla u \cdot \overline{\nabla v} \,\mathrm{d}x. \tag{2.1}$$

Given $F \in \mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^{d\times d})$, its weak divergence is defined, as usual, as an element in $\mathrm{H}^{-1}(\mathbb{R}^d;\mathbb{C}^d) \coloneqq (\mathrm{H}^1(\mathbb{R}^d;\mathbb{C}^d))'$ via

$$\langle \operatorname{div}(F), w\rangle_{\mathrm{H}^{-1},\mathrm{H}^1} \coloneqq -\sum_{\alpha,i=1}^{d} \int_{\mathbb{R}^d} F_{\alpha i}\overline{\partial_\alpha w_i}\,\mathrm{d}x \qquad (w \in \mathrm{H}^1(\mathbb{R}^d;\mathbb{C}^d)).$$

To interprete the function $v \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ in (2.1) as an element in $\mathrm{H}^1(\mathbb{R}^d;\mathbb{C}^d)$ we introduce the canonical inclusion $\iota : \mathrm{H}^1_\sigma(\mathbb{R}^d) \to \mathrm{H}^1(\mathbb{R}^d;\mathbb{C}^d)$. If $\mathcal{P} \coloneqq \iota' : \mathrm{H}^{-1}(\mathbb{R}^d;\mathbb{C}^d) \to \mathrm{H}^{-1}_\sigma(\mathbb{R}^d)$ denotes its adjoint, then for $u, v \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ we have

$$\langle \mathcal{A}u, v\rangle_{\mathrm{H}^{-1}_\sigma,\mathrm{H}^1_\sigma} = \int_{\mathbb{R}^d} \mu\nabla u \cdot \overline{\nabla \iota(v)}\,\mathrm{d}x = \langle -\operatorname{div}(\mu\nabla u), \iota(v)\rangle_{\mathrm{H}^{-1},\mathrm{H}^1} = \langle -\mathcal{P}\operatorname{div}(\mu\nabla u), v\rangle_{\mathrm{H}^{-1}_\sigma,\mathrm{H}^1_\sigma}.$$

We record this representation in the following lemma.

Lemma 2.2. *We have*

$$\mathcal{A}u = -\mathcal{P}\operatorname{div}(\mu\nabla u) \qquad (u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)).$$

Remark 2.3. The representation in Lemma 2.2 was already proven by Mitrea and Monniaux for the classical Stokes operator (with $\operatorname{div}(\mu\nabla u)$ replaced by Δu) in bounded Lipschitz domains, see [37, Prop. 4.5].

The control on the essential supremum of the $\mathcal{L}(\mathbb{C}^{d\times d})$-norm of μ in Assumption 1.1 can precisely be understood as $|\mu(x)G|_{\mathbb{C}^{d\times d}} \leq \mu^\bullet |G|_{\mathbb{C}^{d\times d}}$ for $G \in \mathbb{C}^{d\times d}$ and $x \in \mathbb{R}^d$. Thus

$$|\mathfrak{a}(u,v)| \leq \mu^\bullet \|\nabla u\|_{\mathrm{L}^2}\|\nabla v\|_{\mathrm{L}^2} \qquad (u, v \in \mathrm{H}^1(\mathbb{R}^d;\mathbb{C}^d)),$$

so that $\mathfrak{a}$ is, in particular, a bounded sesquilinear form on $\mathrm{H}^1_\sigma(\mathbb{R}^d)$. The Gårding-type estimate in Assumption 1.1 exactly means that $\mathfrak{a}$ is coercive. As a consequence, the uniform resolvent bounds in the next proposition, in particular the sectoriality of the generalized Stokes operator in $\mathrm{L}^2_\sigma(\mathbb{R}^d)$, follow from classical form theory involving the Lax–Milgram lemma. The proof will be omitted.

For the rest of this section, we will denote a sector in the complex plane with opening angle 2θ, $\theta \in (0,\pi)$, around the positive real line as

$$\mathrm{S}_\theta \coloneqq \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta\}.$$

Proposition 2.4. *Let μ satisfy Assumption 1.1. Then there exists $\omega \in (\pi/2,\pi)$ depending only on d, $\mu_\bullet$ and $\mu^\bullet$ such that $\mathrm{S}_\omega \subseteq \rho(-A) \cap \rho(-\mathcal{A})$ and for all $\theta \in (0,\omega)$ there exists $C > 0$ such that for all $\lambda \in \mathrm{S}_\theta$ and all $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$ we have*

$$|\lambda|\|(\lambda + A)^{-1} f\|_{\mathrm{L}^2} + |\lambda|^{\frac{1}{2}} \|\nabla(\lambda + A)^{-1} f\|_{\mathrm{L}^2} \leq C\|f\|_{\mathrm{L}^2}.$$

Moreover, there exists $C > 0$ *such that for all* $\lambda \in \mathrm{S}_\theta$ *and all* $F \in \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^{d\times d})$ *we have*

$$|\lambda|^{\frac{1}{2}} \|(\lambda + \mathcal{A})^{-1}\mathcal{P} \operatorname{div}(F)\|_{\mathrm{L}^2} + \|\nabla(\lambda + \mathcal{A})^{-1}\mathcal{P} \operatorname{div}(F)\|_{\mathrm{L}^2} \leq C\|F\|_{\mathrm{L}^2}.$$

The constant C *only depends on* θ*,* d*,* $\mu^\bullet$ *and* $\mu_\bullet$*.*

Finally, we present an off-diagonal type estimate for the family $((\lambda + \mathcal{A})^{-1}\mathcal{P} \operatorname{div})_{\lambda\in \mathrm{S}_\omega}$ which plays a key role in the proof of the square root property below.

Proposition 2.5. *There exists* $\omega \in (\pi/2, \pi)$ *such that for all* $\theta \in (0, \omega)$ *and* $\nu \in (0, d + 2)$ *there exists* $C > 0$ *such that for all balls* $B = B(x_0, r)$*,* $\lambda \in \mathrm{S}_\theta$ *and* $F \in \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^{d\times d})$ *the following nonlocal off-diagonal estimates are valid*

$$\int_B |(\lambda + \mathcal{A})^{-1}\mathcal{P} \operatorname{div}(F)|^2 \,\mathrm{d}x \leq \frac{C}{|\lambda|} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Big(\prod_{s=0}^{n-k-1} \frac{C}{|\lambda| r^2 2^{2s}} \Big) 2^{-\nu k} \int_{2^{n+1}B} |F|^2_{\mathbb{C}^{d\times d}} \,\mathrm{d}x.$$

The constant C *only depends on* θ*,* ν*,* d*,* $\mu^\bullet$ *and* $\mu_\bullet$*.*

Proof. For the sake of clarity, we will write $|\cdot| = |\cdot|_{\mathbb{C}^{d\times d}}$ in this proof.

Let $u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ be given by $u := (\lambda + \mathcal{A})^{-1}\mathcal{P} \operatorname{div}(F)$. Then in [44, Thm. 1.2], the following nonlocal Caccioppoli inequality was proven

$$\begin{aligned} |\lambda| \sum_{k=0}^{\infty} 2^{-\nu k} \int_{2^k B} |u|^2 \,\mathrm{d}x + \sum_{k=0}^{\infty} 2^{-\nu k} \int_{2^k B} |\nabla u|^2 \,\mathrm{d}x \\ \leq \sum_{k=0}^{\infty} 2^{-(\nu+2)k} \frac{C}{r^2} \int_{2^{k+1}B} |u|^2 \,\mathrm{d}x + C \sum_{k=0}^{\infty} 2^{-\nu k} \int_{2^{k+1}B} |F|^2 \,\mathrm{d}x \end{aligned}$$

for some $C > 0$ depending only on $\mu_\bullet, \mu^\bullet, \nu$ and dimension d. Dropping the series of the gradients, dividing by $|\lambda|$ and estimating $2^{-(\nu+2)k} \leq 2^{-\nu k}$ turns the estimate into

$$\sum_{k=0}^{\infty} 2^{-\nu k} \int_{2^k B} |u|^2 \,\mathrm{d}x \leq \frac{C}{|\lambda| r^2} \sum_{k=0}^{\infty} 2^{-\nu k} \int_{2^{k+1}B} |u|^2 \,\mathrm{d}x + \frac{C}{|\lambda|} \sum_{k=0}^{\infty} 2^{-\nu k} \int_{2^{k+1}B} |F|^2 \,\mathrm{d}x. \quad (2.2)$$

Observe that on the right-hand side almost the same term as on the left-hand side appears but that B is replaced by $2B$. Thus, this term can be estimated by using (2.2) once again but with B replaced by $2B$. This leads to

$$\sum_{k=0}^{\infty} 2^{-\nu k} \int_{2^k B} |u|^2 \,\mathrm{d}x \leq \frac{C}{|\lambda| r^2} \frac{C}{|\lambda| (2r)^2} \sum_{k=0}^{\infty} 2^{-\nu k} \int_{2^{k+2}B} |u|^2 \,\mathrm{d}x$$

$$+ \frac{C}{|\lambda|} \sum_{\ell=0}^{1} \sum_{k=0}^{\infty} \Big(\prod_{s=0}^{\ell-1} \frac{C}{|\lambda| r^2 2^{2s}} \Big) 2^{-\nu k} \int_{2^{k+\ell+1}B} |F|^2 \, \mathrm{d}x.$$

Iterating this procedure with growing radii leads in the limit to

$$\begin{aligned} \sum_{k=0}^{\infty} 2^{-\nu k} \int_{2^k B} |u|^2 \, \mathrm{d}x &\leq \frac{C}{|\lambda|} \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \Big(\prod_{s=0}^{\ell-1} \frac{C}{|\lambda| r^2 2^{2s}} \Big) 2^{-\nu k} \int_{2^{k+\ell+1}B} |F|^2 \, \mathrm{d}x \\ &= \frac{C}{|\lambda|} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \Big(\prod_{s=0}^{\ell-1} \frac{C}{|\lambda| r^2 2^{2s}} \Big) 2^{-\nu k} \int_{2^{k+\ell+1}B} |F|^2 \, \mathrm{d}x \\ &= \frac{C}{|\lambda|} \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \Big(\prod_{s=0}^{n-k-1} \frac{C}{|\lambda| r^2 2^{2s}} \Big) 2^{-\nu k} \int_{2^{n+1}B} |F|^2 \, \mathrm{d}x \\ &= \frac{C}{|\lambda|} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Big(\prod_{s=0}^{n-k-1} \frac{C}{|\lambda| r^2 2^{2s}} \Big) 2^{-\nu k} \int_{2^{n+1}B} |F|^2 \, \mathrm{d}x. \quad \square \end{aligned}$$

Corollary 2.6. *For all $\nu \in (0, d+2)$ there exists a constant $C > 0$ such that for all $t > 0$, balls $B = B(x_0, r)$ with $r \geq t$ and $F \in \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^{d \times d})$ we have*

$$\int_B |t(1 + t^2 \mathcal{A})^{-1} \mathcal{P} \operatorname{div}(F)|^2 \, \mathrm{d}x \leq C \sum_{n=0}^{\infty} 2^{-\nu n} \int_{C_n(B)} |F|_{\mathbb{C}^{d \times d}}^2 \, \mathrm{d}x.$$

The constant C only depends on ν, d, $\mu^\bullet$ and $\mu_\bullet$.

Proof. For the sake of clarity, we will write $|\cdot| = |\cdot|_{\mathbb{C}^{d \times d}}$ in this proof.

First observe that $\lambda^{\frac{1}{2}} (\lambda + \mathcal{A})^{-1} \mathcal{P} \operatorname{div}(F) = t(1 + t^2 \mathcal{A})^{-1} \mathcal{P} \operatorname{div}(F)$ if $\lambda = t^{-2}$. Next, an application of Proposition 2.5 with $\nu' \in (\nu, d+2)$ yields

$$\begin{aligned} \int_B |t(1 + t^2 \mathcal{A})^{-1} \mathcal{P} \operatorname{div}(F)|^2 \, \mathrm{d}x &\lesssim \sum_{n=0}^{\infty} \sum_{k=0}^{n} \Big(\prod_{s=0}^{n-k-1} \frac{C}{\frac{r^2}{t^2} 2^{2s}} \Big) 2^{-\nu' k} \int_{2^{n+1}B} |F|^2 \, \mathrm{d}x \\ &\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} C^{n-k} 2^{-(n-k-1)(n-k)} 2^{-\nu' k} \int_{2^{n+1}B} |F|^2 \, \mathrm{d}x. \end{aligned}$$

Without loss of generality, assume that $C \geq 1$. Let $k_0 \in \mathbb{N}_0$ be such that $C \leq 2^{k_0 - 1 - \nu'}$, so that $C 2^{-(n-k-1)} \leq 2^{-\nu'}$ whenever $n - k \geq k_0$. Then

$$\sum_{k=0}^{n} C^{n-k} 2^{-(n-k-1)(n-k)} 2^{-\nu' k} \leq \sum_{k=0}^{n-k_0} 2^{-\nu'(n-k)} 2^{-\nu' k} + \sum_{k=n-k_0+1}^{n} C^{k_0 - 1} 2^{-\nu' k} \lesssim 2^{-\nu n}.$$

Consequently,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} C^{n-k}2^{-(n-k-1)(n-k)}2^{-\nu' k}\int_{2^{n+1}B}|F|^2\,\mathrm{d}x \lesssim \sum_{n=0}^{\infty} 2^{-\nu n}\int_{2^{n+1}B}|F|^2\,\mathrm{d}x.$$

Splitting $2^{n+1}B$ into annuli finally yields

$$\sum_{n=0}^{\infty} 2^{-\nu n}\int_{2^{n+1}B}|F|^2\,\mathrm{d}x = \sum_{n=0}^{\infty}\sum_{\ell=0}^{n} 2^{-\nu n}\int_{C_\ell(B)}|F|^2\,\mathrm{d}x \lesssim \sum_{\ell=0}^{\infty} 2^{-\nu \ell}\int_{C_\ell(B)}|F|^2\,\mathrm{d}x. \qquad \square$$

Remark 2.7.

(1) One can replace balls by cubes in the statements of Proposition 2.5 and Corollary 2.6 since one can prove the nonlocal Caccioppoli inequality for cubes as well.

(2) If $\mathrm{supp}(F) \subseteq \overline{C_n(B)}$ for some $n \in \mathbb{N}$, then the estimate in Corollary 2.6 turns into

$$\|t(1+t^2\mathcal{A})^{-1}\mathcal{P}\,\mathrm{div}(F)\|_{\mathrm{L}^2(B)} \leq C2^{-\frac{\nu}{2}n}\|F\|_{\mathrm{L}^2(C_n(B))},$$

which is an off-diagonal estimate of polynomial order $\frac{\nu}{2}$.

3. Reduction to a square function estimate

As for elliptic operators, the square root property for the generalized Stokes operator is equivalent to the square function estimates

$$\int_0^\infty \|tA(1+t^2A)^{-1}u\|_{\mathrm{L}^2}^2\,\frac{\mathrm{d}t}{t} \lesssim \|\nabla u\|_{\mathrm{L}^2}^2 \qquad (u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)) \tag{3.1}$$

and

$$\int_0^\infty \|tA^*(1+t^2A^*)^{-1}u\|_{\mathrm{L}^2}^2\,\frac{\mathrm{d}t}{t} \lesssim \|\nabla u\|_{\mathrm{L}^2}^2 \qquad (u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)),$$

where A^* is the Hilbert space adjoint to A. An abstract argument for this was presented in [27, Prop. 12.7]. We shortly recapitulate the proof of necessity of this equivalence here.

Since A^* is a generalized Stokes operator which coefficients (given by $(\mu^*)^{ij}_{\alpha\beta} := \overline{\mu^{ji}_{\beta\alpha}}$) still satisfy Assumption 1.1 it will suffice to establish (3.1) for the class of all generalized Stokes operators subject to Assumption 1.1.

Consider the holomorphic function $f(z) := \sqrt{z}(1+z)^{-1}$ on S_π. As A is m-accretive it has a bounded H^∞-calculus due to von Neumann's inequality [25, Thm. 7.1.7]. McIntosh's theorem [25, Thm. 7.3.1] implies that A satisfies quadratic estimates, i.e., that

$$\int_0^\infty \|f(t^2 A)v\|_{\mathrm{L}^2}^2 \,\frac{\mathrm{d}t}{t} \simeq \|v\|_{\mathrm{L}^2}^2 \qquad (v \in \mathrm{L}^2_\sigma(\mathbb{R}^d)). \tag{3.2}$$

Given $u \in \mathcal{D}(A)$, one calculates $f(t^2 A)A^{1/2}u = tA(1 + t^2 A)^{-1}u$, so that (3.1) combined with (3.2) applied to $v = A^{1/2}u$ turns into

$$\|A^{1/2}u\|_{\mathrm{L}^2}^2 \lesssim \int_0^\infty \|tA(1 + t^2 A)^{-1}u\|_{\mathrm{L}^2}^2 \,\frac{\mathrm{d}t}{t} \lesssim \|\nabla u\|_{\mathrm{L}^2}^2 \qquad (u \in \mathcal{D}(A)).$$

Since $\mathcal{D}(A)$ is dense in $\mathrm{H}^1_\sigma(\mathbb{R}^d)$ with respect to the $\mathrm{H}^1_\sigma(\mathbb{R}^d)$-norm (this follows abstractly by [34, Thm. VI.2.1]), the closedness of $A^{1/2}$ yields that $\mathrm{H}^1_\sigma(\mathbb{R}^d) \subseteq \mathcal{D}(A^{1/2})$ and that

$$\|A^{1/2}u\|_{\mathrm{L}^2} \lesssim \|\nabla u\|_{\mathrm{L}^2} \qquad (u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)).$$

Replacing A by A^* we deduce the same estimate for A^*. Thus, Assumption 1.1 yields for $u \in \mathcal{D}(A)$

$$\mu_\bullet \|\nabla u\|_{\mathrm{L}^2}^2 \leq \operatorname{Re} \mathfrak{a}(u, u) = \operatorname{Re}\langle Au, u\rangle_{\mathrm{L}^2} \leq \|A^{1/2}u\|_{\mathrm{L}^2} \|(A^*)^{1/2}u\|_{\mathrm{L}^2} \lesssim \|A^{1/2}u\|_{\mathrm{L}^2} \|\nabla u\|_{\mathrm{L}^2}.$$

Since $\mathcal{D}(A)$ is a core for $A^{1/2}$ (see [34, Thm. V.3.35]), we conclude that $\mathcal{D}(A^{1/2}) \subseteq \mathrm{H}^1_\sigma(\mathbb{R}^d)$ and that

$$\|\nabla u\|_{\mathrm{L}^2} \lesssim \|A^{1/2}u\|_{\mathrm{L}^2} \qquad (u \in \mathcal{D}(A^{1/2}))$$

by density. Thus, Theorem 1.2 is proved once the square function estimate (3.1) can be established for all generalized Stokes operators A which coefficients satisfy Assumption 1.1.

4. Principal part approximation

Since the square function estimate (3.1) requires a control by $F := \nabla u$ we rewrite the expression on the left-hand side in terms of ∇u by using Lemma 2.2

$$tA(1 + t^2 A)^{-1}u = t(1 + t^2 \mathcal{A})^{-1}\mathcal{A}u = -t(1 + t^2 \mathcal{A})^{-1}\mathcal{P} \operatorname{div}(\mu \nabla u) =: \Theta_t(\nabla u). \tag{4.1}$$

Definition 4.1. For $t > 0$ define the bounded operator Θ_t by

$$\Theta_t \colon \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^{d\times d}) \to \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^d), \quad \Theta_t F := -t(1 + t^2 \mathcal{A})^{-1}\mathcal{P} \operatorname{div}(\mu F).$$

Recall that $(-t(1 + t^2 \mathcal{A})^{-1}\mathcal{P} \operatorname{div})_{t>0}$ defines a bounded family of operators in $\mathcal{L}(\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^{d\times d}), \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^d))$ and further satisfies the nonlocal off-diagonal bounds from Proposition 2.5 and Corollary 2.6. Since Θ_t arises from that family by multiplication by the bounded, matrix-valued function μ, these estimates are transferred to Θ_t and will be used frequently in the following. A first application is the following definition of Θ_t on bounded matrix-valued functions.

Proposition 4.2. *Let $b \in \mathrm{L}^\infty(\mathbb{R}^d; \mathbb{C}^{d\times d})$. Then for every $t > 0$ and ball $B = B(x,r) \subseteq \mathbb{R}^d$ the limit*

$$\Theta_t b := \lim_{j\to\infty} \Theta_t(\mathbf{1}_{2^j B} b)$$

exists in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{C}^d)$ and is independent of x and r. Additionally, one obtains the same limit if B is replaced by a cube with sides parallel to the coordinate axes.

Proof. Let $K \subseteq \mathbb{R}^d$ be an arbitrary compact set. Then there exists $j_0 \in \mathbb{N}$ such that $K \subseteq 2^{j_0}B$. Assume from now on that $\ell > j > j_0$. Proposition 2.5 implies

$$\begin{aligned}
&\|\Theta_t(\mathbf{1}_{2^\ell B} b) - \Theta_t(\mathbf{1}_{2^j B} b)\|^2_{\mathrm{L}^2(K)} \leq \|\Theta_t(\mathbf{1}_{2^\ell B\setminus 2^j B} b)\|^2_{\mathrm{L}^2(2^{j_0}B)} \\
&\quad\lesssim \sum_{n=0}^{\infty}\sum_{k=0}^{n}\Bigg(\prod_{s=0}^{n-k-1}\frac{C}{\frac{2^{2(s+j_0)}r^2}{t^2}}\Bigg)2^{-\nu k}\cdot\|\mathbf{1}_{2^\ell B\setminus 2^j B} b\|^2_{\mathrm{L}^2(2^{j_0+n+1}B)} \\
&\quad\leq \sum_{n=j-j_0}^{\infty}\sum_{k=0}^{n}\Bigg(\prod_{s=0}^{n-k-1}\frac{C}{\frac{2^{2(s+j_0)}r^2}{t^2}}\Bigg)2^{-\nu k}\cdot\|b\|^2_{\mathrm{L}^2(2^{j_0+n+1}B)} \\
&\quad\leq \sum_{n=j-j_0}^{\infty}\sum_{k=0}^{n}\Bigg(\prod_{s=0}^{n-k-1}\frac{C}{\frac{2^{2(s+j_0)}r^2}{t^2}}\Bigg)2^{-\nu k}\cdot|2^{j_0+n+1}B|\,\|b\|^2_{\mathrm{L}^\infty} \\
&\quad\leq \sum_{n=j-j_0}^{\infty}\sum_{k=0}^{n}\Bigg(\frac{Ct^2}{2^{2j_0}r^2}\Bigg)^{n-k}2^{-(n-k-1)(n-k)}2^{-\nu k}\cdot|2^{j_0+n+1}B|\,\|b\|^2_{\mathrm{L}^\infty}.
\end{aligned}$$

Choosing j_0 large enough, we can assume that $Ct^2/(2^{2j_0}r^2) \leq 1$. Moreover, maximizing the coefficient $2^{-(n-k-1)(n-k)}2^{-\nu k}$ with respect to k shows that

$$\begin{aligned}
\|\Theta_t(\mathbf{1}_{2^\ell B} b) - \Theta_t(\mathbf{1}_{2^j B} b)\|^2_{\mathrm{L}^2(K)} &\leq C(\nu)\sum_{n=j-j_0}^{\infty} n2^{-\nu n}\cdot|2^{j_0+n+1}B|\,\|b\|^2_{\mathrm{L}^\infty} \\
&\leq C(\nu,d,j_0,r)\sum_{n=j-j_0}^{\infty} n2^{(d-\nu)n}\|b\|^2_{\mathrm{L}^\infty},
\end{aligned}$$

which converges to zero for $\ell, j \to \infty$ because we can choose $\nu > d$. Hence, $(\Theta_t(\mathbf{1}_{2^j B} b))_{j\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{L}^2(K)$ for every K, which implies the convergence in $\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{C}^d)$.

B-independence of the limit as well as that B could be replaced by a cube follows by an analogous pattern, see, e.g., [8, Prop. 5.1]. Recall that (as was mentioned in Remark 2.7) the off-diagonal estimates hold for cubes as well. □

Definition 4.3 *(Principal part of* Θ_t*).* Identify the matrices $e_{jk} = (\delta_{jm}\delta_{kn})_{m,n=1}^d \in \mathbb{C}^{d\times d}$, where δ_{mn} denotes Kronecker's delta, with the respective constant functions on $\mathbb{R}^d$. For $t > 0$ we define the *principal part of* Θ_t as

$$(\gamma_t)_{jk} := \Theta_t(e_{jk}) \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d; \mathbb{C}^d)$$

for all $j, k = 1, \dots, d$. In many situations, we will regard $\gamma_t(x)$, for $x \in \mathbb{R}^d$ and $t > 0$ fixed, as a linear operator in $\mathcal{L}(\mathbb{C}^{d\times d}, \mathbb{C}^d)$ defined via

$$\gamma_t(x) \cdot G := \sum_{j,k=1}^d (\gamma_t(x))_{jk} G_{jk} \qquad (G \in \mathbb{C}^{d\times d}).$$

The norm of $\gamma_t(x)\cdot$ will be denoted by $|\gamma_t(x)\cdot|_{\mathcal{L}(\mathbb{C}^{d\times d},\mathbb{C}^d)}$.

The idea of the principal part is to approximate the operator Θ_t in an averaged sense in order to reduce the square function estimate (3.1) to a certain Carleson measure estimate. For this purpose, we next introduce the dyadic averaging operator as in [27, Sec. 9.3].

Definition 4.4.

(1) For each $j \in \mathbb{Z}$ we define *dyadic cubes of generation* 2^j as elements of the set

$$\square_{2^j} := \{2^j x + [0, 2^j)^d : x \in \mathbb{Z}^d\}$$

and denote by $\square := \cup_{j\in\mathbb{Z}} \square_{2^j}$ the collection of all dyadic cubes. Moreover, for $t > 0$ set $\square_t := \square_{2^j}$ for the unique integer with $2^{j-1} < t \leq 2^j$.

(2) For $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $t > 0$ define the *dyadic averaging operator (at scale t)* as

$$(\mathcal{A}_t u)(x) := \fint_Q u(y)\,\mathrm{d}y$$

where $Q \in \square_t$ is the unique dyadic cube which contains $x \in \mathbb{R}^d$.

When a scalar operator, such as the averaging operator, is applied to a vector- or matrix-valued function, we agree from now on to apply the operator to each component separately.

In the following proposition, the principal part will be applied to an average of ∇u. Observe that the trace of the matrix ∇u satisfies $\mathrm{tr}(\nabla u) = 0$ because u is divergence-free. To capture this fact, we introduce the following projection onto trace-free matrices on $\mathbb{C}^{d\times d}$

$$\mathrm{T}_0 : \mathbb{C}^{d\times d} \to \mathbb{C}^{d\times d}, \quad \mathrm{T}_0 G := G - \frac{\operatorname{tr}(G)}{d}\operatorname{Id}_{d\times d}$$

where $\operatorname{Id}_{d\times d} \in \mathbb{C}^{d\times d}$ is the identity matrix. The map $\gamma_t(x)\cdot \mathrm{T}_0 \in \mathcal{L}(\mathbb{C}^{d\times d}, \mathbb{C}^d)$ will be understood as $\gamma_t(x)\cdot \mathrm{T}_0 G := \gamma_t(x)\cdot(\mathrm{T}_0 G)$ for $G \in \mathbb{C}^{d\times d}$.

Proposition 4.5 *(Reduction to a Carleson measure estimate). If*

$$\int_0^\infty \|(\Theta_t - \gamma_t\cdot\mathcal{A}_t)(\nabla u)\|_{\mathrm{L}^2}^2 \frac{\mathrm{d}t}{t} \lesssim \|\nabla u\|_{\mathrm{L}^2}^2 \qquad (u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)), \tag{4.2}$$

and if $|\gamma_t(x)\cdot \mathrm{T}_0|^2_{\mathcal{L}(\mathbb{C}^{d\times d},\mathbb{C}^d)} \frac{\mathrm{d}x\,\mathrm{d}t}{t}$ *is a Carleson measure, then (3.1) is valid for all* $u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$*.*

Proof. Observe that $\gamma_t(x)\cdot(\mathcal{A}_t\nabla u)(x) = \gamma_t(x)\cdot\mathrm{T}_0(\mathcal{A}_t\nabla u)(x)$ since $\operatorname{div}(u) = 0$. Using (4.2) we estimate

$$\begin{aligned}\int_0^\infty \|\Theta_t(\nabla u)\|_{\mathrm{L}^2}^2 \frac{\mathrm{d}t}{t} &\lesssim \int_0^\infty \|(\Theta_t - \gamma_t\cdot\mathcal{A}_t)(\nabla u)\|_{\mathrm{L}^2}^2 \frac{\mathrm{d}t}{t} \\ &\quad + \int_0^\infty\int_{\mathbb{R}^d} |(\mathcal{A}_t\nabla u)(x)|^2_{\mathbb{C}^{d\times d}} |\gamma_t(x)\cdot\mathrm{T}_0|^2_{\mathcal{L}(\mathbb{C}^{d\times d},\mathbb{C}^d)} \frac{\mathrm{d}x\,\mathrm{d}t}{t} \\ &\lesssim \|\nabla u\|_{\mathrm{L}^2}^2 + \int_0^\infty\int_{\mathbb{R}^d} |(\mathcal{A}_t\nabla u)(x)|^2_{\mathbb{C}^{d\times d}} |\gamma_t(x)\cdot\mathrm{T}_0|^2_{\mathcal{L}(\mathbb{C}^{d\times d},\mathbb{C}^d)} \frac{\mathrm{d}x\,\mathrm{d}t}{t}.\end{aligned}$$

Under the assumption that $|\gamma_t(x)\cdot \mathrm{T}_0|^2_{\mathcal{L}(\mathbb{C}^{d\times d},\mathbb{C}^d)} \frac{\mathrm{d}x\,\mathrm{d}t}{t}$ is a Carleson measure, the second term can be estimated by the L^2-norm of ∇u due to Carleson's lemma, see [27, Thm. 9.19]. The relation (4.1) now implies the claim. □

The rest of this section is devoted to establish (4.2). We follow the approach of [27] by first showing uniform bounds of $\gamma_t\cdot\mathcal{A}_t$ and its approximation $\Theta_t - \gamma_t\cdot\mathcal{A}_t$ in $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^d)$. The following representation of $\gamma_t\cdot\mathcal{A}_t F$ for $F \in \mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^{d\times d})$ will play a useful role in their proofs. For a fixed dyadic cube $Q \in \square_t$ we have for $x \in Q$

$$\begin{aligned}\gamma_t(x)\cdot\mathcal{A}_t F(x) &= \sum_{j,k=1}^d [\Theta_t(e_{jk})](x)\,(F_{jk})_Q = \Big[\Theta_t\Big(\sum_{j,k=1}^d e_{jk}\,(F_{jk})_Q\Big)\Big](x) \\ &= [\Theta_t\big(\mathbf{1}_{\mathbb{R}^d}(F)_Q\big)](x).\end{aligned}$$

Thus, by virtue of Proposition 4.2, we obtain

$$\begin{aligned}\gamma_t \cdot \mathcal{A}_t F &= \lim_{j\to\infty} \Theta_t\big(\mathbf{1}_{2^j Q}(F)_Q\big)\\ &= \lim_{j\to\infty} \sum_{\ell=0}^{j-1} \Theta_t\big(\mathbf{1}_{C_\ell(Q)}(F)_Q\big) = \sum_{\ell=0}^{\infty} \Theta_t\big(\mathbf{1}_{C_\ell(Q)}(F)_Q\big).\end{aligned} \tag{4.3}$$

Lemma 4.6. *For all $t > 0$ we have*

$$\|\gamma_t \cdot \mathcal{A}_t F\|_{\mathrm{L}^2} \lesssim \|F\|_{\mathrm{L}^2} \qquad (F \in \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^{d\times d})).$$

Proof. It is enough to show

$$\|\gamma_t \cdot \mathcal{A}_t F\|_{\mathrm{L}^2(Q)}^2 \lesssim \|F\|_{\mathrm{L}^2(Q)}^2 \tag{4.4}$$

for all cubes $Q \in \square_t$. As for (4.4), we use (4.3) and apply the off-diagonal type estimates for Θ_t from Corollary 2.6 with $\ell(Q) \geq t$ in order to estimate

$$\begin{aligned}\|\gamma_t \cdot \mathcal{A}_t F\|_{\mathrm{L}^2(Q)} &\leq \sum_{\ell=0}^{\infty} \big\|\Theta_t\big(\mathbf{1}_{C_\ell(Q)}(F)_Q\big)\big\|_{\mathrm{L}^2(Q)}\\ &\leq C\sum_{\ell=0}^{\infty} \Big(\sum_{n=0}^{\infty} 2^{-\nu n}\|\mathbf{1}_{C_\ell(Q)}(F)_Q\|_{\mathrm{L}^2(C_n(Q))}^2\Big)^{\frac{1}{2}}\\ &= C\sum_{\ell=0}^{\infty} 2^{-\frac{\nu}{2}\ell}\|(F)_Q\|_{\mathrm{L}^2(C_\ell(Q))}.\end{aligned} \tag{4.5}$$

By Hölder's inequality, it follows

$$\|(F)_Q\|_{\mathrm{L}^2(C_\ell(Q))} = \frac{|C_\ell(Q)|^{\frac{1}{2}}}{|Q|}\Big|\int_Q F \,\mathrm{d}x\Big|_{\mathbb{C}^{d\times d}} \leq \frac{|C_\ell(Q)|^{\frac{1}{2}}}{|Q|^{\frac{1}{2}}}\|F\|_{\mathrm{L}^2(Q)} = 2^{\frac{d}{2}\ell}\|F\|_{\mathrm{L}^2(Q)}.$$

Hence, we get

$$\|\gamma_t \cdot \mathcal{A}_t F\|_{\mathrm{L}^2(Q)} \leq C\sum_{\ell=0}^{\infty} 2^{\frac{d-\nu}{2}\ell}\|F\|_{\mathrm{L}^2(Q)} \leq C\|F\|_{\mathrm{L}^2(Q)}$$

where we used in the last inequality that the choice $\nu > d$ is admissible. This proves (4.4). □

The following lemma provides already an estimate in the direction of (4.2) but with a fractional Sobolev norm instead of an L^2-norm on the right-hand side. Observe, that the information that F is a gradient is not (yet) used in this estimate. Note that such a control by a fractional Sobolev norm was also used in the resolution of the parabolic Kato problem, see [9, Sec. 7.3].

Lemma 4.7. *For all $t > 0$ and $\alpha \in (0,1)$ we have*

$$\|(\Theta_t - \gamma_t \cdot \mathcal{A}_t)F\|_{\mathrm{L}^2} \lesssim t^\alpha \|F\|_{\dot{\mathrm{H}}^\alpha} \qquad (F \in \mathrm{H}^\alpha(\mathbb{R}^d; \mathbb{C}^{d\times d})).$$

Proof. Similar to the proof in [27, Lem. 13.6] we argue in two steps: First we work on a fixed dyadic cube and then sum over a partition of $\mathbb{R}^d$. So fix a cube $Q \in \square_t$. Since Θ_t is bounded from $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^{d\times d})$ to $\mathrm{L}^2_\sigma(\mathbb{R}^d)$, we have

$$\Theta_t F = \sum_{\ell=0}^\infty \Theta_t\big(\mathbf{1}_{C_\ell(Q)} F\big) \qquad (F \in \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^{d\times d}))$$

with convergence in $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^d)$. Subtracting (4.3) on Q and applying the off-diagonal estimates from Corollary 2.6, we find

$$\begin{aligned}
\|(\Theta_t - \gamma_t \cdot \mathcal{A}_t)F\|_{\mathrm{L}^2(Q)} &\le \sum_{\ell=0}^\infty \big\|\Theta_t\big(\mathbf{1}_{C_\ell(Q)}(F - (F)_Q)\big)\big\|_{\mathrm{L}^2(Q)} \\
&\lesssim \sum_{\ell=0}^\infty \Big(\sum_{n=0}^\infty 2^{-\nu n} \big\|\mathbf{1}_{C_\ell(Q)}(F - (F)_Q)\big\|^2_{\mathrm{L}^2(C_n(Q))}\Big)^{\frac{1}{2}} \\
&= \sum_{\ell=0}^\infty 2^{-\frac{\nu}{2}\ell} \|F - (F)_Q\|_{\mathrm{L}^2(2^{\ell+1}Q)}.
\end{aligned}$$

Next, the aim is to control the right-hand side by a fractional Sobolev–Poincaré inequality. For a sufficiently good control on the dependency of the implicit constant on ℓ, we introduce a telescopic sum and estimate

$$\begin{aligned}
&\|F - (F)_Q\|_{\mathrm{L}^2(2^{\ell+1}Q)} \\
&\quad\le \|F - (F)_{2^{\ell+1}Q}\|_{\mathrm{L}^2(2^{\ell+1}Q)} + \sum_{j=1}^{\ell+1} \|(F)_{2^jQ} - (F)_{2^{j-1}Q}\|_{\mathrm{L}^2(2^{\ell+1}Q)} \\
&\quad= \|F - (F)_{2^{\ell+1}Q}\|_{\mathrm{L}^2(2^{\ell+1}Q)} + |2^{\ell+1}Q|^{\frac{1}{2}} \sum_{j=1}^{\ell+1} |(F)_{2^jQ} - (F)_{2^{j-1}Q}|_{\mathbb{C}^{d\times d}} \\
&\quad= \|F - (F)_{2^{\ell+1}Q}\|_{\mathrm{L}^2(2^{\ell+1}Q)} + |2^{\ell+1}Q|^{\frac{1}{2}} \sum_{j=1}^{\ell+1} \frac{1}{|2^{j-1}Q|} \Big| \int_{2^{j-1}Q} F - (F)_{2^jQ} \,\mathrm{d}x \Big|_{\mathbb{C}^{d\times d}} \\
&\quad\le \|F - (F)_{2^{\ell+1}Q}\|_{\mathrm{L}^2(2^{\ell+1}Q)} + |2^{\ell+1}Q|^{\frac{1}{2}} \sum_{j=1}^{\ell+1} \frac{1}{|2^{j-1}Q|^{\frac{1}{2}}} \|F - (F)_{2^jQ}\|_{\mathrm{L}^2(2^jQ)}
\end{aligned}$$

where we used Hölder's inequality in the last step. Now, apply the fractional Sobolev–Poincaré inequality from [33, Lem. 2.2] componentwise with $p, q = 2$ and $\alpha \in (0,1)$: There is a constant C, depending only on d, such that

$$\|F-(F)_{2^jQ}\|_{\mathrm{L}^2(2^jQ)} \leq C|2^jQ|^{\frac{\alpha}{d}}\|F\|_{\dot{\mathrm{H}}^\alpha(2^jQ)} \leq Ct^\alpha 2^{\alpha j}\|F\|_{\dot{\mathrm{H}}^\alpha(2^jQ)} \tag{4.6}$$

for all $j \in \mathbb{N}$. This leads to

$$\begin{aligned}
&\|F-(F)_Q\|_{\mathrm{L}^2(2^{\ell+1}Q)} \\
&\quad\lesssim t^\alpha 2^{\alpha(\ell+1)}\|F\|_{\dot{\mathrm{H}}^\alpha(2^{\ell+1}Q)} + |2^{\ell+1}Q|^{\frac{1}{2}}\sum_{j=1}^{\ell+1}\frac{t^\alpha 2^{\alpha j}}{|2^{j-1}Q|^{\frac{1}{2}}}\|F\|_{\dot{\mathrm{H}}^\alpha(2^jQ)} \\
&\quad\simeq t^\alpha 2^{\alpha\ell}\|F\|_{\dot{\mathrm{H}}^\alpha(2^{\ell+1}Q)} + t^\alpha 2^{\frac{d}{2}\ell}\sum_{j=1}^{\ell+1}2^{\alpha j-\frac{d}{2}j}\|F\|_{\dot{\mathrm{H}}^\alpha(2^jQ)}.
\end{aligned}$$

Summarizing, we showed for $0 < \nu' < \nu$

$$\begin{aligned}
&\|(\Theta_t-\gamma_t\cdot\mathcal{A}_t)F\|_{\mathrm{L}^2(Q)}^2 \\
&\quad\lesssim t^{2\alpha}\sum_{\ell=0}^\infty 2^{-\nu'\ell}\Big(2^{\alpha\ell}\|F\|_{\dot{\mathrm{H}}^\alpha(2^{\ell+1}Q)} + 2^{\frac{d}{2}\ell}\sum_{j=1}^{\ell+1}2^{\alpha j-\frac{d}{2}j}\|F\|_{\dot{\mathrm{H}}^\alpha(2^jQ)}\Big)^2 \\
&\quad\lesssim t^{2\alpha}\sum_{\ell=0}^\infty 2^{-\nu'\ell}\Big(2^{2\alpha\ell}\|F\|_{\dot{\mathrm{H}}^\alpha(2^{\ell+1}Q)}^2 + (\ell+1)2^{d\ell}\sum_{j=1}^{\ell+1}2^{2\alpha j-dj}\|F\|_{\dot{\mathrm{H}}^\alpha(2^jQ)}^2\Big).
\end{aligned}$$

Now, summing over all $Q \in \square_t$ yields

$$\begin{aligned}
&\|(\Theta_t-\gamma_t\cdot\mathcal{A}_t)F\|_{\mathrm{L}^2}^2 \\
&\lesssim t^{2\alpha}\sum_{Q\in\square_t}\sum_{\ell=0}^\infty 2^{-\nu'\ell}\Big(2^{2\alpha\ell}\|F\|_{\dot{\mathrm{H}}^\alpha(2^{\ell+1}Q)}^2 + (\ell+1)2^{d\ell}\sum_{j=1}^{\ell+1}2^{2\alpha j-dj}\|F\|_{\dot{\mathrm{H}}^\alpha(2^jQ)}^2\Big) \\
&= t^{2\alpha}\sum_{\ell=0}^\infty 2^{-\nu'\ell}\Big(2^{2\alpha\ell}\sum_{Q\in\square_t}\|F\|_{\dot{\mathrm{H}}^\alpha(2^{\ell+1}Q)}^2 + (\ell+1)2^{d\ell}\sum_{j=1}^{\ell+1}2^{2\alpha j-dj}\sum_{Q\in\square_t}\|F\|_{\dot{\mathrm{H}}^\alpha(2^jQ)}^2\Big).
\end{aligned}$$

Observe that

$$\begin{aligned}
\sum_{Q\in\square_t}\|F\|_{\dot{\mathrm{H}}^\alpha(2^jQ)}^2 &= \sum_{Q\in\square_t}\int_{2^jQ}\int_{2^jQ}\frac{|F(x)-F(y)|_{\mathbb{C}^{d\times d}}^2}{|x-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big(\sum_{Q\in\square_t}\mathbf{1}_{2^jQ}(x)\Big)\frac{|F(x)-F(y)|_{\mathbb{C}^{d\times d}}^2}{|x-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y \\
&\lesssim 2^{dj}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|F(x)-F(y)|_{\mathbb{C}^{d\times d}}^2}{|x-y|^{d+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y \\
&= 2^{dj}\|F\|_{\dot{\mathrm{H}}^\alpha(\mathbb{R}^d)}^2
\end{aligned}$$

for all $j \in \mathbb{N}$. Thus, we conclude

$$\begin{aligned}
&\|(\Theta_t - \gamma_t \cdot \mathcal{A}_t)F\|_{\mathrm{L}^2}^2 \\
&\lesssim t^{2\alpha} \sum_{\ell=0}^{\infty} 2^{-\nu'\ell} \Big(2^{(2\alpha+d)\ell} \|F\|_{\dot{\mathrm{H}}^\alpha(\mathbb{R}^d)}^2 + (\ell+1) 2^{d\ell} \sum_{j=1}^{\ell+1} 2^{2\alpha j} \|F\|_{\dot{\mathrm{H}}^\alpha(\mathbb{R}^d)}^2 \Big) \\
&\lesssim t^{2\alpha} \|F\|_{\dot{\mathrm{H}}^\alpha(\mathbb{R}^d)}^2 \sum_{\ell=0}^{\infty} 2^{-\nu'\ell} \Big(2^{(2\alpha+d)\ell} + (\ell+1) 2^{(2\alpha+d)\ell} \Big) \\
&\lesssim t^{2\alpha} \|F\|_{\dot{\mathrm{H}}^\alpha(\mathbb{R}^d)}^2
\end{aligned}$$

where the series converges for every $\alpha \in (0,1)$ since we can choose $d+2\alpha < \nu' < d+2$. □

To prove (4.2), we will first smooth out the “harsh” approximation to get a control via quadratic estimates of some well-behaved operator and afterwards remove the smoothing again to gain back the full estimate. For this purpose, we introduce the following smoothing operator in the spirit of [27].

Definition 4.8. For $t > 0$ define the following *smoothing* operators on $\mathrm{L}^2(\mathbb{R}^d)$:

$$P_t \coloneqq (1 - t^2\Delta)^{-1} \quad \text{and} \quad Q_t \coloneqq t\nabla(1 - t^2\Delta)^{-1}.$$

As a reminder, the reader should keep in mind that we agreed to use the same notation for extensions of operators on vector fields by acting componentwise.

Proposition 4.9. *For $F \in \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^{d\times d})$ the following smoothed principal part approximation holds*

$$\int_0^\infty \|(\Theta_t - \gamma_t \cdot \mathcal{A}_t)P_t F\|_{\mathrm{L}^2}^2 \, \frac{\mathrm{d}t}{t} \lesssim \|F\|_{\mathrm{L}^2}^2.$$

Proof. Due to Lemma 4.7 and the characterization of fractional Sobolev-spaces through the fractional Laplacian (see [26, Prop. 3.6]), we have

$$\|(\Theta_t - \gamma_t \cdot \mathcal{A}_t)P_t F\|_{\mathrm{L}^2} \lesssim t^\alpha \|P_t F\|_{\dot{\mathrm{H}}^\alpha} = \|t^\alpha (-\Delta)^{\frac{\alpha}{2}} (1 - t^2\Delta)^{-1} F\|_{\mathrm{L}^2}$$

for $\alpha \in (0,1)$. We conclude by quadratic estimates of the form (3.2) with the function $f(z) = z^{\frac{\alpha}{2}}(1+z)^{-1}$ which are valid for the Laplacian by McIntosh's theorem [25, Thm. 7.3.1], since the Laplacian admits a bounded H^∞-calculus on $\mathrm{L}^2(\mathbb{R}^d)$. □

To remove the smoothing, we split

$$(\Theta_t - \gamma_t \cdot \mathcal{A}_t) = (\Theta_t - \gamma_t \cdot \mathcal{A}_t)P_t + \Theta_t(1 - P_t) - \gamma_t \cdot \mathcal{A}_t(1 - P_t) \tag{4.7}$$

and elaborate square function estimates for the last two terms.

Proposition 4.10. *For every $u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ we have the square function estimate*

$$\int_0^\infty \|\Theta_t(1-P_t)\nabla u\|^2_{\mathrm{L}^2} \frac{\mathrm{d}t}{t} \lesssim \|\nabla u\|^2_{\mathrm{L}^2}.$$

Proof. First observe that for every $u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ we also have $-t^2\Delta(1-t^2\Delta)^{-1}u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ for all $t > 0$ since the Laplacian commutes with derivatives. Thus, we can compute by virtue of (4.1)

$$\begin{aligned} \Theta_t(1-P_t)\nabla u &= (\Theta_t\nabla)(1-P_t)u \\ &= \big(tA(1+t^2A)^{-1}\big)\big(-t^2\Delta(1-t^2\Delta)^{-1}u\big) \\ &= \big(1-(1+t^2A)^{-1}\big)\big(-t\Delta(1-t^2\Delta)^{-1}u\big) \\ &= \big(1-(1+t^2A)^{-1}\big)Q_t^*\nabla u, \end{aligned}$$

where $Q_t^*\nabla u = (Q_t^*\nabla u_1, \dots, Q_t^*\nabla u_d)^\top$. By the sectoriality of A, we conclude

$$\|\Theta_t(1-P_t)\nabla u\|_{\mathrm{L}^2} \lesssim \|Q_t^*\nabla u\|_{\mathrm{L}^2}$$

such that the claim follows now from [27, Lem. 13.9]. □

Proposition 4.11. *For all $u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ we have*

$$\int_0^\infty \|\gamma_t \cdot \mathcal{A}_t(1-P_t)\nabla u\|^2_{\mathrm{L}^2} \frac{\mathrm{d}t}{t} \lesssim \|\nabla u\|^2_{\mathrm{L}^2}.$$

Proof. Observe that $\mathcal{A}_t = \mathcal{A}_t^2$. Then the uniform L^2-bounds of $\gamma_t \cdot \mathcal{A}_t$ by Lemma 4.6 reduces the square function estimate to

$$\int_0^\infty \|\mathcal{A}_t(1-P_t)\nabla u\|^2_{\mathrm{L}^2} \frac{\mathrm{d}t}{t} \lesssim \|\nabla u\|^2_{\mathrm{L}^2} \qquad (u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)).$$

This estimate is now independent of the operator A and well-known, see, e.g., [27, Prop. 13.13]. □

The proof of the above proposition uses an interpolation inequality, which will be of use in the last section. Let us state it at this point and refer to [27, Lem. 13.12] for a detailed proof.

Lemma 4.12. *There is a constant* $C > 0$ *such that for all dyadic cubes* $Q \in \square$ *and all* $u \in \mathrm{H}^1(\mathbb{R}^d; \mathbb{C}^d)$ *we have*

$$\Big| \fint_Q \nabla u \,\mathrm{d}x \Big|^2_{\mathbb{C}^{d\times d}} \leq \frac{C}{\ell(Q)} \Big(\fint_Q |u|^2 \,\mathrm{d}x \Big)^{\frac{1}{2}} \Big(\fint_Q |\nabla u|^2_{\mathbb{C}^{d\times d}} \,\mathrm{d}x \Big)^{\frac{1}{2}}.$$

A combination of Propositions 4.9, 4.10 and 4.11 with the splitting (4.7) establishes the full principal part approximation.

Proposition 4.13 *(Principal part approximation). We have the square function estimate*

$$\int_0^\infty \|(\Theta_t - \gamma_t \cdot \mathcal{A}_t)\nabla u\|^2_{\mathrm{L}^2} \frac{\mathrm{d}t}{t} \lesssim \|\nabla u\|^2_{\mathrm{L}^2} \qquad (u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)).$$

5. The Carleson measure argument

In order to finish the proof of Theorem 1.2 it suffices, by virtue of Propositions 4.5 and 4.13, to show that

$$|\gamma_t(x) \cdot \mathrm{T}_0|^2_{\mathcal{L}(\mathbb{C}^{d\times d}, \mathbb{C}^d)} \frac{\mathrm{d}x \,\mathrm{d}t}{t}$$

is a Carleson measure. We shortly recapitulate the notion of a Carleson measure.

Definition 5.1. A Borel measure ν on $\mathbb{R}^{d+1}_+$ is called a *Carleson measure* if there exists $C > 0$ such that

$$\nu(R(Q)) \leq C|Q| \qquad (Q \in \square),$$

where $R(Q) \coloneqq Q \times (0, \ell(Q)]$ denotes a *Carleson box*. The infimum over all constants C is called the *Carleson norm* of ν and denoted by $\|\nu\|_{\mathcal{C}}$.

For elliptic operators in divergence form a key idea in the proof of the Kato property is to use a sectorial decomposition of the range of the principal part. In our situation we have to decompose the range of $\gamma_t \cdot \mathrm{T}_0$ which is a proper subset of the range of $\gamma_t \cdot$. Since $\gamma_t(x)$ is a $(d \times d)$-matrix with entries in $\mathbb{C}^d$ we have for $x \in \mathbb{R}^d$ and $t > 0$

$$\gamma_t(x) \cdot \mathrm{T}_0 \subseteq \mathcal{V} \coloneqq \{\zeta \cdot \mathrm{T}_0 \colon \zeta \in (\mathbb{C}^d)^{d\times d}\} \subseteq \mathcal{L}(\mathbb{C}^{d\times d}, \mathbb{C}^d)$$

where $\zeta \cdot \mathrm{T}_0$ is defined in the same way as $\gamma_t(x) \cdot \mathrm{T}_0$.

Lemma 5.2. *$\mathcal{V}$ is a closed subspace of $\mathcal{L}(\mathbb{C}^{d\times d},\mathbb{C}^d)$.*

Proof. Let $(\xi_n)_{n\in\mathbb{N}}\subseteq\mathcal{V}$ converge to some $\xi\in\mathcal{L}(\mathbb{C}^{d\times d},\mathbb{C}^d)$. Since $\xi_n=z_n\cdot\mathrm{T}_0$ for some $z_n\in(\mathbb{C}^d)^{d\times d}$ and T_0 is a projection we get

$$\xi_n G=z_n\cdot\mathrm{T}_0G=z_n\cdot\mathrm{T}_0^2G=\xi_n\mathrm{T}_0G\qquad(G\in\mathbb{C}^{d\times d}).$$

As a consequence we find that

$$\xi G=\lim_{n\to\infty}\xi_n\mathrm{T}_0G=\xi\mathrm{T}_0G=\sum_{j,k=1}^{\infty}\xi(e_{jk})(\mathrm{T}_0G)_{jk}=\zeta\cdot\mathrm{T}_0G,$$

where $\zeta_{jk}\coloneqq\xi(e_{jk})$ and e_{jk} was defined in Definition 4.3. □

The sectorial decomposition will now be defined with respect to the space $\mathcal{V}$. For the sake of brevity, we will write the $\mathcal{L}(\mathbb{C}^{d\times d},\mathbb{C}^d)$-norm of elements in $\mathcal{V}$ as $|\cdot|_{\mathcal{V}}$.

Definition 5.3. Let $\varepsilon>0$. For $\xi\in\mathcal{V}$ with $|\xi|_{\mathcal{V}}=1$, define the open cone with central axis ξ by

$$\Gamma_\xi^\varepsilon\coloneqq\Big\{w\in\mathcal{V}\setminus\{0\}\;\Big|\;\Big|\tfrac{w}{|w|_{\mathcal{V}}}-\xi\Big|_{\mathcal{V}}<\varepsilon\Big\}.$$

Our goal is now to prove the following Carleson measure estimate, where we restrict the values of the principal part to a cone Γ_ξ^ε.

Proposition 5.4 *(Directional Carleson estimate). There is a choice of $\varepsilon>0$, depending only on d, $\mu_\bullet$ and $\mu^\bullet$, such that for every $\xi\in\mathcal{V}$ with $|\xi|_{\mathcal{V}}=1$ we have*

$$\iint_{R(Q)}|\gamma_t(x)\cdot\mathrm{T}_0|_{\mathcal{V}}^2\,\mathbf{1}_{\Gamma_\xi^\varepsilon}(\gamma_t(x)\cdot\mathrm{T}_0)\,\frac{\mathrm{d}x\,\mathrm{d}t}{t}\lesssim|Q|\qquad(Q\in\square).$$

Taking the directional Carleson measure estimate from Proposition 5.4 for granted, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Since $\mathcal{V}\subseteq\mathcal{L}(\mathbb{C}^{d\times d},\mathbb{C}^d)$ is closed, by Lemma 5.2, its unit sphere is compact in the subspace topology. Thus, we can cover $\mathcal{V}\setminus\{0\}$ by finitely many cones $\Gamma_{\xi_1}^\varepsilon,\dots,\Gamma_{\xi_N}^\varepsilon$, where N depends only on d and ε. The estimate from Proposition 5.4 yields

$$\iint_{R(Q)}|\gamma_t(x)\cdot\mathrm{T}_0|_{\mathcal{V}}^2\,\frac{\mathrm{d}x\,\mathrm{d}t}{t}\le\iint_{R(Q)}\sum_{j=1}^{N}|\gamma_t(x)\cdot\mathrm{T}_0|_{\mathcal{V}}^2\,\mathbf{1}_{\Gamma_{\xi_j}^\varepsilon}(\gamma_t(x)\cdot\mathrm{T}_0)\,\frac{\mathrm{d}x\,\mathrm{d}t}{t}\le CN|Q|$$

for every cube $Q\in\square$ as required. Thus, $|\gamma_t(x)\cdot\mathrm{T}_0|_{\mathcal{V}}^2\frac{\mathrm{d}x\mathrm{d}t}{t}$ defines a Carleson measure. Combined with Propositions 4.13 and 4.5, we obtain (3.1) which, thanks to Section 3, proves Theorem 1.2. □

The remainder of this section is devoted to the proof or the directional Carleson measure estimate which bases on the construction of a $T(b)$-type test function. More precisely, for given $Q \in \square$ and $\xi \in \mathcal{V}$ with $|\xi|_{\mathcal{V}} = 1$, we strive for a construction of functions b_Q^ε such that

$$|w|_{\mathcal{V}} \lesssim \big|w\big((\mathcal{A}_t b_Q^\varepsilon)(x)\big)\big| \qquad (w \in \Gamma_\xi^\varepsilon)$$

for a hopefully large set of $(x,t) \in R(Q)$. The test function b_Q^ε should have two properties:

(1) It should be a gradient of an $\mathrm{H}^1_\sigma(\mathbb{R}^d)$-function in order to make the principal part approximation accessible.
(2) It should point in the direction of the unit matrix $\overline{\xi} \in \mathbb{C}^{d\times d}$ that realizes the norm of ξ.

The first point will be achieved by making an ansatz for b_Q^ε as the gradient of a suitable resolvent of A, i.e.,

$$b_Q^\varepsilon \simeq \nabla(1 + \varepsilon^2 \ell(Q)^2 A)^{-1}\Phi.$$

Since $(1 + \varepsilon^2\ell(Q)^2 A)^{-1}$ is an approximation of the identity if ε is sufficiently small, the second point roughly holds if Φ is chosen such that $\nabla\Phi = \overline{\xi}$, i.e., $\Phi(x) \coloneqq \overline{\xi}^\top (x - x_Q)$, where x_Q is the center of Q. A problem is, that Φ is neither L^2-integrable nor divergence-free. It will suffice to modify Φ by a multiplication with a bump function to achieve the L^2-integrability. The divergence-freeness should be achieved by a Bogovskiĭ correction. However, this only works if $x \mapsto \overline{\xi}^\top (x - x_Q)$ is itself divergence-free which is only the case if the trace of the matrix $\overline{\xi}^\top$ vanishes. We will show, that there will be now harm if we replace $\overline{\xi}^\top$ by $(\mathrm{T}_0\overline{\xi})^\top$.

From now on, the map $\xi \in \mathcal{V}$ with $|\xi|_{\mathcal{V}} = 1$ is fixed and we write $\Gamma^\varepsilon \coloneqq \Gamma^\varepsilon_\xi$. Let $\overline{\xi} \in \mathbb{C}^{d\times d}$ be such that

$$|\overline{\xi}|_{\mathbb{C}^{d\times d}} = 1 \quad \text{and} \quad |\xi\overline{\xi}| = |\xi|_{\mathcal{V}}.$$

We will fix this choice of $\overline{\xi}$ in the following. Before we present the construction of b_Q^ε in detail, let us introduce our final tool, the so-called Bogovskiĭ operator.

Fix a cube $Q_0 \coloneqq (-1,1)^d$ and define $\mathcal{C} \coloneqq 2Q_0 \setminus \overline{Q_0}$. A Bogovskiĭ operator $\mathcal{B}_{\mathcal{C}} \colon \mathrm{L}^2_0(\mathcal{C}) \to \mathrm{H}^1_0(\mathcal{C};\mathbb{C}^d)$ denotes a solution operator to the divergence equation for functions $f \in \mathrm{L}^2_0(\mathcal{C})$

$$\begin{cases} \operatorname{div}(u) = f & \text{in } \mathcal{C}, \\ \qquad u = 0 & \text{on } \partial\mathcal{C}, \end{cases}$$

that is bounded from $\mathrm{L}^2_0(\mathcal{C})$ to $\mathrm{H}^1_0(\mathcal{C};\mathbb{C}^d)$. Here, L^2_0 denotes the subspace of L^2 of average-free functions on $\mathcal{C}$. Here, we use the operator constructed in [28, Sec. III.3], which satisfies

$$\operatorname{div}(\mathcal{B}_{\mathcal{C}} f) = f \quad \text{and} \quad \|\mathcal{B}_{\mathcal{C}} f\|_{\mathrm{H}^1(\mathcal{C})} \leq C_{Bog} \|f\|_{\mathrm{L}^2(\mathcal{C})} \qquad (f \in \mathrm{L}^2_0(\mathcal{C}))$$

for some constant depending only on d. On the sets $\alpha\mathcal{C}$, for $\alpha > 0$, one can construct Bogovskiĭ operators by rescaling as follows: If $f \in \mathrm{L}^2_0(\alpha\mathcal{C})$, then $f_\alpha(x) := \alpha f(\alpha x)$ lies in $\mathrm{L}^2_0(\mathcal{C})$. Define

$$[\mathcal{B}_{\alpha\mathcal{C}} f](x) := [\mathcal{B}_{\mathcal{C}} f_\alpha](\alpha^{-1} x) \qquad (f \in \mathrm{L}^2_0(\alpha\mathcal{C}),\, x \in \alpha\mathcal{C}).$$

Clearly, $\mathcal{B}_{\alpha\mathcal{C}}$ is bounded from $\mathrm{L}^2_0(\alpha\mathcal{C})$ onto $\mathrm{H}^1_0(\alpha\mathcal{C}; \mathbb{C}^d)$ and satisfies $\operatorname{div}(\mathcal{B}_{\alpha\mathcal{C}} f) = f$. Furthermore, we have

$$\|\nabla \mathcal{B}_{\alpha\mathcal{C}} f\|_{\mathrm{L}^2(\alpha\mathcal{C})} \leq C_{Bog} \|f\|_{\mathrm{L}^2(\alpha\mathcal{C})} \qquad (f \in \mathrm{L}^2_0(\alpha\mathcal{C})) \tag{5.1}$$

with the same constant $C_{Bog} > 0$ as above. Finally, by translation we might define Bogovskiĭ operators for translated annuli as well and these will again satisfy (5.1) with the same constant.

Now, we are prepared for the construction of the $T(b)$-type test function.

Proposition 5.5. *There exists* $\varepsilon_0 \in (0,1]$ *such that for all* $0 < \varepsilon \leq \varepsilon_0$, *all* $\xi \in \mathcal{V}$ *satisfying* $|\xi|_{\mathcal{V}} = 1$ *and all cubes* $Q \in \square$ *there exists* $b_Q^\varepsilon \in \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^{d \times d})$ *with the following properties:*

(1) $\|b_Q^\varepsilon\|_{\mathrm{L}^2} \lesssim |Q|^{1/2}$,

(2) $\Big|\xi \fint_Q b_Q^\varepsilon \, \mathrm{d}x\Big| \geq 1$,

(3) $\displaystyle\iint_{R(Q)} |\gamma_t(x) \cdot \mathrm{T}_0(\mathcal{A}_t b_Q^\varepsilon)(x)|^2 \, \frac{\mathrm{d}x \, \mathrm{d}t}{t} \lesssim \frac{|Q|}{\varepsilon^2}$.

Proof. We fix $Q \in \square$ and abbreviate $\ell := \ell(Q)$. Similarly, we simplify notation by omitting ε and Q when constructing the test function $b = b_Q^\varepsilon$.

We start by fixing $\eta \in \mathrm{C}_c^\infty(2Q)$ such that

$$\eta = 1 \quad \text{in} \quad Q \quad \text{and} \quad \|\eta\|_{\mathrm{L}^\infty} + \ell \|\nabla \eta\|_{\mathrm{L}^\infty} \leq C. \tag{5.2}$$

With x_Q the center of Q, we construct a smooth function with compact support, whose gradient is equal to $\overline{\xi}$ on Q as follows. First, note that

$$\operatorname{div}\big((\mathrm{T}_0 \overline{\xi})^\top (x - x_Q)\big) = 0.$$

Let $\mathcal{B}$ be the Bogovskiĭ operator in $2Q^\circ \setminus \overline{Q}$ as introduced above and define

$$\Phi(x) := \eta(x) (\mathrm{T}_0 \overline{\xi})^\top (x - x_Q) - \mathcal{B}\big(\nabla \eta \cdot (\mathrm{T}_0 \overline{\xi})^\top (\cdot - x_Q)\big)(x), \tag{5.3}$$

where we regard $\mathcal{B}\big(\nabla\eta\cdot(\mathrm{T}_0\overline{\xi})^\top(\cdot-x_Q)\big)$ to be extended by zero outside of $2Q^\circ\setminus\overline{Q}$. Finally, we define the desired test function as

$$b := 2\nabla(1+\varepsilon^2\ell^2 A)^{-1}\Phi. \tag{5.4}$$

Then, we have

$$\begin{aligned}\frac{1}{2}b-\nabla\Phi &= \nabla\big((1+\varepsilon^2\ell^2 A)^{-1}-1\big)\Phi\\ &= \nabla\big(-(1+\varepsilon^2\ell^2\mathcal{A})^{-1}\varepsilon^2\ell^2\mathcal{A}\big)\Phi\\ &= \varepsilon^2\ell^2\nabla(1+\varepsilon^2\ell^2\mathcal{A})^{-1}\mathcal{P}\operatorname{div}(\mu\nabla\Phi).\end{aligned} \tag{5.5}$$

Let us prove that we can pick ε small enough such that b has the stated properties.

In order to prove (1), we begin by calculating

$$\begin{aligned}\big|\nabla\Phi(x)\big|_{\mathbb{C}^{d\times d}} \leq \big|\nabla\eta(x)\otimes(\mathrm{T}_0\overline{\xi})^\top(x-x_Q)\big|_{\mathbb{C}^{d\times d}} + \big|\eta(x)\mathrm{T}_0\overline{\xi}\big|_{\mathbb{C}^{d\times d}}\\ + \big|\nabla\mathcal{B}\big(\nabla\eta\cdot(\mathrm{T}_0\overline{\xi})^\top(\cdot-x_Q)\big)(x)\big|_{\mathbb{C}^{d\times d}}.\end{aligned}$$

Taking L^2-norms, we obtain by (5.1) and (5.2) together with the boundedness of T_0 and the property $|\overline{\xi}|_{\mathbb{C}^{d\times d}}=1$

$$\|\nabla\Phi\|_{\mathrm{L}^2}^2 \lesssim \frac{|\mathrm{T}_0\overline{\xi}|_{\mathbb{C}^{d\times d}}}{\ell^2}\int_{2Q}|x-x_Q|^2\,\mathrm{d}x + |Q||\mathrm{T}_0\overline{\xi}|_{\mathbb{C}^{d\times d}} \lesssim |Q||\mathrm{T}_0\overline{\xi}|_{\mathbb{C}^{d\times d}} \lesssim |Q|. \tag{5.6}$$

Combining (5.5) and Proposition 2.4, we find

$$\|b-2\nabla\Phi\|_{\mathrm{L}^2}^2 \lesssim \|\mu\nabla\Phi\|_{\mathrm{L}^2}^2 \lesssim |Q|$$

and together with (5.6) we arrive at (1).

We turn to the proof of statement (2). As $\nabla\Phi=\mathrm{T}_0\overline{\xi}$ on Q, we can write

$$\fint_Q\big(b-2\mathrm{T}_0\overline{\xi}\big)\,\mathrm{d}x = \fint_Q\big(b-2\nabla\Phi\big)\,\mathrm{d}x = 2\varepsilon^2\ell^2\fint_Q\nabla u\,\mathrm{d}x, \tag{5.7}$$

where in the second step we have used (5.5) and $u := (1+\varepsilon^2\ell^2\mathcal{A})^{-1}\mathcal{P}\operatorname{div}(\mu\nabla\Phi)$. The uniform L^2-bounds in Proposition 2.4 in combination with (5.6) yield

$$\|u\|_{\mathrm{L}^2}\leq\frac{C}{\varepsilon\ell}|Q|^{1/2}\quad\text{and}\quad\|\nabla u\|_{\mathrm{L}^2}\leq\frac{C}{\varepsilon^2\ell^2}|Q|^{1/2},$$

so that estimating the average on the right-hand side of (5.7) by means of Lemma 4.12 leads us to

$$\Bigl|\fint_Q \bigl(b - 2\mathrm{T}_0\overline{\xi}\bigr)\,\mathrm{d}x\Bigr|^2_{\mathbb{C}^{d\times d}} \le \frac{C\varepsilon^4\ell^4}{\ell|Q|}\|u\|_{\mathrm{L}^2}\|\nabla u\|_{\mathrm{L}^2} \le C\varepsilon,$$

where C varies from step to step. Since $\xi \in \mathcal{V}$ and T_0 is a projection, we have $\xi\overline{\xi} = \xi\mathrm{T}_0\overline{\xi}$ so that

$$\xi\fint_Q b\,\mathrm{d}x = 2\xi\overline{\xi} + \xi\fint_Q \bigl(b - 2\mathrm{T}_0\overline{\xi}\bigr)\,\mathrm{d}x.$$

Recall that $\overline{\xi}$ was chosen in such a way, that $|\xi\overline{\xi}| = |\xi|_{\mathcal{V}} = 1$ which yields

$$\Bigl|\xi\fint_Q b\,\mathrm{d}x\Bigr| \ge 2 - |\xi|_{\mathcal{V}}\Bigl|\fint_Q \bigl(b - 2\mathrm{T}_0\overline{\xi}\bigr)\,\mathrm{d}x\Bigr|_{\mathbb{C}^{d\times d}} \ge 2 - \sqrt{C\varepsilon}.$$

Now, (2) follows by taking $\varepsilon \le C^{-1}$.

For the proof of (3), we start with the principal part approximation from Proposition 4.13 and then use (1) to bound

$$\begin{aligned}\frac{1}{2}\iint_{R(Q)} |\gamma_t(x)\cdot\mathrm{T}_0(\mathcal{A}_t b)(x)|^2\,\frac{\mathrm{d}x\,\mathrm{d}t}{t} &\le \int_0^\ell \Bigl(\|(\gamma_t\cdot\mathcal{A}_t - \Theta_t)b\|^2_{\mathrm{L}^2} + \|\Theta_t b\|^2_{\mathrm{L}^2}\Bigr)\,\frac{\mathrm{d}t}{t} \\ &\lesssim \|b\|^2_{\mathrm{L}^2} + \int_0^\ell \|\Theta_t b\|^2_{\mathrm{L}^2}\,\frac{\mathrm{d}t}{t} \\ &\lesssim |Q| + \int_0^\ell \|\Theta_t b\|^2_{\mathrm{L}^2}\,\frac{\mathrm{d}t}{t}.\end{aligned} \tag{5.8}$$

Recall from (5.4) that b is by definition the gradient of a function in $\mathrm{H}^1_\sigma(\mathbb{R}^d)$ and so that applying Proposition 4.13 was allowed. For the last term, we compute $\Theta_t b$ as

$$\begin{aligned}\Theta_t b &= -2t(1 + t^2\mathcal{A})^{-1}\mathcal{P}\operatorname{div}\bigl(\mu\nabla(1 + \varepsilon^2\ell^2 A)^{-1}\Phi\bigr) \\ &= 2t(1 + t^2\mathcal{A})^{-1}\mathcal{A}(1 + \varepsilon^2\ell^2\mathcal{A})^{-1}\Phi \\ &= -t\bigl((1 + t^2 A)^{-1}\bigr)\bigl((1 + \varepsilon^2\ell^2\mathcal{A})^{-1}\mathcal{P}\operatorname{div}\bigr)2\mu\nabla\Phi.\end{aligned}$$

We use Proposition 2.4 once again and (5.6) in order to control

$$\|\Theta_t b\|^2_{\mathrm{L}^2} \lesssim \frac{t^2}{\varepsilon^2\ell^2}\|2\mu\nabla\Phi\|^2_{\mathrm{L}^2} \lesssim \frac{t^2}{\varepsilon^2\ell^2}|Q|.$$

Finally, integration in t yields

$$\int_0^{\ell} \|\Theta_t b\|_{\mathrm{L}^2}^2 \frac{\mathrm{d}t}{t} \lesssim \frac{1}{\varepsilon^2}|Q|,$$

which we use back in (5.8) to conclude provided we take $\varepsilon \leq 1$. □

Having a $T(b)$-type test function at our disposal, the Carleson measure property will follow by a combination of a stopping time argument with a John–Nirenberg lemma for Carleson measures. Concerning the first, we refer to [14, Lem. 5.11] or [27, Lem. 14.8] for a proof.

Lemma 5.6. *Let $\varepsilon_0 \in (0, 1]$ be the parameter from Proposition 5.5. Then there exists $0 < \varepsilon \leq \varepsilon_0$, depending only on d, $\mu_\bullet$ and $\mu^\bullet$, such that for each dyadic cube $Q \in \square$ there exists a collection of pairwise disjoint dyadic subcubes $(Q_j)_j$ of Q for which the sets*

$$E(Q) := Q \setminus \bigcup_j Q_j \quad \text{and} \quad E^*(Q) := R(Q) \setminus \bigcup_j R(Q_j) \tag{5.9}$$

have the following properties:

(1) $\bigl|E(Q)\bigr| \geq \eta|Q|$, *for some $\eta > 0$ depending only on d, $\mu_\bullet$ and $\mu^\bullet$,*
(2) $\bigl|w(\mathcal{A}_t b_Q^\varepsilon)(x)\bigr| \geq \dfrac{1}{2}|w|$, *whenever $(x, t) \in E^*(Q)$ and $w \in \Gamma^\varepsilon$.*

Lemma 5.6 (2) guarantees that the Borel measure ν on $\mathbb{R}^{d+1}_+$ defined on Borel sets $E \subseteq \mathbb{R}^{d+1}_+$ by

$$\nu(E) := \iint_E |\gamma_t(x) \cdot \mathrm{T}_0|_{\mathcal{V}}^2 \, \mathbf{1}_{\Gamma_\xi^\varepsilon}(\gamma_t(x) \cdot \mathrm{T}_0) \frac{\mathrm{d}x \, \mathrm{d}t}{t}$$

satisfies for cubes $Q \in \square$

$$\begin{aligned} \nu(E^*(Q)) &= \iint_{E^*(Q)} |\gamma_t(x) \cdot \mathrm{T}_0|_{\mathcal{V}}^2 \, \mathbf{1}_{\Gamma_\xi^\varepsilon}(\gamma_t(x) \cdot \mathrm{T}_0) \frac{\mathrm{d}x \, \mathrm{d}t}{t} \\ &\leq 4 \iint_{E^*(Q)} |\gamma_t(x) \cdot \mathrm{T}_0(\mathcal{A}_t b_Q^\varepsilon)(x)|^2 \frac{\mathrm{d}x \, \mathrm{d}t}{t}. \end{aligned}$$

The right-hand side can be controlled by virtue of Proposition 5.5 (3) leading to the estimate

$$\nu(E^*(Q)) \leq \frac{4C}{\varepsilon^2}|Q| \qquad (Q \in \square), \tag{5.10}$$

where C is the constant from the very proposition. The Carleson measure property of ν and thus the statement of Proposition 5.4 finally follows by a combination of (5.10) with Lemma 5.6 (1) and the following John–Nirenberg lemma for Carleson measures. We refer to [27, Lem. 14.10] for its proof. The argument is nowadays standard in the verification of the square root property of elliptic operators.

Lemma 5.7. *Let ν be a Borel measure on $\mathbb{R}^{d+1}_+$ and suppose that there exist constants $\kappa, \eta > 0$ with the following properties. For every dyadic cube $Q \in \square$ there exist pairwise disjoint dyadic subcubes Q_j of Q such that the sets $E(Q)$ and $E^*(Q)$ defined in (5.9) satisfy*

(1) $\bigl|E(Q)\bigr| \geq \eta |Q|$,
(2) $\nu\bigl(E^*(Q)\bigr) \leq \kappa |Q|$.

Then ν is a Carleson measure with $\|\nu\|_{\mathcal{C}} \leq \kappa \eta^{-1}$.

6. Holomorphic dependency

In this section we present the proof of Theorem 1.3 and follow an argument of Auscher and Tchamitchian for elliptic operators in divergence form [13, Sec. 0.5]. Even though the necessary modifications to the generalized Stokes operator are very small, we give the full argument for the sake of completeness. We start by proving that the set $\mathcal{O}$ in Theorem 1.3 is open.

Lemma 6.1. *The set $\mathcal{O} := \{\mu : \mu$ satisfies Assumption 1.1 for some $\mu_\bullet, \mu^\bullet > 0\}$ is open in the L^∞-topology.*

Proof. Let μ satisfy Assumption 1.1 with constants $\mu^\bullet, \mu_\bullet > 0$ and $M = (M^{ij}_{\alpha\beta})^d_{\alpha,\beta,i,j=1}$ with $M^{ij}_{\alpha\beta} \in \mathrm{L}^\infty(\mathbb{R}^d; \mathbb{C})$ for all $1 \leq \alpha, \beta, i, j \leq d$ be such that

$$\|\mu - M\|_{\mathrm{L}^\infty(\mathbb{R}^d; \mathcal{L}(\mathbb{C}^{d\times d}))} < \frac{\mu_\bullet}{2}.$$

Then M is obviously bounded and fulfills for $u \in \mathrm{H}^1(\mathbb{R}^d; \mathbb{C}^d)$

$$\begin{aligned} \operatorname{Re} \sum_{\alpha,\beta,i,j=1}^d \int_{\mathbb{R}^d} M^{ij}_{\alpha\beta} \partial_\beta u_j \overline{\partial_\alpha u_i} \, \mathrm{d}x &= \operatorname{Re} \sum_{\alpha,\beta,i,j=1}^d \int_{\mathbb{R}^d} \mu^{ij}_{\alpha\beta} \partial_\beta u_j \overline{\partial_\alpha u_i} \, \mathrm{d}x \\ &\quad + \operatorname{Re} \sum_{\alpha,\beta,i,j=1}^d \int_{\mathbb{R}^d} (M^{ij}_{\alpha\beta} - \mu^{ij}_{\alpha\beta}) \partial_\beta u_j \overline{\partial_\alpha u_i} \, \mathrm{d}x \\ &\geq \mu_\bullet \|\nabla u\|^2_{\mathrm{L}^2} - \|M - \mu\|_{\mathrm{L}^\infty(\mathbb{R}^d; \mathcal{L}(\mathbb{C}^{d\times d}))} \|\nabla u\|^2_{\mathrm{L}^2} \\ &\geq \frac{\mu_\bullet}{2} \|\nabla u\|^2_{\mathrm{L}^2}. \end{aligned}$$

Hence, M satisfies Assumption 1.1 and the claim follows. $\square$

As in the introduction, we emphasize the dependency of A on the coefficients μ by writing A_μ. The corresponding weak Stokes operator will be denoted by $\mathcal{A}_\mu$.

Proof of Theorem 1.3. Let μ_0 satisfy Assumption 1.1 and let $M = (M^{ij}_{\alpha\beta})^d_{\alpha,\beta,i,j=1}$ with $M^{ij}_{\alpha\beta} \in \mathrm{L}^\infty(\mathbb{R}^d;\mathbb{C})$ for all $1 \leq \alpha,\beta,i,j \leq d$. By means of the previous lemma, the holomorphic map $\mathbb{C} \ni z \mapsto \mu_z := \mu_0 + zM$ maps into $\mathcal{O}$ whenever it is restricted to a small ball $|z| < \frac{\varepsilon}{\|M\|_\infty}$. For $u \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$ we have

$$\begin{aligned}
&((1+t^2A_{\mu_z})^{-1} - (1+t^2A_{\mu_0})^{-1})u \\
&\qquad = ((1+t^2\mathcal{A}_{\mu_z})^{-1} - (1+t^2\mathcal{A}_{\mu_0})^{-1})u \\
&\qquad = (1+t^2\mathcal{A}_{\mu_0})^{-1}\big[(1+t^2\mathcal{A}_{\mu_0}) - (1+t^2\mathcal{A}_{\mu_z})\big](1+t^2\mathcal{A}_{\mu_z})^{-1}u \\
&\qquad = t^2(1+t^2\mathcal{A}_{\mu_0})^{-1}\big[\mathcal{P}\operatorname{div}((\mu_0+zM)\nabla\boldsymbol{\cdot}) - \mathcal{P}\operatorname{div}(\mu_0\nabla\boldsymbol{\cdot})\big](1+t^2\mathcal{A}_{\mu_z})^{-1}u \\
&\qquad = t^2\big((1+t^2\mathcal{A}_{\mu_0})^{-1}\mathcal{P}\operatorname{div}\big)zM\nabla(1+t^2A_{\mu_z})^{-1}u.
\end{aligned}$$

Iterating this identity yields

$$(1+t^2A_{\mu_z})^{-1}u = \sum_{n=0}^\infty \Big[t^2\big((1+t^2\mathcal{A}_{\mu_0})^{-1}\mathcal{P}\operatorname{div}\big)zM\nabla\Big]^n(1+t^2A_{\mu_0})^{-1}u$$

which converges in $\mathrm{L}^2_\sigma(\mathbb{R}^d)$ whenever $\varepsilon > 0$ is small enough. Indeed, the families $(t^2\nabla(1+t^2\mathcal{A}_{\mu_0})^{-1}\mathcal{P}\operatorname{div})_{t>0}$ and $(t(1+t^2\mathcal{A}_{\mu_0})^{-1}\mathcal{P}\operatorname{div})_{t>0}$ are uniformly bounded in L^2 by Proposition 2.4 such that

$$\begin{aligned}
&\Big\|\Big[t^2\big((1+t^2\mathcal{A}_{\mu_0})^{-1}\mathcal{P}\operatorname{div}\big)zM\nabla\Big]^n(1+t^2A_{\mu_0})^{-1}u\Big\|_{\mathrm{L}^2} \\
&\qquad\qquad \leq C_1^n|z|^n\|M\|^n_{\mathrm{L}^\infty}\|t\nabla(1+t^2A_{\mu_0})^{-1}u\|_{\mathrm{L}^2} \\
&\qquad\qquad \leq C_1^n|z|^n\|M\|^n_{\mathrm{L}^\infty}C\|u\|_{\mathrm{L}^2} \\
&\qquad\qquad \leq 2^{-n}C\|u\|_{\mathrm{L}^2}
\end{aligned} \tag{6.1}$$

for $\varepsilon \leq (2C_1)^{-1}$ and where $C_1 > 0$ denotes the boundedness constant from Proposition 2.4. Now, using the Balakrishnan representation for square roots (see [27, Prop. 6.18]) we have for all $u \in \mathcal{D}(A^{1/2}_{\mu_z}) = \mathrm{H}^1_\sigma(\mathbb{R}^d)$,

$$\begin{aligned}
A^{\frac{1}{2}}_{\mu_z}u &= \frac{2}{\pi}\int_0^\infty A_{\mu_z}(1+t^2A_{\mu_z})^{-1}u\,\mathrm{d}t \\
&= \frac{2}{\pi}\int_0^\infty \frac{1}{t^2}[\mathrm{Id} - (1+t^2A_{\mu_z})^{-1}]u\,\mathrm{d}t \\
&= \frac{2}{\pi}\int_0^\infty \frac{1}{t^2}\Big[\mathrm{Id} - \sum_{n=0}^\infty\Big[t^2\big((1+t^2\mathcal{A}_{\mu_0})^{-1}\mathcal{P}\operatorname{div}\big)zM\nabla\Big]^n(1+t^2A_{\mu_0})^{-1}\Big]u\,\mathrm{d}t
\end{aligned}$$

$$= \frac{2}{\pi} \int_0^\infty A_{\mu_0}(1 + t^2 A_{\mu_0})^{-1} u$$

$$- \frac{1}{t^2} \sum_{n=1}^\infty \Big[t^2 \big((1 + t^2 \mathcal{A}_{\mu_0})^{-1} \mathcal{P} \operatorname{div} \big) z M \nabla \Big]^n (1 + t^2 A_{\mu_0})^{-1} u \, \mathrm{d}t$$

where the integrals have to be understood as improper integrals in $\mathrm{L}^2_\sigma(\mathbb{R}^d)$. Focusing on

$$\int_0^\infty \frac{1}{t^2} \sum_{n=1}^\infty \Big[t^2 \big((1 + t^2 \mathcal{A}_{\mu_0})^{-1} \mathcal{P} \operatorname{div} \big) z M \nabla \Big]^n (1 + t^2 A_{\mu_0})^{-1} u \, \mathrm{d}t$$

$$= \lim_{\delta \to 0} \int_\delta^{\frac{1}{\delta}} \frac{1}{t^2} \sum_{n=1}^\infty \Big[t^2 \big((1 + t^2 \mathcal{A}_{\mu_0})^{-1} \mathcal{P} \operatorname{div} \big) z M \nabla \Big]^n (1 + t^2 A_{\mu_0})^{-1} u \, \mathrm{d}t,$$

we first interchange integration and summation due to (6.1) to get

$$= \lim_{\delta \to 0} \sum_{n=1}^\infty \int_\delta^{\frac{1}{\delta}} \frac{1}{t^2} \Big[t^2 \big((1 + t^2 \mathcal{A}_{\mu_0})^{-1} \mathcal{P} \operatorname{div} \big) z M \nabla \Big]^n (1 + t^2 A_{\mu_0})^{-1} u \, \mathrm{d}t.$$

Next, we want to interchange the limit with the series by dominated convergence but not in the space $\mathrm{L}^2_\sigma(\mathbb{R}^d)$. Instead we will work in $\mathrm{H}^{-1}_\sigma(\mathbb{R}^d)$ to guarantee existence of a limit in the first place. For this purpose, set for $0 < \delta \leq 1$

$$f_\delta : \mathbb{N} \to \mathrm{H}^{-1}_\sigma(\mathbb{R}^d), \quad f_\delta(n) = \int_\delta^{\frac{1}{\delta}} \frac{1}{t^2} \Big[t^2 \big((1 + t^2 \mathcal{A}_{\mu_0})^{-1} \mathcal{P} \operatorname{div} \big) z M \nabla \Big]^n (1 + t^2 A_{\mu_0})^{-1} u \, \mathrm{d}t,$$

then one has existence of a pointwise limit by the following argument: For $0 < \delta' < \delta \leq 1$ and $v \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ one has

$$\langle f_\delta(n) - f_{\delta'}(n), v \rangle_{\mathrm{H}^{-1}_\sigma, \mathrm{H}^1_\sigma}$$

$$= \int_{\frac{1}{\delta}}^{\frac{1}{\delta'}} \frac{1}{t^2} \Big\langle \Big[t^2 \big((1 + t^2 \mathcal{A}_{\mu_0})^{-1} \mathcal{P} \operatorname{div} \big) z M \nabla \Big]^n (1 + t^2 A_{\mu_0})^{-1} u, v \Big\rangle_{\mathrm{L}^2_\sigma, \mathrm{L}^2_\sigma} \mathrm{d}t$$

$$+ \int_{\delta'}^{\delta} \frac{1}{t^2} \Big\langle \Big[t^2 \big((1 + t^2 \mathcal{A}_{\mu_0})^{-1} \mathcal{P} \operatorname{div} \big) z M \nabla \Big]^n (1 + t^2 A_{\mu_0})^{-1} u, v \Big\rangle_{\mathrm{L}^2_\sigma, \mathrm{L}^2_\sigma} \mathrm{d}t.$$

Recall, that the adjoint of A_{μ_0} is the generalized Stokes operator with coefficients μ_0^*. Thus, by duality and uniform boundedness of the families $(t^2 \nabla (1 + t^2 \mathcal{A}_{\mu_0})^{-1} \mathcal{P} \operatorname{div})_{t>0}$

and $(t(1+t^2\mathcal{A}_{\mu_0})^{-1}\mathcal{P}\operatorname{div})_{t>0}$ in L^2 from Proposition 2.4 we estimate the integrand as follows

$$\begin{aligned}
&\frac{1}{t^2}\Big|\Big\langle\Big[t^2\big((1+t^2\mathcal{A}_{\mu_0})^{-1}\mathcal{P}\operatorname{div}\big)zM\nabla\Big]^n(1+t^2A_{\mu_0})^{-1}u,v\Big\rangle_{\mathrm{L}^2_\sigma,\mathrm{L}^2_\sigma}\Big|\\
&=\frac{1}{t^2}\Big|\Big\langle tzM\nabla\Big[t^2\big((1+t^2\mathcal{A}_{\mu_0})^{-1}\mathcal{P}\operatorname{div}\big)zM\nabla\Big]^{n-1}(1+t^2A_{\mu_0})^{-1}u,t\nabla(1+t^2A^*_{\mu_0})^{-1}v\Big\rangle_{\mathrm{L}^2_\sigma,\mathrm{L}^2_\sigma}\Big|\\
&\le\frac{1}{t^2}\Big\|tzM\nabla\Big[t^2\big((1+t^2\mathcal{A}_{\mu_0})^{-1}\mathcal{P}\operatorname{div}\big)zM\nabla\Big]^{n-1}(1+t^2A_{\mu_0})^{-1}u\Big\|_{\mathrm{L}^2}\cdot\|t\nabla(1+t^2A^*_{\mu_0})^{-1}v\|_{\mathrm{L}^2}\\
&\le\frac{1}{t^2}C_1^{n-1}|z|^n\|M\|^n_{\mathrm{L}^\infty}\cdot\|t\nabla(1+t^2A_{\mu_0})^{-1}u\|_{\mathrm{L}^2}\cdot\|t\nabla(1+t^2A^*_{\mu_0})^{-1}v\|_{\mathrm{L}^2}.
\end{aligned}$$

Depending on the size of t we choose two different ways of controlling the gradients of the resolvents. On the one hand, Proposition 2.4 implies for $u,v\in\mathrm{L}^2_\sigma(\mathbb{R}^d)$

$$\|t\nabla(1+t^2A_{\mu_0})^{-1}u\|_{\mathrm{L}^2}\cdot\|t\nabla(1+t^2A^*_{\mu_0})^{-1}v\|_{\mathrm{L}^2}\lesssim\|u\|_{\mathrm{L}^2}\cdot\|v\|_{\mathrm{L}^2}$$

and, on the other hand, the square root property yields for $u,v\in\mathrm{H}^1_\sigma(\mathbb{R}^d)$

$$\begin{aligned}
&\|t\nabla(1+t^2A_{\mu_0})^{-1}u\|_{\mathrm{L}^2}\cdot\|t\nabla(1+t^2A^*_{\mu_0})^{-1}v\|_{\mathrm{L}^2}\\
&\qquad\simeq\|tA^{\frac{1}{2}}_{\mu_0}(1+t^2A_{\mu_0})^{-1}u\|_{\mathrm{L}^2}\cdot\|t(A^*_{\mu_0})^{\frac{1}{2}}(1+t^2A^*_{\mu_0})^{-1}v\|_{\mathrm{L}^2}\\
&\qquad=\|t(1+t^2A_{\mu_0})^{-1}A^{\frac{1}{2}}_{\mu_0}u\|_{\mathrm{L}^2}\cdot\|t(1+t^2A^*_{\mu_0})^{-1}(A^*_{\mu_0})^{\frac{1}{2}}v\|_{\mathrm{L}^2}\\
&\qquad\lesssim t^2\|A^{\frac{1}{2}}_{\mu_0}u\|_{\mathrm{L}^2}\cdot\|(A^*_{\mu_0})^{\frac{1}{2}}v\|_{\mathrm{L}^2}\\
&\qquad\simeq t^2\|\nabla u\|_{\mathrm{L}^2}\cdot\|\nabla v\|_{\mathrm{L}^2}.
\end{aligned}$$

Hence, the integrand can be controlled by $C_1^{n-1}|z|^n\|M\|^n_{\mathrm{L}^\infty}\min\{1,t^{-2}\}\cdot\|u\|_{\mathrm{H}^1}\|v\|_{\mathrm{H}^1}$ which yields

$$|\langle f_\delta(n)-f_{\delta'}(n),v\rangle_{\mathrm{H}^{-1}_\sigma,\mathrm{H}^1_\sigma}|\lesssim C_1^{n-1}|z|^n\|M\|^n_{\mathrm{L}^\infty}\|u\|_{\mathrm{H}^1}\|v\|_{\mathrm{H}^1}(\delta-\delta').$$

This shows pointwise convergence of f_δ as $\delta\to0$ and simultaneously gives a summable majorant for the sequence for $\delta\in(0,1]$, namely

$$\|f_\delta(n)\|_{\mathrm{H}^{-1}_\sigma(\mathbb{R}^d)}\lesssim C_1^{n-1}|z|^n\|M\|^n_{\mathrm{L}^\infty}\|u\|_{\mathrm{H}^1}\le\frac{2^{-n}}{C_1}\|u\|_{\mathrm{H}^1}=:g(n).$$

Thus, by dominated convergence it follows

$$\lim_{\delta\to0}\sum_{n=1}^\infty\int_\delta^{\frac{1}{\delta}}\frac{1}{t^2}\Big[t^2\big((1+t^2\mathcal{A}_{\mu_0})^{-1}\mathcal{P}\operatorname{div}\big)zM\nabla\Big]^n(1+t^2A_{\mu_0})^{-1}u\,\mathrm{d}t$$

$$= \sum_{n=1}^{\infty} \lim_{\delta \to 0} \int_{\delta}^{\frac{1}{\delta}} \frac{1}{t^2} \Big[t^2 \big((1 + t^2 \mathcal{A}_{\mu_0})^{-1} \mathcal{P} \operatorname{div} \big) z M \nabla \Big]^n (1 + t^2 A_{\mu_0})^{-1} u \, \mathrm{d}t.$$

Again due to Balakrishnan representation for square roots and the above identity we conclude that

$$A_{\mu_z}^{\frac{1}{2}} u = A_{\mu_0}^{\frac{1}{2}} u + \sum_{n=1}^{\infty} z^n T_n u \quad \text{in } \mathrm{H}_\sigma^{-1}(\mathbb{R}^d) \tag{6.2}$$

holds for all $u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ and appropriate operators $(T_n)_{n \in \mathbb{N}} \subseteq \mathcal{L}(\mathrm{H}^1_\sigma(\mathbb{R}^d), \mathrm{H}^{-1}_\sigma(\mathbb{R}^d))$. This readily proves holomorphy of the map $z \mapsto A_{\mu_z}^{1/2} u$ in the H^{-1}_σ-topology. To show convergence in $\mathrm{L}^2_\sigma(\mathbb{R}^d)$ we use Taylor's theorem (see, e.g., [2, Prop. A.1]) in $\mathrm{H}^{-1}_\sigma(\mathbb{R}^d)$ to represent the coefficients as

$$T_n u = \frac{1}{2\pi \mathrm{i}} \int_{|w|=r} A_{\mu_w}^{\frac{1}{2}} u \, \frac{\mathrm{d}w}{w^{n+1}}$$

for $r > 0$ small enough. Now, due to the Kato property it is

$$\|T_n u\|_{\mathrm{L}^2} \leq \frac{1}{2\pi} \int_{|w|=r} \|A_{\mu_w}^{\frac{1}{2}} u\|_{\mathrm{L}^2} \frac{|\mathrm{d}w|}{|w|^{n+1}} \lesssim \frac{1}{2\pi} \int_{|w|=r} \|\nabla u\|_{\mathrm{L}^2} \frac{|\mathrm{d}w|}{|w|^{n+1}} = r^{-n} \|\nabla u\|_{\mathrm{L}^2}.$$

Thus, the right-hand side of (6.2) actually converges absolutely in $\mathrm{L}^2_\sigma(\mathbb{R}^d)$ if $|z| < r$ and is equal to $A_{\mu_z}^{1/2} u$. Since each of the operators $A_{\mu_z}^{1/2}$ lies in $\mathcal{L}(\mathrm{H}^1_\sigma(\mathbb{R}^d), \mathrm{L}^2_\sigma(\mathbb{R}^d))$ the property [2, Prop. A.3] allows to conclude holomorphy with respect to the operator norm by strong holomorphy. This proves the claim. □

As mentioned in the introduction, that holomorphy implies a local Lipschitz property, is well known and treated, e.g., in [39, Thm. 3.24] (see also [6, Thm. 2.3]) in the context of holomorphic functional calculus for perturbed Dirac operators. Although, as above, the changes to the square root of the generalized Stokes operator are very small, we present the full argument for the sake of completeness.

Proof of Corollary 1.4. Let $\mu_1 \in \mathcal{O}$ and $0 < \delta < (\mu_1)_\bullet$. Then for all $\mu_2 \in \mathcal{O}$ satisfying $0 < \|\mu_2 - \mu_1\|_{\mathrm{L}^\infty(\mathbb{R}^d; \mathcal{L}(\mathbb{C}^{d \times d}))} < \delta$ we have that

$$\mu_z := \mu_1 + z\delta \cdot \frac{\mu_2 - \mu_1}{\|\mu_1 - \mu_2\|_{\mathrm{L}^\infty(\mathbb{R}^d; \mathcal{L}(\mathbb{C}^{d \times d}))}}$$

belongs to $\mathcal{O}$ if $z \in \mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$. In this case, Theorem 1.2 and Theorem 1.3 imply that

$$C_\delta^{-1}\|\nabla u\|_{\mathrm{L}^2} \leq \|A_{\mu_z}^{1/2} u\|_{\mathrm{L}^2} \leq C_\delta \|\nabla u\|_{\mathrm{L}^2} \qquad (u \in \mathrm{H}^1_\sigma(\mathbb{R}^d))$$

for some constant $C_\delta > 0$ depending only on $d, (\mu_1)_\bullet, (\mu_1)^\bullet$ and δ and that the map

$$\mathbb{D} \ni z \mapsto A_{\mu_z}^{1/2} \in \mathcal{L}(\mathrm{H}^1_\sigma(\mathbb{R}^d), \mathrm{L}^2_\sigma(\mathbb{R}^d))$$

is holomorphic. Next, define the function

$$G : \mathbb{D} \to \mathrm{L}^2_\sigma(\mathbb{R}^d), \quad G(z) := \frac{A_{\mu_z}^{1/2} u - A_{\mu_1}^{1/2} u}{C' \|\nabla u\|_{\mathrm{L}^2}}$$

for $u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ and some constant $C' > 0$ to be chosen such that

$$\|G(z)\|_{\mathrm{L}^2} \leq \frac{\|A_{\mu_z}^{1/2} u\|_{\mathrm{L}^2} + \|A_{\mu_1}^{1/2} u\|_{\mathrm{L}^2}}{C' \|\nabla u\|_{\mathrm{L}^2}} \leq \frac{C_\delta + C_0}{C'} < 1.$$

Then for $v \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$ with $\|v\|_{\mathrm{L}^2} = 1$ the scalar-valued function

$$\mathbb{D} \ni z \mapsto \langle G(z), v \rangle_{\mathrm{L}^2_\sigma, \mathrm{L}^2_\sigma}$$

is holomorphic, maps onto the unit disk $\mathbb{D}$ and is equal to 0 if $z = 0$. This allows us to apply Schwarz lemma from complex analysis to get the estimate

$$\|G(z)\|_{\mathrm{L}^2} = \sup_{\substack{v \in \mathrm{L}^2_\sigma(\mathbb{R}^d) \\ \|v\|_{\mathrm{L}^2} = 1}} |\langle G(z), v \rangle_{\mathrm{L}^2_\sigma, \mathrm{L}^2_\sigma}| \leq |z|$$

for all $z \in \mathbb{D}$. Using the definition of $G(z)$ and choosing $z = \frac{\|\mu_2 - \mu_1\|_{\mathrm{L}^\infty(\mathbb{R}^d; \mathcal{L}(\mathbb{C}^{d \times d}))}}{\delta} \in \mathbb{D}$ we get

$$\|A_{\mu_2}^{1/2} u - A_{\mu_1}^{1/2} u\|_{\mathrm{L}^2} \leq \frac{C'}{\delta} \|\nabla u\|_{\mathrm{L}^2} \cdot \|\mu_2 - \mu_1\|_{\mathrm{L}^\infty(\mathbb{R}^d; \mathcal{L}(\mathbb{C}^{d \times d}))}$$

for all $u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ which proves the claim. □

Data availability

No data was used for the research described in the article.

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