Rejection Sampling from Arbitrary Multivariate Distributions Using Generalized Fibonacci Lattices

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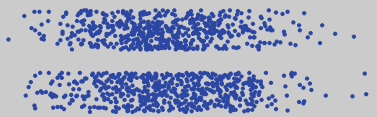
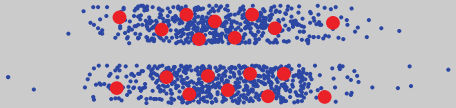
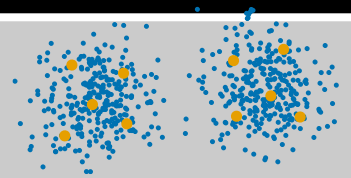
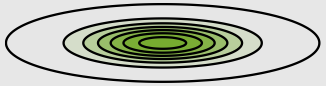
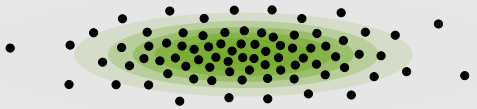
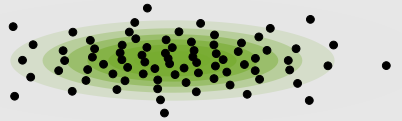
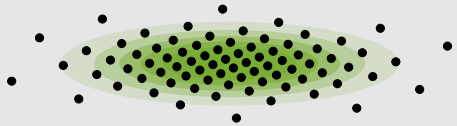
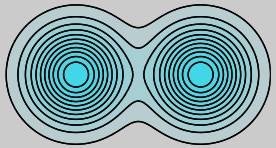
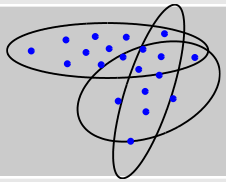
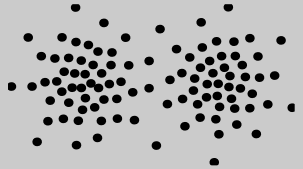
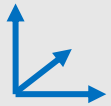
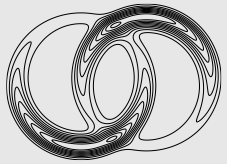
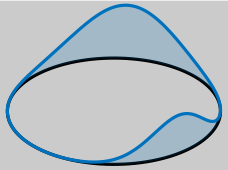
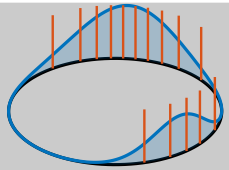
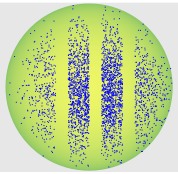
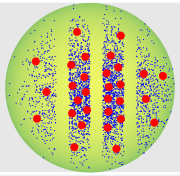
Karlsruhe Institute of Technology

Intelligent Sensor-Actuator-Systems

Deterministic Sampling – State of Art

Domain	Reference	LCD	PCD	Fibonacci
$\mathbb{R}^d$ 	DM 			
$\mathbb{R}^d$ 	Gauss 			
$\mathbb{R}^d$ 	GM 			
$\mathbb{R}^d$ 	Arbitrary 			
S^1 	Arbitrary 			
S^n 	DM 			

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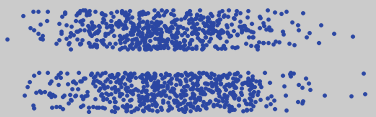
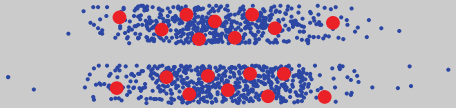
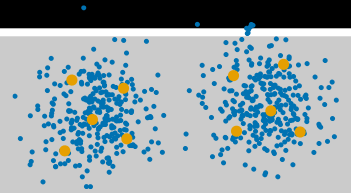
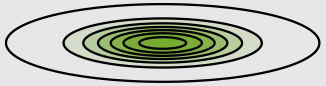
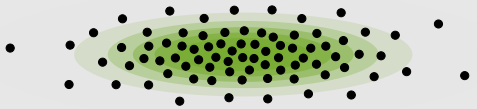
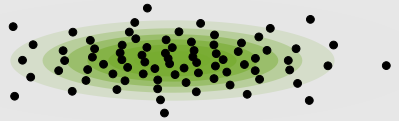
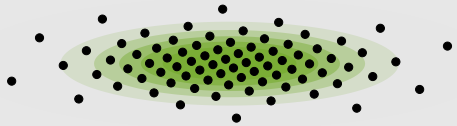
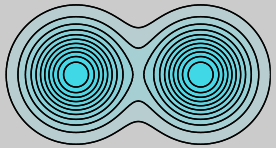
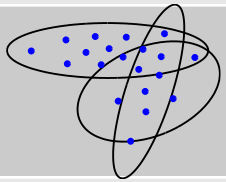
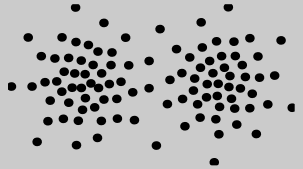
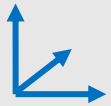
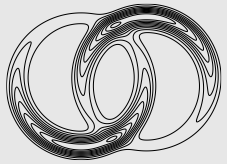
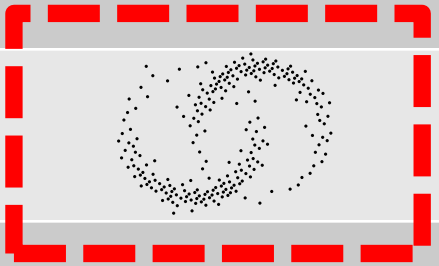
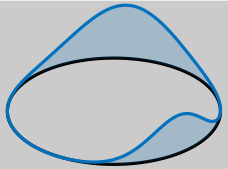
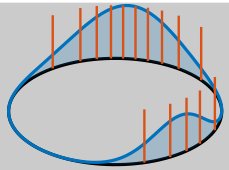
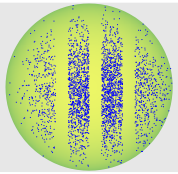
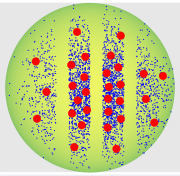
$\mathbb{R}^d$ 	DM 			
$\mathbb{R}^d$ 	Gauss 			
$\mathbb{R}^d$ 	GM 			
$\mathbb{R}^d$ 	Arbitrary 			
S^1 	Arbitrary 			
S^n 	DM 			

Overview

Input

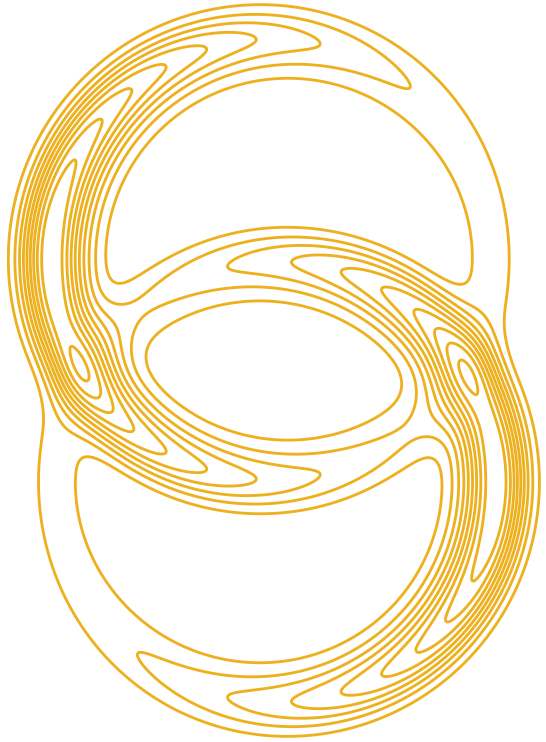
Density Function

Overview

Input

Density Function

- Arbitrary function handle

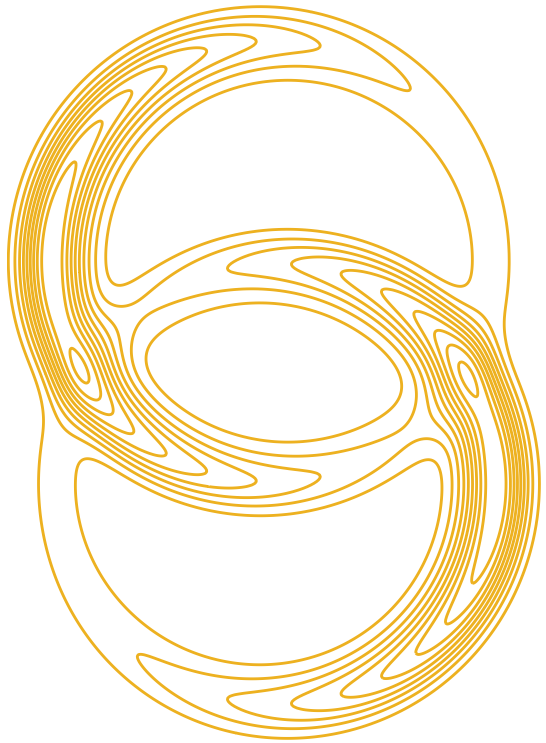


Overview

Input

Density Function

- Arbitrary function handle
- Proposal



Overview

Input

Density Function

- Arbitrary function handle
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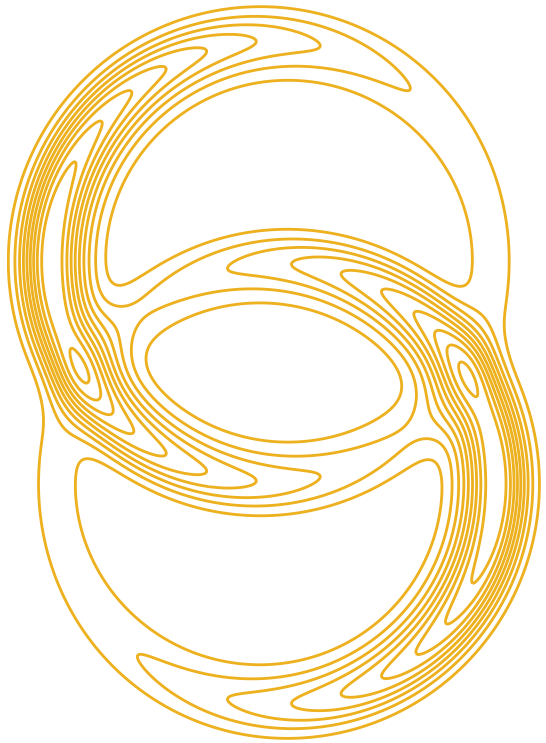
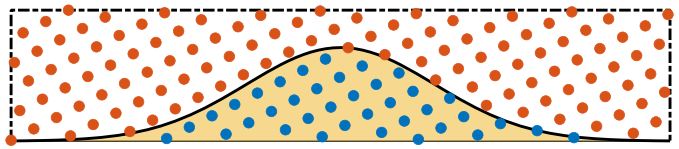
Rejection Sampling





Overview

Input

Density Function

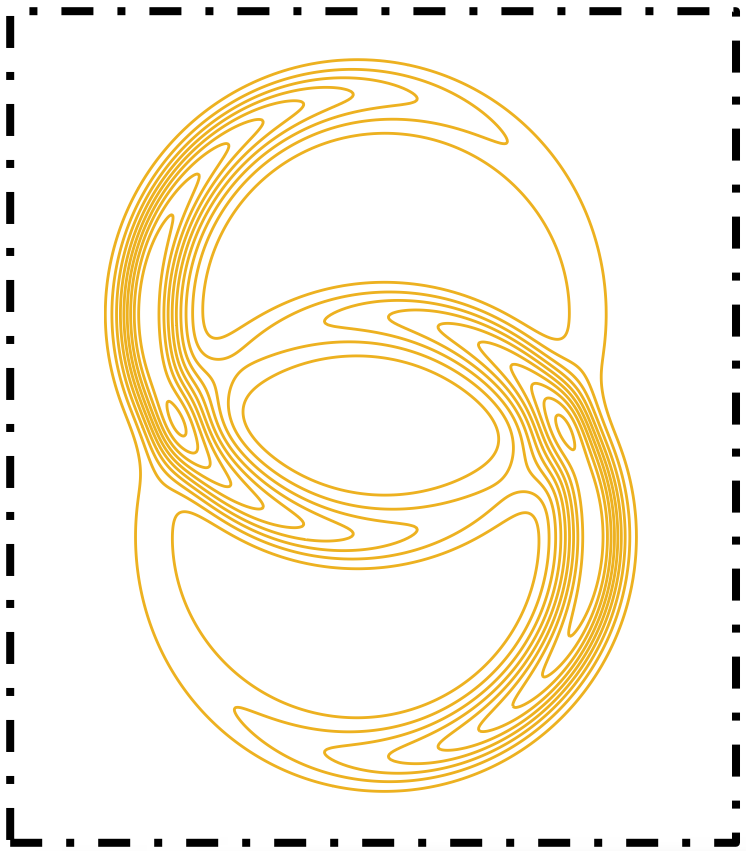
- Arbitrary function handle
- Proposal

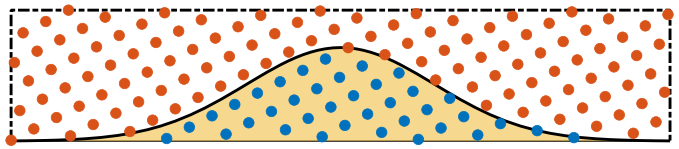
Rejection Sampling

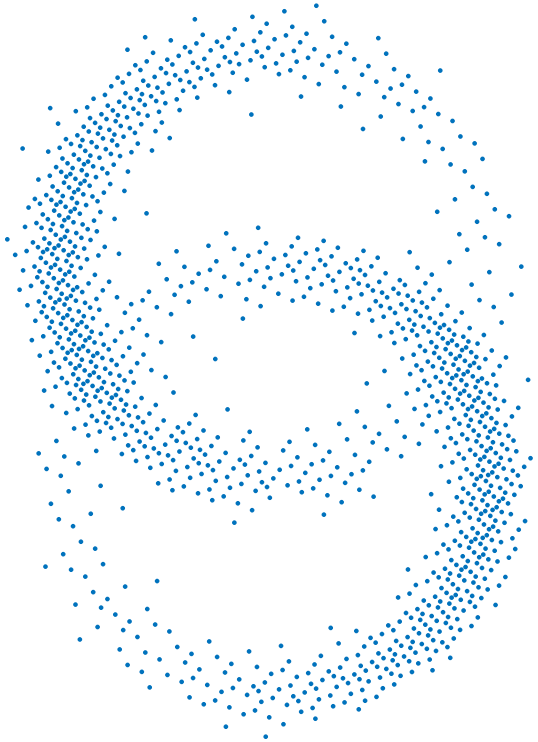
Output

Dirac Mixture Approximation

- Sample locations







Rejection Sampling

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Rejection Sampling

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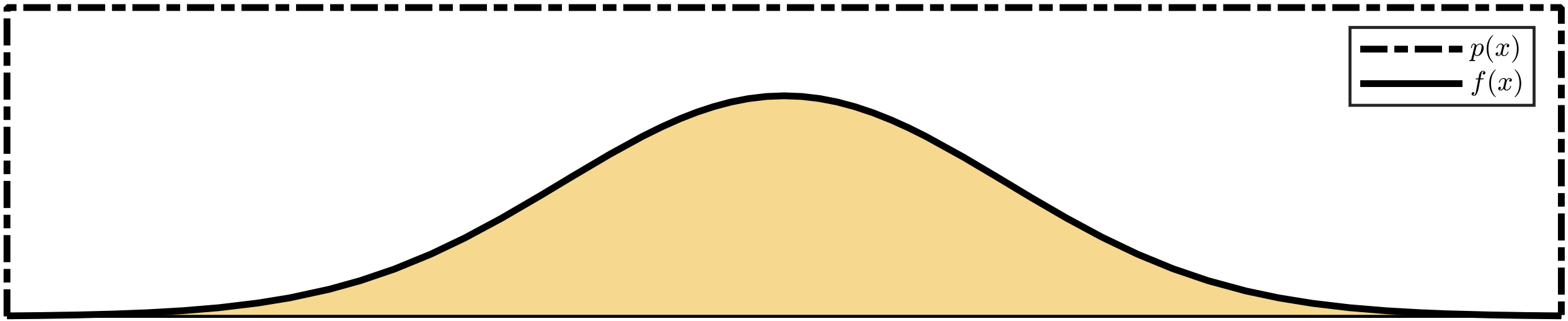
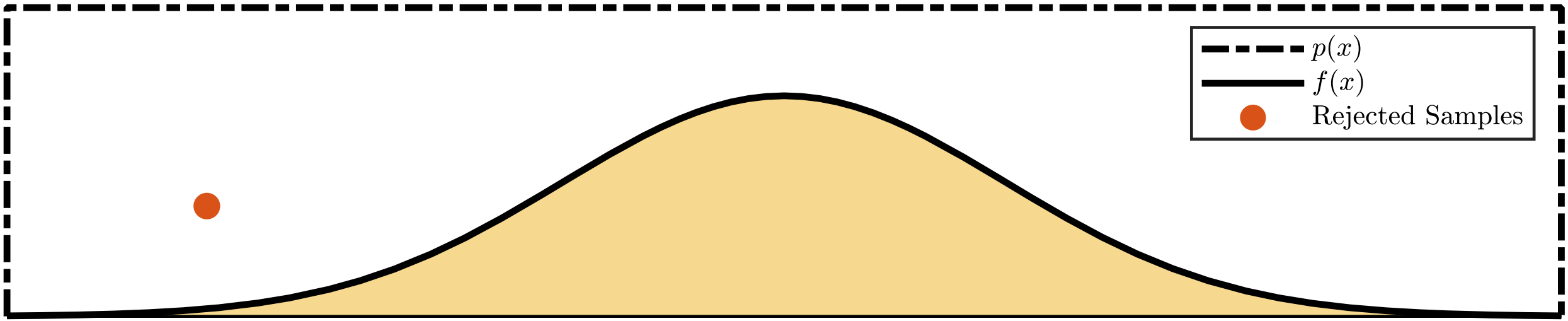


Rejection Sampling

- $p(x)$: Proposal (easy to sample)
- $f(x)$: Arbitrary (difficult to sample)
- Sample from $p(x)$



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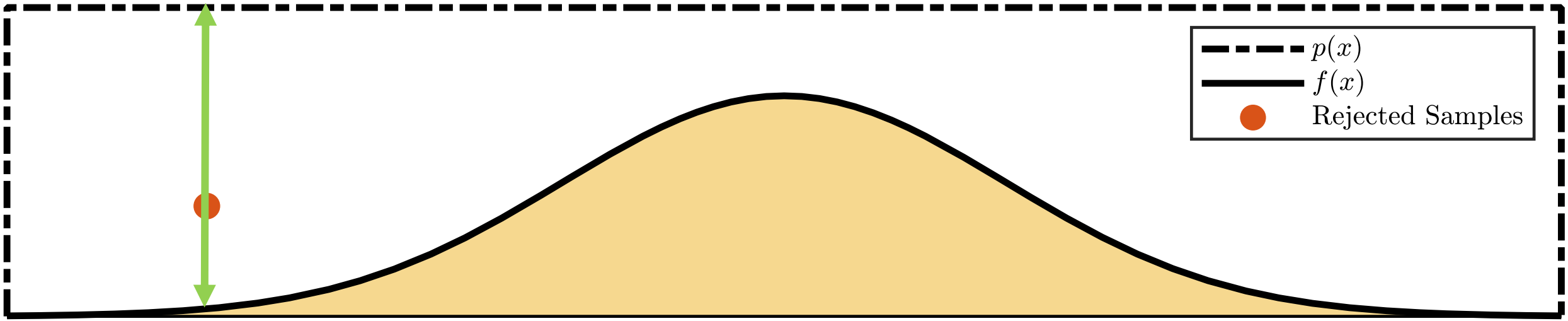

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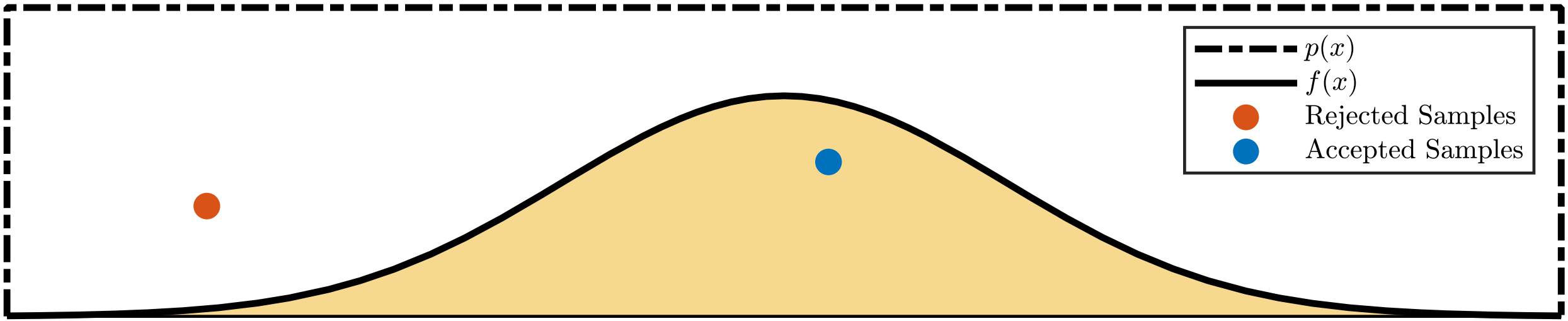

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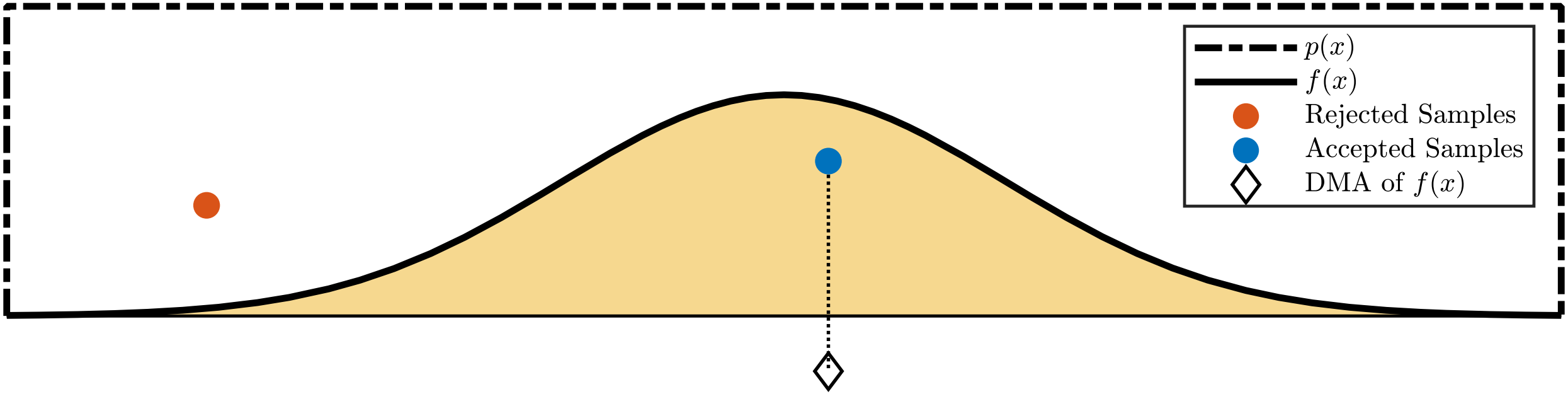

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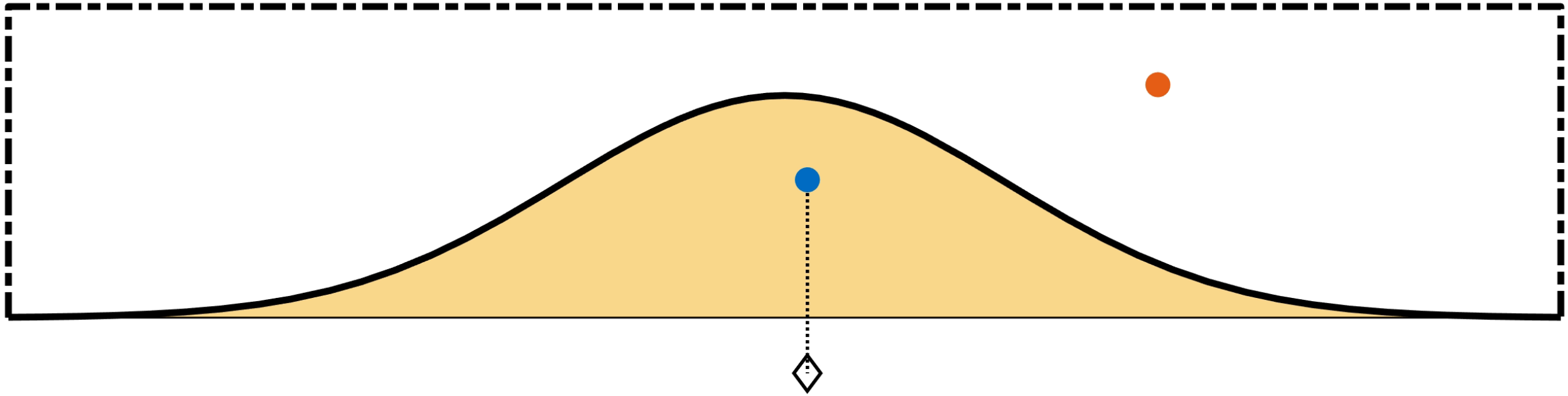
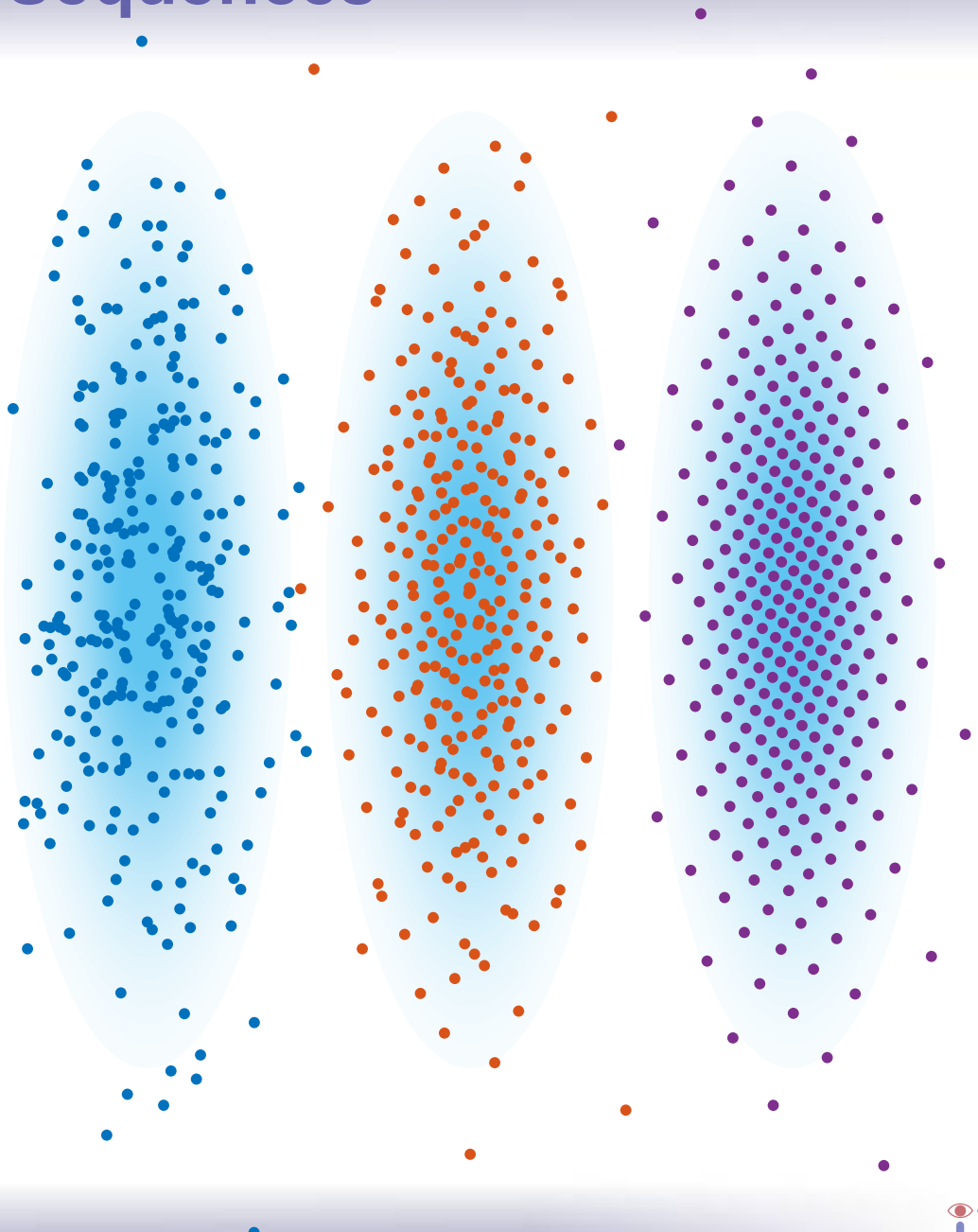
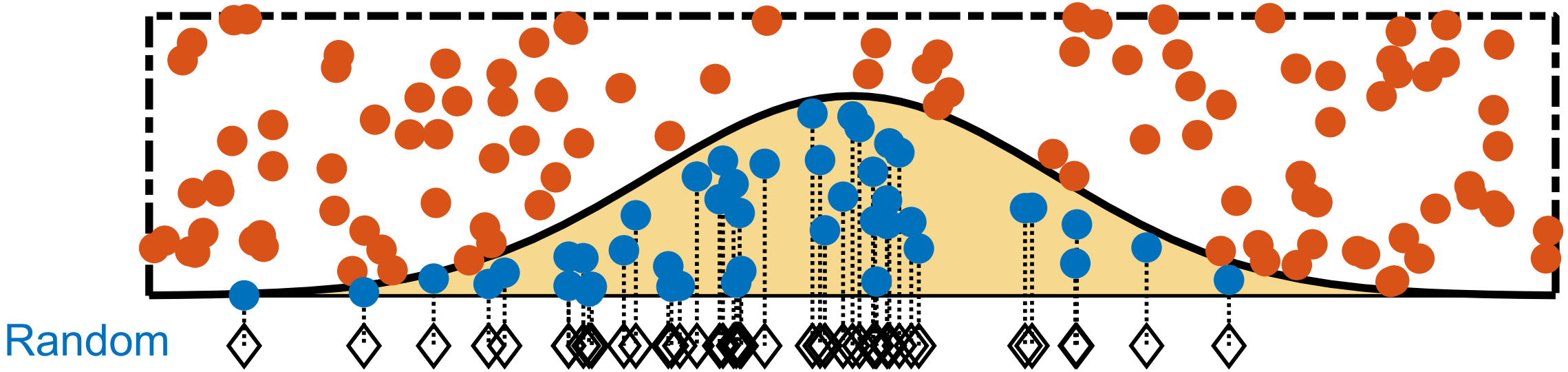


Low-Discrepancy Sequences

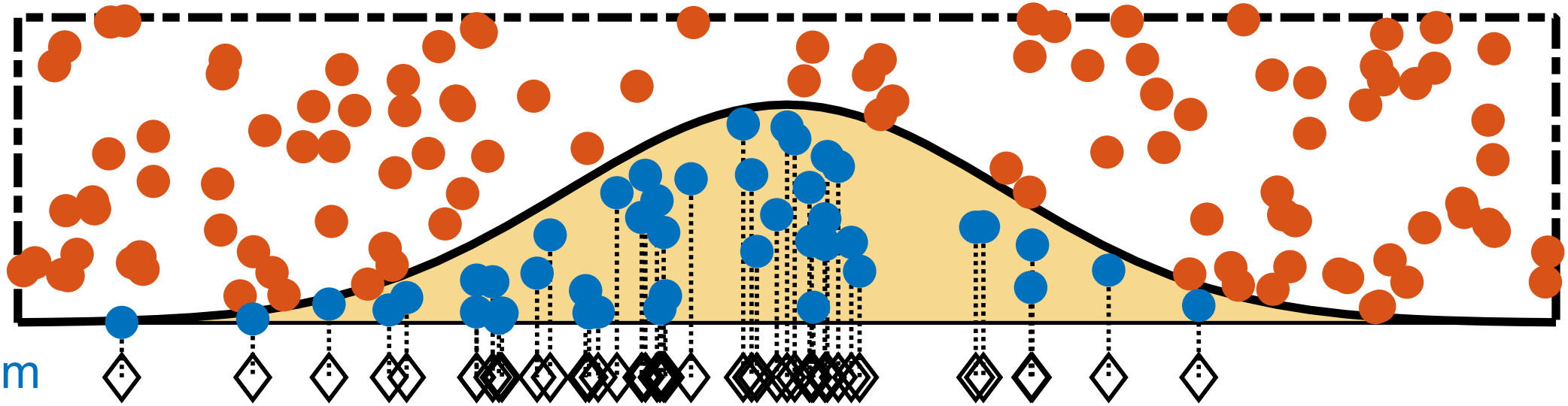
Sampling	Discrepancy
Random (iid)	High
Quasi-random	Low
Fibonacci	Lowest



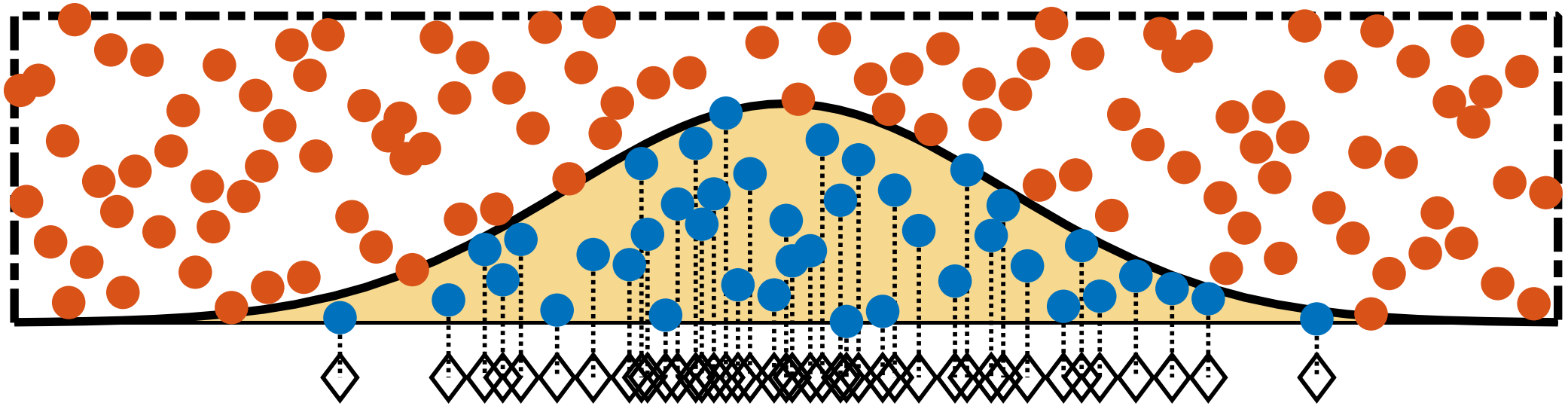
Random vs Halton



Random vs Halton



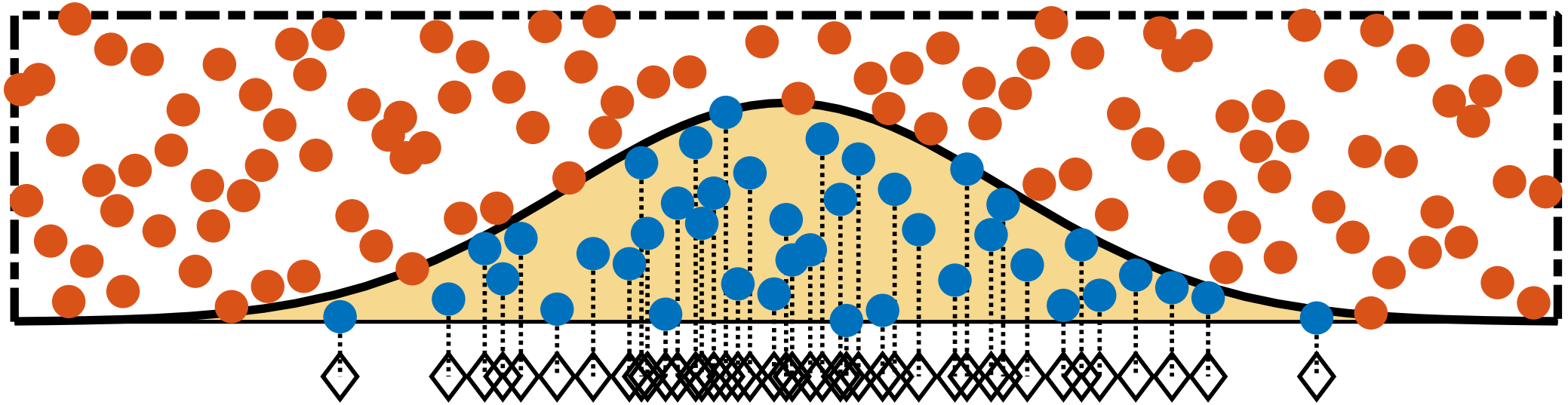
Random



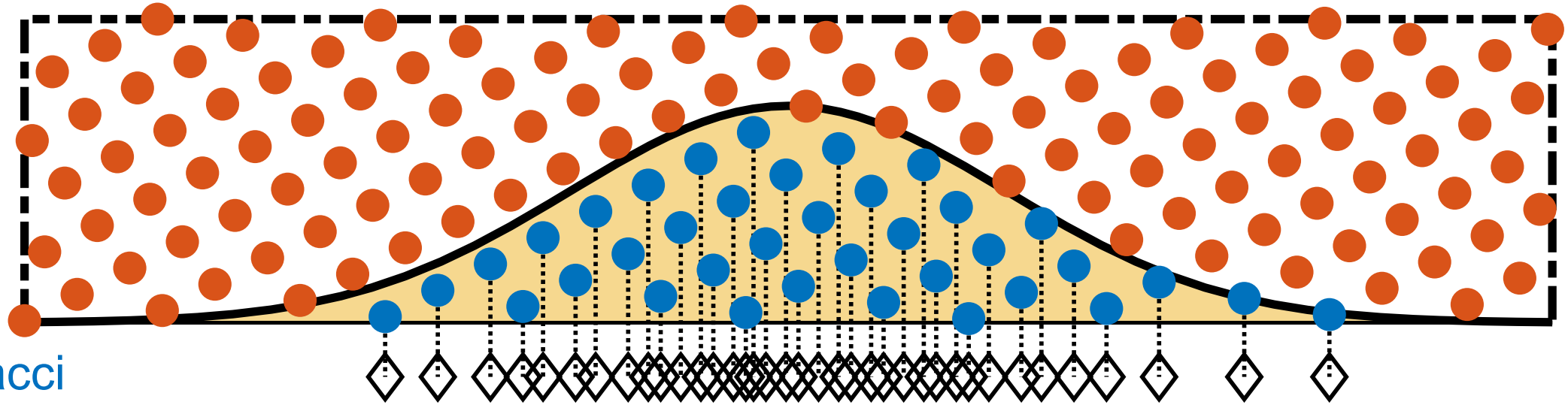
Halton

Random vs Fibonacci

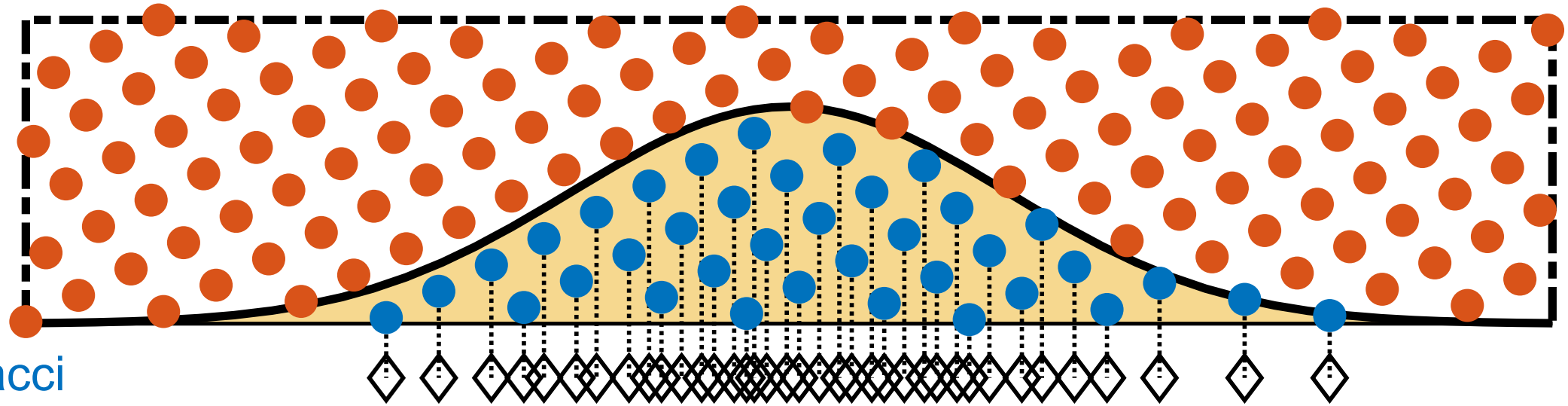
Halton



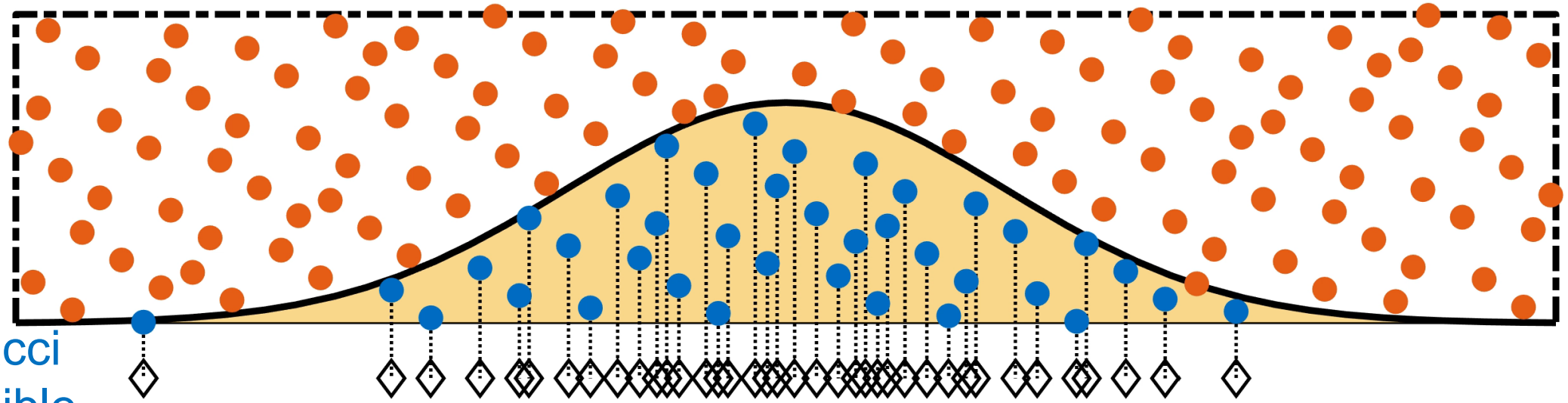
Fibonacci



Extensible Fibonacci

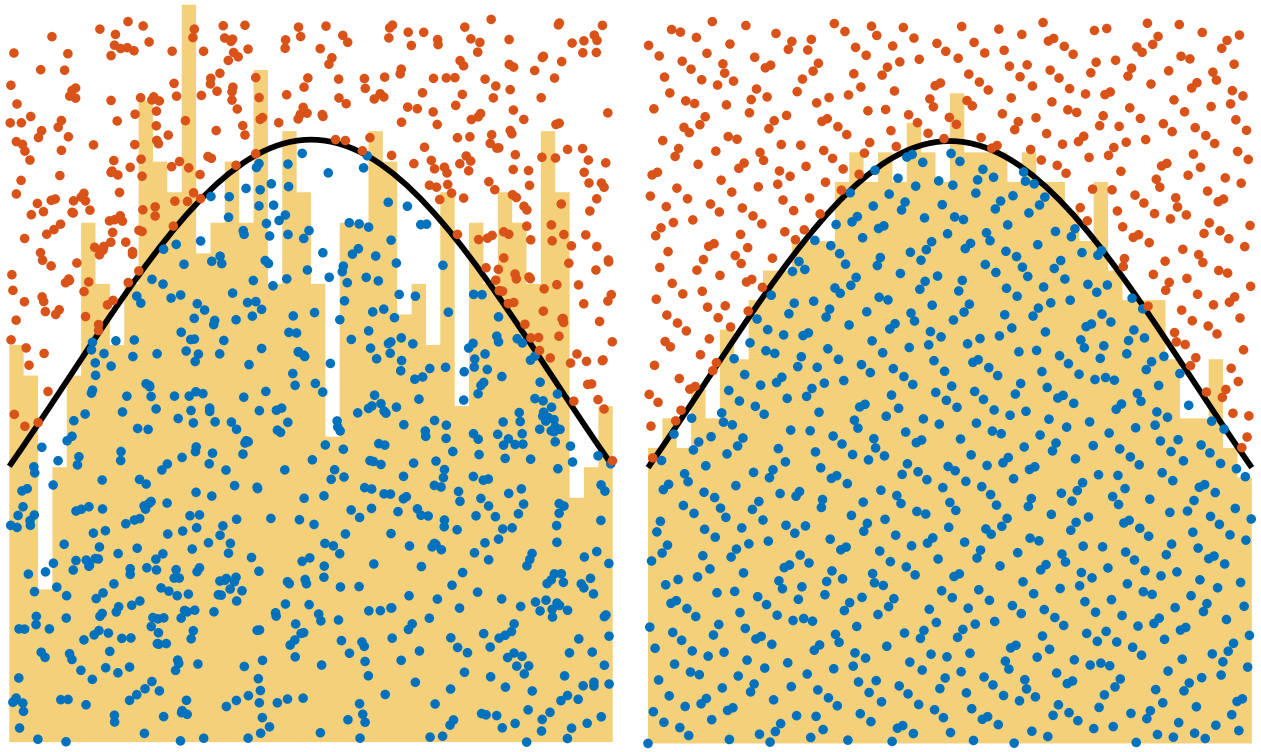


Fibonacci



Fibonacci
Extensible

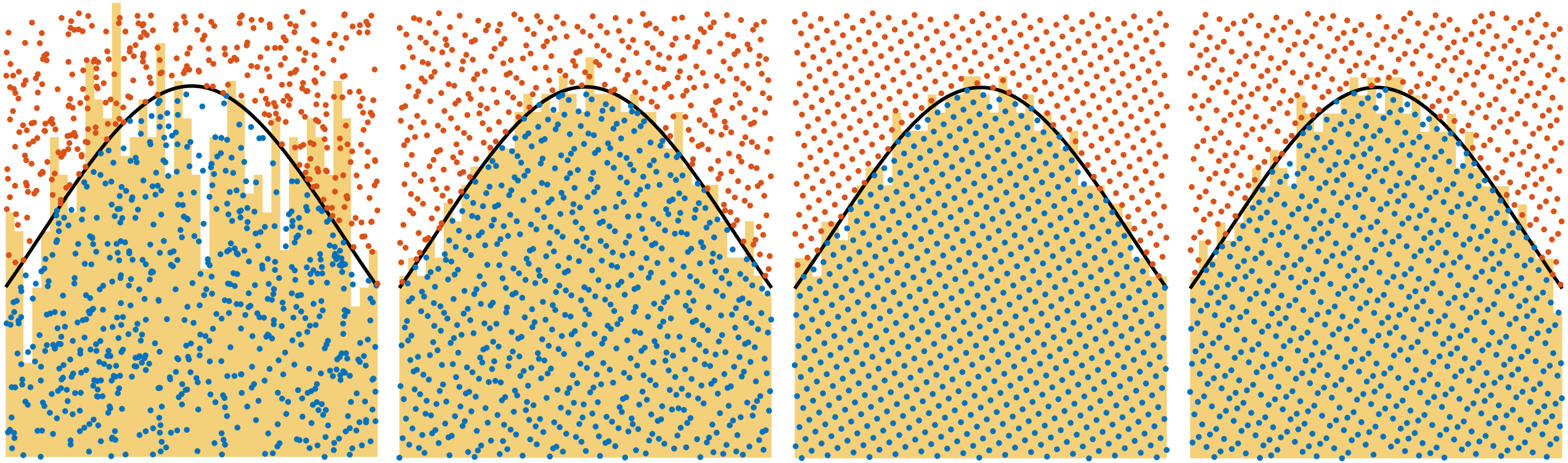
Histograms



Random Halton

State of Art

Histograms



Random

Halton

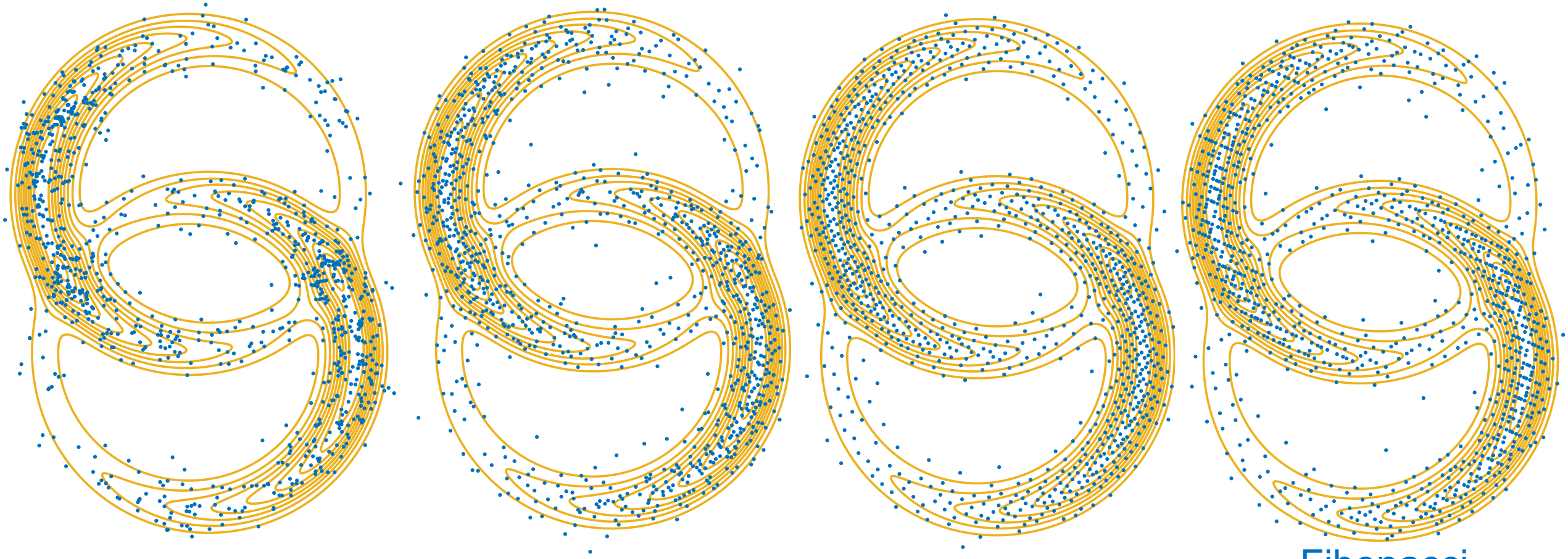
State of Art

Fibonacci

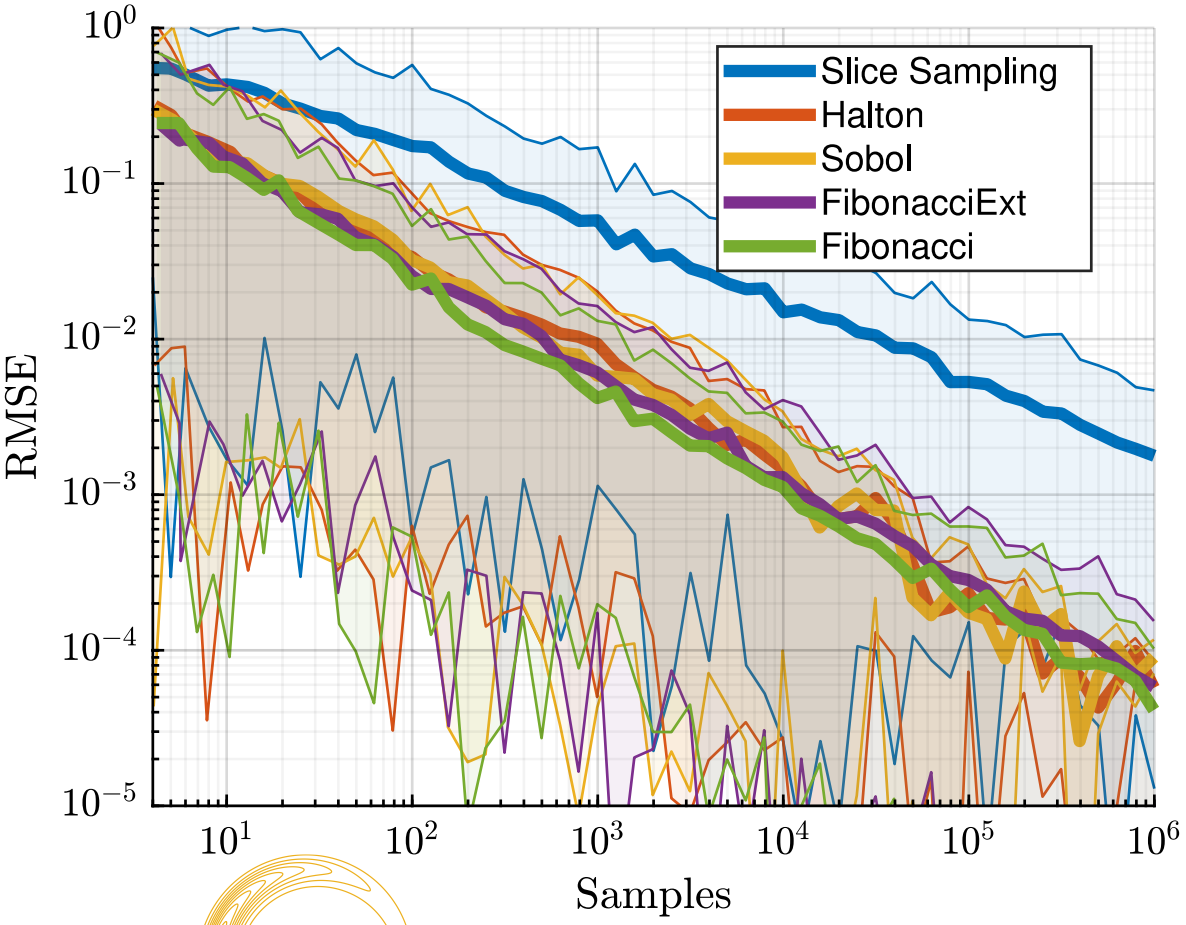
Fibonacci Extensible

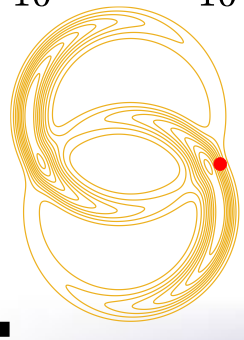
Proposed

Multivariate Density

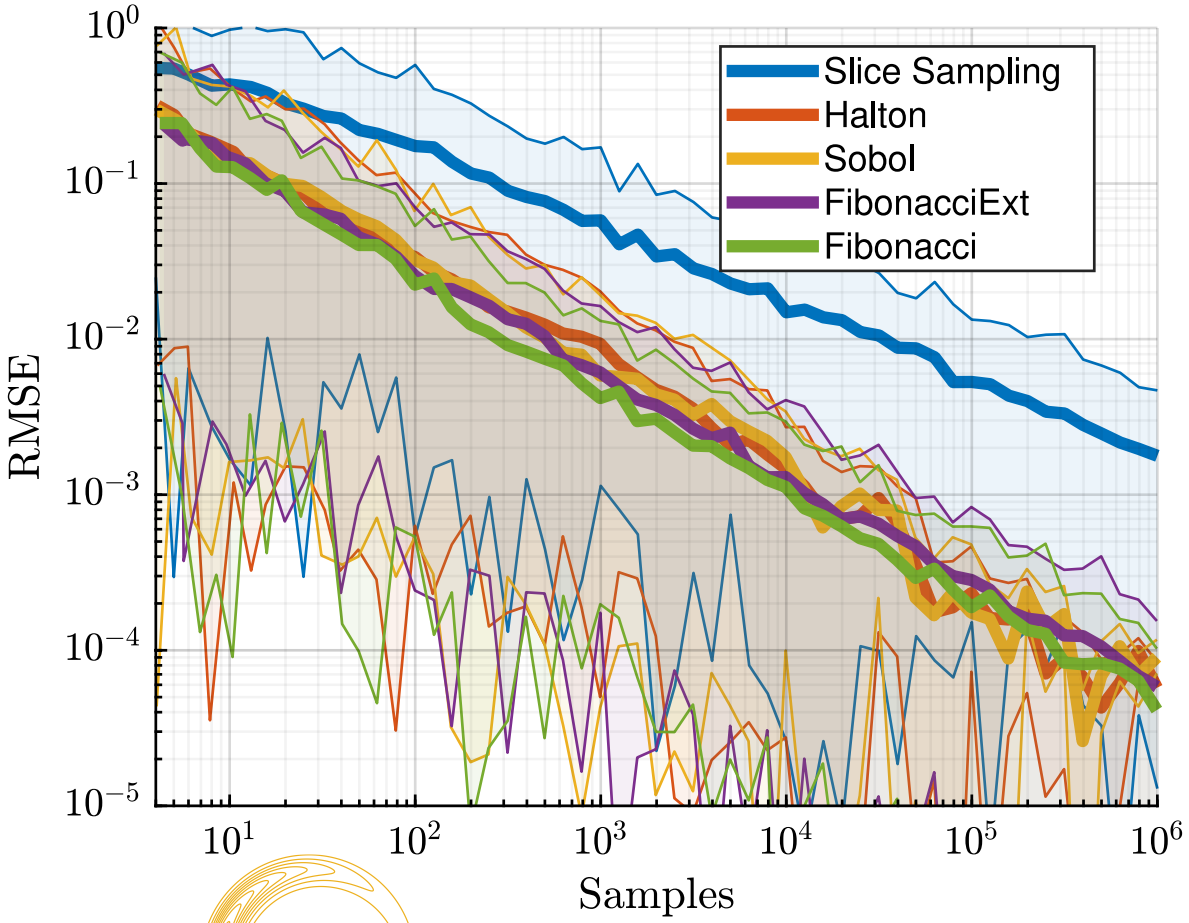


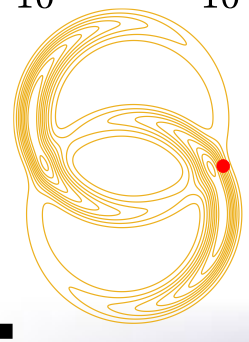
Evaluation

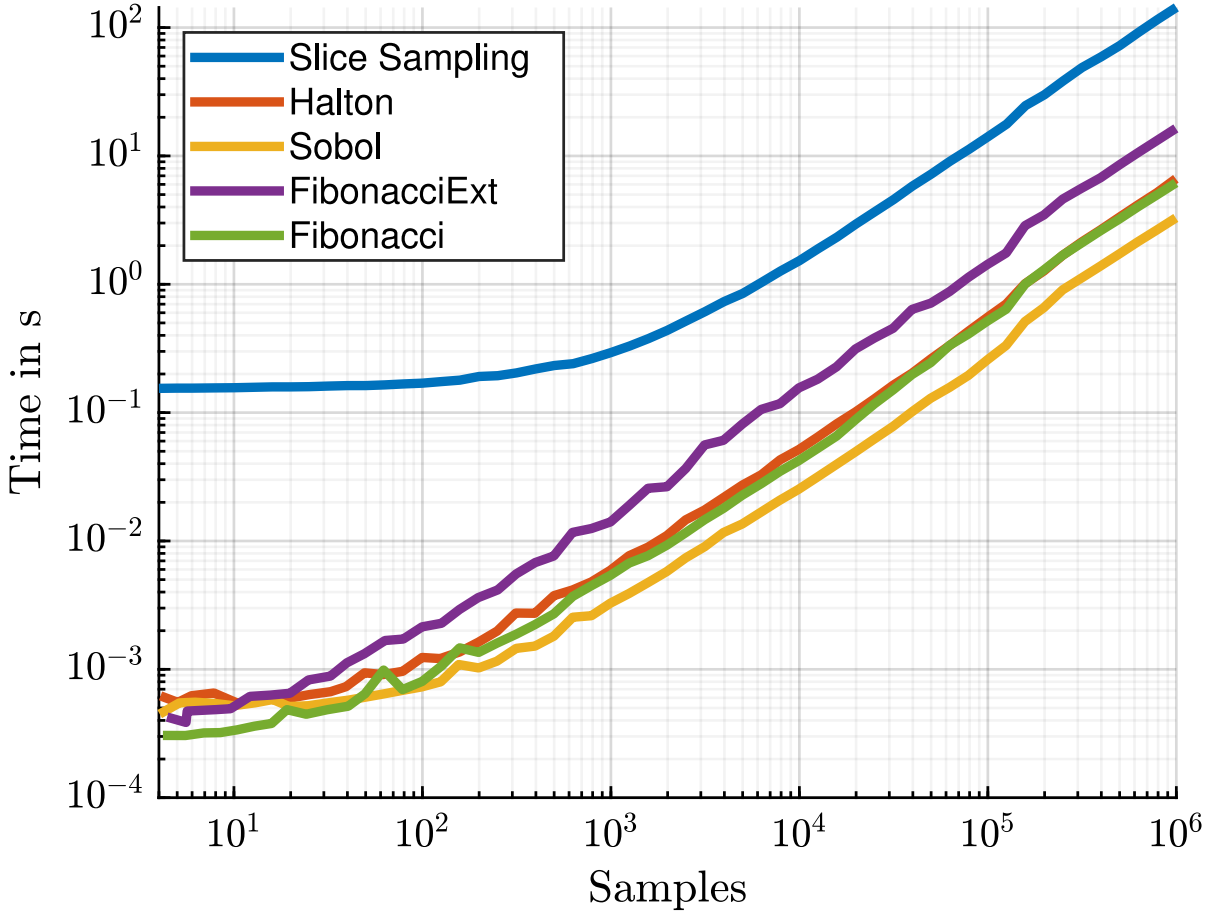

Slice Sampling
Halton
Sobol
FibonacciExt
Fibonacci
RMSE
Samples
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^1
10^2
10^3
10^4
10^5
10^6



Evaluation


Slice Sampling
Halton
Sobol
FibonacciExt
Fibonacci
RMSE
Samples




Slice Sampling
Halton
Sobol
FibonacciExt
Fibonacci
Time in s
Samples

Fibonacci-Matrix

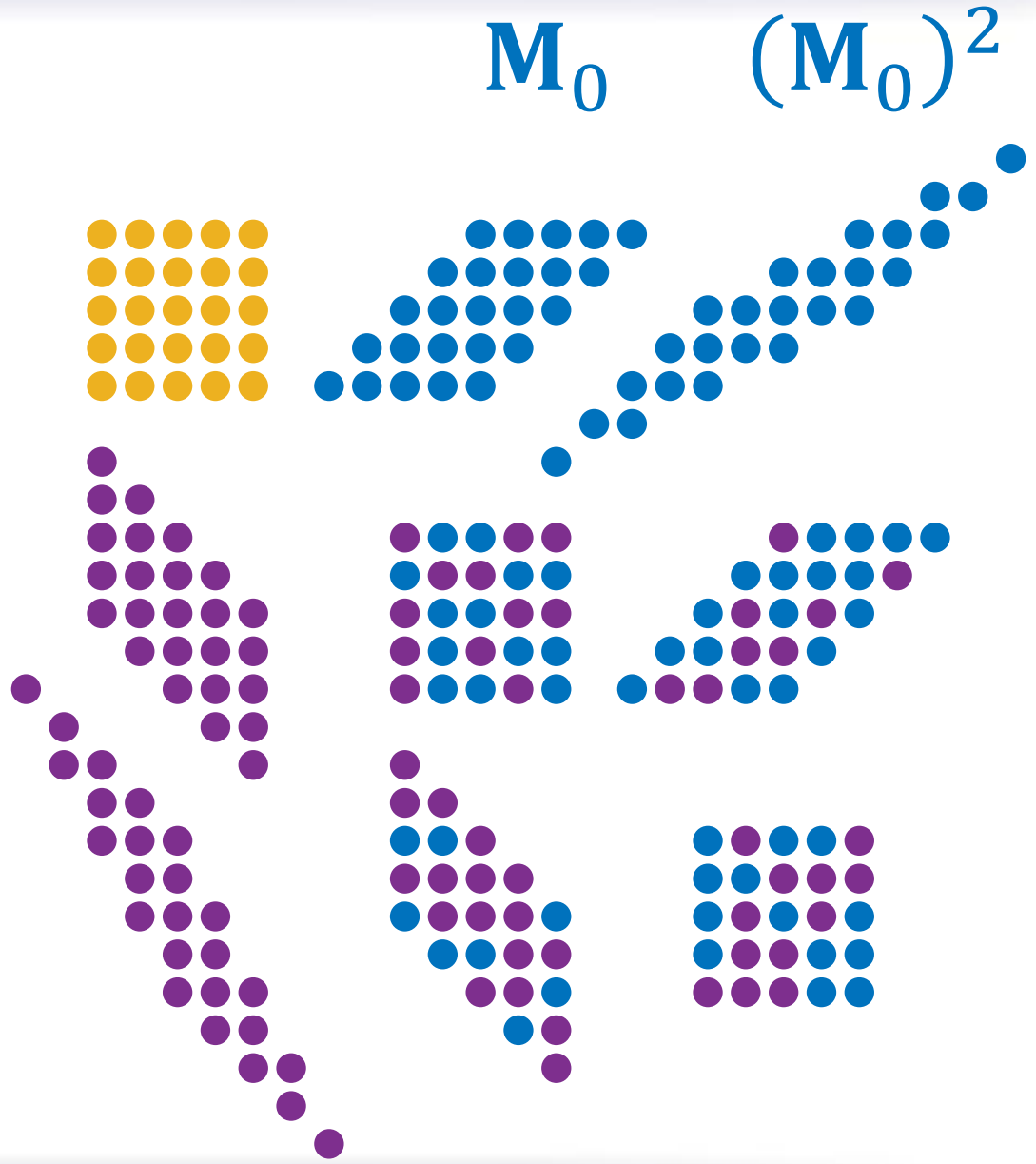
Fibonacci-Matrix $\mathbf{M}$

- Symmetric
- Unimodular
  - $\mathbf{M} \in \mathbb{Z}^2$
  - $\det(\mathbf{M}) = \pm 1$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

2D Fibonacci-Grid

- $\mathbf{M}_0 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$
- $\mathbf{M}_1 = \begin{bmatrix} 0 & -1 \\ -1 & 1 \end{bmatrix}$
- $\mathbf{M}_0\mathbf{M}_1 = \mathbf{M}_1\mathbf{M}_0$



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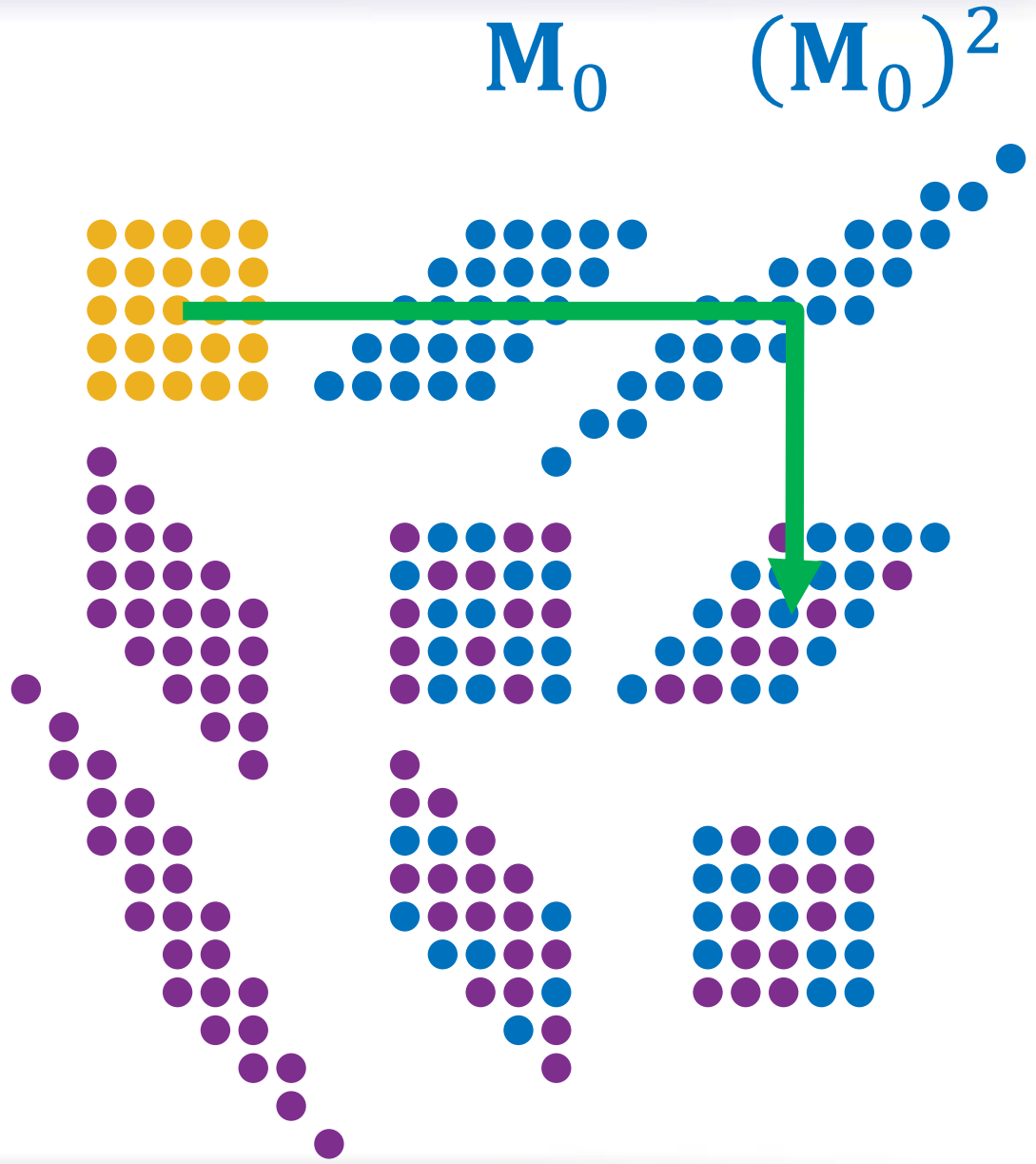

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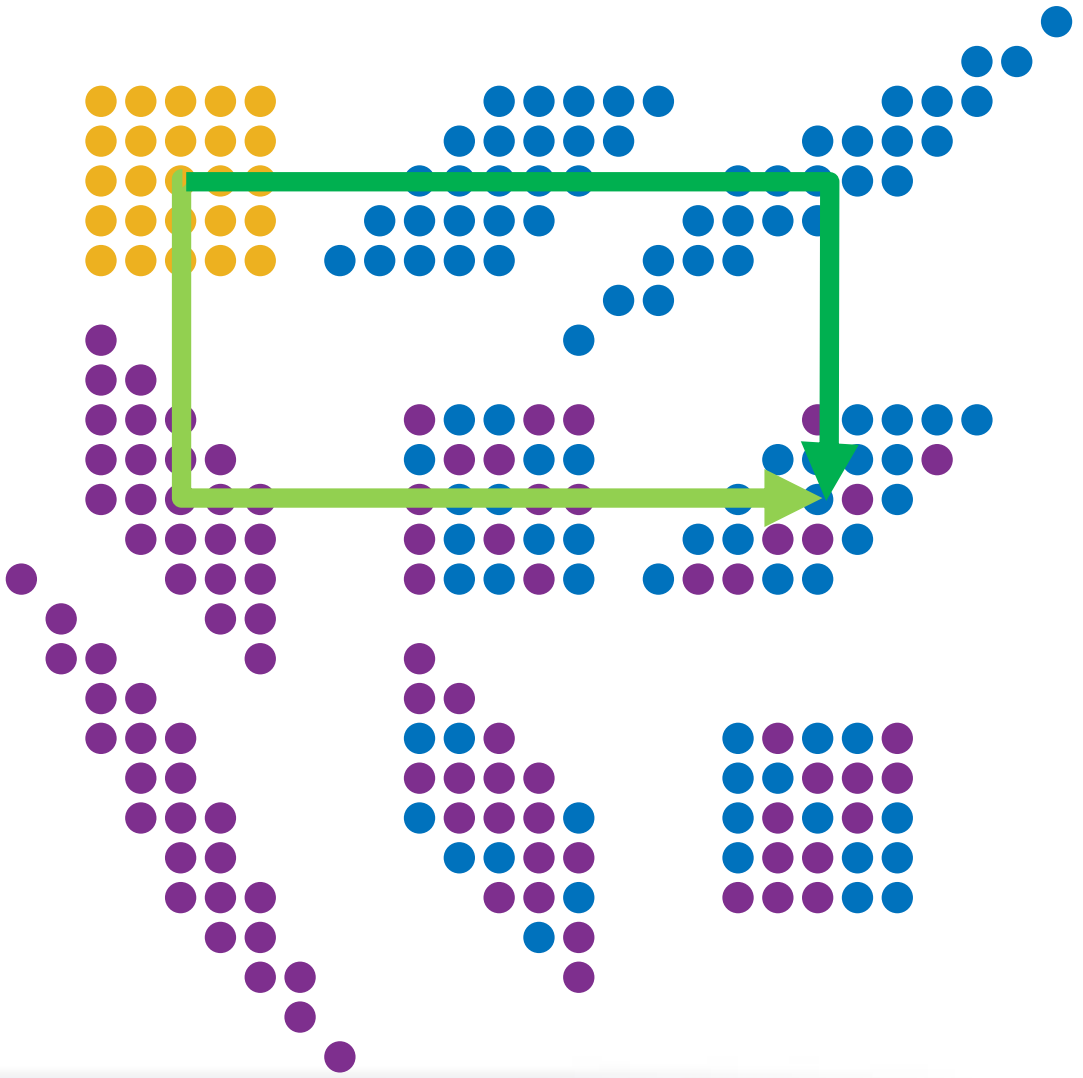
$\mathbf{M_0}$ $(\mathbf{M_0})^2$

$\mathbf{M_1}$

$(\mathbf{M_1})^2$



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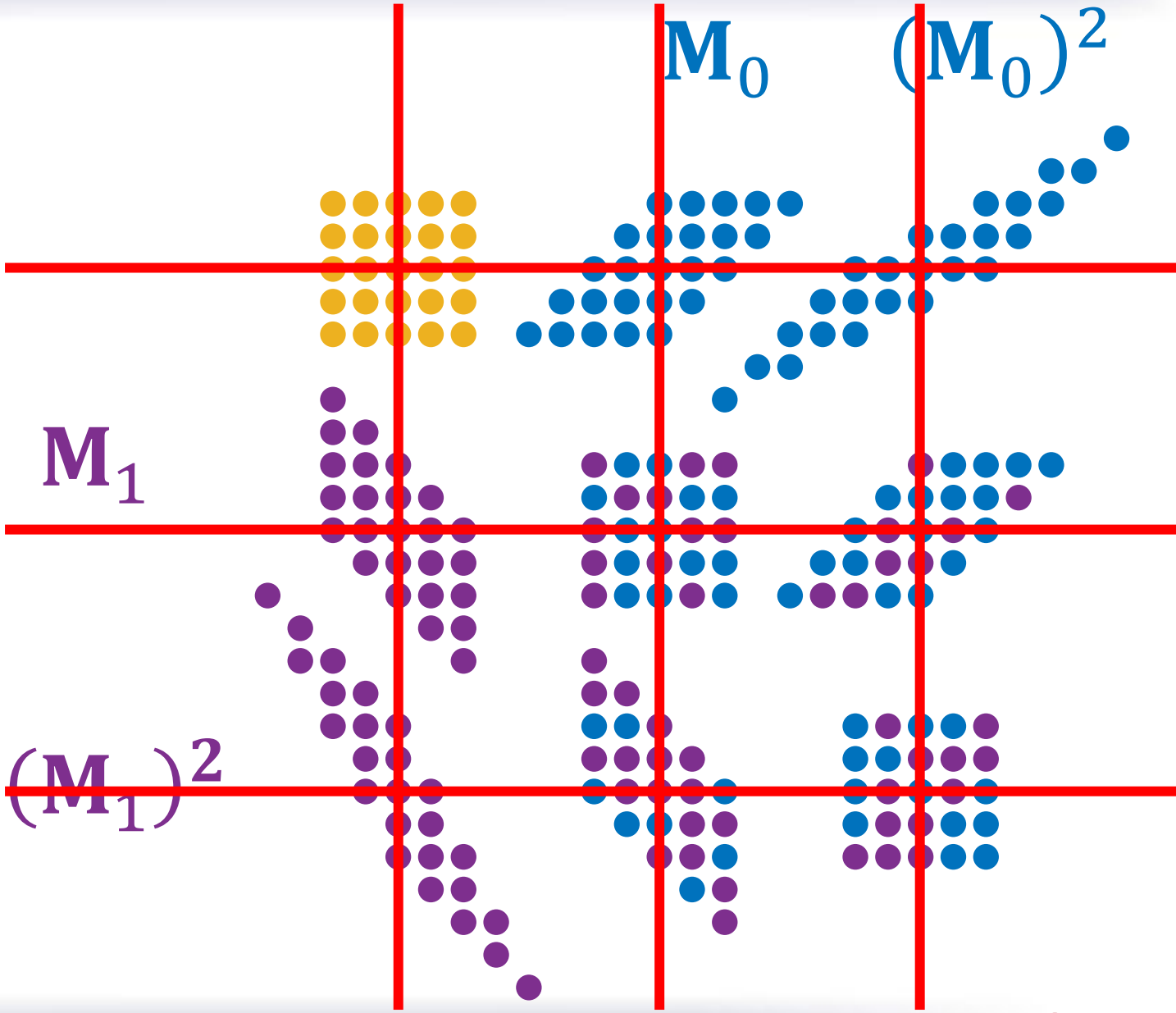

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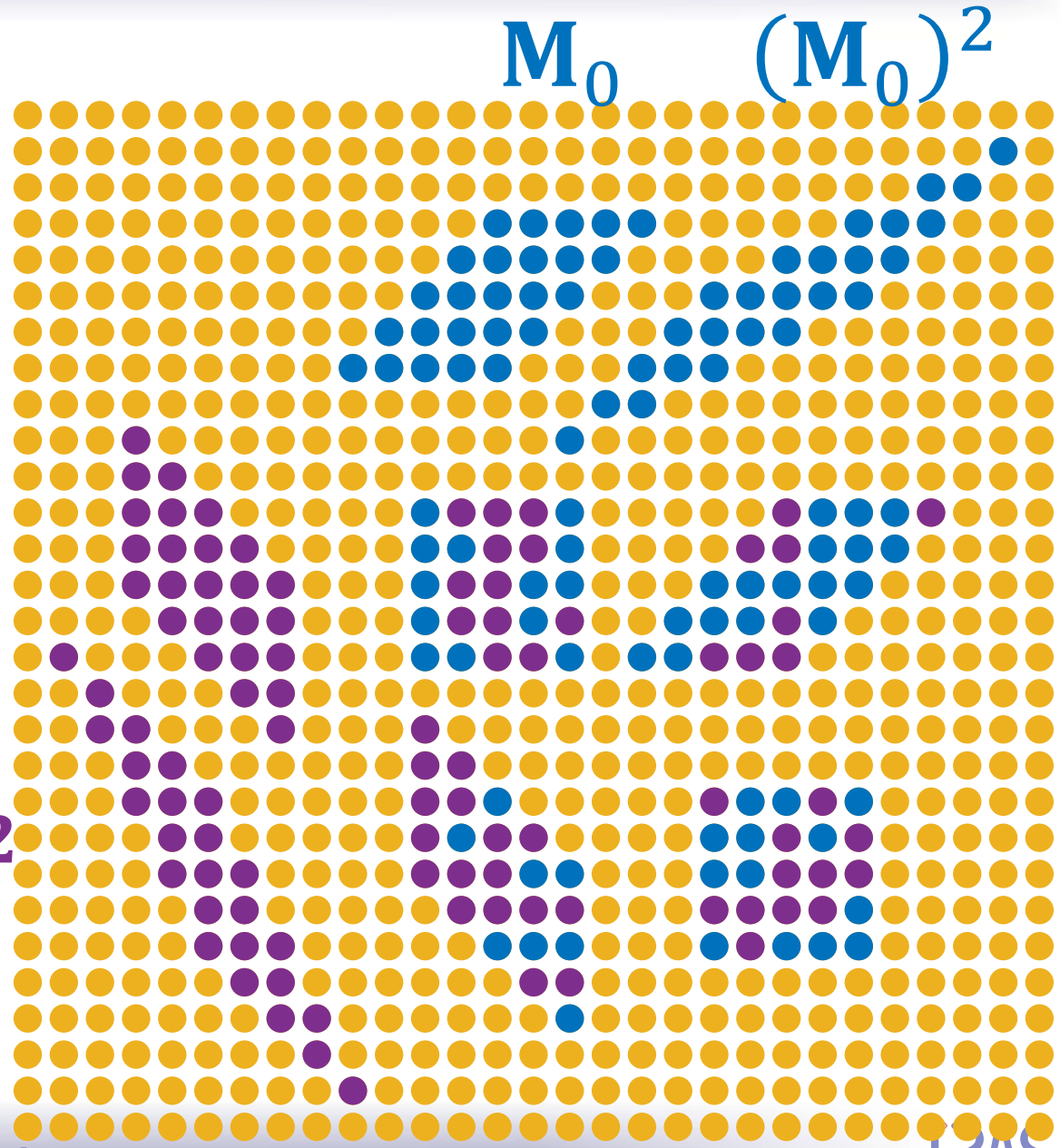
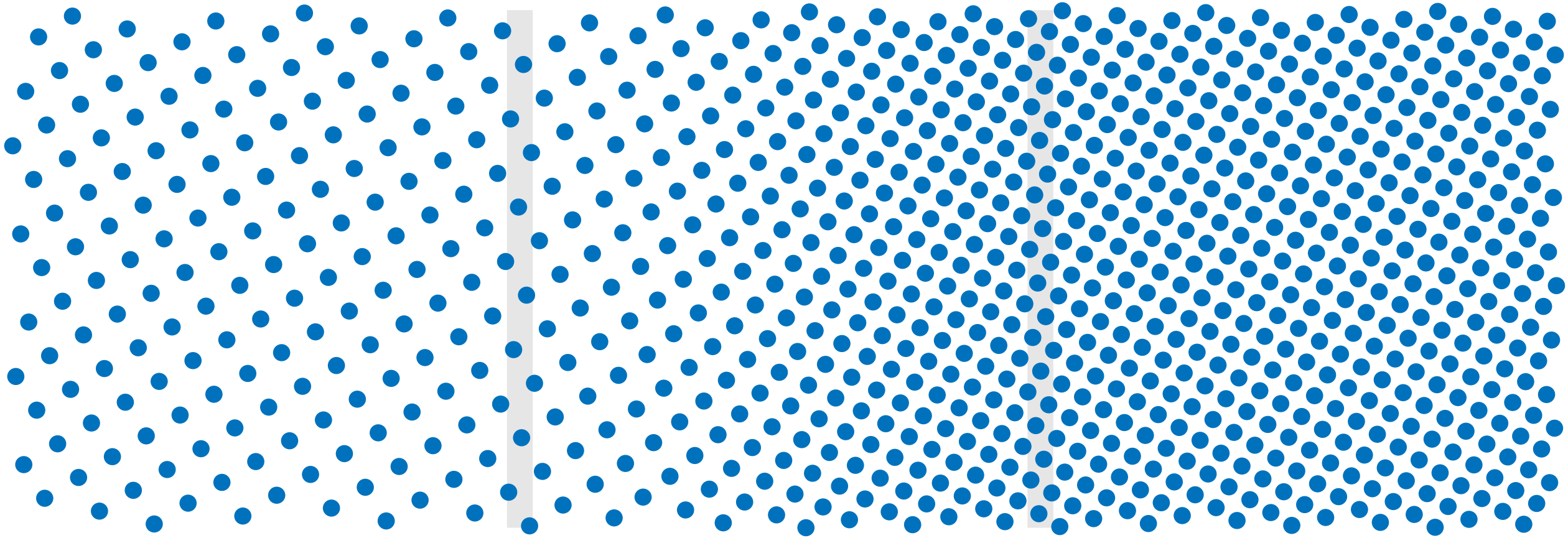
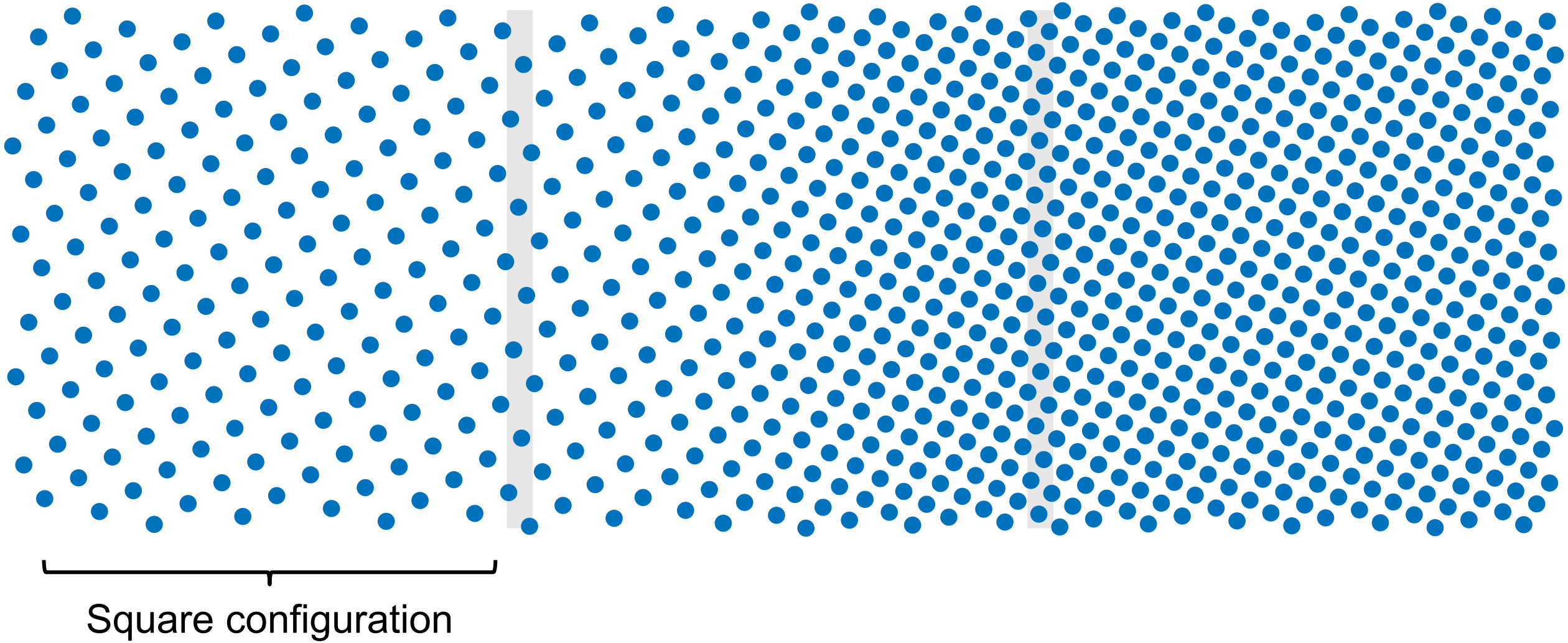


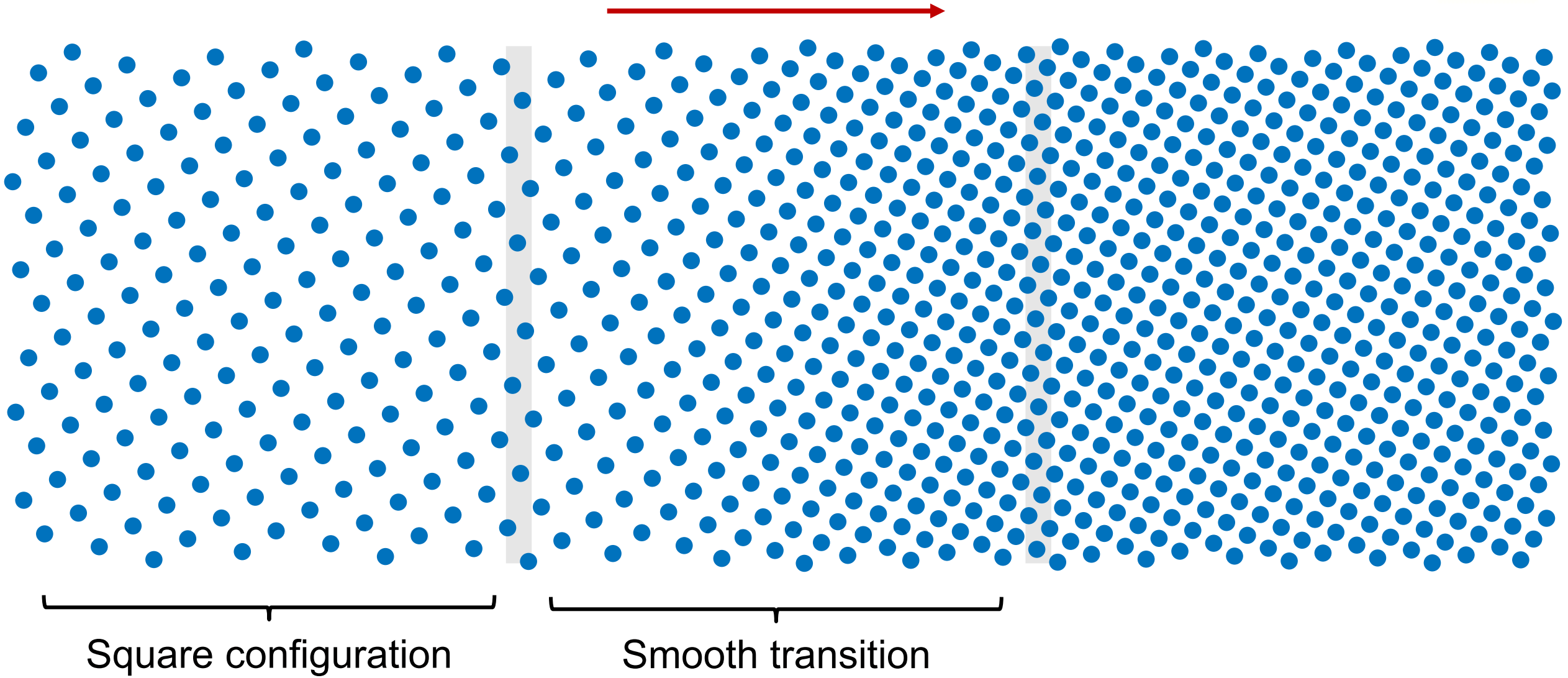
Anisotropic Rescaling



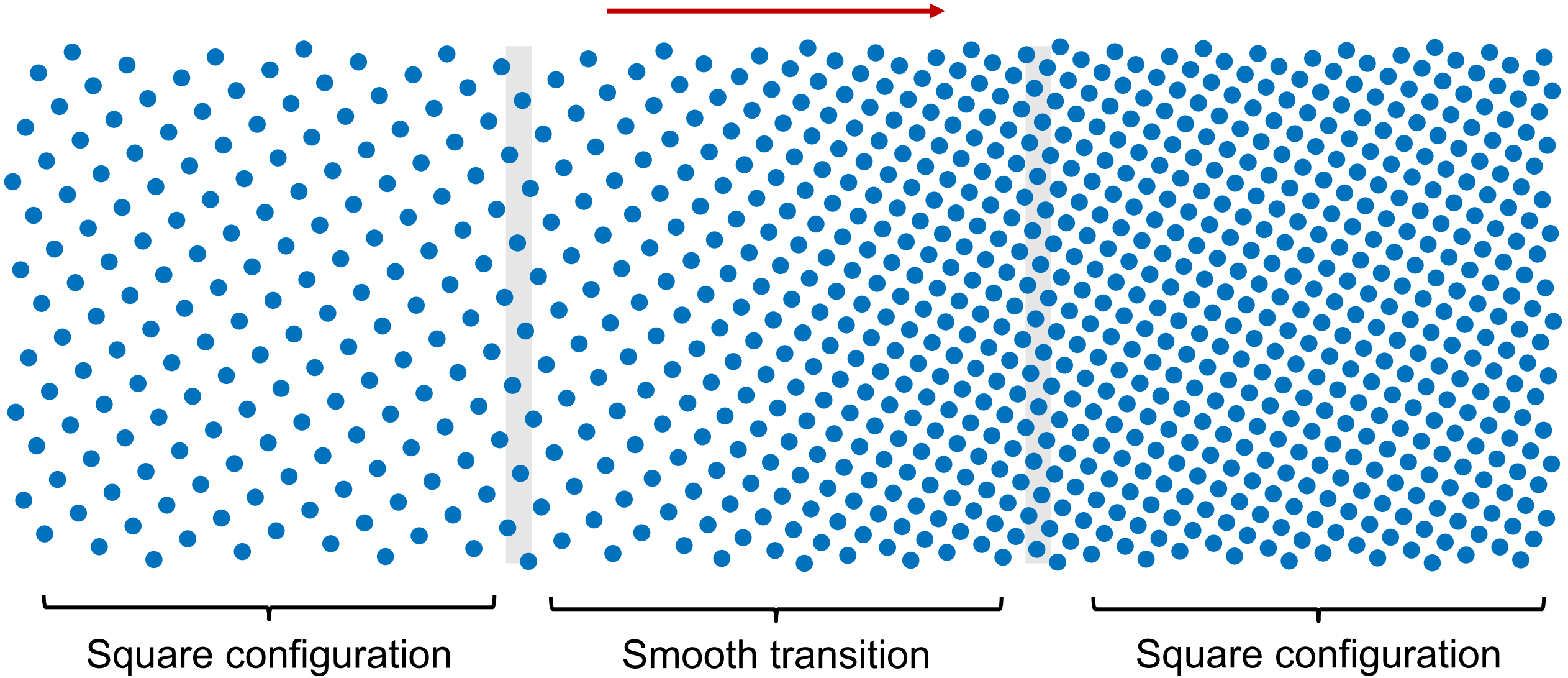
Anisotropic Rescaling



Anisotropic Rescaling



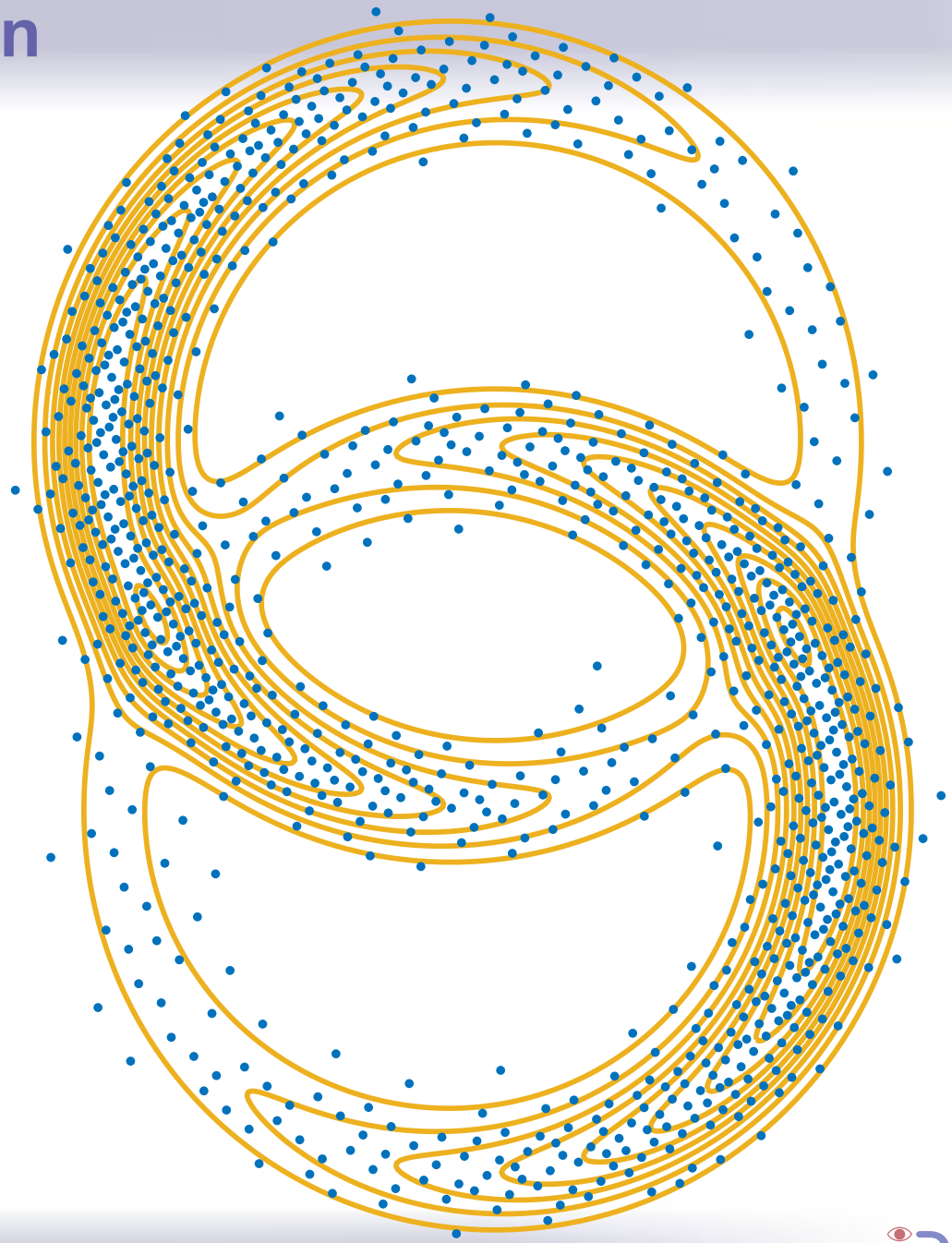
Anisotropic Rescaling



Conclusion

Achieved

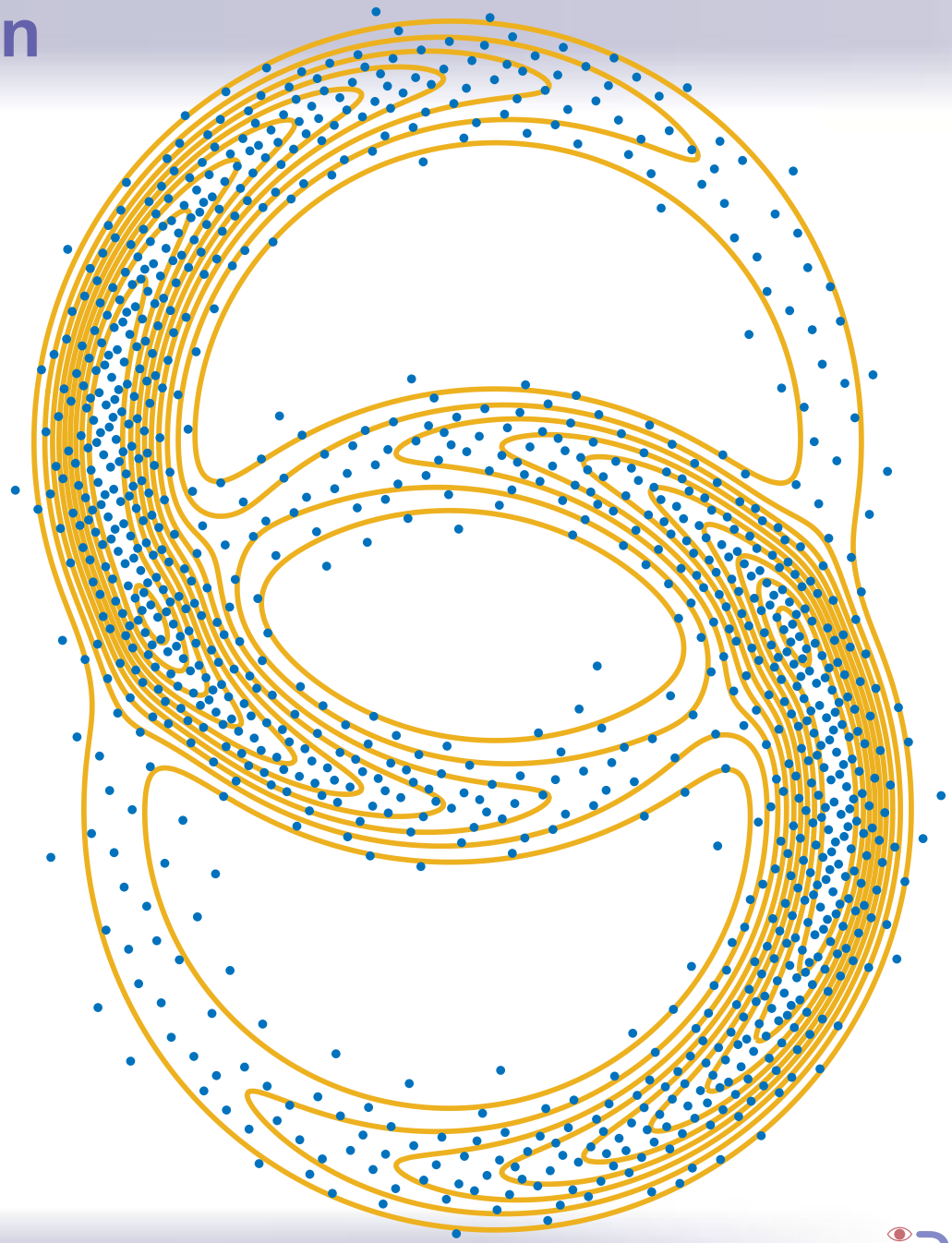
- Rejection sampling



Conclusion

Achieved

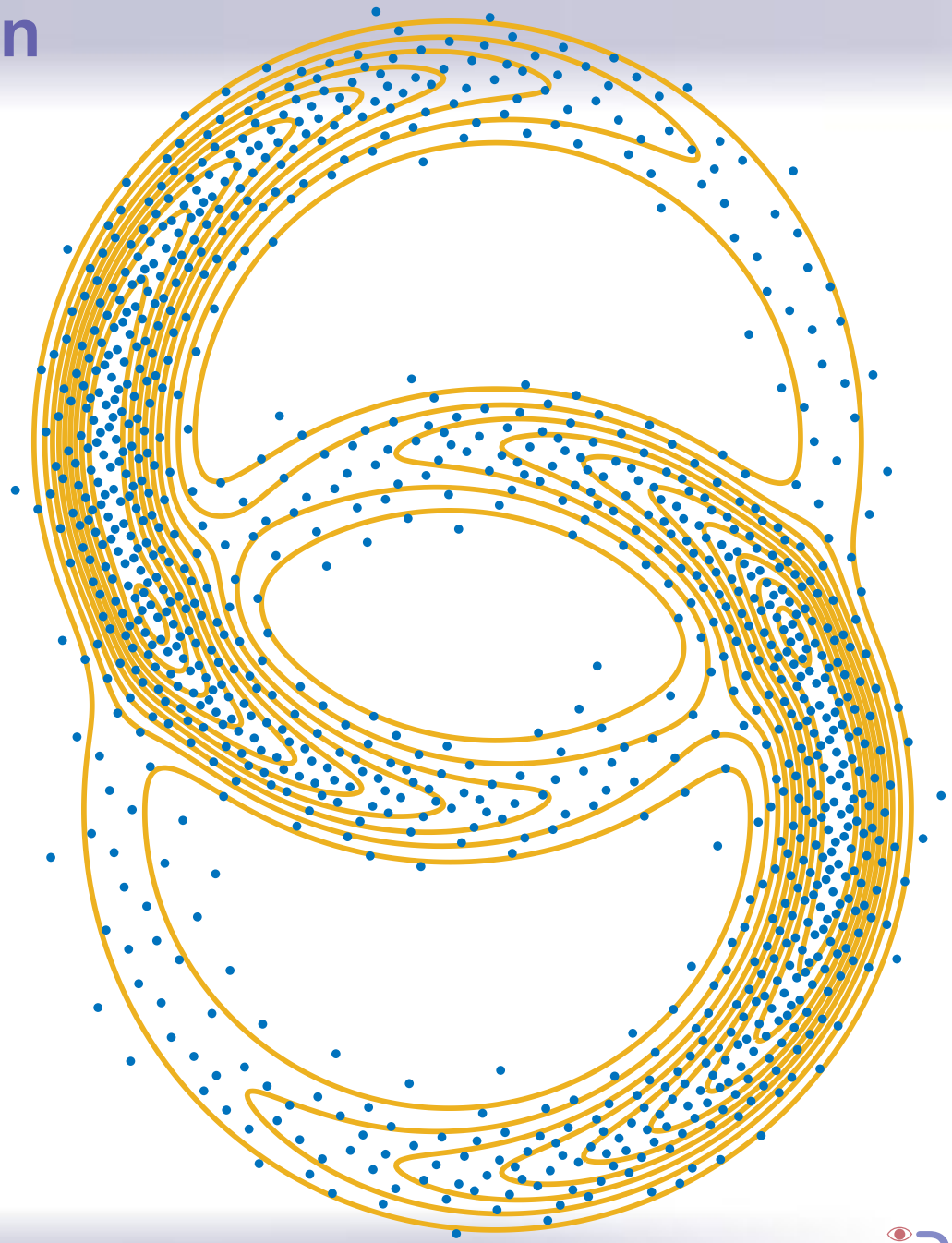
- Rejection sampling
- Arbitrary multivariate densities



Conclusion

Achieved

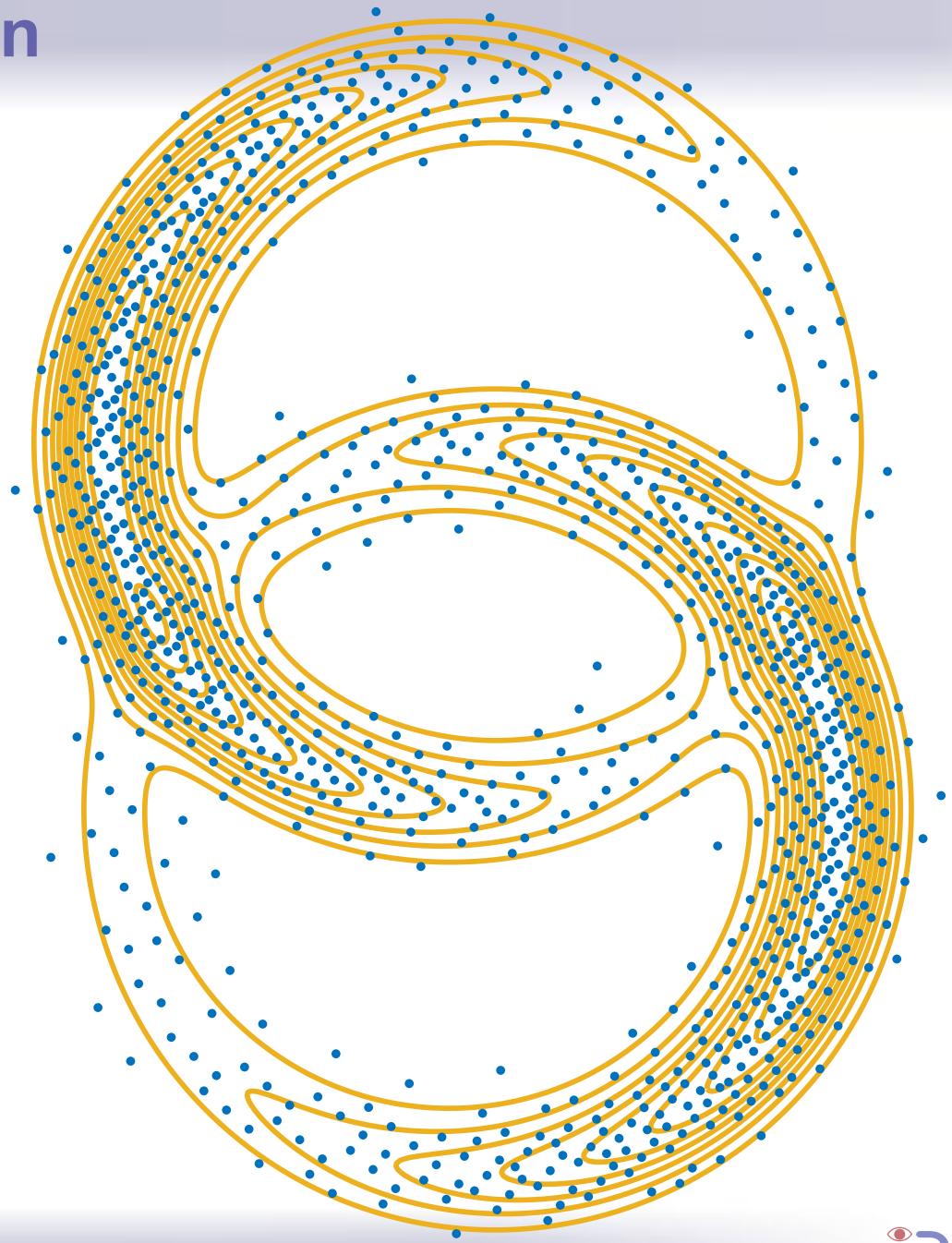
- Rejection sampling
- Arbitrary multivariate densities
- Proposal



Conclusion

Achieved

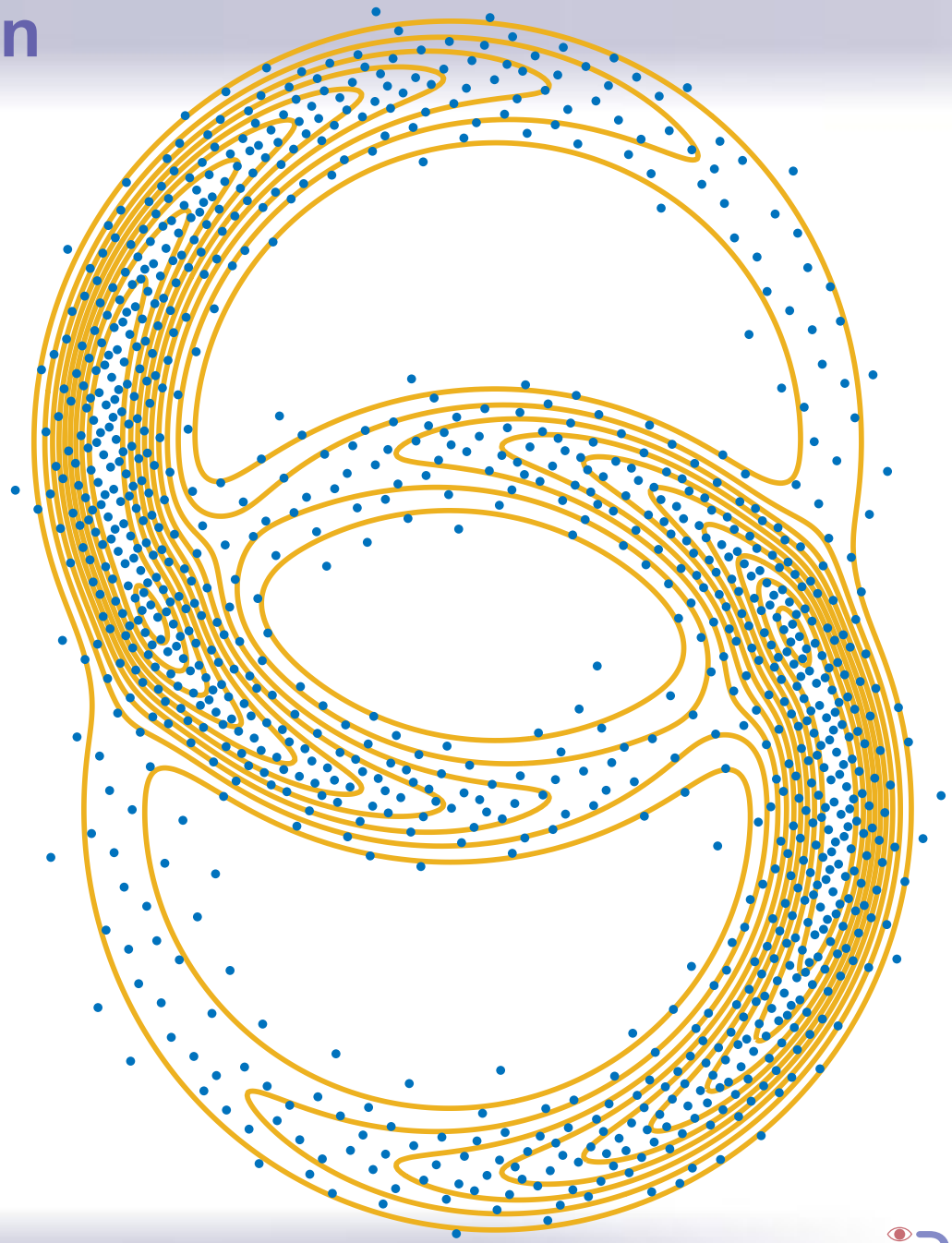
- Rejection sampling
- Arbitrary multivariate densities
- Proposal
  - Uniform



Conclusion

Achieved

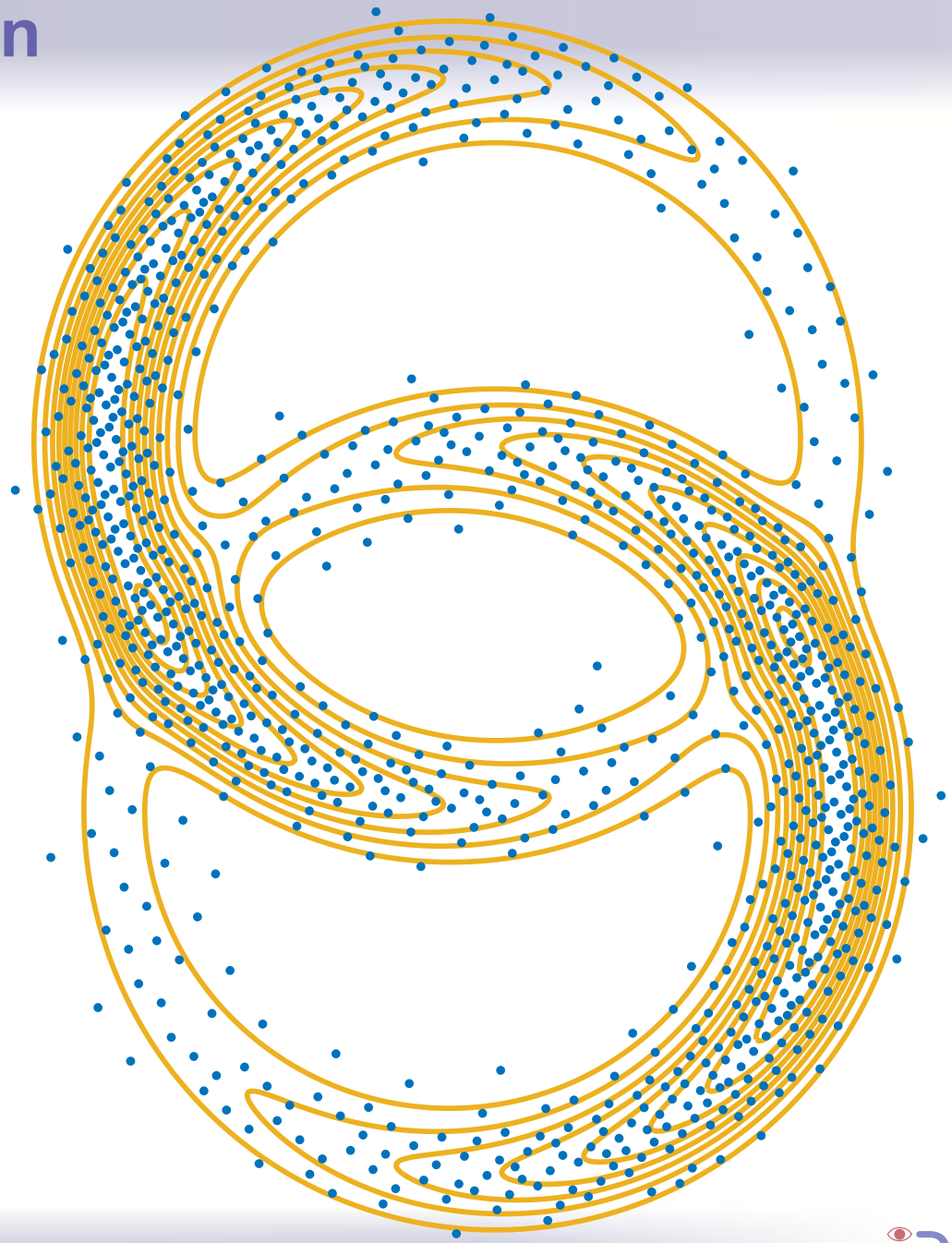
- Rejection sampling
- Arbitrary multivariate densities
- Proposal
  - Uniform
  - Gaussian



Conclusion

Achieved

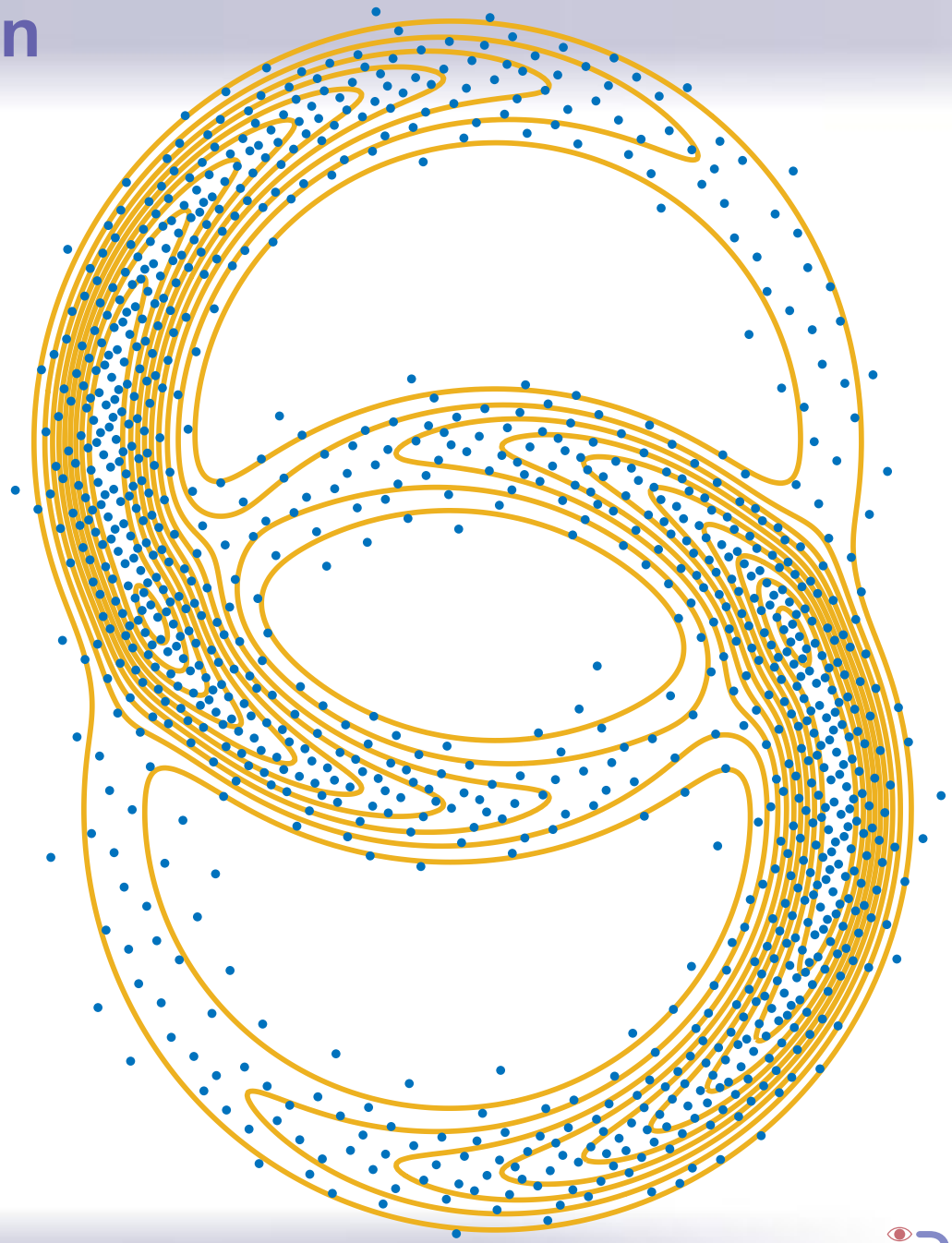
- Rejection sampling
- Arbitrary multivariate densities
- Proposal
  - Uniform
  - Gaussian
- Generalized Fibonacci Grids



Conclusion

Achieved

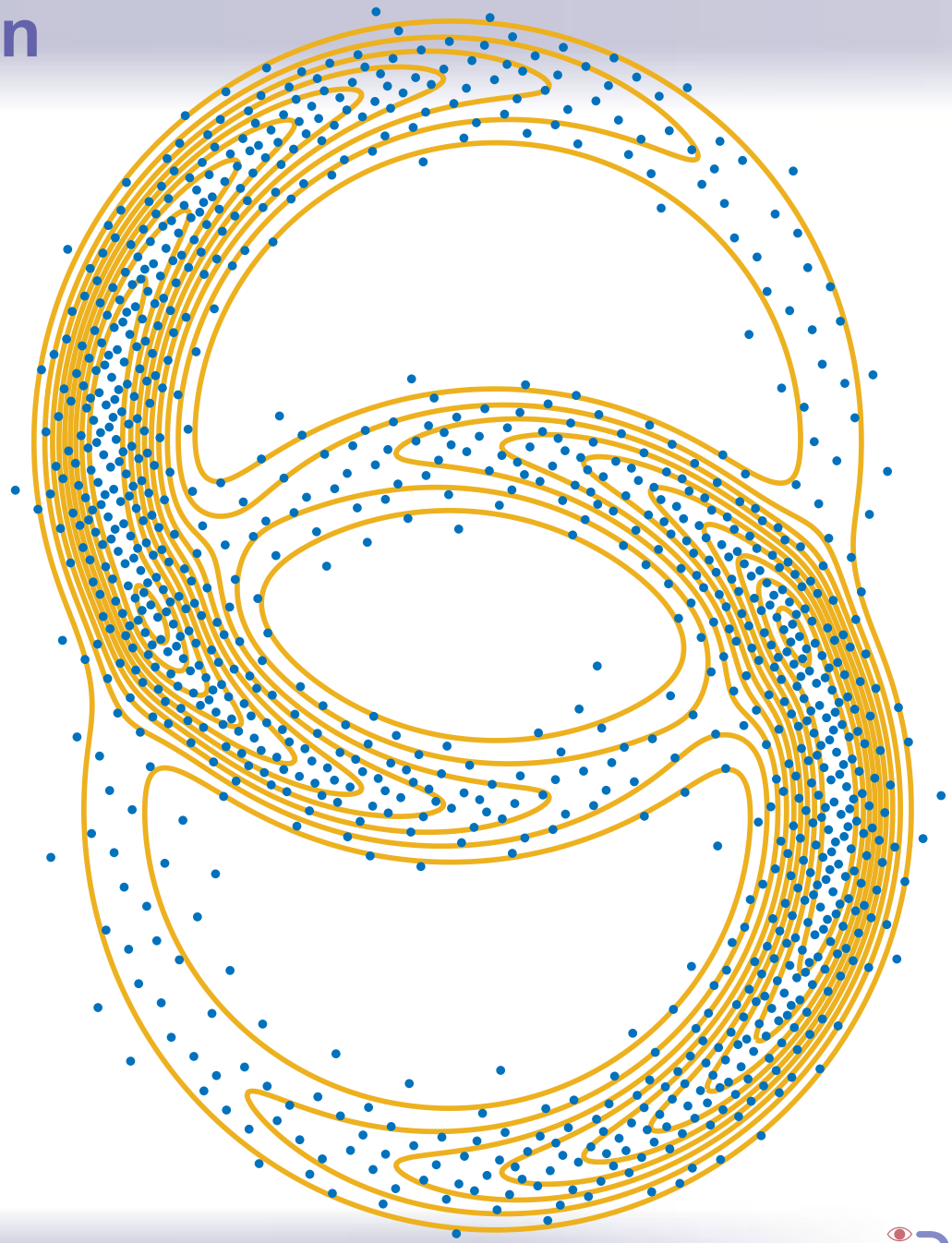
- Rejection sampling
- Arbitrary multivariate densities
- Proposal
  - Uniform
  - Gaussian
- Generalized Fibonacci Grids
  - Fixed number of points



Conclusion

Achieved

- Rejection sampling
- Arbitrary multivariate densities
- Proposal
  - Uniform
  - Gaussian
- Generalized Fibonacci Grids
  - Fixed number of points
  - Extensible



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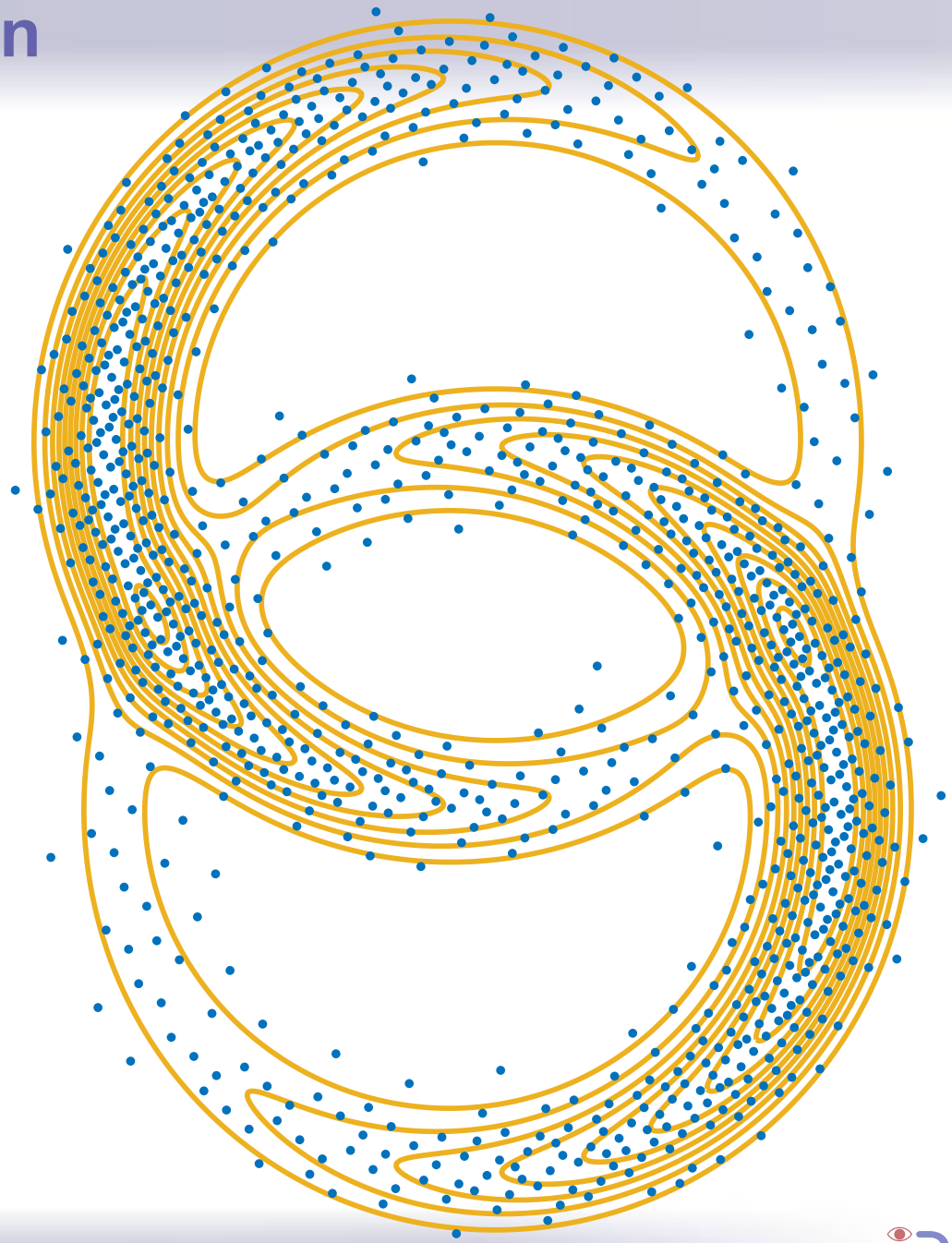
Future Work

- Fast Generalized Fibonacci Grids



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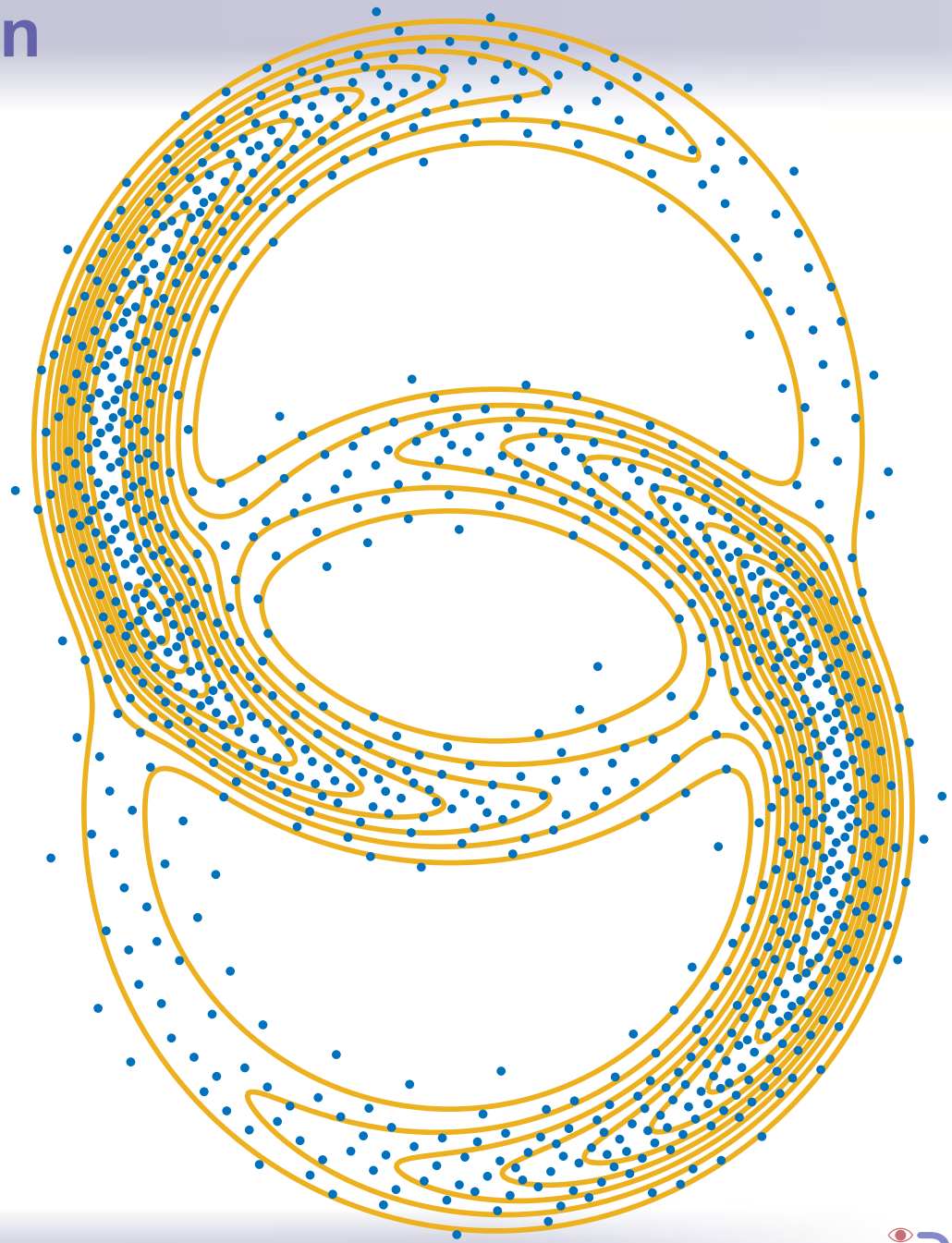
Future Work

- Fast Generalized Fibonacci Grids
- Filtering



Thank you for your attention