Deterministic Gaussian Sampling With Generalized Fibonacci Grids

Daniel Frisch and Uwe D. Hanebeck

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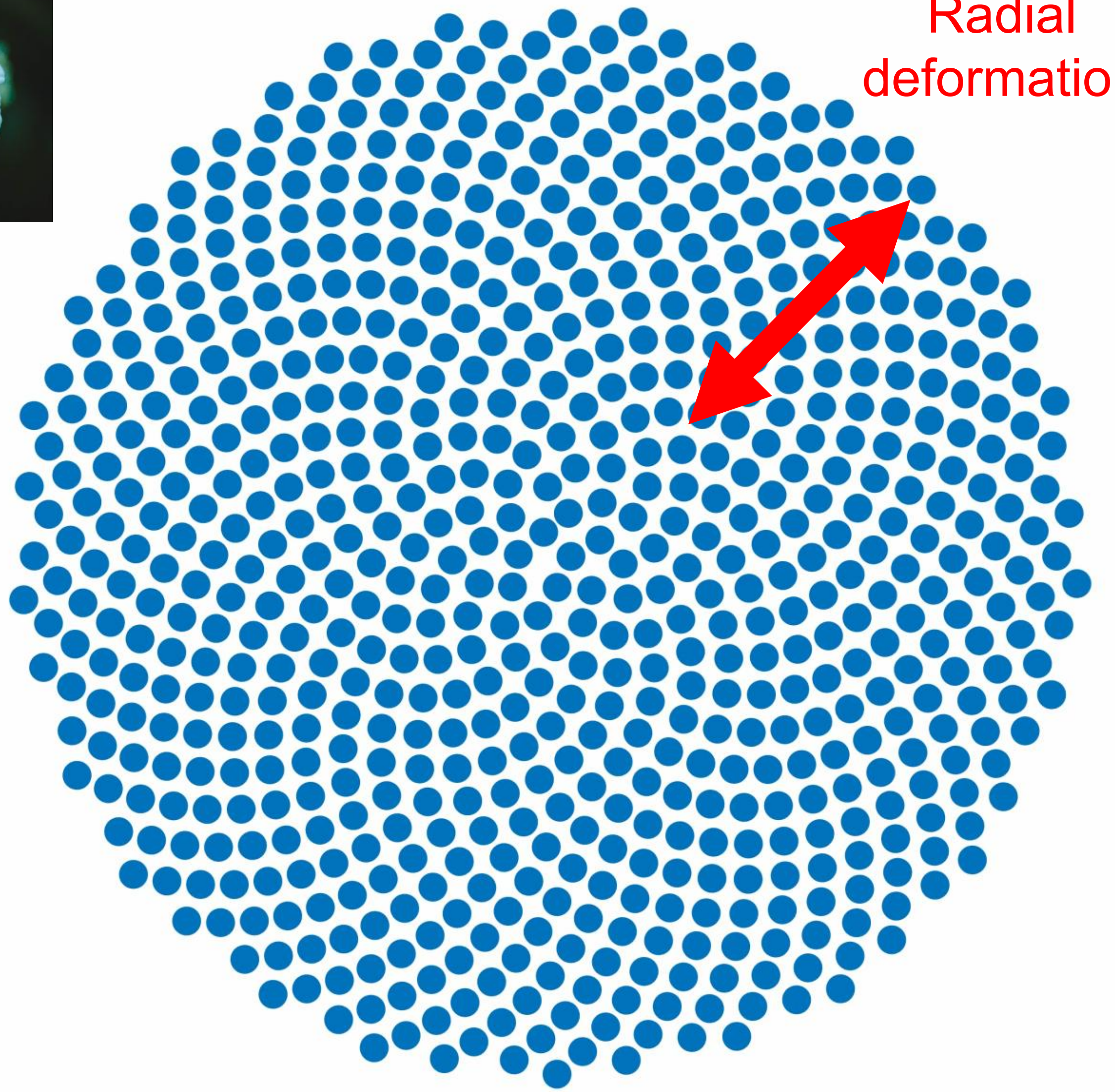
Intelligent Sensor-Actuator-Systems Laboratory (ISAS)
Institute for Anthropomatics and Robotics
Karlsruhe Institute of Technology (KIT)
Karlsruhe, Germany

Karlsruhe Institute of Technology

https://isas.iar.kit.edu

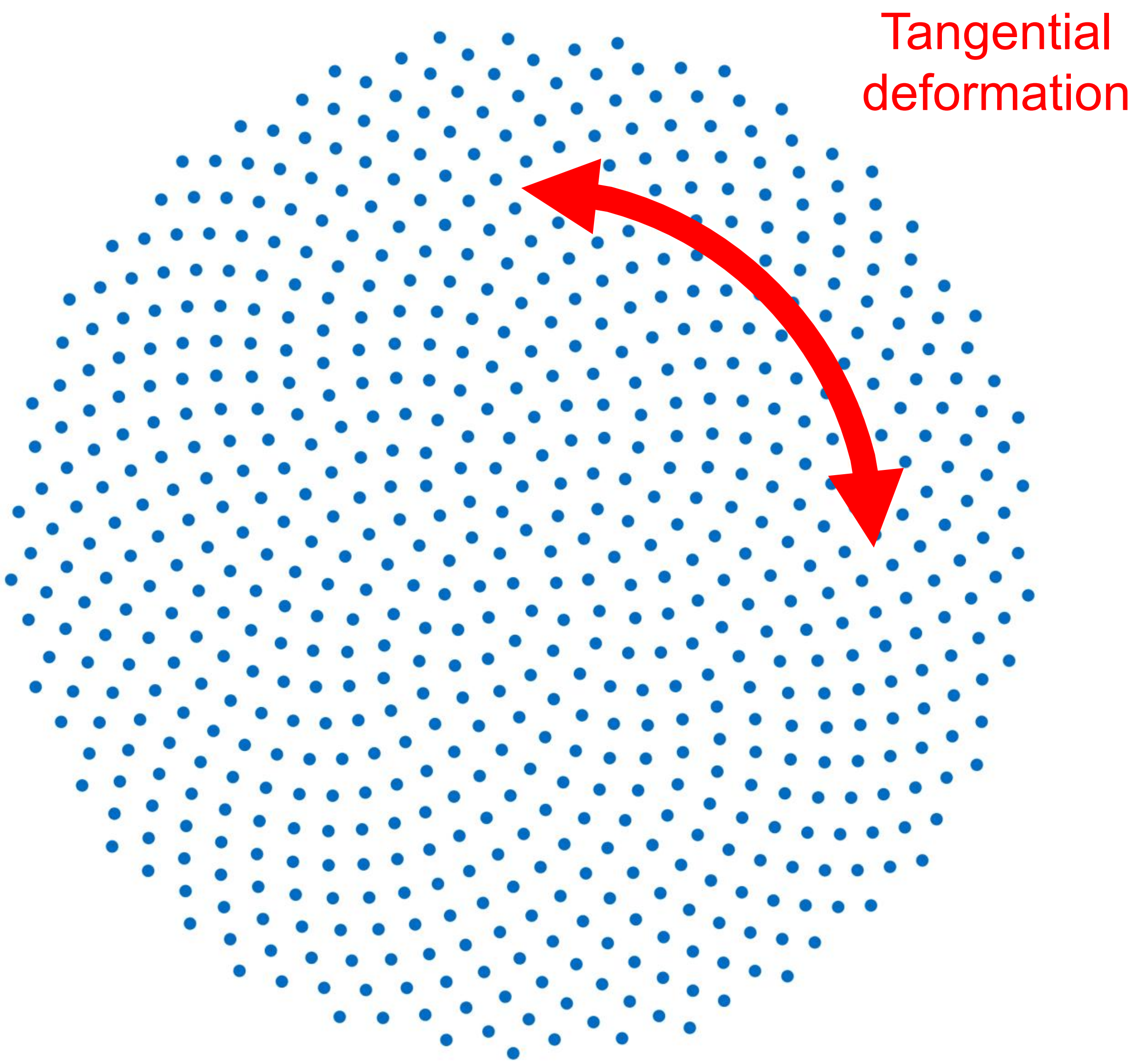
Intelligent Sensor-Actuator-Systems

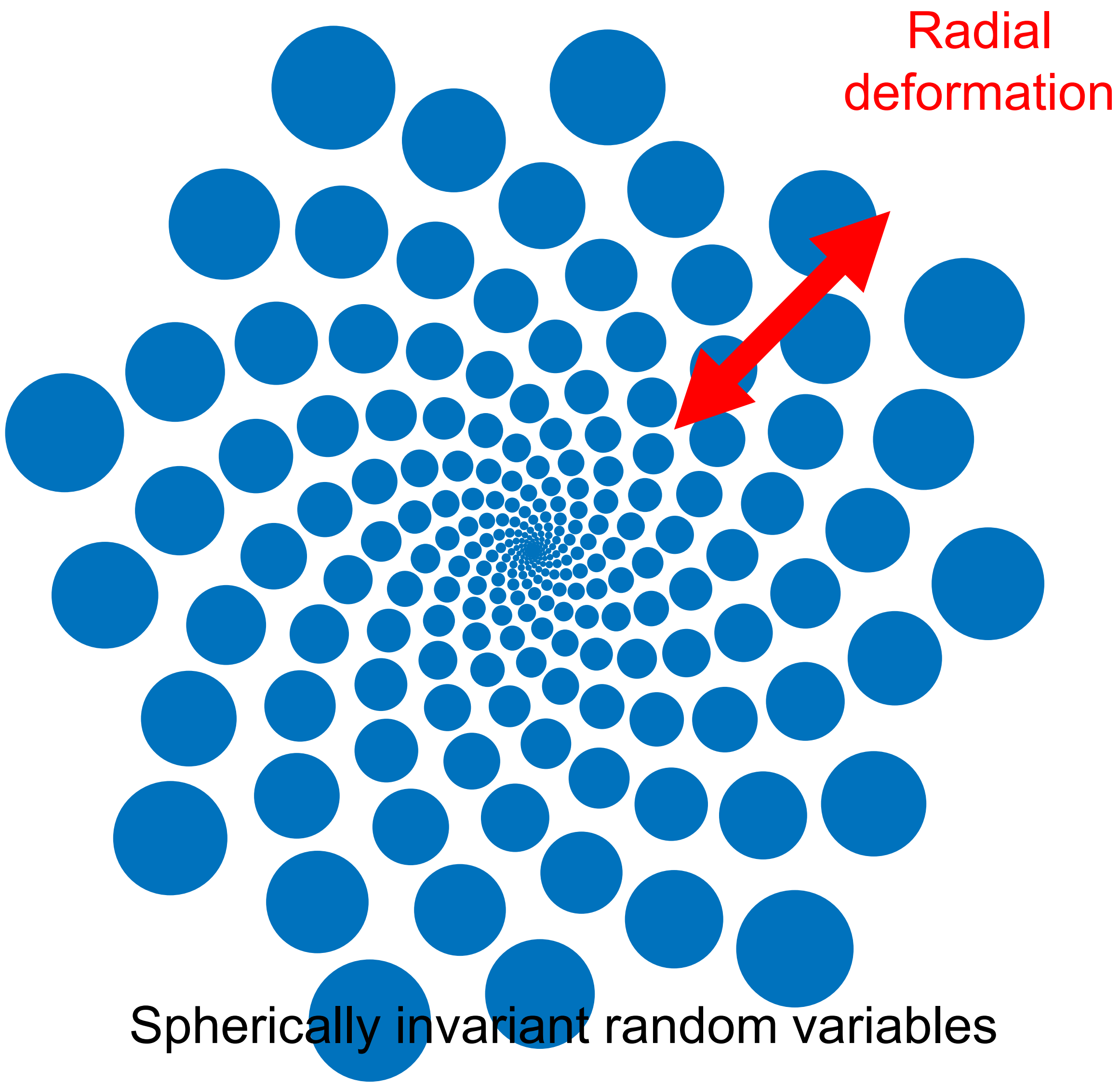
Polar Fibonacci Grid



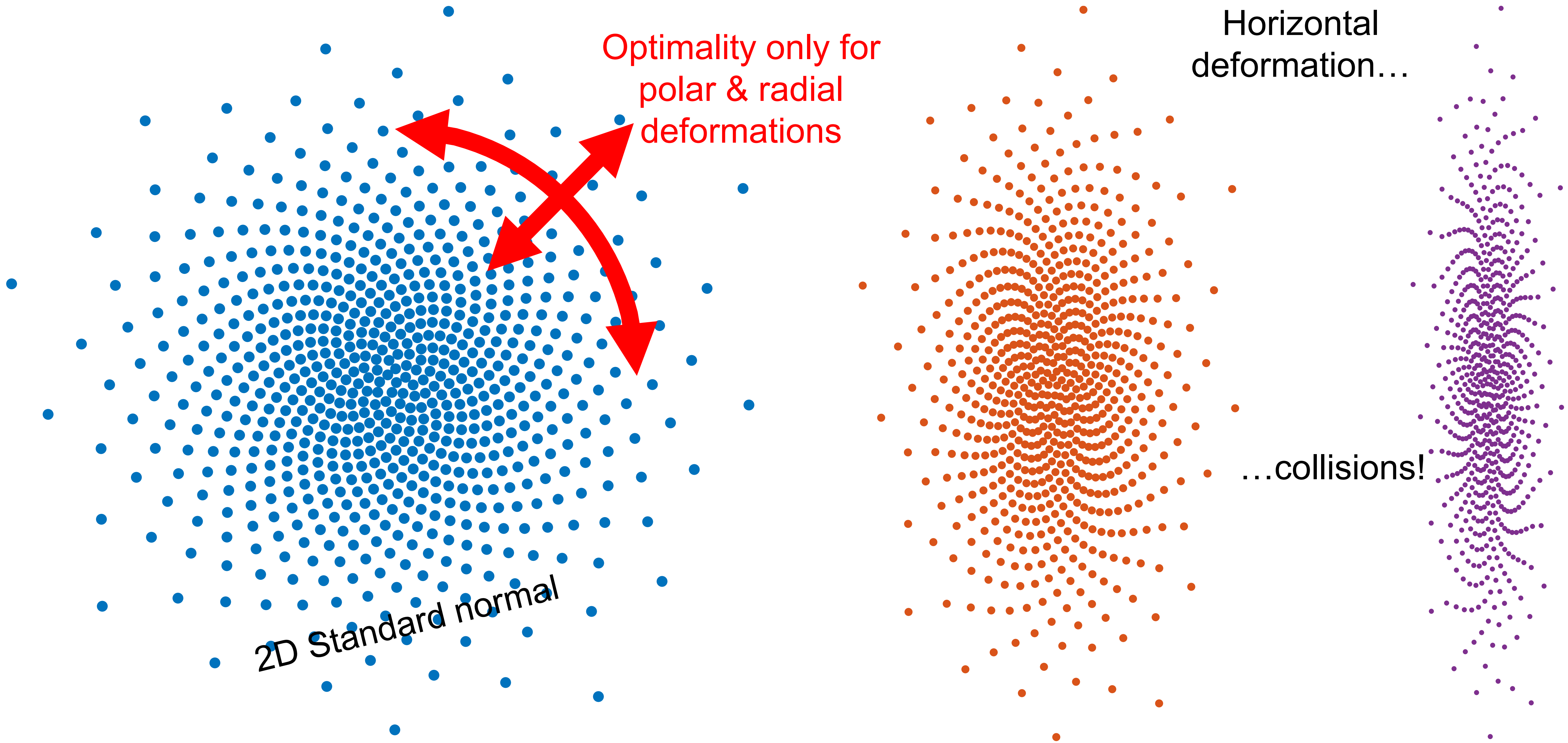
Radial deformation

Polar Fibonacci Grid

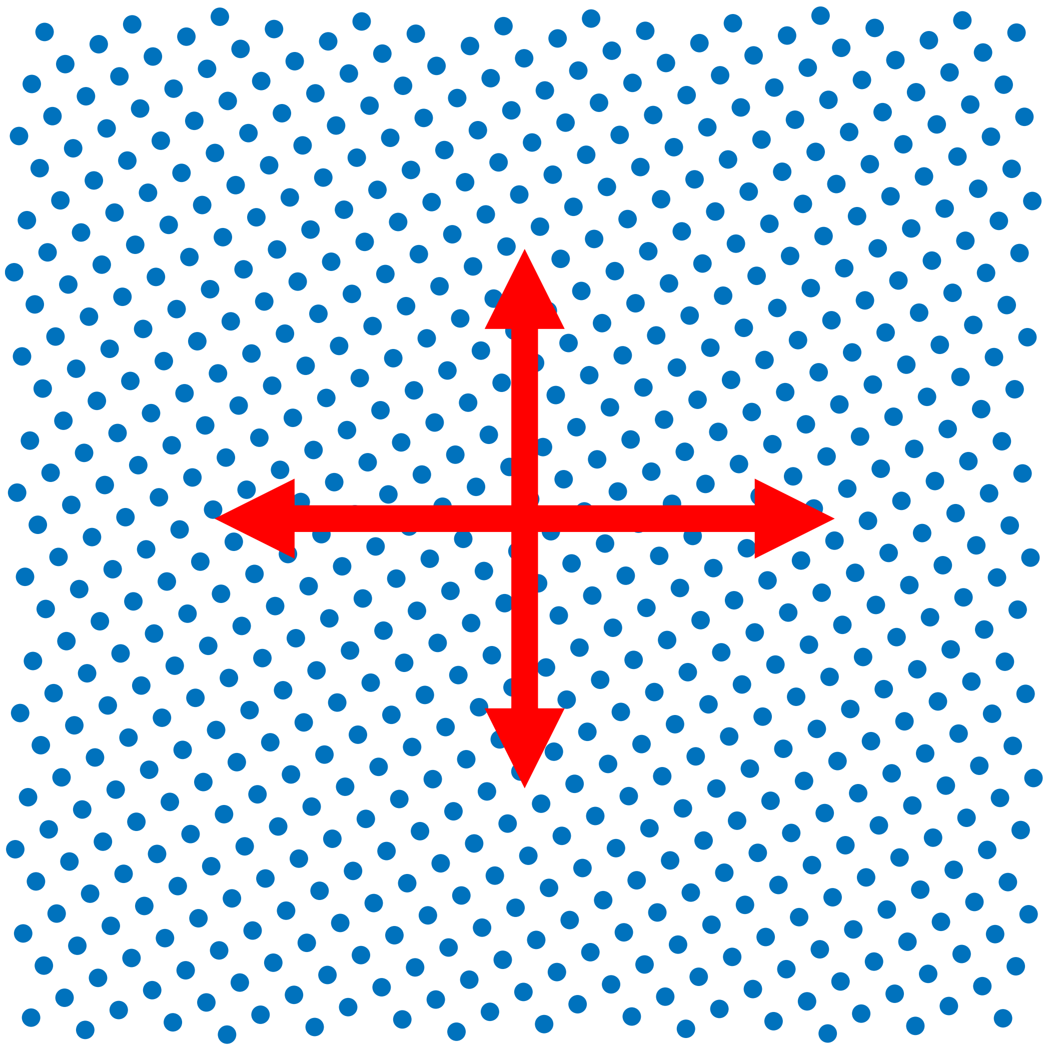


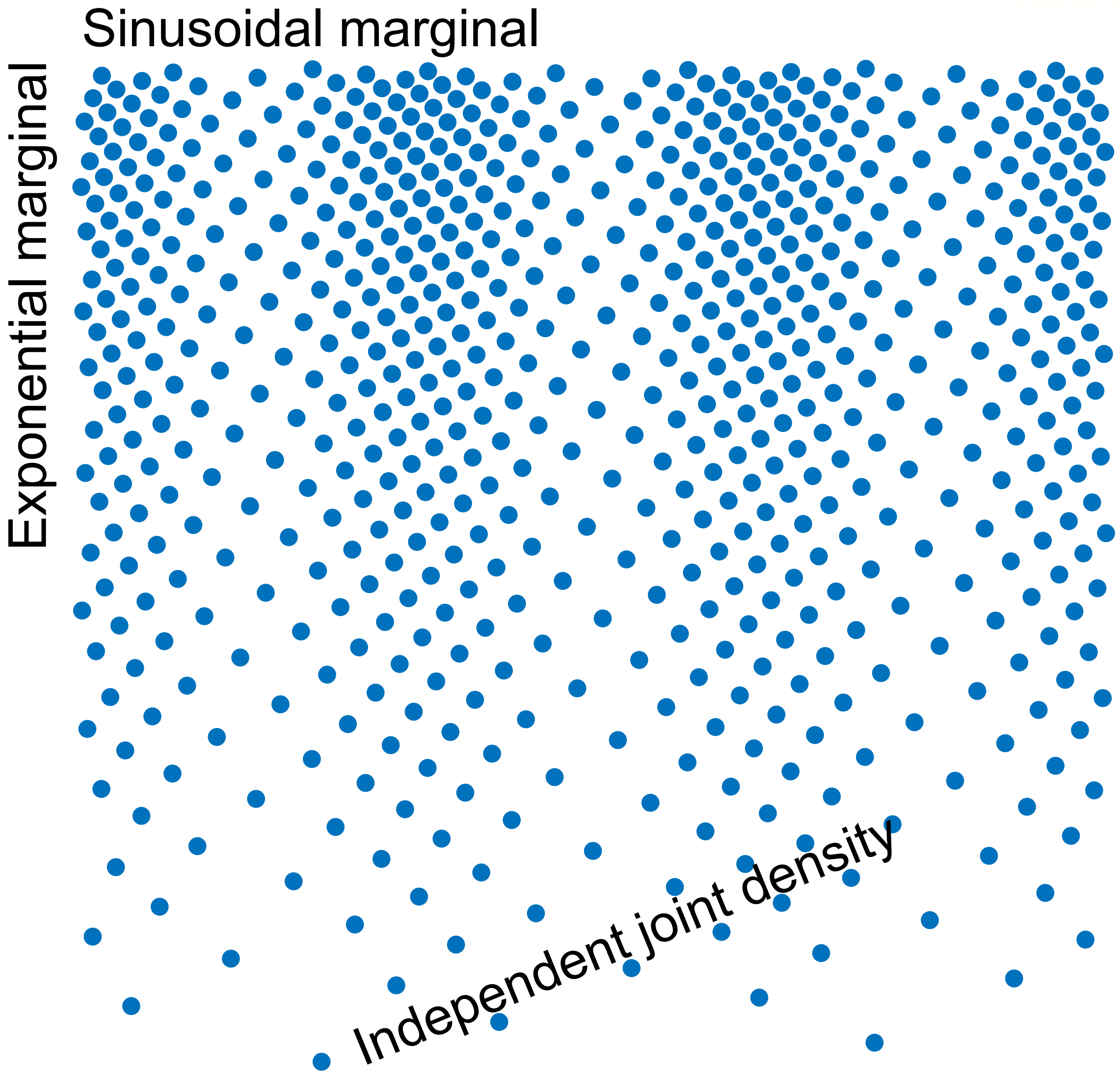


Polar Fibonacci Grid → Gaussian

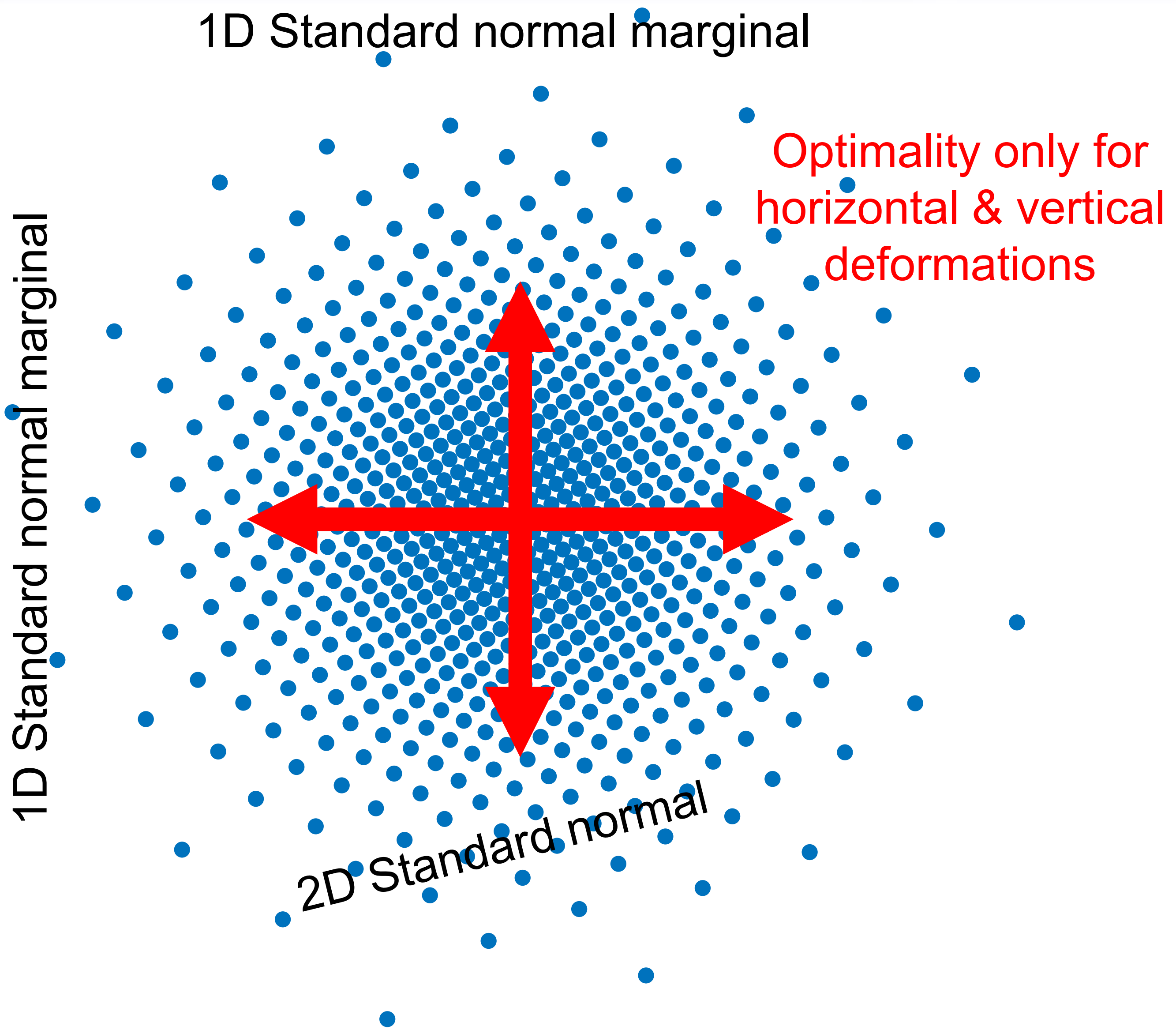


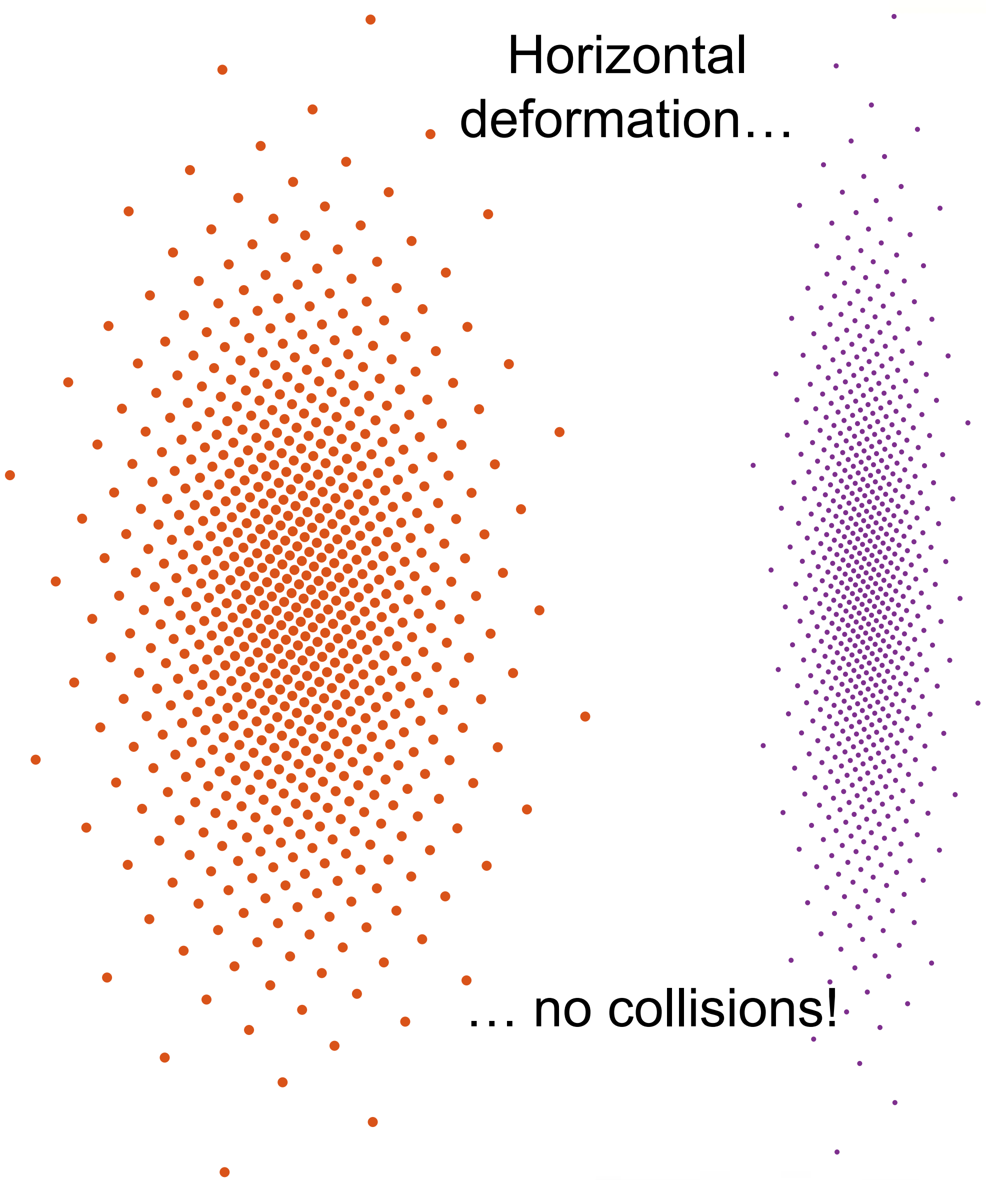
Cartesian Fibonacci Grid





Cartesian Fibonacci Grid → Gaussian

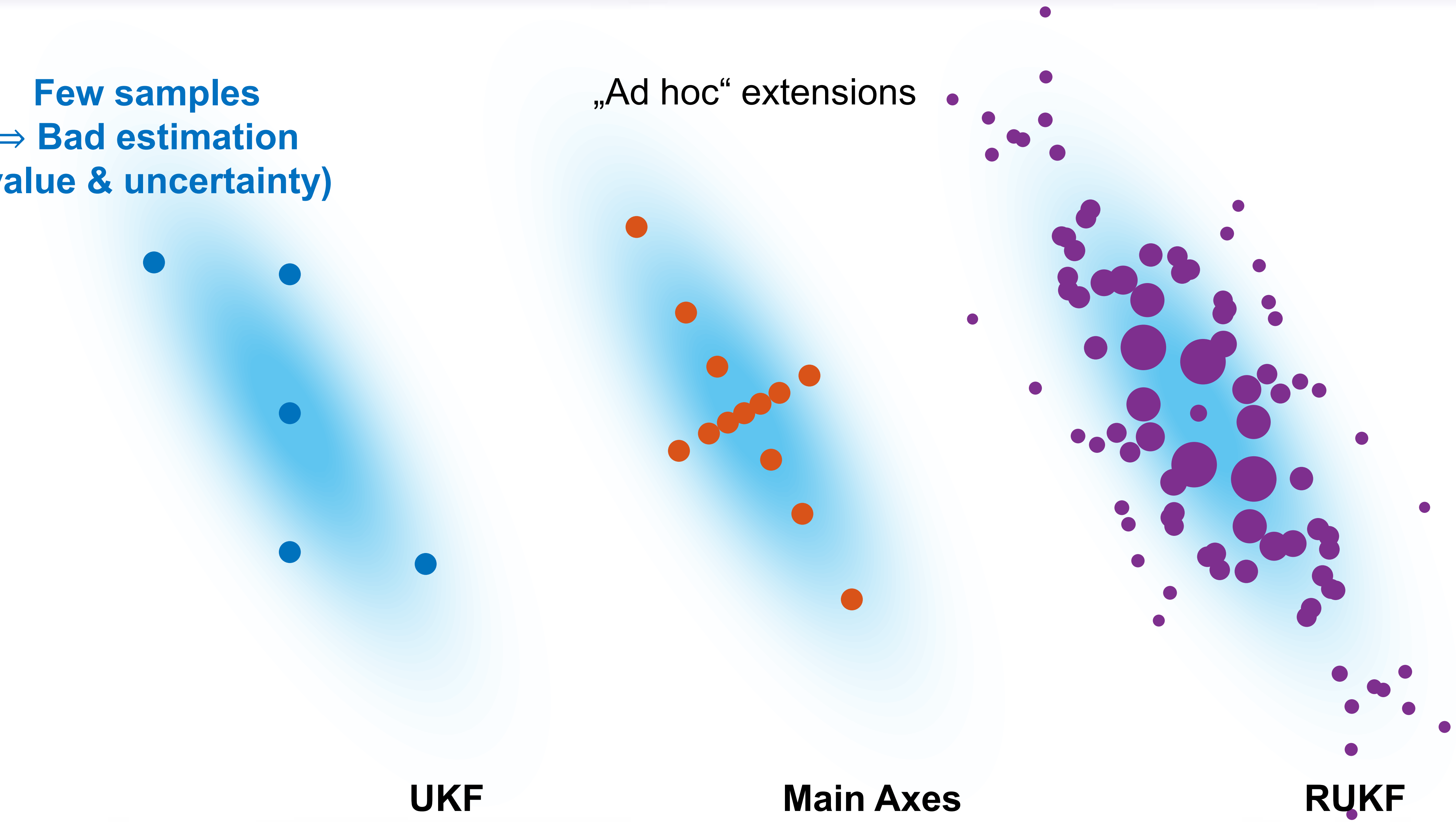




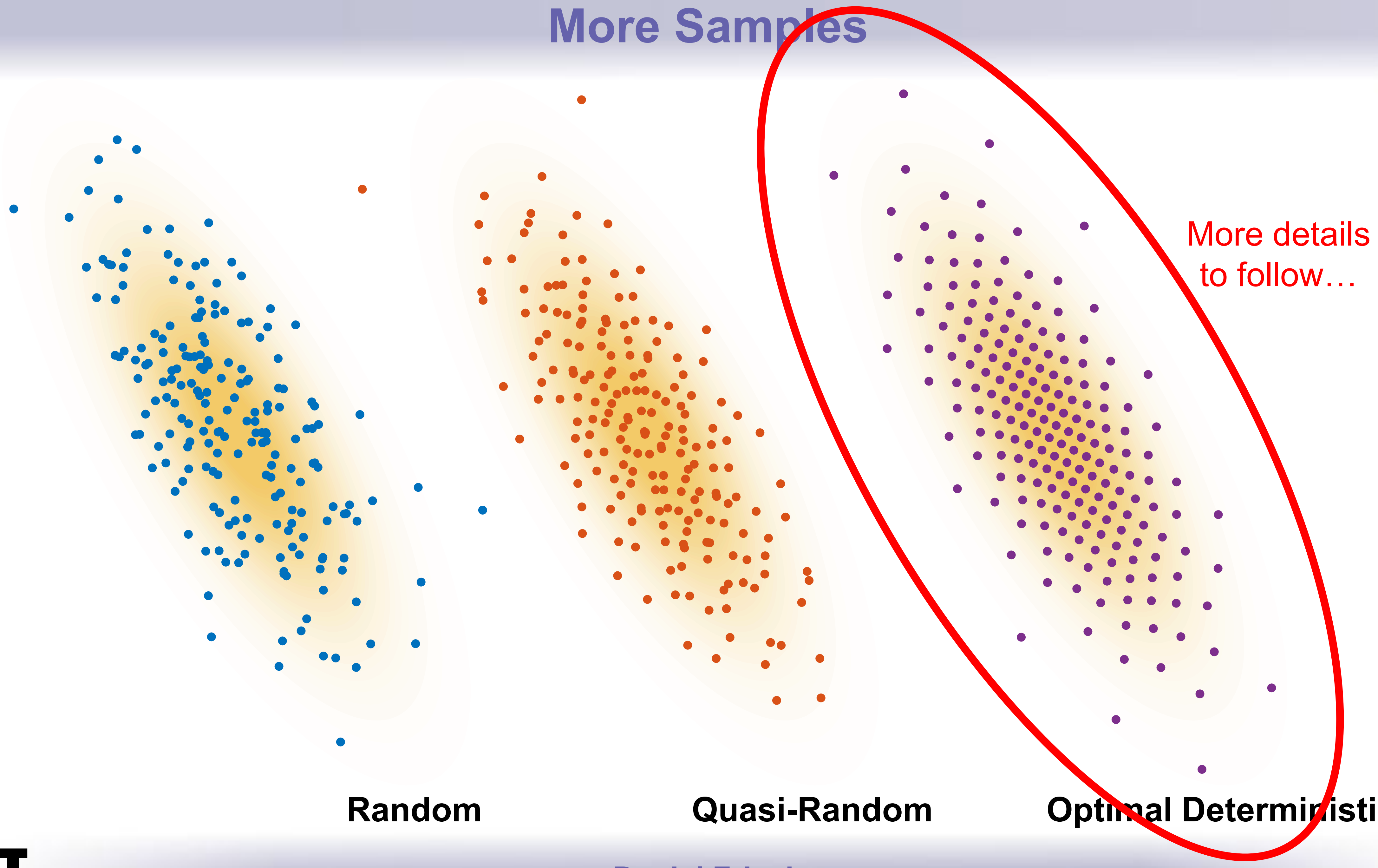
UKF

Few samples
⇒ Bad estimation
(value & uncertainty)

„Ad hoc“ extensions

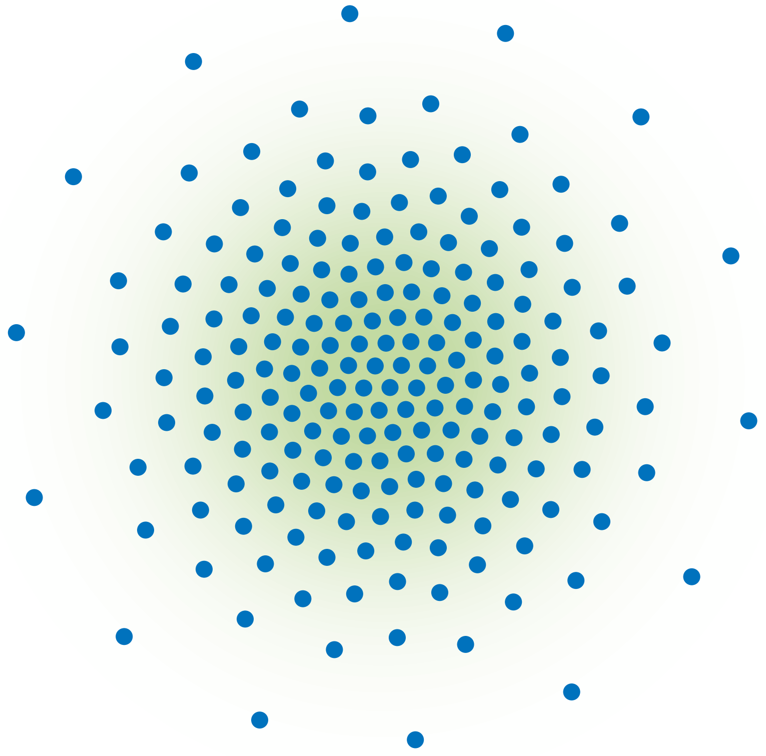


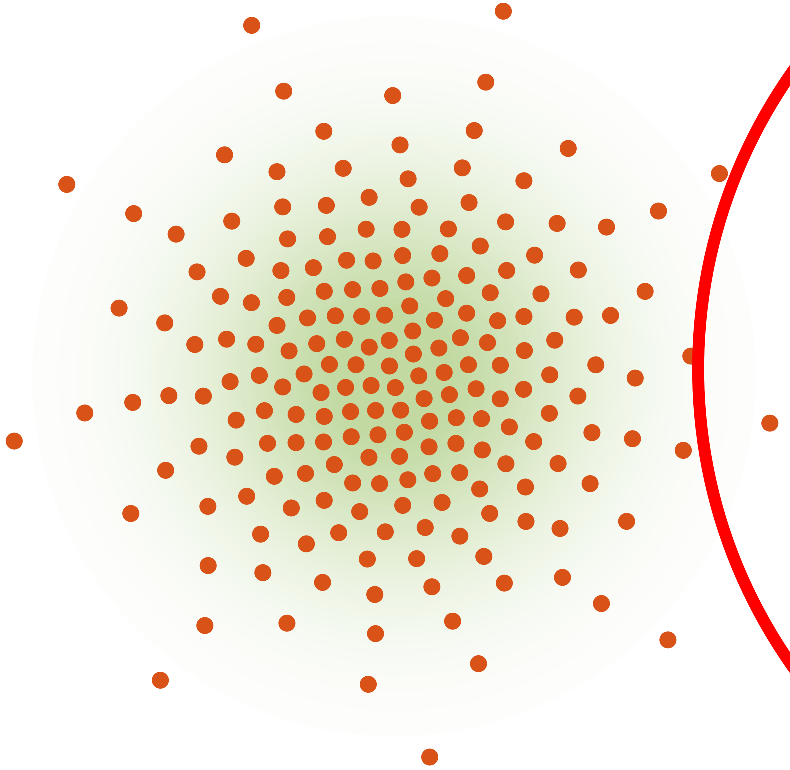
More Samples



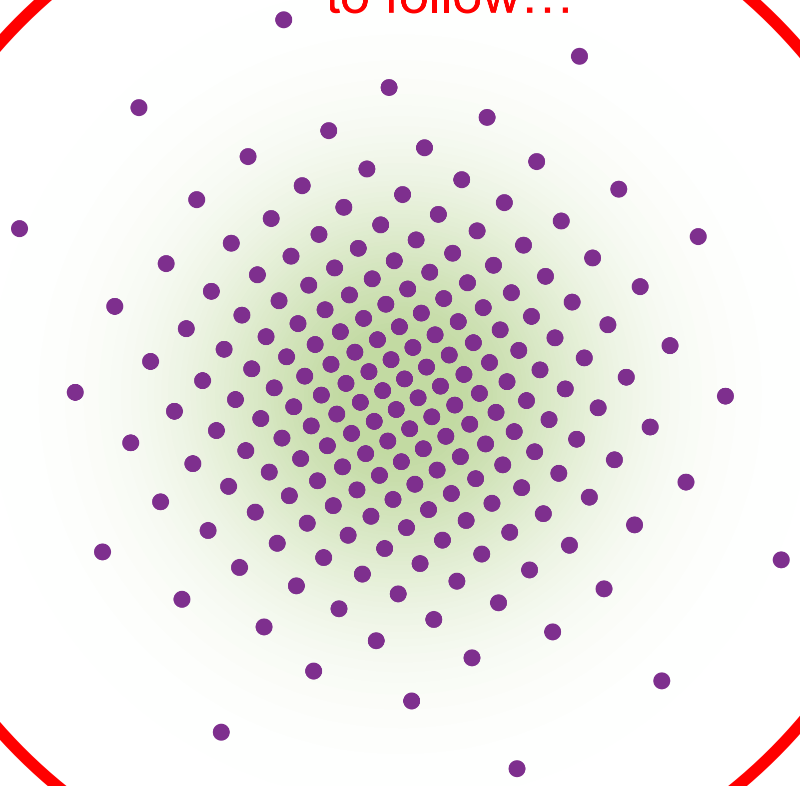
Random Quasi-Random Optimal Deterministic

Optimal Deterministic Sampling





More details to follow…



Localized Cumulative Distribution (LCD)

Projected Cumulative Distribution (PCD)

Fibonacci

Fibonacci Dimensions – State of Art

„Fibonacci Grid only for 2D"

- In the case of more than two dimensions no recipe other than trial and error is known for finding good lattice points.
  [1969 Zaremba: "A Remarkable Lattice Generated by Fibonacci Numbers"]
- In any case, explicit constructions analogous to the Fibonacci lattice are not known in higher dimensions
  [1994 Niederreiter, Sloan: "Integration of nonperiodic functions of two variables by Fibonacci lattice rules"]
- … this lattice rule has proven optimal discrepancy in two dimensions only
  [2011 Schretter, Kobbelt: "Golden Ratio Sequences For Low-Discrepancy Sampling"]
- Fibonacci lattices in two dimensions have a certain optimality property, but there is no obvious generalization to higher dimensions that retains the optimality property.
  [2013 Dick, Kuo, Sloan: "High-dimensional integration: The quasi-Mote Carlo way"]

Purser's Generalized Fibonacci Grid

Fibonacci-type Grid for higher dimensions!

- A system of $D - 1$ mutually commuting and multiplicatively independent integer-component symmetric matrices with shared eigenvectors aligned obliquely to the basic Cartesian directions guarantees the existence of an associated 'Fibonacci'-type grid.

[2008 R. James Purser: „Generalized Fibonacci Grids; A New Class of Structured, Smoothly Adaptive Multi-Dimensional Computational Lattices"]

Fibonacci Matrix

Linear system generating Fibonacci numbers

$$\underline{x}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \underline{x}_k \,, \qquad \underline{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

2D $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **3D**

Overview

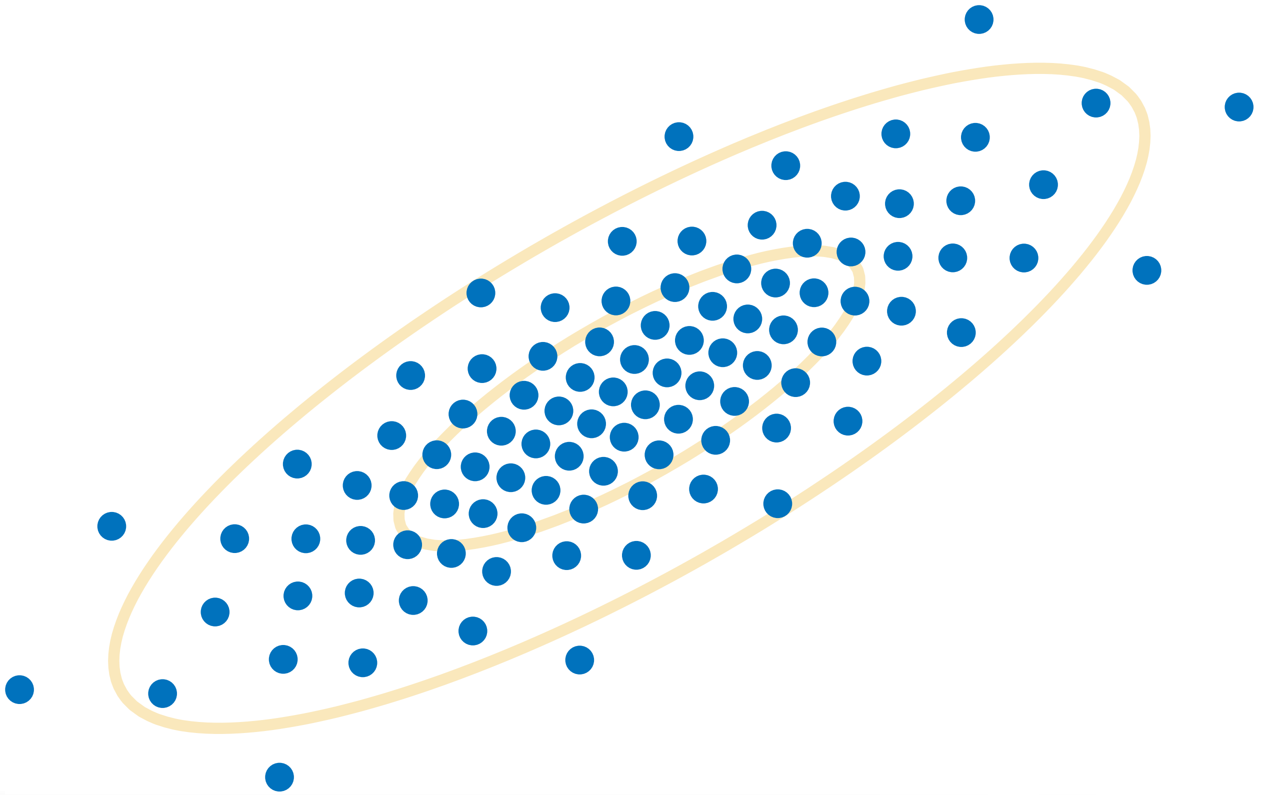
Input

- No. Samples
- Dimension
- Covariance matrix

Deterministic Sampling

Output

- Sample locations



Notation

Input

- No. Samples : L
- Dimension : D
- Covariance matrix : $\mathbf{C}$

Deterministic Sampling

$\underline{x} \in \mathbb{R}^D$

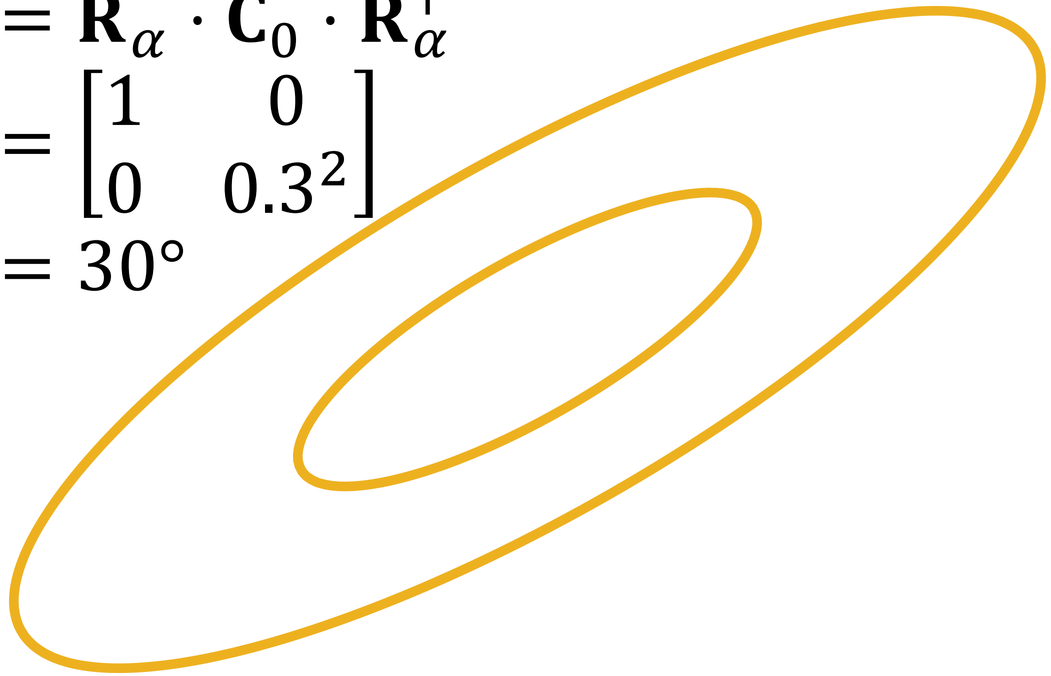
Output

- Sample locations :
  $\underline{x}_i$, $\mathbf{X}_{\text{Gauss}}$

$$\mathbf{C} = \mathbf{R}_\alpha \cdot \mathbf{C}_0 \cdot \mathbf{R}_\alpha^\top$$

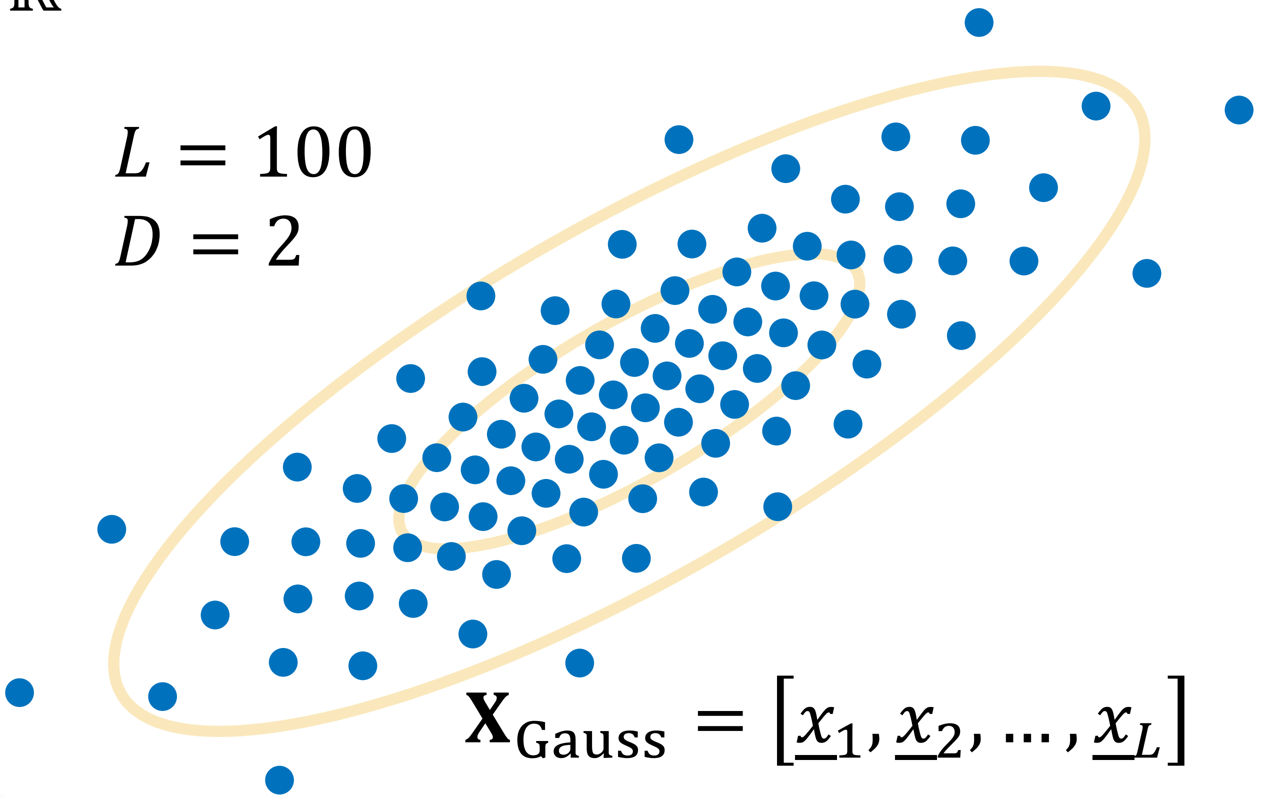
$$\mathbf{C}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0.3^2 \end{bmatrix}$$

$$\alpha = 30°$$



$$L = 100$$

$$D = 2$$



$$\mathbf{X}_{\text{Gauss}} = [\underline{x}_1, \underline{x}_2, \dots, \underline{x}_L]$$

3D Projection

Input
- No. Samples : L
- Dimension : D
- Covariance matrix : $\mathbf{C}$

→ **Deterministic Sampling** →

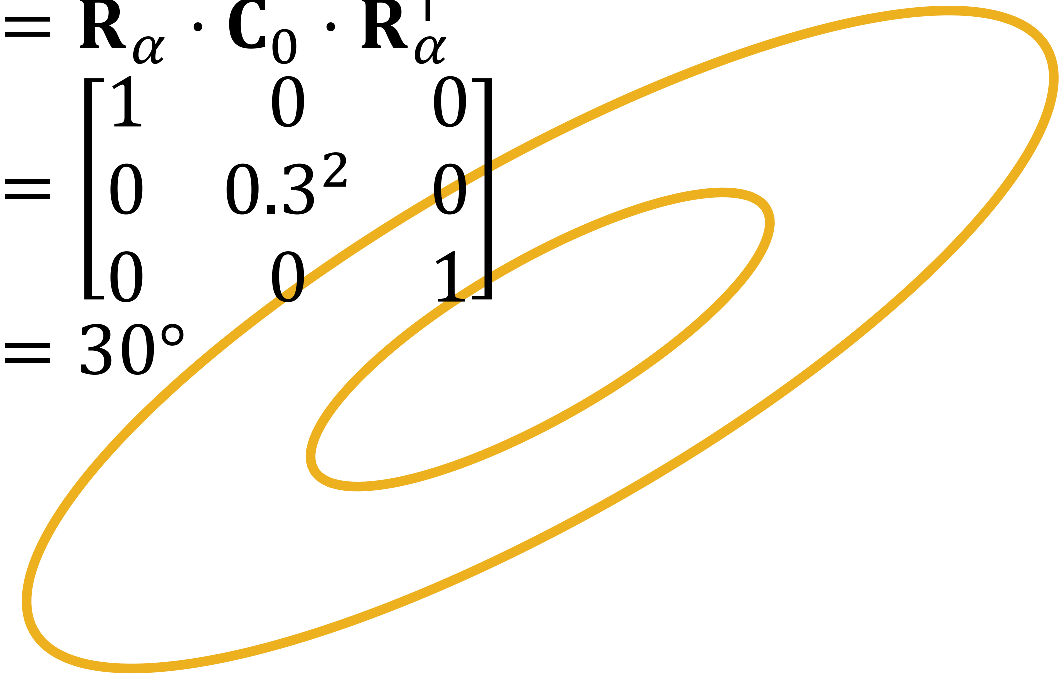
$\underline{x} \in \mathbb{R}^D$

Output
- Sample locations :
  $\underline{x}_i$, $\mathbf{X}_{\text{Gauss}}$

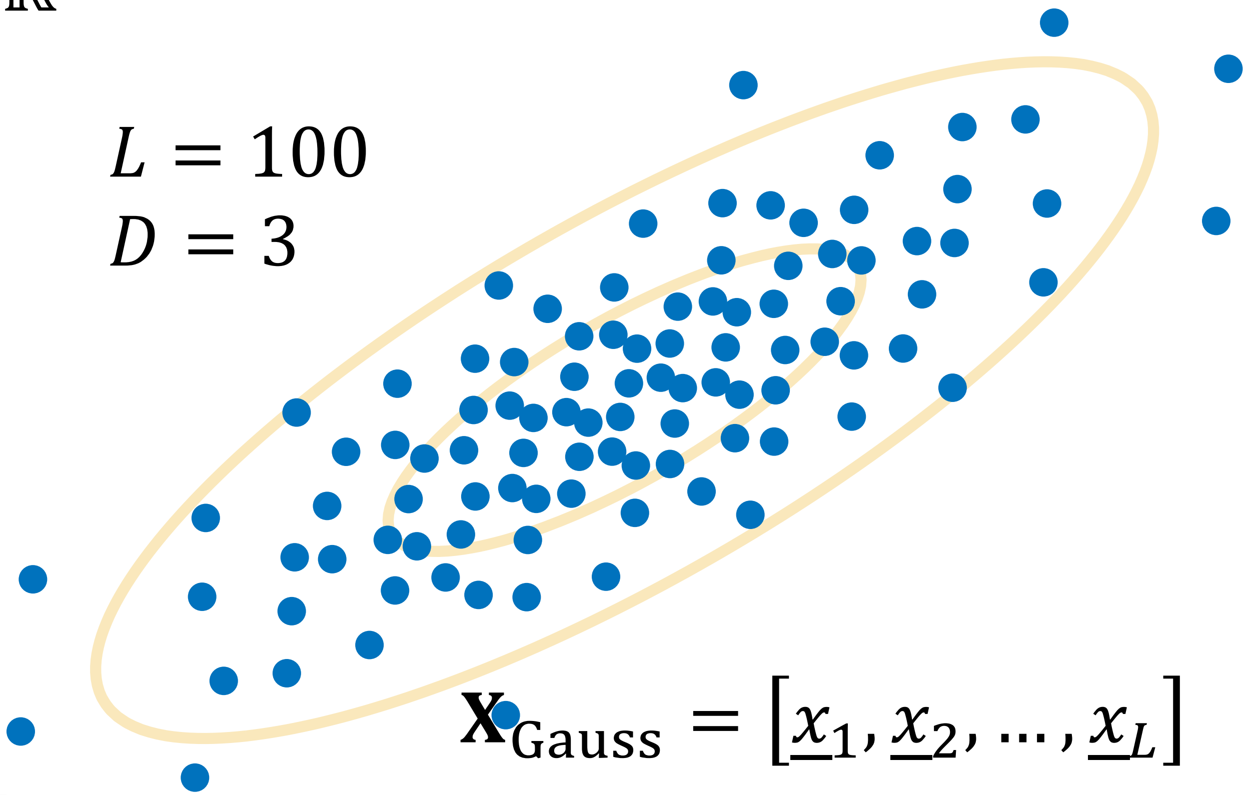
$$\mathbf{C} = \mathbf{R}_\alpha \cdot \mathbf{C}_0 \cdot \mathbf{R}_\alpha^\top$$

$$\mathbf{C}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.3^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

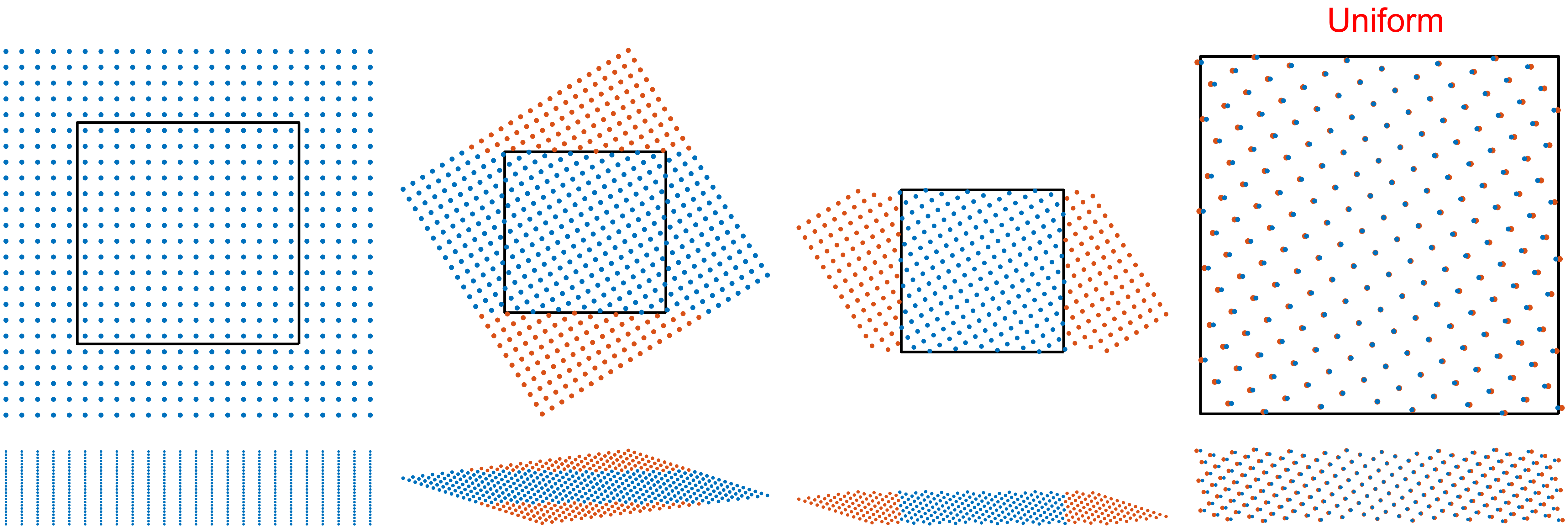
$$\alpha = 30°$$



$$L = 100$$
$$D = 3$$



$$\mathbf{X}_{\text{Gauss}} = [\underline{x}_1, \underline{x}_2, \dots, \underline{x}_L]$$

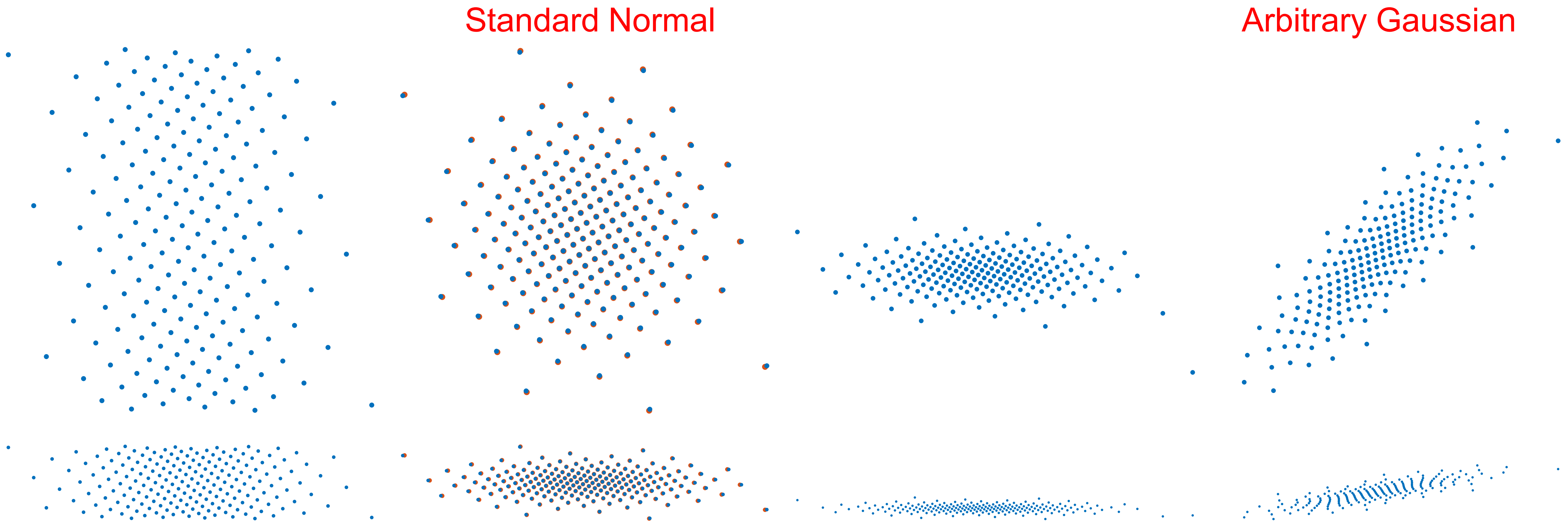


(a) Regular Lattice
→ Rotate

(b) Rotated
→ Crop dimension $2, 3, \ldots$

(c) Removed points (part one)
→ Sort along x and crop to L

(d) Fibonacci Grid on $(0,1)^2$
→ Gauss along x direction



(e) Gaussian along x direction → Gauss along y direction

(f) Standard Normal → Anisotropic Rescale

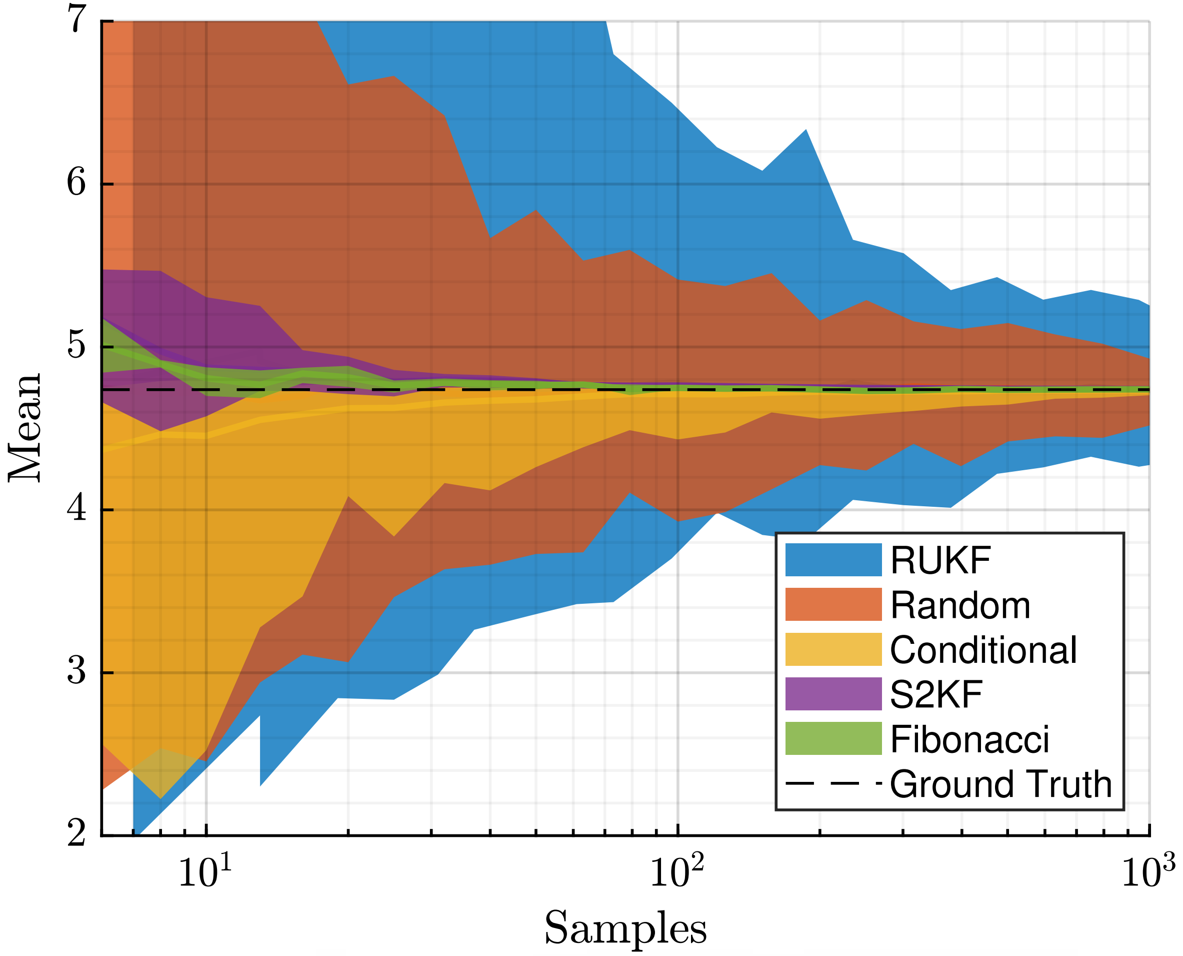
(g) Uncorrelated Gaussian → Rotate

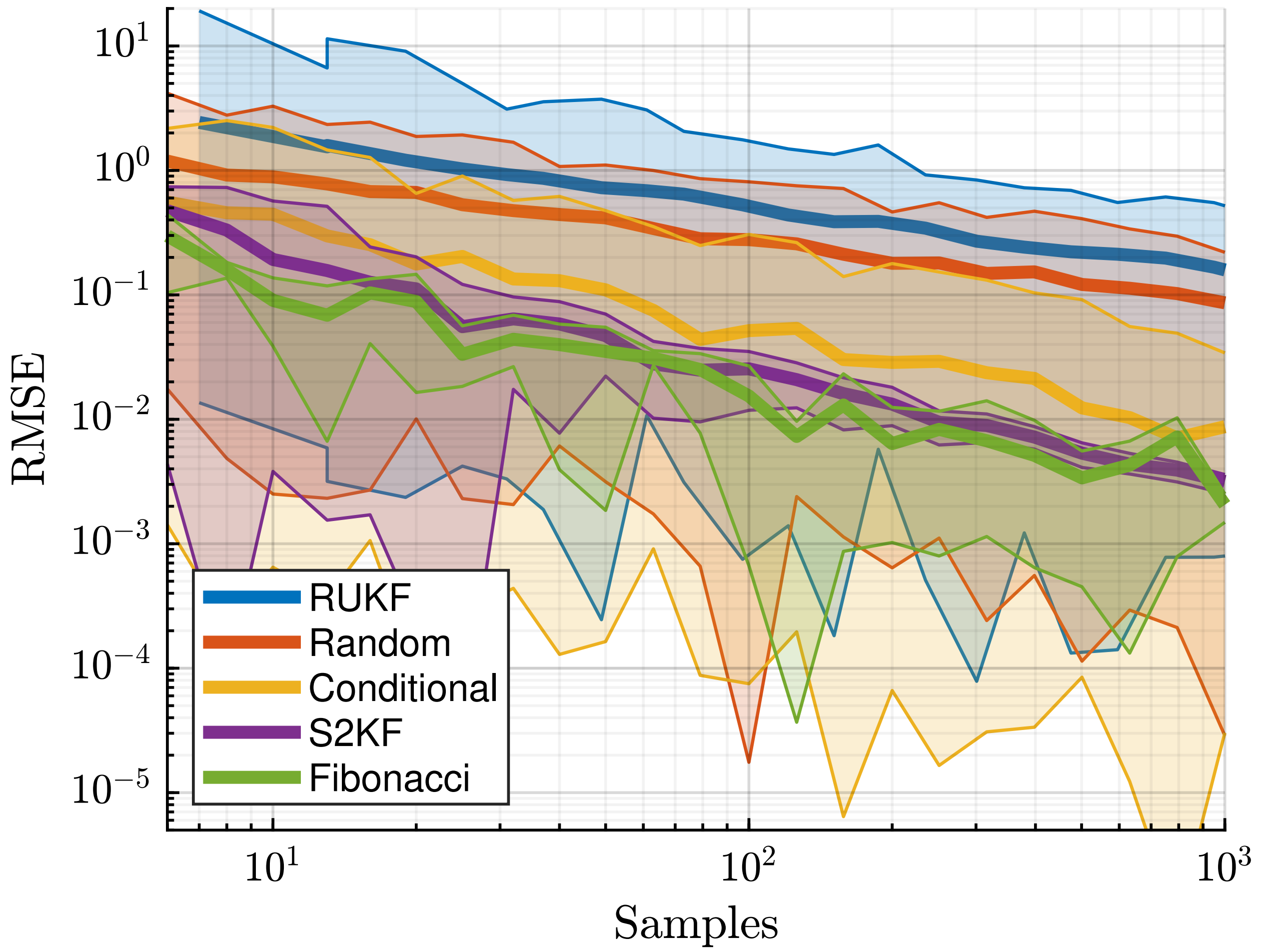
(h) Arbitrary Gaussian → Use in filter

Gaussian Samples

Evaluation

$$y = \sqrt{(x^{(1)})^2 + (x^{(2)})^2 + (x^{(3)})^2}\,, \qquad \underline{x} \sim \mathcal{N}(\underline{0}, \mathbf{C})\,, \qquad \mathbf{C} = \mathbf{R} \cdot \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \boldsymbol{R}^\top$$





Conclusion

Achieved

- Optimal deterministic samples
  - Uniform
  - Gaussian
- No numerical optimization
- Dimensions: 2, 3, 5, 6, 8, 9

Future Work

- More density types
- Lattice rules & Kronecker sequences
  - Higher dimensions
  - Periodic manifolds

Thank you for your attention

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Gaussian Samples