The Generalized Fibonacci Grid as Low-Discrepancy Point Set for Optimal Deterministic Gaussian Sampling

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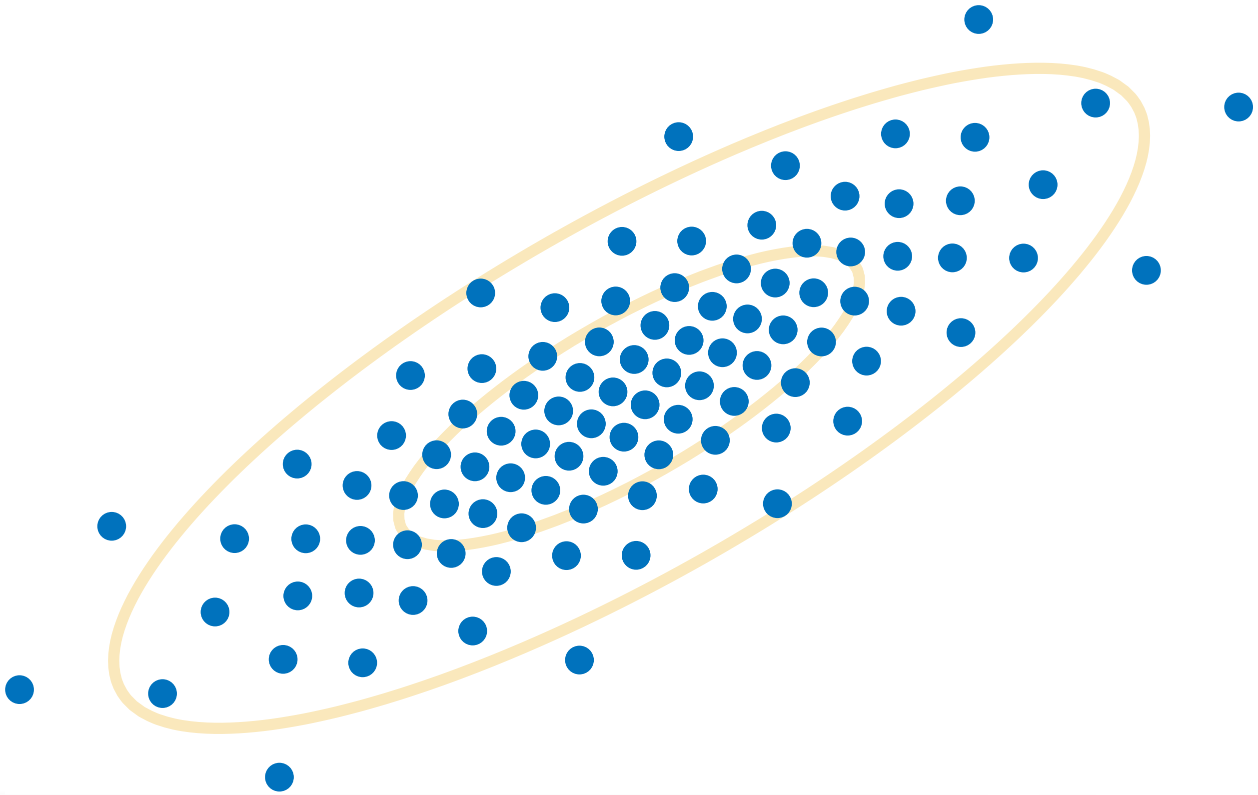
Intelligent Sensor-Actuator-Systems

Overview

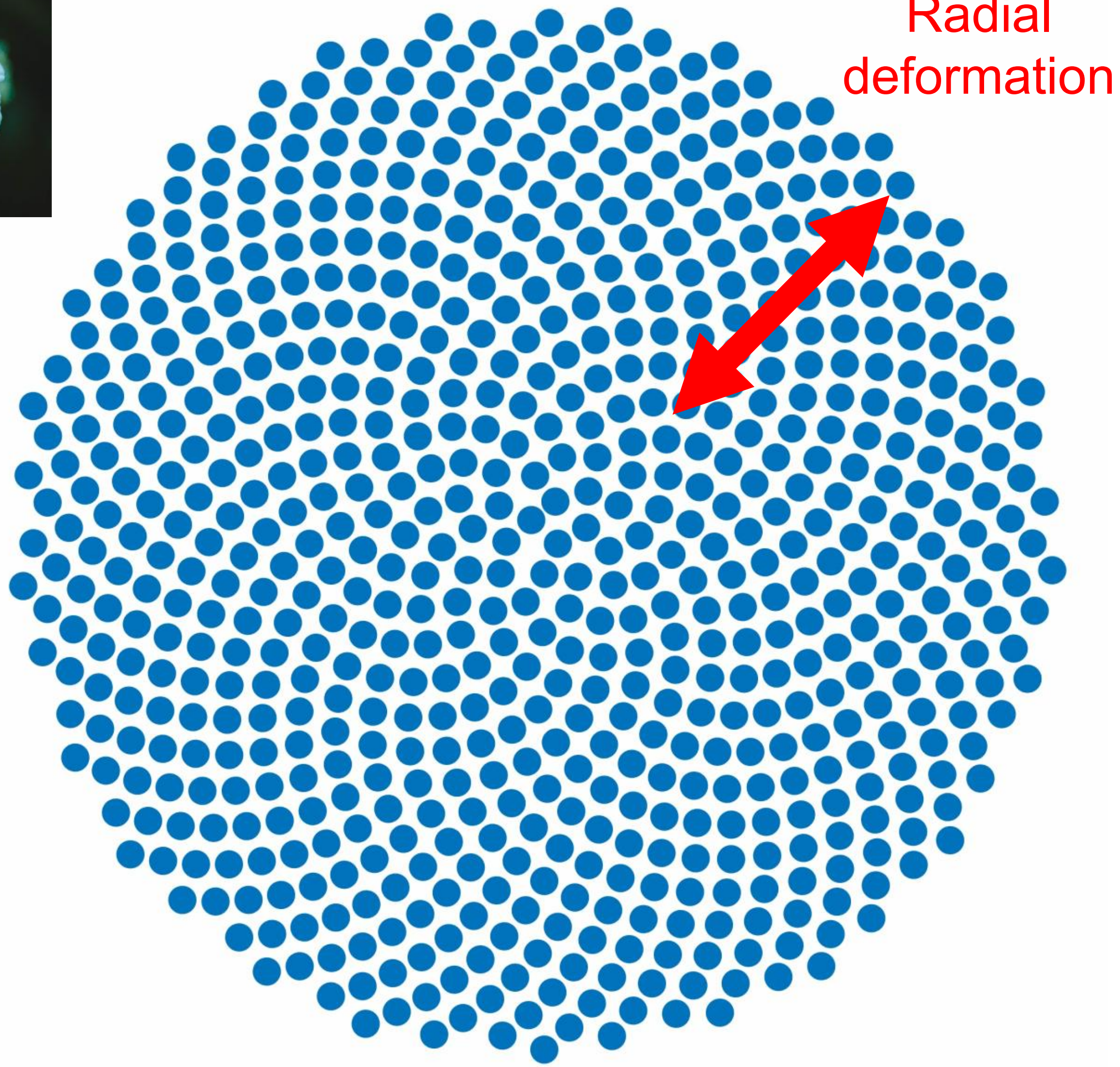
Input
- No. Samples
- Dimension
- Covariance matrix

→ **Deterministic Sampling** →

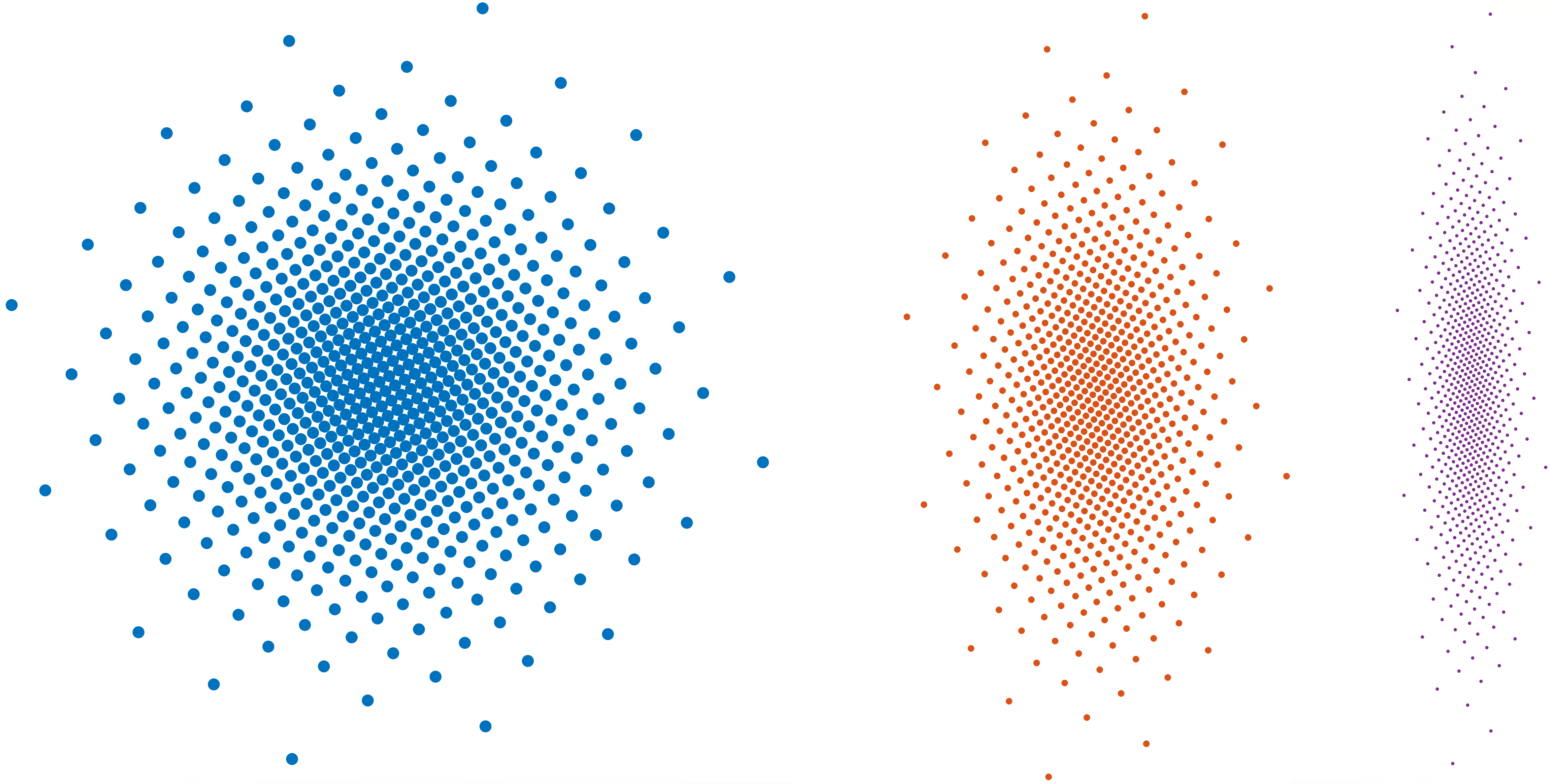
Output
- Sample locations



Polar Fibonacci Grid



Cartesian Fibonacci Grid → Gaussian



Fibonacci Matrix

Linear system generating Fibonacci numbers

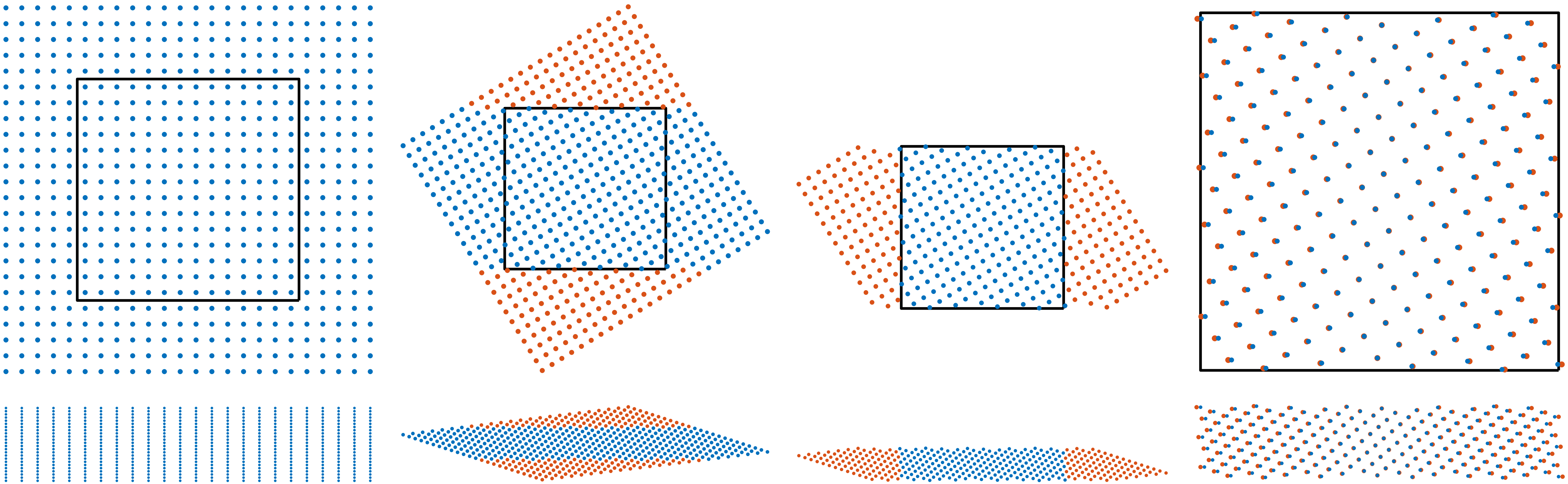
$$\underline{x}_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \underline{x}_k \,, \qquad \underline{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

2D $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **3D**

Eigenvalues as Rotation Matrix

Valid generalization if $2\,\mathrm{dim} + 1$ prime

Uniform Construction



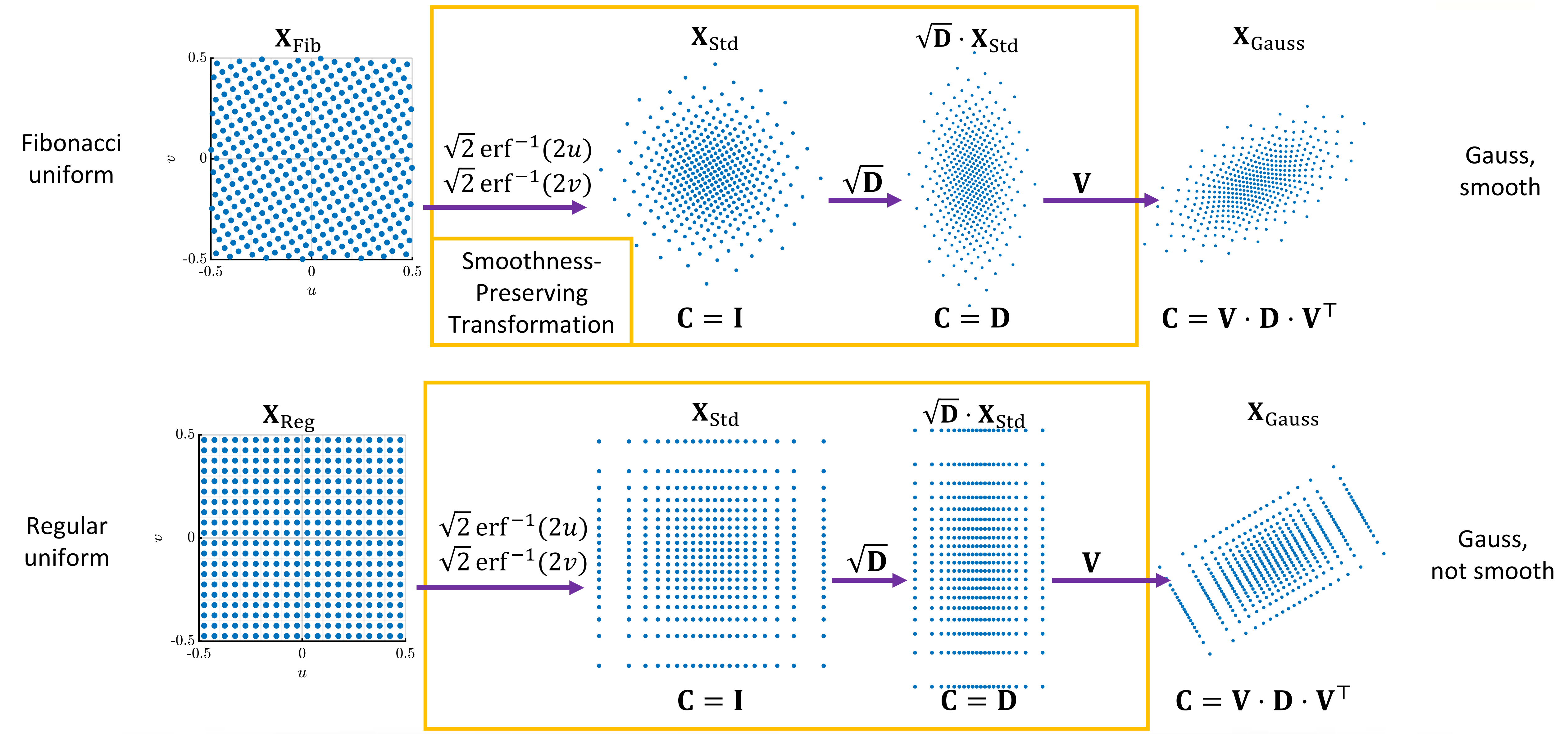
(a) Regular Lattice
→ Rotate

(b) Rotated
→ Crop dimension $2, 3, \ldots$

(c) Removed points (part one)
→ Sort along x and crop to L

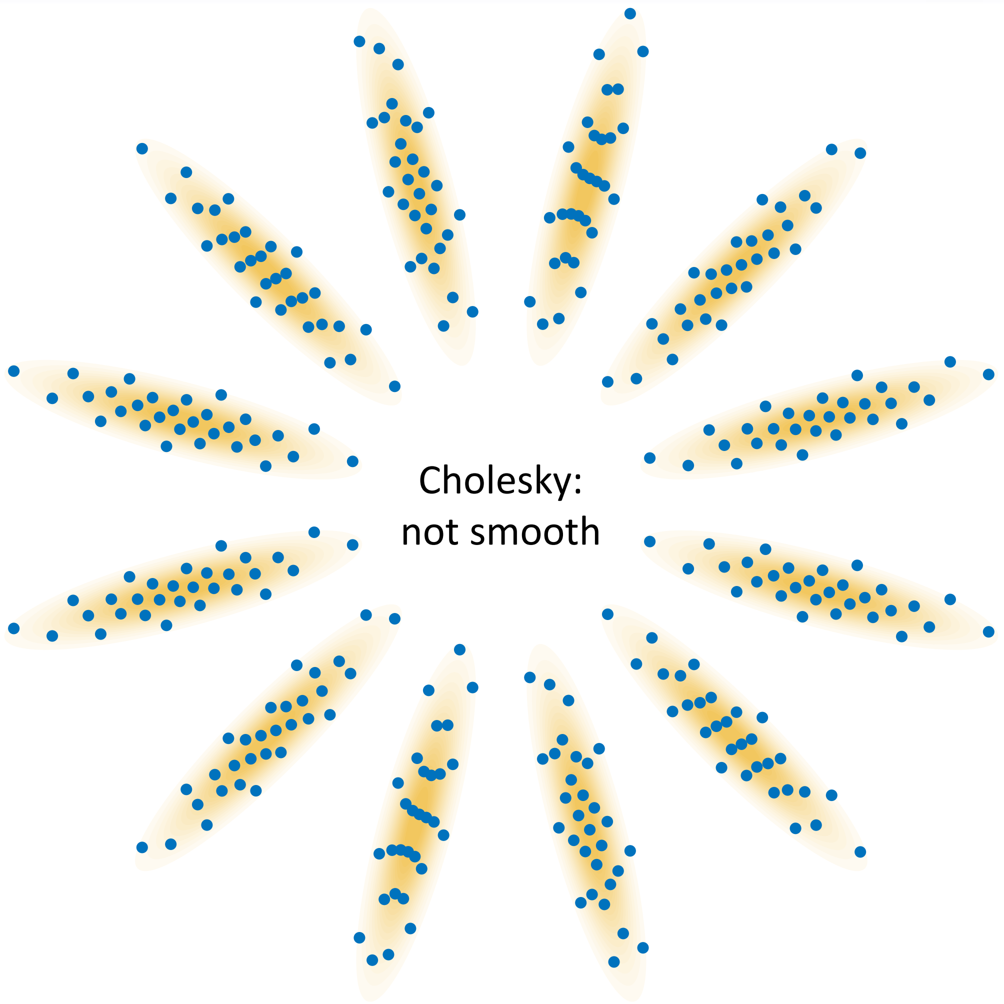
(d) Fibonacci Grid on $(0,1)^2$
→ Gauss along x direction

Uniform to Gauss

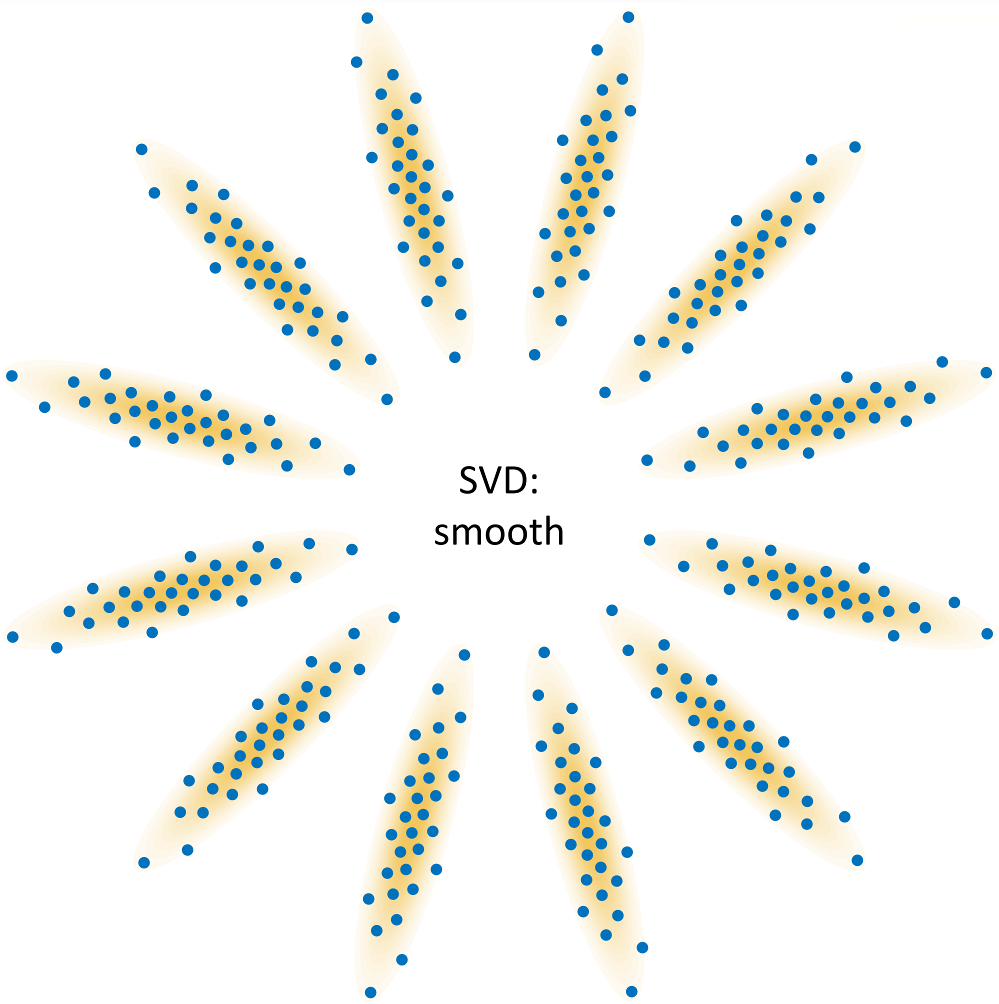


Transformable Gaussian Samples

Matrix Roots: Cholesky vs SVD



$$\mathbf{C} = \mathbf{L} \cdot \mathbf{L}^\top$$
$$\mathbf{X}_{\text{Gauss}} = \mathbf{L} \cdot \mathbf{X}_{\text{SND}}$$

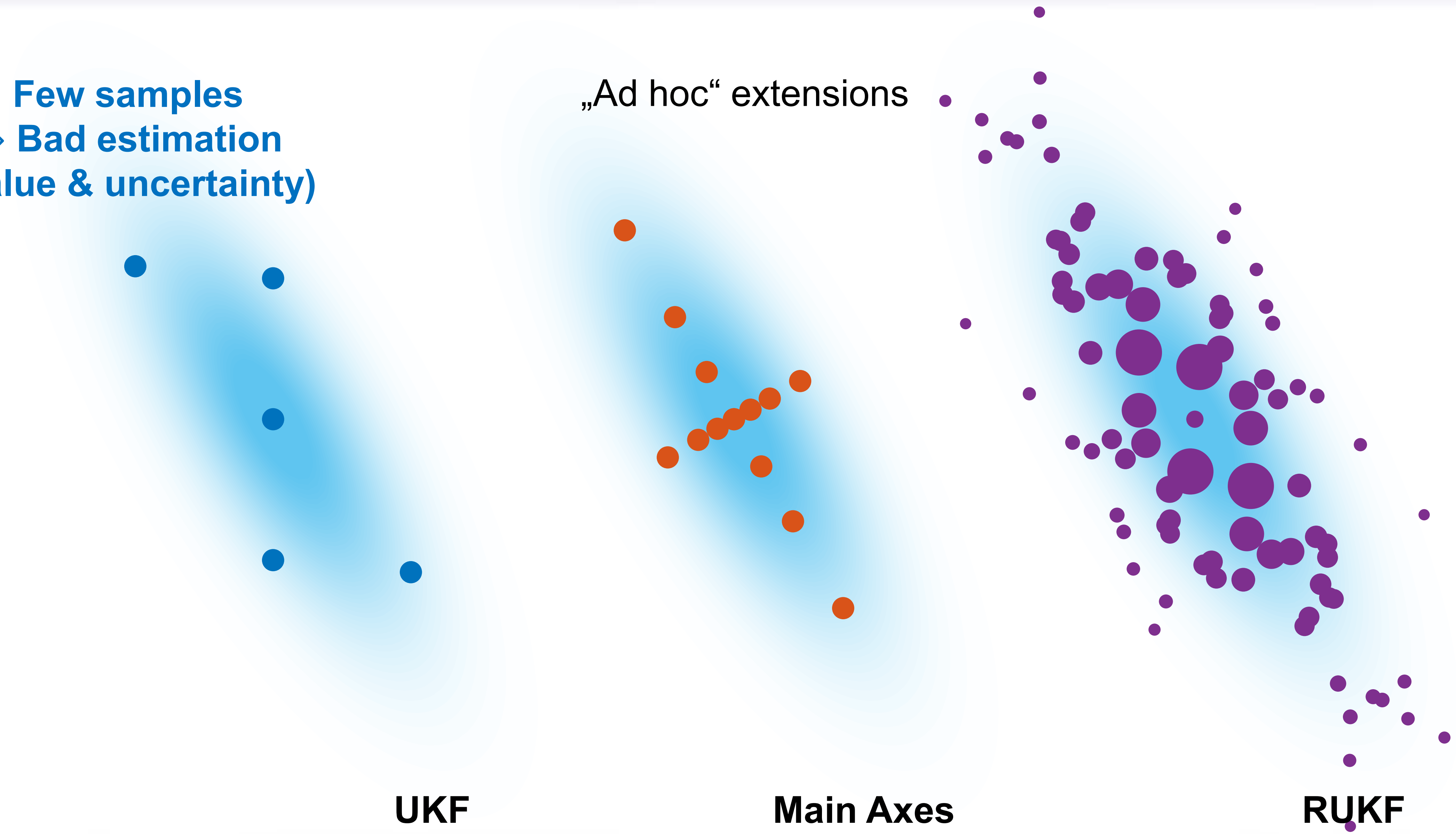


$$\mathbf{C} = \mathbf{V} \cdot \mathbf{D} \cdot \mathbf{V}^\top$$
$$\mathbf{X}_{\text{Gauss}} = \mathbf{V} \cdot \mathbf{D} \cdot \mathbf{X}_{\text{SND}}$$

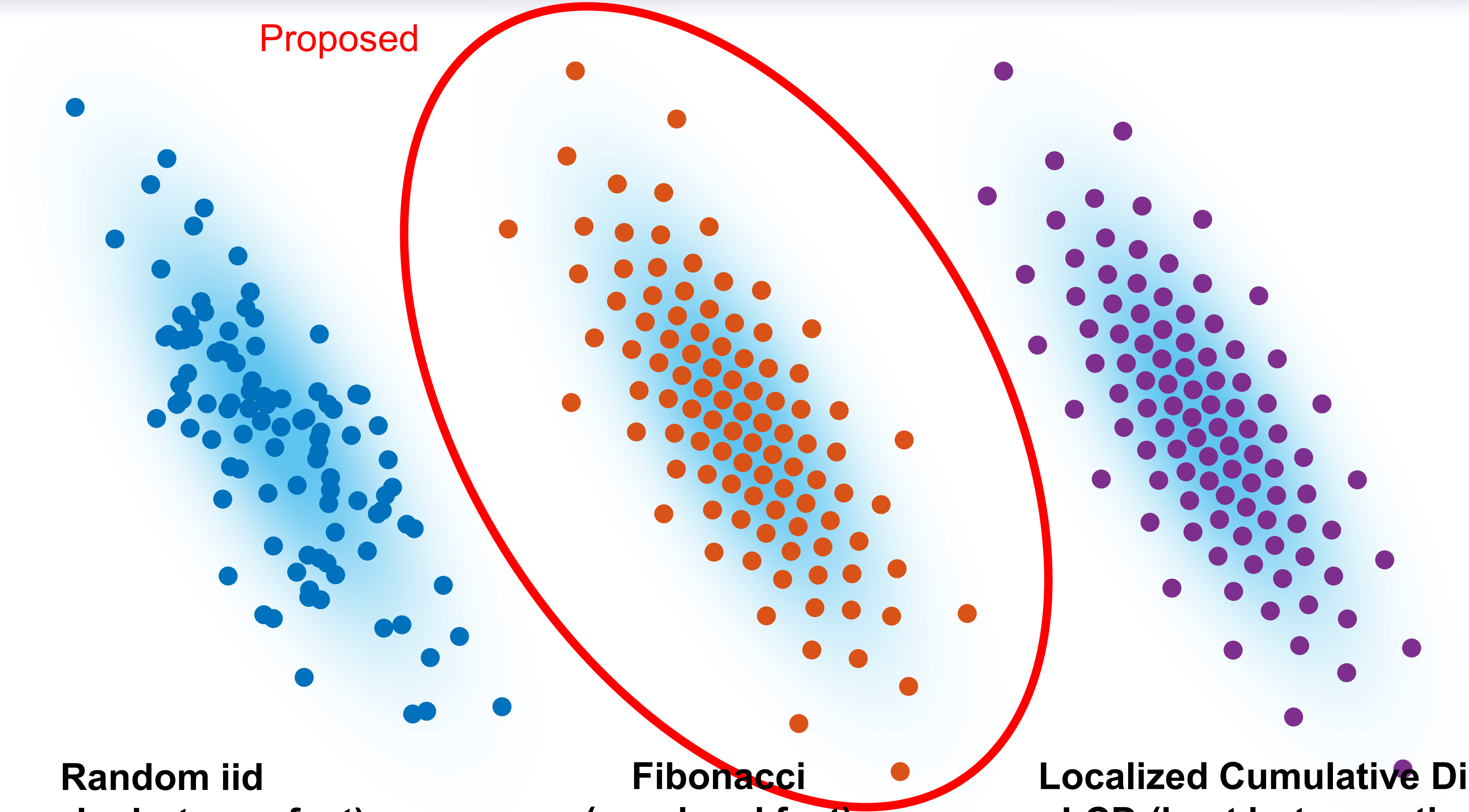
Deterministic Sampling: State of Art

Few samples
⇒ Bad estimation
(value & uncertainty)

„Ad hoc" extensions



Optimal Deterministic Sampling

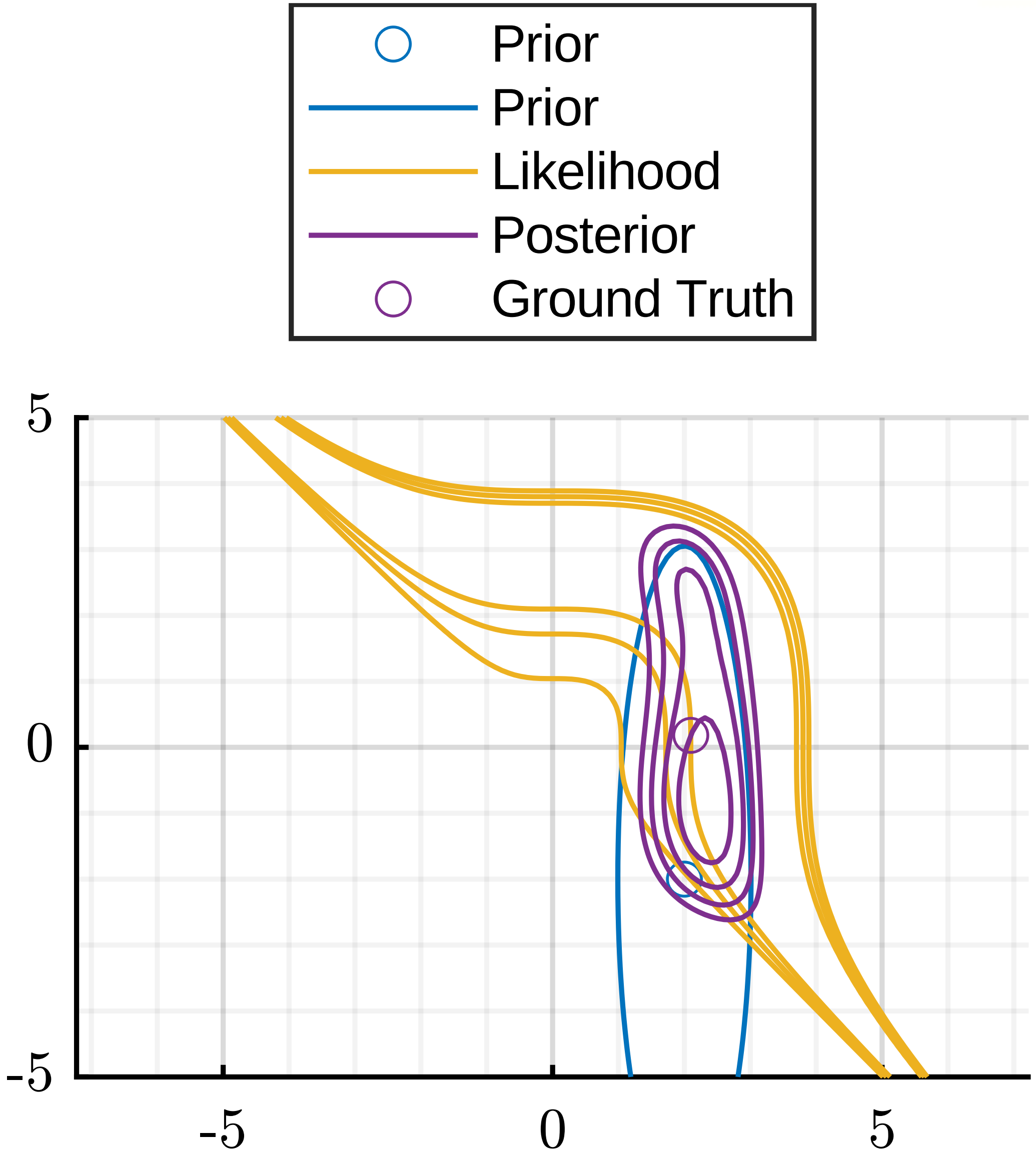


Random iid
(irregular but very fast)

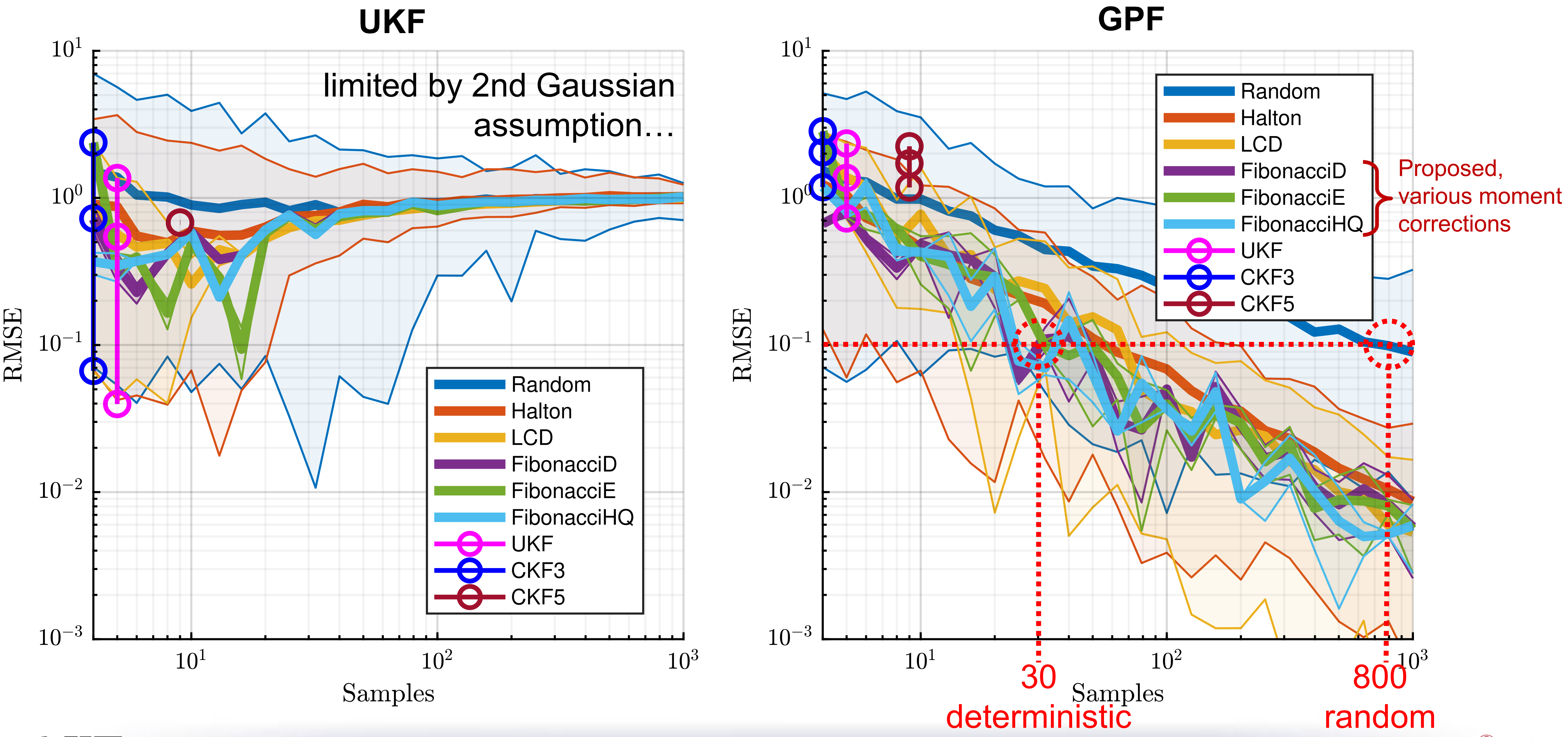
Fibonacci
(good and fast)

Localized Cumulative Distribution
LCD (best but sometimes slow)

- L samples $\underline{x}_i$ from prior
  - $\underline{x}_i \sim \mathcal{N}\left(\underline{x}, \underline{x}^{\mathrm{p}}, \mathbf{C}^{\mathrm{p}}\right)$
  - $\underline{x}^{\mathrm{p}} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$
  - $\mathbf{C}^{\mathrm{p}} = \begin{bmatrix} 1 & 0 \\ 0 & 5^2 \end{bmatrix}$
- Measurement model
  - $\hat{y} = \left\|\underline{x}\right\|_3^3 + v$
  - $\mathbf{C}^{v} = 30^2$
- Posterior mean
  - UKF: $\underline{x}^{\mathrm{e}} = \underline{x}^{\mathrm{p}} + \mathbf{C}_{xy}\mathbf{C}_{yy}^{-1}(\hat{y} - y^{\mathrm{p}})$
  - GPF: $\underline{x}^{\mathrm{e}} = \frac{1}{c}\sum_i \underline{x}_i \cdot \mathcal{N}\left(\hat{y}, \left\|\underline{x}_i\right\|_3^3, \mathbf{C}^{v}\right)$
  - Ground truth: Matlab `integral2()`



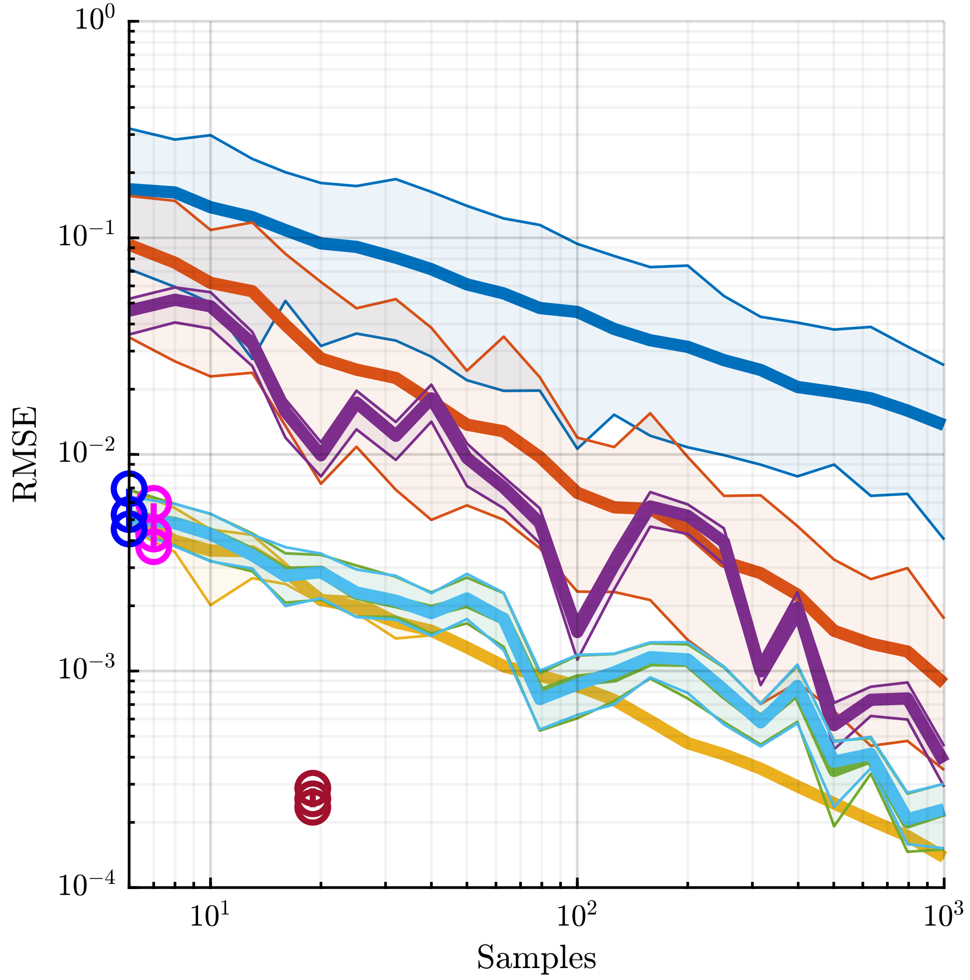
Simple Evaluation: UKF & GPF



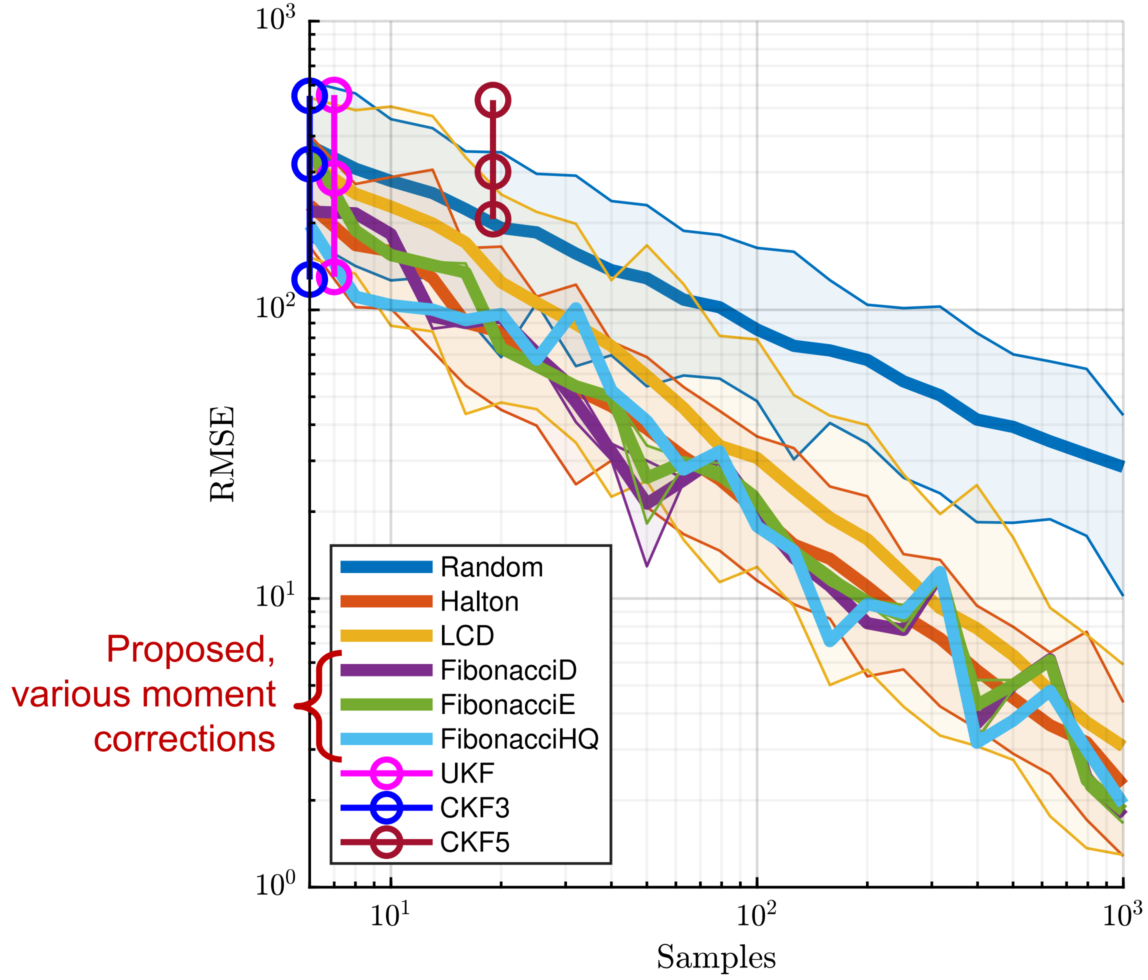
- Specific integrand → Average over multiple integrands
- Spatial harmonics: $g_{\underline{t}}(\underline{x}) = \exp\{i\,\underline{t}\cdot\underline{x}\}$
  - True expectation: $\mathrm{E}\{g_{\underline{t}}(\underline{x})\} = \int_{\mathbb{R}^D} g_{\underline{t}}(\underline{x}) \cdot f(\underline{x})\, d\underline{x}$
  - Sample approx: $\hat{\mathrm{E}}\{g_{\underline{t}}(\underline{x})\} = \frac{1}{L}\sum_i g_{\underline{t}}(\underline{x}_i)\,, \quad \underline{x}_i \sim f$
- Distance measure: $\mathrm{RMSE} = \int_{\mathcal{T}} \left|\mathrm{E}\{g_{\underline{t}}(\underline{x})\} - \hat{\mathrm{E}}\{g_{\underline{t}}(\underline{x})\}\right|^2 d\underline{t}$

Advanced Evaluation

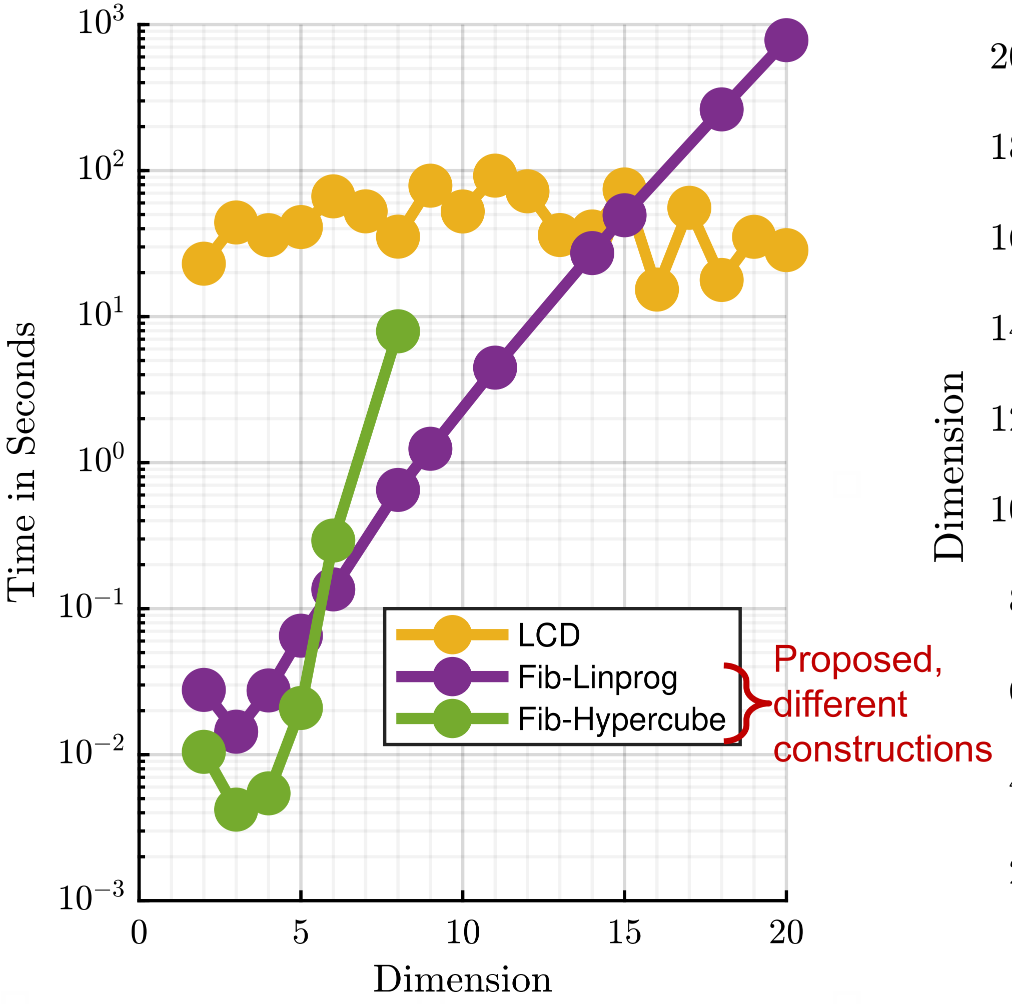
Isotropic: $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



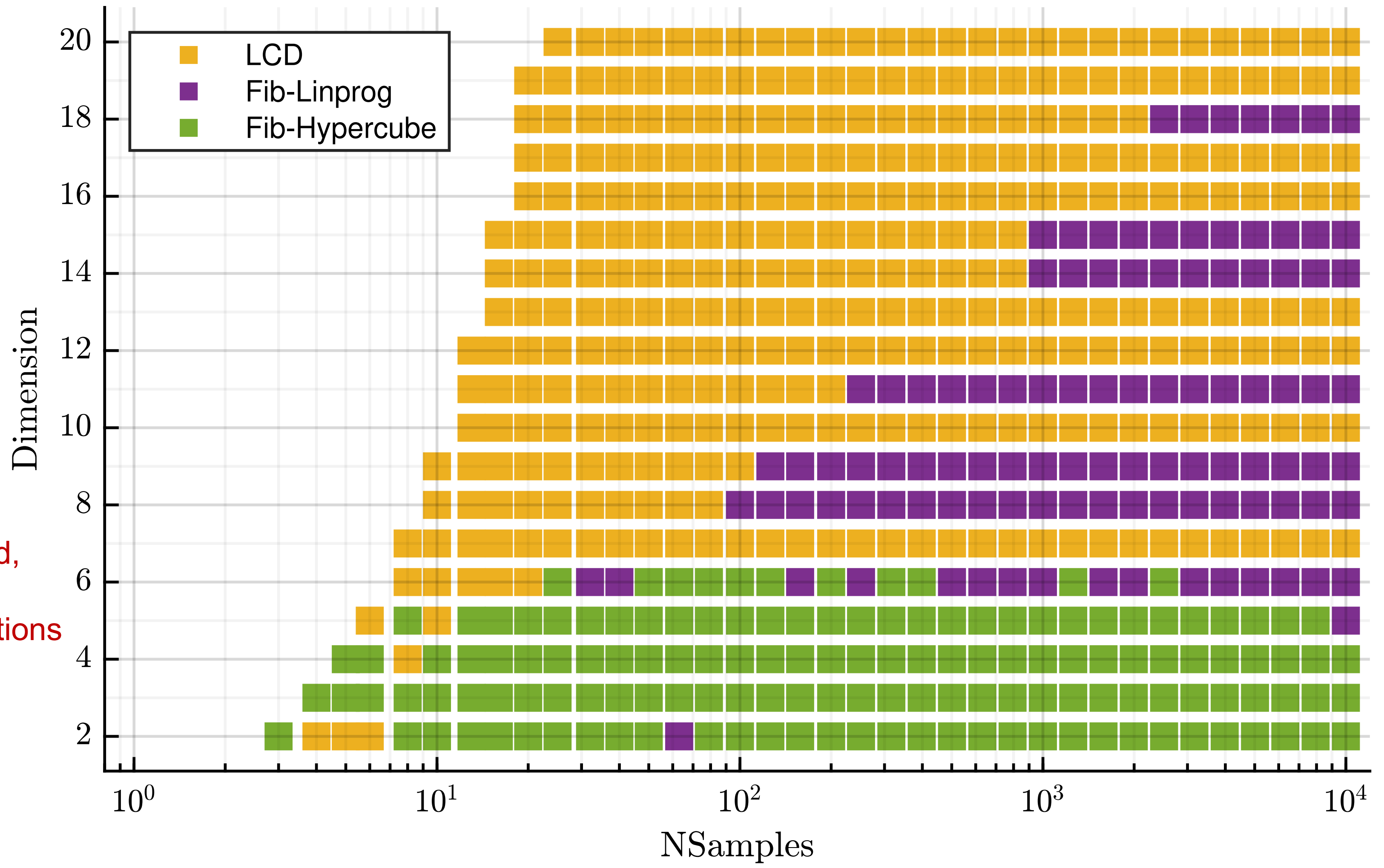
Anisotropic: $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.1^2 & 0 \\ 0 & 0 & 0.01^2 \end{bmatrix}$



Runtime

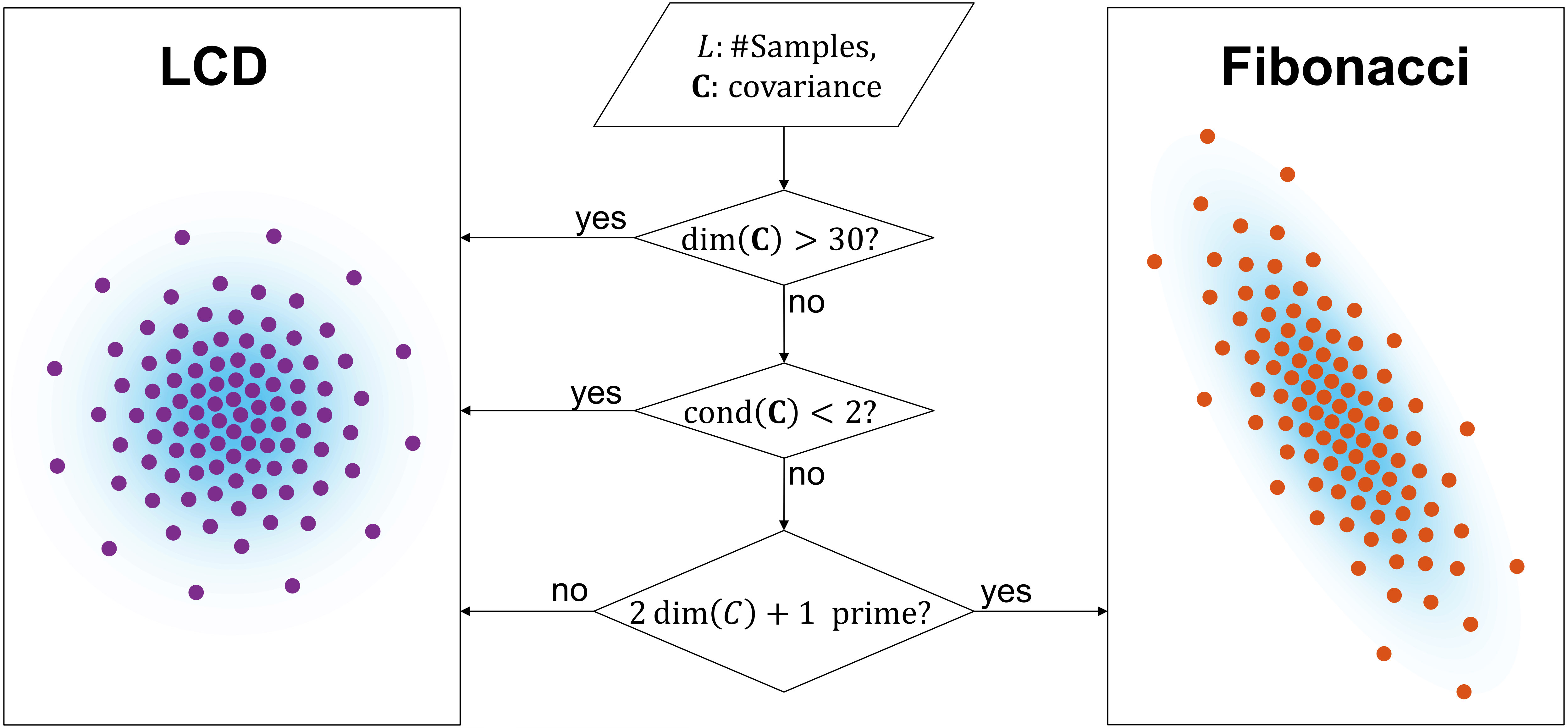


Time in Seconds
Dimension
LCD
Fib-Linprog
Fib-Hypercube
Proposed, different constructions

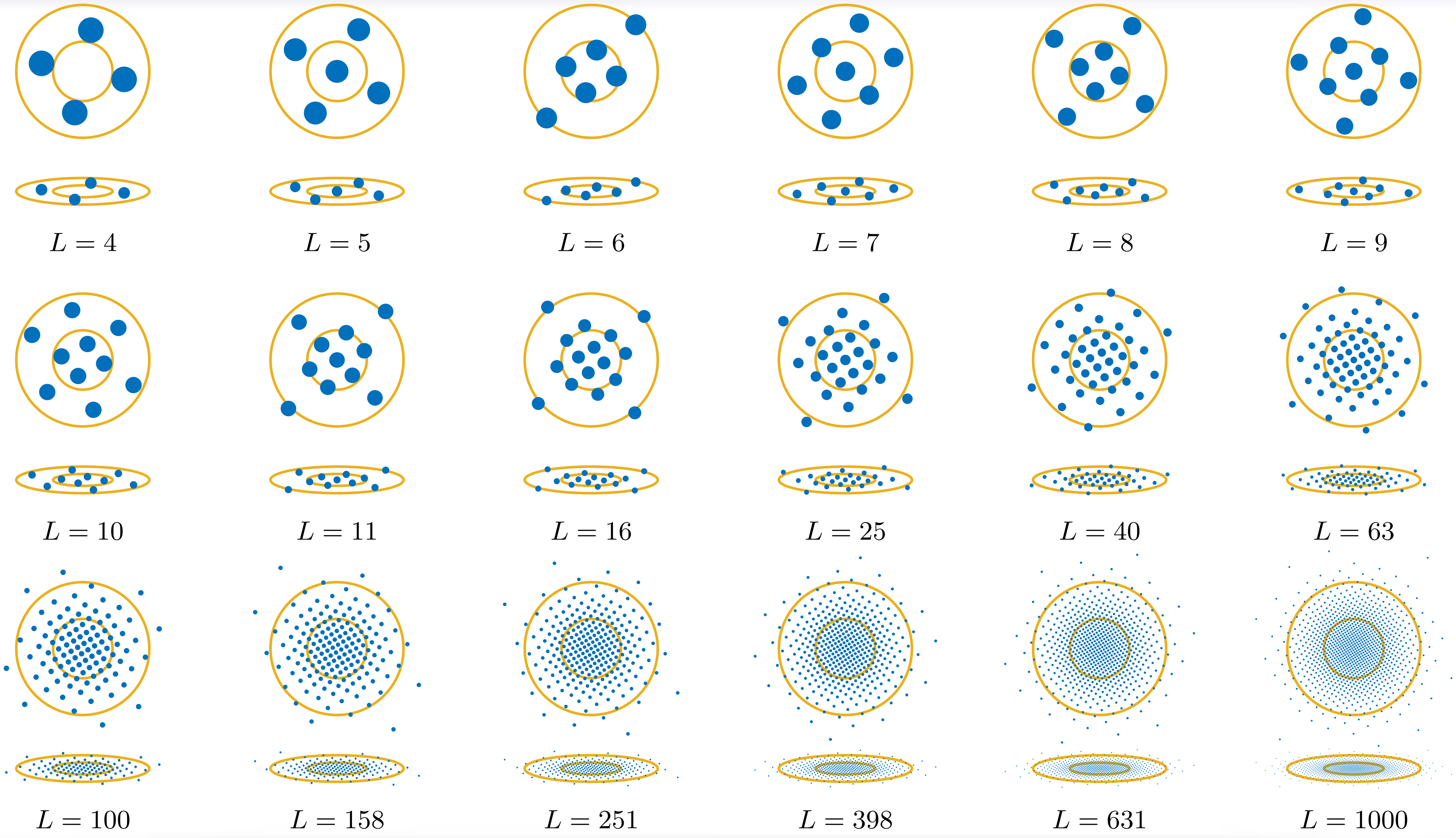


Dimension
NSamples
LCD
Fib-Linprog
Fib-Hypercube

Best Gaussian Sampling



Fibonacci → Gaussian Samples



Thank you for your attention

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