# Optimal Reduction of Dirac Mixture Densities on the 2-Sphere

**Daniel Frisch, Kailai Li, and Uwe D. Hanebeck**

**IFAC 2020 Conference Presentation**

Intelligent Sensor-Actuator-Systems Laboratory (ISAS)
Institute for Anthropomatics and Robotics
Karlsruhe Institute of Technology (KIT)
Karlsruhe, Germany

Karlsruhe Institute of Technology

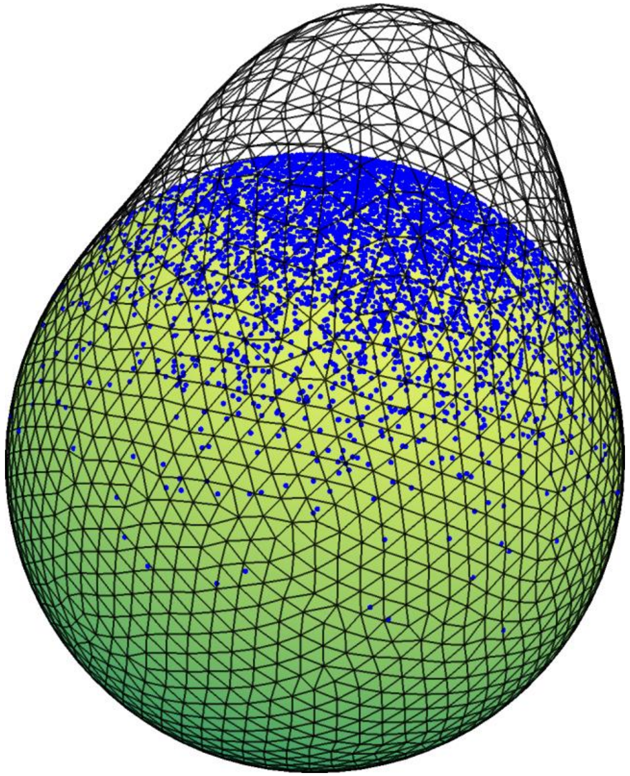
https://isas.iar.kit.edu

Intelligent Sensor-Actuator-Systems

**Input**
Dirac Mixture Density (DMD)
- Random

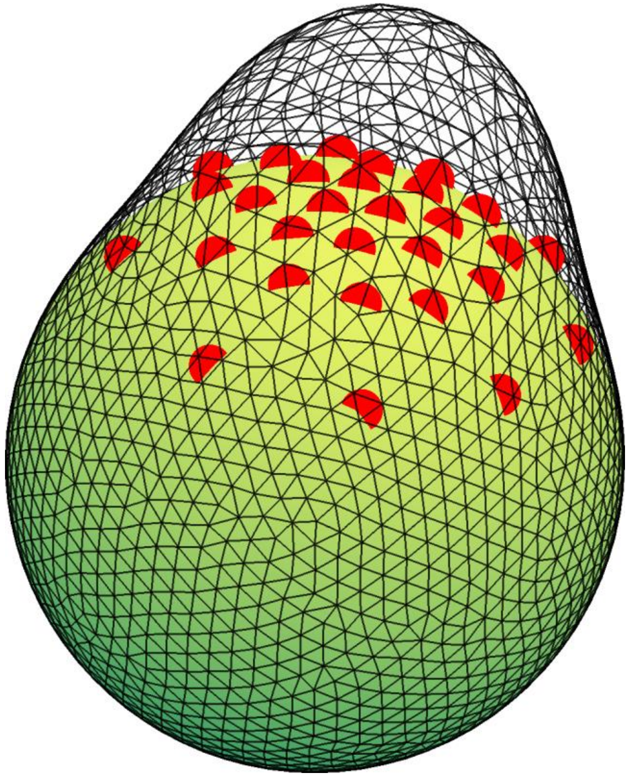
→ **Optimal Reduction** →

**Output**
Dirac Mixture Density (DMD)
- Deterministic



**Methods**

- **Distance for DMDs**
  - Localized Cumulative Distribution (LCD)
  - Modified Cramér-von Mises Distance
- Analytical integration
- Numerical optimization



# Notation

**Input**

Dirac Mixture Density (DMD)

- Random

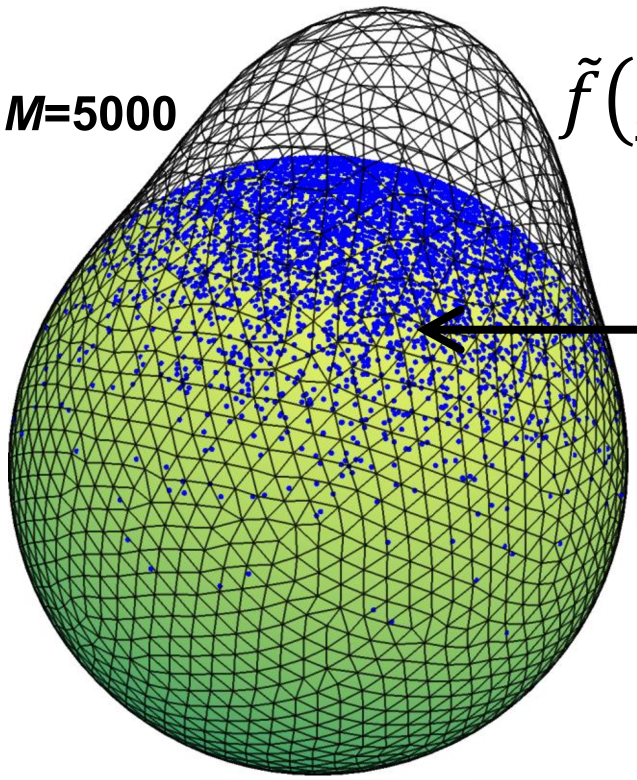
$\underline{x} \in \mathbb{S}^2 \subset \mathbb{R}^3$

**Output**

Dirac Mixture Density (DMD)

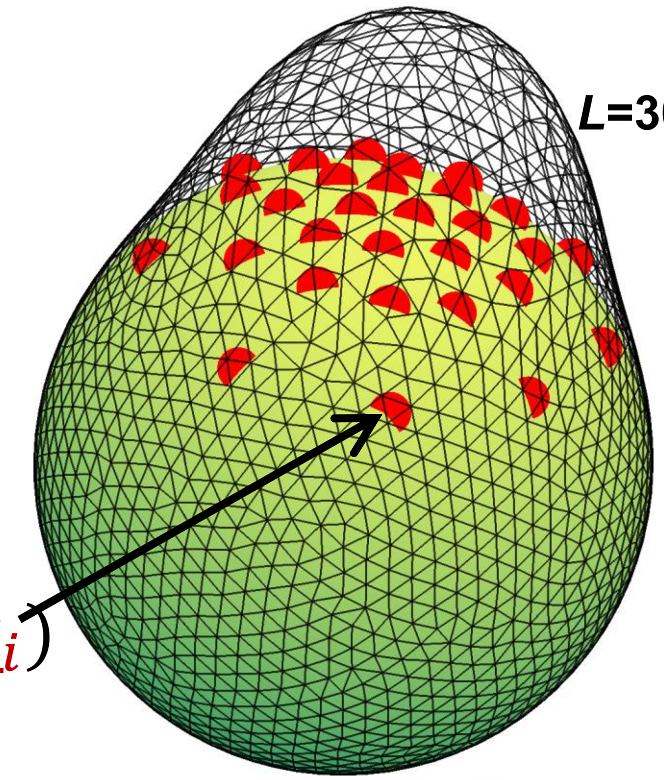
- Deterministic

*M*=5000



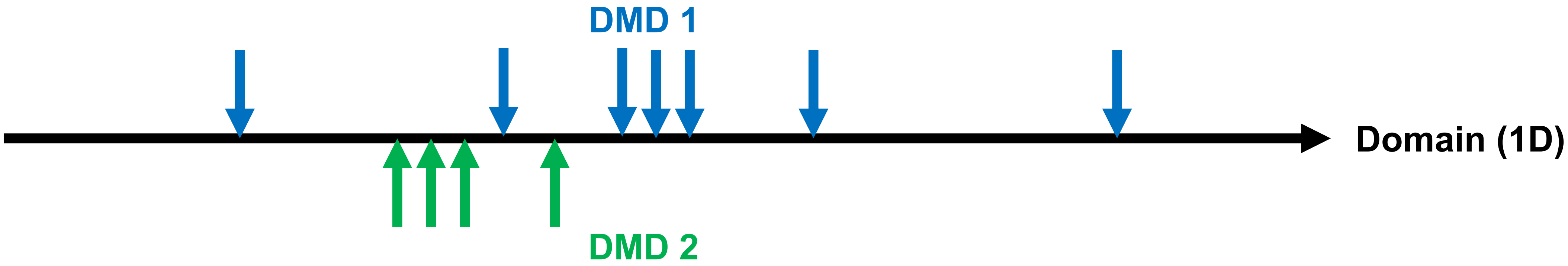
$$\tilde{f}(\underline{x}) = \frac{1}{M} \sum_{j=1}^{M} \delta(\underline{x} - \underline{y}_j)$$

*L*=30



$$f(\underline{x}) = \frac{1}{L} \sum_{i=1}^{L} \delta(\underline{x} - \underline{x}_i)$$

# Distance Measure for DMDs I




- $L_2$ norm of PDFs?
  - No common support

$$D^2 = \int_{-\infty}^{\infty} \left( \tilde{f}(x) - f(x) \right)^2 \mathrm{d}x \ = 0$$

- $L_2$ norm of CDFs?
  - Works **for 1D**

$$D^2 = \int_{-\infty}^{\infty} \left( \tilde{F}(x) - F(x) \right)^2 \mathrm{d}x \ , \qquad F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t$$

# CDF in Higher Dimensions

**1D**

$$\int_{-\infty}^{x} f(t)\,\mathrm{d}t$$

$$\int_{x}^{+\infty} f(t)\,\mathrm{d}t$$

**2D**

$$\int_{-\infty}^{y}\int_{-\infty}^{x} f(t,u)\,\mathrm{d}t\,\mathrm{d}u$$

$$\int_{+\infty}^{y}\int_{-\infty}^{x} f(t,u)\,\mathrm{d}t\,\mathrm{d}u$$

$$\int_{+\infty}^{y}\int_{+\infty}^{x} f(t,u)\,\mathrm{d}t\,\mathrm{d}u$$

$$\int_{-\infty}^{y}\int_{+\infty}^{x} f(t,u)\,\mathrm{d}t\,\mathrm{d}u$$

**3D**

$$\int_{-\infty}^{z}\int_{-\infty}^{y}\int_{-\infty}^{x} f(t,u)\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v$$

$$\int_{-\infty}^{z}\int_{-\infty}^{y}\int_{+\infty}^{x} f(t,u)\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v$$

$$\int_{-\infty}^{z}\int_{+\infty}^{y}\int_{-\infty}^{x} f(t,u)\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v$$

$$\int_{-\infty}^{z}\int_{+\infty}^{y}\int_{+\infty}^{x} f(t,u)\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v$$

$$\int_{+\infty}^{z}\int_{-\infty}^{y}\int_{-\infty}^{x} f(t,u)\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v$$

$$\int_{+\infty}^{z}\int_{-\infty}^{y}\int_{+\infty}^{x} f(t,u)\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v$$

$$\int_{+\infty}^{z}\int_{+\infty}^{y}\int_{-\infty}^{x} f(t,u)\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v$$

$$\int_{+\infty}^{z}\int_{+\infty}^{y}\int_{+\infty}^{x} f(t,u)\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}v$$

# Localized Cumulative Distribution (LCD)

$$F(x, y, b) = \int_{y-b}^{y+b} \int_{x-b}^{x+b} f(t, u)\,\mathrm{d}t\,\mathrm{d}u$$

$$F(\underline{m}, b) = \iint_{\|\underline{x}-\underline{m}\| \leq b} f(\underline{x})\,\mathrm{d}\underline{x}$$

$$F(\underline{m}, b) = \iint_{X} f(\underline{x})\,K(\underline{x}, \underline{m}, b)\,\mathrm{d}\underline{x}$$

**LCD**

# Modified Cramér-von Mises Distance

$$D^2 = \int_{\mathbb{R}^+} w(b) \int_{\mathcal{X}} \left( \tilde{F}(\underline{m}, b) - F(\underline{m}, b) \right)^2 \mathrm{d}\underline{m} \, \mathrm{d}b$$

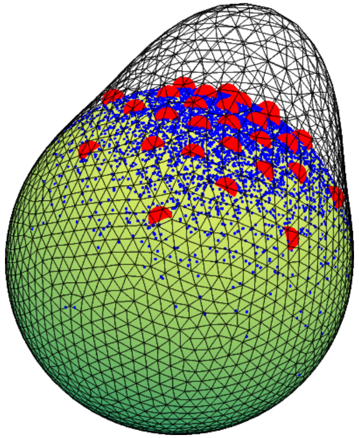
Distance measure for comparing all types of densities:

- **discrete vs discrete**
- discrete vs continuous
- continuous vs continuous

# Overview: DMD-DMD Distance

**DMD1**

$$\tilde{f}(\underline{x}) = \frac{1}{M}\sum_{j=1}^{M} \delta(\underline{x} - \underline{y}_j)$$



**DMD2**

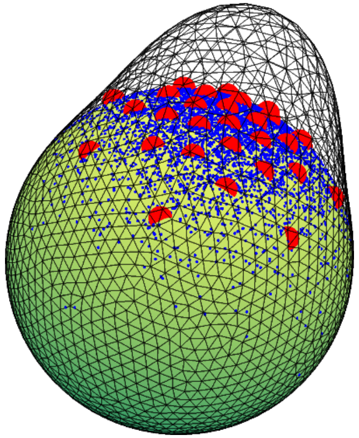
$$f(\underline{x}) = \frac{1}{L}\sum_{i=1}^{L} \delta(\underline{x} - \underline{x}_i)$$

$$\tilde{F}(\underline{m}, b) = \int_{X} \tilde{f}(\underline{x})\, K(\underline{x}, \underline{m}, b)\, \mathrm{d}\underline{x}$$

$$F(\underline{m}, b) = \int_{X} f(\underline{x})\, K(\underline{x}, \underline{m}, b)\, \mathrm{d}\underline{x}$$

**DMD1**

$$\tilde{f}(\underline{x}) = \frac{1}{M}\sum_{j=1}^{M}\delta(\underline{x}-\underline{y}_j)$$



**DMD2**

$$f(\underline{x}) = \frac{1}{L}\sum_{i=1}^{L}\delta(\underline{x}-\underline{x}_i)$$

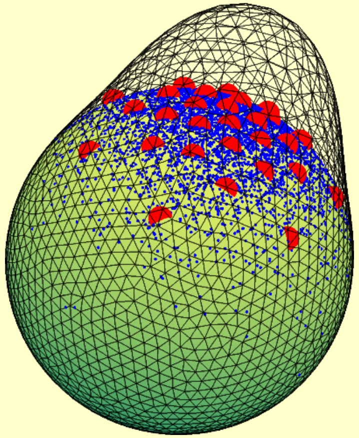
**LCD1**

$$\tilde{F}(\underline{m},b) = \int_{\mathcal{X}} \tilde{f}(\underline{x})\,K(\underline{x},\underline{m},b)\,\mathrm{d}\underline{x}$$

**LCD2**

$$F(\underline{m},b) = \int_{\mathcal{X}} f(\underline{x})\,K(\underline{x},\underline{m},b)\,\mathrm{d}\underline{x}$$

$$D^2 = \int_{\mathbb{R}^+} w(b)\int_{\mathcal{X}}\left(\tilde{F}(\underline{m},b) - F(\underline{m},b)\right)^2 \mathrm{d}\underline{m}\,\mathrm{d}b$$



**DMD1**

$$\tilde{f}(\underline{x}) = \frac{1}{M} \sum_{j=1}^{M} \delta(\underline{x} - \underline{y}_j)$$

**DMD2**

$$f(\underline{x}) = \frac{1}{L} \sum_{i=1}^{L} \delta(\underline{x} - \underline{x}_i)$$

**LCD1**

$$\tilde{F}(\underline{m}, b) = \int_{\mathcal{X}} \tilde{f}(\underline{x})\, K(\underline{x}, \underline{m}, b)\, \mathrm{d}\underline{x}$$

**LCD2**

$$F(\underline{m}, b) = \int_{\mathcal{X}} f(\underline{x})\, K(\underline{x}, \underline{m}, b)\, \mathrm{d}\underline{x}$$

$$D^2 = \int_{\mathbb{R}^+} w(b) \int_{\mathcal{X}} \left( \tilde{F}(\underline{m}, b) - F(\underline{m}, b) \right)^2 \mathrm{d}\underline{m}\, \mathrm{d}b$$

**Modified Cramér-von Mises Distance**

# Overview: Deterministic Sampling

**Quasi-Newton Optimizer**

**Initial guess** $\{\underline{x}_i\}_{i=1}^{L}$

**Continuous Nonlinear Optimization**

**Hypothesize DMD2** $\{\underline{x}_i\}_{i=1}^{L}$

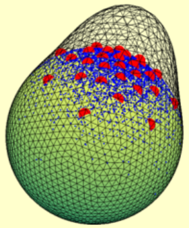
**Optimal result** $\{\underline{x}_i\}_{i=1}^{L}$

**constant**

**Objective Function**

DMD1

$$\tilde{f}(\underline{x}) = \frac{1}{M}\sum_{j=1}^{M}\delta(\underline{x}-\underline{y}_j)$$



DMD2

$$f(\underline{x}) = \frac{1}{L}\sum_{i=1}^{L}\delta(\underline{x}-\underline{x}_i)$$

LCD1

$$\tilde{F}(\underline{m},b) = \int_{\mathcal{X}} \tilde{f}(\underline{x})\, K(\underline{x},\underline{m},b)\, \mathrm{d}\underline{x}$$

LCD2

$$F(\underline{m},b) = \int_{\mathcal{X}} f(\underline{x})\, K(\underline{x},\underline{m},b)\, \mathrm{d}\underline{x}$$

$$D^2 = \int_{\mathbb{R}^+} w(b) \int_{\mathcal{X}} \left(\tilde{F}(\underline{m},b) - F(\underline{m},b)\right)^2 \mathrm{d}\underline{m}\, \mathrm{d}b$$

**Modified Cramér-von Mises Distance**

- Gaussian Kernel

$$K(\underline{x}, \underline{m}, \sigma) = \exp\left\{-\frac{(\underline{x}-\underline{m})^\top(\underline{x}-\underline{m})}{2\,\sigma}\right\}$$

$$K(d, \sigma) = \exp\left\{\frac{1}{\sigma}\cdot\left(-\frac{d^2}{2}\right)\right\}$$

$\mathbb{R}^2$

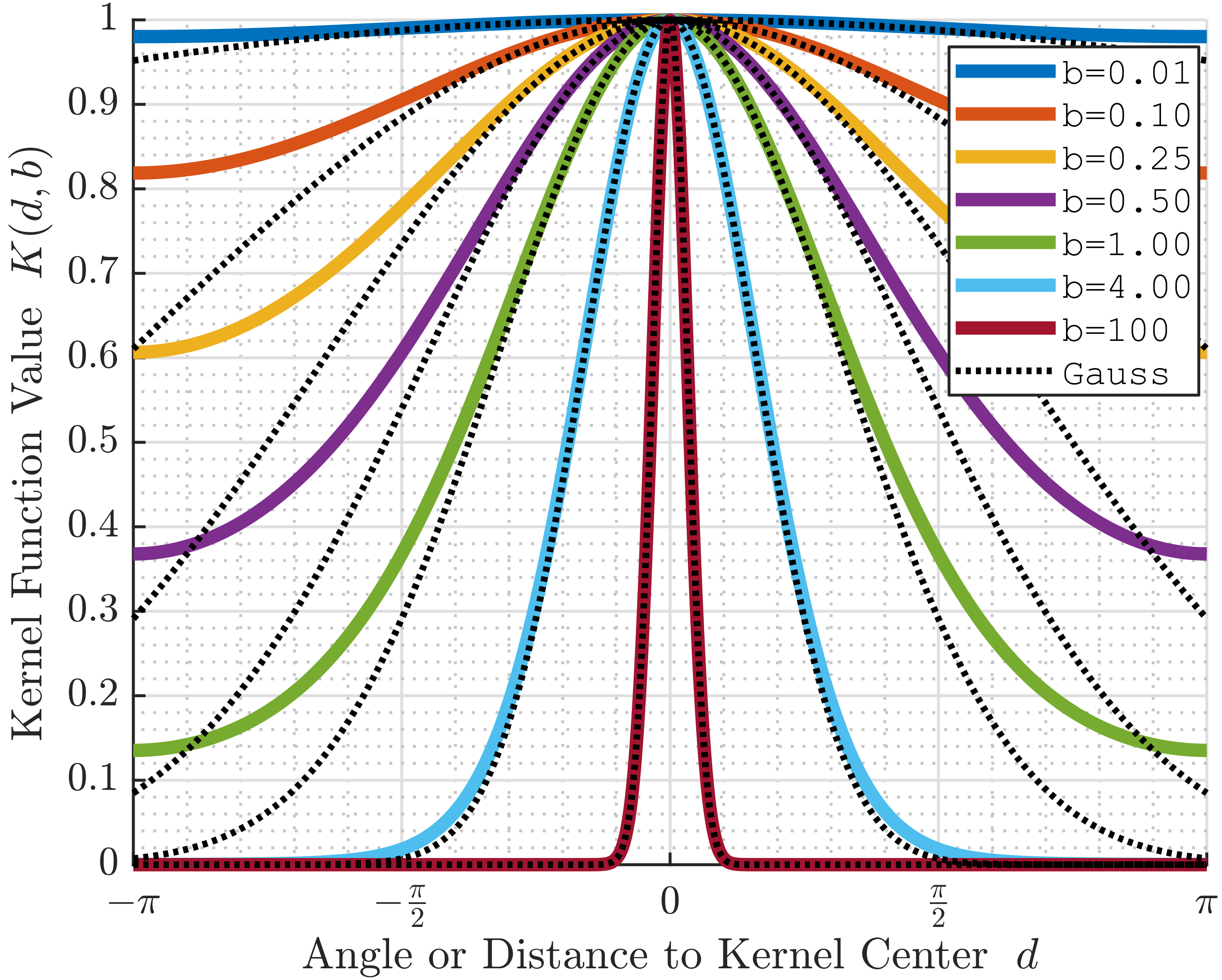
- Von Mises–Fisher Kernel

$$K(d, b) = \exp\{b \cdot \cos d\}$$

$\mathbb{S}^2$

$$K(\underline{x}, \underline{m}, b) = \exp\{b \cdot \underline{x}^\top \underline{m}\}$$

# Kernel Functions II



# Computational Complexity I

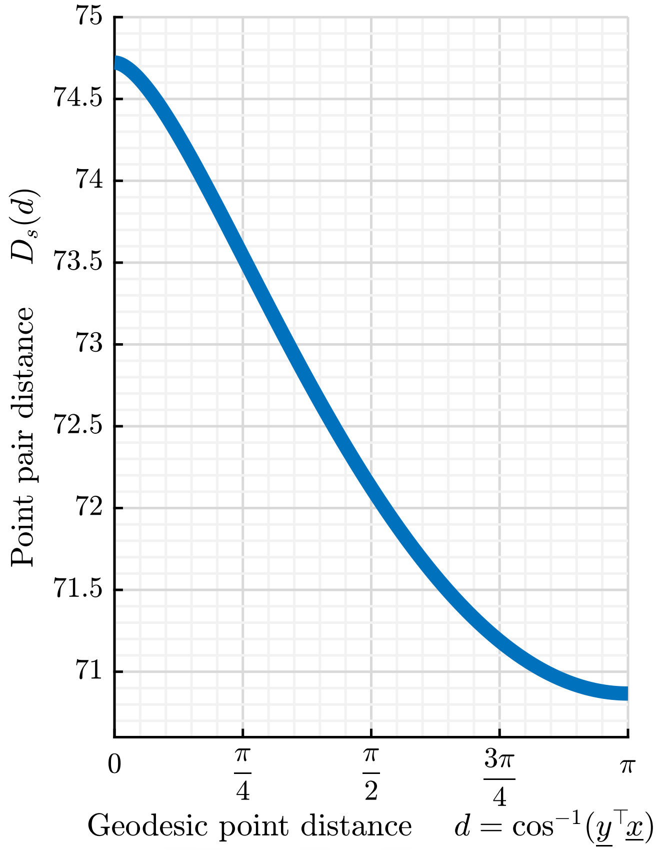
$$D^2 = \frac{1}{M^2}\sum_{j=1}^{M}\sum_{i=1}^{M} D_s\left(\underline{y}_i, \underline{y}_j\right) - 2\frac{1}{ML}\sum_{j=1}^{M}\sum_{i=1}^{L} D_s\left(\underline{y}_i, \underline{x}_j\right) + \frac{1}{L^2}\sum_{j=1}^{L}\sum_{i=1}^{L} D_s\left(\underline{x}_i, \underline{x}_j\right)$$

$$d = \cos^{-1}\left(\underline{y}^\top \underline{x}\right)$$

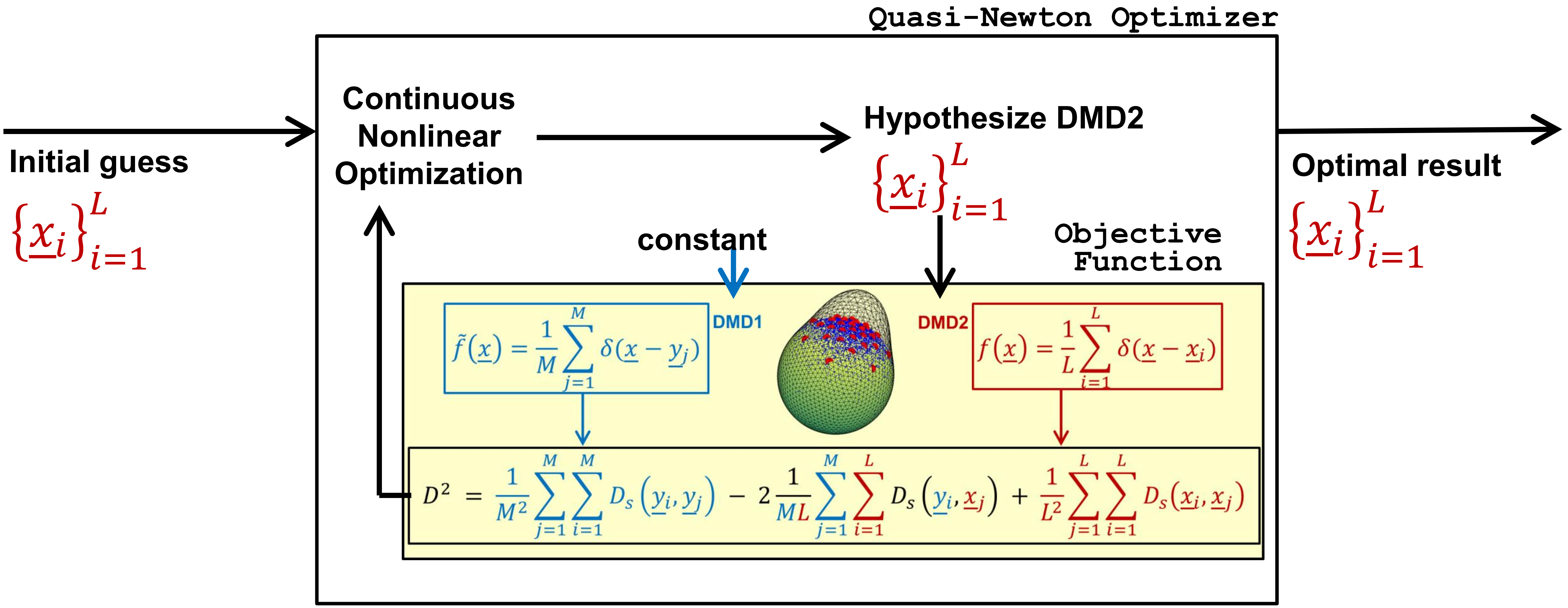
$$D_s(d) = \frac{\pi}{\cos {}^{d}/_{2}}\Big[c_1 \operatorname{Ei}(c_1 b) - c_2 \operatorname{Ei}(c_2 b) + \frac{1}{b}\left(e^{c_2 b}\right.$$

$$c_1 = +2 \cos {}^{d}/_{2} - 2$$

$$c_1 = -2 \cos {}^{d}/_{2} - 2$$

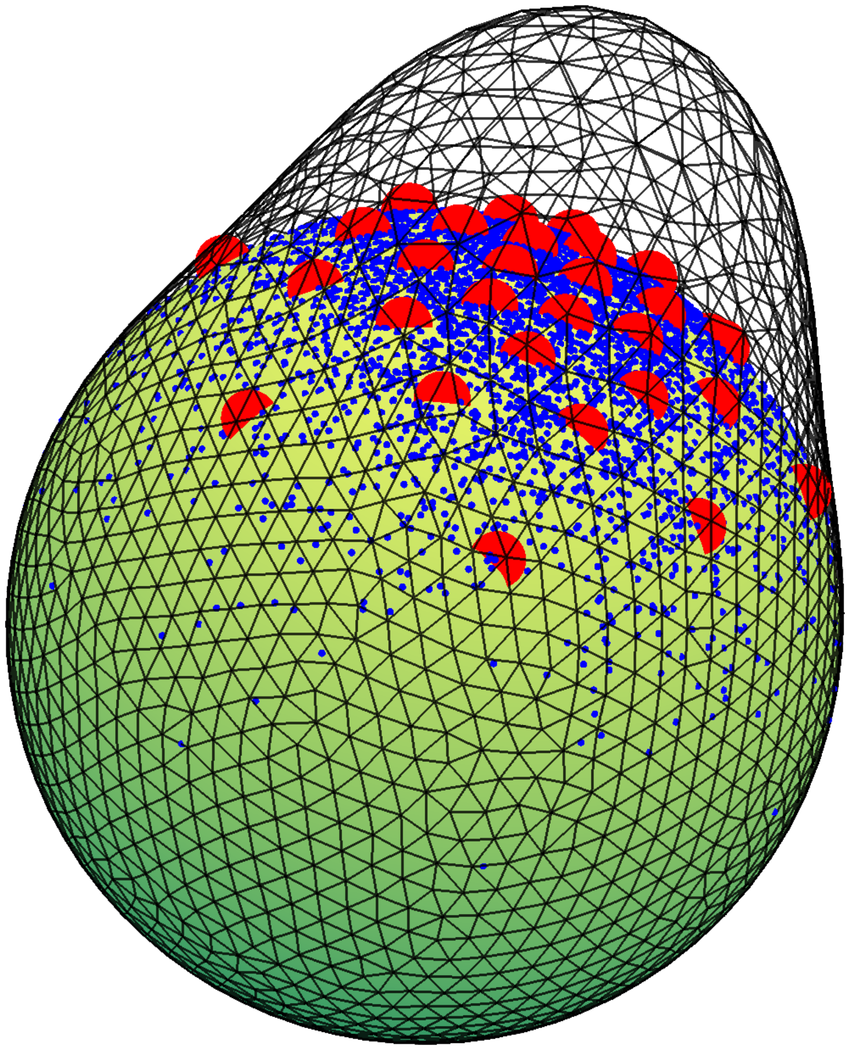


# Computational Complexity II

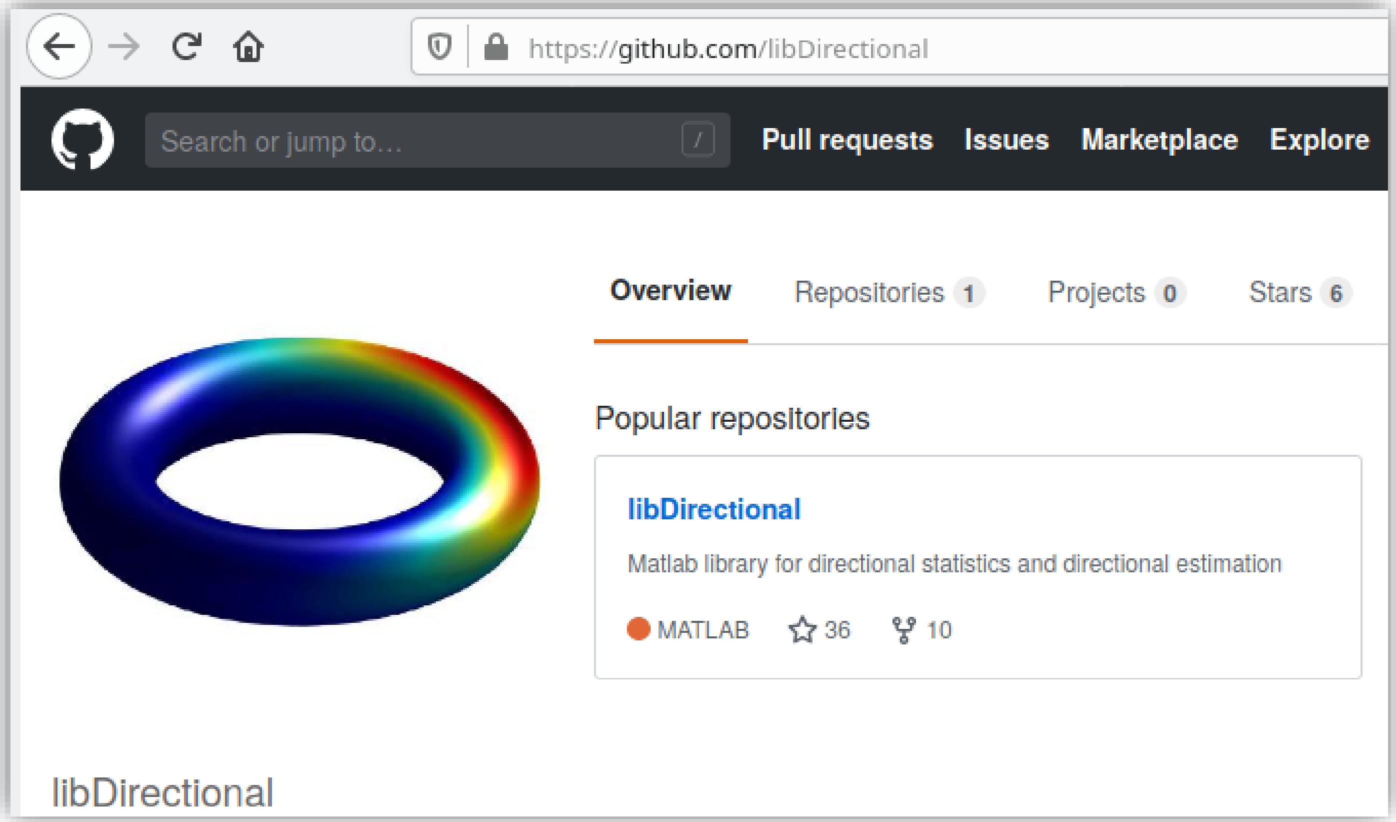


**General procedure**

1. Select spherical distribution
2. Draw ***M*** random samples $\underline{y}_j$
3. Draw ***L*** initial guess samples $\underline{x}_i$
4. Optimize ***L*** sample positions
   - Minimize LCD-based distance measure
5. Use result as deterministic samples



# libDirectional




*Journal of Statistical Software*

*May 2019, Volume 89, Issue 4.* *doi: 10.18637/jss.v089.i04*


## Directional Statistics and Filtering Using libDirectional


**Gerhard Kurz**, Karlsruhe Institute of Technology (KIT)

**Igor Gilitschenski**, Massachusetts Institute of Technology

**Florian Pfaff**, Karlsruhe Institute of Technology (KIT)

**Lukas Drude**, University of Paderborn

**Uwe D. Hanebeck**, Karlsruhe Institute of Technology (KIT)

**Reinhold Haeb-Umbach**, University of Paderborn

**Roland Y. Siegwart**, ETH Zürich

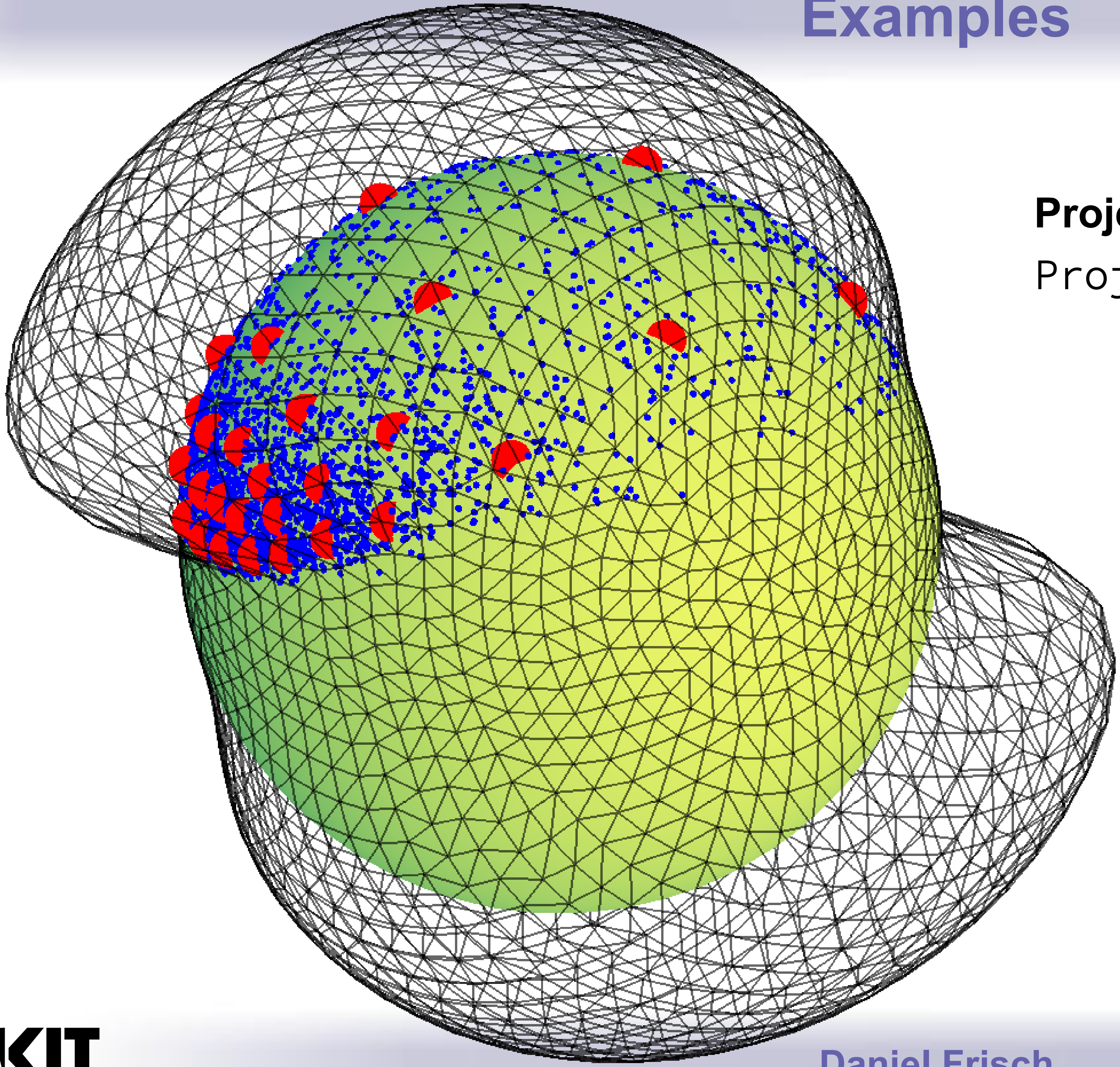


**Abstract**

In this paper, we present **libDirectional**, a MATLAB library for directional statistics and directional estimation. It supports a variety of commonly used distributions on the unit circle, such as the von Mises, wrapped normal, and wrapped Cauchy distributions. Furthermore, various distributions on higher-dimensional manifolds such as the unit hypersphere and the hypertorus are available. Based on these distributions, several recursive filtering algorithms in **libDirectional** allow estimation on these manifolds. The functionality is implemented in a clear, well-documented, and object-oriented structure that is both easy to use and easy to extend.

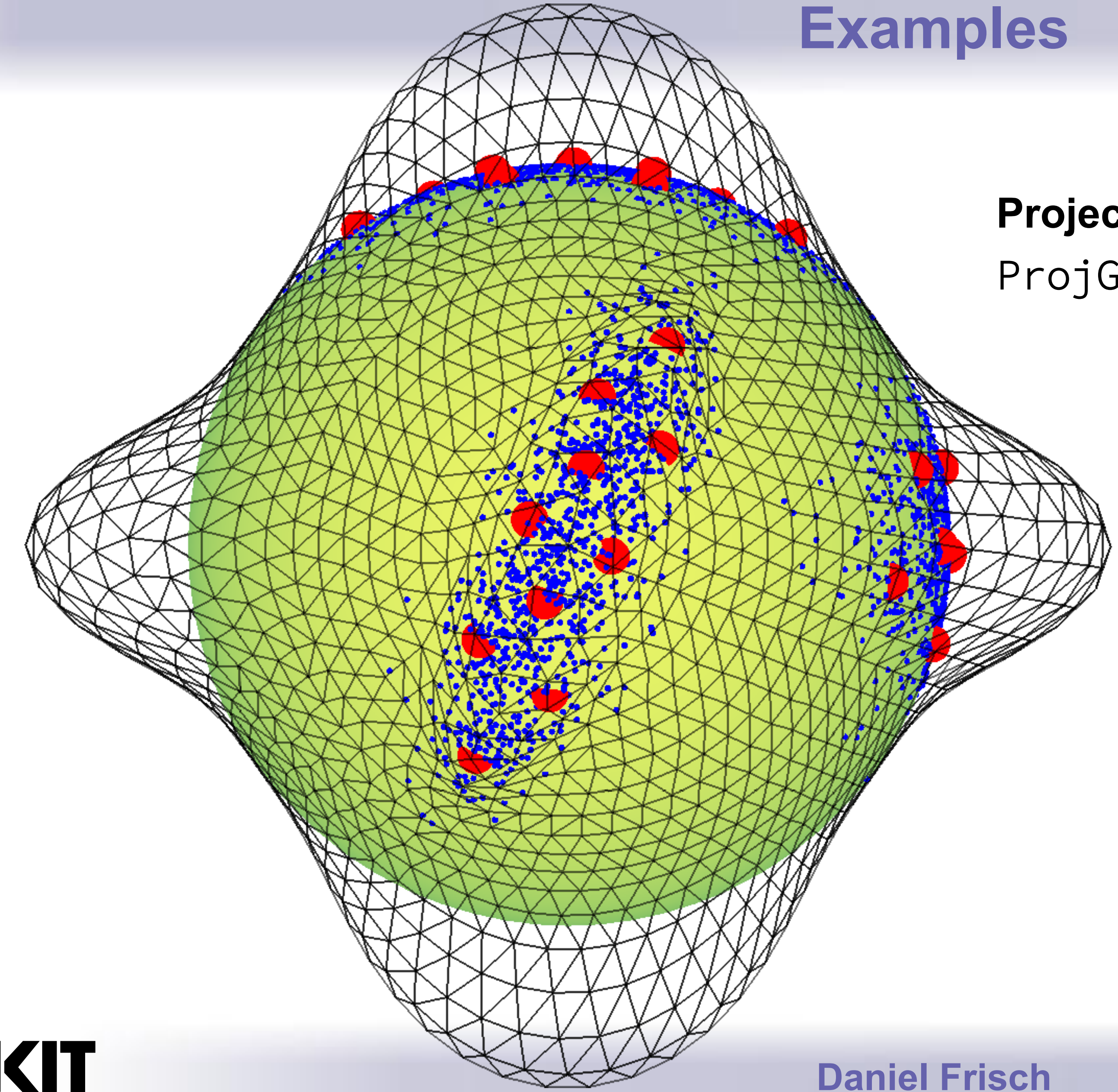


*Keywords*: recursive filtering, phase estimation, orientation estimation, circle, hypertorus, hypersphere.



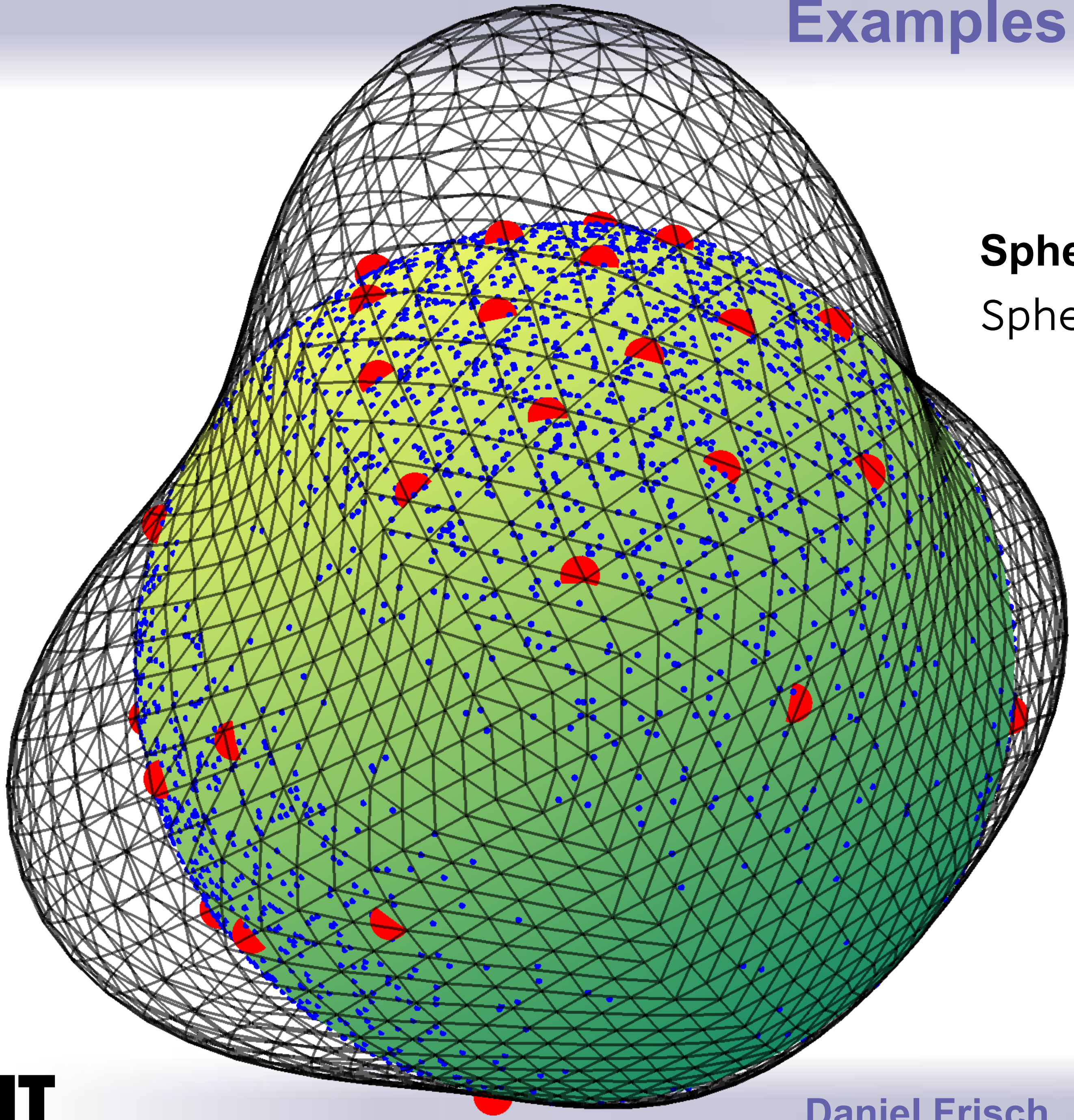
**Projected Gaussian Distribution**

`ProjGaussianDist()`



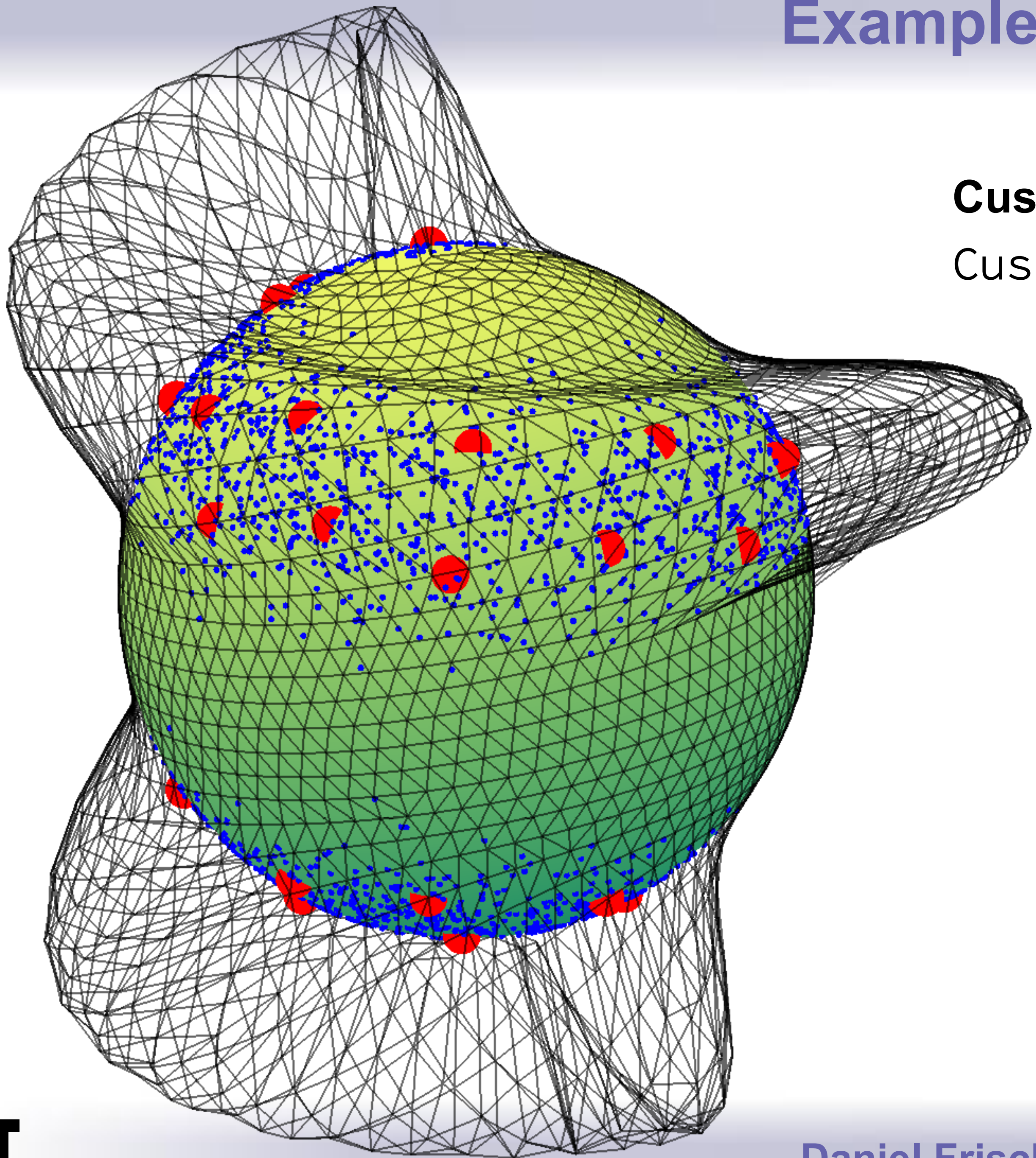
**Projected Gaussian Mixture Distribution**

`ProjGaussianMixtureDist()`



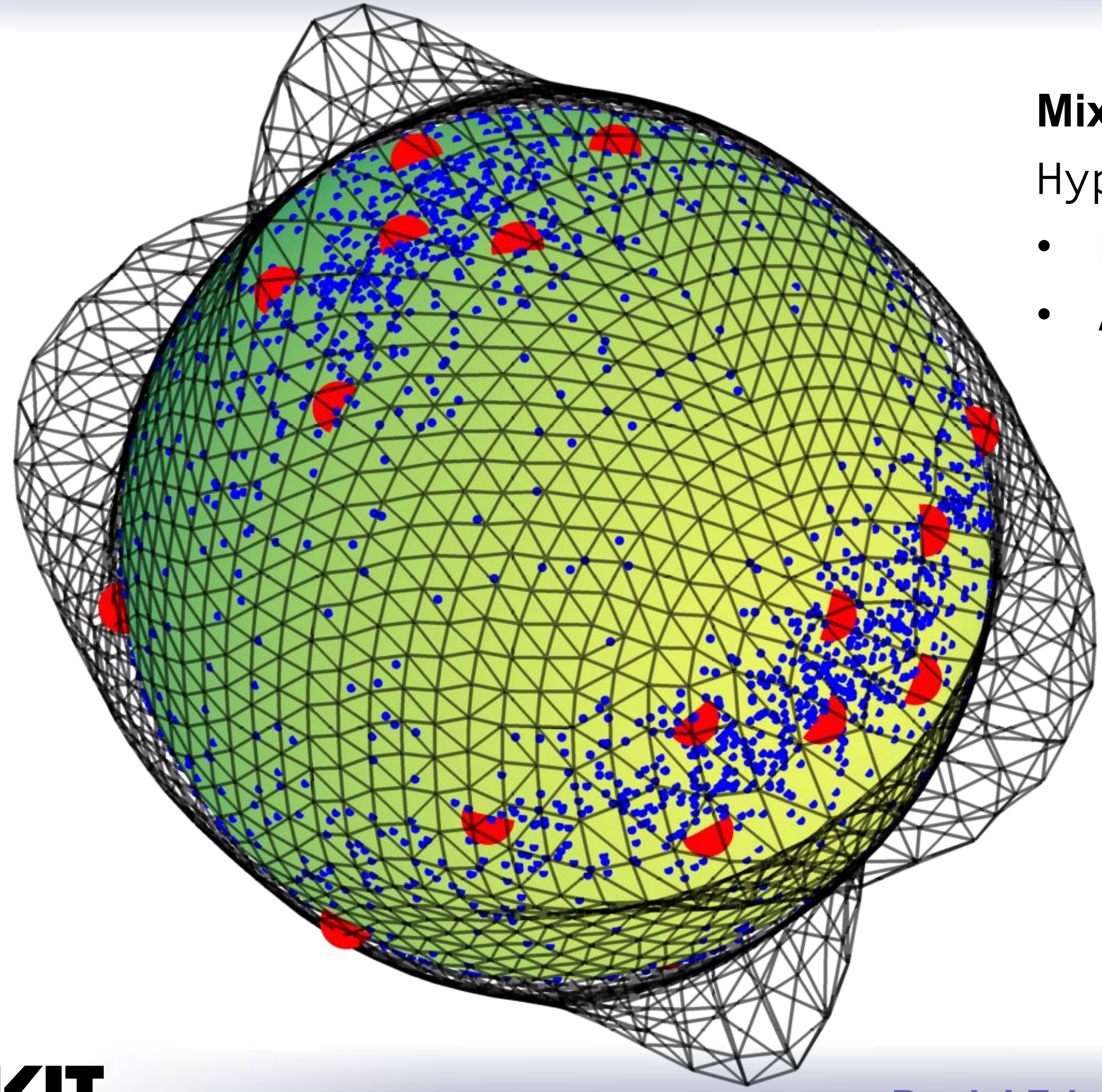
**Spherical Harmonics Distribution**

`SphericalHarmonicsDistributionReal()`



**Custom Distribution**

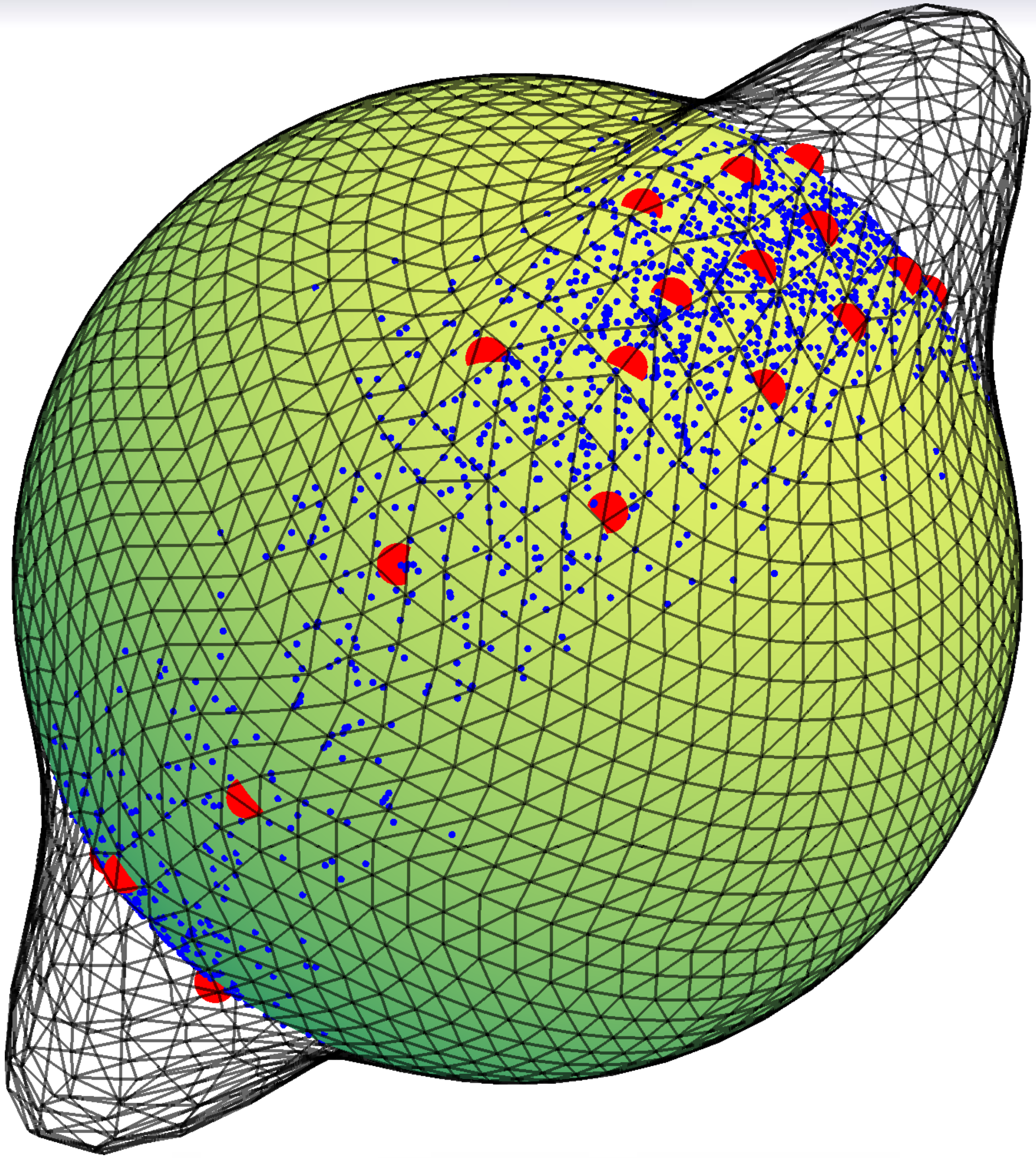
`CustomHypersphericalDistribution()`



**Mixture Distribution**

`HypersphericalMixture()`

- `BinghamDistribution()`
- `AngularCentralGaussianDistribution()`



**Bingham Distribution**

```
BinghamDistribution()
```

# Conclusion

## Deterministic Sampling on Sphere

- Application scenarios
  - Monte Carlo experiments
  - Proposal density
  - Importance sampling
  - State estimation
  - Control
- Universally applicable
  - DMD → DMD
  - Weighted DMD → DMD

## Outlook

- Hyperspheres
  - Higher dimensions
- Riemannian manifolds
  - Torus
  - Cylinder
  - Special Euclidean Groups
- Continuous → Discrete deterministic sampling
  - For certain densities

# Thank you for your attention

Intelligent

Sensor-Actuator-Systems