Deterministic Von Mises–Fisher Sampling on the Sphere Using Fibonacci Lattices

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Von Mises–Fisher Sampling

Input

von Mises–Fisher (VMF) Density

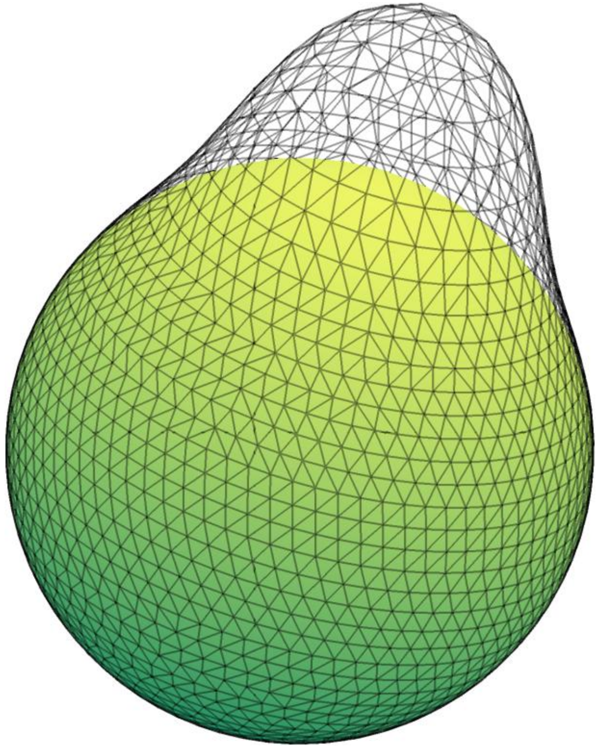
- κ, μ : parameters
- L : #samples

Deterministic Sampling

Output

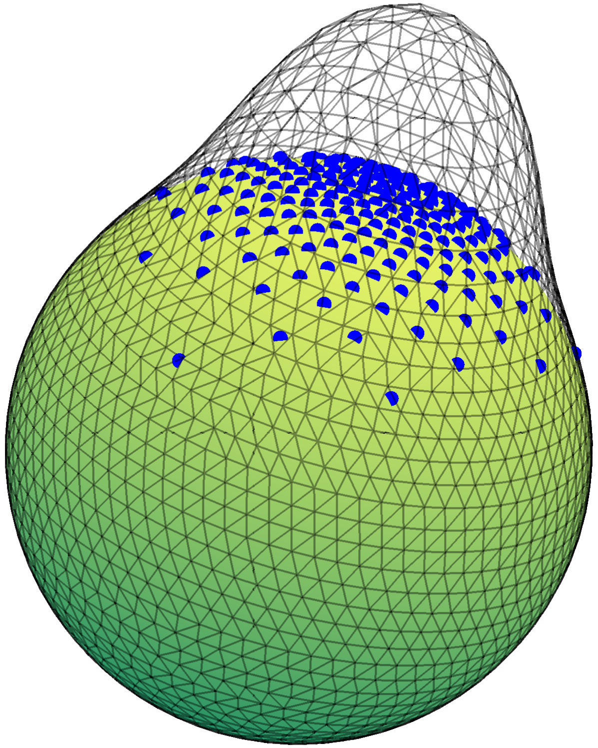
Dirac Mixture Density (DMD)

- Sample Positions, locally homogeneous



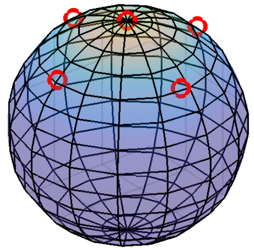
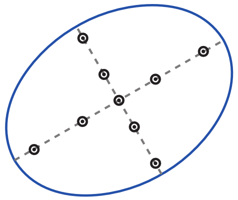
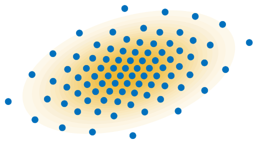
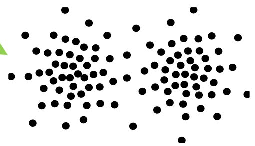
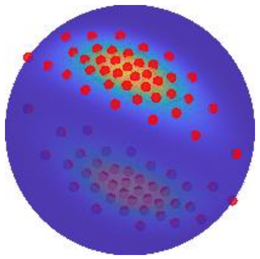
Methods

- Fibonacci Grid
- Transformation

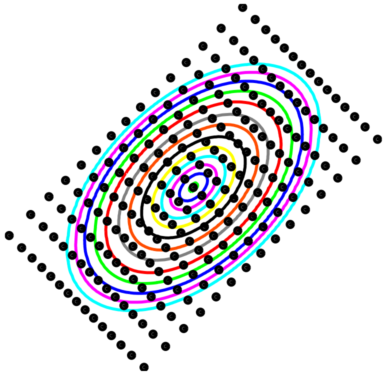


Sampling – Zwicky Box

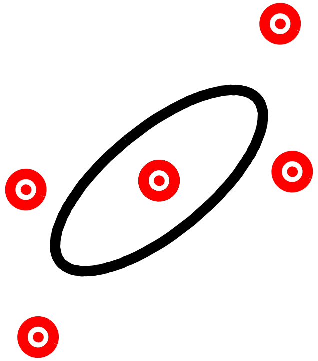
Dimension	Domain	Density	Method	#Samples
1	$\mathbb{R}^d$	Uniform	Random iid	5
2	$\mathbb{S}^d$	Gaussian	Unscented	9
3	$\mathbb{T}^d$	Gaussian Mixture	Main Axes	30
4		Dirac Mixture	Regular Grid	100
5		Wrapped normal	Fibonacci / Low-Discrepancy	300
6		Von Mises–Fisher	LCD	1000
7		Bingham	PCD	3000

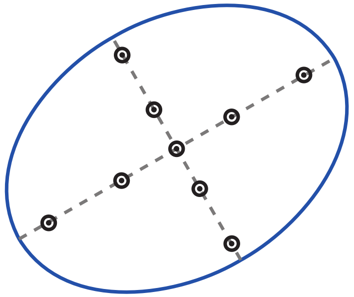
Deterministic Sampling



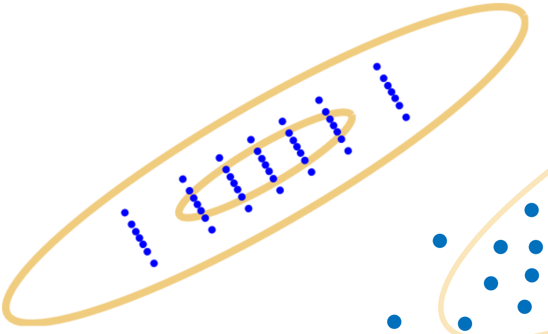
Gauss-Hermite
[Jaeckel 2005]



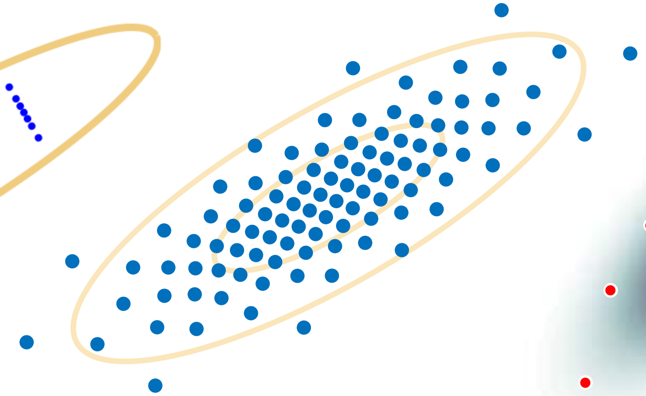
Unscented
[Julier, Uhlmann, 1997]



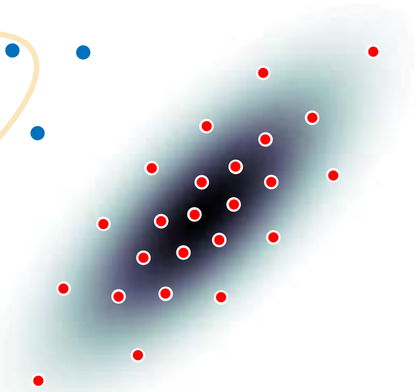
Main Axes
[Huber, Hanebeck, 2008]



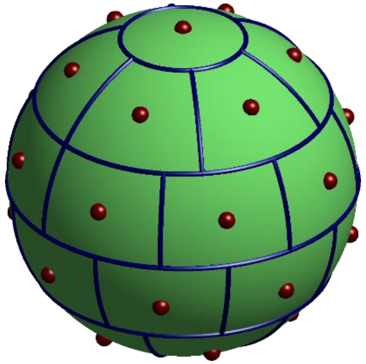
Cartesian



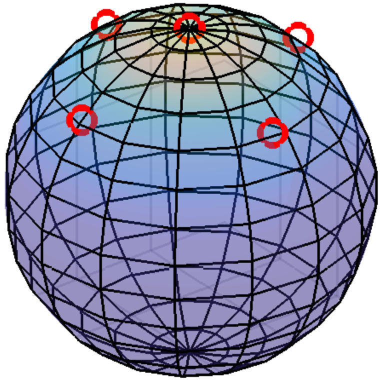
Fibonacci
[Frisch, Hanebeck, 2023]



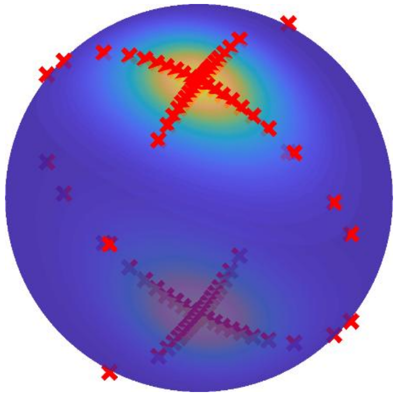
Localized Cumulative Distr (LCD)
[Hanebeck, Huber, Klumpp, 2009]



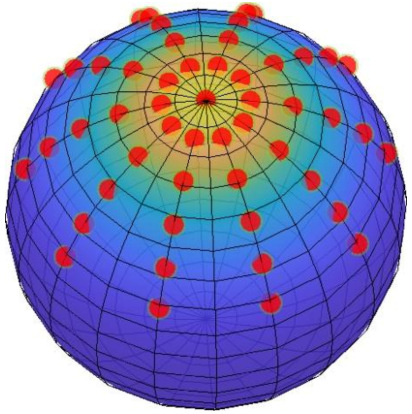
Grid
[Leopardi 2006]



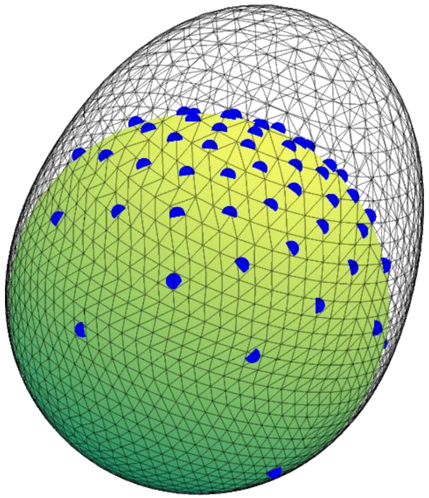
Unscented
[Kurz, Gilitschenski, Hanebeck, 2016]



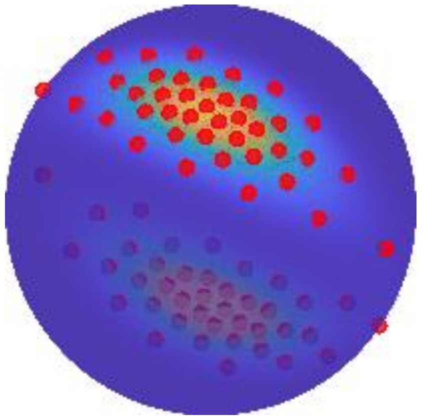
Main Axes
[Li, Frisch, Noack, Hanebeck, 2019]



Orbit-Planet
[Li, Pfaff, Hanebeck, 2021]



Fibonacci
[Proposed]



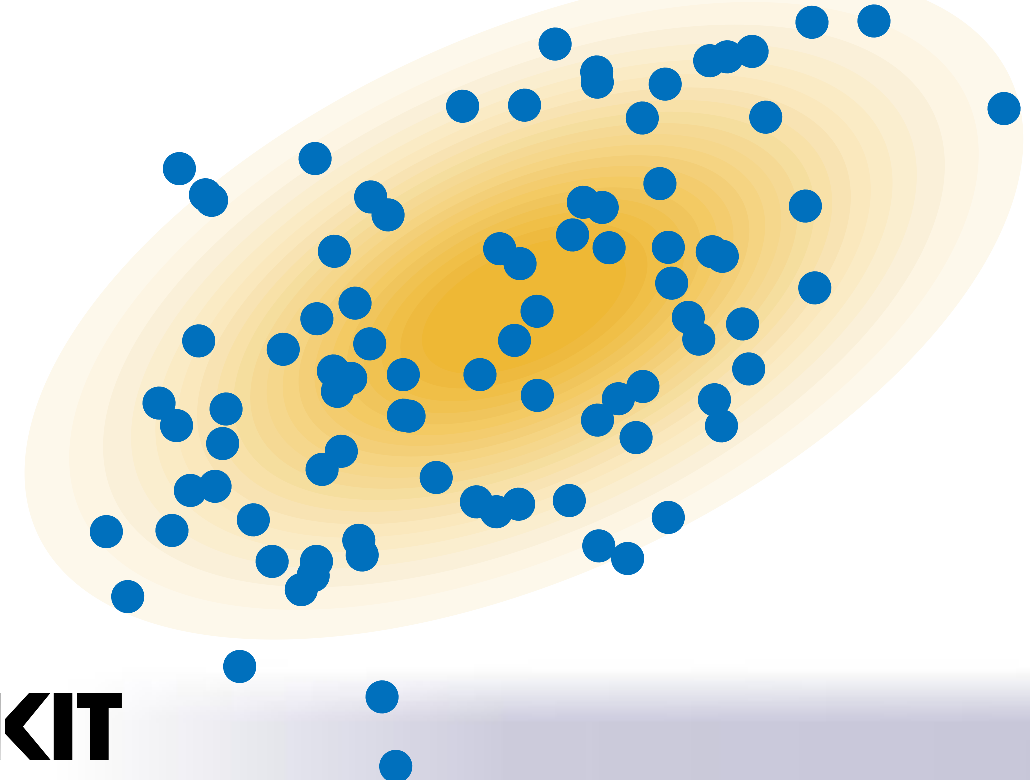
Localized Cumulative Distr (LCD)
[Li, Pfaff, Hanebeck, 2021]

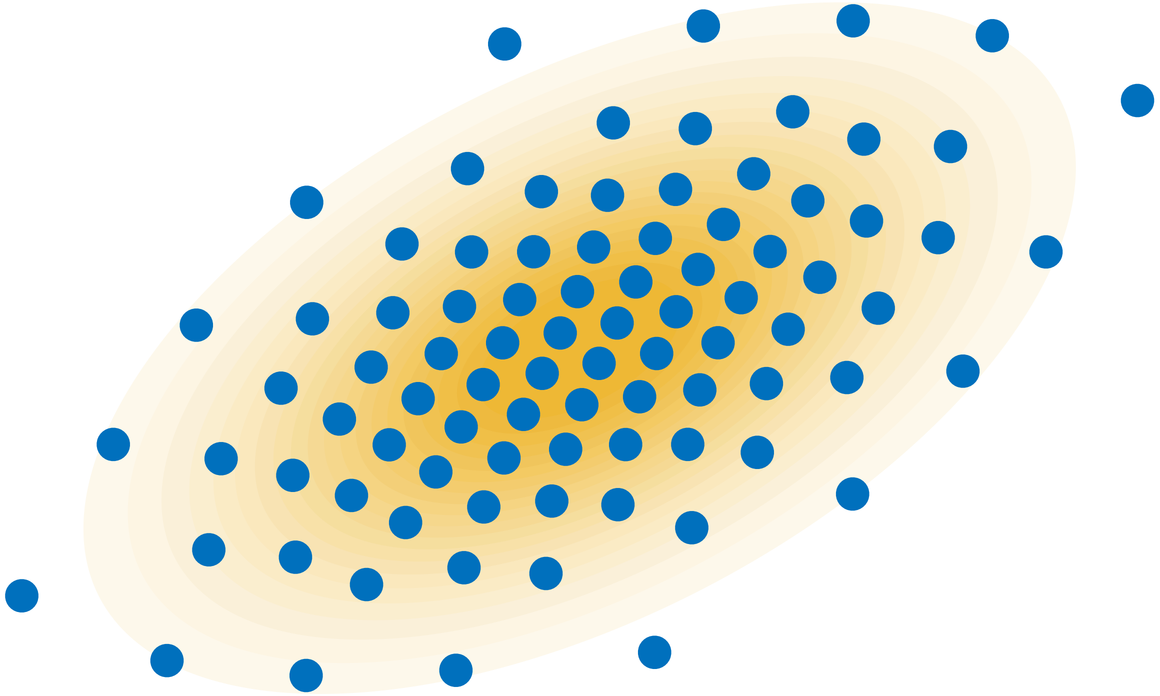
Random Sampling

- Unbiased
- Arbitrary dimension
- Smoothness irrelevant
- Practical error estimate

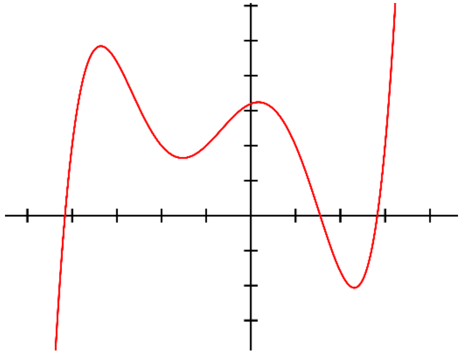
Deterministic Sampling

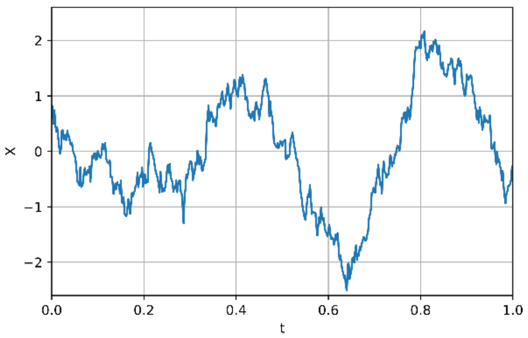
- Superior convergence
- Reproducible
- Locally homogeneous





Numerical Integration Error

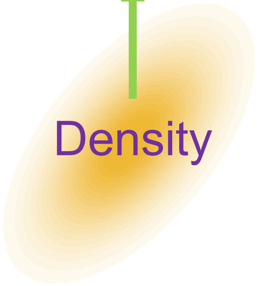




Integrand

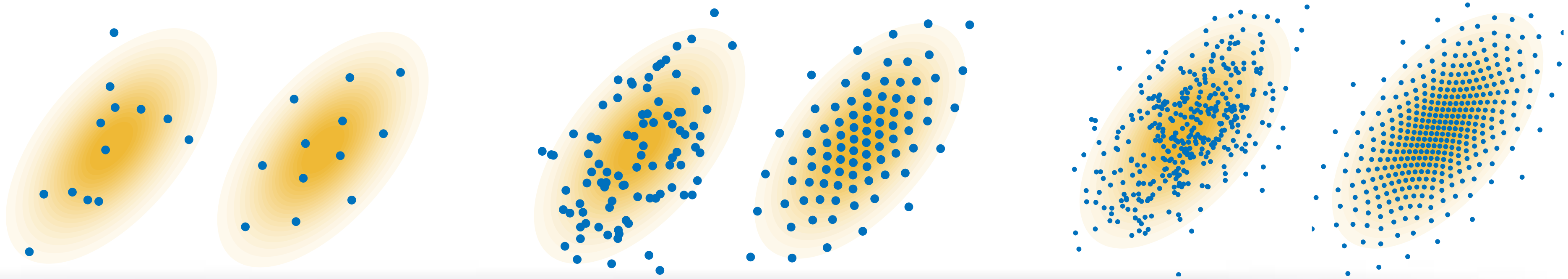
Integration Error

$$\epsilon = \left| \int g(\underline{x})\, f(\underline{x})\, \mathrm{d}\underline{x} - \frac{1}{L} \sum_{\underline{x} \in \mathcal{P}} g(\underline{x}) \right|$$



Density

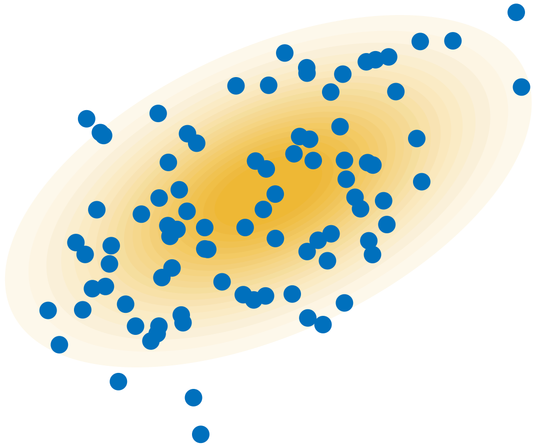
Samples



(Quasi-) Monte Carlo Error

- Central Limit Theorem (CLT)
  - For iid samples
  - Convergence: $L^{-\frac{1}{2}}$
  - Depends on
    - L
    - σ_g

$$\epsilon \sim \mathcal{N}\left(0, \sigma_g \cdot \frac{1}{\sqrt{L}}\right)$$

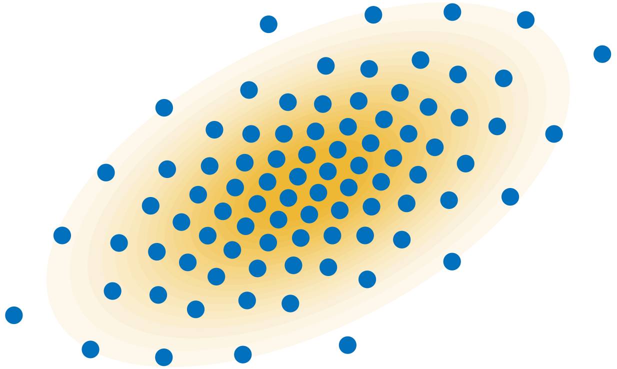


- Koksma-Hlawka
  - Convergence: $\log(L)^{s-1} \cdot L^{-1}$
  - Depends on
    - $\operatorname{discr}(\mathcal{P})$
      - L
      - s
    - $V(g)$

$$\epsilon \leq \underbrace{\operatorname{discr}(\mathcal{P})} \cdot V(g)$$

„Low discrepancy“:

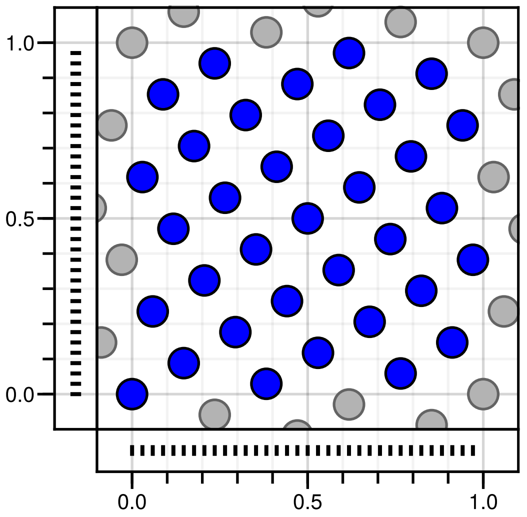
$$\mathcal{O}\left(\log(L)^{s-1} \cdot \frac{1}{L}\right)$$



Fibonacci – Rank-1 Lattice

- $L = F_k \in \{1, 2, 3, 5, 8, 13, 21, \ldots\}$
- Periodic: 2 dimensions

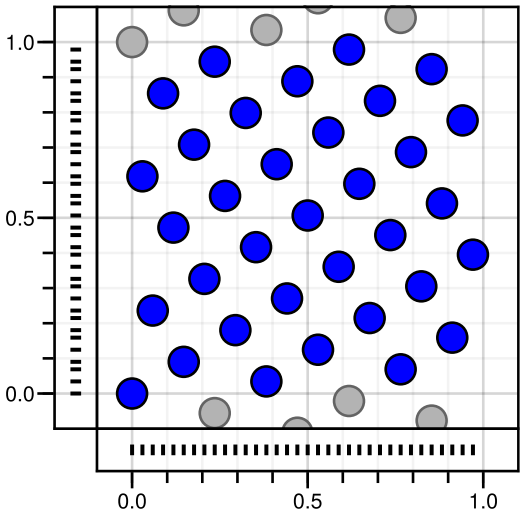
$$\underline{x}_i = i \cdot \begin{bmatrix} \frac{1}{F_k} \\ \frac{F_{k-1}}{F_k} \end{bmatrix} \bmod 1$$



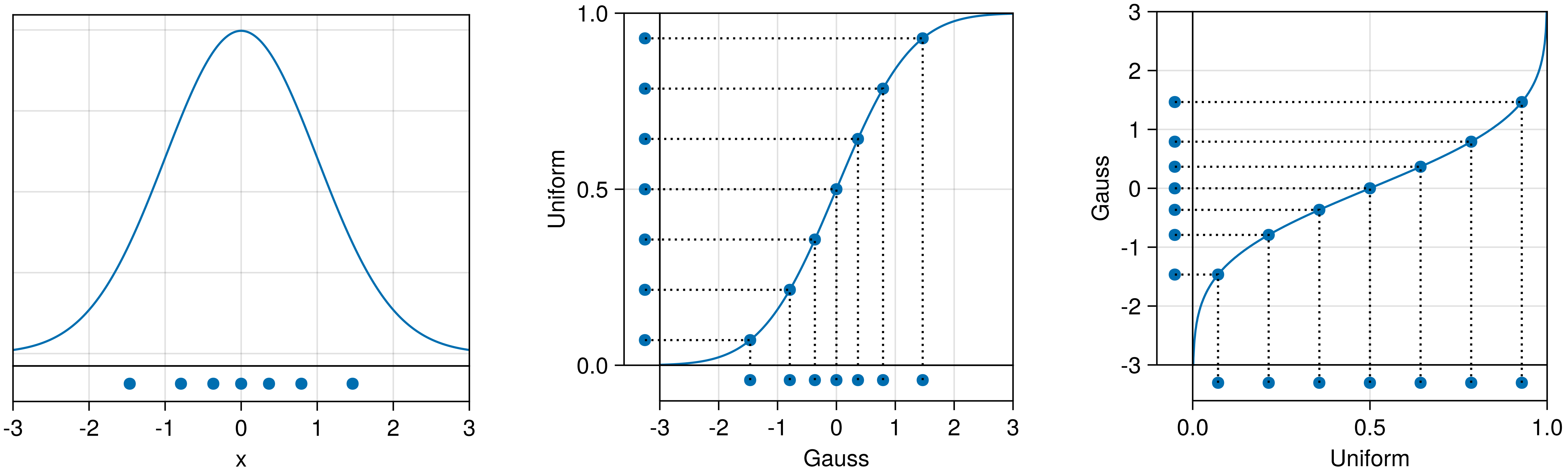
Fibonacci – Kronecker Lattice

- $L \in \mathbb{N}$
- Periodic: 1 dimension

$$\underline{x}_i = i \cdot \begin{bmatrix} \frac{1}{L} \\ \frac{1}{\Phi} \end{bmatrix} \bmod 1$$



Low discrepancy



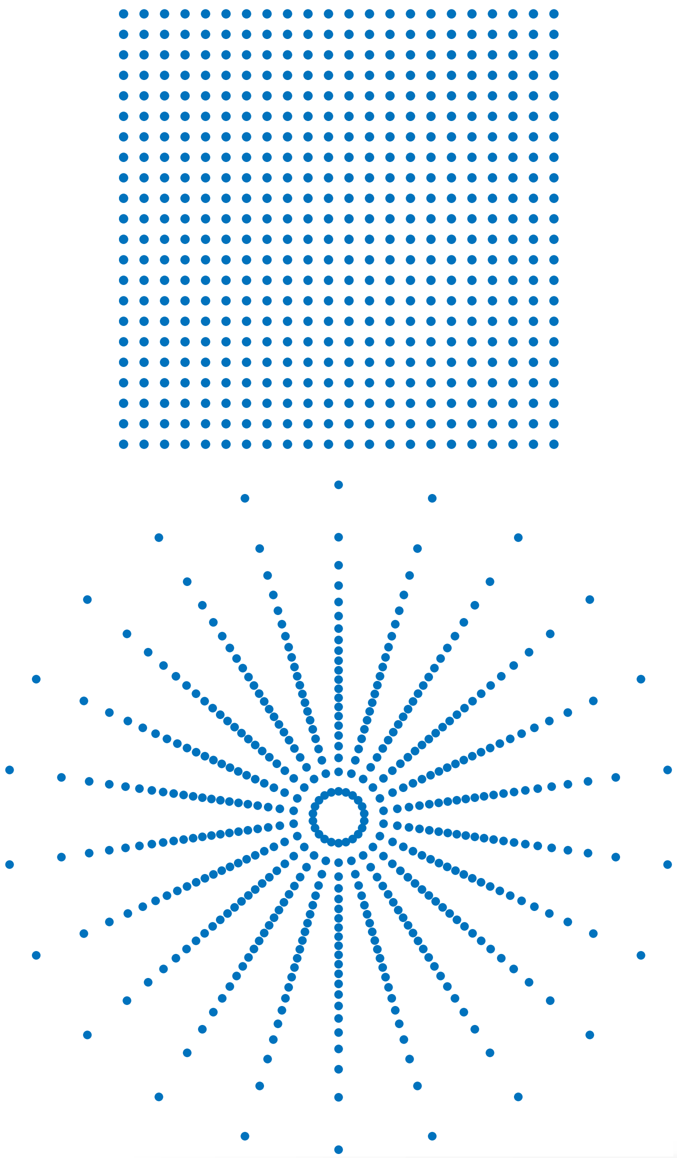
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$F_\theta(\theta) = \frac{1}{2}\left(1 + \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right)$$

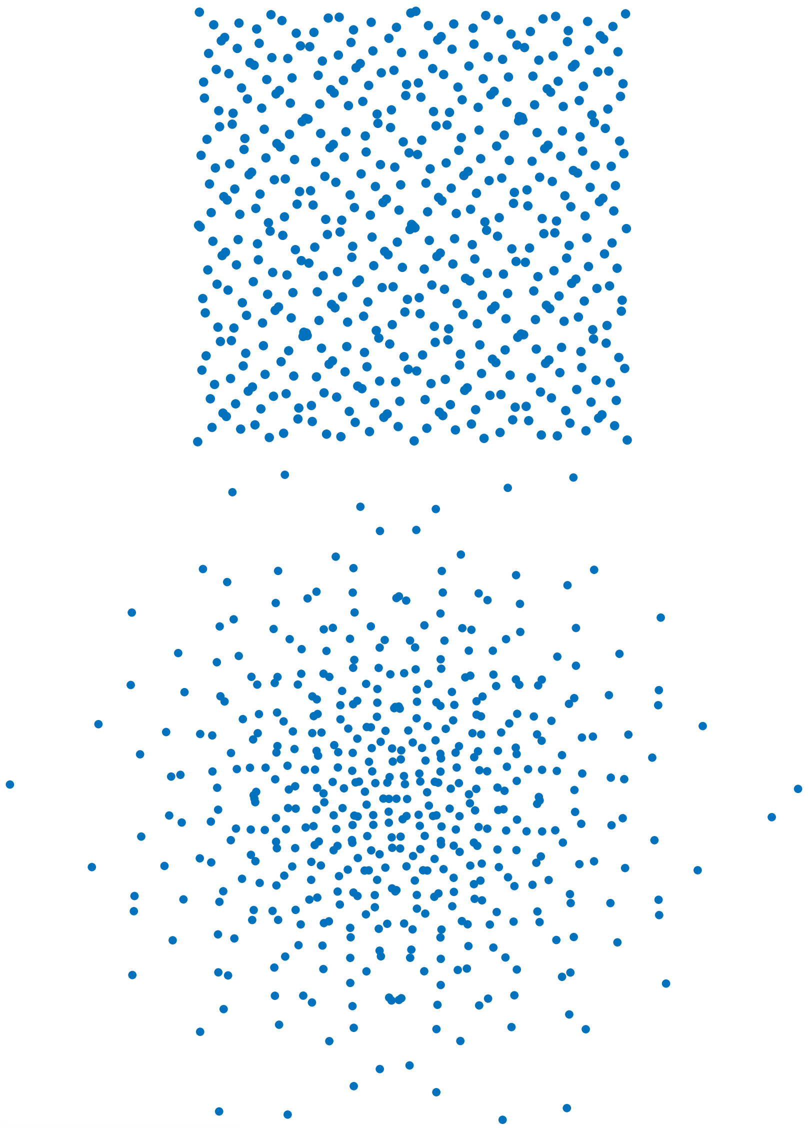
$$F_\theta^{-1}(p) = \sqrt{2}\,\mathrm{erf}^{-1}(2p-1)$$

Inverse Transform Sampling – 2D Gauss

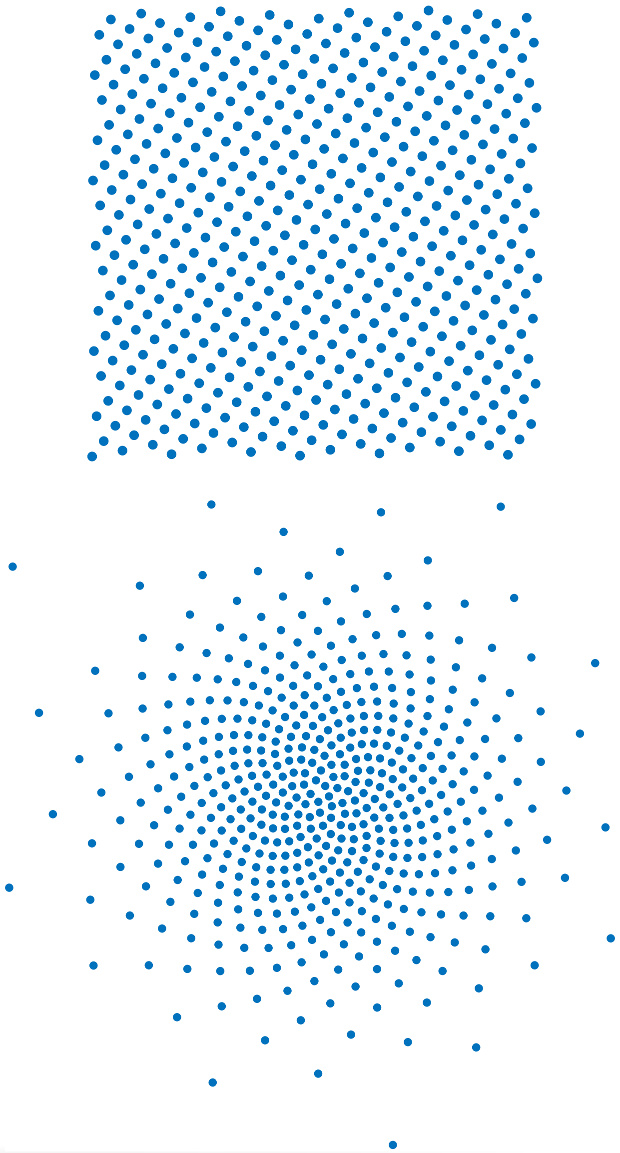
Regular Grid



Sobol



Fibonacci



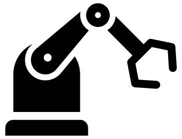
Why Riemannian Manifolds?

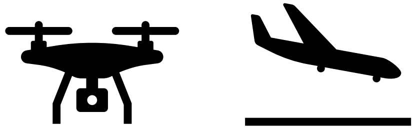
Riemannian State Spaces

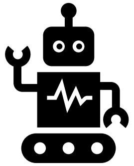
- Robots
- Drones
- Wind direction
- Wireless AoA

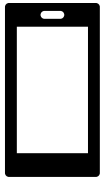
Uncertainty ↑

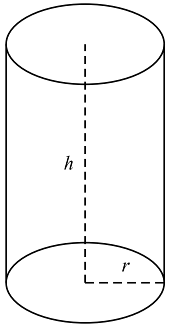
- Predictions
  - Simulations: fine-grained
  - Drones: small & versatile
- Measurements
  - Sensors: cheap, mass market

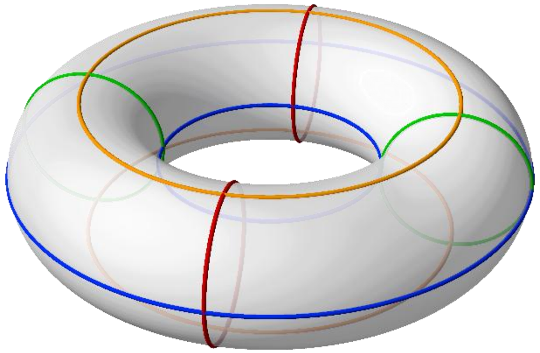


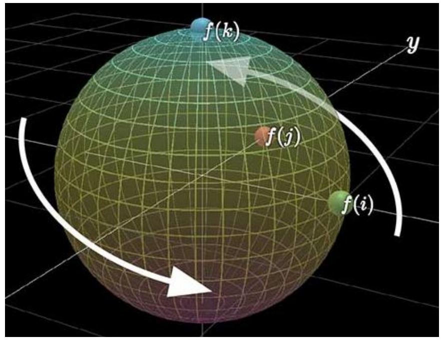








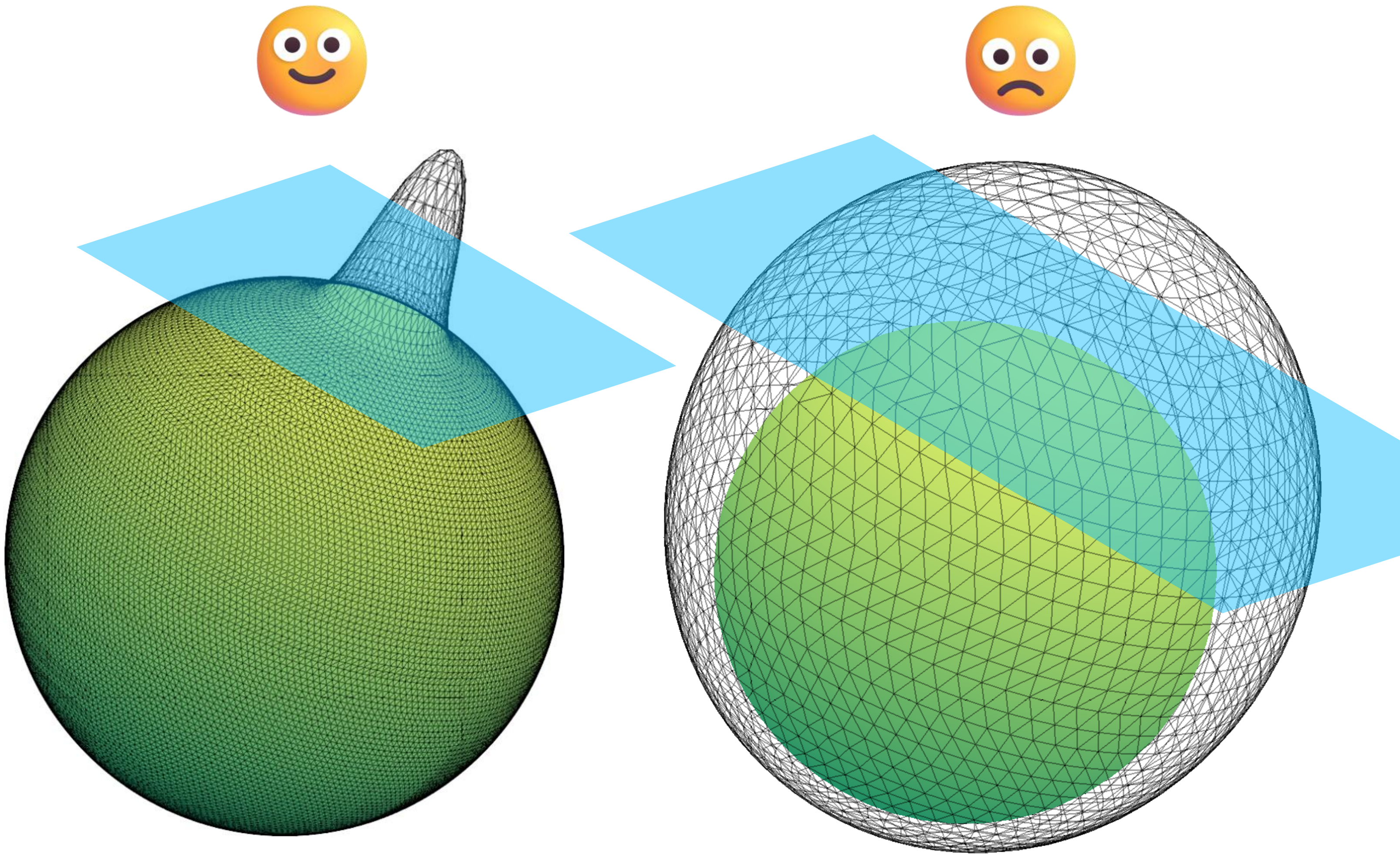




Linearizing State Spaces

Typical solution

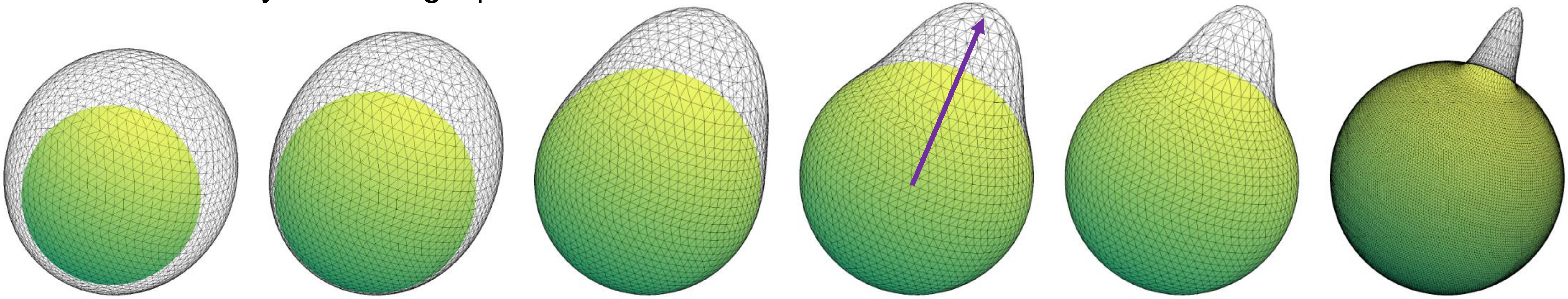
- Space Linearization
- Gauss on tangent plane



Why von Mises–Fisher Density?

- „Normal“ density of the sphere
- Wrapped Gauss for $\kappa \to \infty$
- Isotropic
- Applications:
  - DoA measurements
  - Fiber tracking
  - Weakly interacting dipoles

$$f\left(\underline{x}; \underline{\mu}, \kappa\right) = c \cdot \exp(\kappa \cdot \underline{\mu}^T \underline{x})$$



(a) $\kappa = 1$ (b) $\kappa = 2$ (c) $\kappa = 5$ (d) $\kappa = 10$ (e) $\kappa = 20$ (f) $\kappa = 100$

- Gaussian

$$f(\underline{x}, \underline{m}, \sigma) = \exp\left\{-\frac{(\underline{x} - \underline{m})^\top (\underline{x} - \underline{m})}{2\,\sigma}\right\}$$

$$f(d, \sigma) = \exp\left\{\frac{1}{\sigma} \cdot \left(-\frac{d^2}{2}\right)\right\}$$

$\mathbb{R}^2$

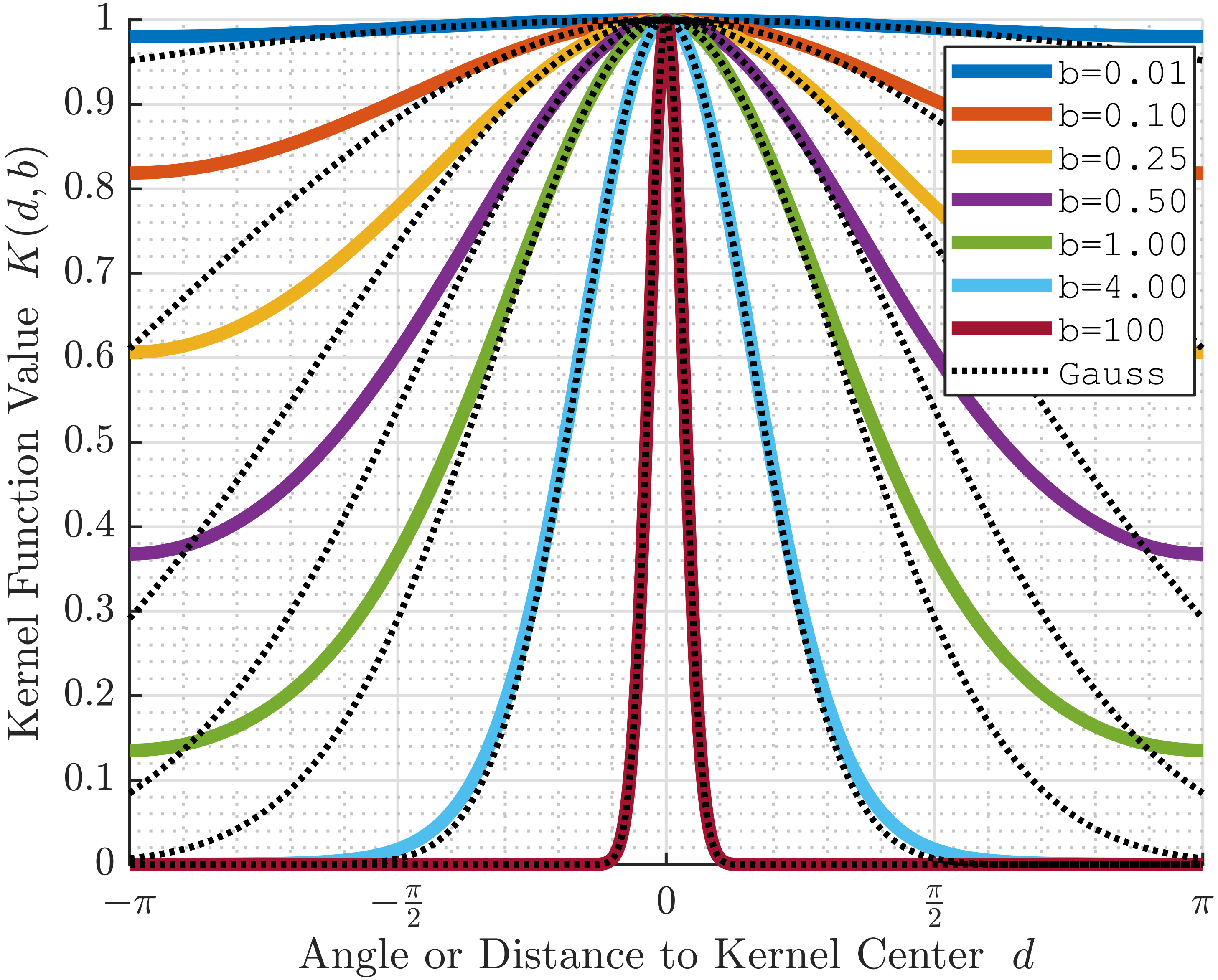
- Von Mises–Fisher

$$f(d, \kappa) = \exp\{\kappa \cdot \cos d\}$$

$\mathbb{S}^2$

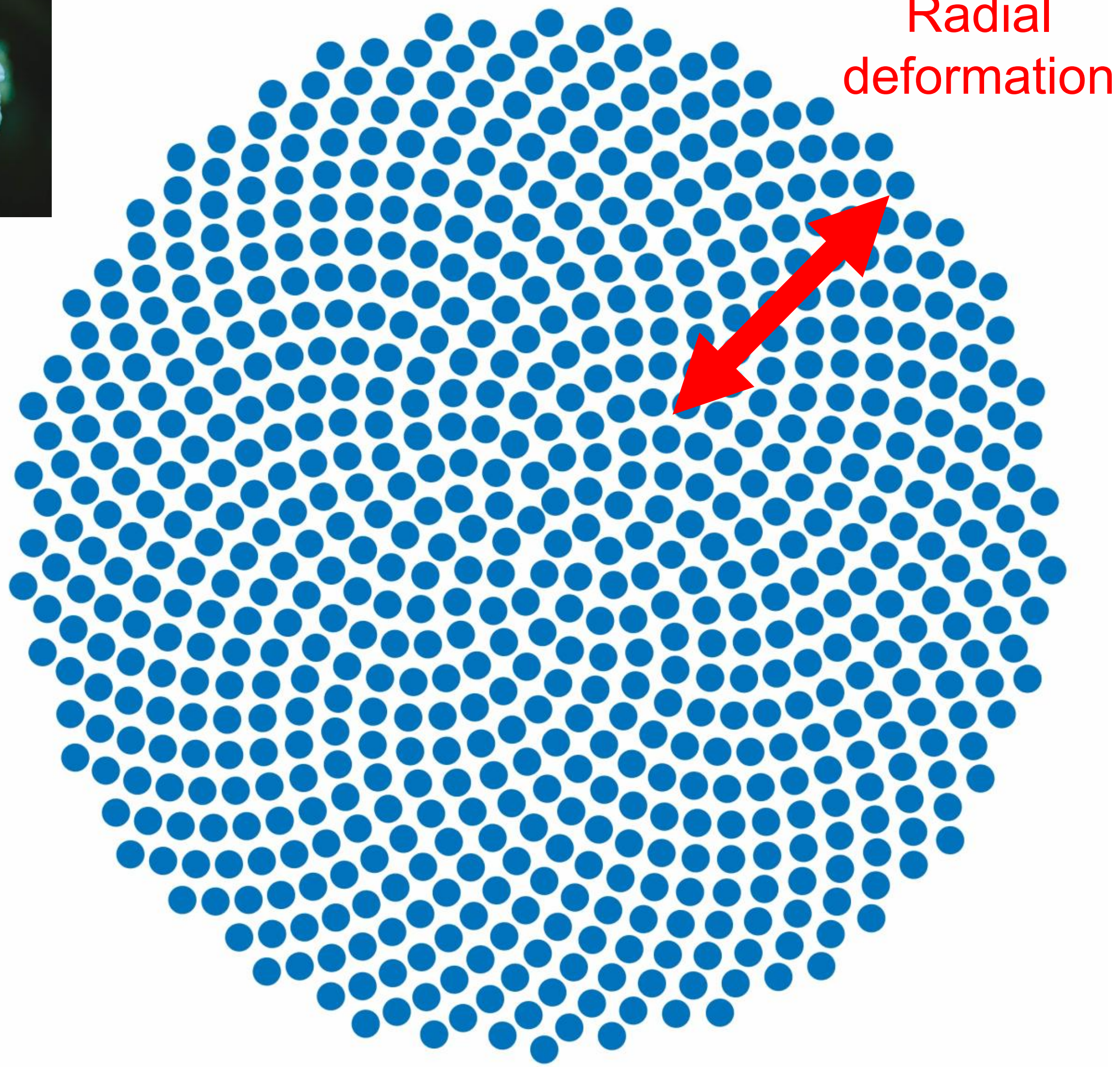
$$f(\underline{x}, \underline{m}, \kappa) = \exp\{\kappa \cdot \underline{x}^\top \underline{m}\}$$

Gauss ↔ VMF

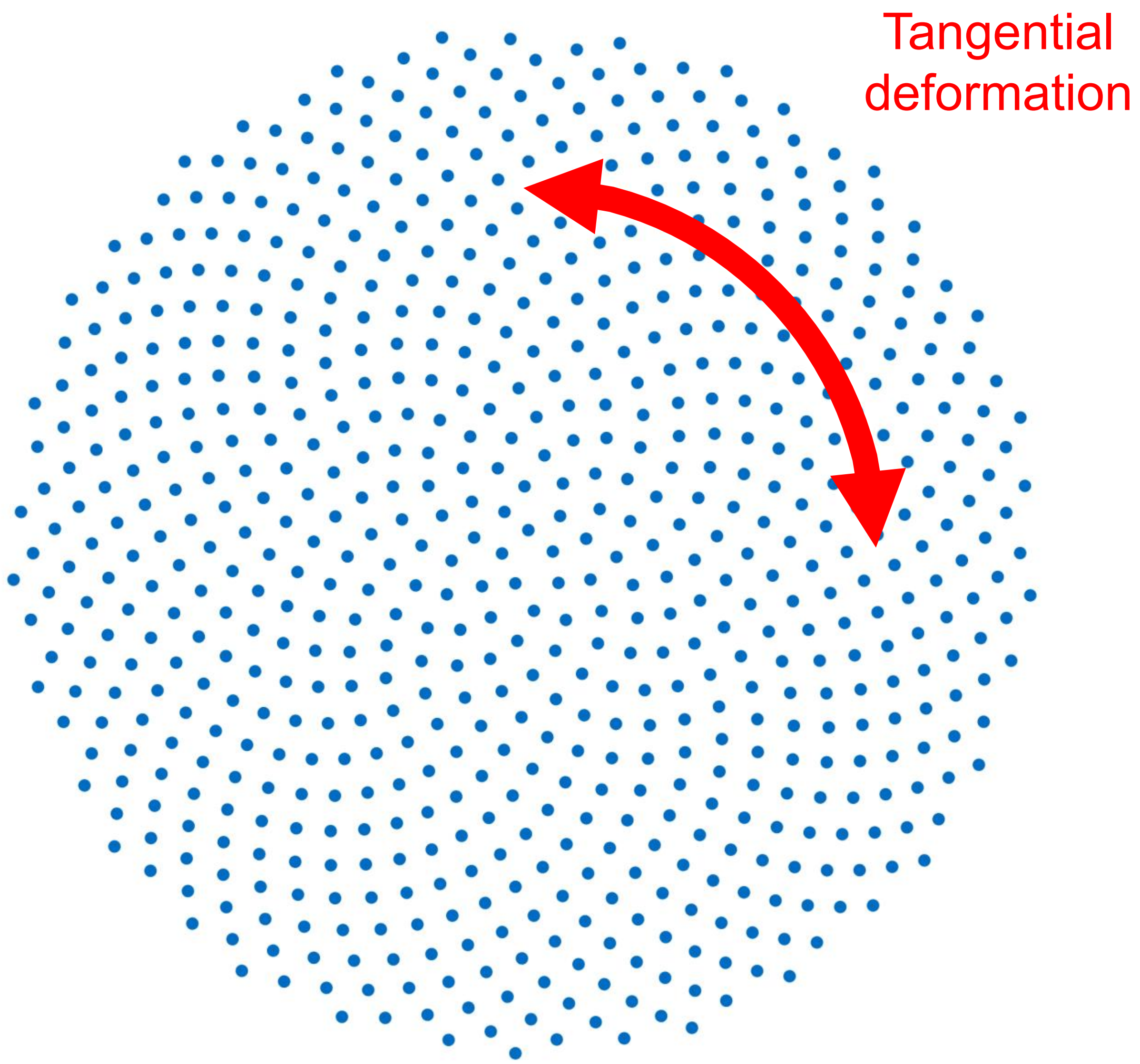


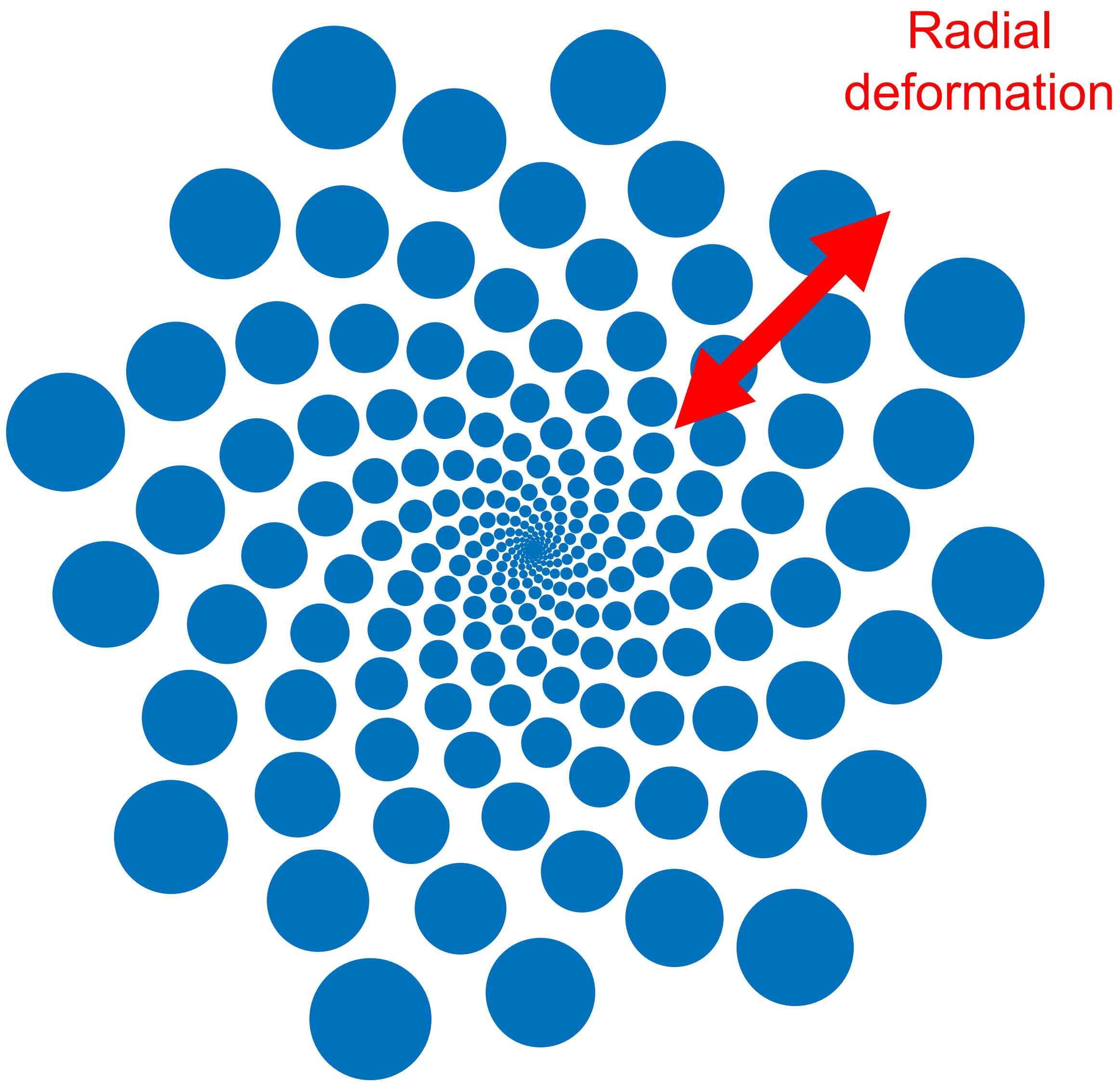
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Kernel Function Value $K(d, b)$
$-\pi$
$-\frac{\pi}{2}$
0
$\frac{\pi}{2}$
π
Angle or Distance to Kernel Center d
b=0.01
b=0.10
b=0.25
b=0.50
b=1.00
b=4.00
b=100
Gauss

Polar Fibonacci Grid

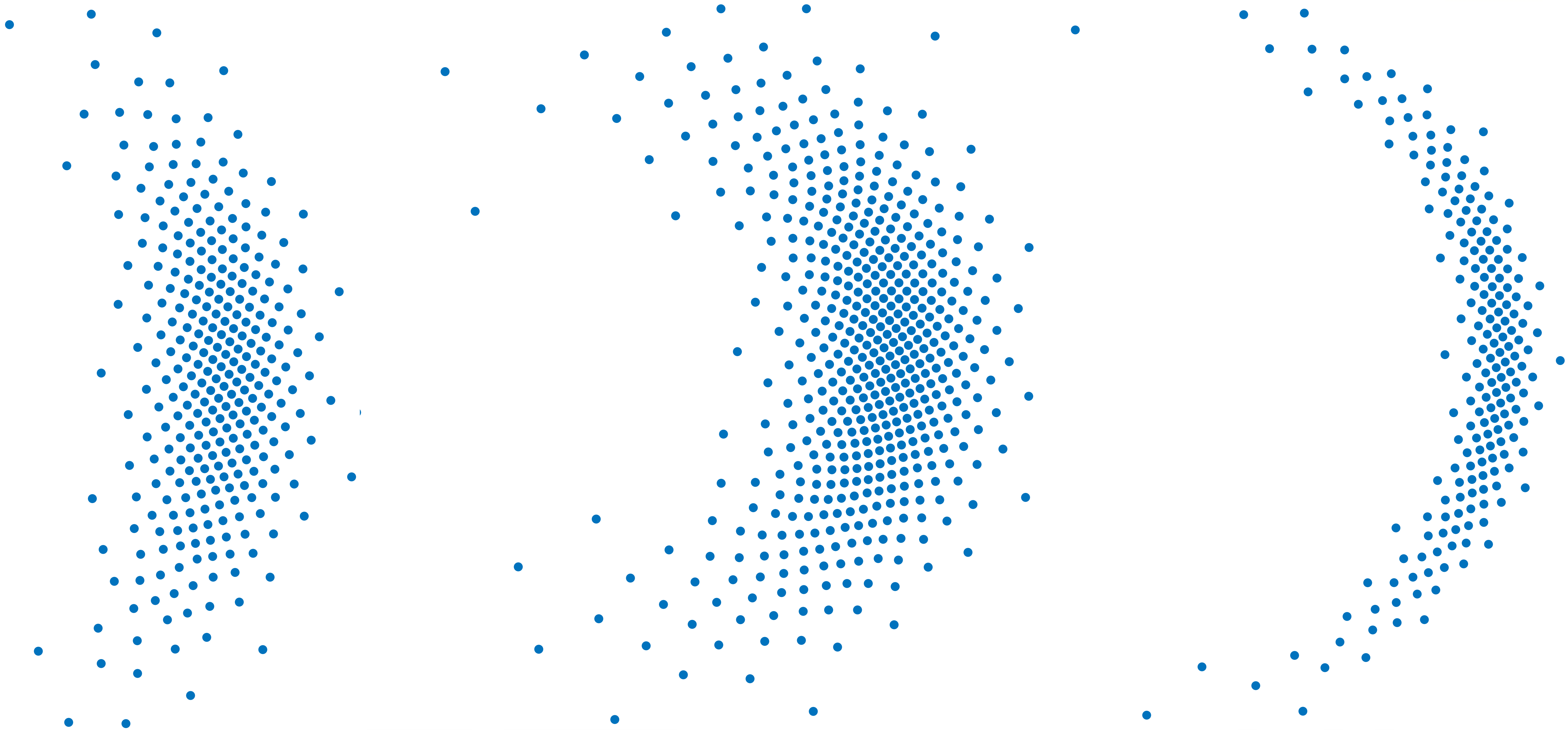


Polar Fibonacci Grid

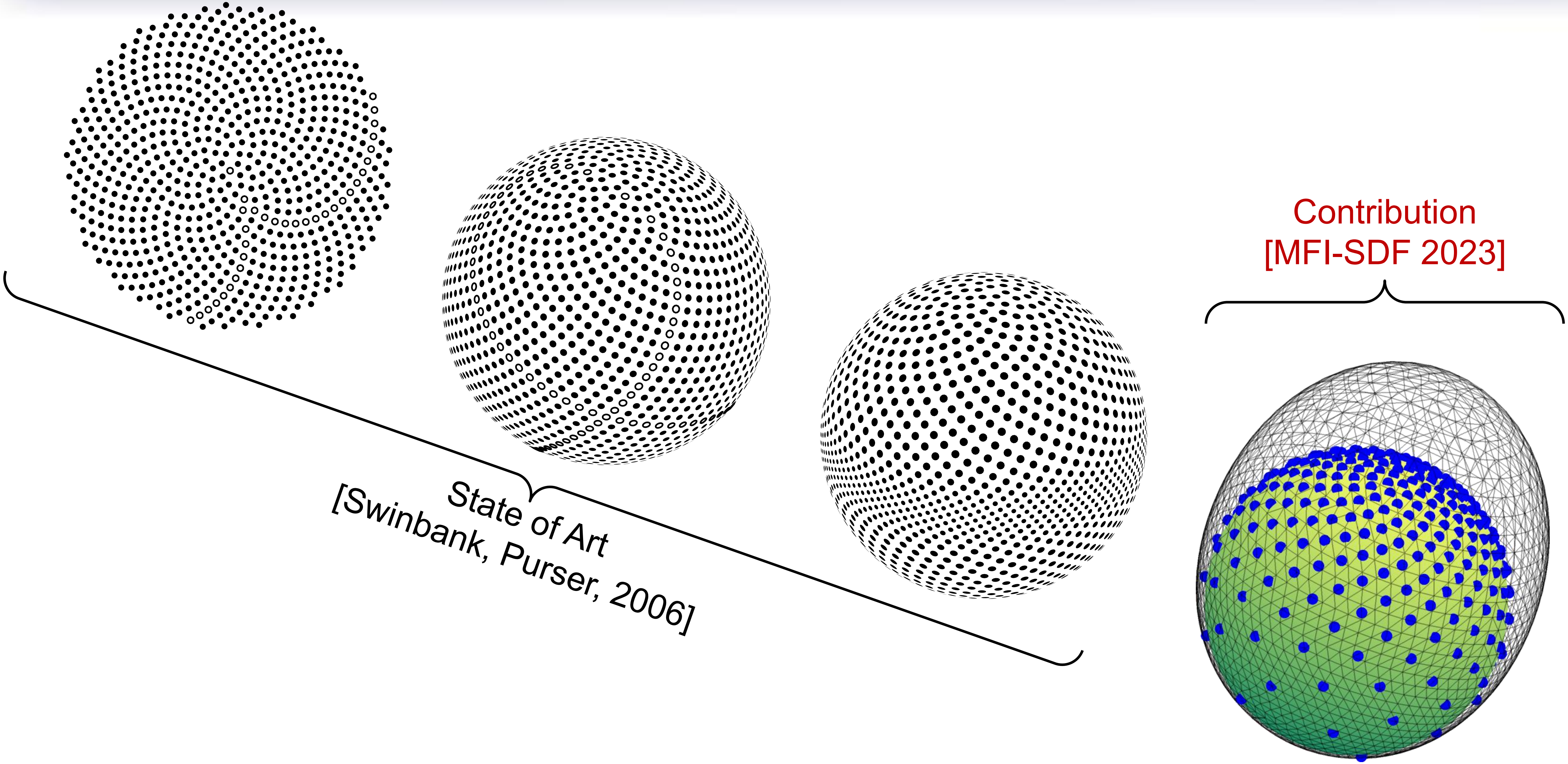


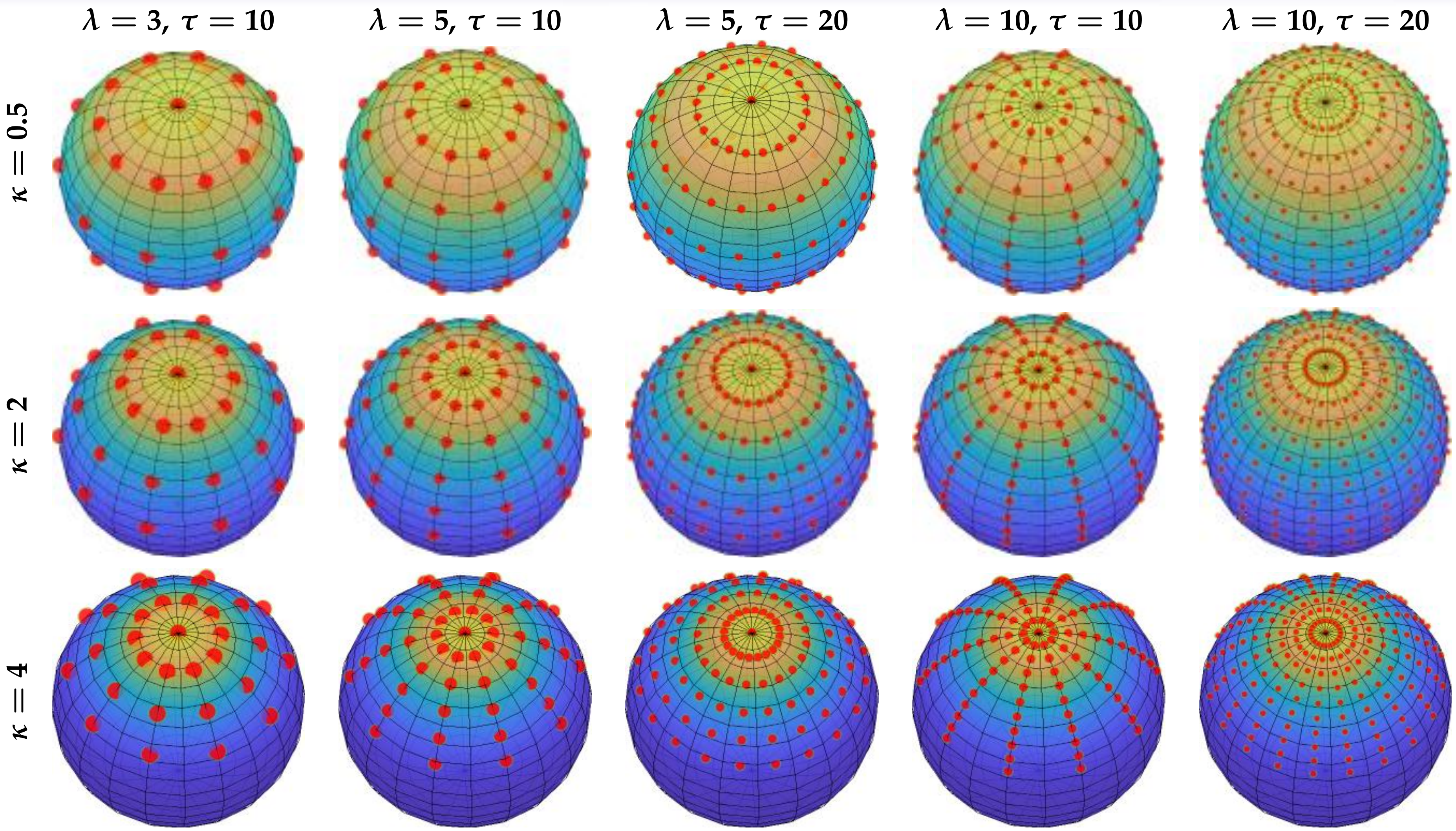


„Radar-Tracking“ Density

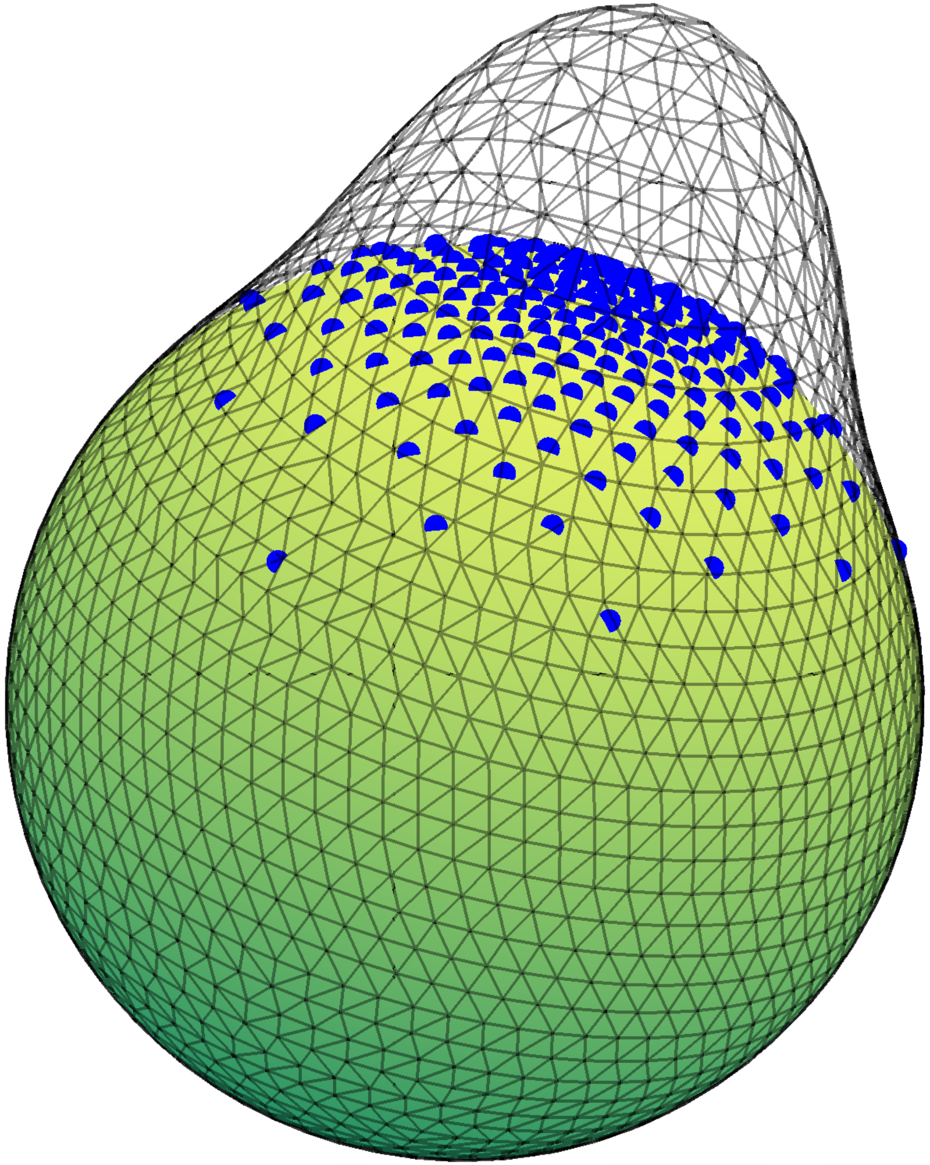


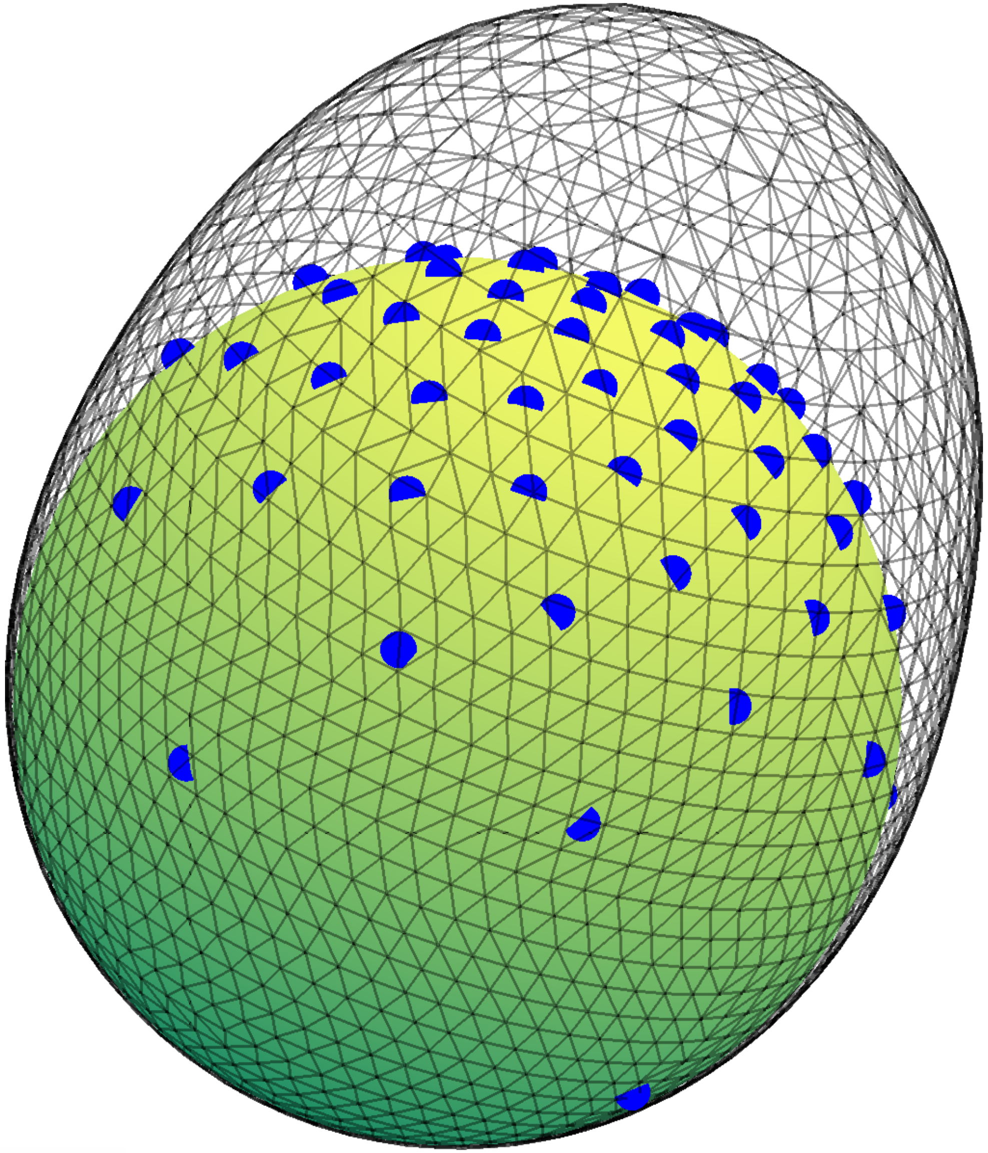
Polar Fibonacci Grid → Sphere

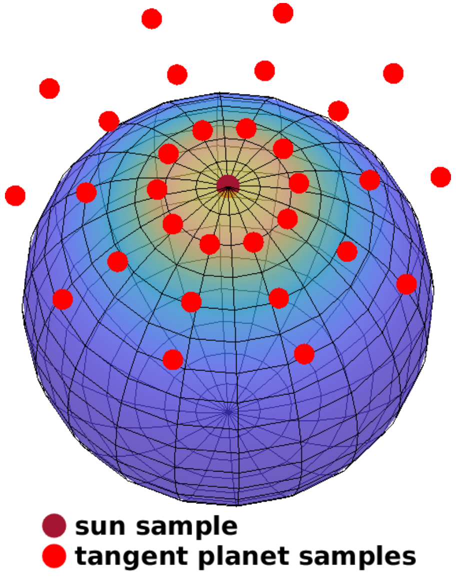


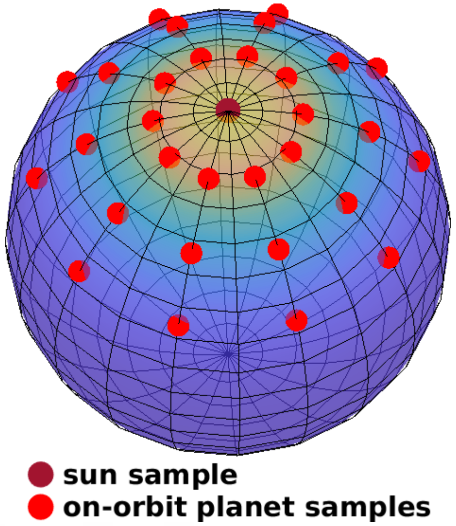


- Kailai Li and Florian Pfaff and Uwe D. Hanebeck, Nonlinear von Mises--Fisher Filtering Based on Isotropic Deterministic Sampling, MFI 2021
- Kailai Li and Florian Pfaff and Uwe D. Hanebeck, Progressive von Mises--Fisher Filtering Using Isotropic Sample Sets for Nonlinear Hyperspherical Estimation, Sensors 2021



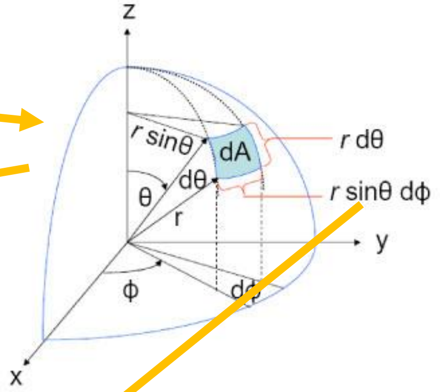



sun sample
tangent planet samples


sun sample
on-orbit planet samples

Uniform → VMF

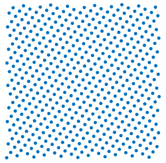
$$f(\underline{x}) = c\, e^{\kappa \cdot \underline{\mu}^\top \underline{x}}$$



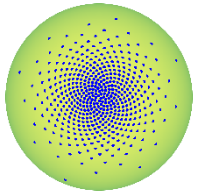
$$f_\theta(\theta) = c\, e^{\kappa \cdot \cos(\theta)}$$

$$F_\theta(\theta) = 2\pi c \int_0^\theta e^{\kappa \cdot \cos(\theta)} \cdot \sin(\theta) \cdot d\theta = -\frac{2\pi c}{\kappa}\left(e^{\kappa \cdot \cos(\theta)} - e^{\kappa}\right)$$

$$F_\theta^{-1}(p) = \cos^{-1}\left(\frac{1}{\kappa}\log\left[e^{\kappa} - p \cdot \left(e^{\kappa} - e^{-\kappa}\right)\right]\right)$$



$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} \theta \\ \varphi \end{bmatrix} = \begin{bmatrix} F^{-1}(x) \\ 2\pi \cdot y \end{bmatrix}$$



Implementation

$$i \cdot \begin{bmatrix} \frac{1}{L} \\ \frac{1}{\Phi} \end{bmatrix} \bmod 1 \to Q(\cdot) \to \mathrm{spher2cart}(\cdot)$$

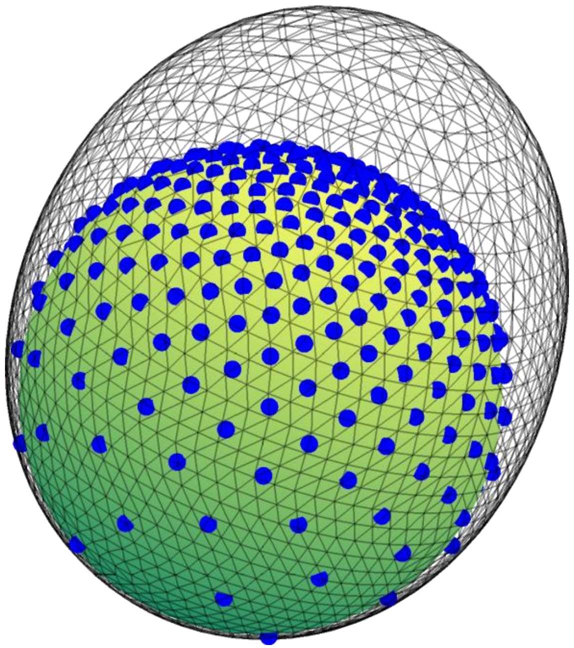
L, κ →

$$\underline{x}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} w \\ \sqrt{1-w^2} \cdot \cos\left(\frac{2\pi i}{\Phi}\right) \\ \sqrt{1-w^2} \cdot \sin\left(\frac{2\pi i}{\Phi}\right) \end{bmatrix},$$

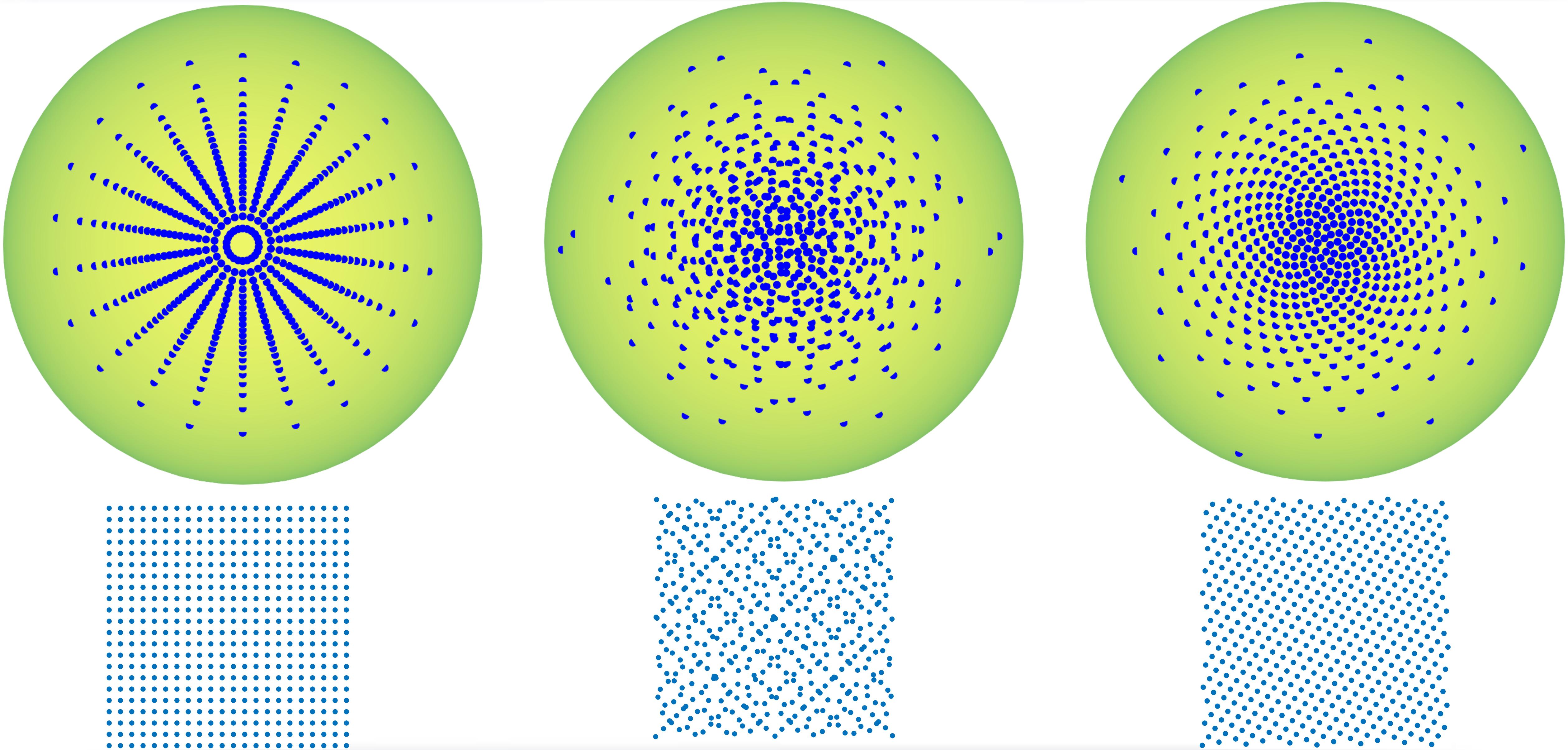
$$w = 1 + \frac{1}{\kappa} \cdot \log\left(1 + \frac{2i-1}{2L} \cdot (\exp(-2\kappa) - 1)\right),$$

$$i \in \{1,2,3,\dots,L\}$$

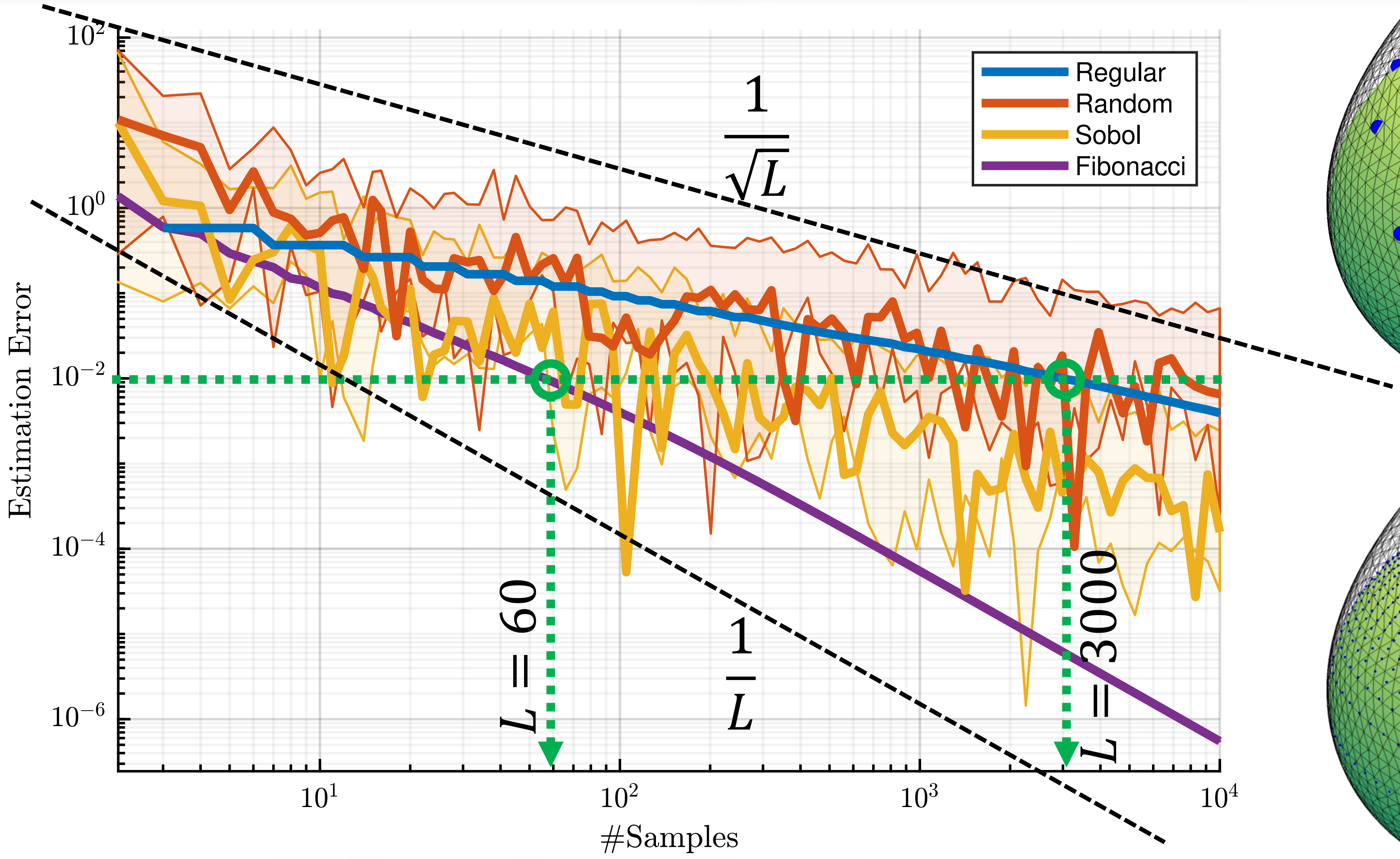
→ $\underline{x}_i$

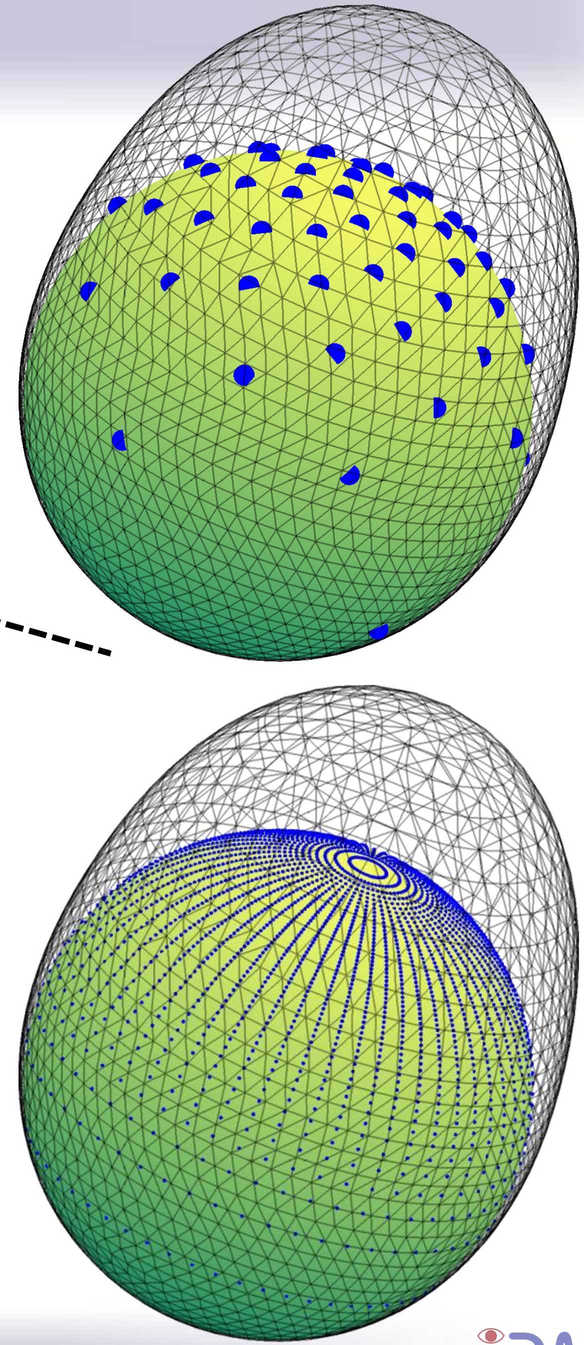


VMF Sampling



Evaluation: κ Estimation



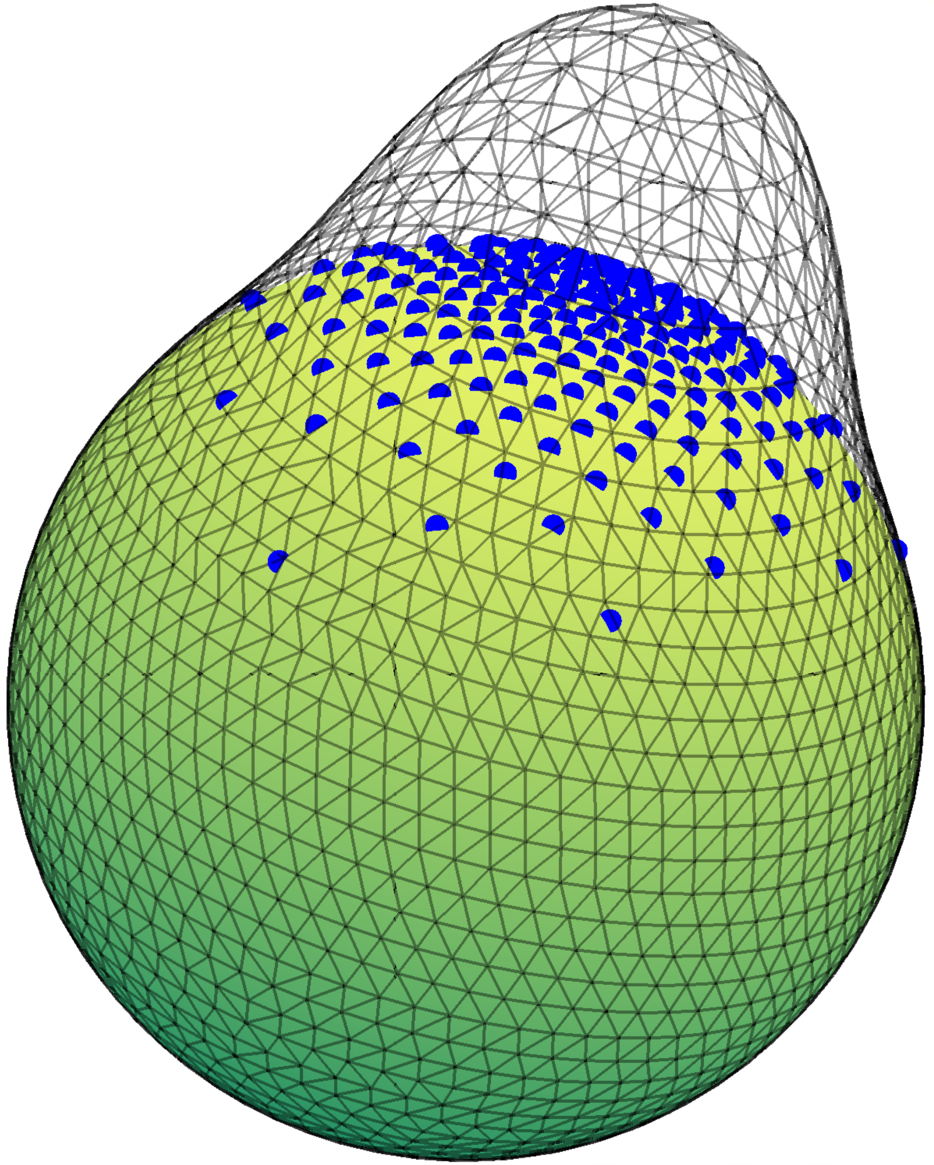


Conclusion

Contribution
- vMF Sampling
- 2-Sphere
- Locally homogeneous

Extensions
- Densities (Bingham, …)
- Dimensions

Moment Correction
- κ : concentration parameter
- μ : mean direction



Thank you for your attention

Intelligent Sensor-Actuator-Systems