


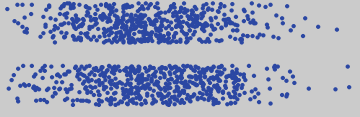
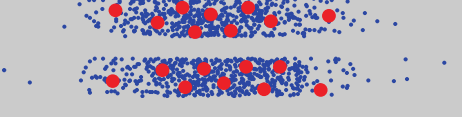
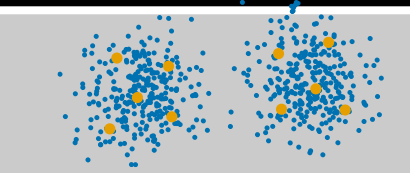

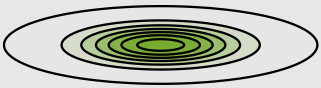
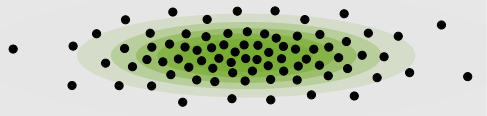
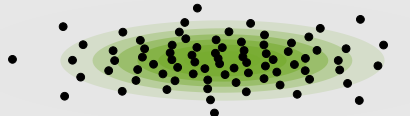
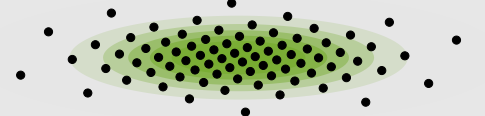

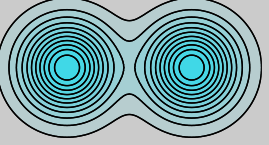
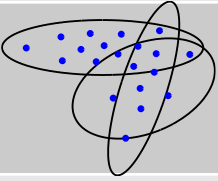
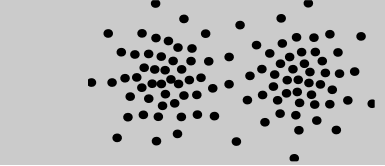


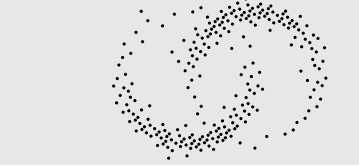

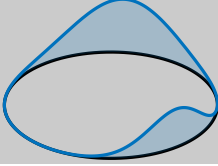

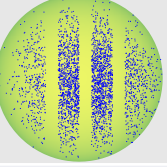
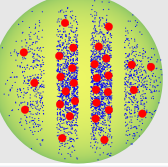
Deterministic Sampling on the Circle Using Projected Cumulative Distributions

Daniel Frisch and **Uwe D. Hanebeck**


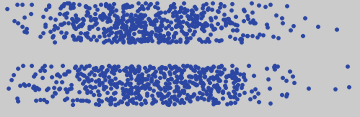
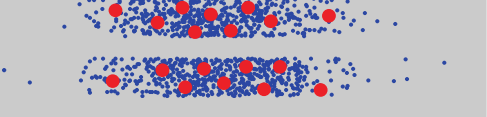
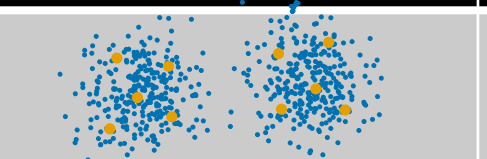

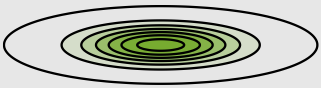
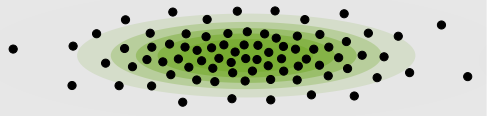
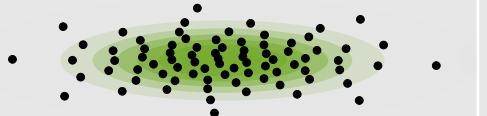
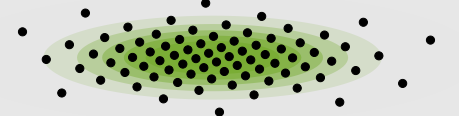

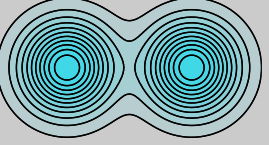
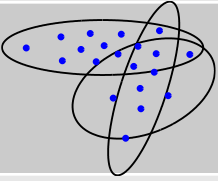
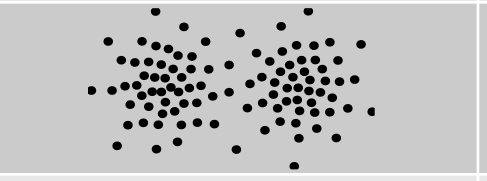


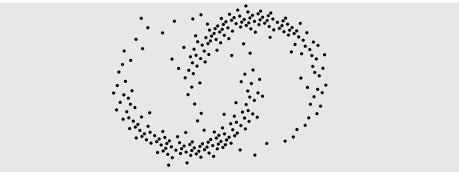

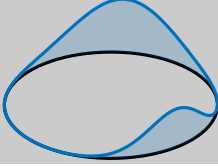
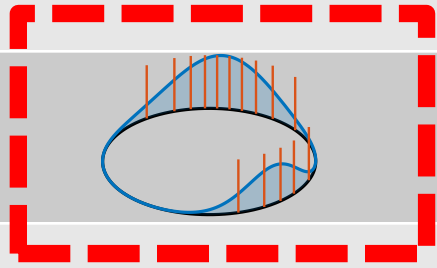

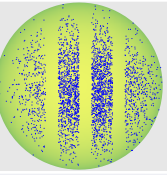
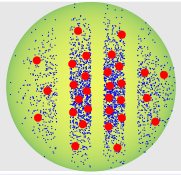
Fusion 2022 Conference Presentation

Intelligent Sensor-Actuator-Systems Laboratory (ISAS)
Institute for Anthropomatics and Robotics
Karlsruhe Institute of Technology (KIT)
Karlsruhe, Germany

Deterministic Sampling – State of Art

Domain		Reference	LCD	PCD	Fibonacci
\mathbb{R}^d		DM 			
\mathbb{R}^d		Gauss 			
\mathbb{R}^d		GM 			
\mathbb{R}^d		Arbitrary 			
S^1		Arbitrary 			
S^n		DM 			

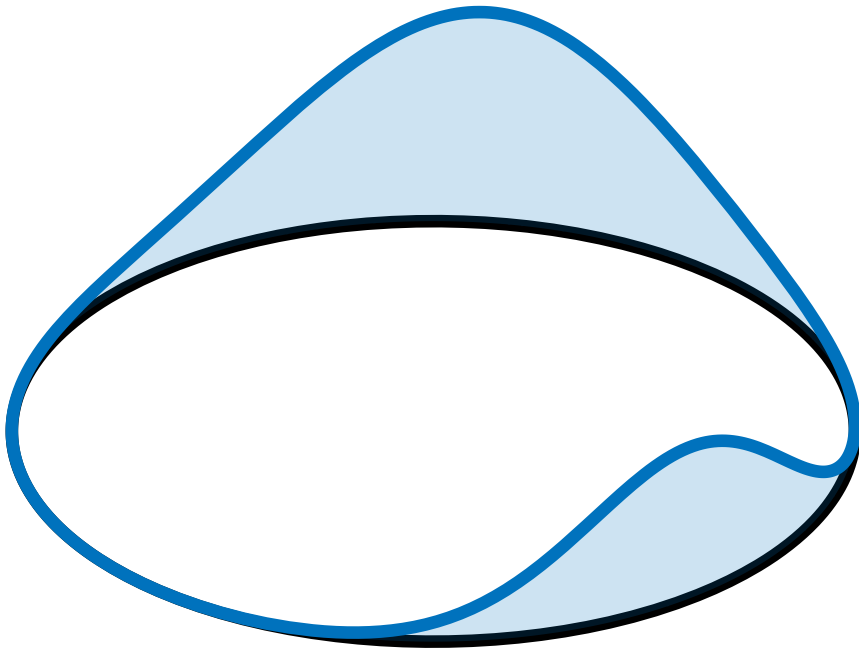
Deterministic Sampling – State of Art

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\mathbb{R}^d		DM 			
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\mathbb{R}^d		GM 			
\mathbb{R}^d		Arbitrary 			
S^1		Arbitrary 			
S^n		DM 			

Input

Circular Density Function

- Arbitrary function handle

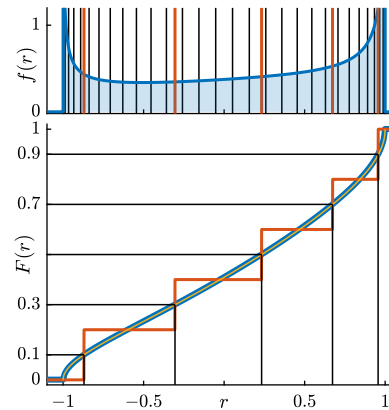
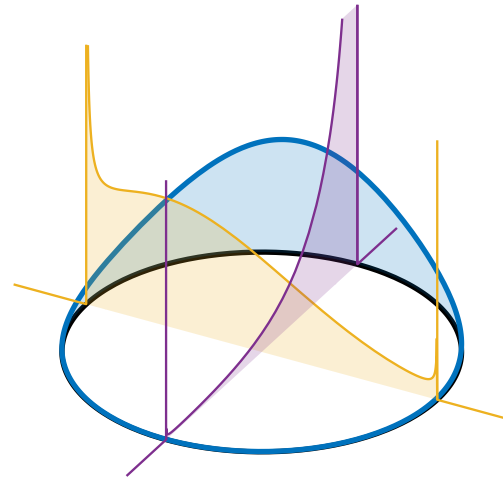
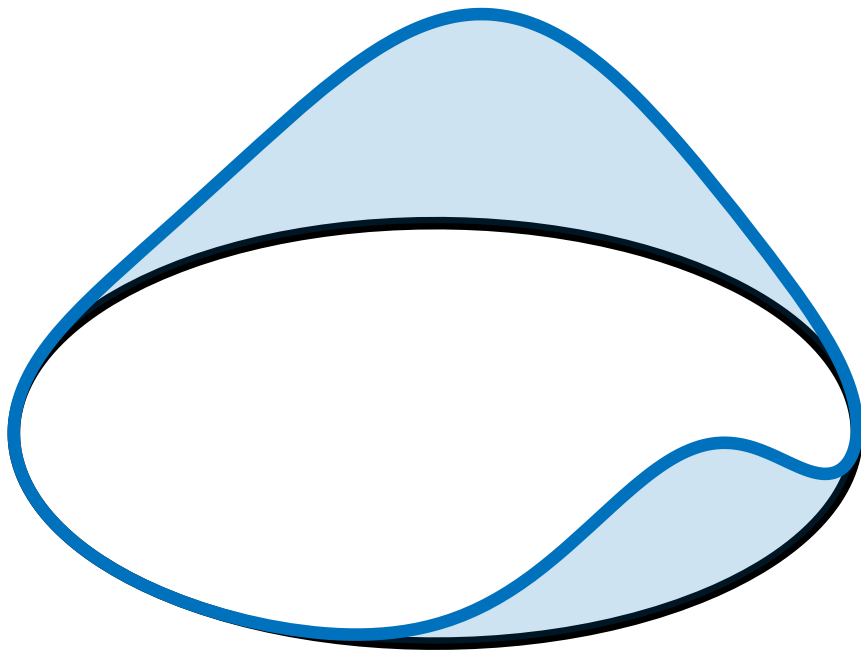


Overview

Input

- Circular Density Function
- Arbitrary function handle

PCD Sampling



Overview

Input

Circular Density Function

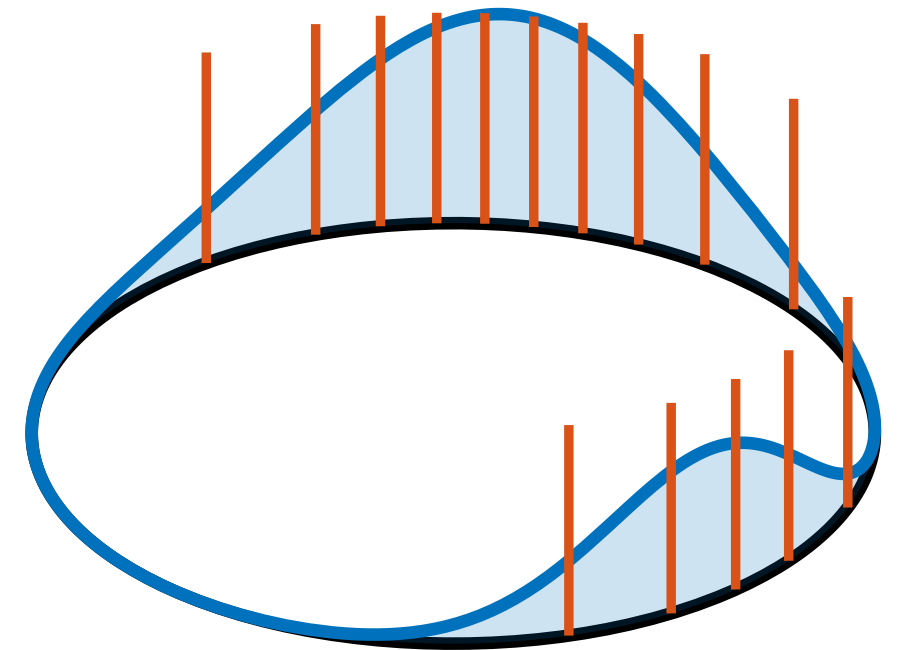
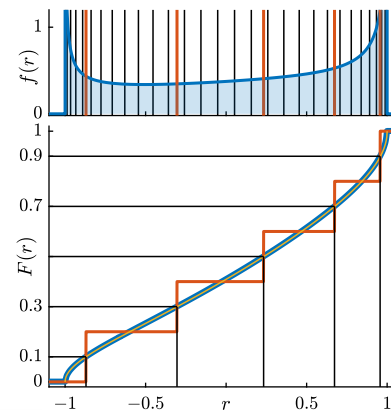
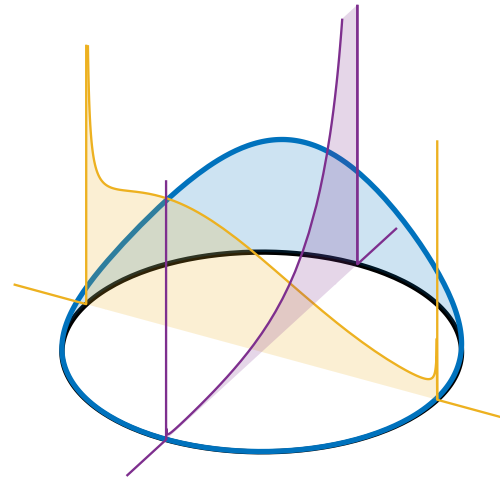
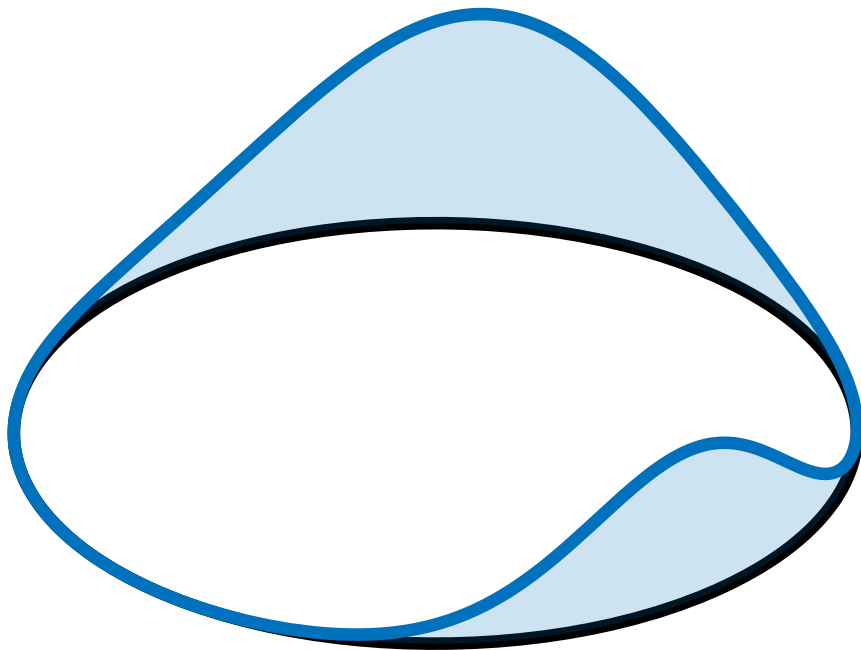
- Arbitrary function handle

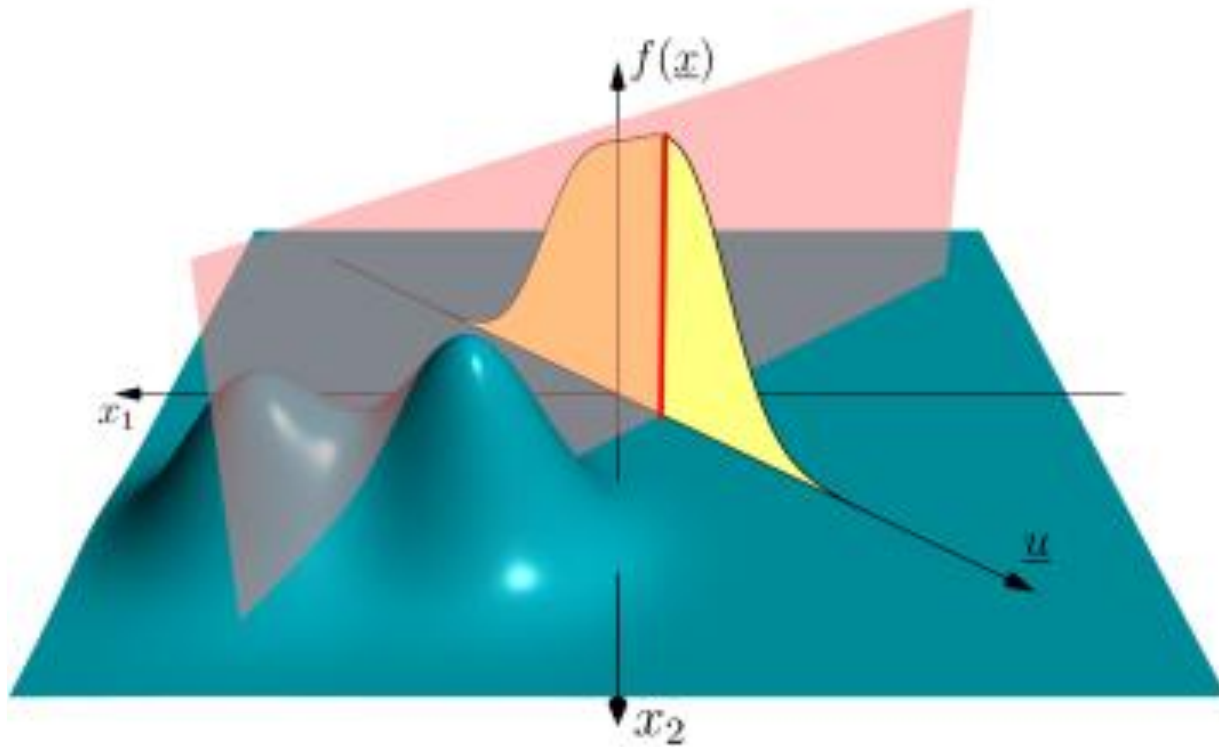
PCD
Sampling

Output

Dirac Mixture Approximation

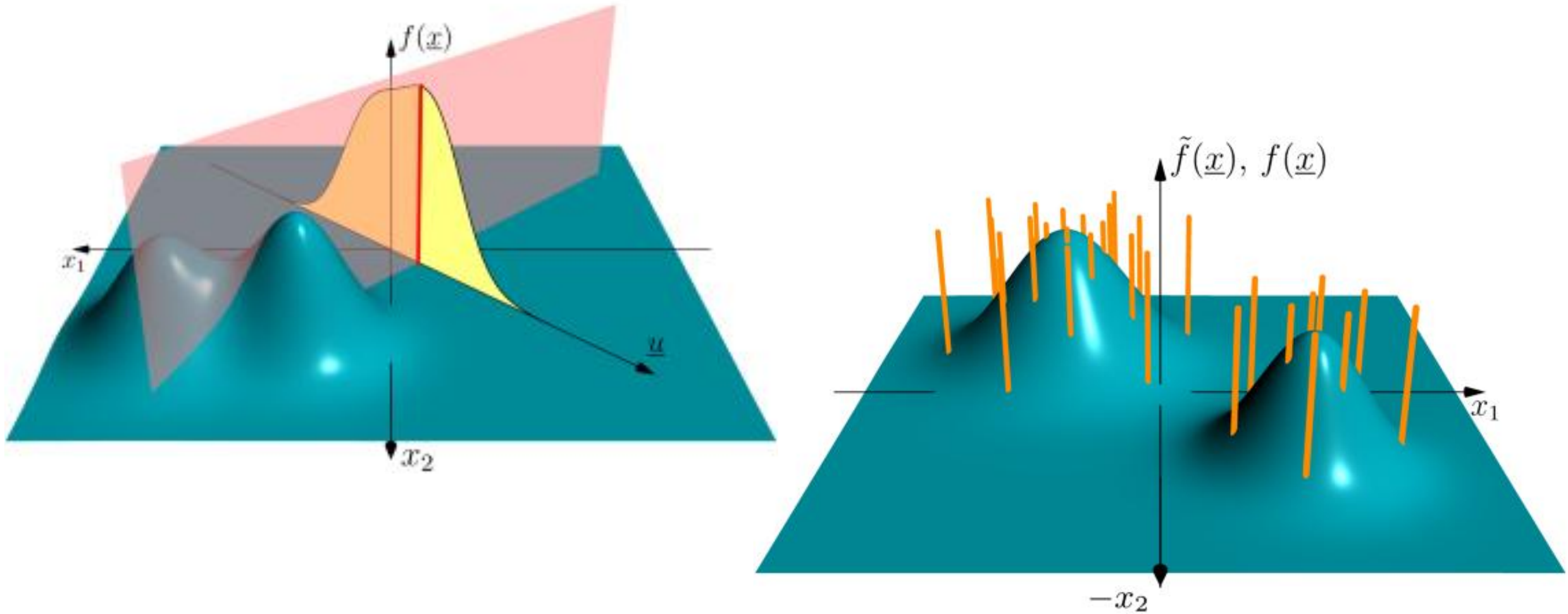
- Sample locations (angles)





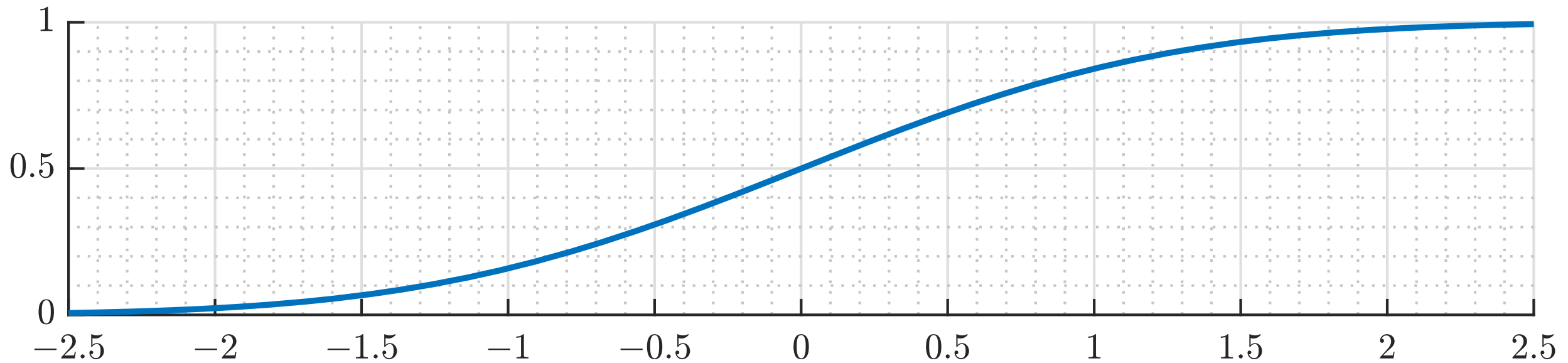
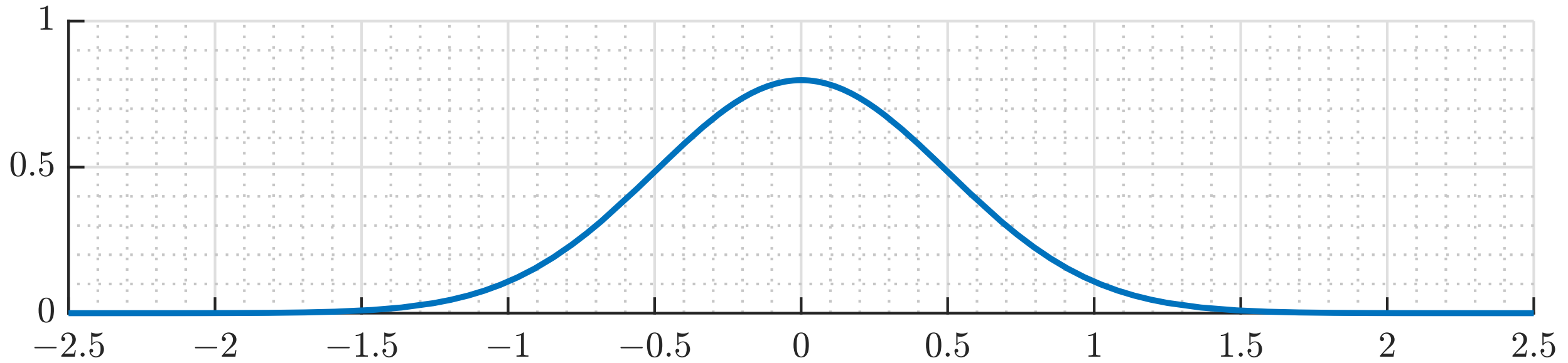
Hanebeck, CISS 2020, Deterministic Sampling of Multivariate Densities based on Projected Cumulative Distributions

PCD – State of the Art

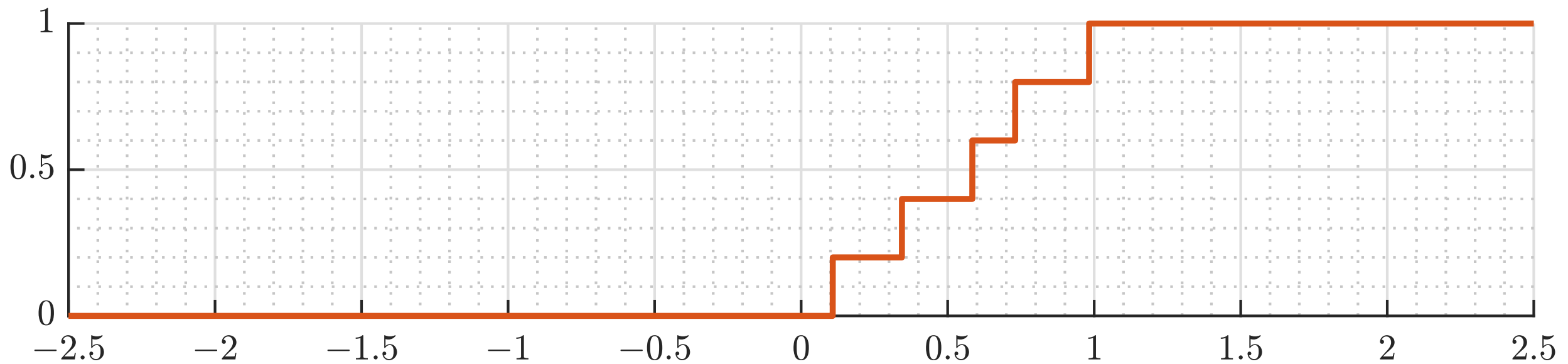
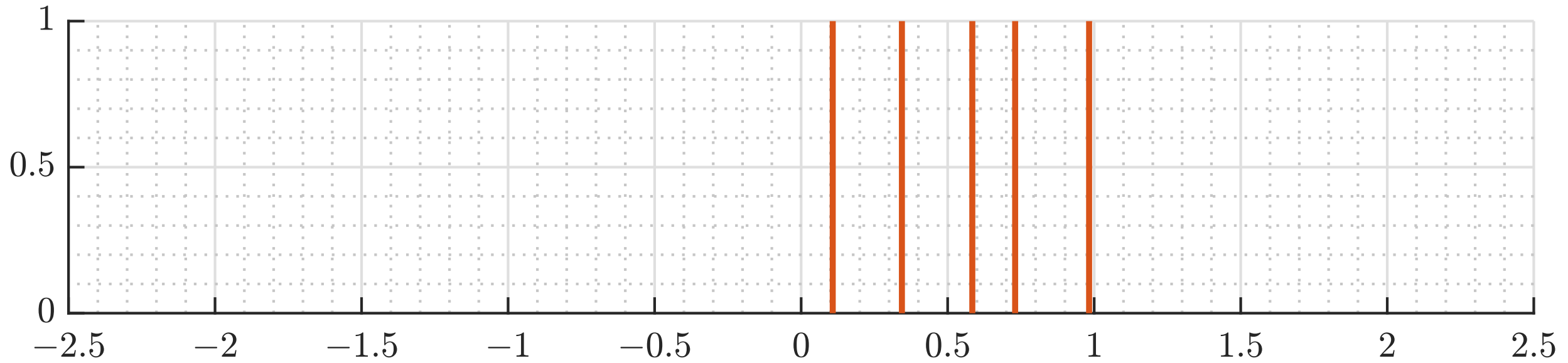


Hanebeck, CISS 2020, Deterministic Sampling of Multivariate Densities based on Projected Cumulative Distributions

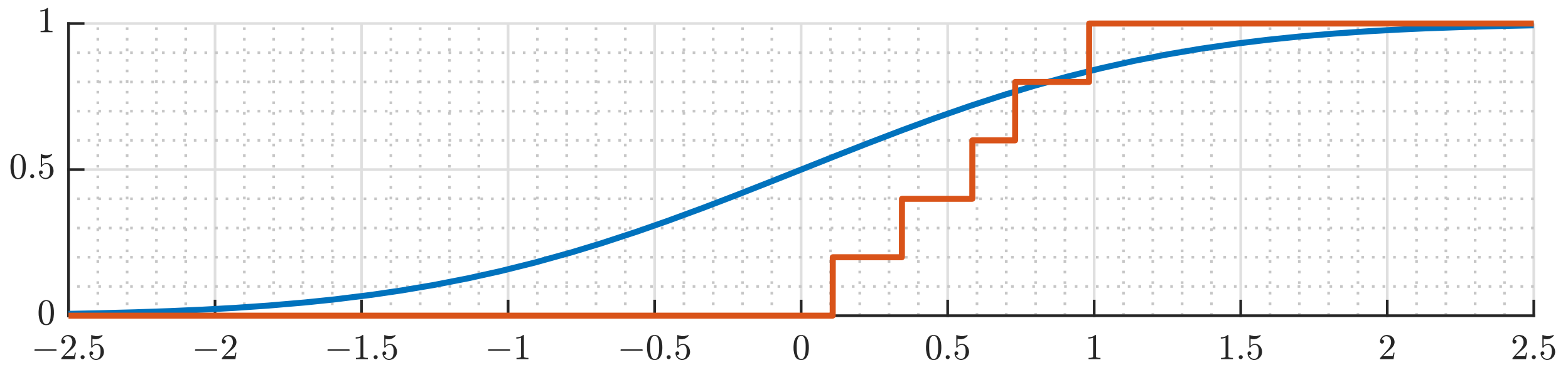
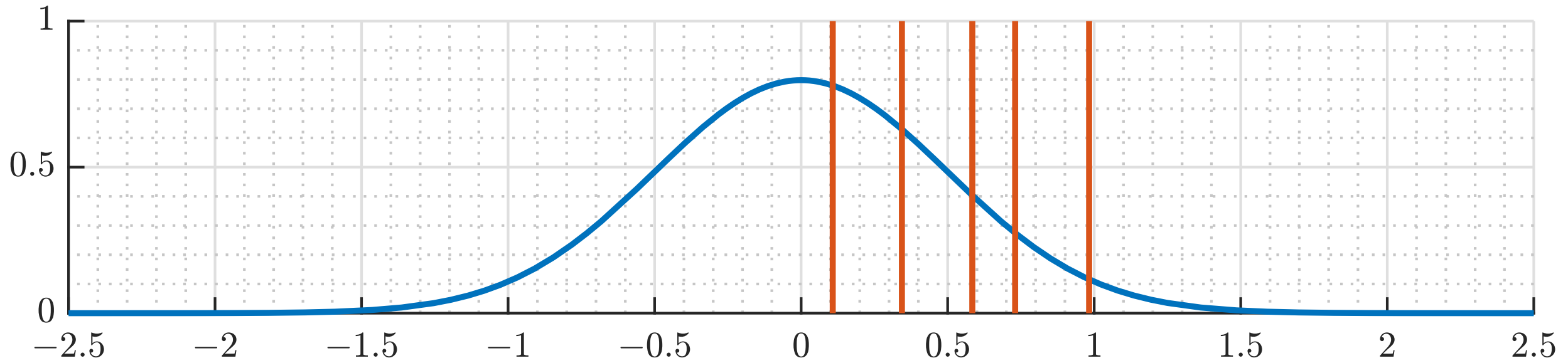
Continuous Density \rightarrow Cumulative Distribution



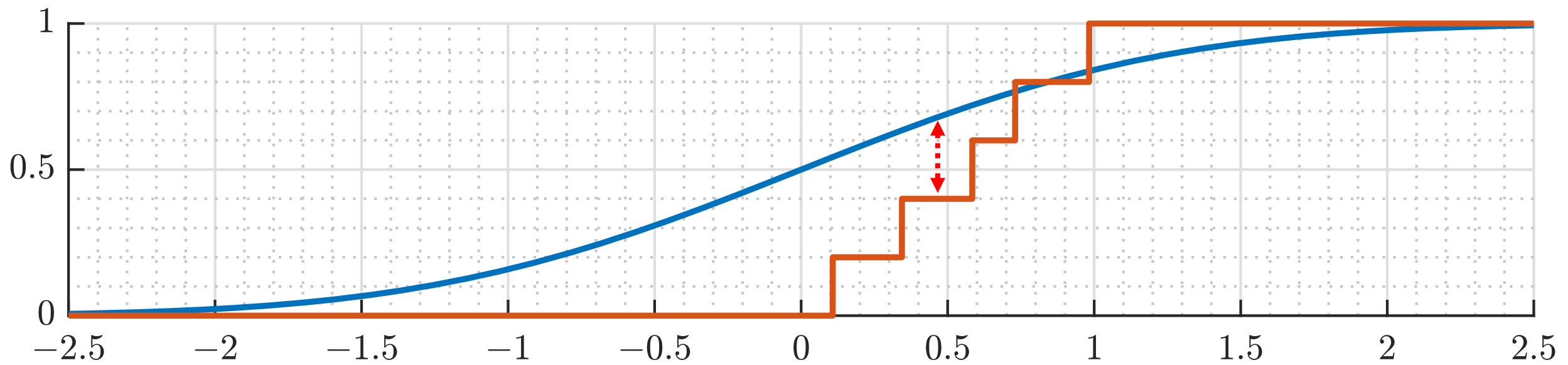
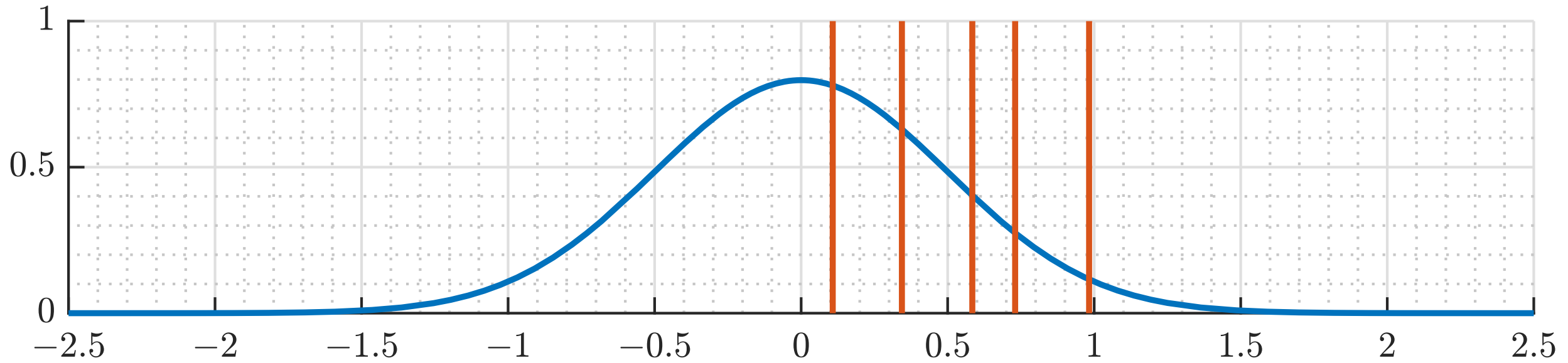
Discrete Density \rightarrow Cumulative Distribution



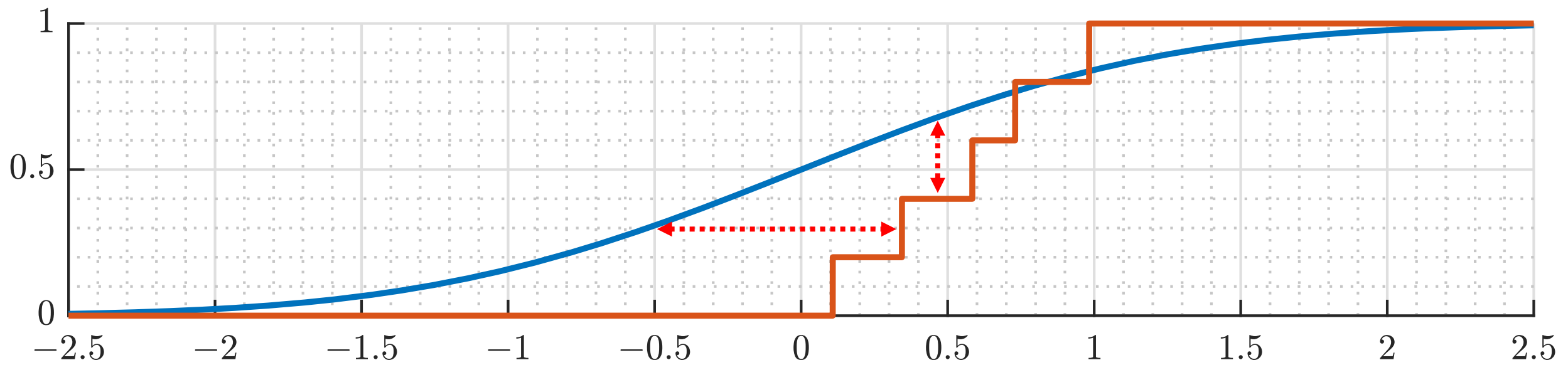
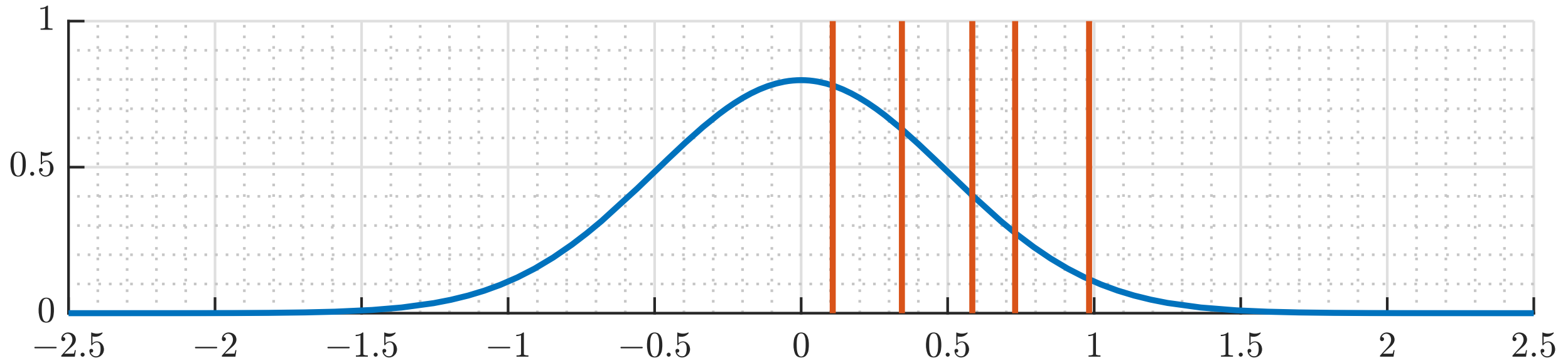
Univariate DMA



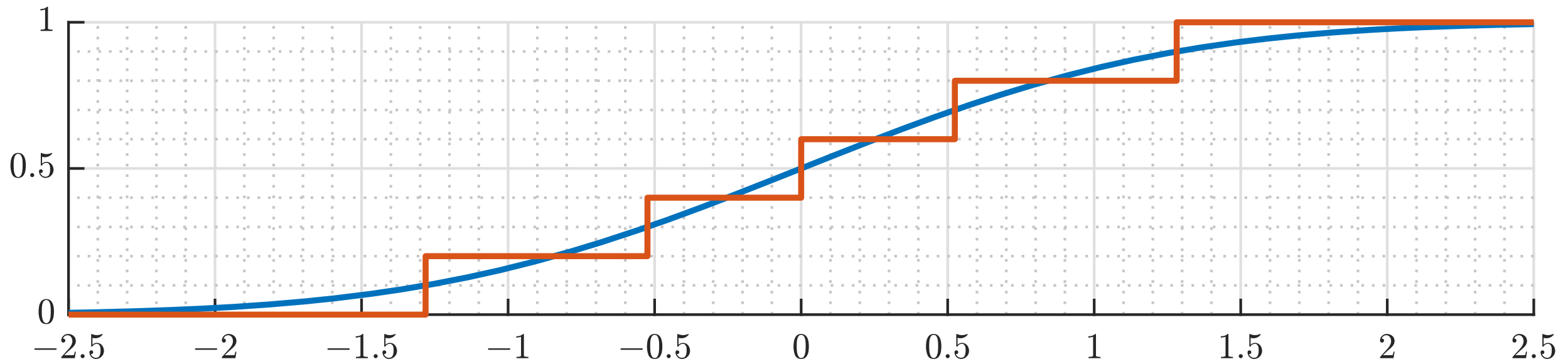
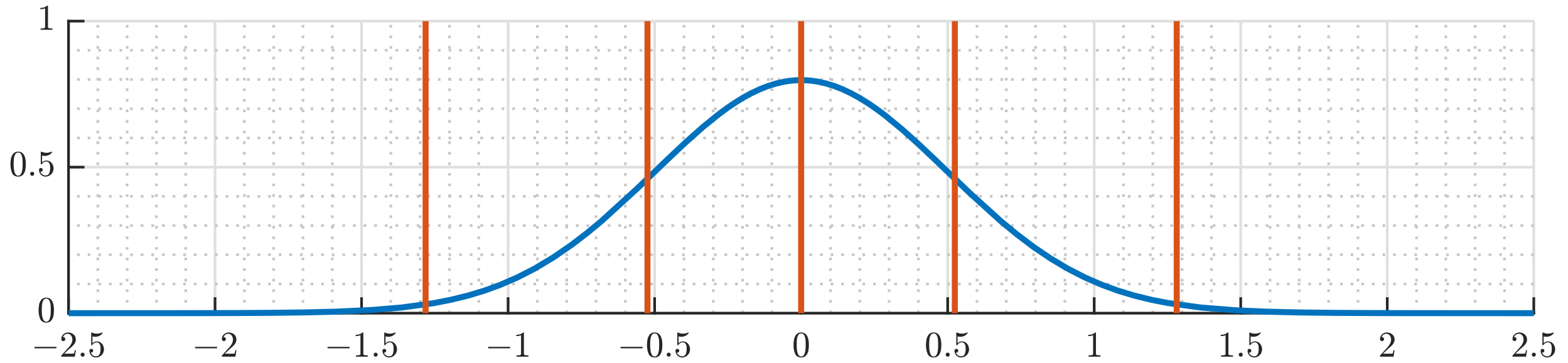
Univariate DMA



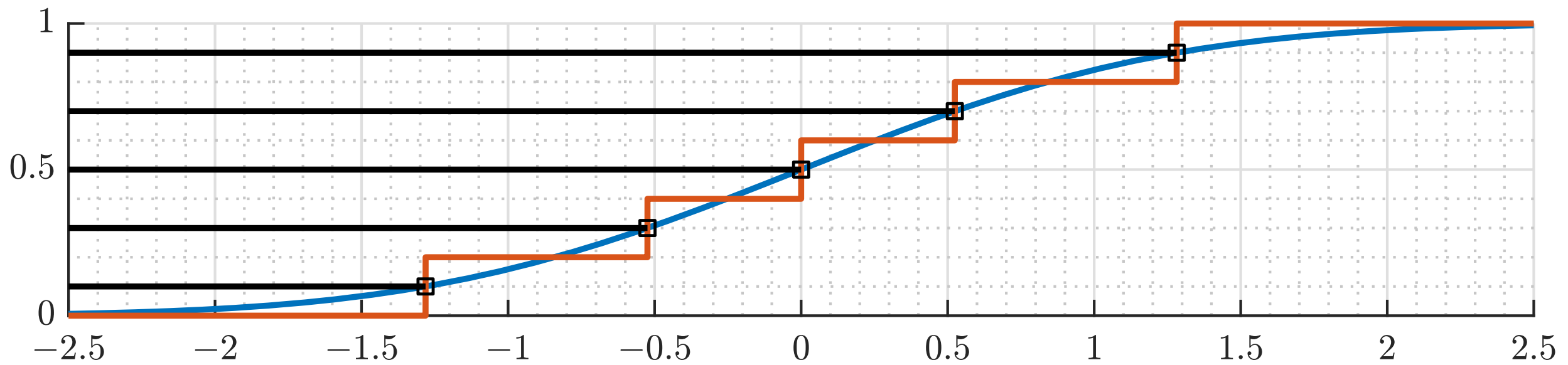
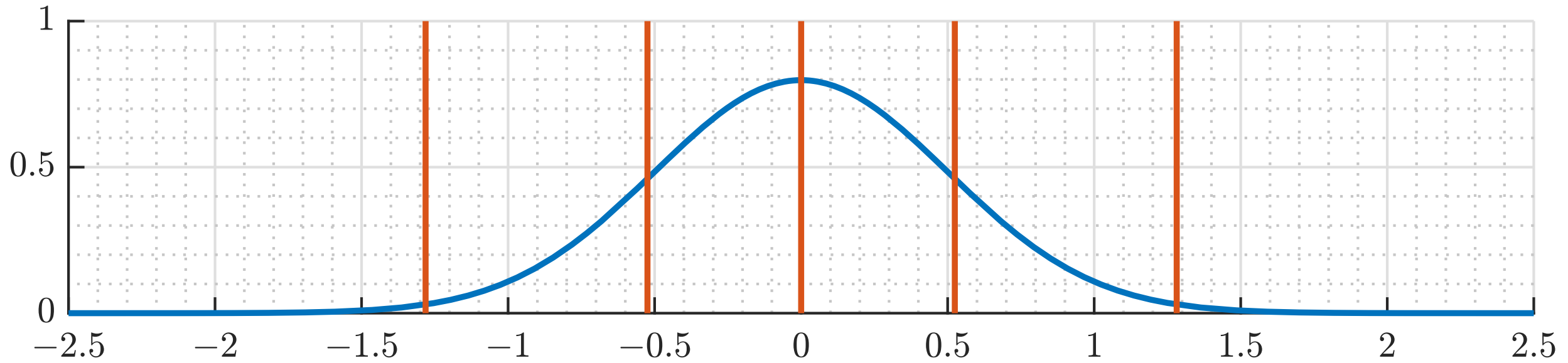
Univariate DMA



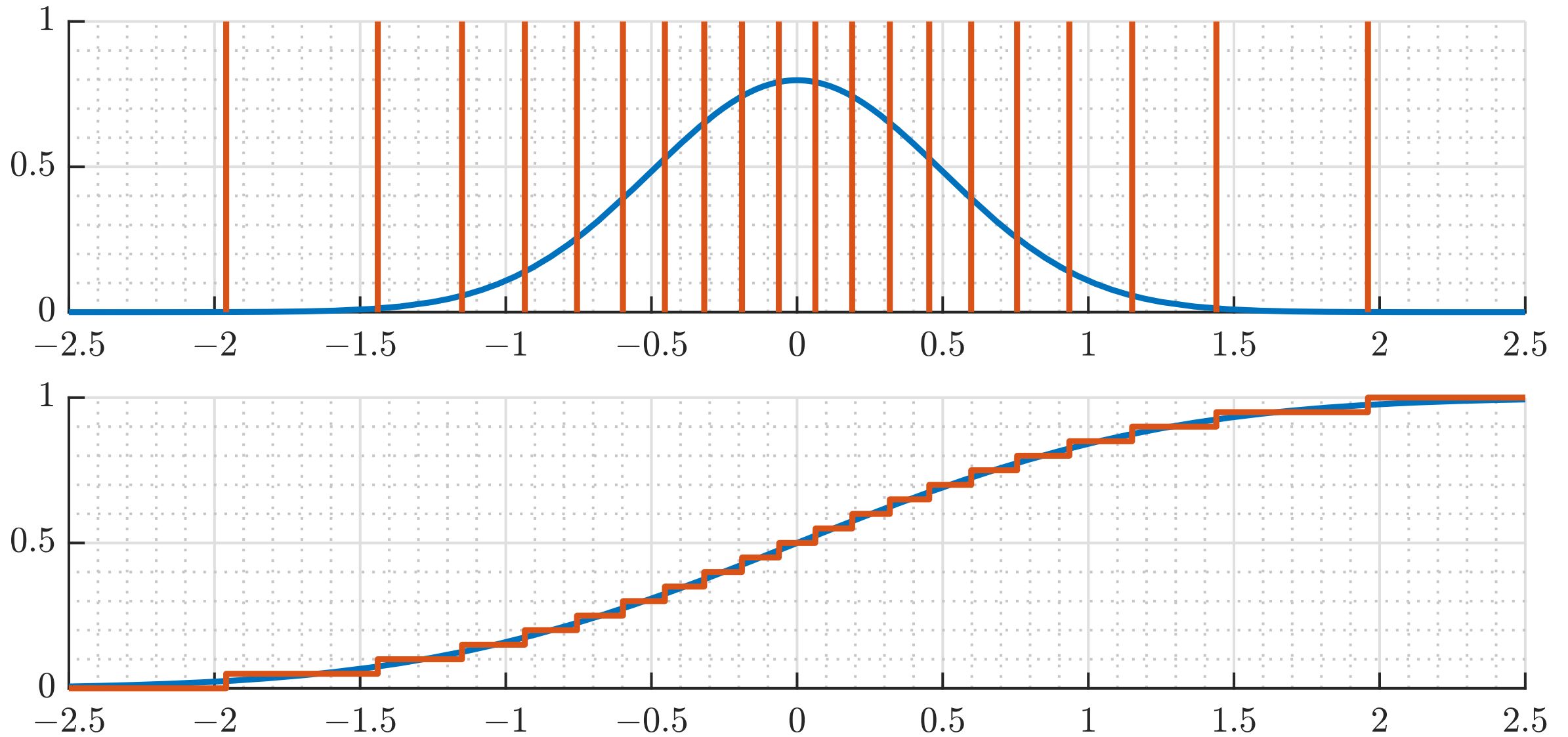
Univariate DMA



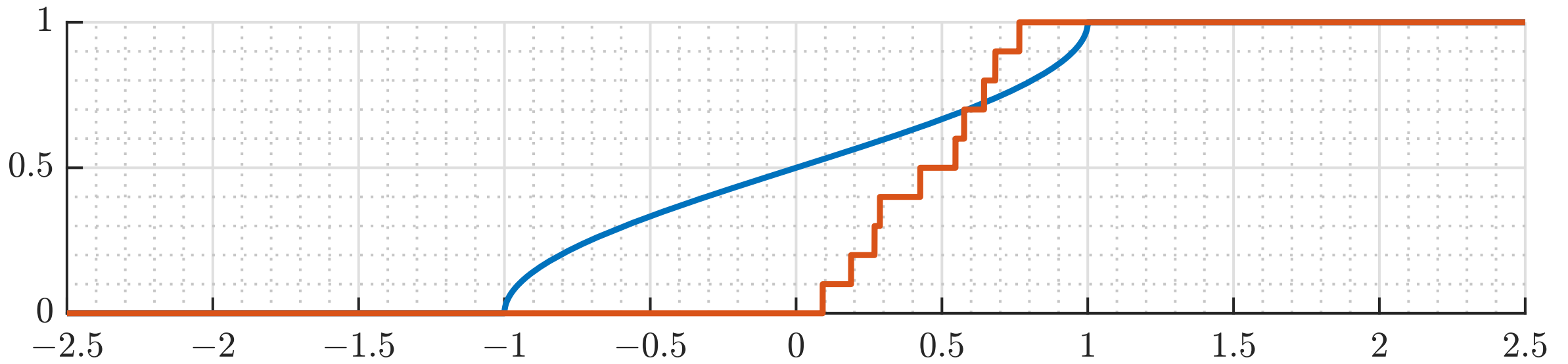
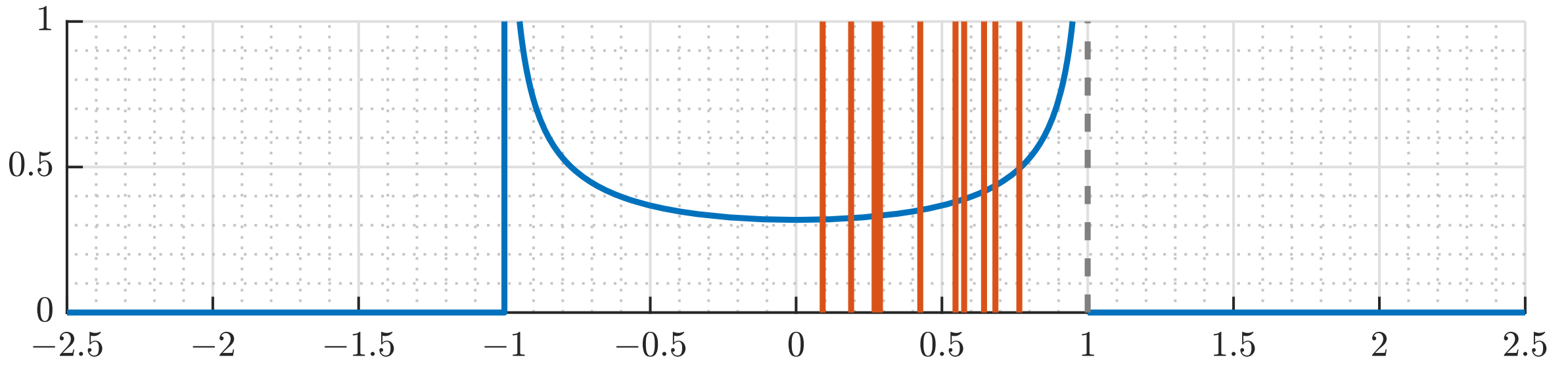
Univariate DMA



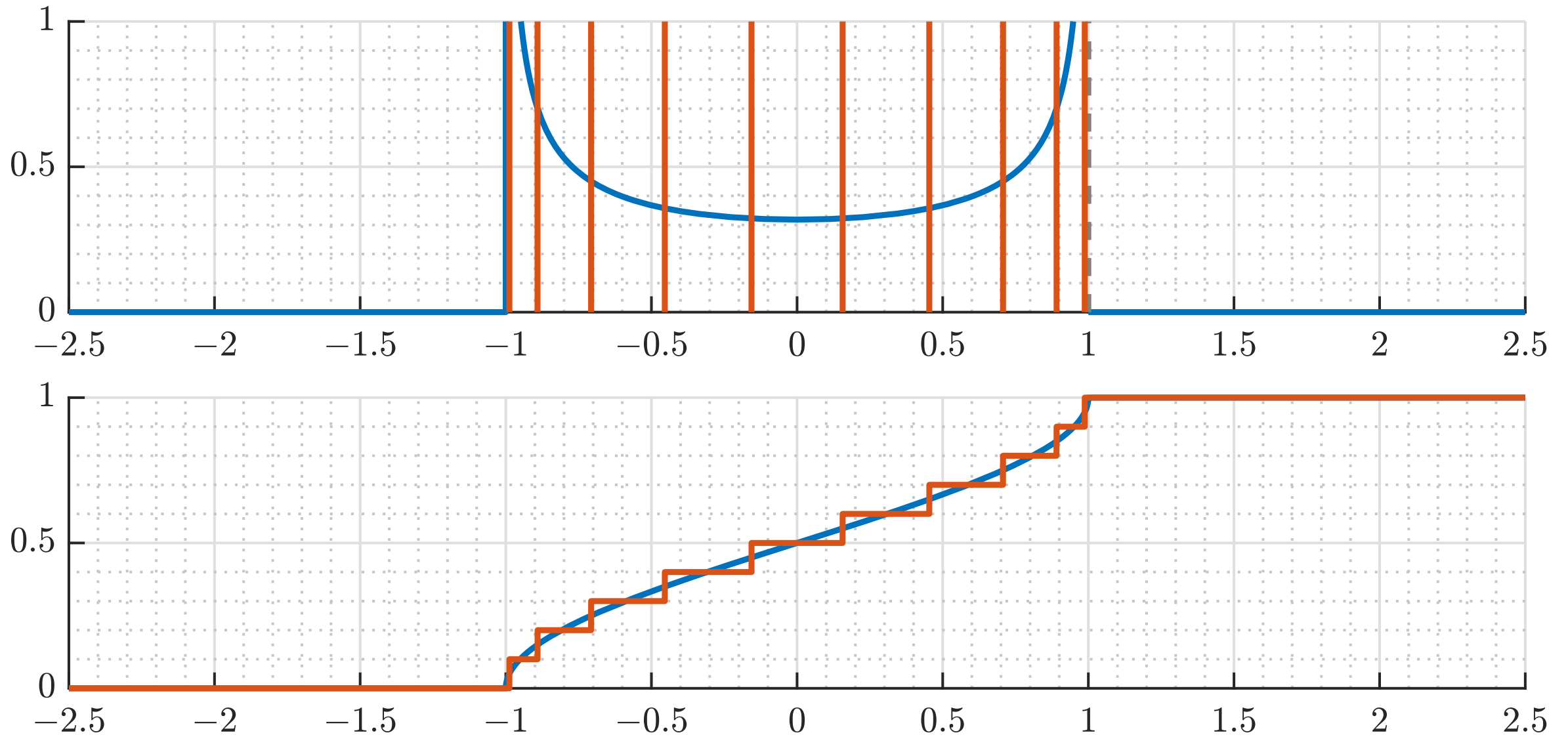
Univariate DMA



Univariate DMA

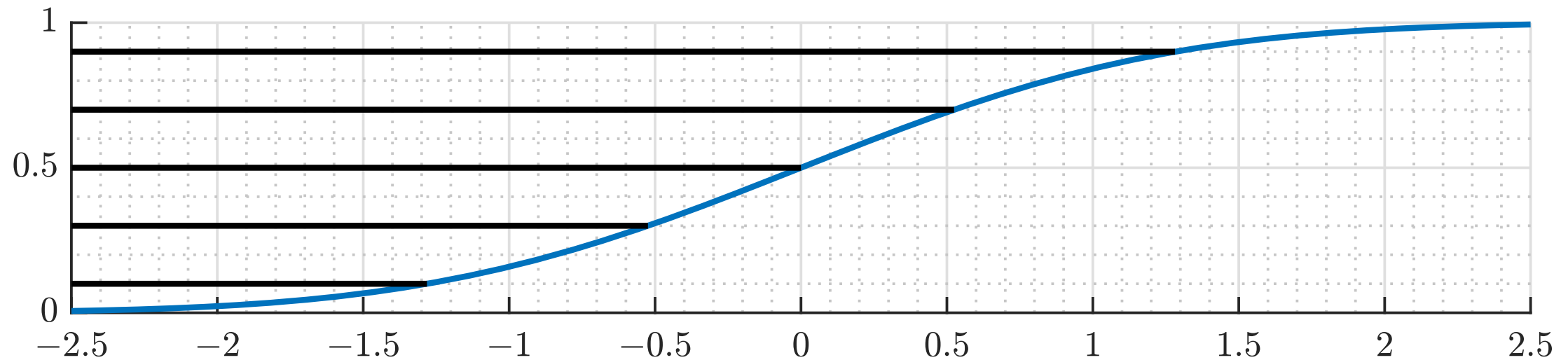


Univariate DMA



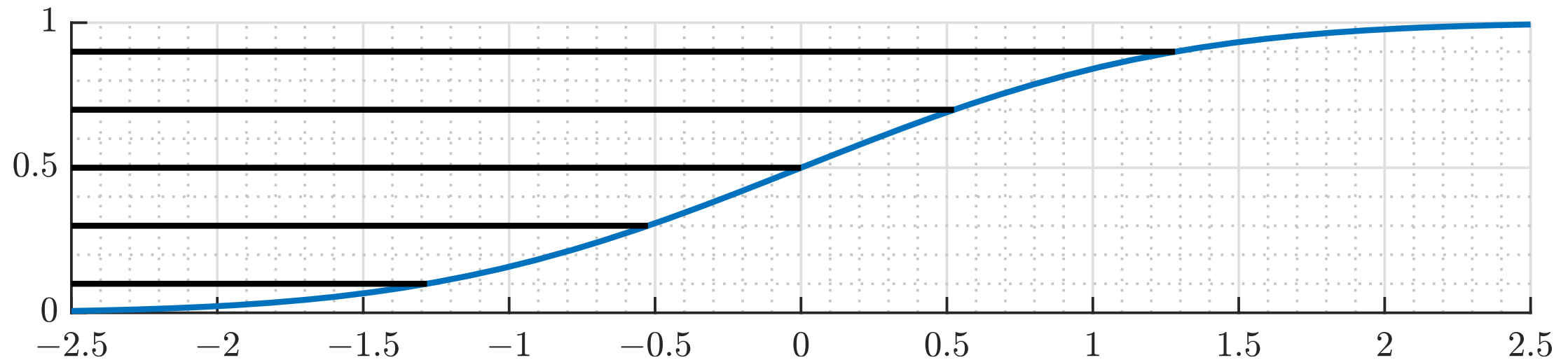
What we need for Univariate DMA

- Number of samples



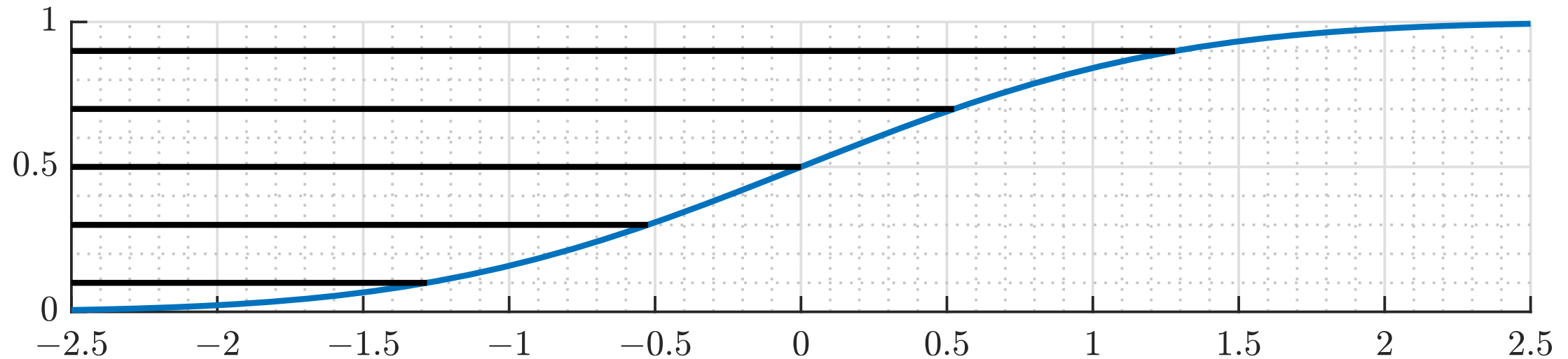
What we need for Univariate DMA

- Number of samples
- CDF



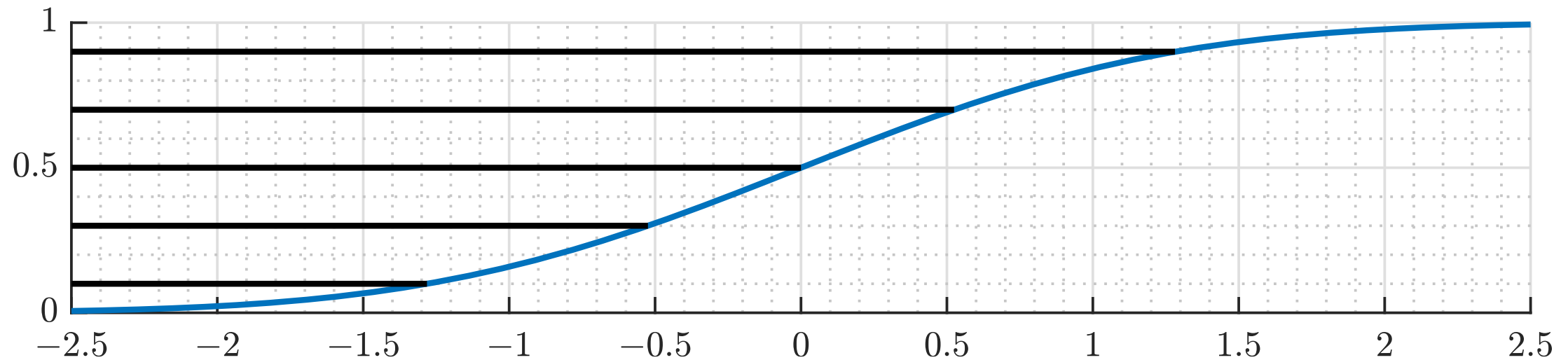
What we need for Univariate DMA

- Number of samples
- CDF
- Inverse CDF



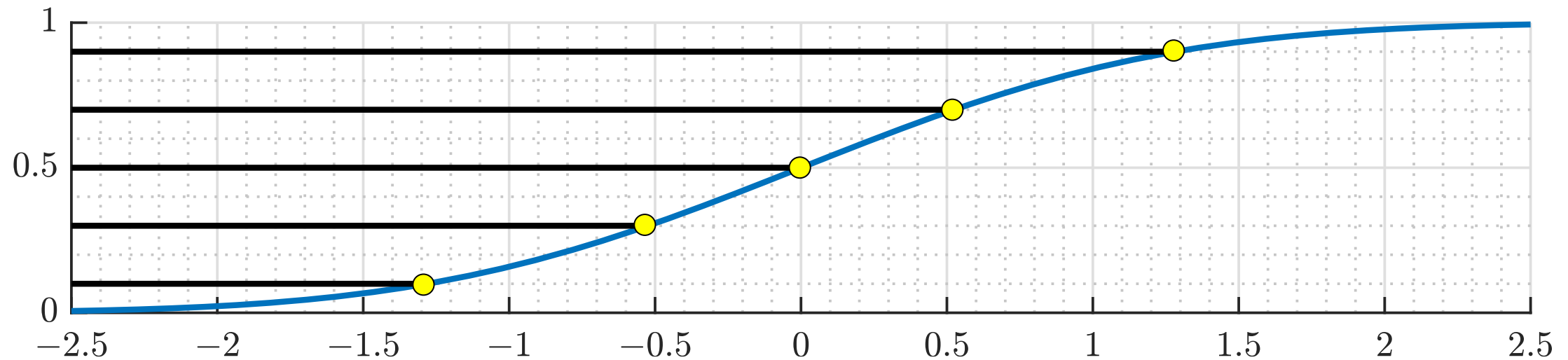
What we need for Univariate DMA

- Number of samples
- CDF
- Inverse CDF
 - Analytical



What we need for Univariate DMA

- Number of samples
- CDF
- Inverse CDF
 - Analytical
 - Nested intervals



Here: Arbitrary Density Function

Here: Arbitrary Density Function

- Number of samples

Here: Arbitrary Density Function

- Number of samples
- PDF

Here: Arbitrary Density Function

- Number of samples
- PDF
- no CDF

Here: Arbitrary Density Function

- Number of samples
- PDF
- no CDF

CDF: Numerical integration

Here: Arbitrary Density Function

- Number of samples
- PDF
- no CDF

CDF: Numerical integration

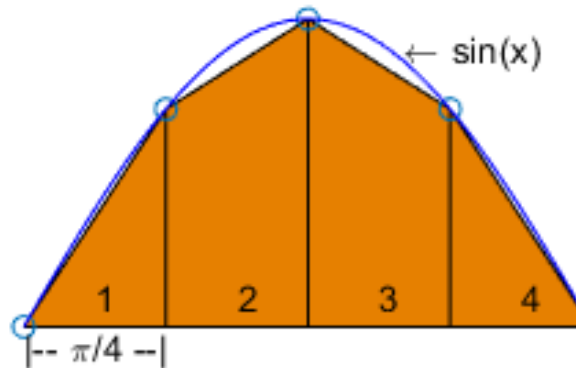
- ODE solver (ode45)
 - Accurate

Here: Arbitrary Density Function

- Number of samples
- PDF
- no CDF

CDF: Numerical integration

- ODE solver (ode45)
 - Accurate
- Trapezoidal rule

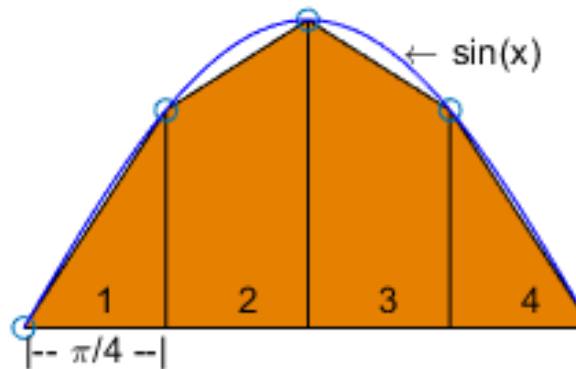


Here: Arbitrary Density Function

- Number of samples
- PDF
- no CDF

CDF: Numerical integration

- ODE solver (ode45)
 - Accurate
- Trapezoidal rule
 - Very fast

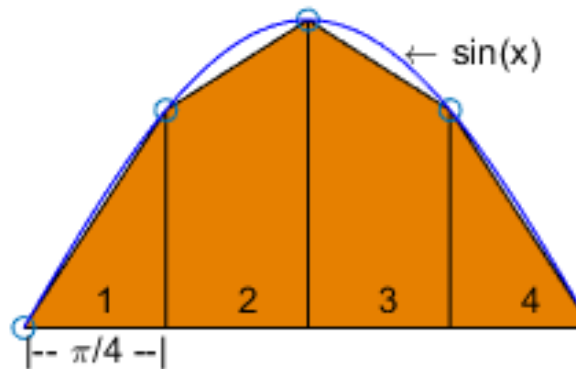


Here: Arbitrary Density Function

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- PDF
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CDF: Numerical integration

- ODE solver (ode45)
 - Accurate
- Trapezoidal rule
 - Very fast
 - Easily invertible

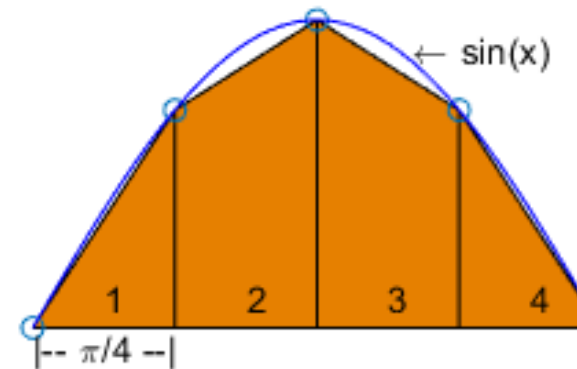


Here: Arbitrary Density Function

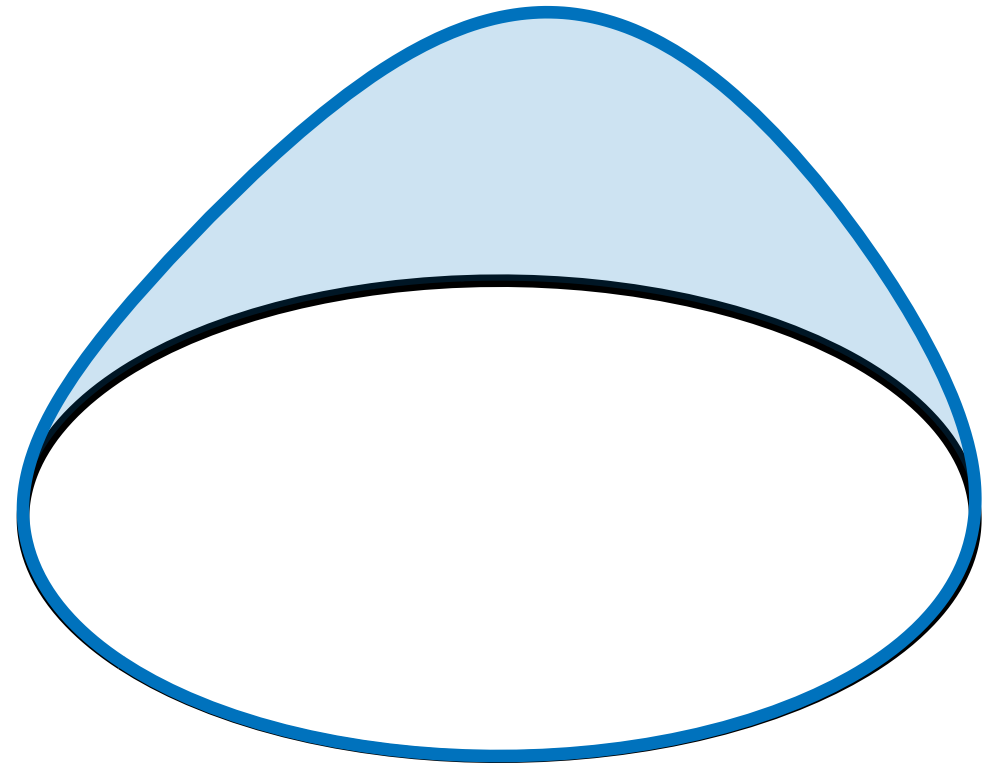
- Number of samples
- PDF
- no CDF

CDF: Numerical integration

- ODE solver (ode45)
 - Accurate
- Trapezoidal rule
 - Very fast
 - Easily invertible
 - Enough eval points important



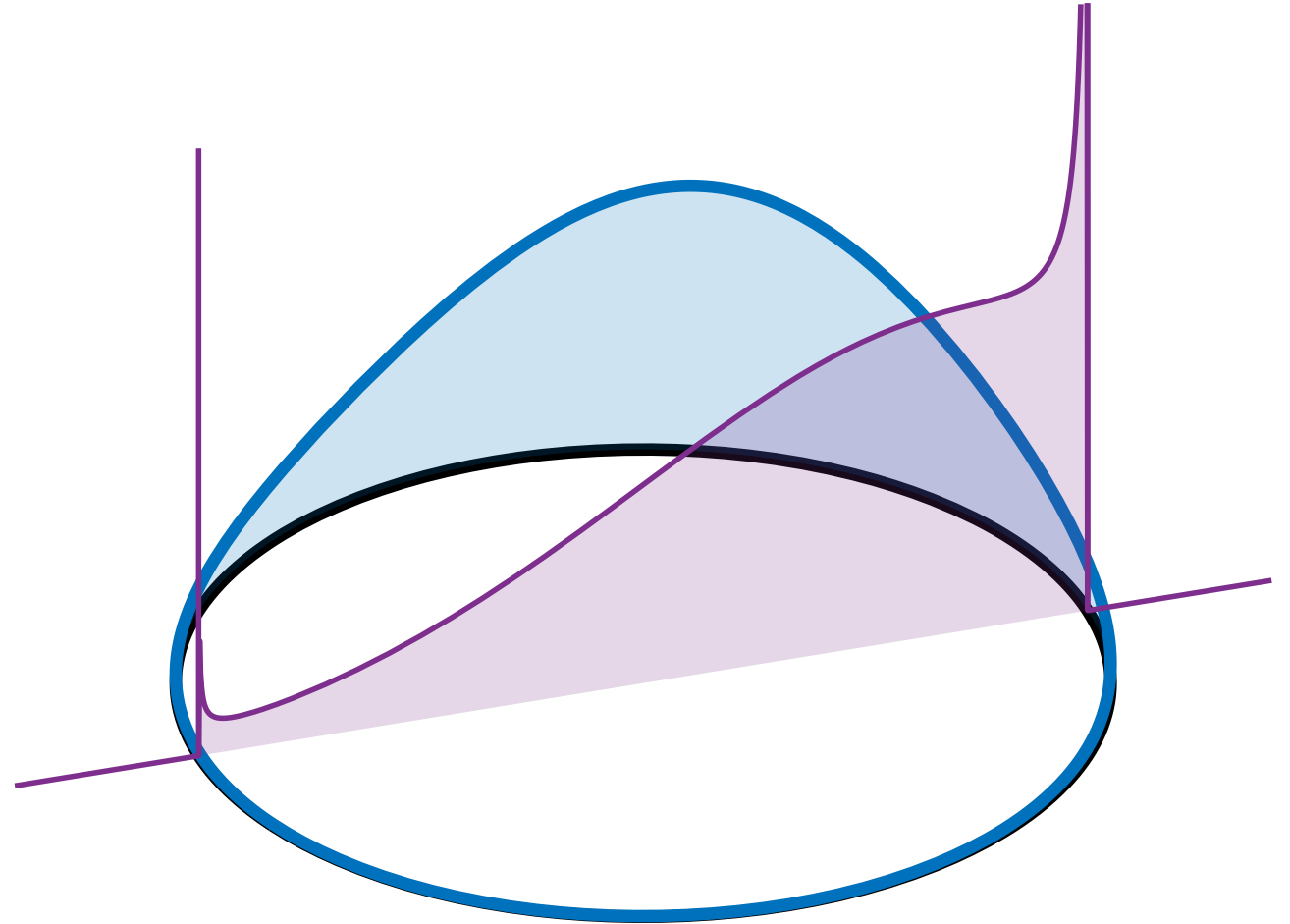
Project Circular Distributions



Project Circular Distributions

Projections $S^1 \rightarrow \mathbb{R}^1$

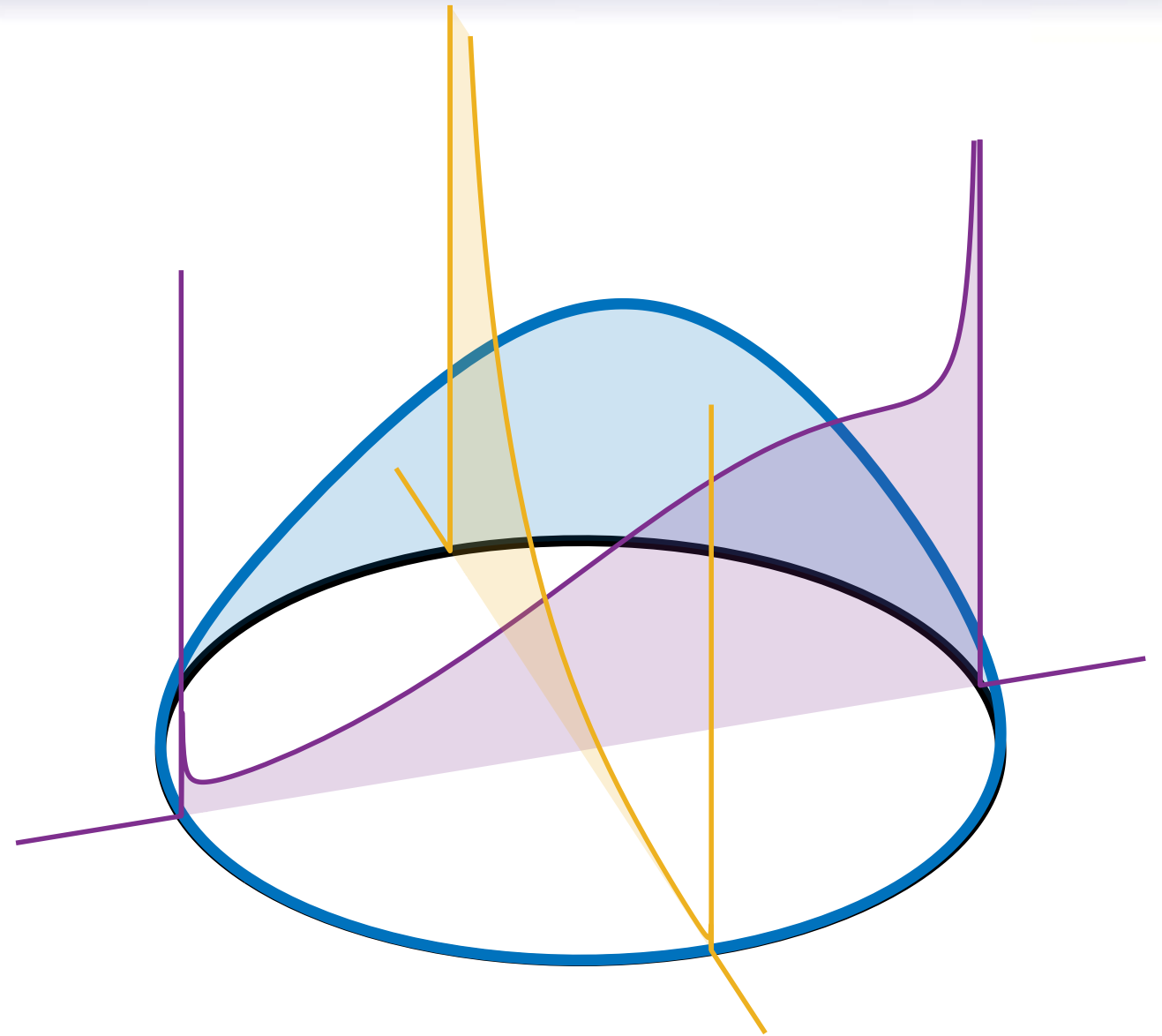
$$\begin{array}{ccc} \mathbb{R}^1 & S^1 & S^1 \\ \downarrow & \downarrow & \downarrow \\ \mathbf{r} & = & \underline{\mathbf{u}}_1^T \underline{\mathbf{x}} \end{array}$$



Project Circular Distributions

Projections $S^1 \rightarrow \mathbb{R}^1$

$$\begin{array}{ccc} \mathbb{R}^1 & S^1 & S^1 \\ \downarrow & \downarrow & \downarrow \\ \mathbf{r} & = & \underline{\mathbf{u}}_1^T \underline{\mathbf{x}} \\ \mathbf{r} & = & \underline{\mathbf{u}}_2^T \underline{\mathbf{x}} \end{array}$$

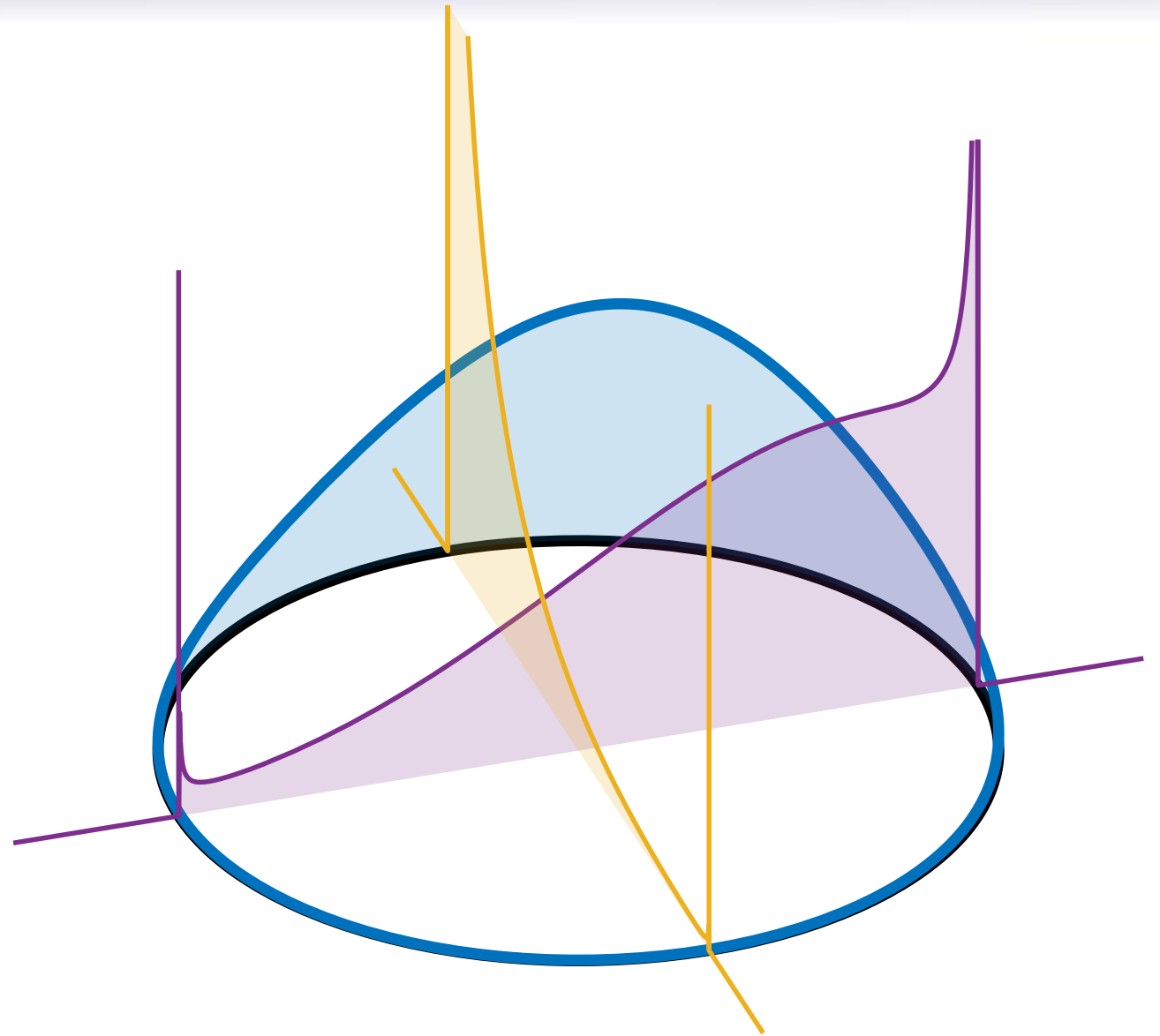


Project Circular Distributions

Projections $S^1 \rightarrow \mathbb{R}^1$

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Radon: Many 1D projections fully describe arbitrary density!

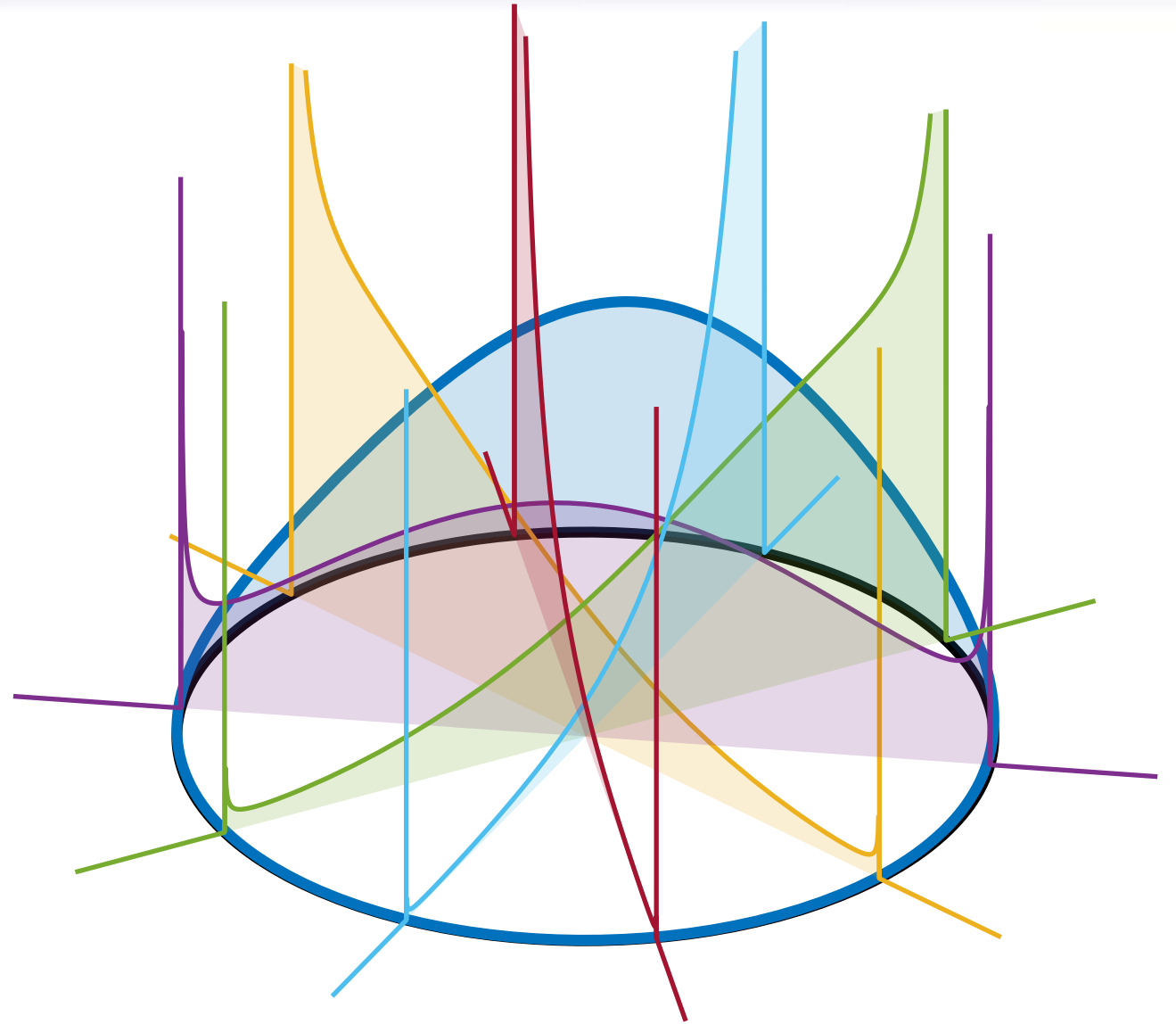


Project Circular Distributions

Projections $S^1 \rightarrow \mathbb{R}^1$

$$\begin{array}{ccc} \mathbb{R}^1 & S^1 & S^1 \\ \downarrow & \downarrow & \downarrow \\ \mathbf{r} & = & \underline{\mathbf{u}}_1^T \underline{\mathbf{x}} \\ \mathbf{r} & = & \underline{\mathbf{u}}_2^T \underline{\mathbf{x}} \end{array}$$

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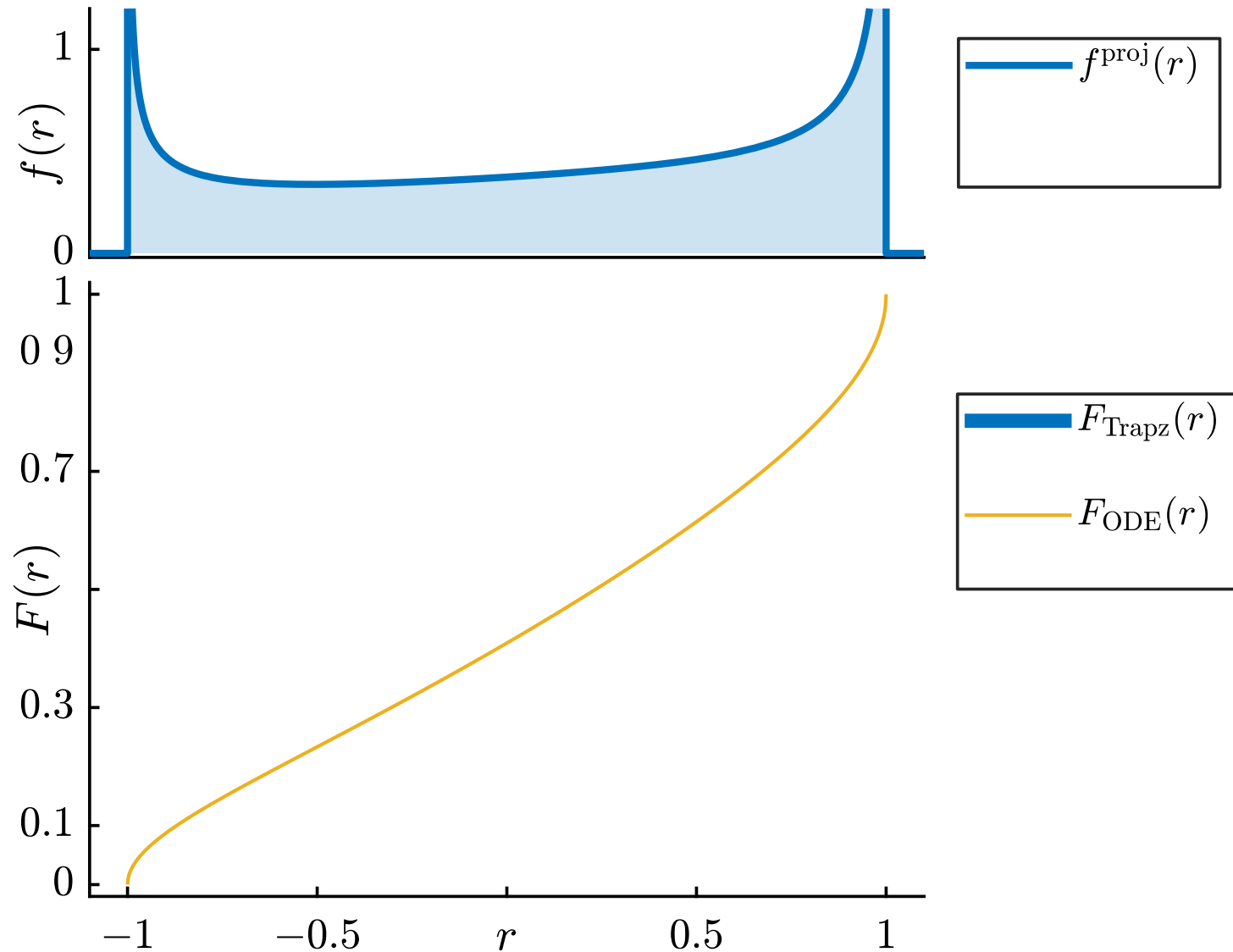


Project Circular Distributions

Projection $S^1 \rightarrow 1D$

- Known support $[-1, 1]$
- Strong weighting at $\{-1, 1\}$

→ evaluation points?



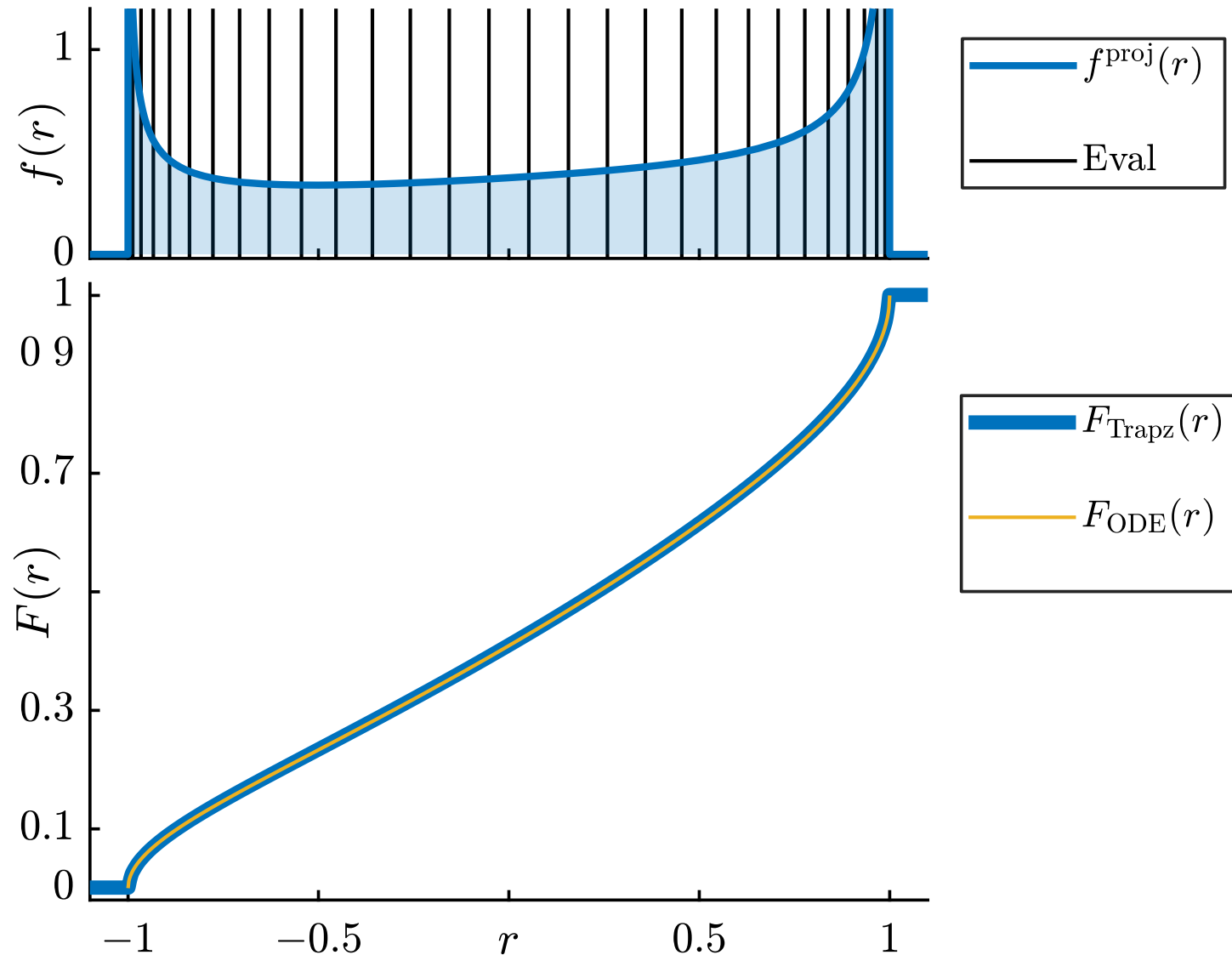
Project Circular Distributions

Projection $S^1 \rightarrow 1D$

- Known support $[-1, 1]$
- Strong weighting at $\{-1, 1\}$

Therefore:

- Fixed evaluation points
- Good trapezoidal



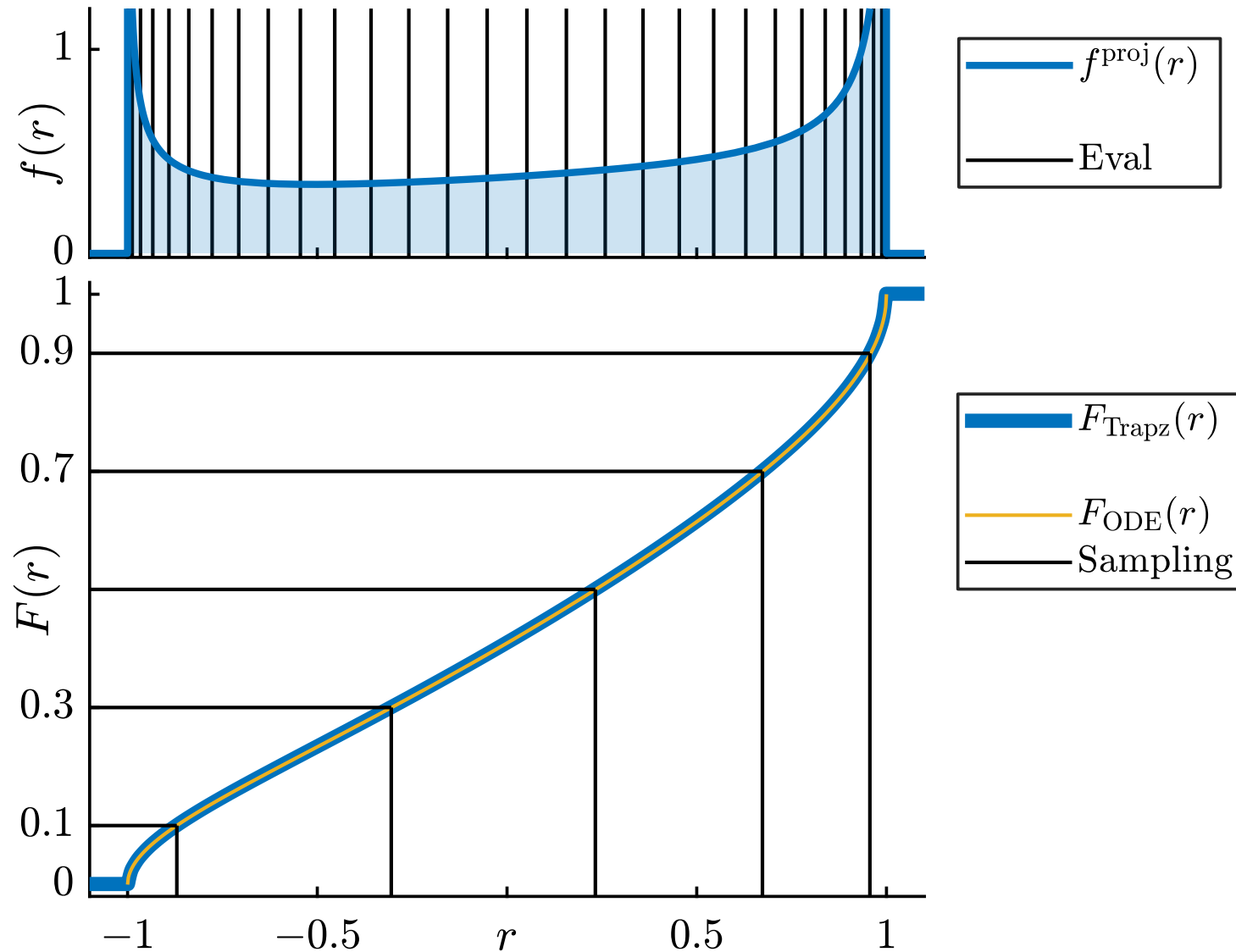
Project Circular Distributions

Projection $S^1 \rightarrow 1D$

- Known support $[-1, 1]$
- Strong weighting at $\{-1, 1\}$

Therefore:

- Fixed evaluation points
- Good trapezoidal



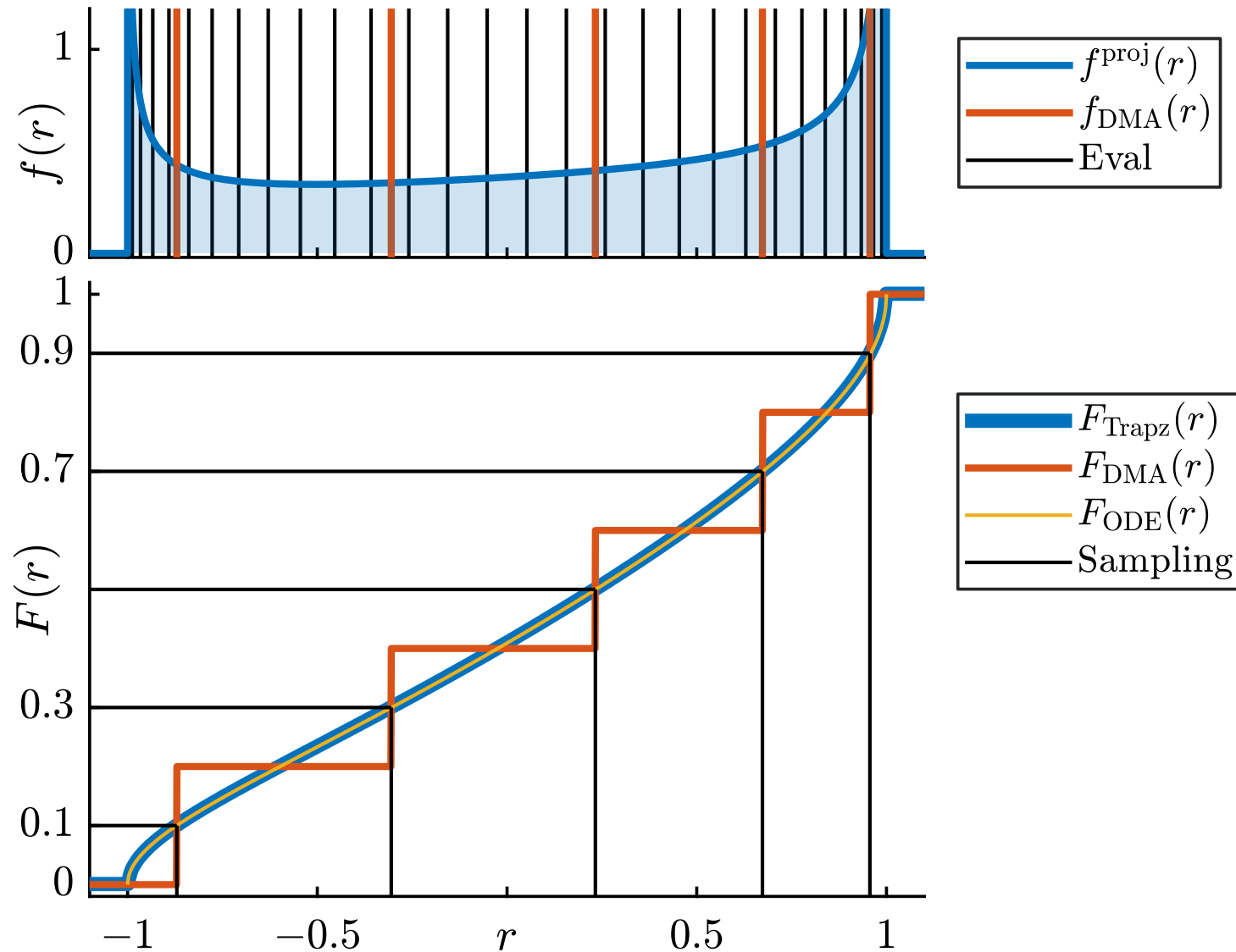
Project Circular Distributions

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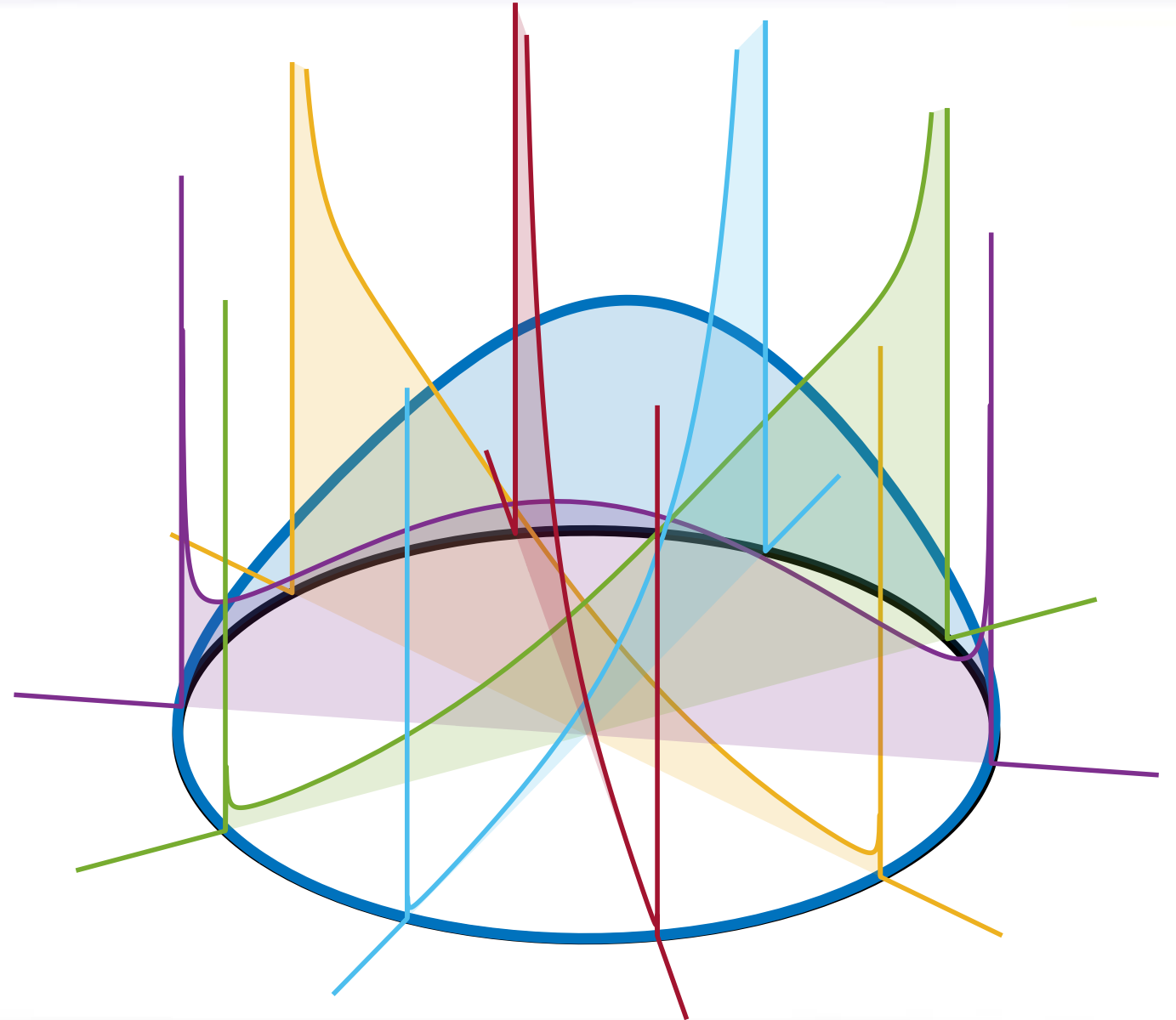
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Therefore:

- Fixed evaluation points
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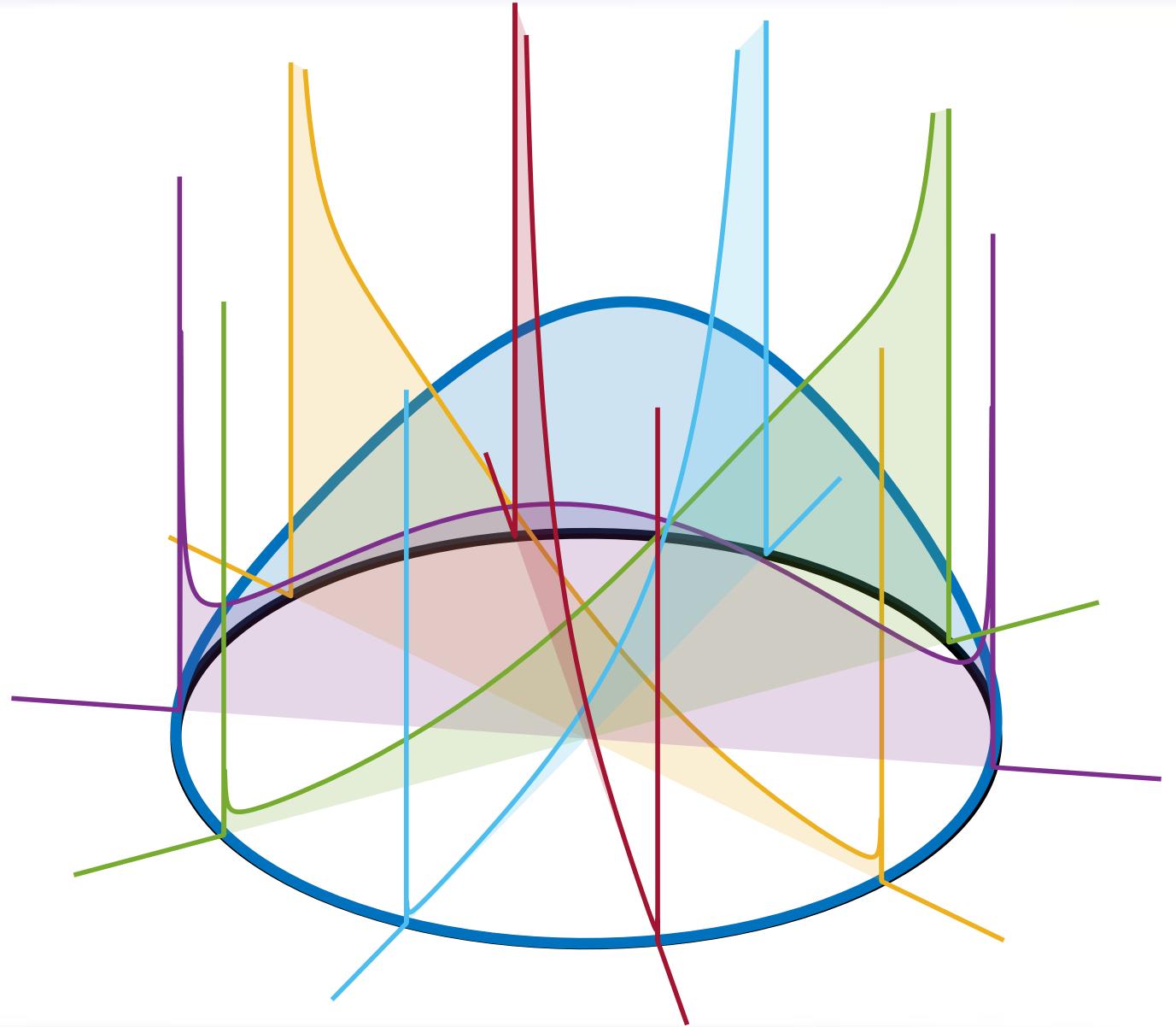


- Consider **many** projections



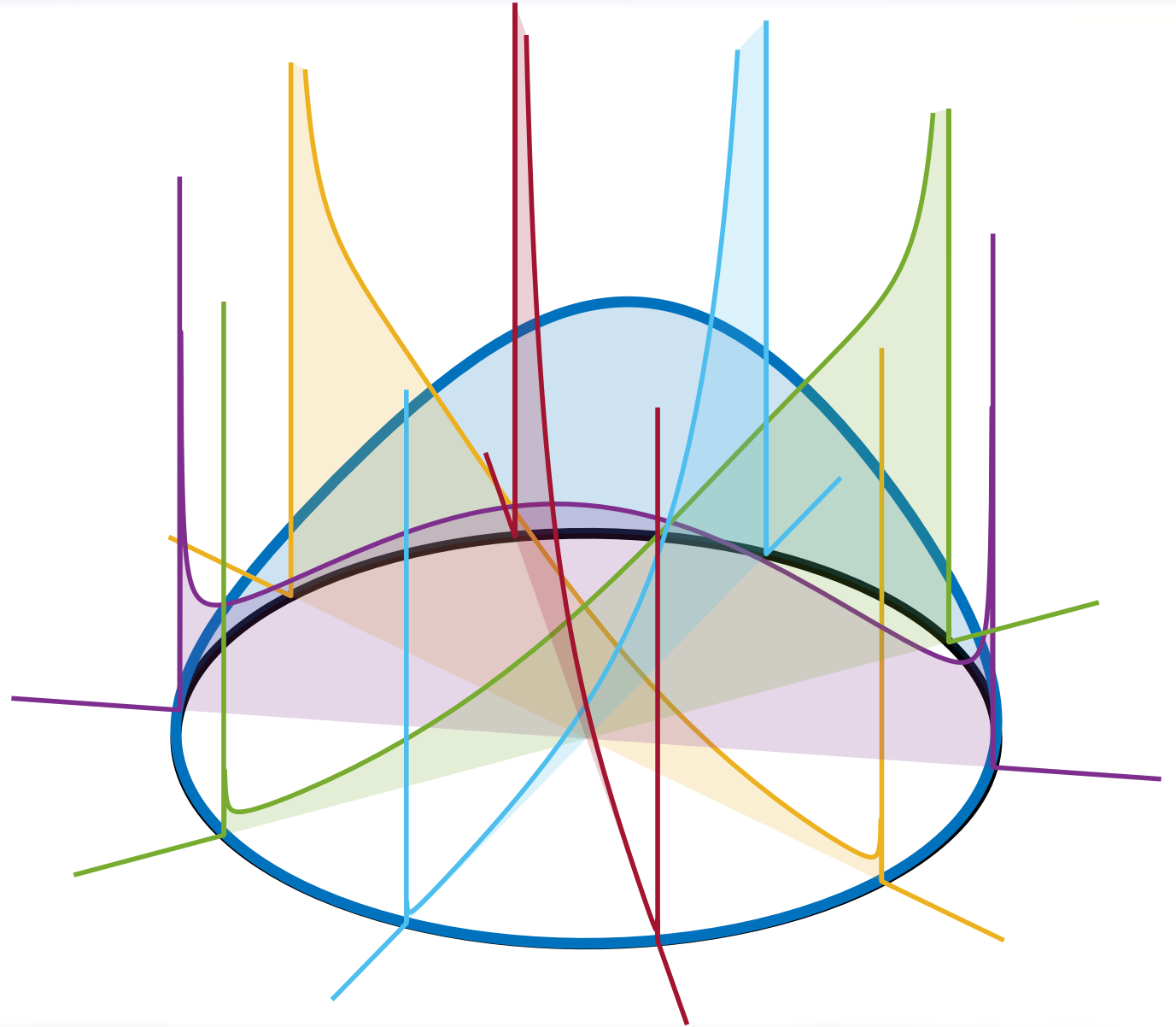
Iterations

- Consider **many** projections
- Iterate
 - Current set of samples



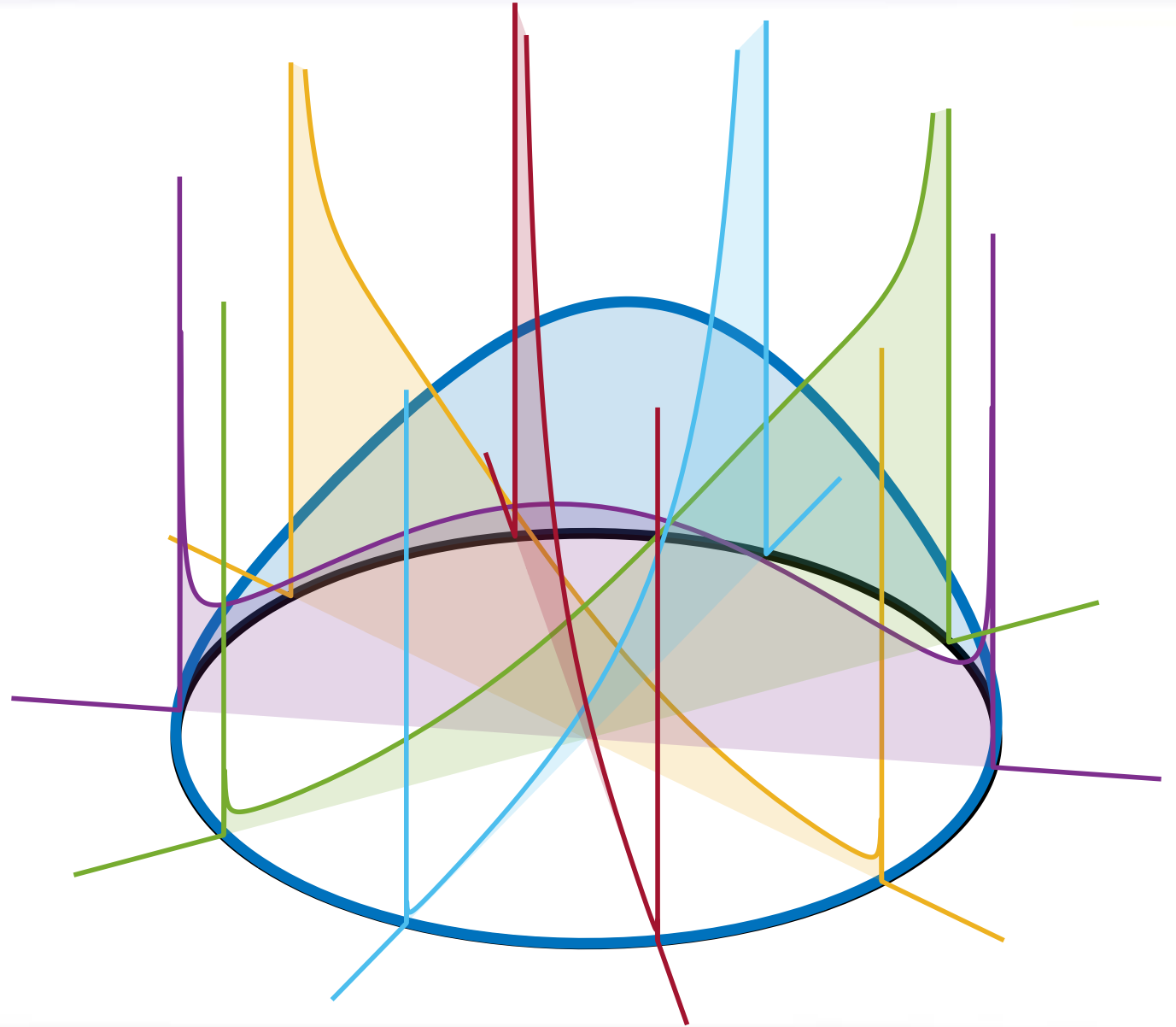
Iterations

- Consider **many** projections
- Iterate
 - Current set of samples
 - New projection



Iterations

- Consider **many** projections
- Iterate
 - Current set of samples
 - New projection
 - Sample update



In Projected Space

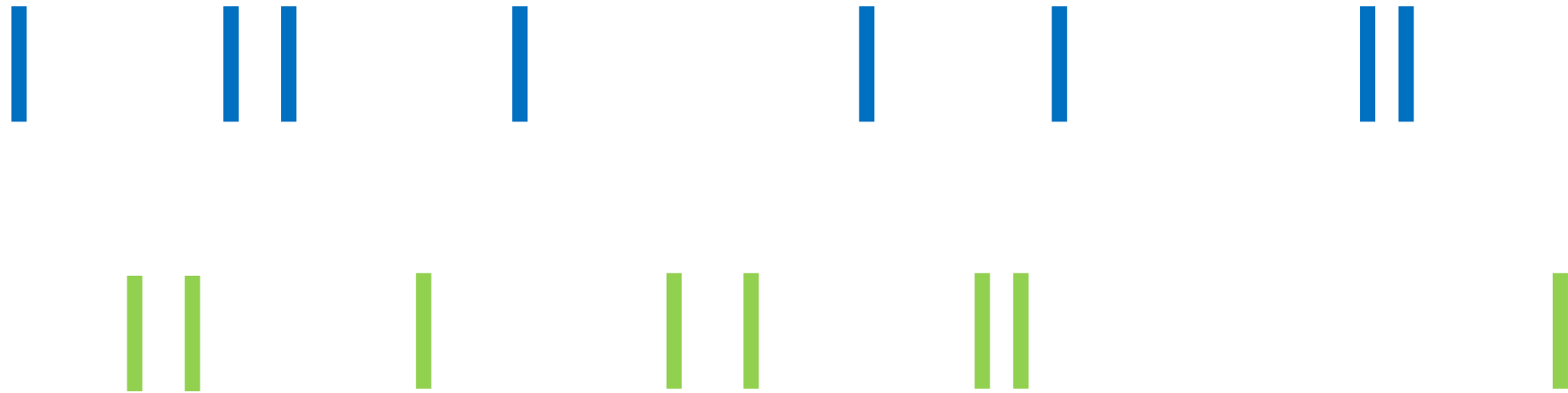
In Projected Space

- Current samples



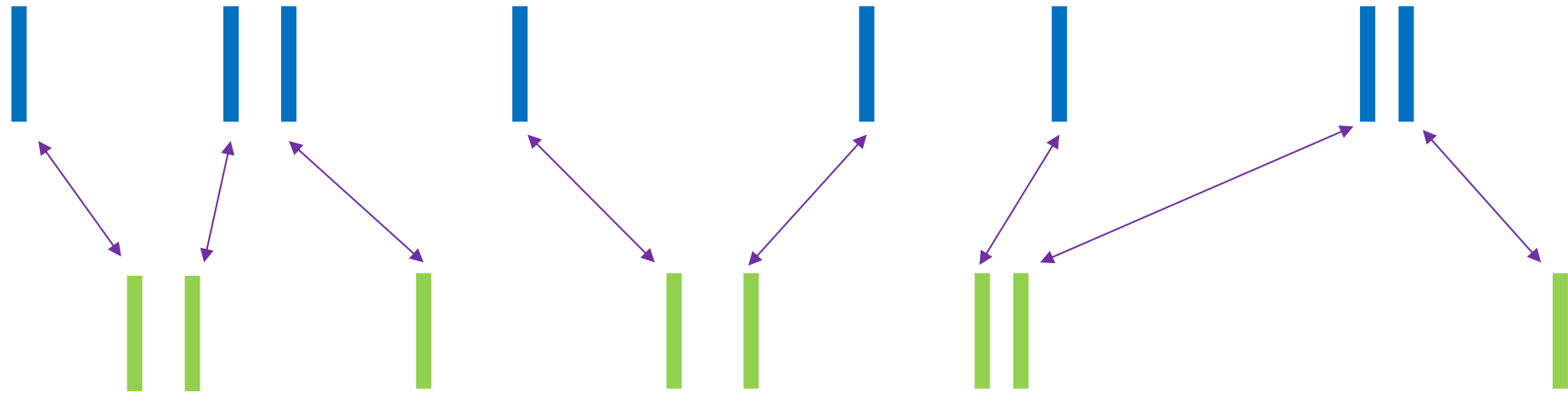
In Projected Space

- Current samples
- From CDF



In Projected Space

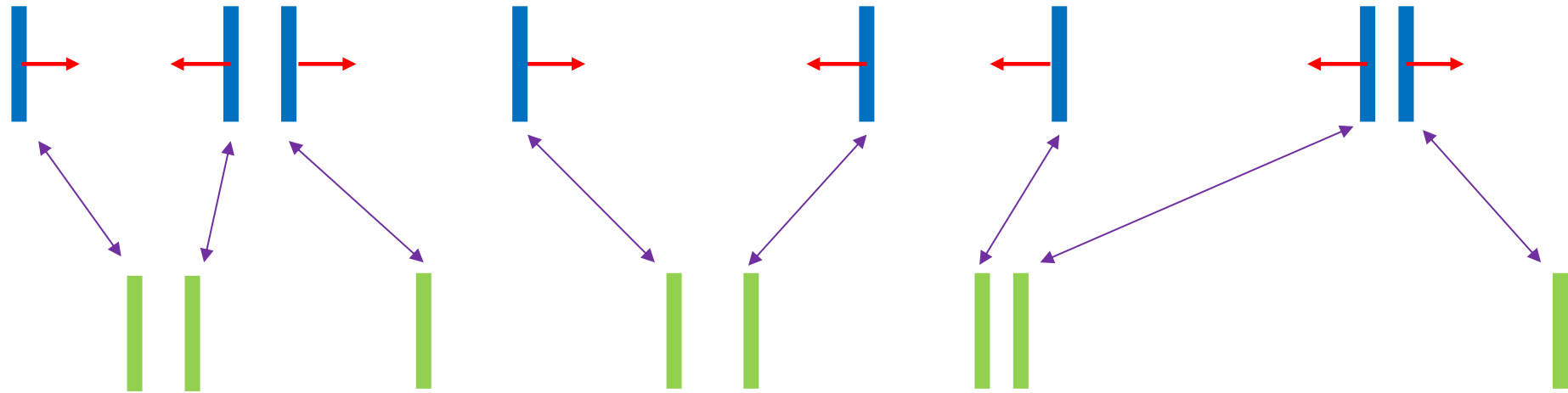
- Current samples
- From CDF
- Wasserstein



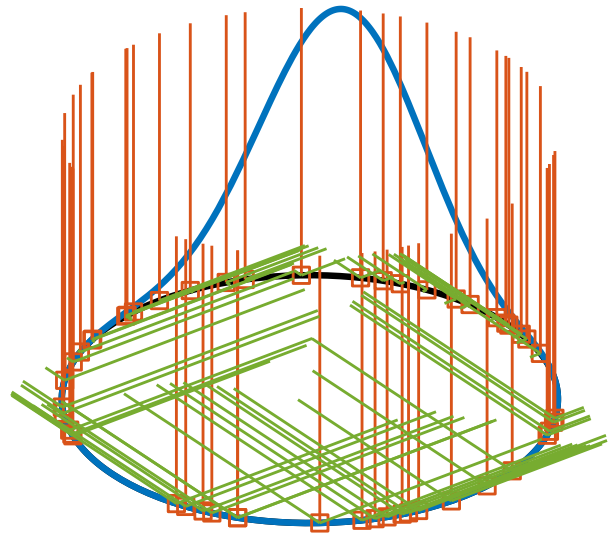
Sample Update

In Projected Space

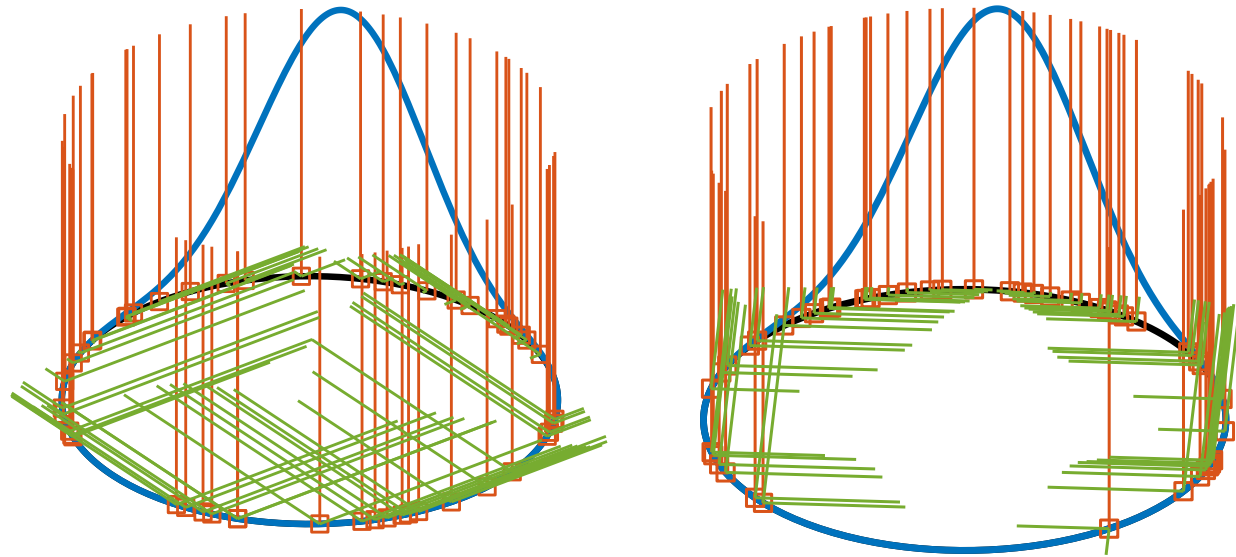
- Current samples
- From CDF
- Wasserstein
- Update



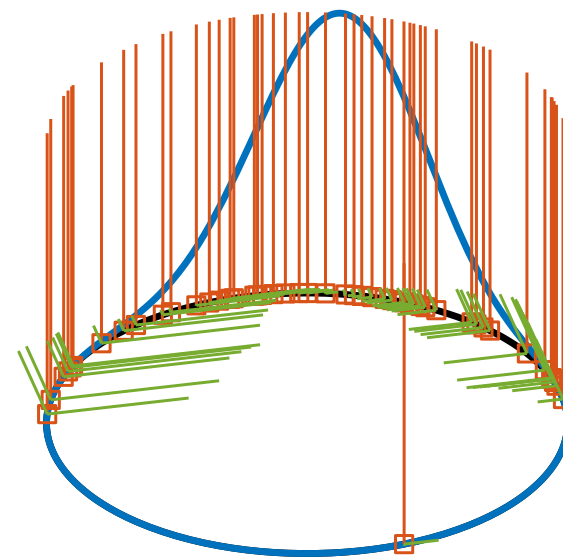
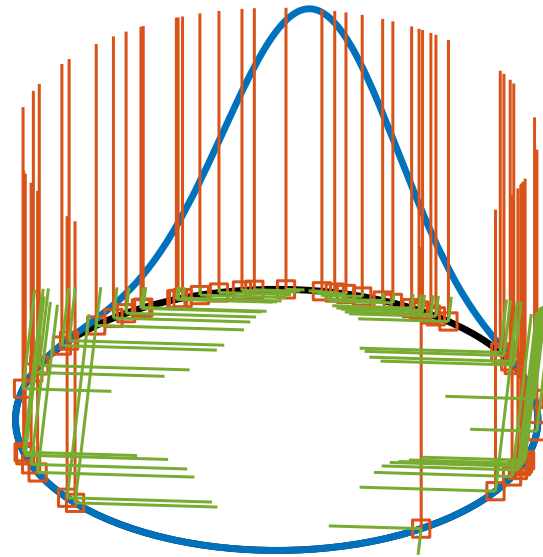
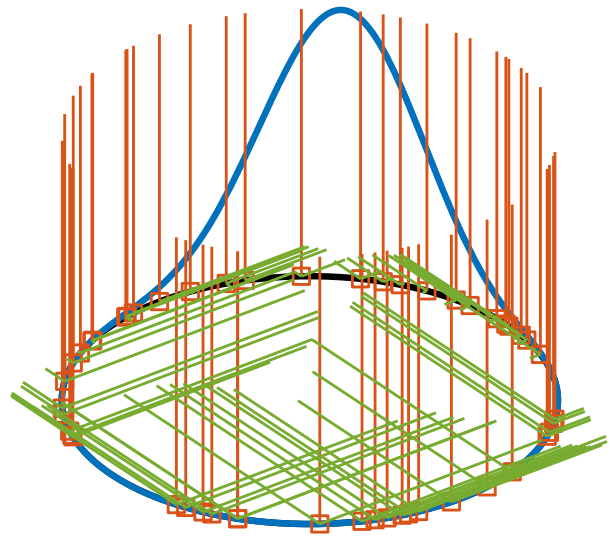
Arbitrary Circular Distribution \rightarrow DMA (Iterations)



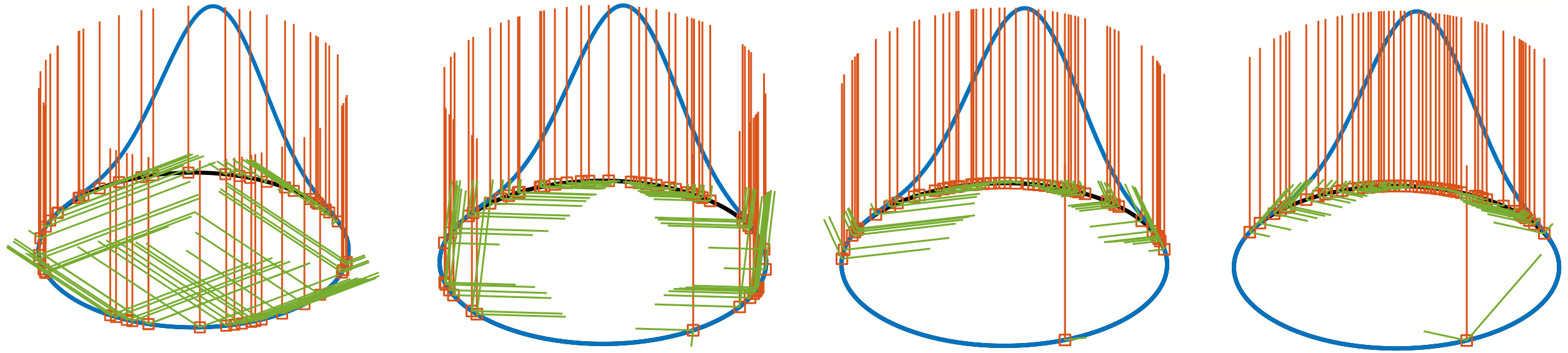
Arbitrary Circular Distribution \rightarrow DMA (Iterations)



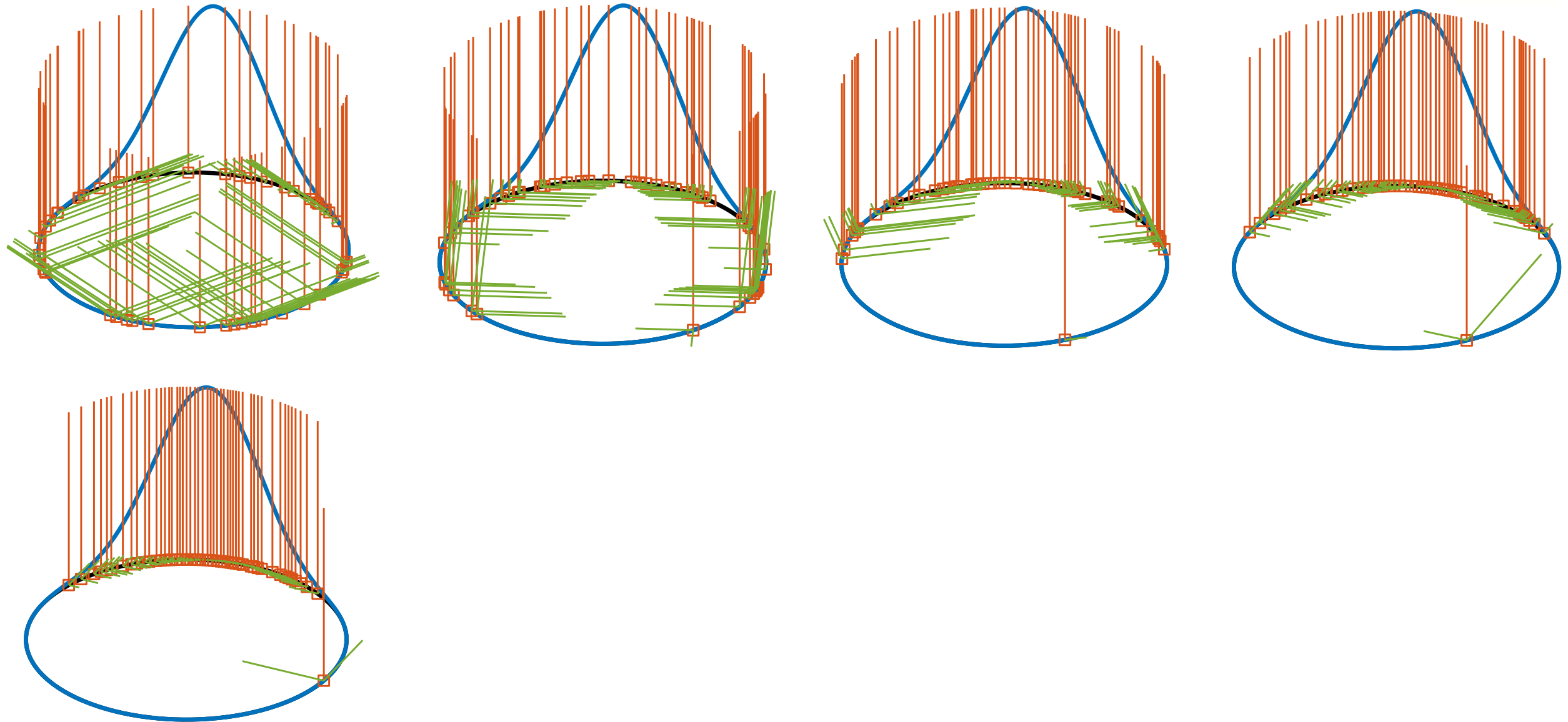
Arbitrary Circular Distribution \rightarrow DMA (Iterations)



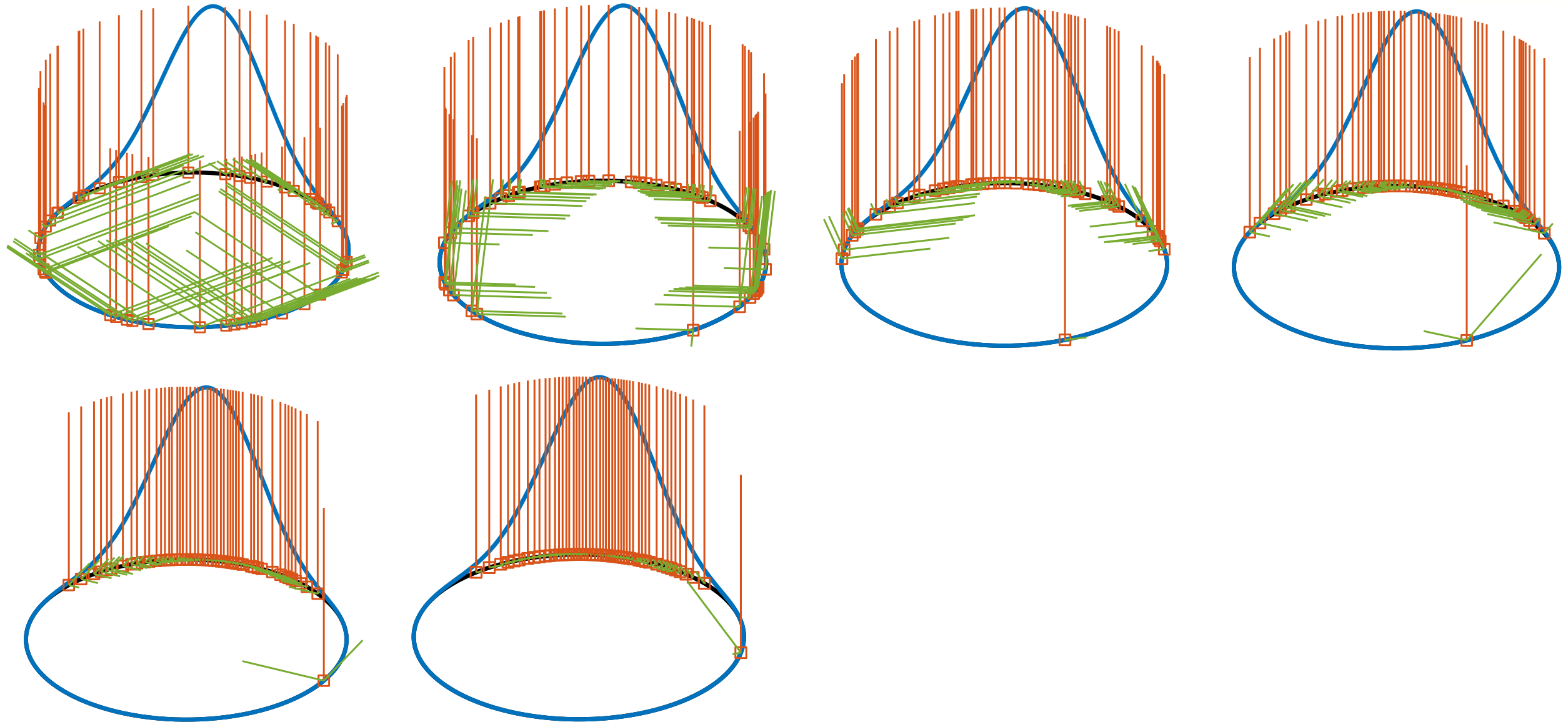
Arbitrary Circular Distribution \rightarrow DMA (Iterations)



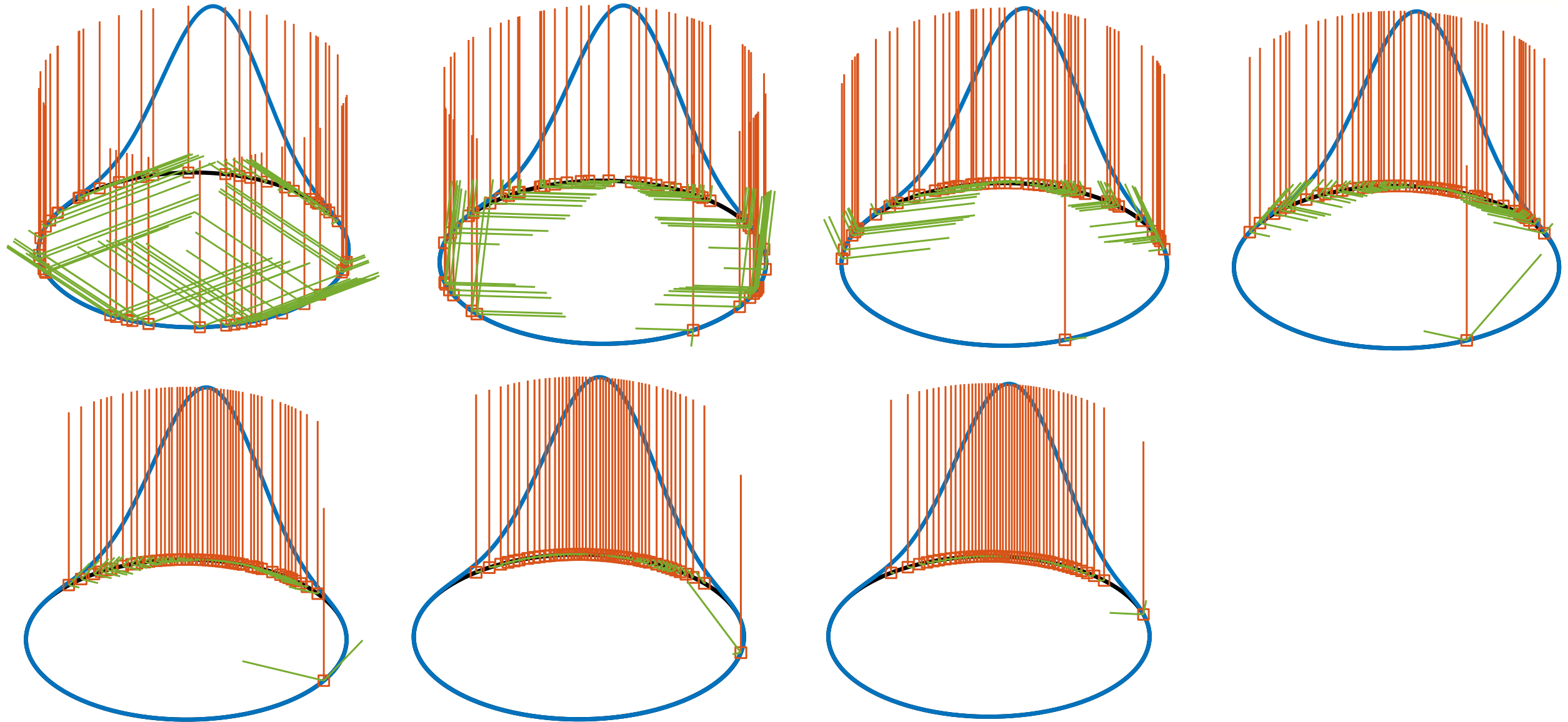
Arbitrary Circular Distribution \rightarrow DMA (Iterations)



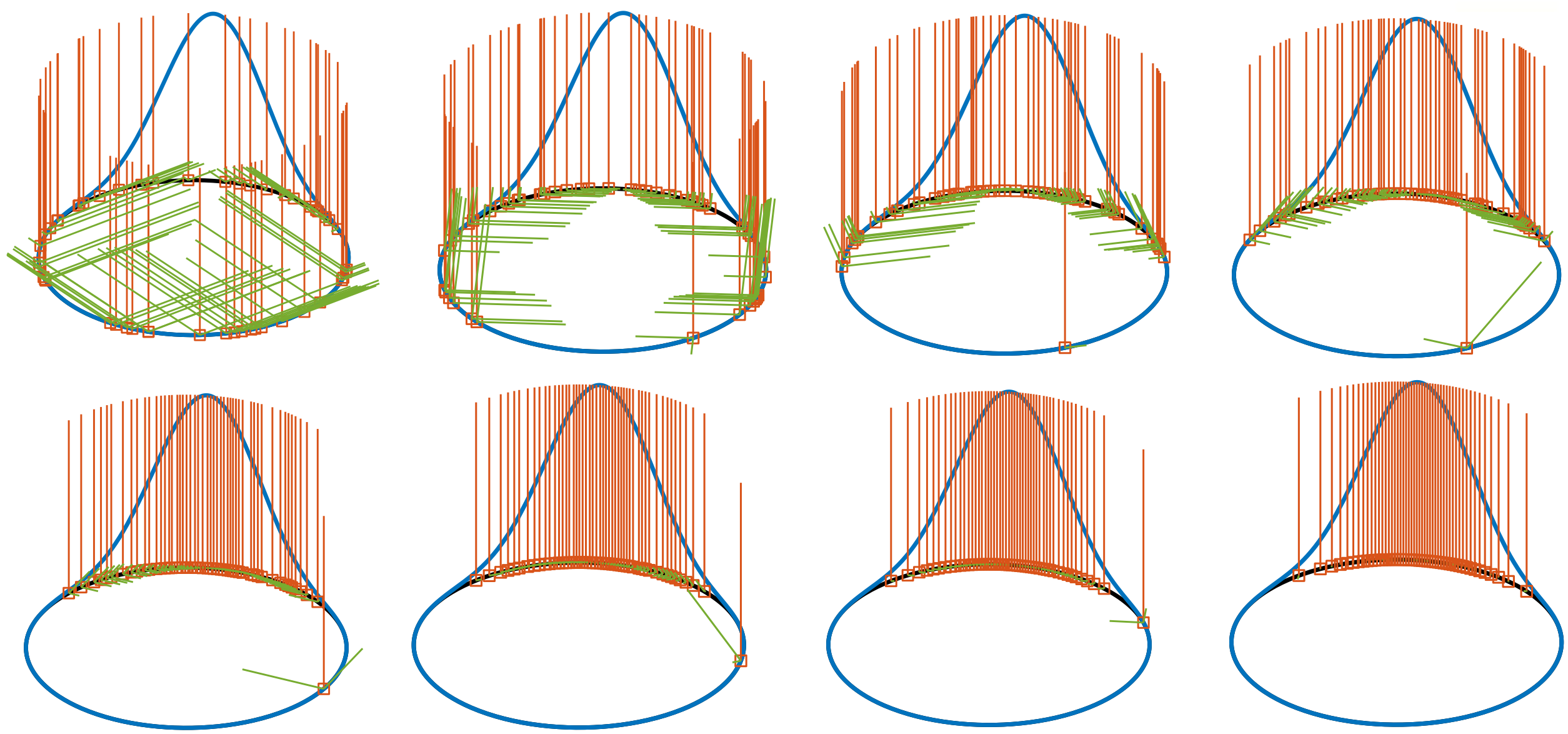
Arbitrary Circular Distribution \rightarrow DMA (Iterations)



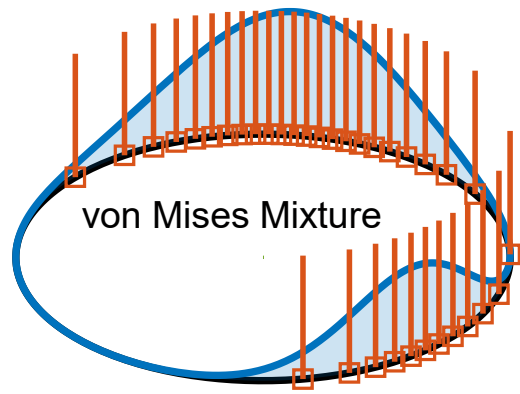
Arbitrary Circular Distribution \rightarrow DMA (Iterations)



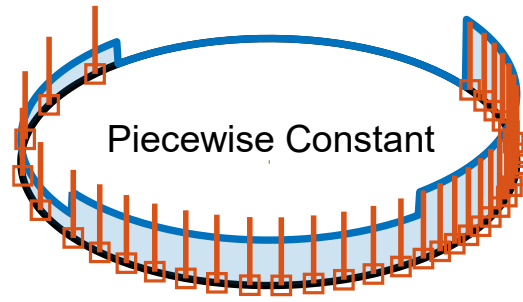
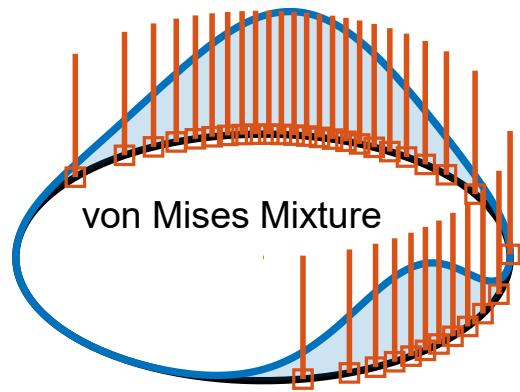
Arbitrary Circular Distribution → DMA (Iterations)



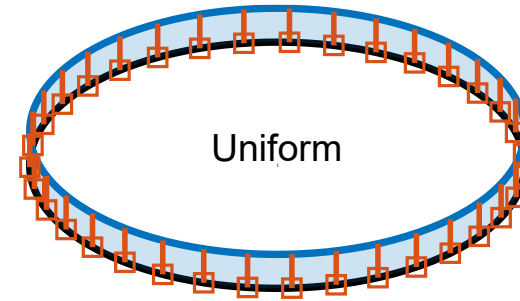
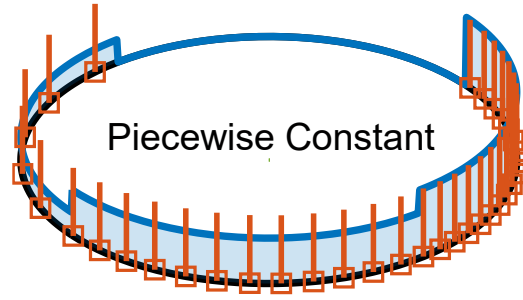
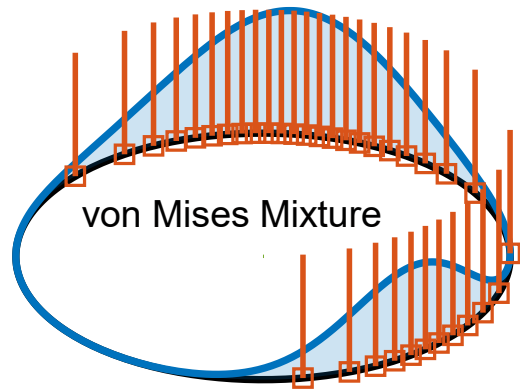
Arbitrary Circular Distribution \rightarrow DMA (Examples)



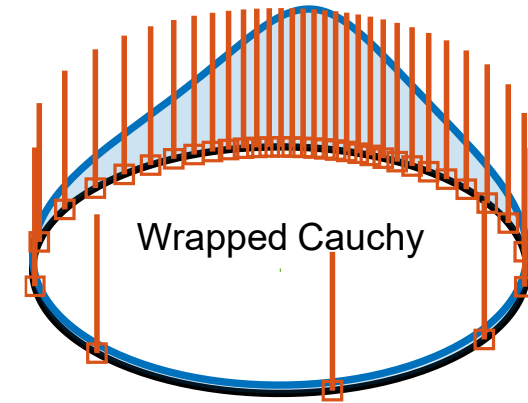
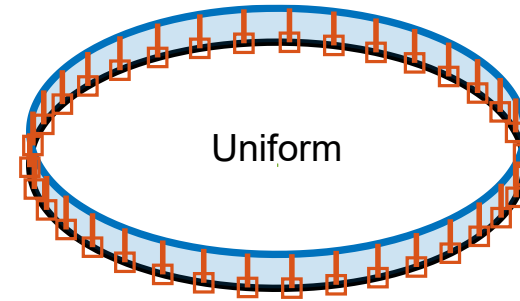
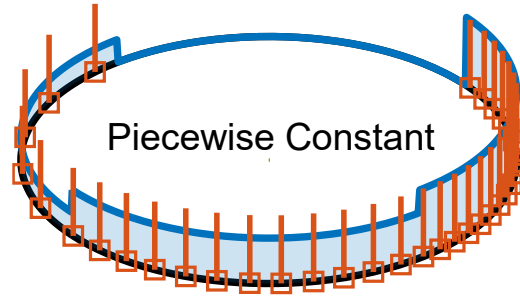
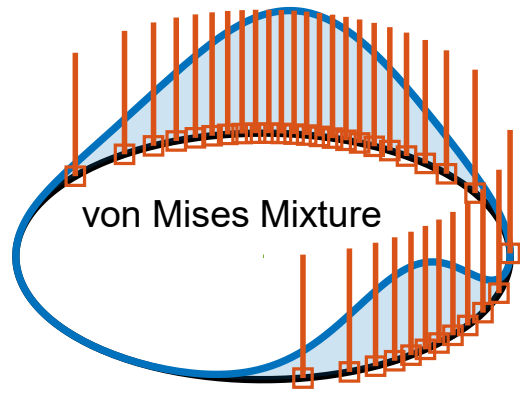
Arbitrary Circular Distribution \rightarrow DMA (Examples)



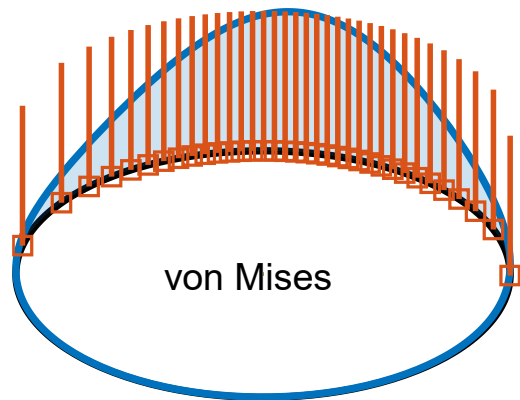
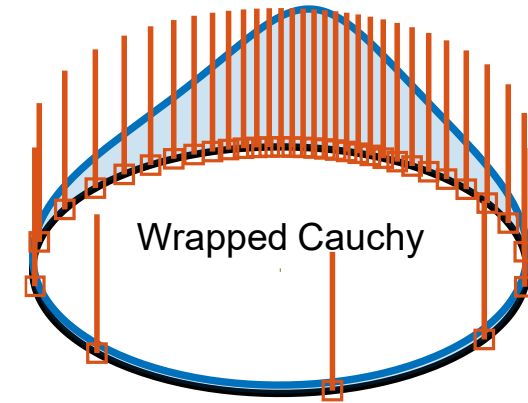
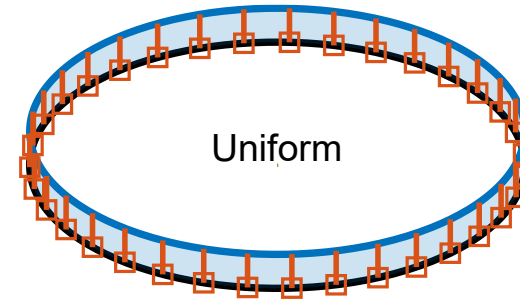
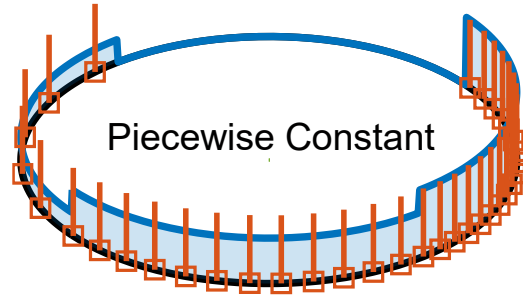
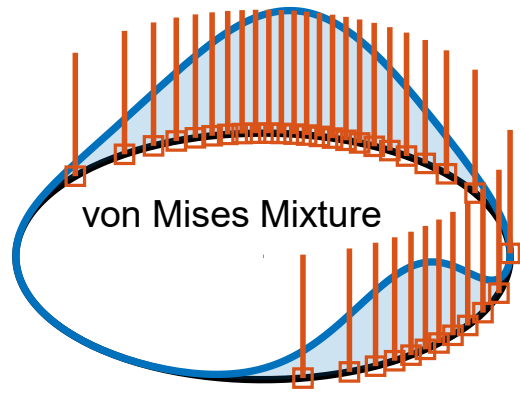
Arbitrary Circular Distribution \rightarrow DMA (Examples)



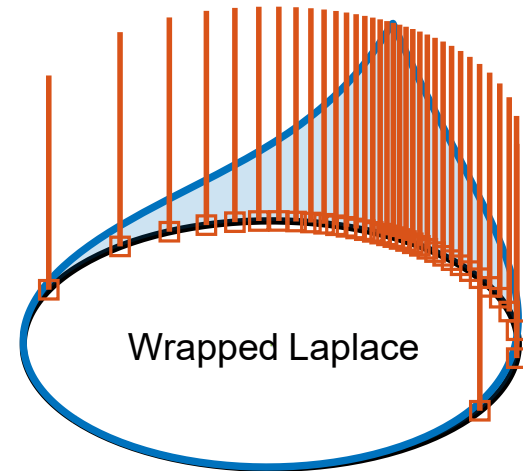
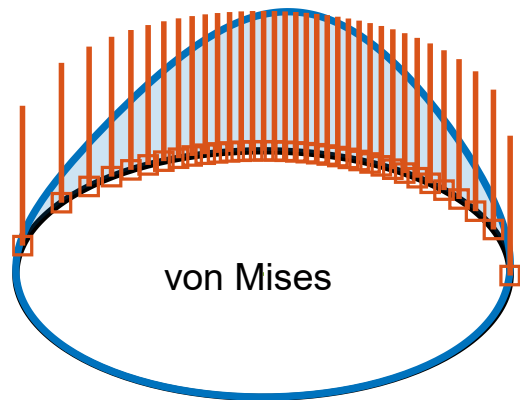
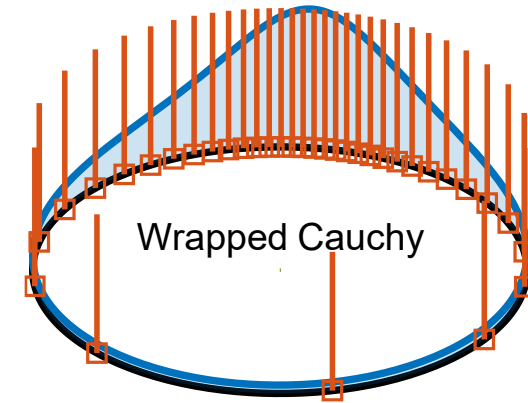
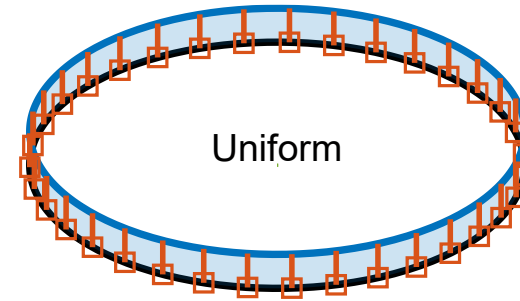
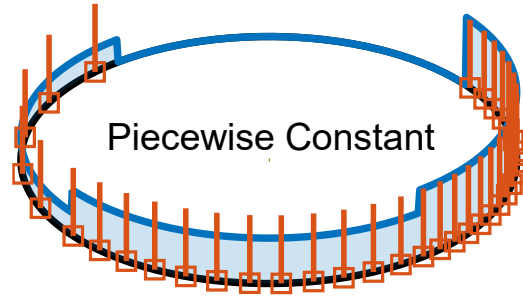
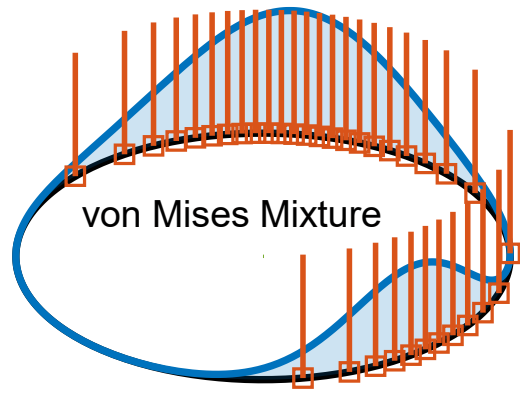
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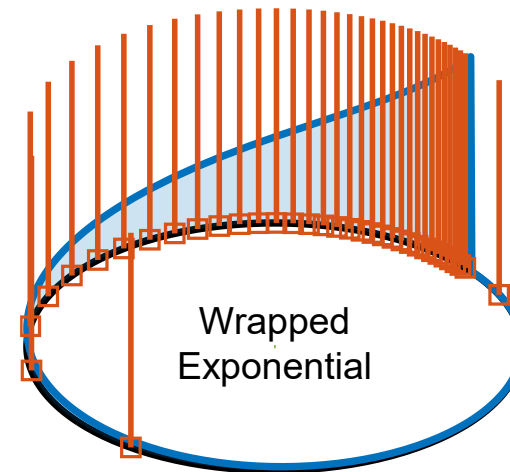
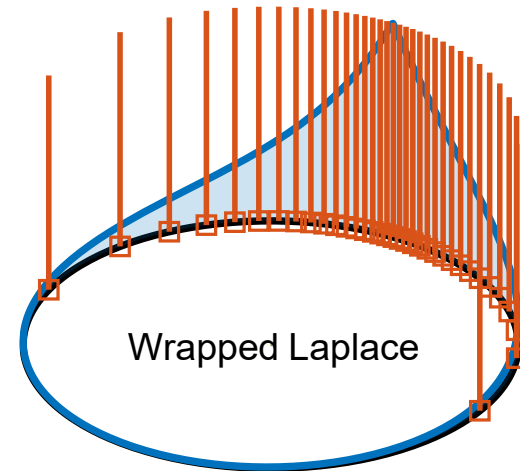
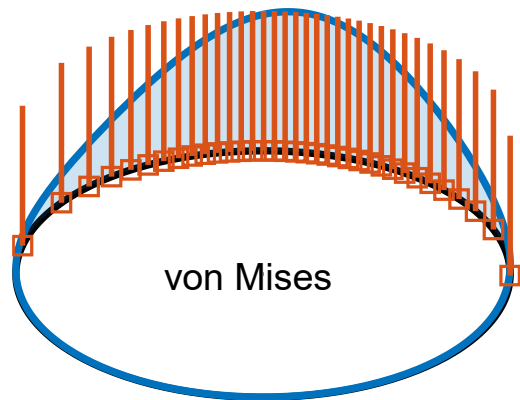
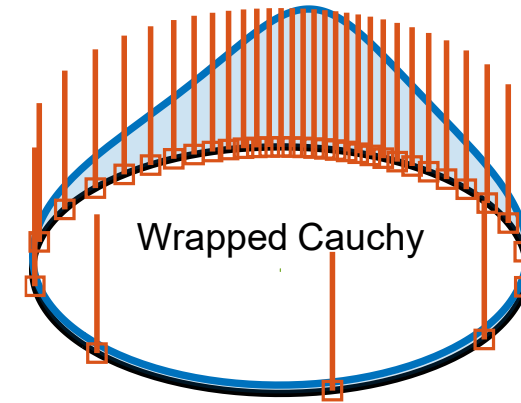
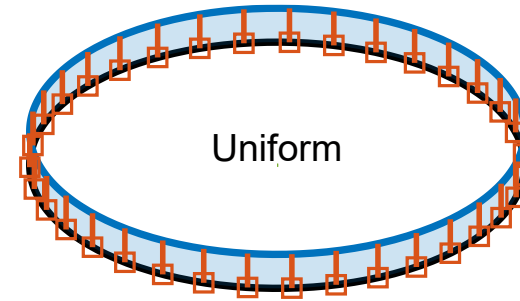
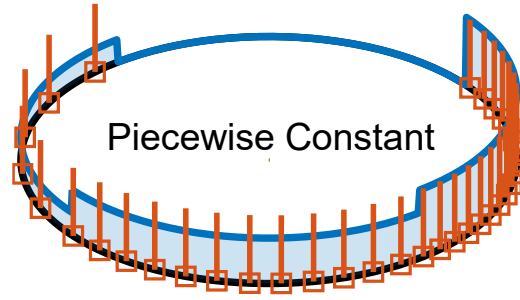
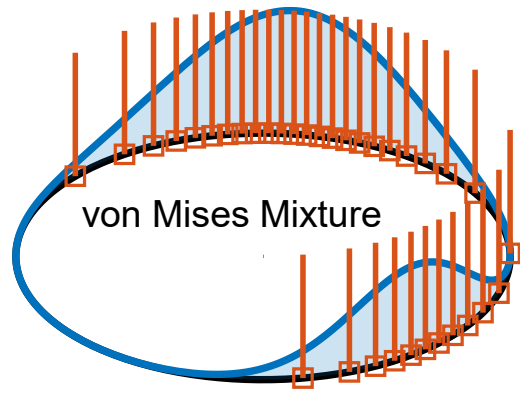
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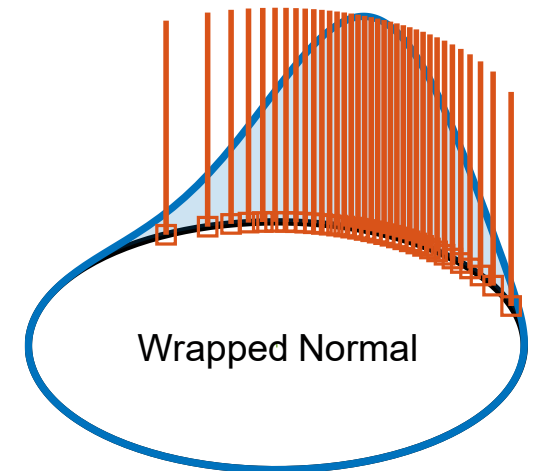
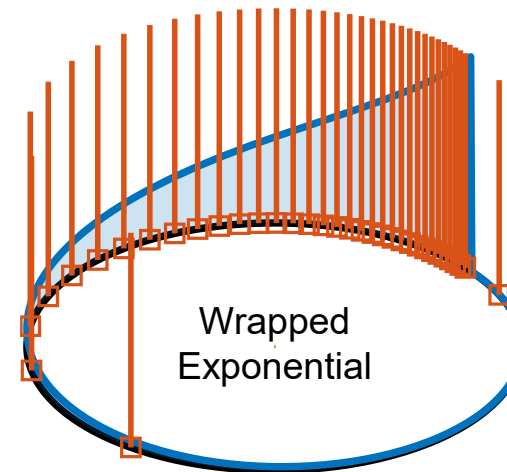
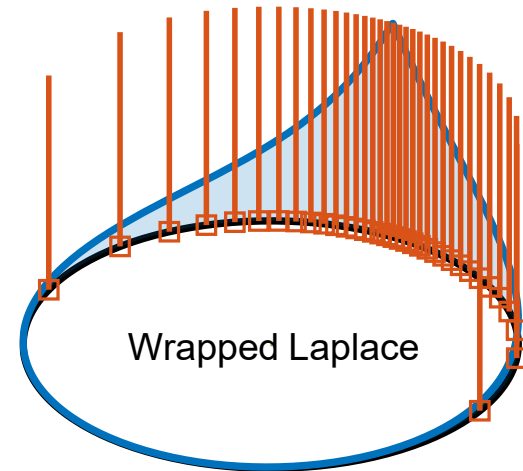
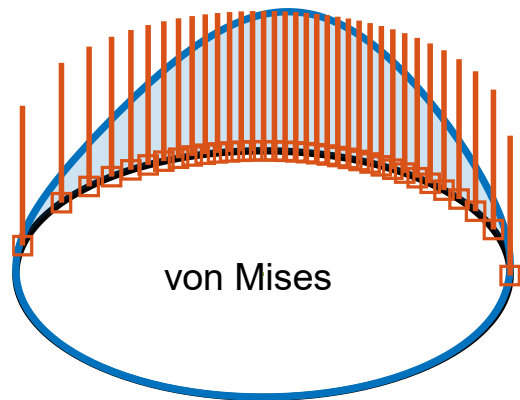
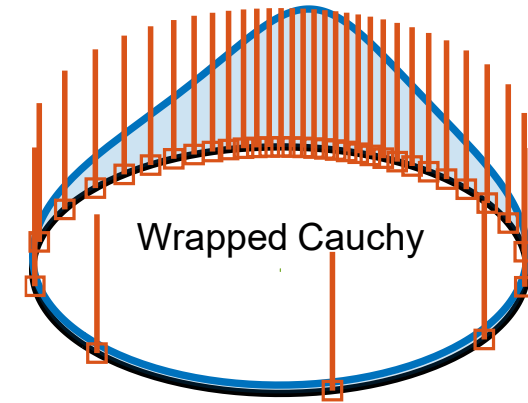
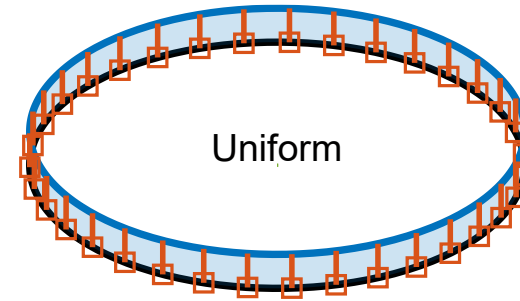
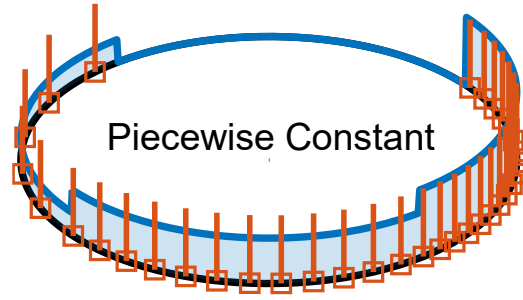
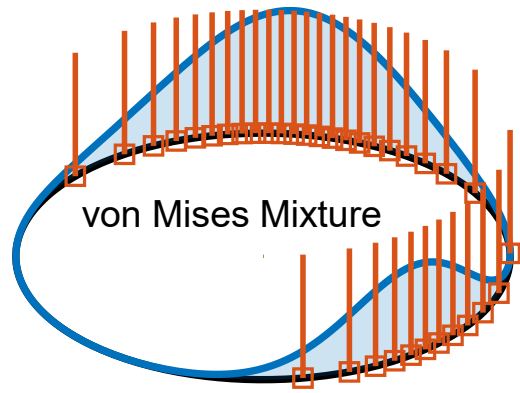
Arbitrary Circular Distribution \rightarrow DMA (Examples)



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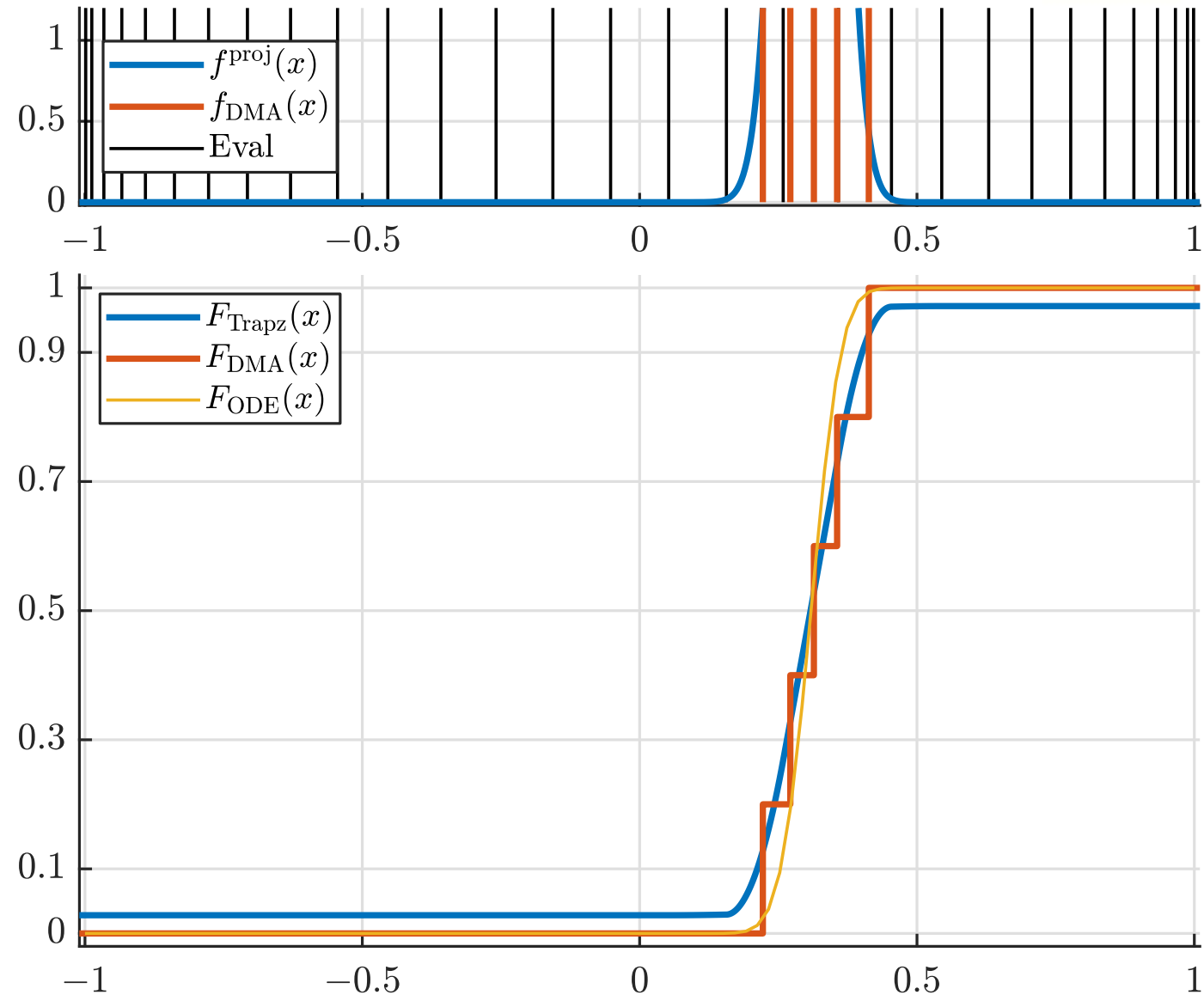


Arbitrary Circular Distribution \rightarrow DMA (Examples)



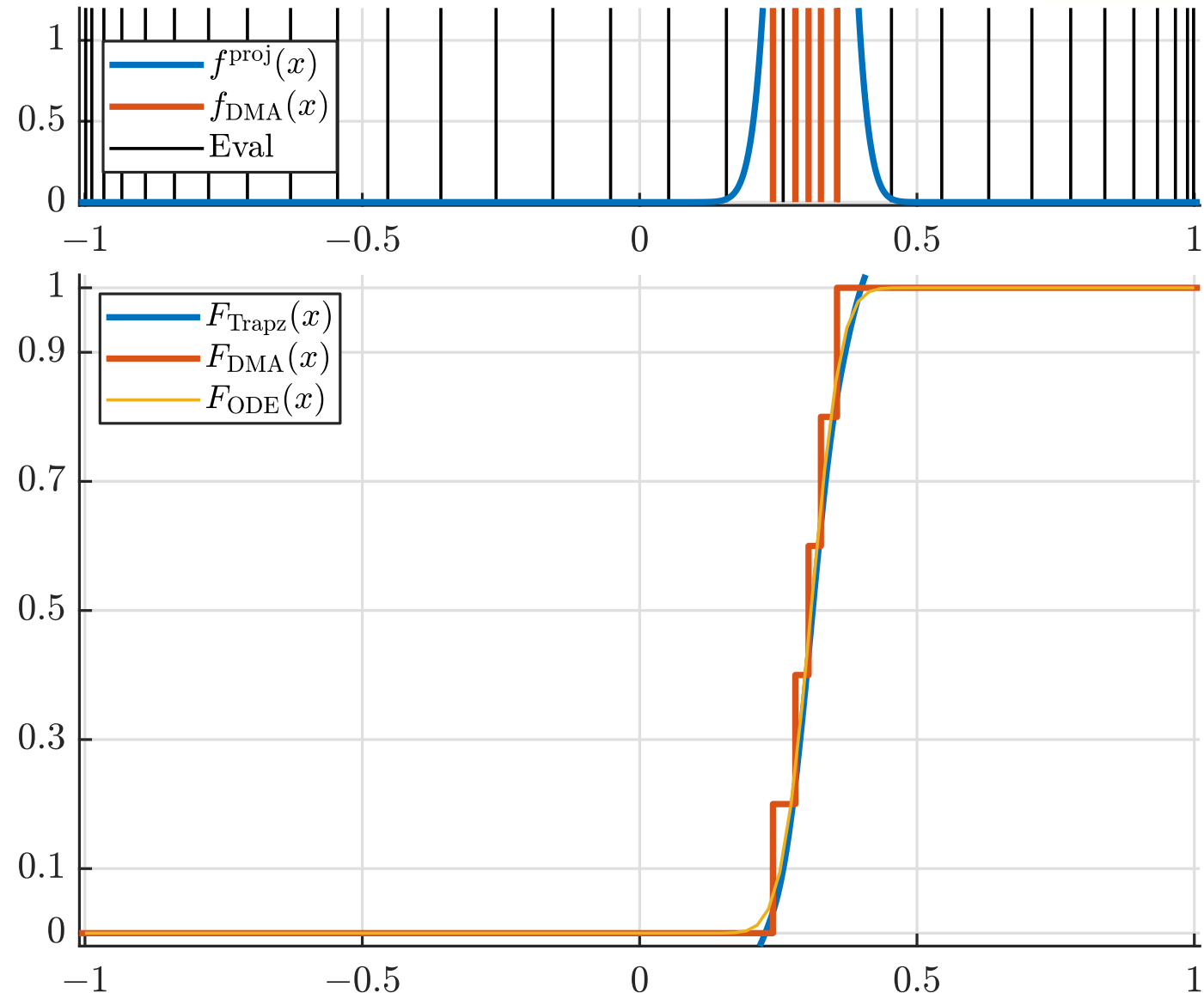
Concentrated Densities

- Small uncertainty
- Fixed eval points not enough



Concentrated Densities

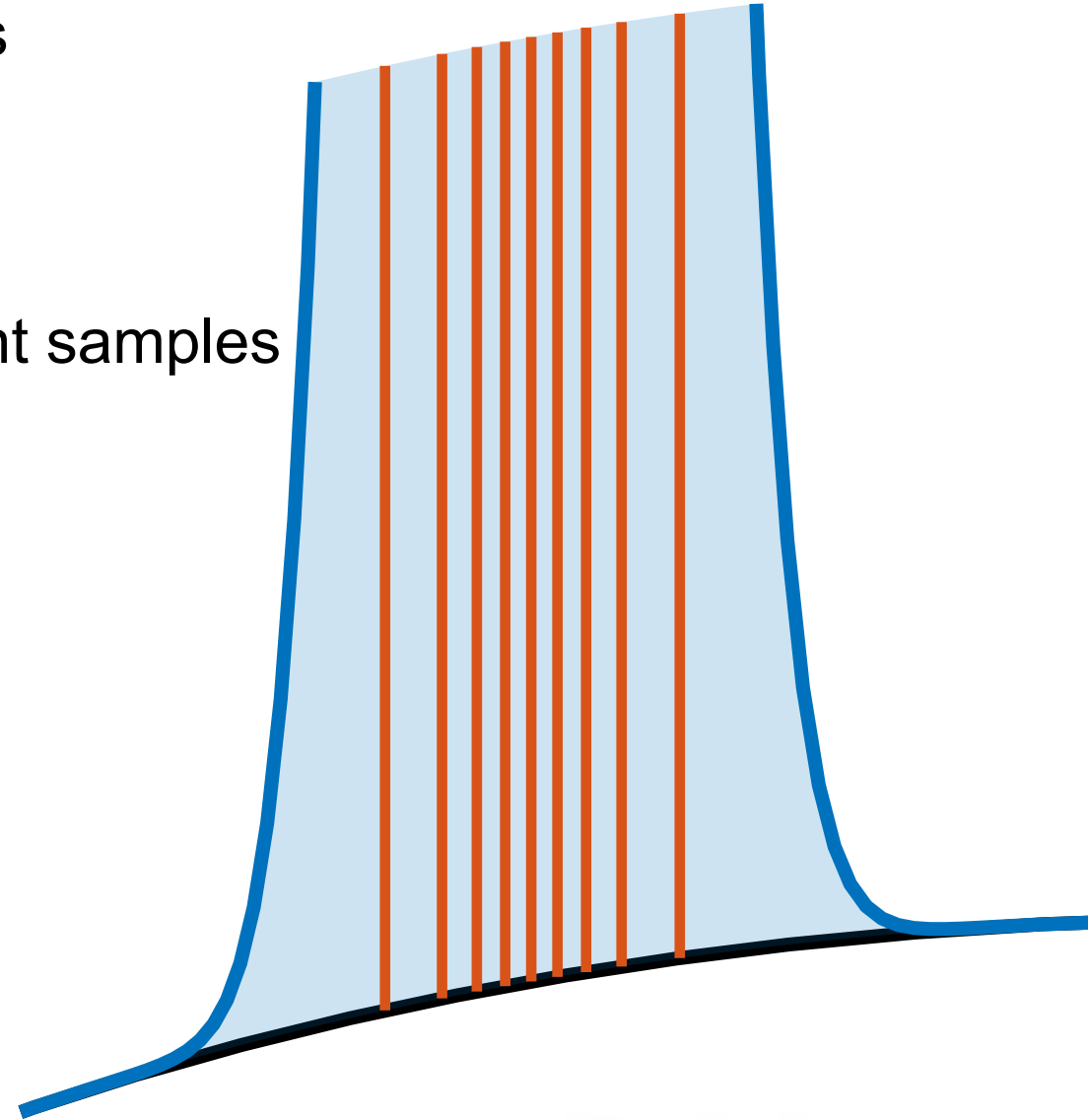
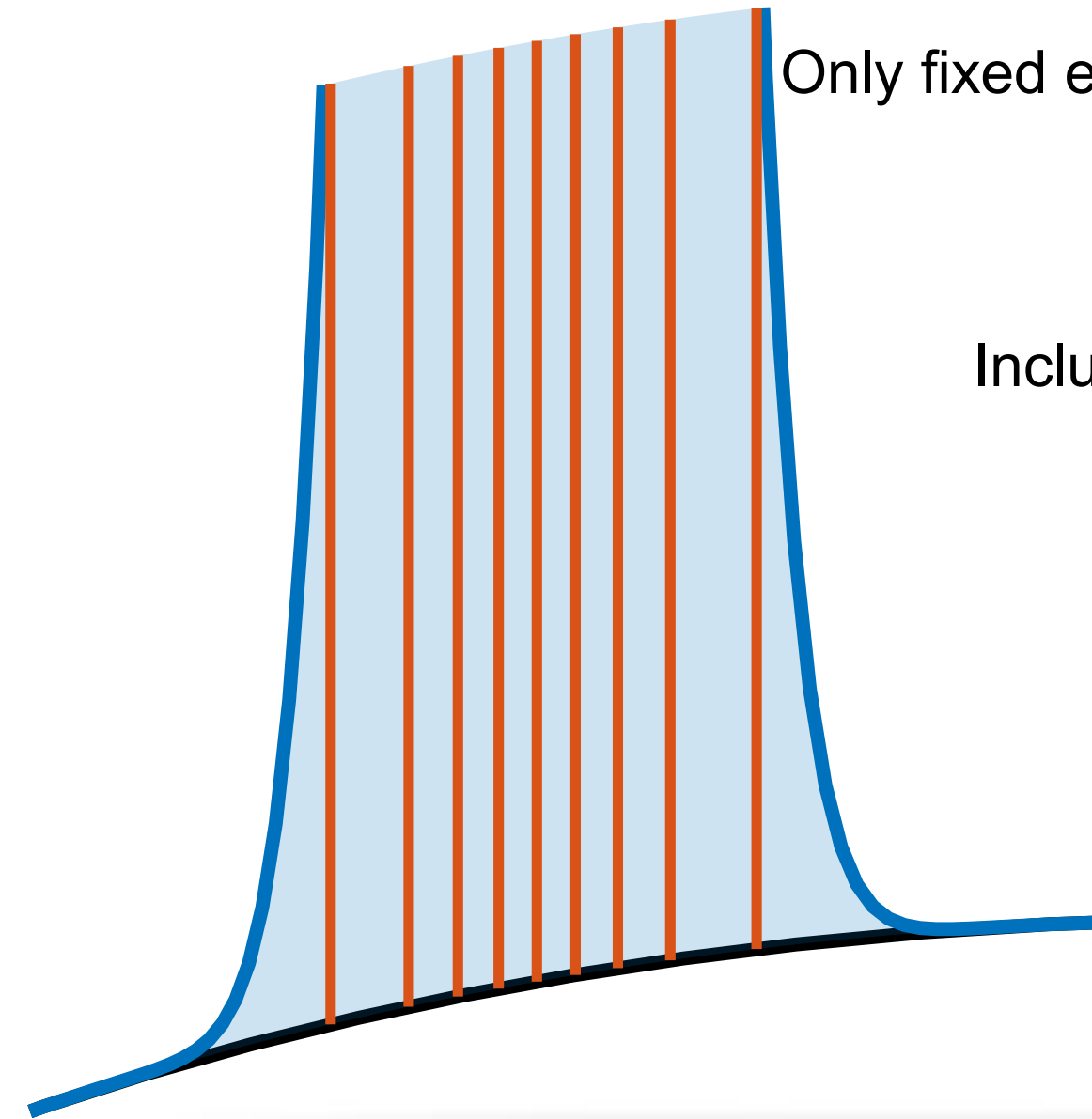
- Small uncertainty
- Fixed eval points not enough
- Use current samples as well for
 - PDF evaluation
 - CDF Trapezoidal integration



Concentrated Densities

Only fixed eval points

Include current samples



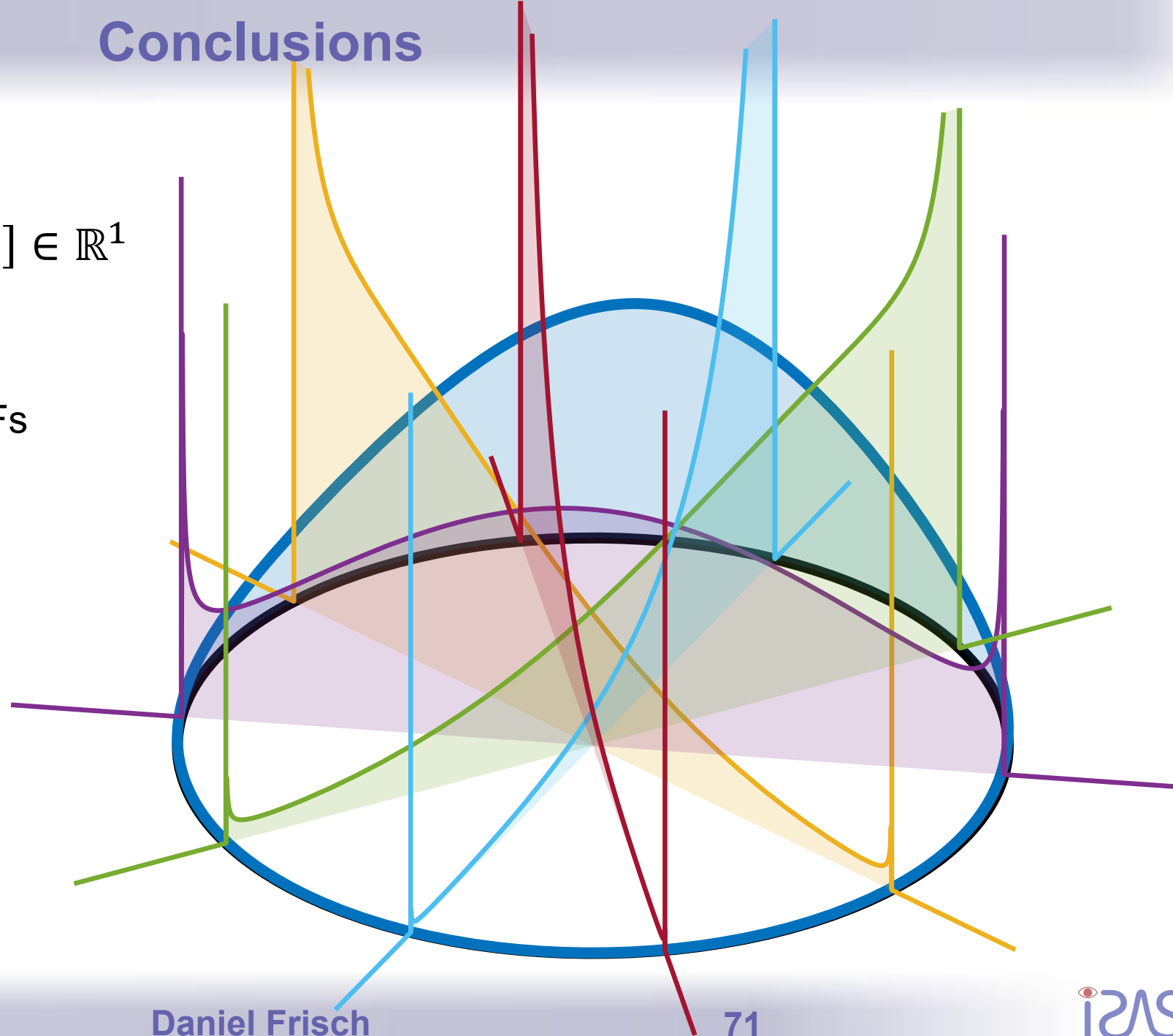
Conclusions

Achieved

- PCD projects \mathcal{S}^1 to $[-1,1] \in \mathbb{R}^1$
- Univariate CDFs
 - Easily compare continuous & discrete PDFs
 - Easily compute from PDF

Outlook

- Hypersphere
 - Sample reduction
 - Von Mises–Fisher
 - Bingham
 - etc.



Thank you for your attention

Intelligent
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Sensor-Actuator-Systems