Deterministic Sampling of Arbitrary Densities Using Equal Sphere Packing of Volume under the Density (PoVuD)

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Rejection Sampling

- $p(x)$: Proposal (easy to sample)

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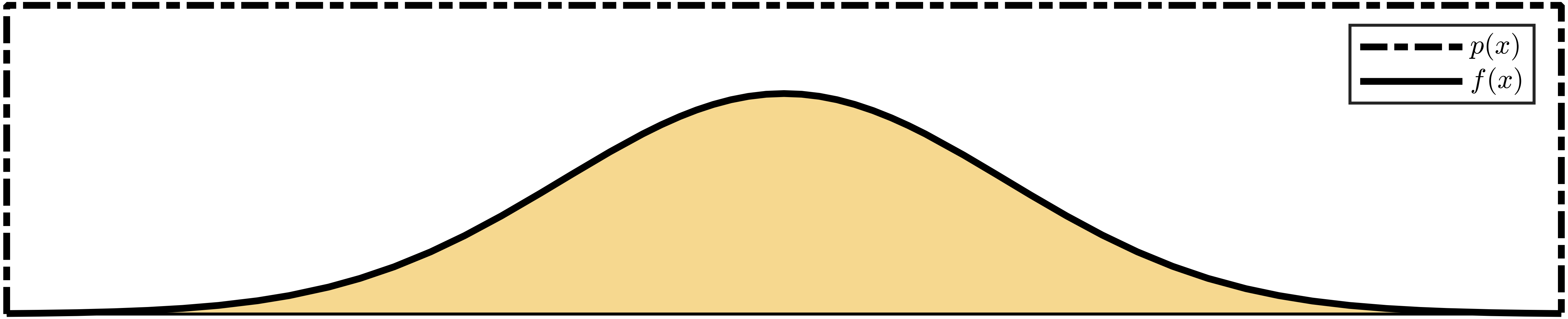
- $p(x)$: Proposal (easy to sample)
- $f(x)$: Arbitrary (difficult to sample)



Rejection Sampling

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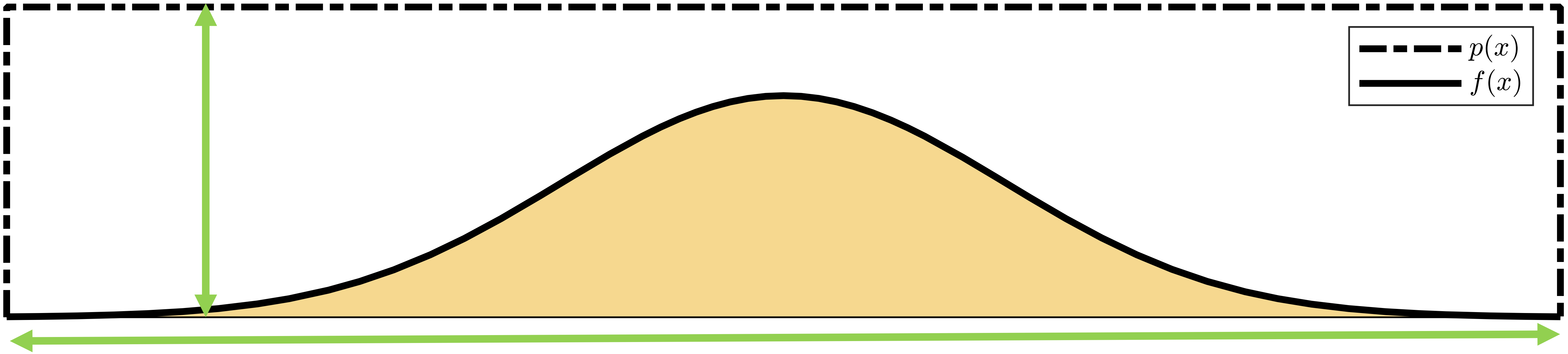
- Sample from $p(x)$



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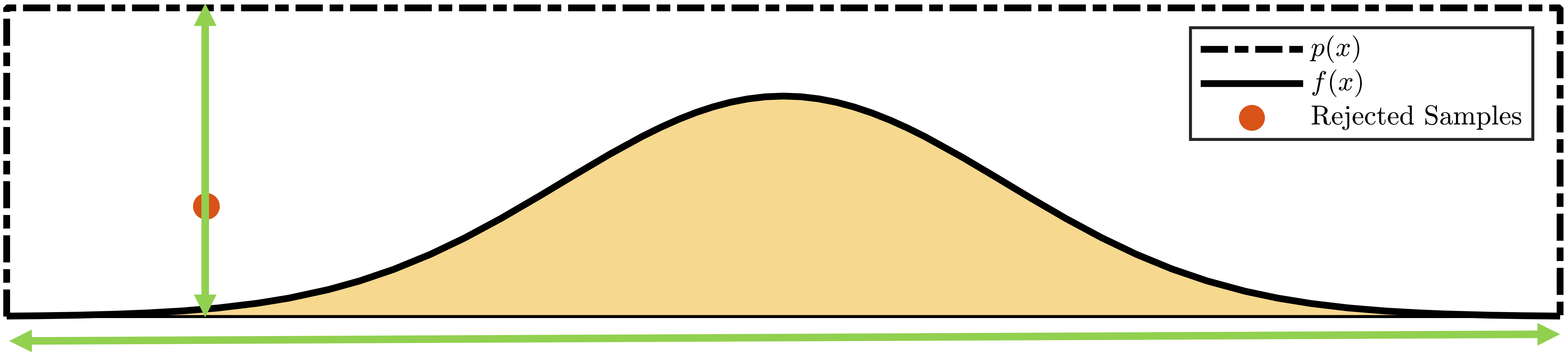

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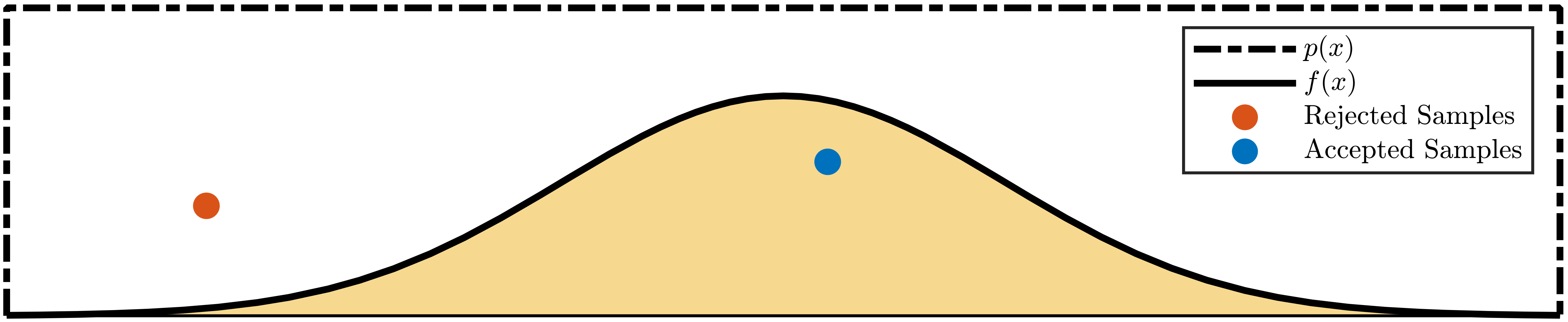

- $p(x)$: Proposal (easy to sample)
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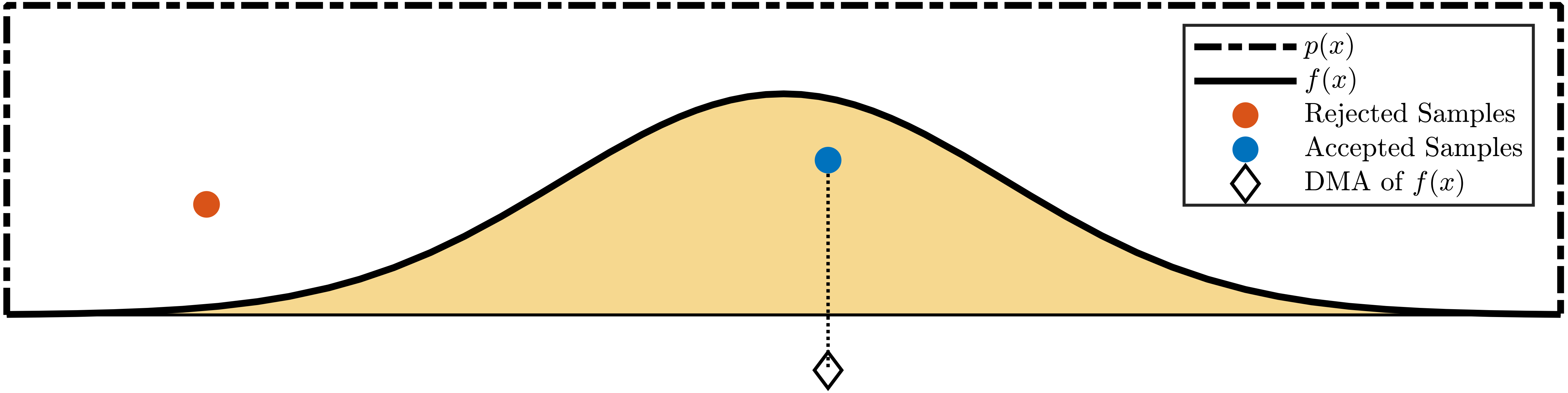
- Sample from $p(x)$
- Accepted $\rightarrow$ DMA of $f(x)$



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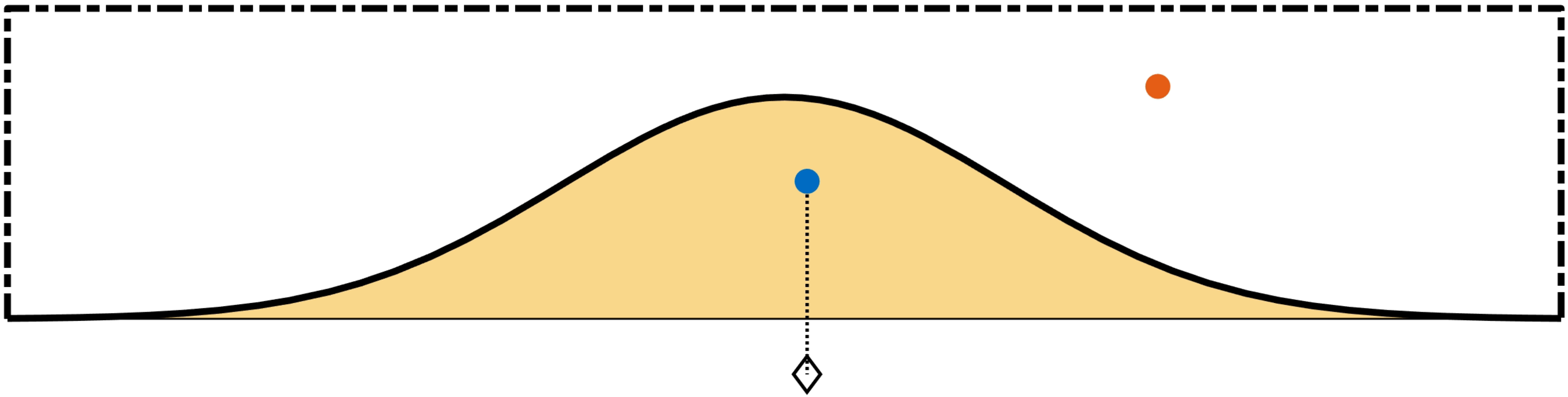
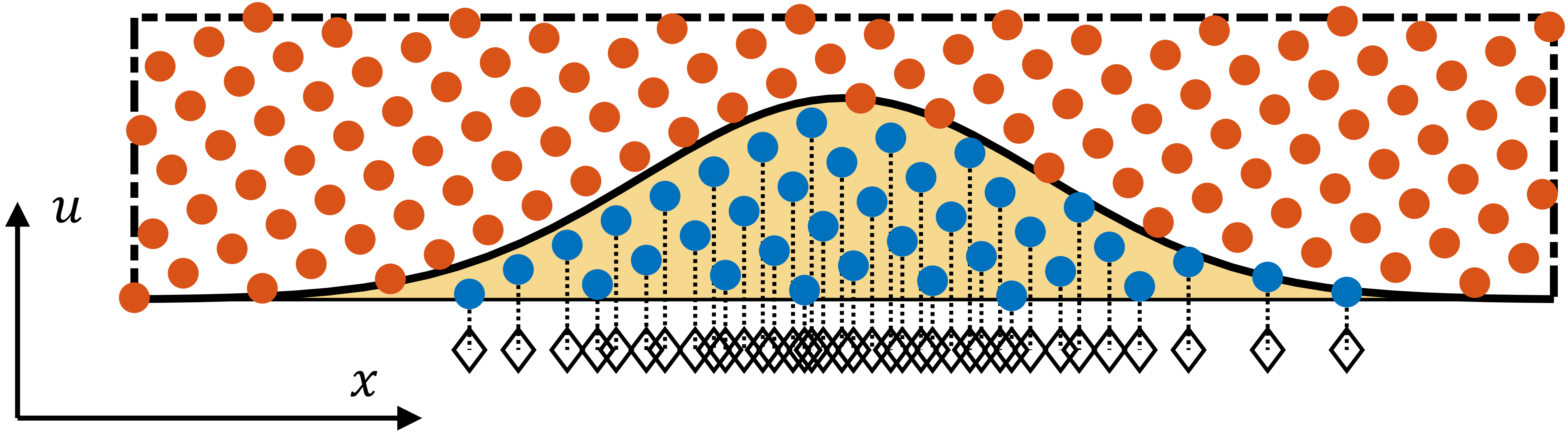


Rejection Sampling

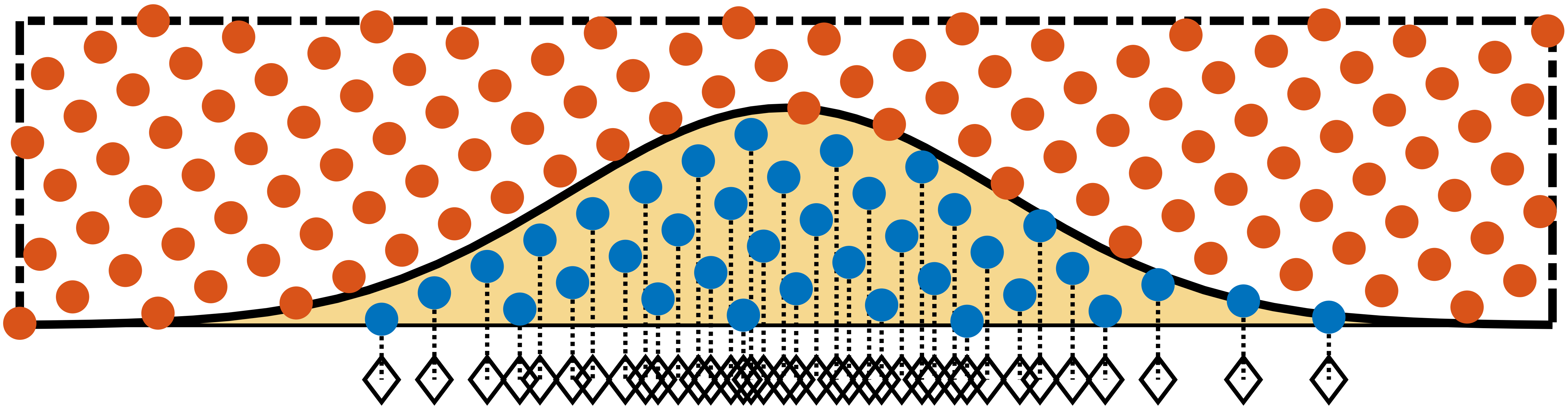
$$\underline{\xi} = \begin{bmatrix} x \\ u \end{bmatrix}$$

$$f_s\left(\underline{\xi}\right) = f_s(x, u) = c$$



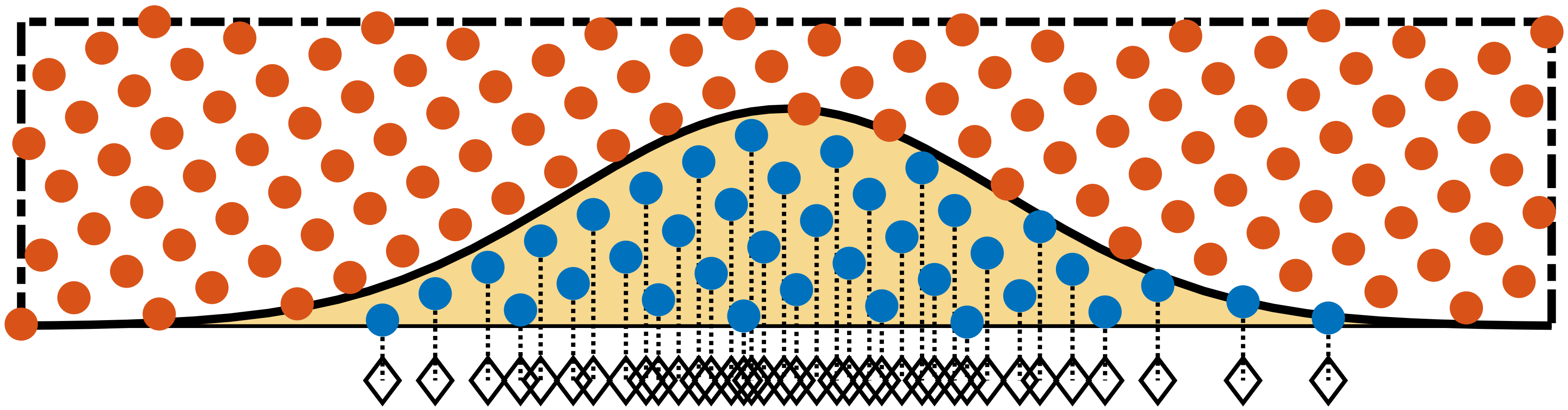
Pros & Cons:

- Arbitrary Densities
- $\mathcal{O}(e^D)$

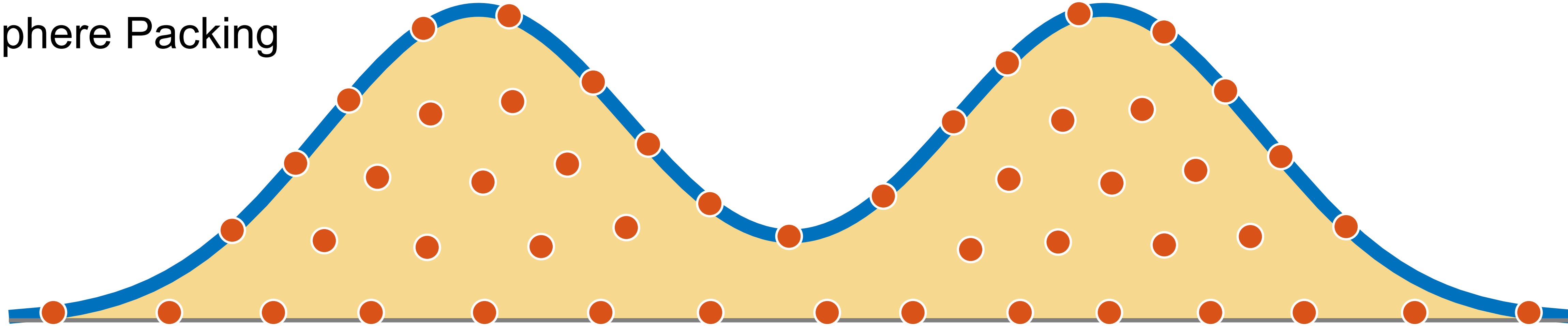


Rejection vs Volume-Filling Sampling

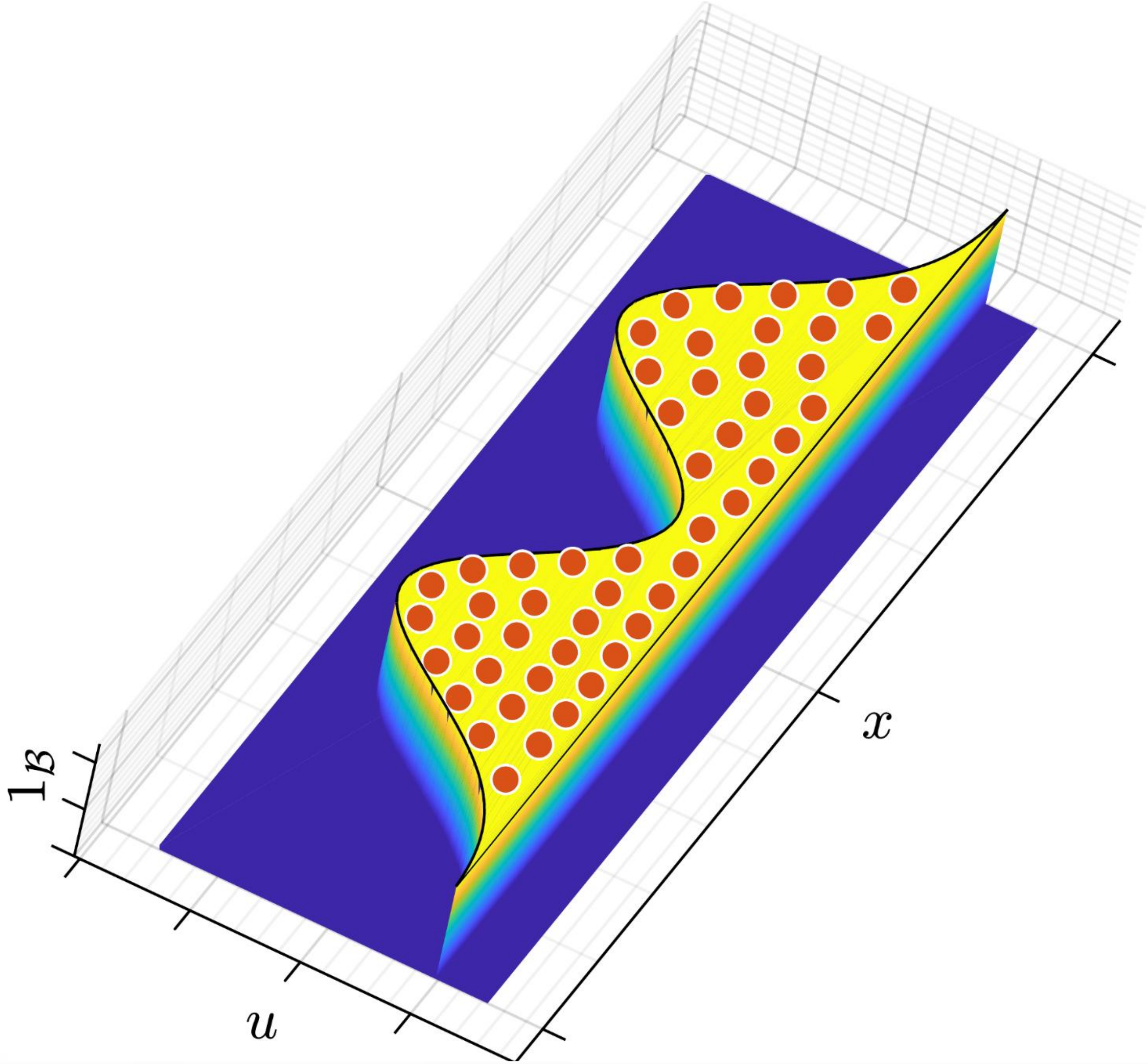
- Rejection



- Sphere Packing



Volume under Density (VuD)



Uniformity Measures

- Coulomb Energy

$$\underline{x}_{1:L}^{\text{opt}} = \arg\min_{\underline{x}_{1:L}} \left\{ \sum_{i=1}^{L} \sum_{j=1}^{L} \frac{1}{\left\| \underline{x}_i - \underline{x}_j \right\|} \right\}$$

- Dispersion

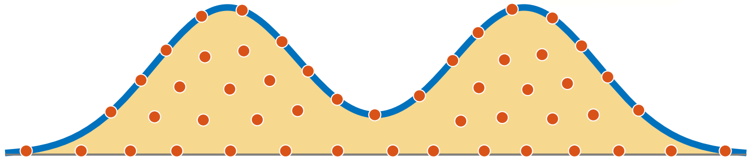
$$\underline{x}_{1:L}^{\text{opt}} = \arg\max_{\underline{x}_{1:L}} \left\{ \min_{\substack{i,j \\ i \neq j}} \left\| \underline{x}_i - \underline{x}_j \right\| \right\}$$

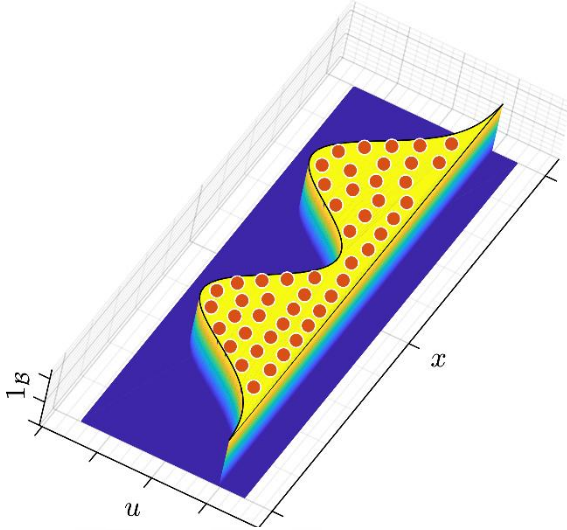
- Closest Distance Relaxed

$$\underline{x}_{1:L}^{\text{opt}} = \arg\min_{\underline{x}_{1:L}} \left\{ \sum_{i=1}^{L} \sum_{j=1}^{L} \frac{1}{\left\| \underline{x}_i - \underline{x}_j \right\|^p} \right\}$$

- Nearest Neigbor

$$\underline{x}_{1:L}^{\text{opt}} = \arg\min_{\underline{x}_{1:L}} \left\{ \sum_{i=1}^{L} \frac{1}{\min_{\substack{j \\ j \neq i}} \left\| \underline{x}_i - \underline{x}_j \right\|} \right\}$$

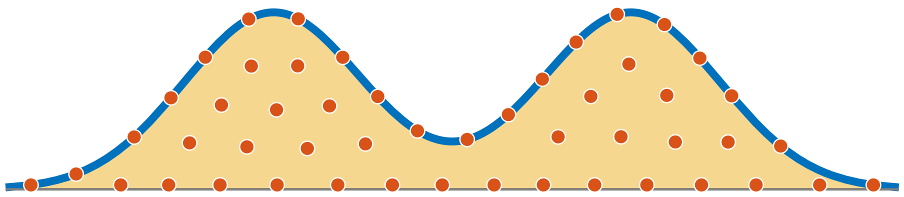


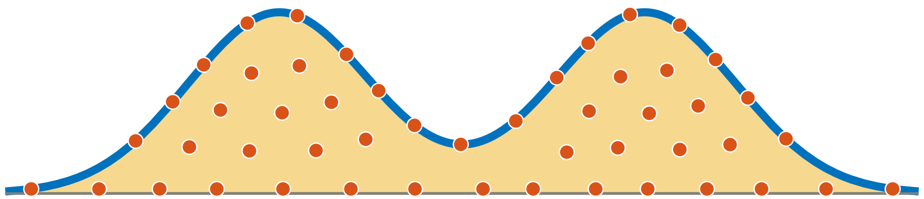


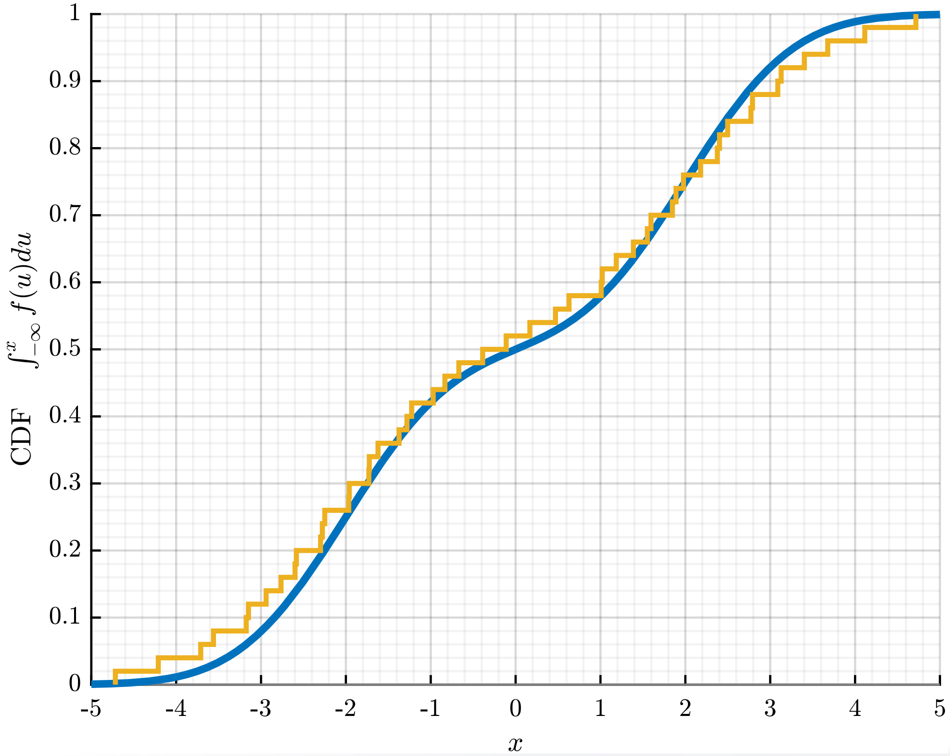
Uniformity Measures

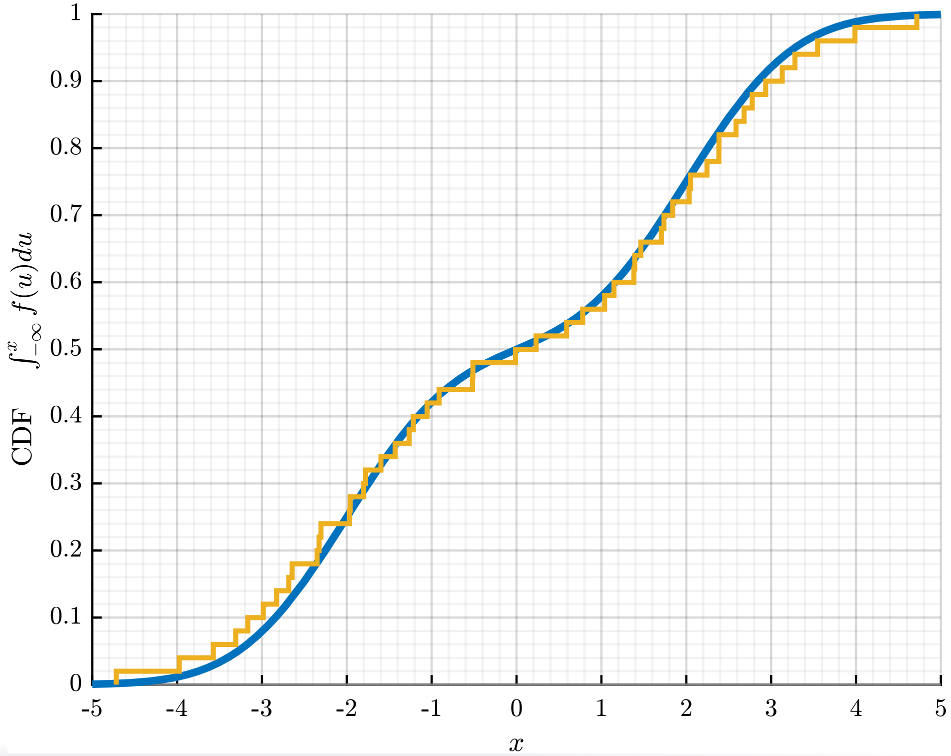
$$\underset{\underline{\xi}_{1:L}}{\arg\min}\left\{\sum_{i=1}^{L}\sum_{j=1}^{L}\frac{1}{\left\|\underline{\xi}_i-\underline{\xi}_j\right\|}\right\}$$

$$\underset{\underline{\xi}_{1:L}}{\arg\min}\left\{\sum_{i=1}^{L}\frac{1}{\min\limits_{j}\left\|\underline{\xi}_i-\underline{\xi}_j\right\|}\right\}$$

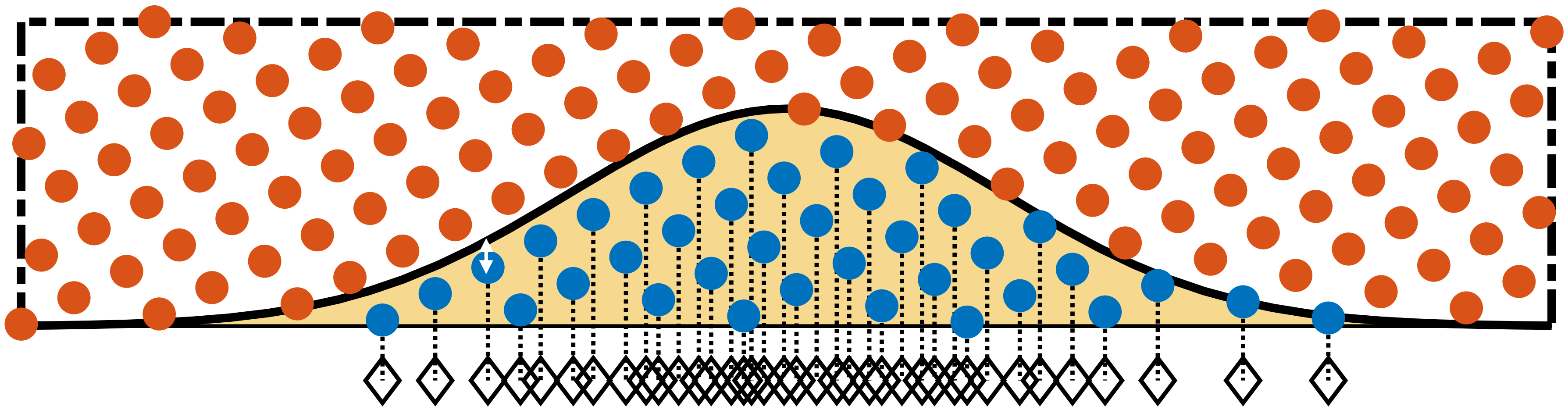




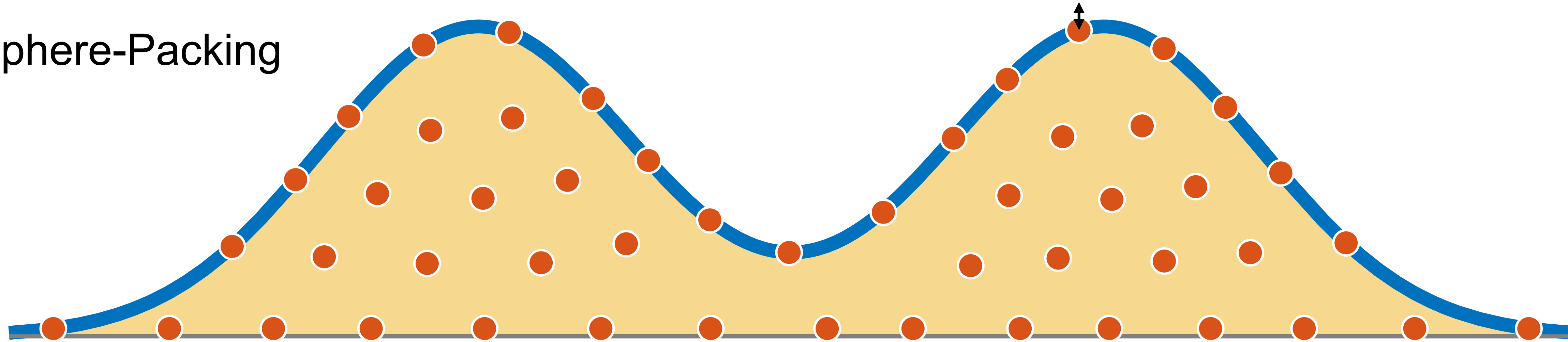




- Rejection

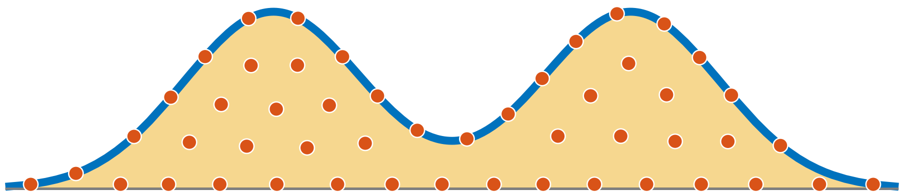


- Sphere-Packing

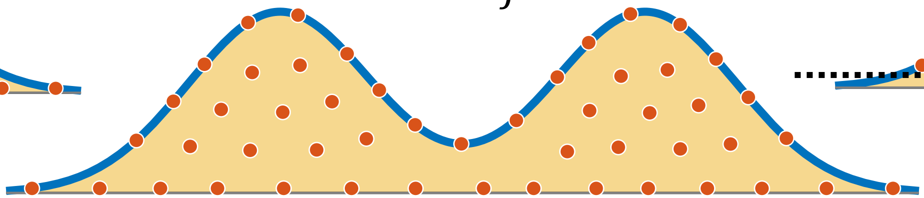


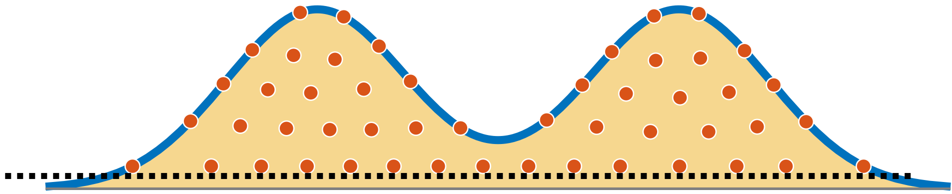
Uniformity Measures

$$\Theta\left(\underline{\xi}_{1:L}\right) = \sum_{i=1}^{L}\sum_{j=1}^{L} \frac{1}{\left\| \underline{\xi}_i - \underline{\xi}_j \right\|}$$

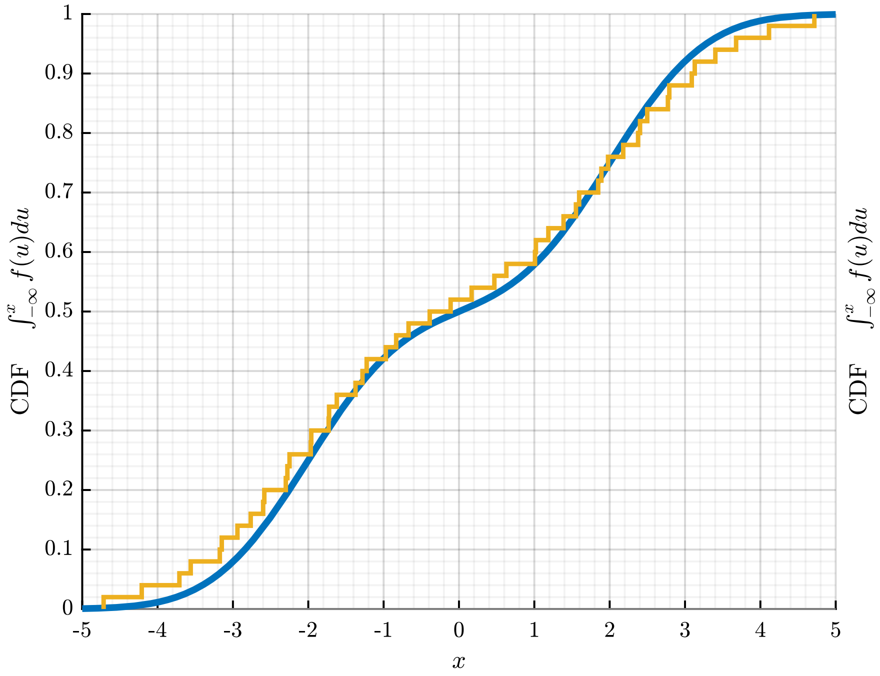


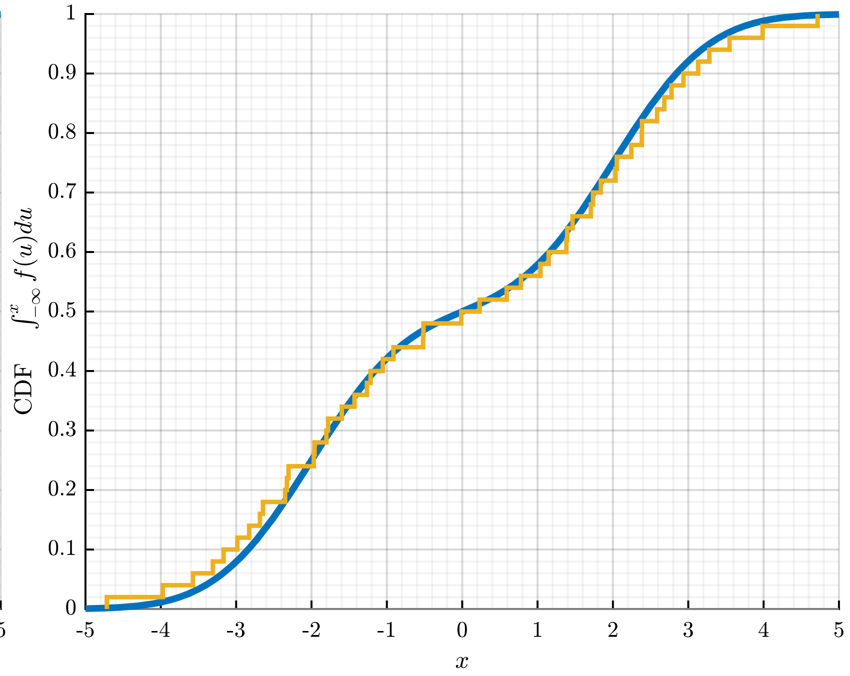
$$\Theta\left(\underline{\xi}_{1:L}\right) = \sum_{i=1}^{L} \frac{1}{\min\limits_{j} \left\| \underline{\xi}_i - \underline{\xi}_j \right\|}$$

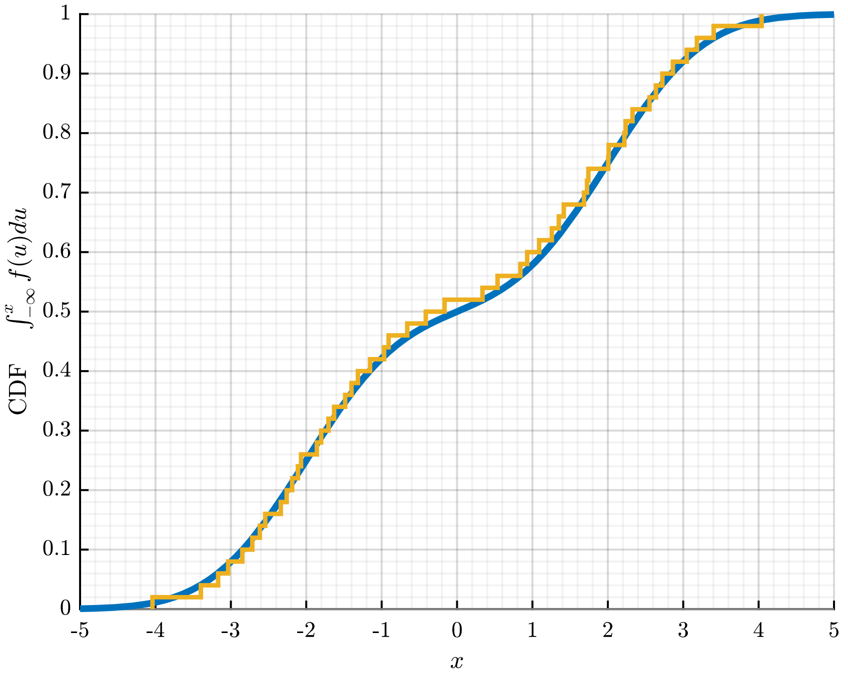




parameter 😟

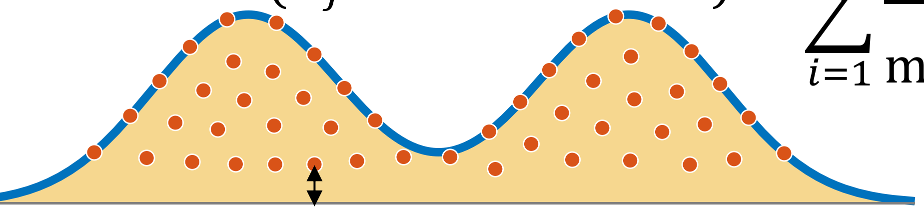






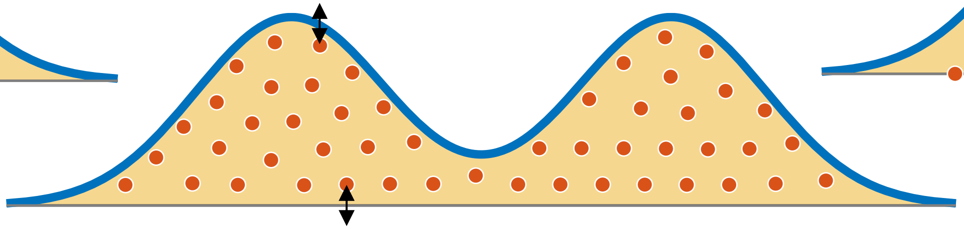
Distance to Border

$$\sum_{i=1}^{L} \frac{1}{\min\left\{\min_j\{\|\underline{x}_i - \underline{x}_j\|\}, u_i\right\}}$$

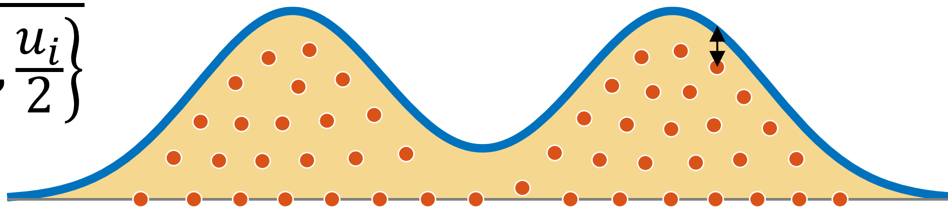


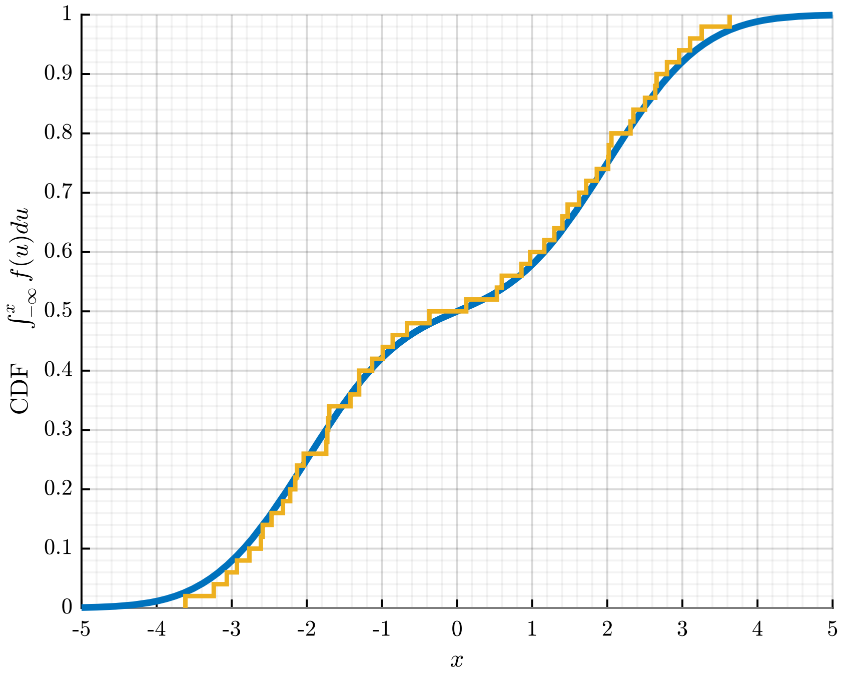
parameter-free 🙂

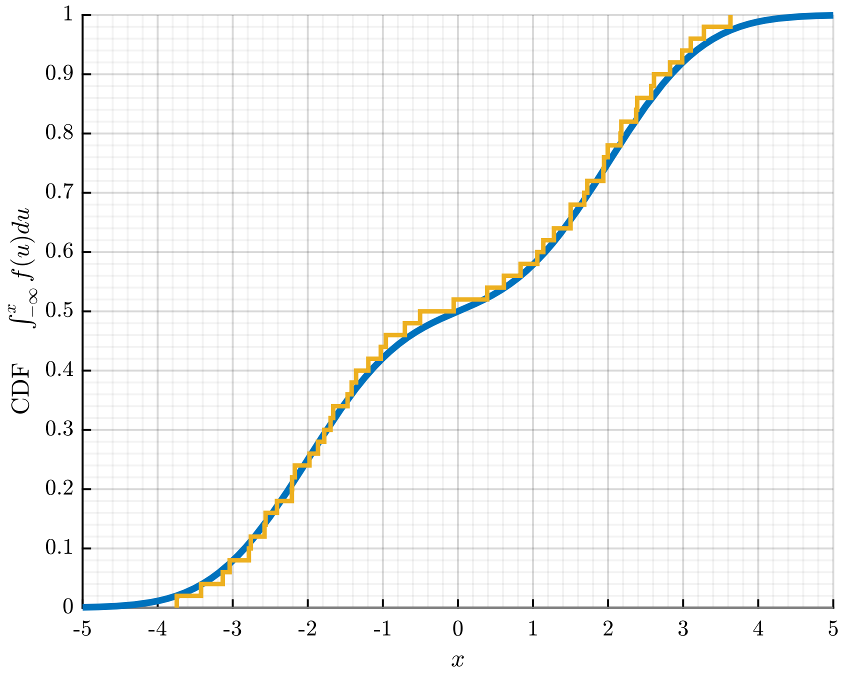
$$\sum_{i=1}^{L} \frac{1}{\min\left\{\min_j\{\|\underline{x}_i - \underline{x}_j\|\}, \frac{f(x_i) - u_i}{2}, \frac{u_i}{2}\right\}}$$

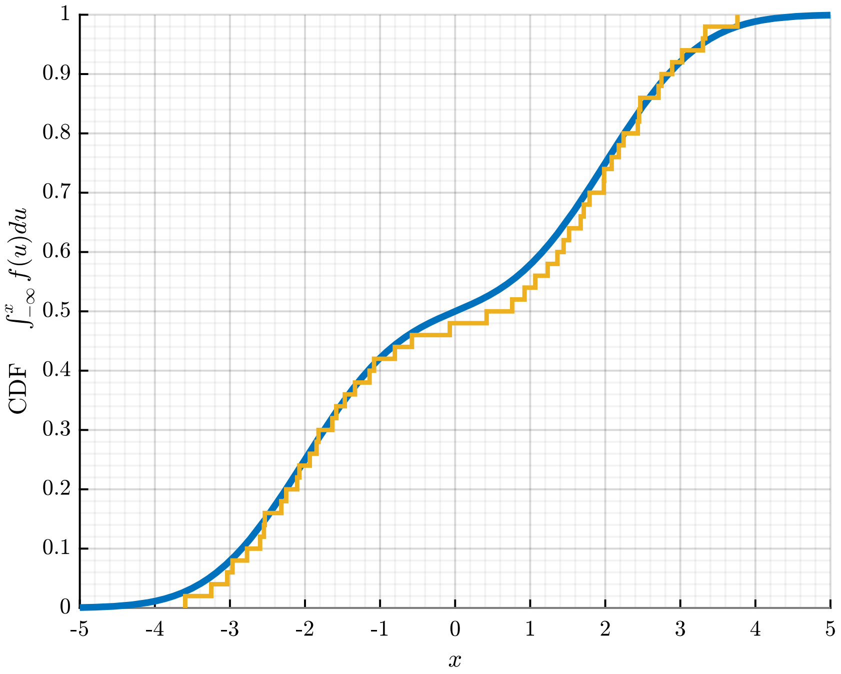


$$\sum_{i=1}^{L} \frac{1}{\min\left\{\min_j\{\|\underline{x}_i - \underline{x}_j\|\}, f(x_i) - u_i\right\}}$$



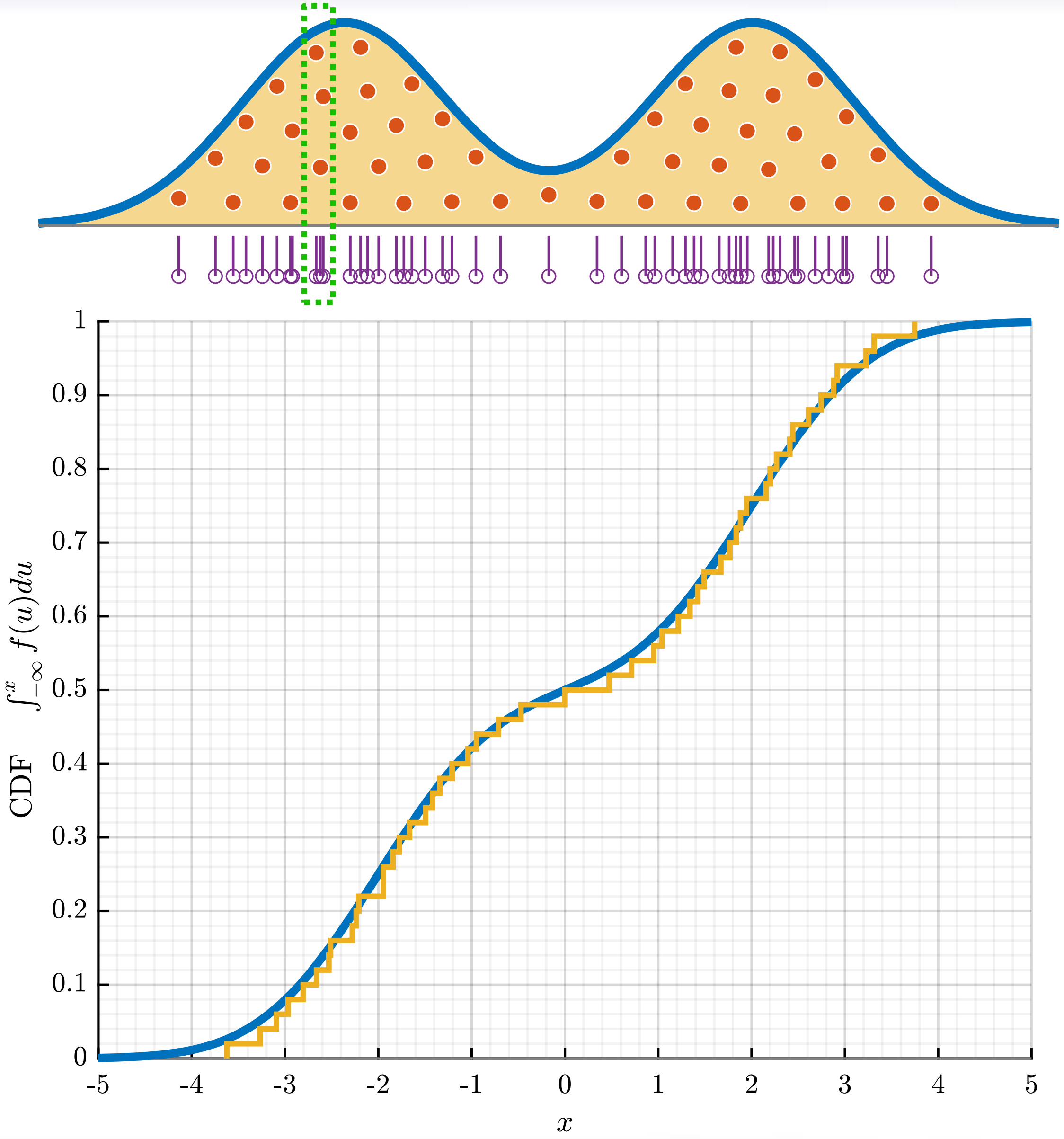






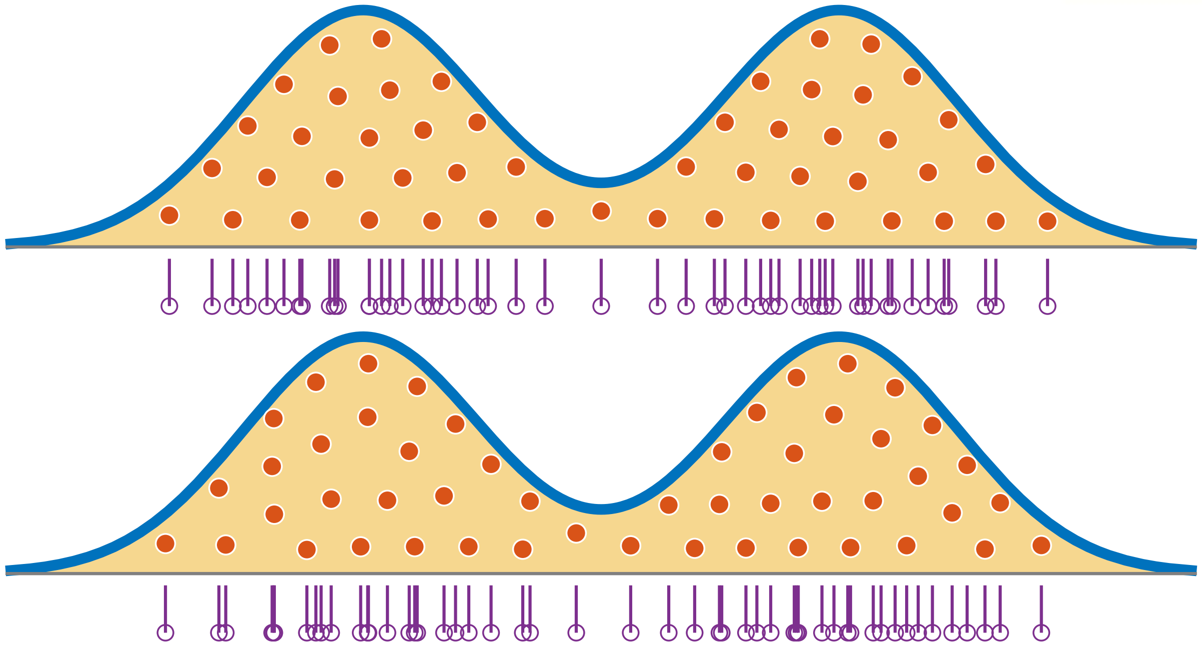
Projection

- Result: Projections
- Locally inhomogeneous
- Similar to random



Multi-Objective

- Null Space
  - Various distributions of $\underline{\xi}$
  - Ambiguity → Smooth Projection
- Uniformity measure x2
  - Joint space $\underline{\xi}$
  - State space x
- Parameter α 😳
  - Choose appropriately
  - Pareto front

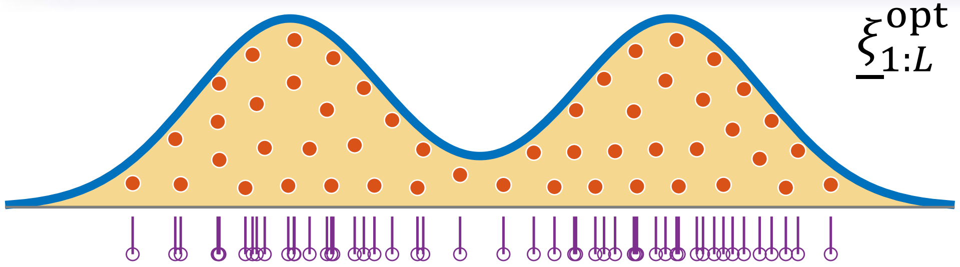


$$\underline{\xi}_{1:L}^{\text{opt}} = \arg\min_{\underline{\xi}_{1:L}} \left\{ \Theta\left(\underline{\xi}_{1:L}\right) + \alpha \cdot \Theta(x_{1:L}) \right\}$$

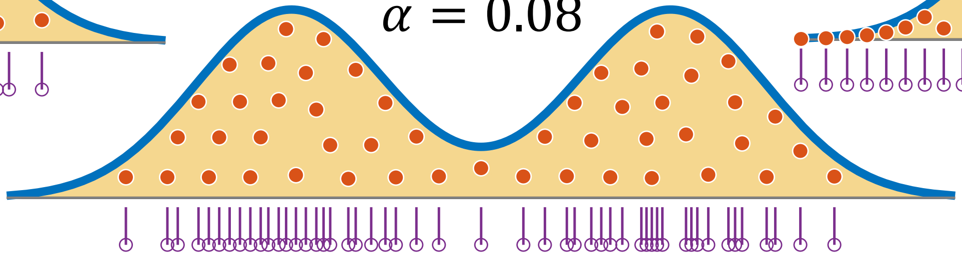
Parameter Choice

$$\underline{\xi}_{1:L}^{\text{opt}} = \underset{\underline{\xi}_{1:L}}{\arg\min} \left\{ \Theta\left(\underline{\xi}_{1:L}\right) + \alpha \cdot \Theta(x_{1:L}) \right\}$$

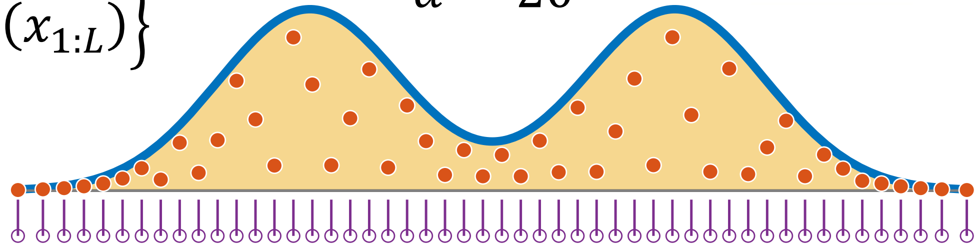
$\alpha = 0.00002$

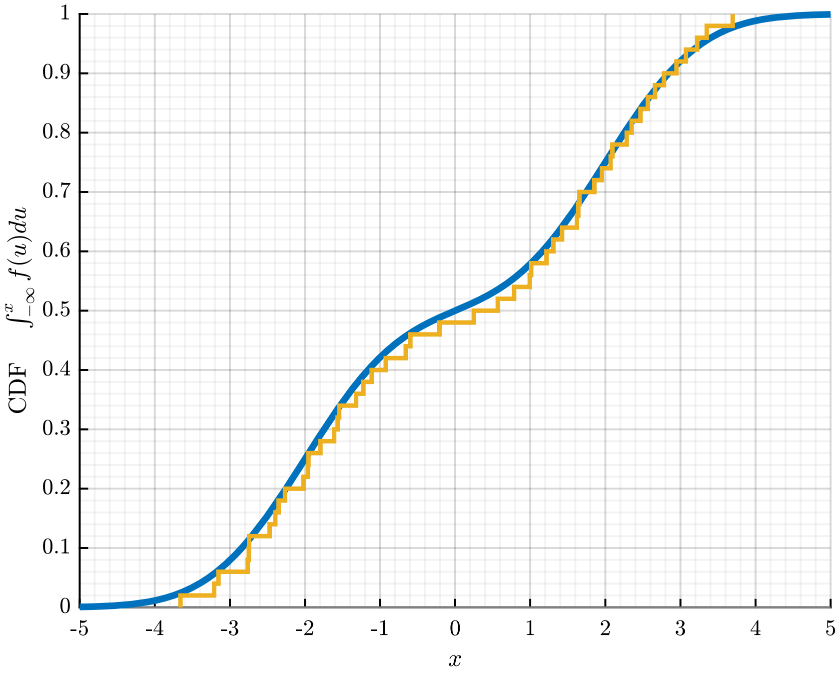


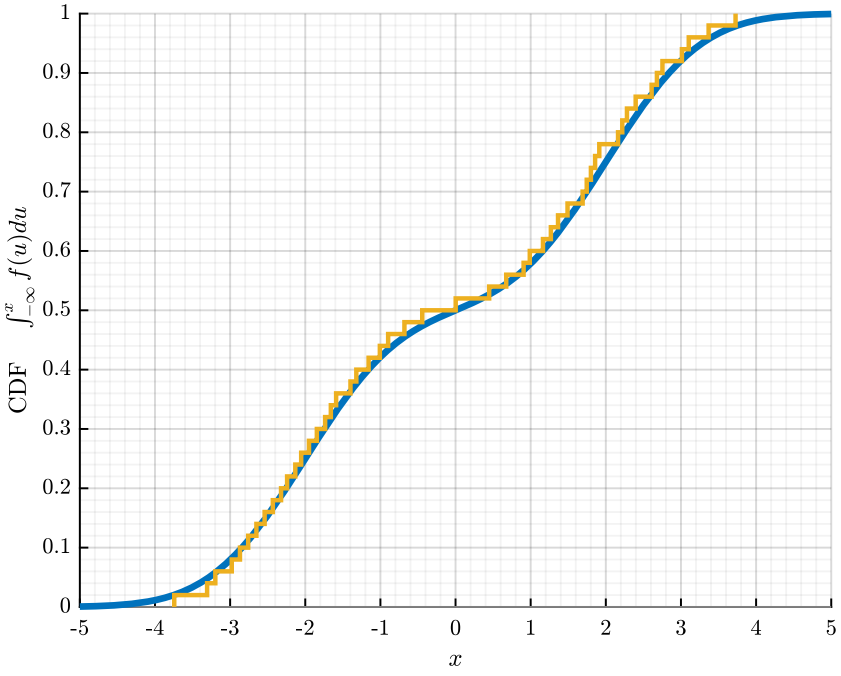
$\alpha = 0.08$

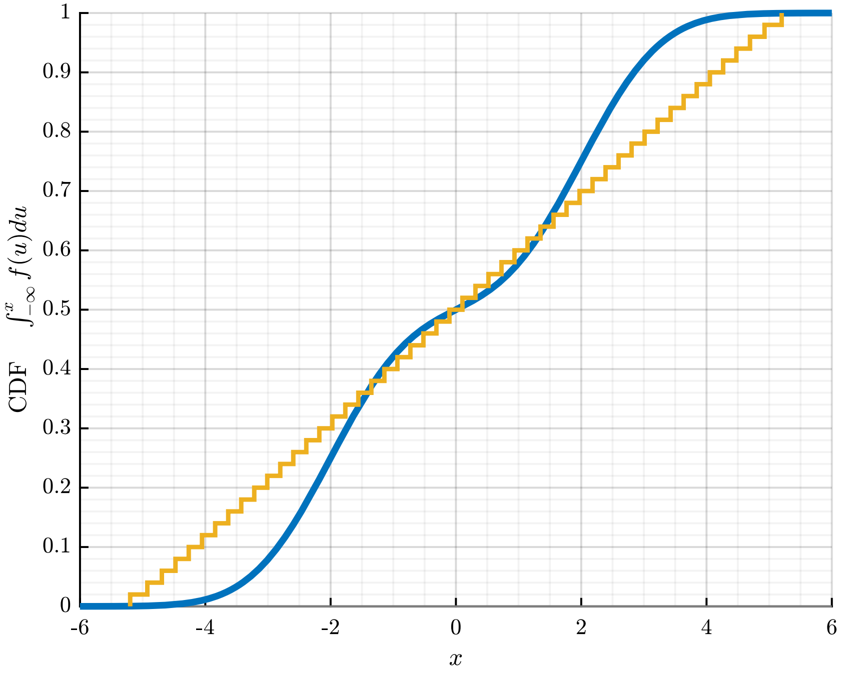


$\alpha = 20$





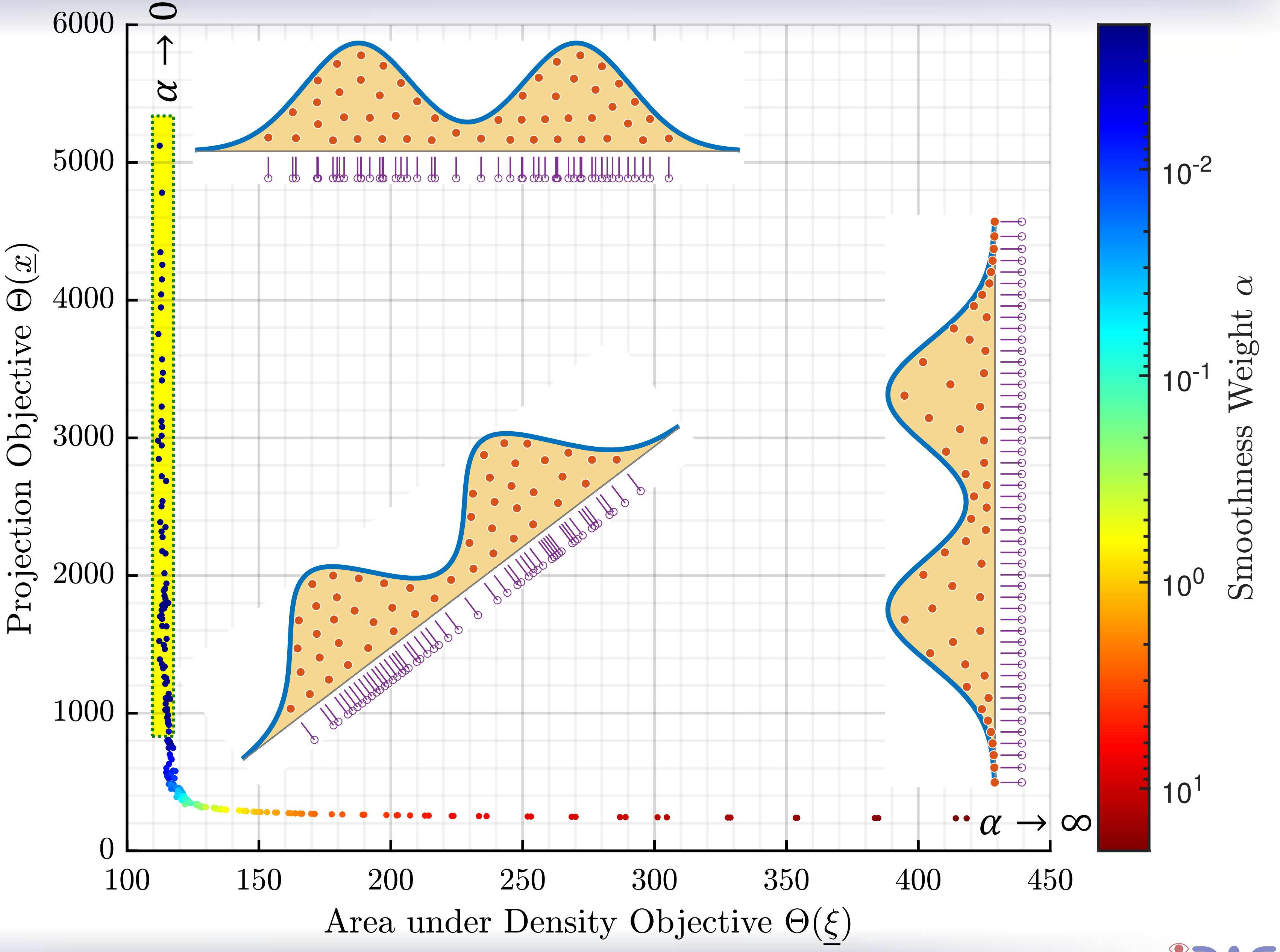




Pareto Front

$$\Theta\left(\underline{\xi}_{1:L}\right) + \alpha \cdot \Theta(x_{1:L})$$

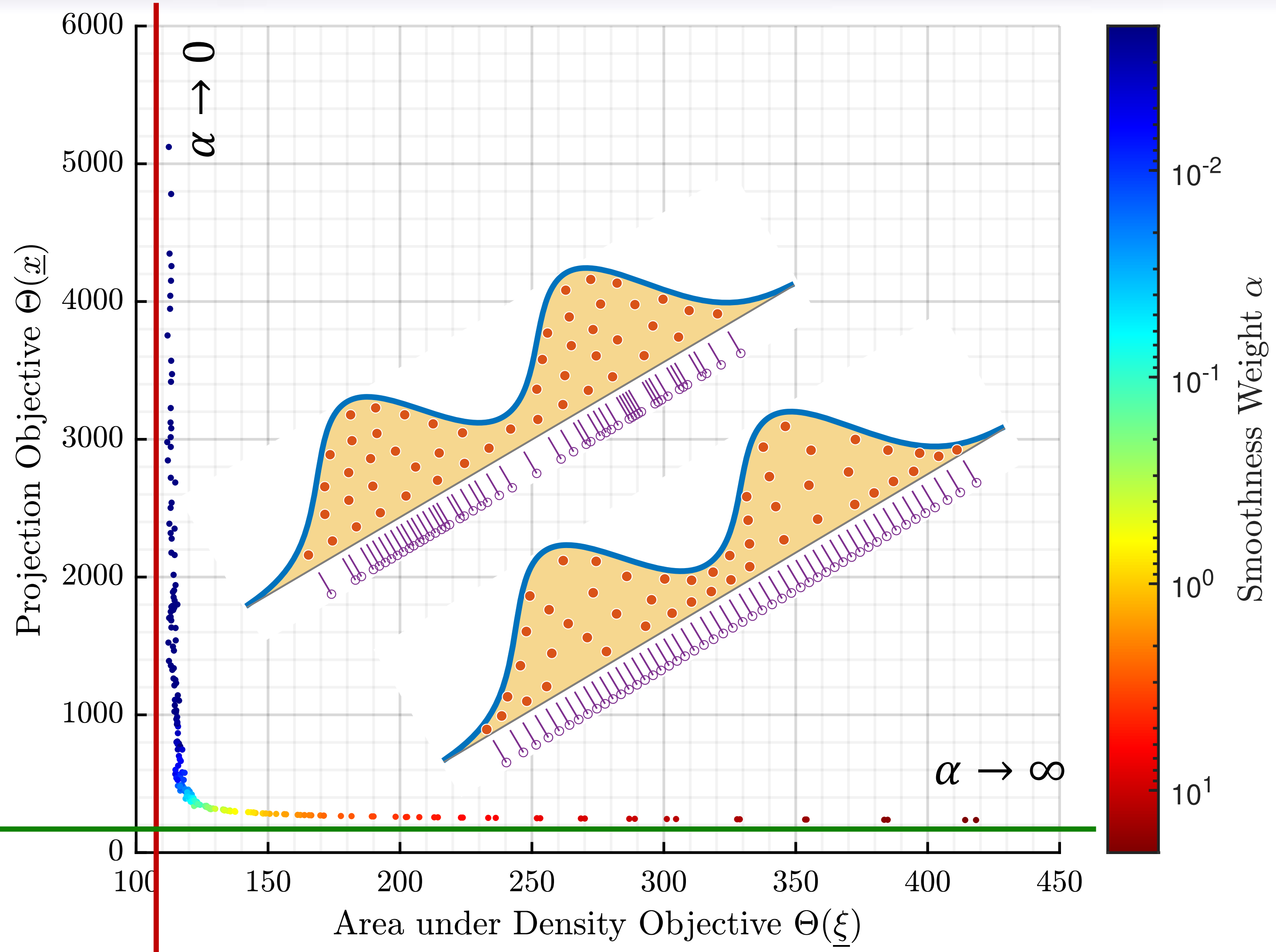
- Pareto front
  - Vaious α
- Null space
  - Uniform $\underline{\xi}$
  - Uniform x
- Tradeoff



Determine Parameter?

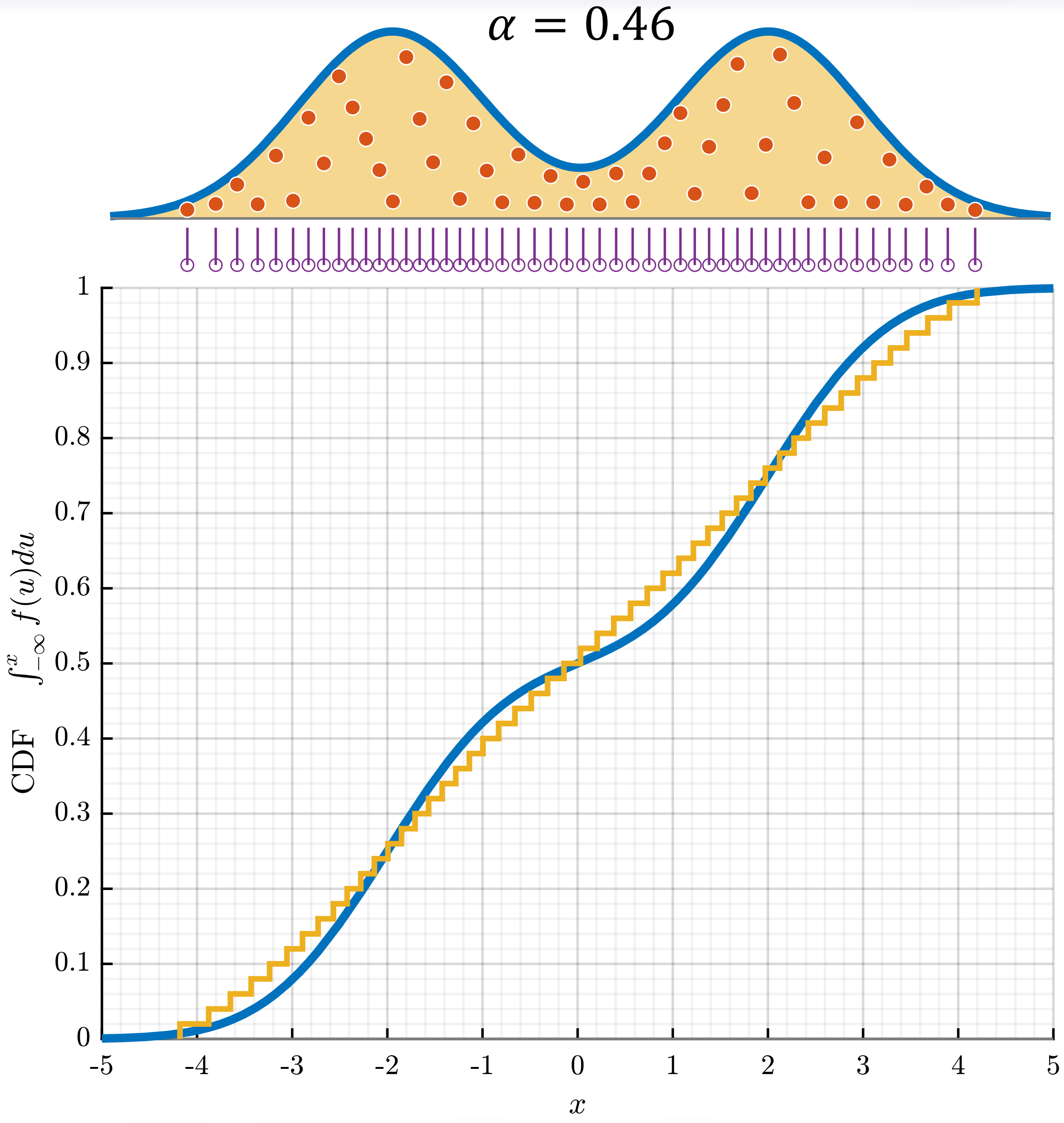
$$\underbrace{\Theta\left(\underline{\xi}_{1:L}\right)}_{\approx 112} + \alpha\, \underbrace{\Theta\left(\underline{x}_{1:L}\right)}_{\approx 236}$$

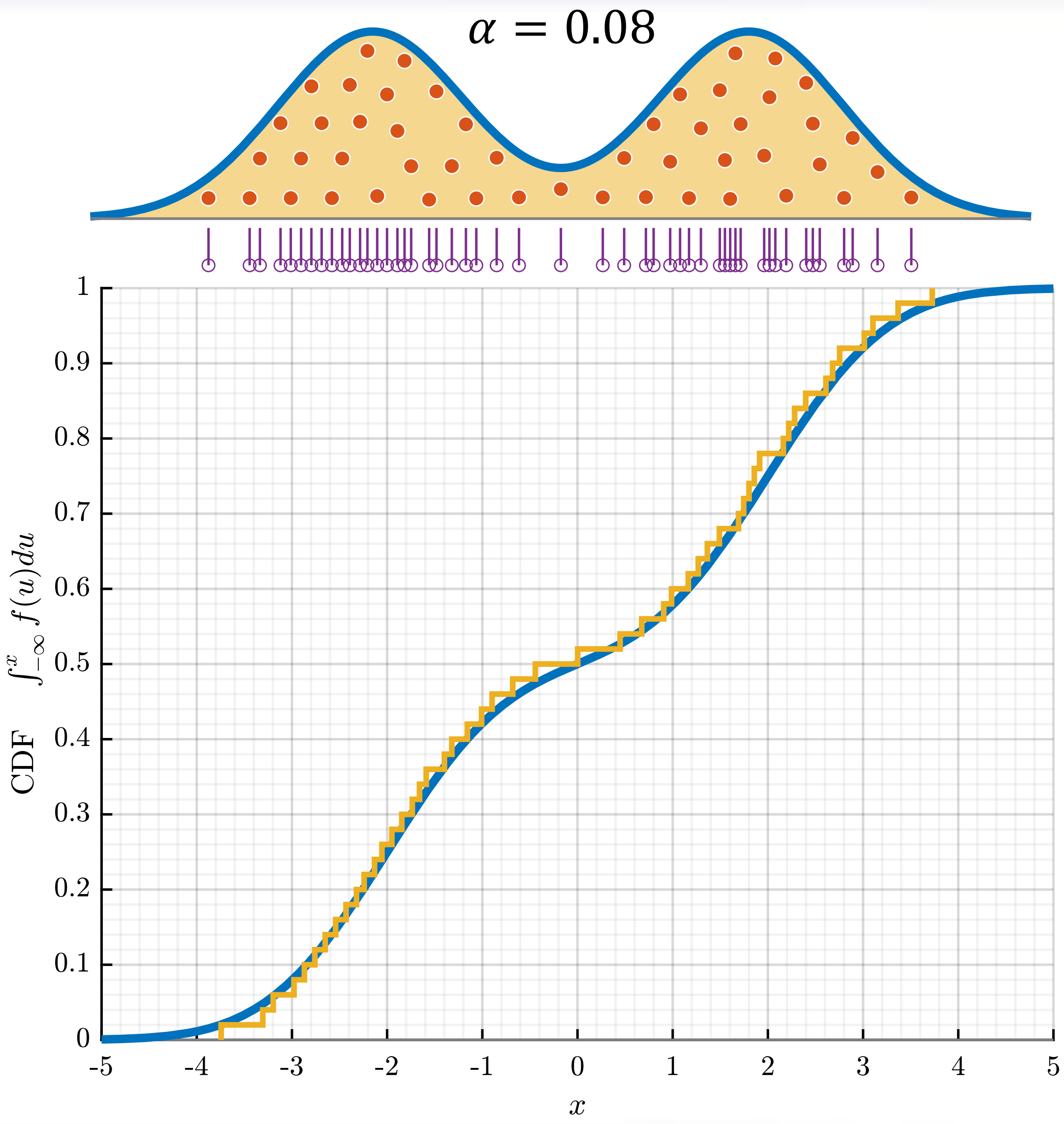
$\min_{\underline{\xi}_{1:L}} \{\Theta(\underline{x}_{1:L})\} \approx 236$



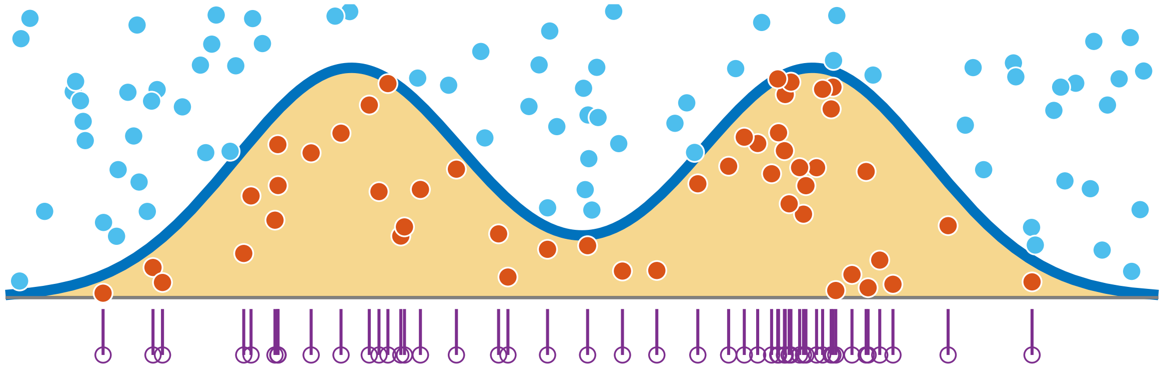
$112 \approx \min_{\underline{\xi}_{1:L}} \left\{\Theta\left(\underline{\xi}_{1:L}\right)\right\}$

Fair-Weight Parameter

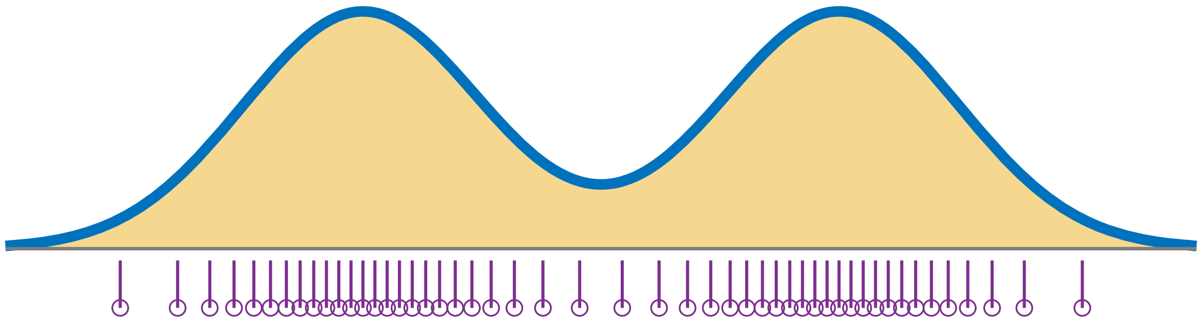




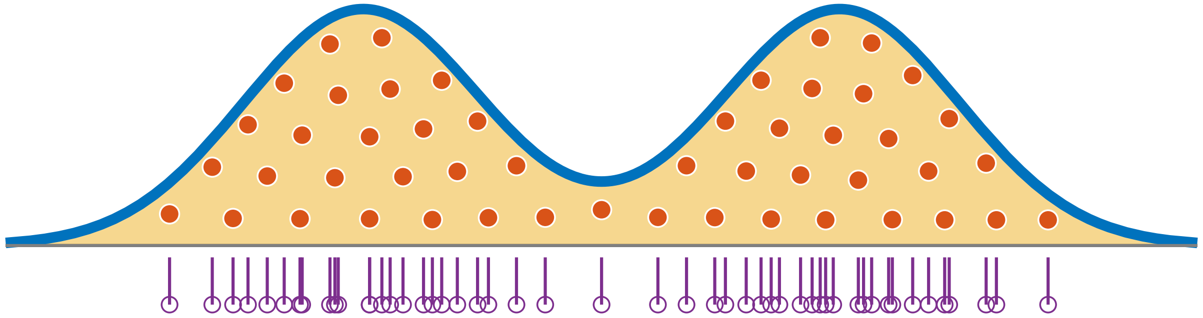
Summary



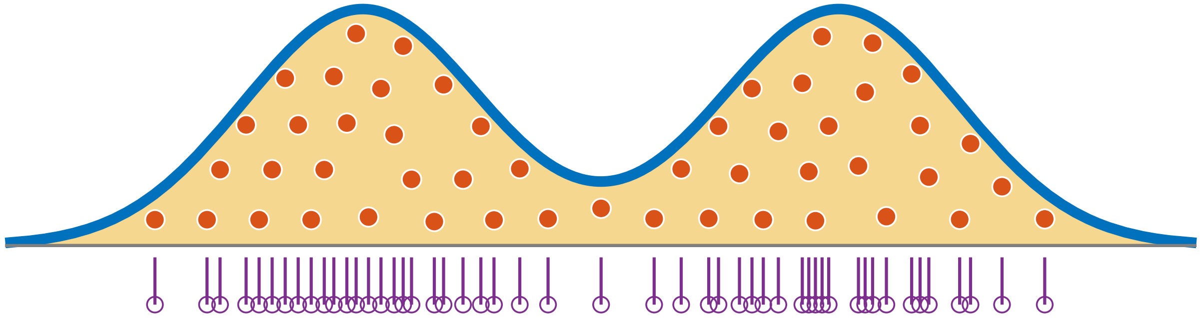
Random Rejection



Inverse Transform Sampling



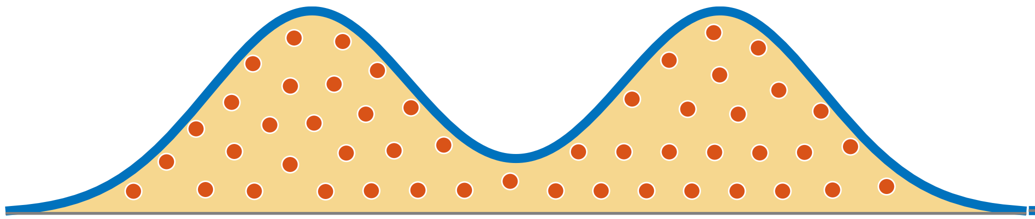
PoVuD

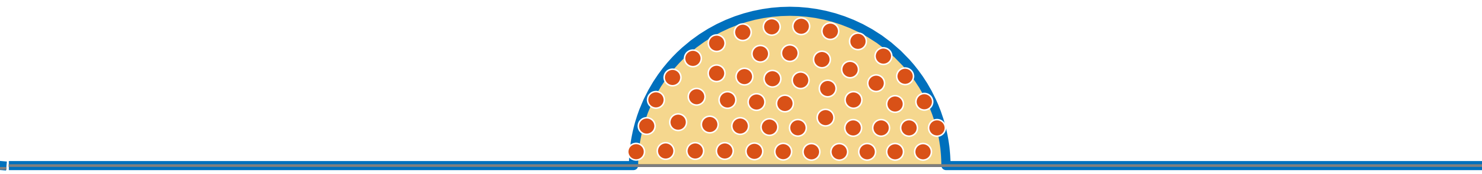


PoVuD + Smoothing

Outlook

- Eliminate parameter dependency on
  - Other parameters
  - Density function
  - Number of samples
  - Dimension
- Optimization
  - More uniformity measures
  - Different one for projection
  - More optimizers
  - Implement Julia
- Implement
  - Higher dimensions
  - More densities
  - Evaluation





Thank you for your attention

Intelligent Sensor-Actuator-Systems