Gaussian Mixture Estimation from Weighted Samples

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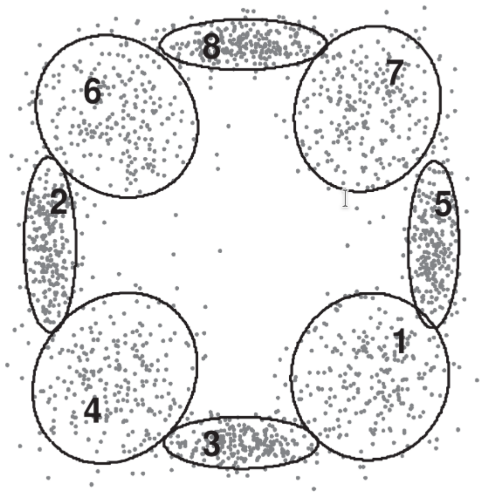
Intelligent Sensor-Actuator-Systems

	k-Means	EM
Unweighted	1,500,000	40,000
Weighted	1,500	1 + 1



Applications

- Clustering
- Machine learning
- Multi-Target tracking
- Density representation

Why Weights?

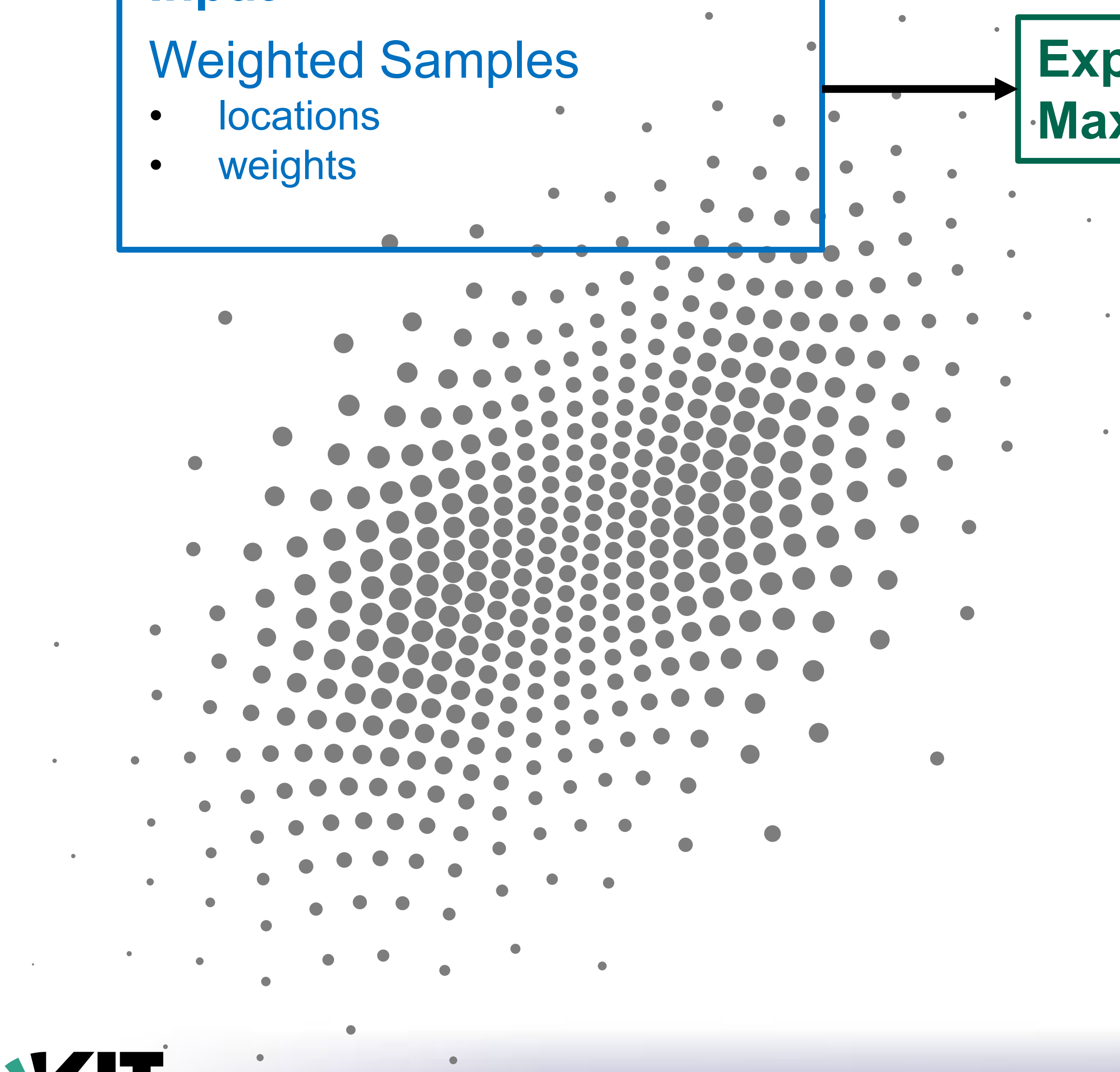
- Density-Biased Sampling
- Boosting
- Down-weigh scatter
- Likelihood
- Integer instances

Overview

Input

Weighted Samples

- locations
- weights

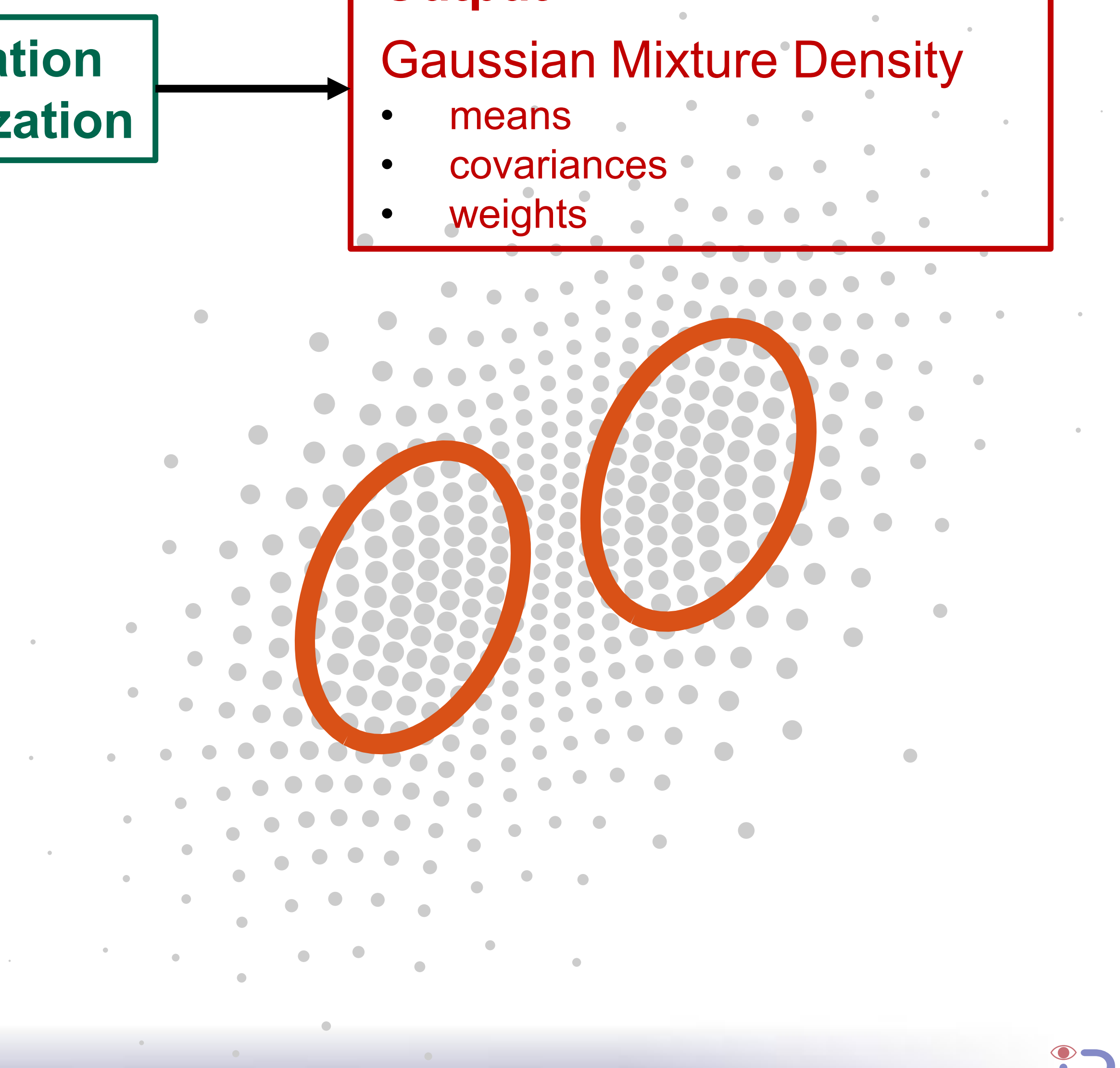


Expectation Maximization

Output

Gaussian Mixture Density

- means
- covariances
- weights



Notation

Input

Weighted Samples

- locations
- weights

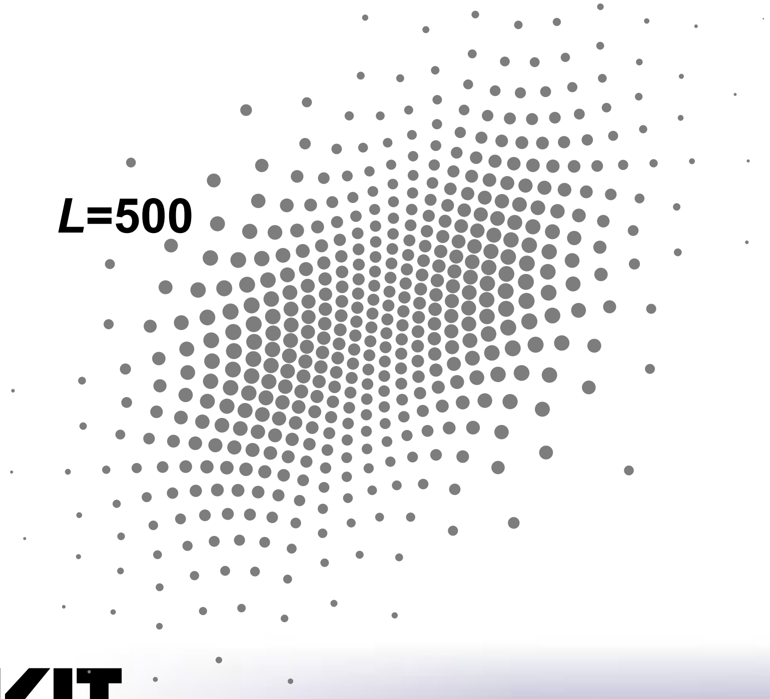
$\underline{x} \in \mathbb{R}^D$

Output

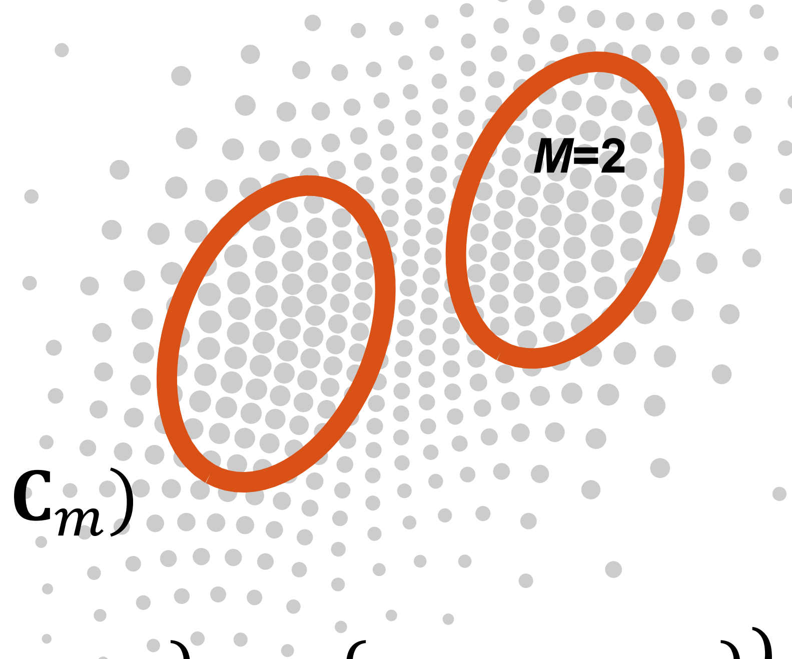
Gaussian Mixture Density

- means
- covariances
- weights

$$\mathbf{Y} = \left\{ \{\alpha_1, \underline{s}_1\}, \{\alpha_2, \underline{s}_2\}, \dots, \{\alpha_L, \underline{s}_L\} \right\}$$



$$f(\underline{x}|\boldsymbol{\Theta}) = \sum_{m=1}^{L} w_m \cdot \mathcal{N}(\underline{x}, \underline{\mu}_m, \mathbf{C}_m)$$



$$\boldsymbol{\Theta} = \left\{ \{w_1, \underline{\mu}_1, \mathbf{C}_1\}, \{w_2, \underline{\mu}_2, \mathbf{C}_2\}, \dots, \{w_M, \underline{\mu}_M, \mathbf{C}_M\} \right\}$$

Maximum Likelihood Solution

$$\boldsymbol{\Theta}^{\mathrm{ML}} = \max_{\boldsymbol{\Theta}} \{ \, \mathrm{f}(\mathbf{Y}|\boldsymbol{\Theta}) \, \}$$

Gradient-Based Optimization

- Gradients complicated
- Safeguards for invalid $\boldsymbol{\Theta}$

Expectation-Maximization

- Statistical standard tasks
- Always valid $\boldsymbol{\Theta}$

Split-Sample Linearity

Should give identical result:

- one double-weight sample
- two single-weight samples at the same location

Essence of meaning of weight.

Is it kept?

- Proposed: **yes**
- Gebru 2016 PAMI: **no**

Expectation

$$\eta_{i,m}^{(r+1)} = \frac{w_m^{(r)}\,\mathcal{N}\!\left(\underline{s}_i - \underline{\mu}_m^{(r)},\ \mathbf{C}_m^{(r)}\right)}{\sum_{\widetilde{m}=1}^{M} w_{\widetilde{m}}^{(r)}\,\mathcal{N}\!\left(\underline{s}_i - \underline{\mu}_{\widetilde{m}}^{(r)},\ \mathbf{C}_{\widetilde{m}}^{(r)}\right)}$$

Proposed

$$\eta_{i,m}^{(r+1)} = \frac{w_m^{(r)}\,\mathcal{N}\!\left(\underline{s}_i - \underline{\mu}_m^{(r)},\ \mathbf{C}_m^{(r)}/\alpha_i\right)}{\sum_{\widetilde{m}=1}^{M} w_{\widetilde{m}}^{(r)}\,\mathcal{N}\!\left(\underline{s}_i - \underline{\mu}_{\widetilde{m}}^{(r)},\ \mathbf{C}_{\widetilde{m}}^{(r)}/\alpha_i\right)}$$

Gebru 2016 PAMI

Maximization

Proposed

$$w_m^{(r+1)} = \frac{\sum_{i=1}^{L} \eta_{i,m}^{(r+1)} \alpha_i}{\sum_{\widetilde{m}=1}^{M} \sum_{\tilde{i}=1}^{L} \eta_{\tilde{i},\widetilde{m}}^{(r+1)} \alpha_{\tilde{i}}} = \frac{\sum_{i=1}^{L} \eta_{i,m}^{(r+1)} \alpha_i}{\sum_{\tilde{i}=1}^{L} \alpha_{\tilde{i}}}$$

$$\underline{\mu}_m^{(r+1)} = \frac{\sum_{i=1}^{L} \eta_{i,m}^{(r+1)} \alpha_i \, \underline{s}_i}{\sum_{\tilde{i}=1}^{L} \eta_{\tilde{i},m}^{(r+1)} \alpha_{\tilde{i}}}$$

$$\mathbf{C}_m^{(r+1)} = \frac{\sum_{i=1}^{L} \eta_{i,m}^{(r+1)} \alpha_i \left(\underline{s}_i - \underline{\mu}_m^{(r+1)}\right) \left(\underline{s}_i - \underline{\mu}_m^{(r+1)}\right)^\top}{\sum_{\tilde{i}=1}^{L} \eta_{\tilde{i},m}^{(r+1)} \alpha_{\tilde{i}}}$$

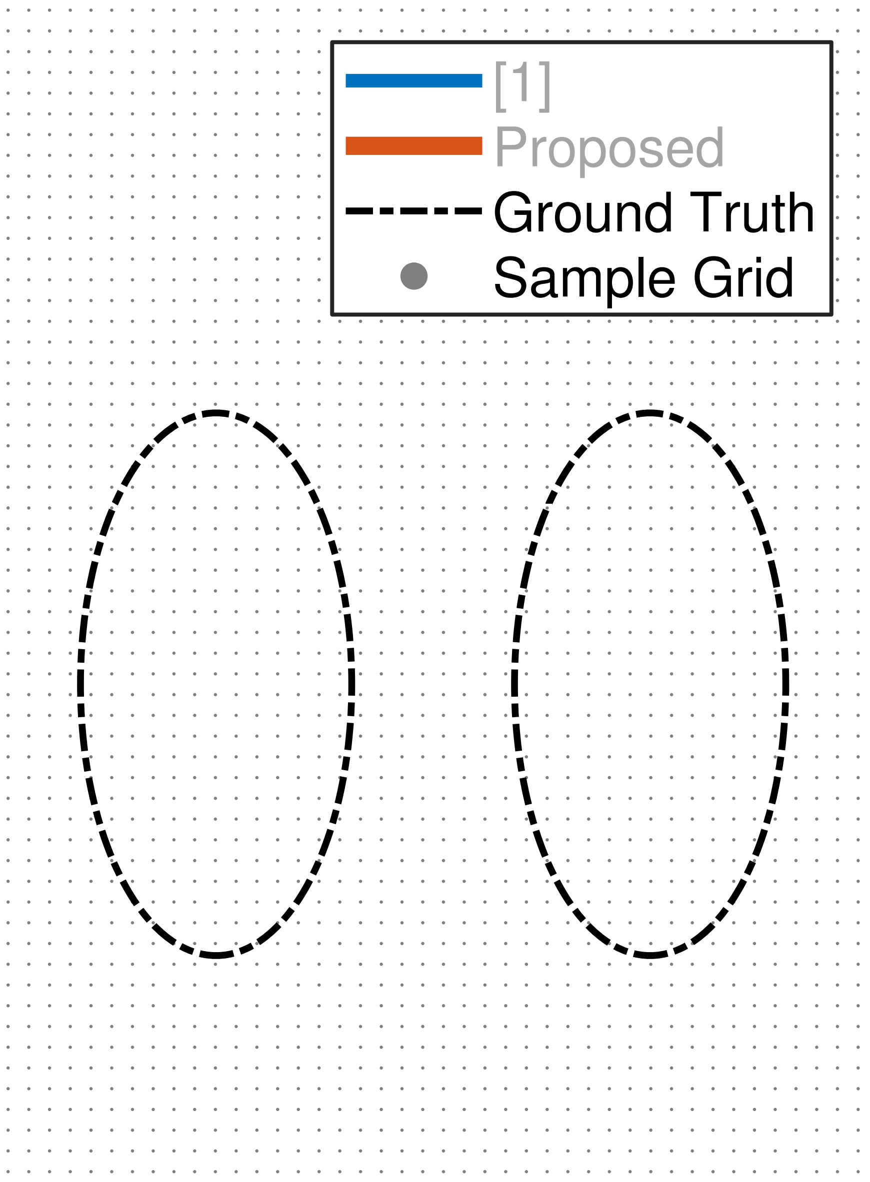
$$w_m^{(r+1)} = \frac{1}{L} \sum_{i=1}^{L} \eta_{i,m}^{(r+1)}$$

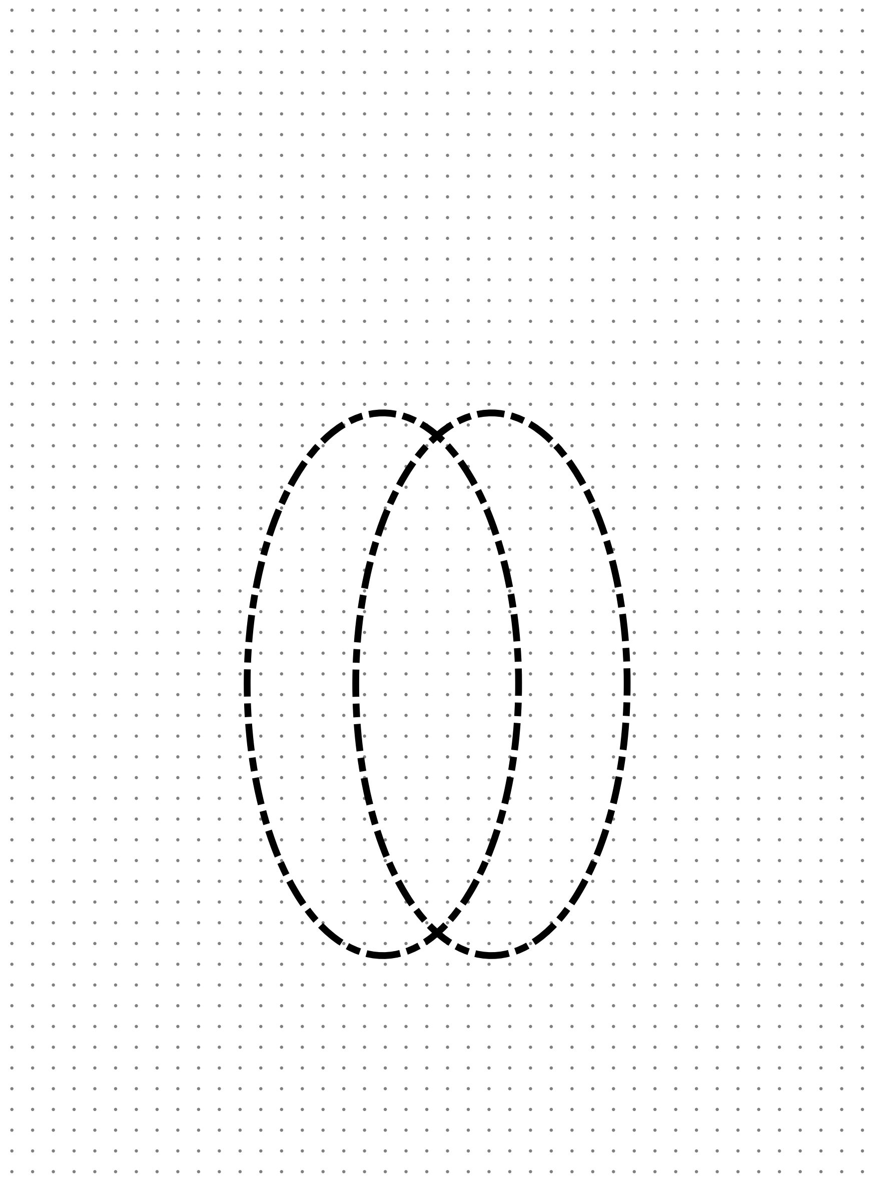
$$\underline{\mu}_m^{(r+1)} = \frac{\sum_{i=1}^{L} \eta_{i,m}^{(r+1)} \alpha_i \underline{s}_i}{\sum_{\tilde{i}=1}^{L} \eta_{\tilde{i},m}^{(r+1)} \alpha_{\tilde{i}}}$$

$$\mathbf{C}_m^{(r+1)} = \frac{\sum_{i=1}^{L} \eta_{i,m}^{(r+1)} \alpha_i \left(\underline{s}_i - \underline{\mu}_m^{(r+1)}\right) \left(\underline{s}_i - \underline{\mu}_m^{(r+1)}\right)^\top}{\sum_{\tilde{i}=1}^{L} \eta_{\tilde{i},m}^{(r+1)}}$$

Gebru 2016 PAMI

2D Examples – Separated and Overlapping





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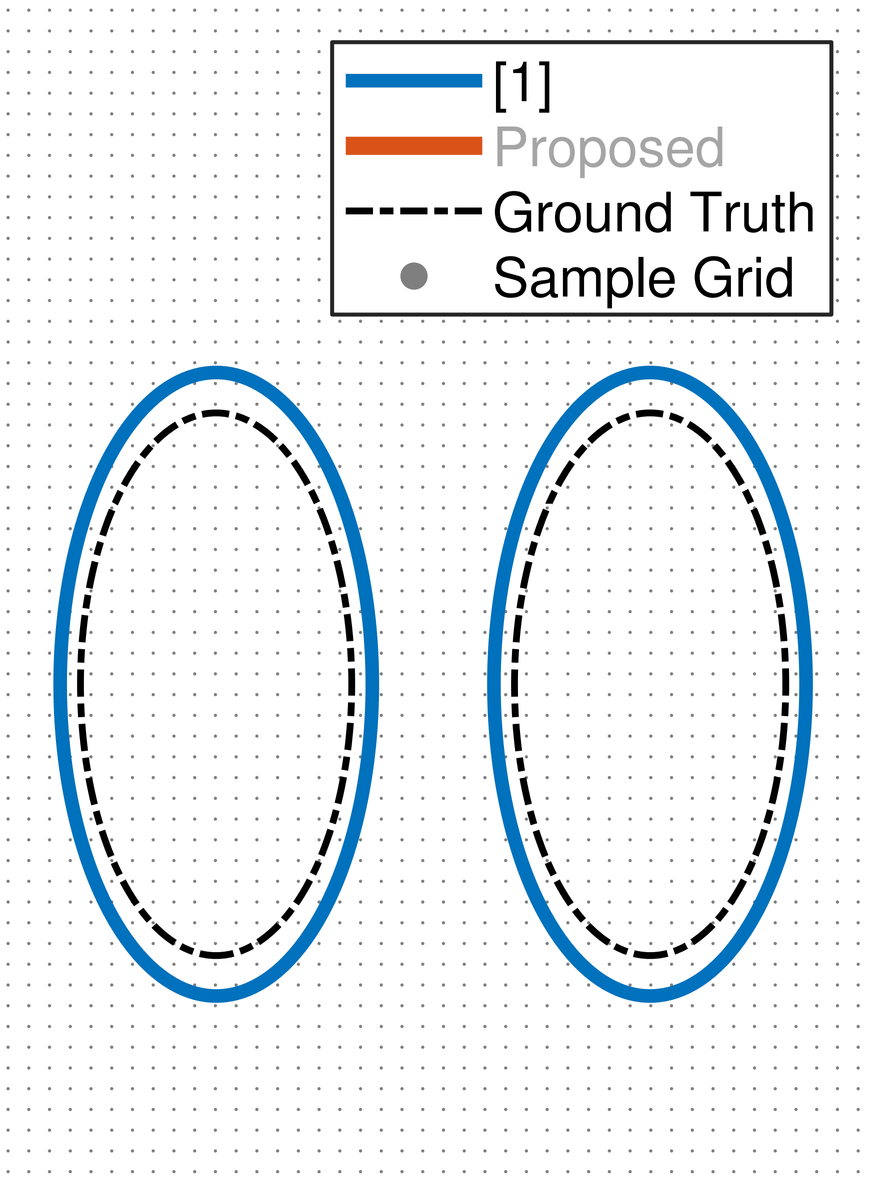

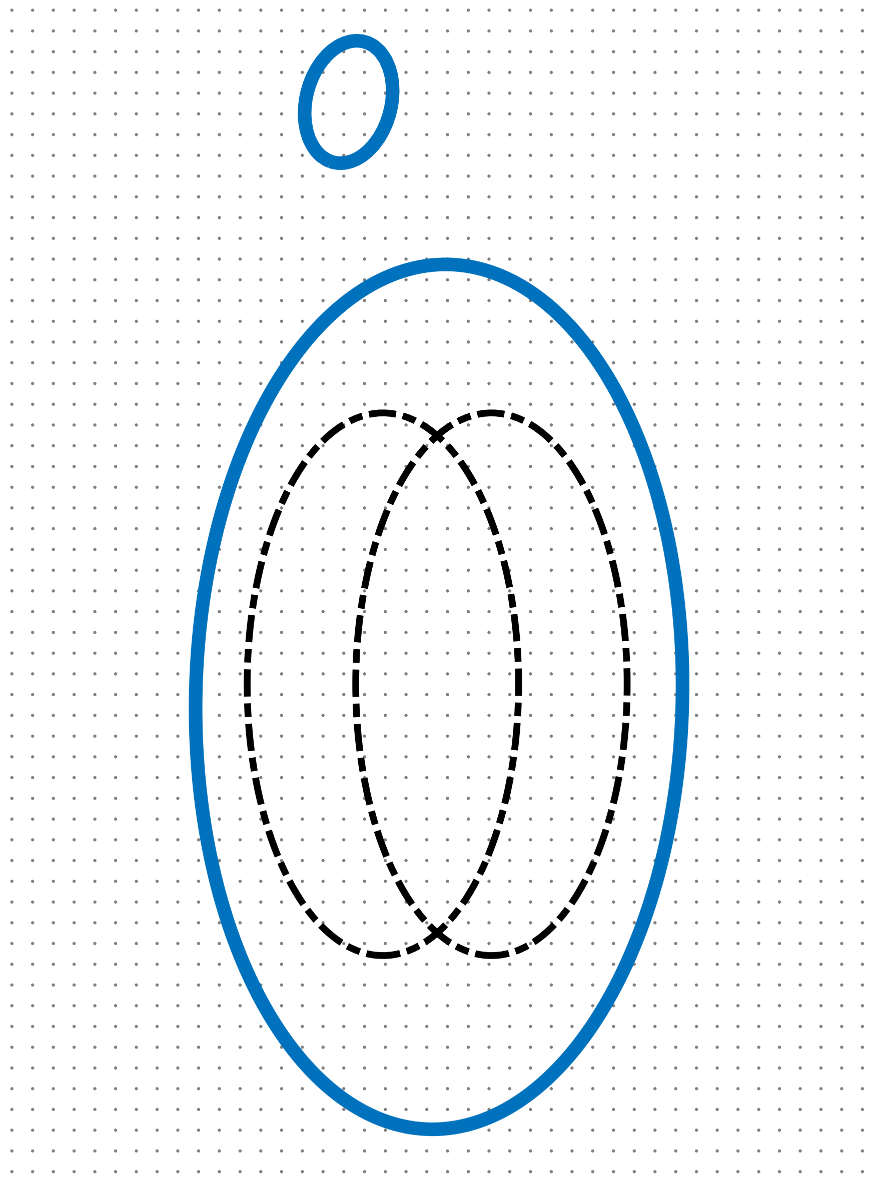

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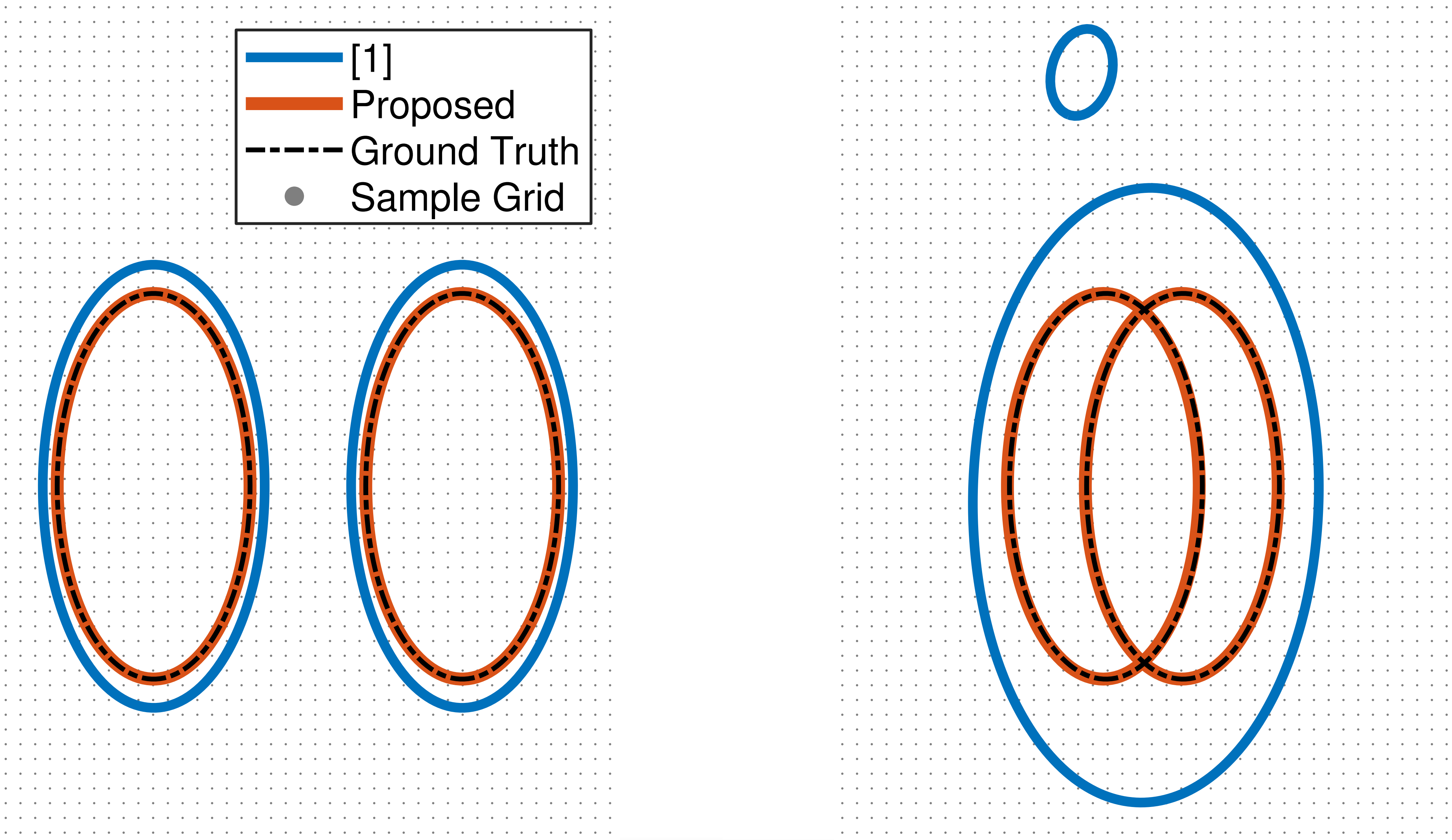

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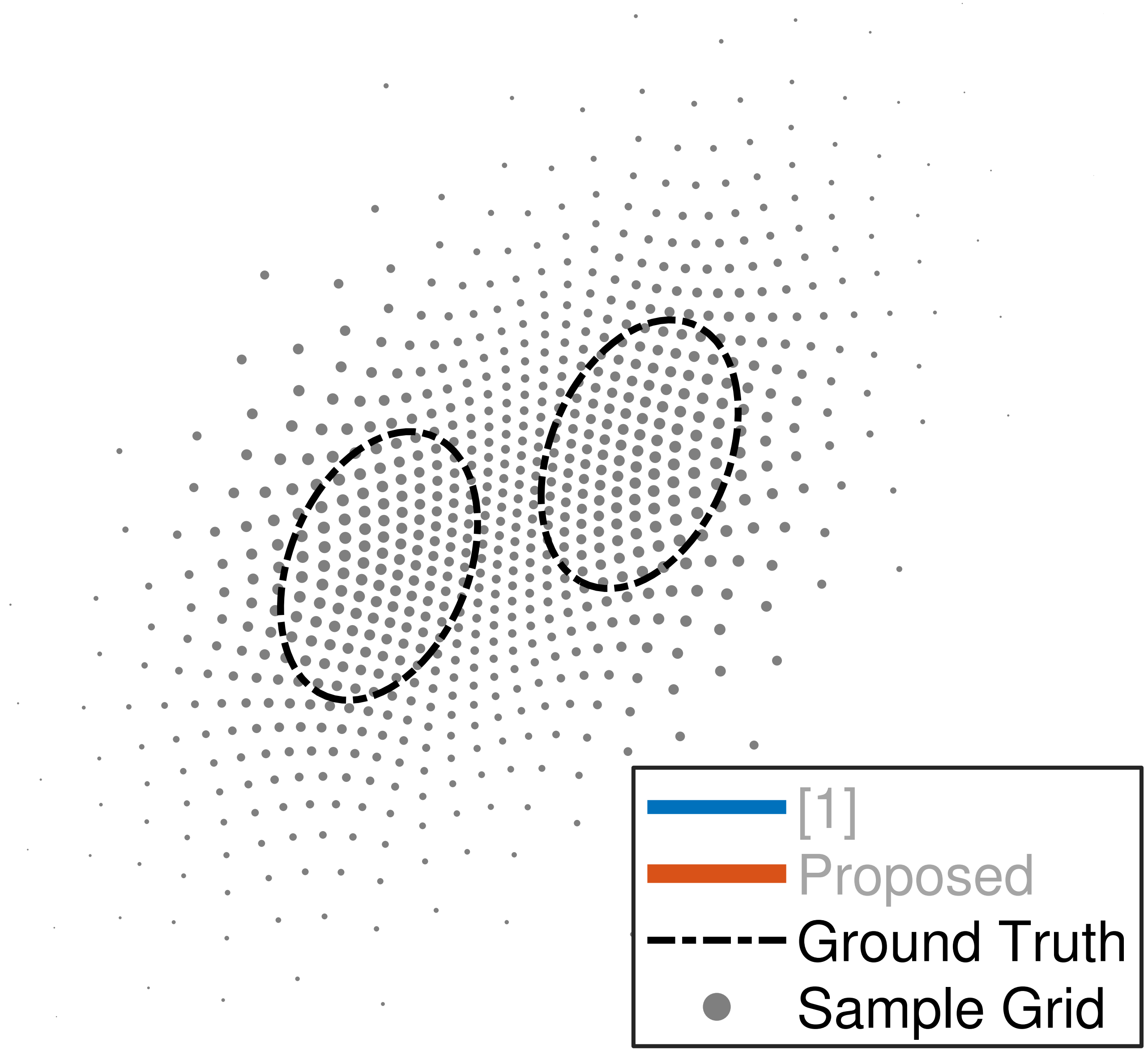

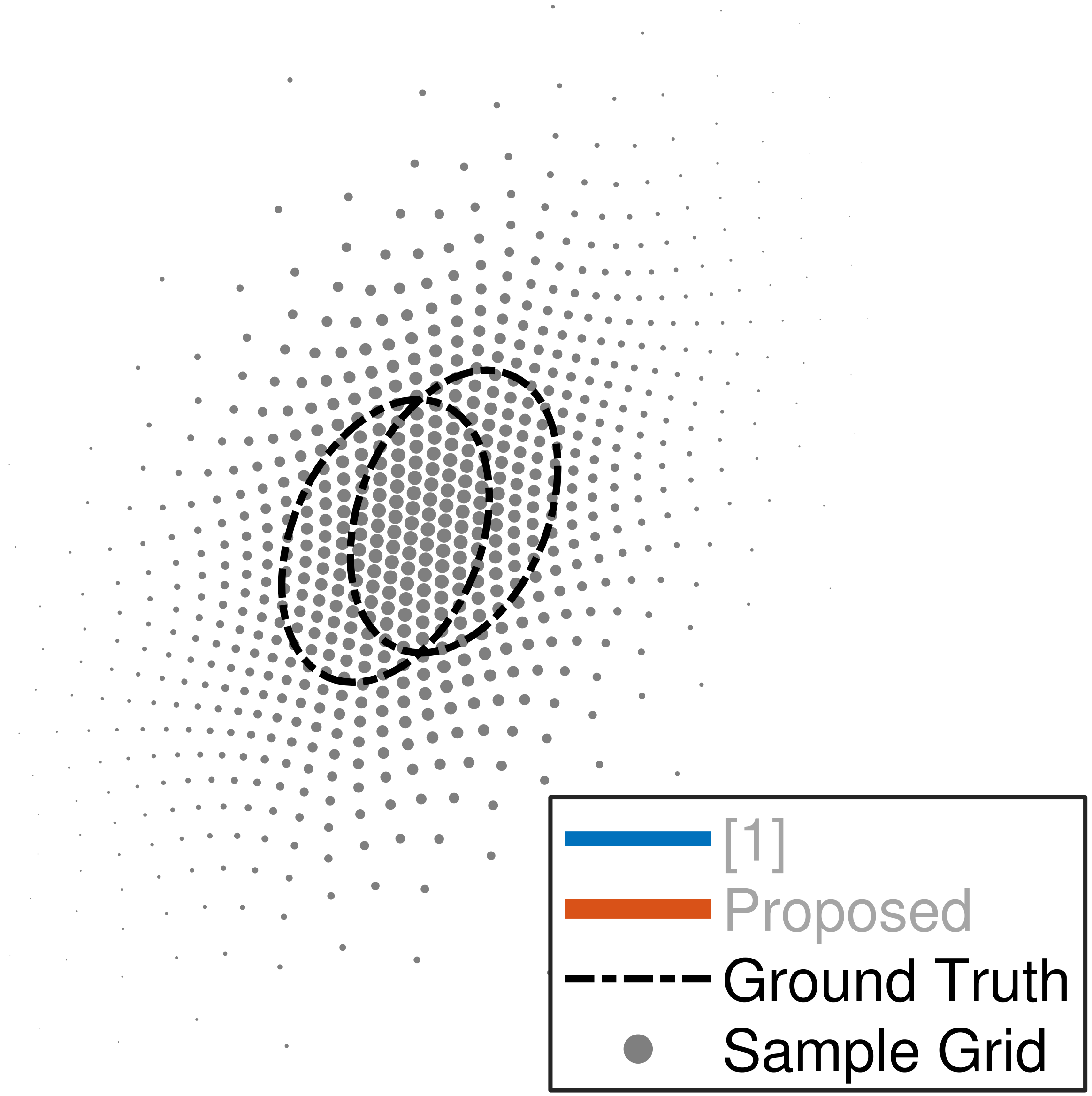

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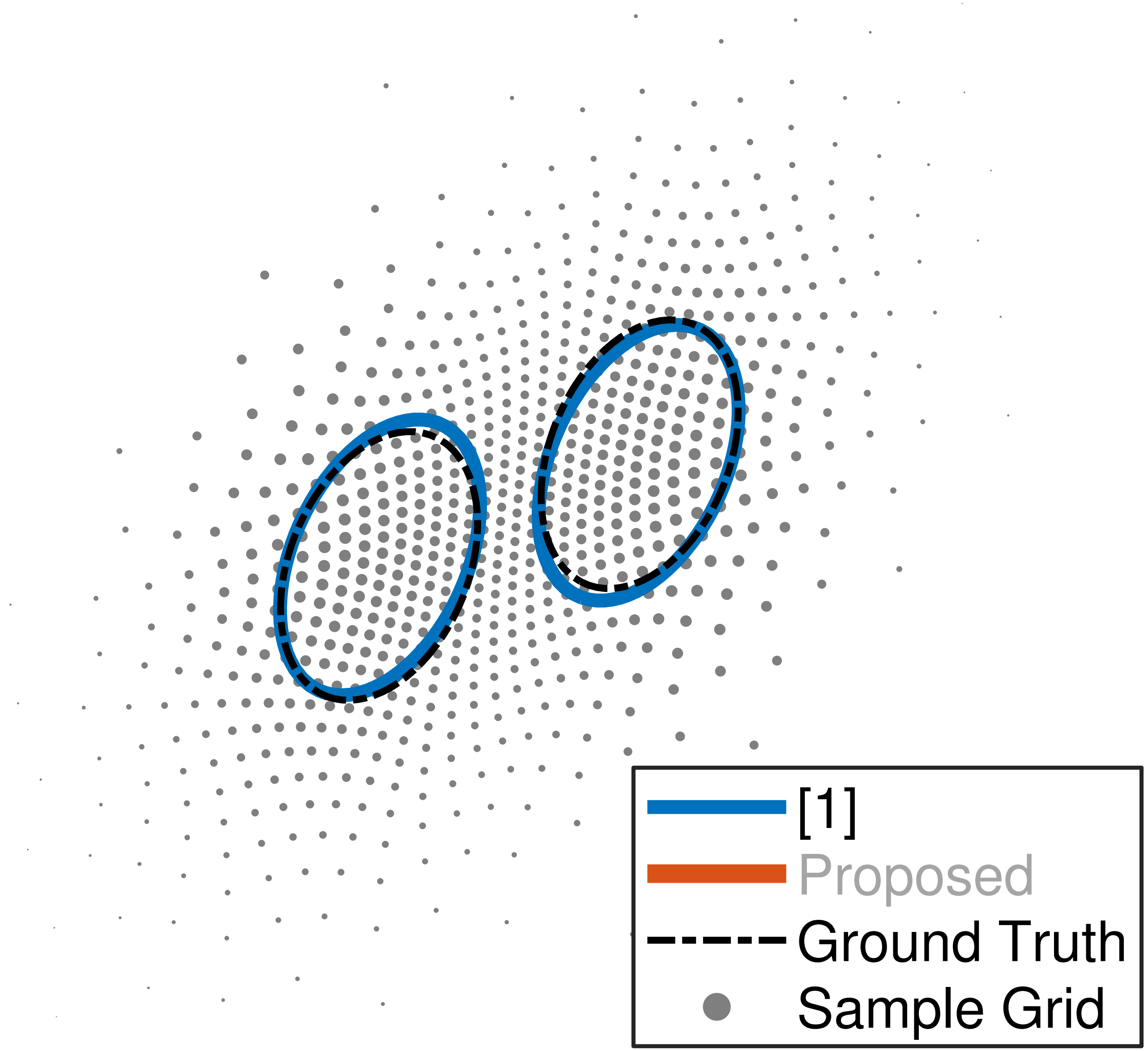

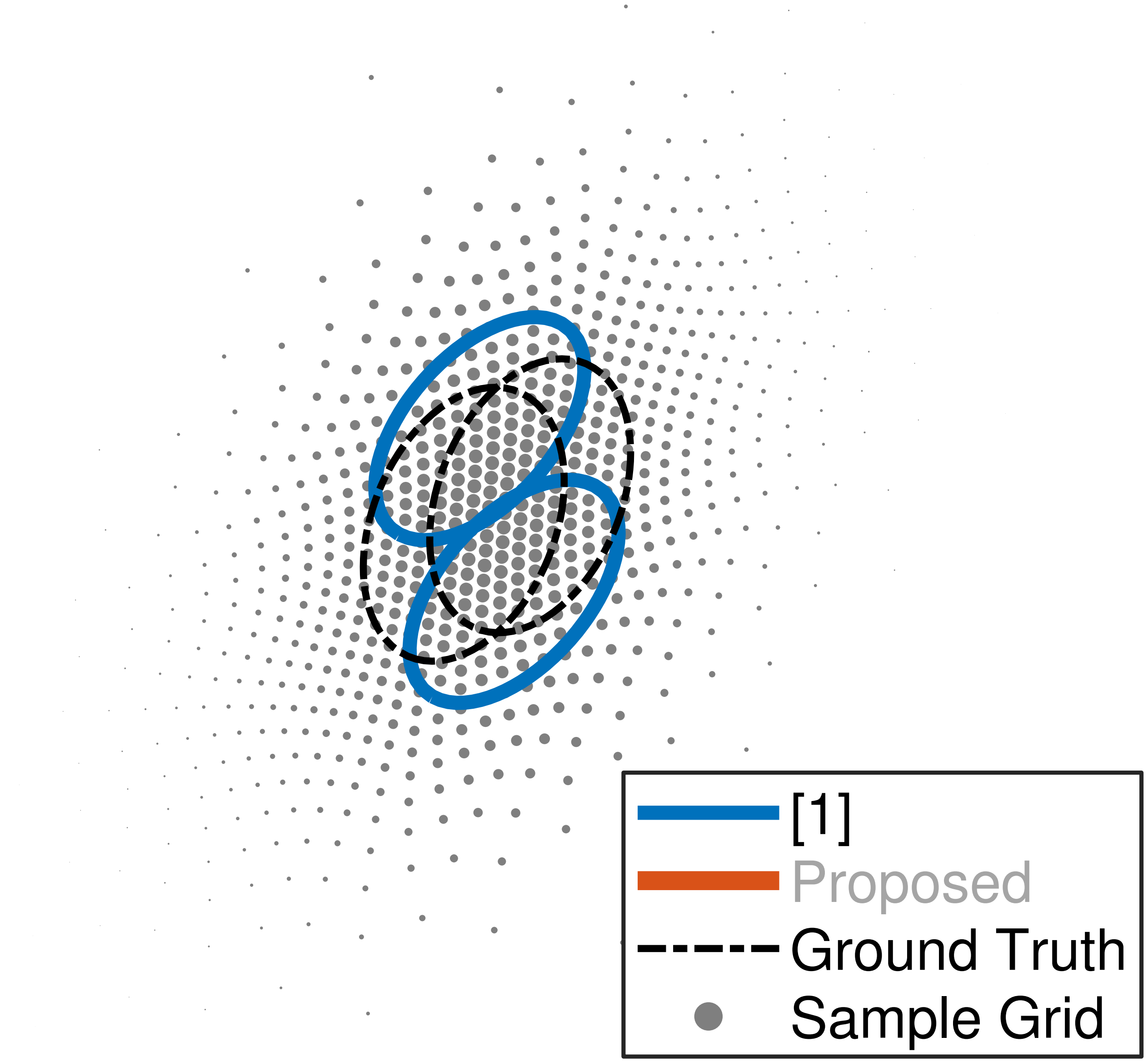

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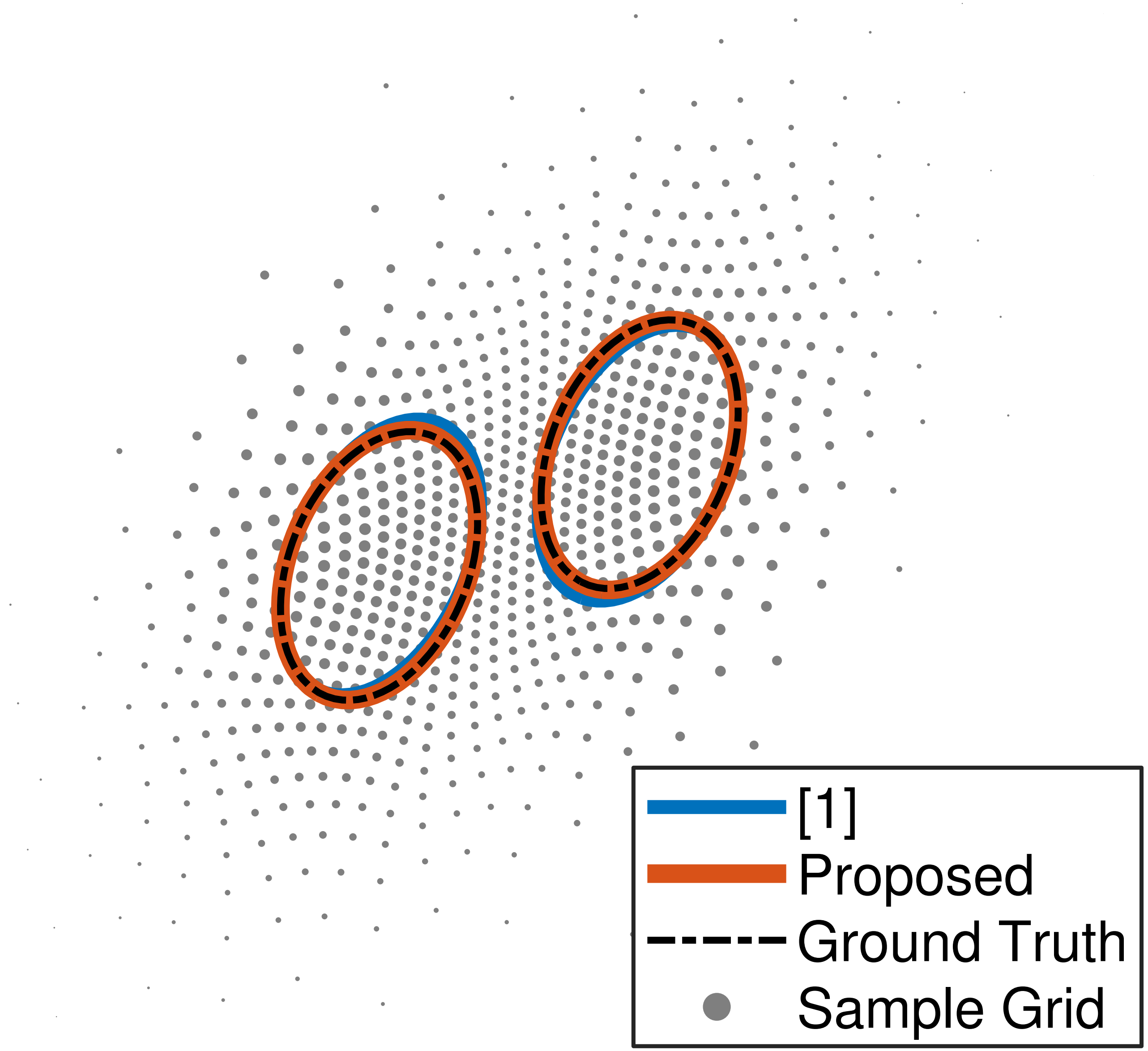

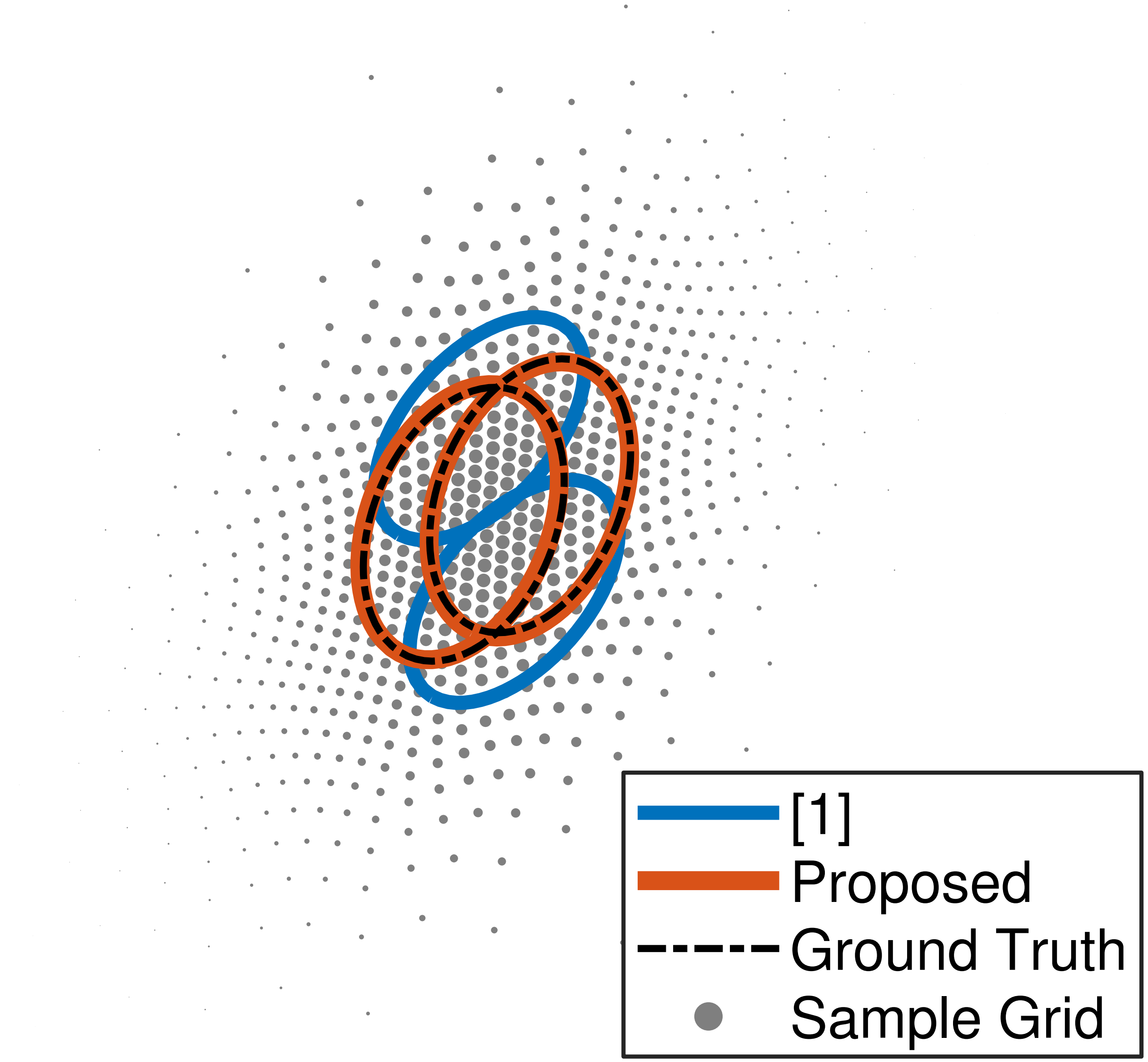

Conclusion

Achieved

- GM Estimation
- Correct treatment of sample weights

Future work

- Number of Gaussian components automatically
- Initial guess from weighted k-means

Thank you for your attention

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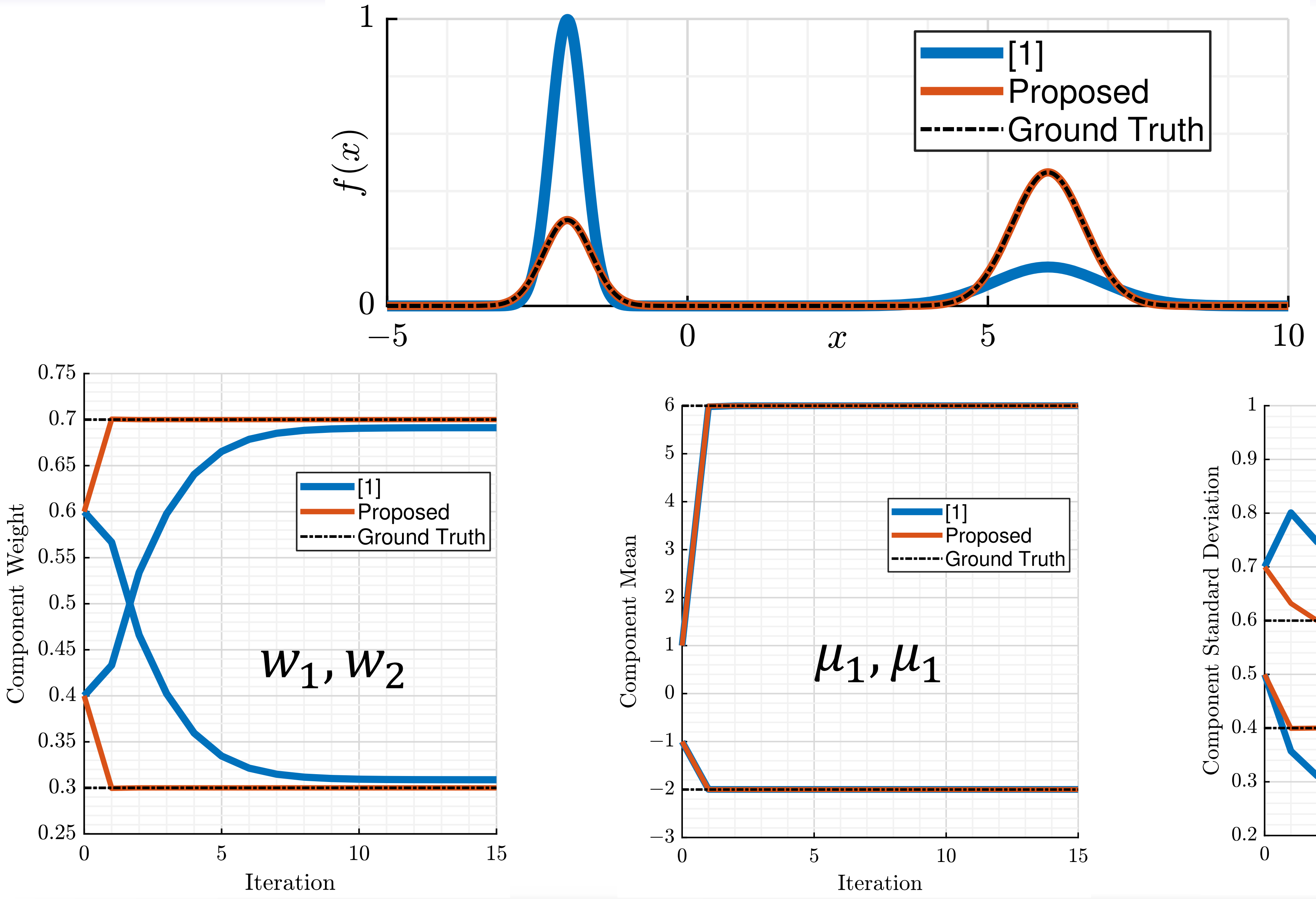
1D Evaluation – Separated



1D Evaluation – Overlapping