Gaussian Mixture Particle Filter Step based on Method of Moments

Daniel Frisch and **Uwe D. Hanebeck**

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Overview: Processing Pipeline

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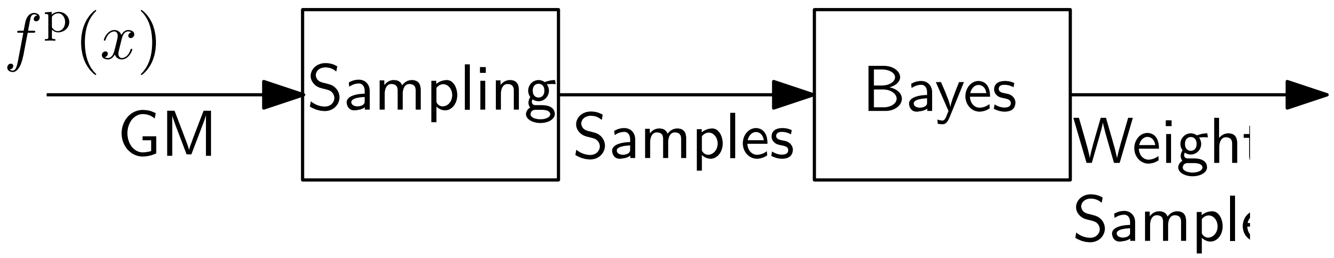
Overview: Processing Pipeline



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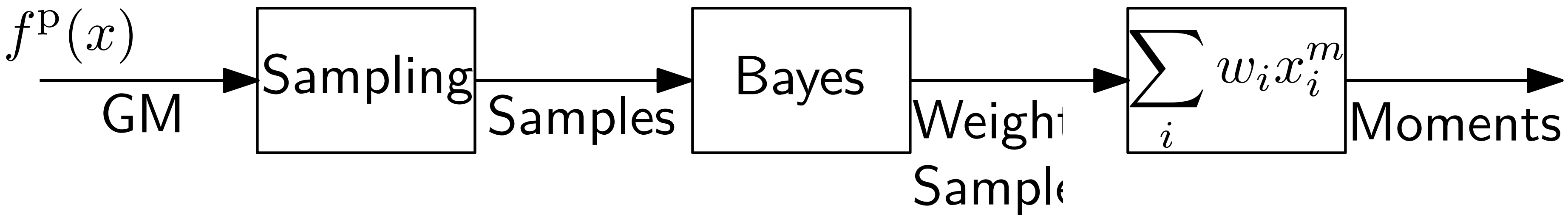

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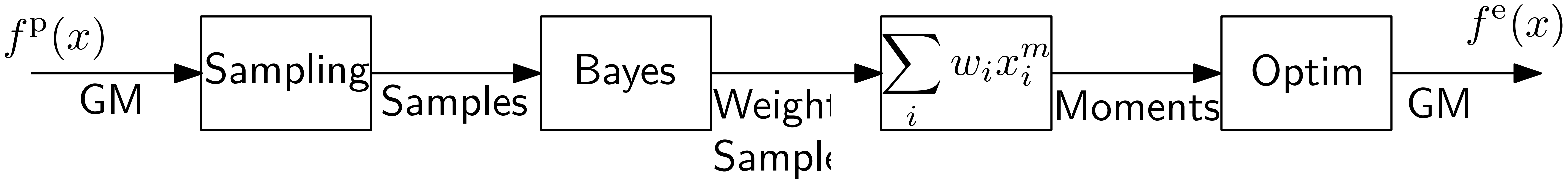

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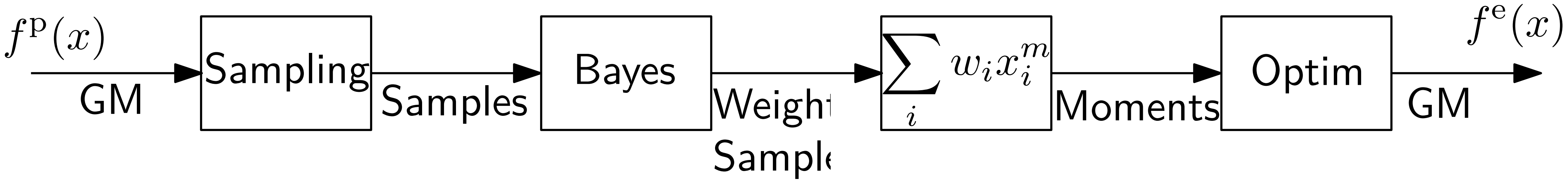

Contribution 1:

- Optimal deterministic GM sampling
- Empirical moments converge with $1/L$ instead of $1/\sqrt{L}$

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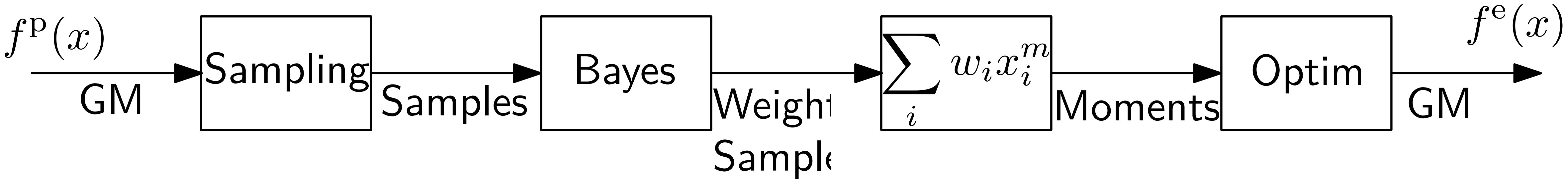


Contribution 1:

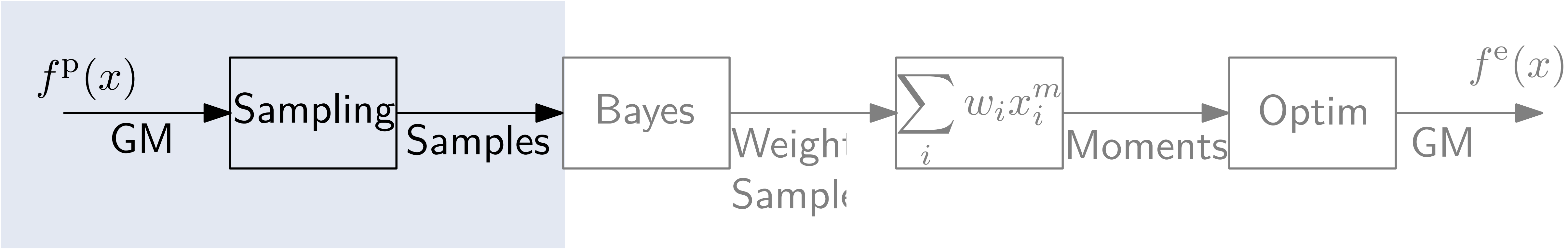
- Optimal deterministic GM sampling
- Empirical moments converge with $1/L$ instead of $1/\sqrt{L}$

Contribution 2:

- Moments $\rightarrow$ GM
- I_F Regularization

$$f^{\mathrm{e}}(x) = \arg\min_{f(\cdot)} I_F\{f(\cdot)\}$$

$$\text{s.t.} \int_{\mathbb{R}} x^m f^e(x) dx = \sum_i w_i x_i^m$$



Contribution 1: Sampling

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Monte Carlo Quadrature / Random Samples

Integration error: $\epsilon = \left| \int\limits_{-\infty}^{\infty} g(u)\,\mathrm{d}u - \sum_{i=1}^{L} g(u_i) \right|$

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$$\Big\downarrow$$

CLT: $\epsilon \sim \mathcal{N}\left(\epsilon; 0, \Sigma \cdot L^{-1/2}\right)$

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$$\Sigma = \int_0^1 \left(g(u) - \left(\int_0^1 g(\tilde{u})\,\mathrm{d}\tilde{u} \right) \right)^2 \mathrm{d}u$$

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$\Rightarrow$ Convergence with only $L^{-1/2}$.

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Quasi-MC Quadrature / Deterministic Samples

Integration error: $\epsilon = \left| \int_{-\infty}^{\infty} g(u)\,\mathrm{d}u - \sum_{i=1}^{L} g(u_i) \right|$

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Koksma-Hlawka Inequality: $\epsilon \leq \mathrm{discr}(\{u_i\}) \cdot V(g)$

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"Stair function"

Discrepancy: $\mathrm{discr}(\{u_i\}) = \sup_{t \in [0,1]} \left| x - F_{\{u_i\}}(x) \right|$

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$\Rightarrow$ Ideal: "lowest-discrepancy" point set, $\{u_i\}^* = \arg\min\limits_{\{u_i\}} \{\mathrm{discr}(\{u_i\})\}$

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Deterministic Sampling

1D samples with lowest discrepancy: Equidistant!

$$u_i^* = \frac{2i-1}{2L} \quad , \quad i \in \{1, 2, \ldots, L\}$$

$$\downarrow$$

Then: $\mathrm{discr}(\{u_i\}^*) = L^{-1}$

$\Rightarrow$ Convergence with L^{-1} instead of $L^{-1/2}$

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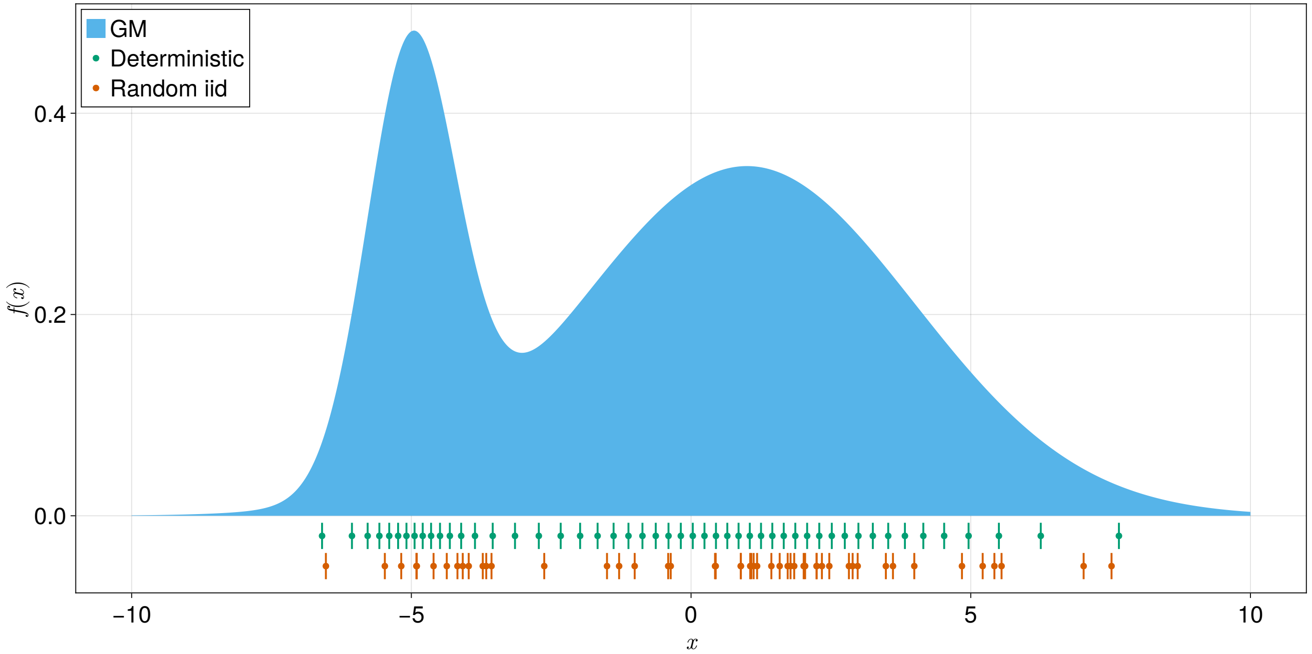
Non-Uniform: Substitution / Inverse Transform

Gaussian: $x_i^* = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x-\mu}{\sqrt{2\Sigma}}\right)\right]$ → "closed form"

Gaussian Mixture: $\frac{2i-1}{2L} - \sum_{k=1}^{K} \frac{w_k}{2}\left[1 + \mathrm{erf}\left(\frac{x_i^* - \mu_k}{\sqrt{2\,\Sigma_k}}\right)\right] \overset{!}{=} 0$ → need bisection



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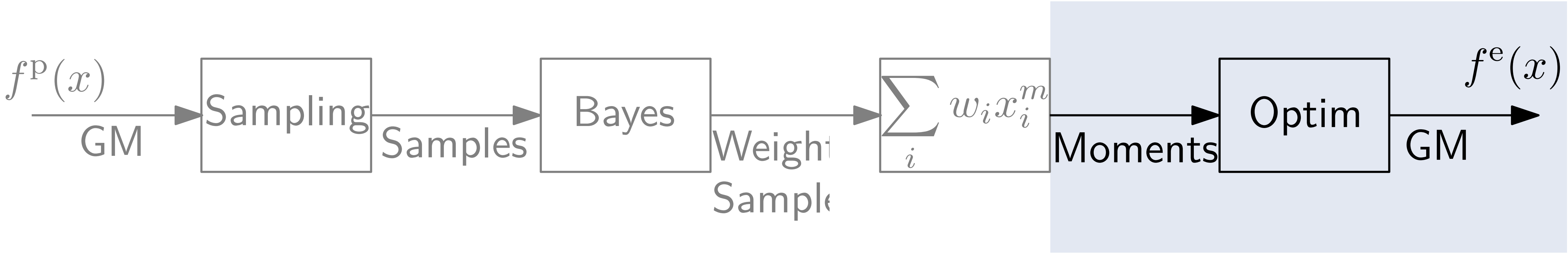

Contribution 2: Fisher Information Optimization

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Review: Fisher Information with Respect to Parameter (1)

Assumptions: Scalar latent variable x, scalar parameter θ

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Likelihood function $f(x|\theta)$

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BTW: Setting score to zero gives maximum likelihood estimate of θ

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$$\mathrm{E}\{s(x|\theta)\} = \int_{\mathbb{R}} s(x|\theta)\, f(x|\theta) dx$$

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Review: Fisher Information with Respect to Parameter (2)

Fisher information is variance of score function

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Review: Fisher Information with Respect to Parameter (2)

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Fisher information is variance of score function

$$I_F(\theta) = \mathrm{var}\{s(x|\theta)\}$$

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Fisher information is variance of score function

$$
\begin{aligned}
I_F(\theta) &= \mathrm{var}\{s(x|\theta)\} \\
&= \mathrm{E}\left\{\Big(s(x|\theta) - \mathrm{E}\{s(x|\theta)\}\Big)^2\right\}
\end{aligned}
$$

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Review: Fisher Information with Respect to Parameter (3)

For **location parameter** θ, the likelihood is given by $f(x|\theta) = f(x+\theta)$, the score is

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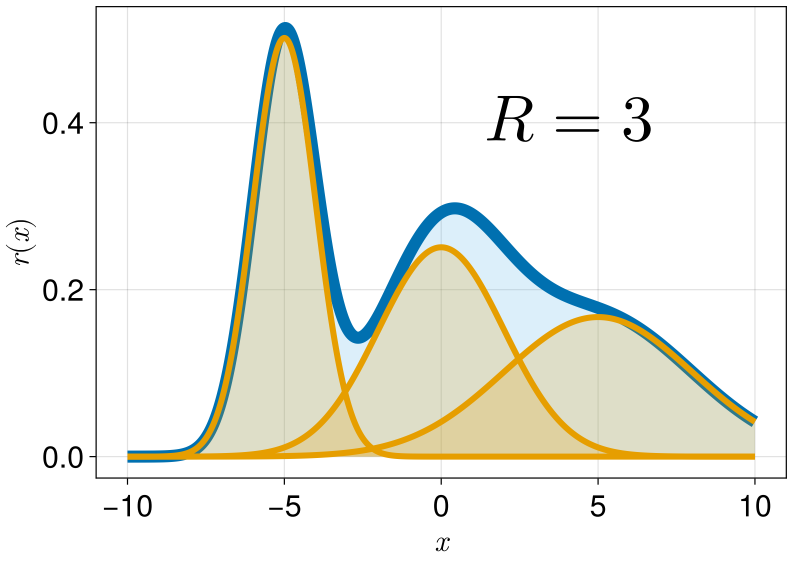
Gaussian Root Mixture

$$\text{GRM:} \qquad r(x) = \sum_{i=1}^{R} v_i \, \mathcal{N}(x; \rho_i, P_i)$$



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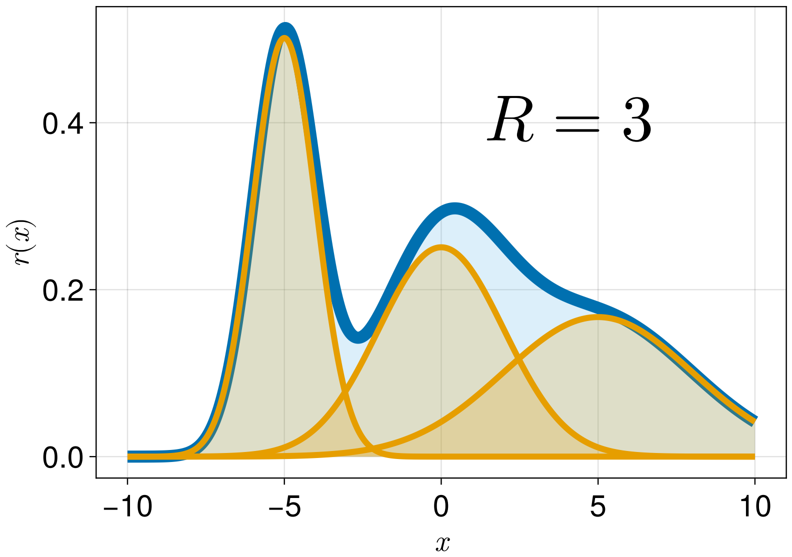

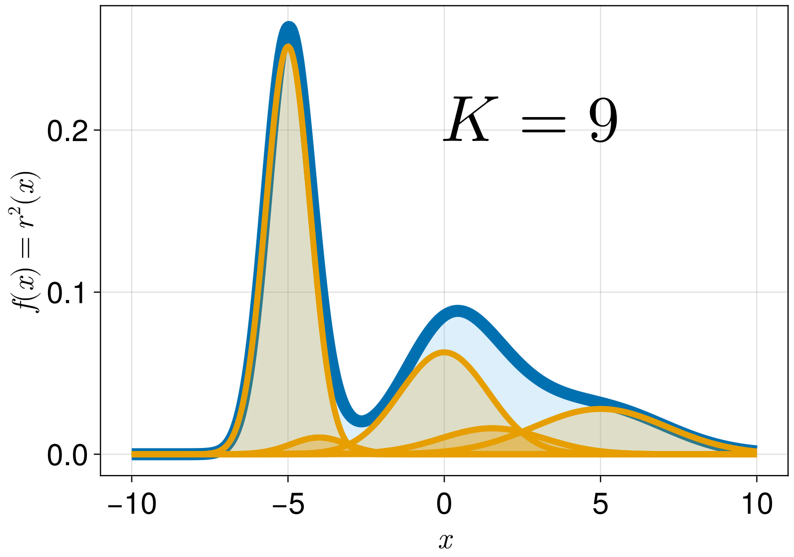

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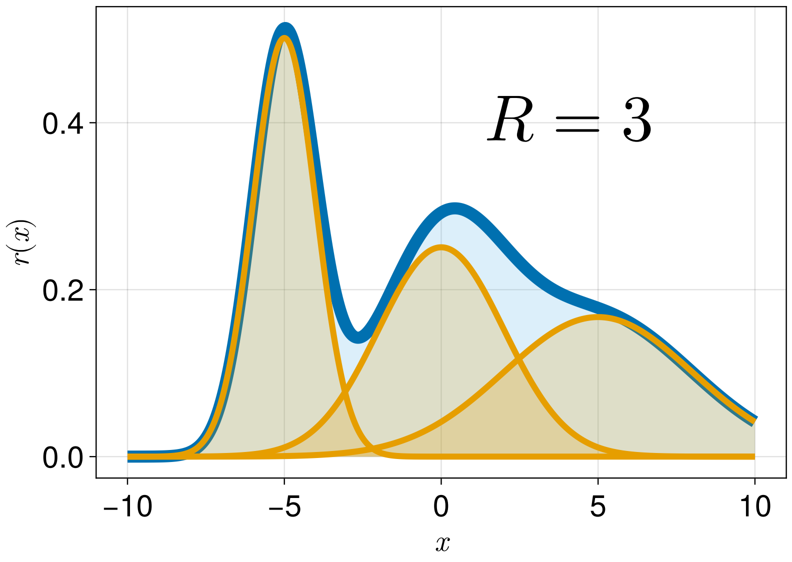
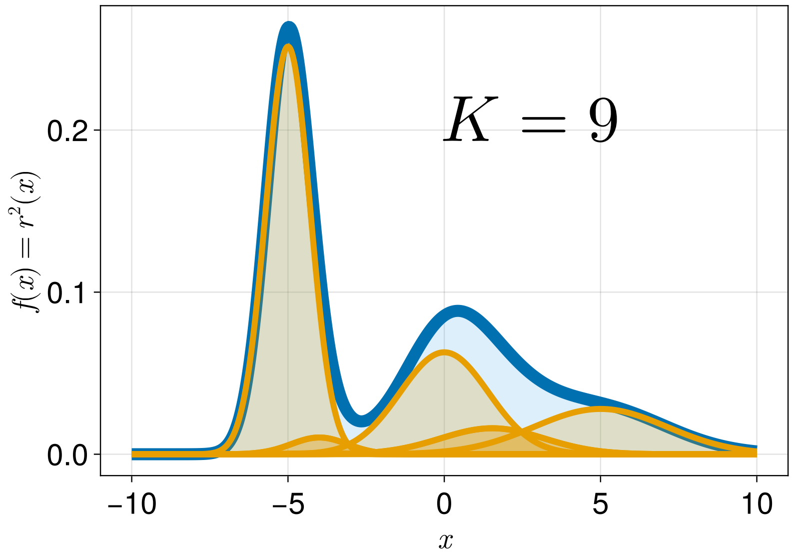
$\rightarrow$ Closed form expression of f in the parameters of r





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Fisher Information

$$I_F(f) = \int_{x \in \mathbb{R}, f>0} \frac{\big(f'(x)\big)^2}{f(x)} \, \mathrm{d}x$$

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bad for mixtures

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$$I_F(f) = \int_{x \in \mathbb{R}, f>0} \frac{\big(f'(x)\big)^2}{f(x)} \, \mathrm{d}x \qquad = \qquad I_F^{\mathcal{R}}(r) = 4 \int_{x \in \mathbb{R}} \big(r'(x)\big)^2 \, \mathrm{d}x$$

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GM Moment Matching

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params of $r(x)$:
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GM Moment Matching

$$I_F^{\mathcal{R}}(r) = I_F(f) \longrightarrow \text{minimize}$$

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Constrained nonlinear optimization

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Optimization (2)

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Tandem processing in root mixture space space $\mathcal{R}$ and original mixture space $\mathcal{M}$

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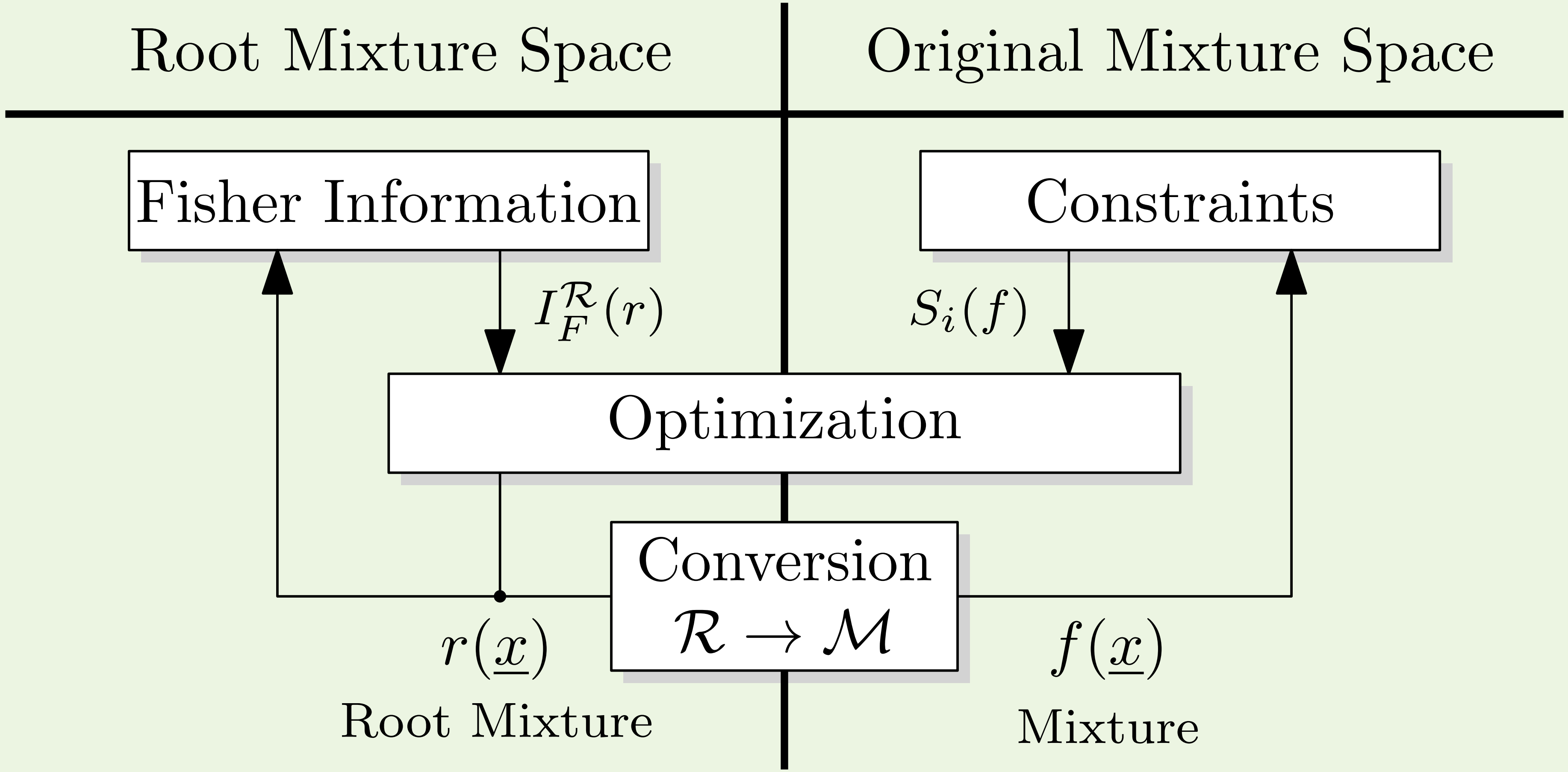

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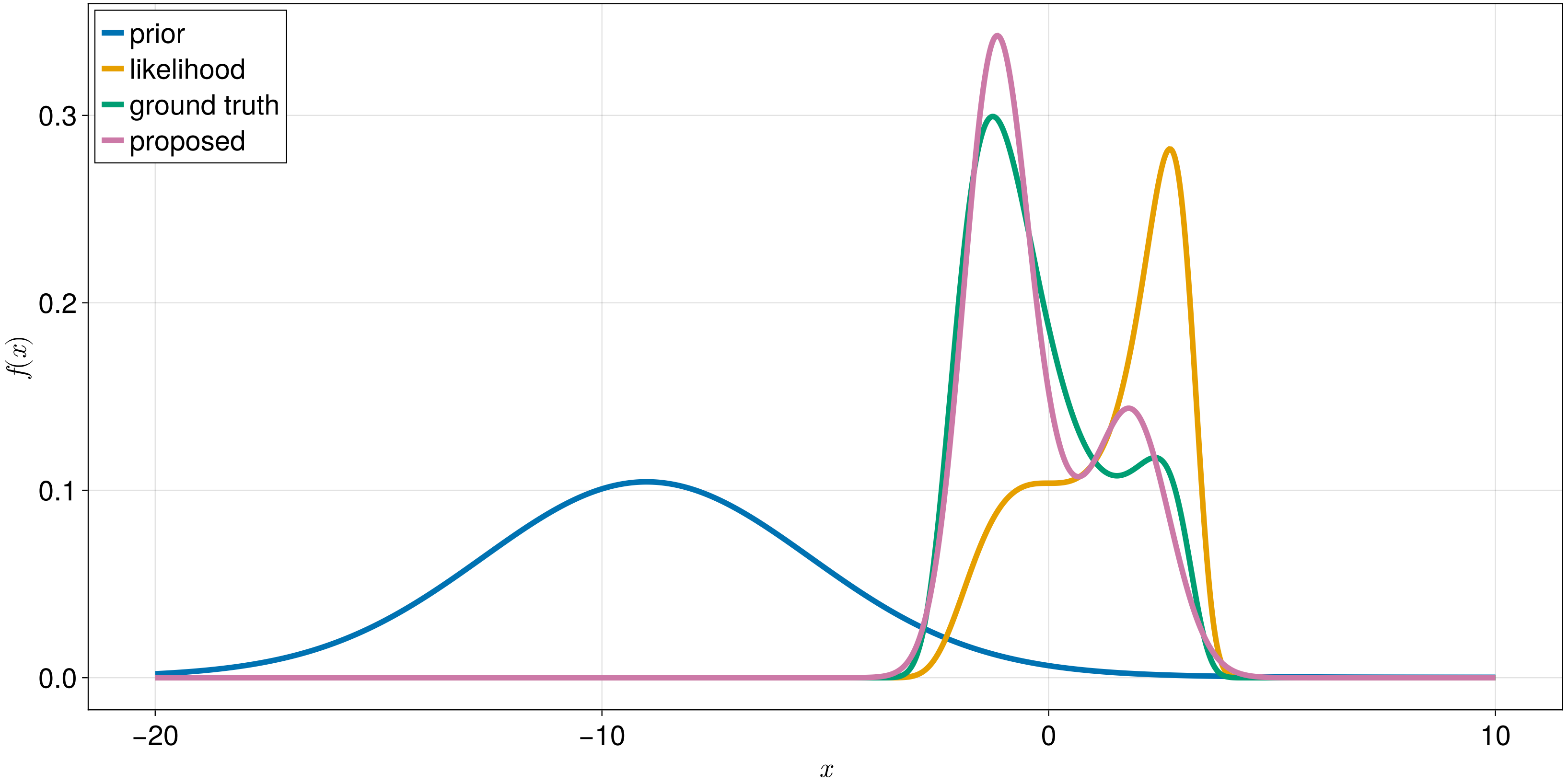
Demonstration

Measurement model: $\quad y = h(x) + v \;, \quad h(x) = \frac{1}{10}x^3 - x \;, \quad \Sigma_v = 2$

$\Rightarrow$ Likelihood: $\quad \Lambda(x) = \mathcal{N}(x; \hat{y} - h(x), \Sigma_v)$

Measurement: $\quad \hat{y} = 2$



Context & Motivation (3)

More typical use cases:

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Context & Motivation (3)

More typical use cases:

- Estimating continuous probability density function from given samples

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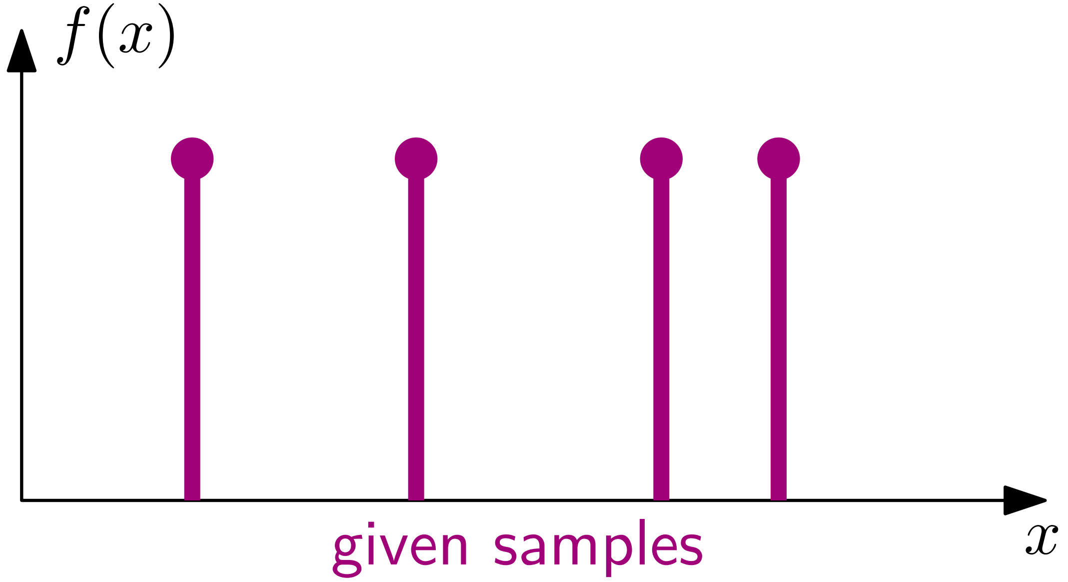

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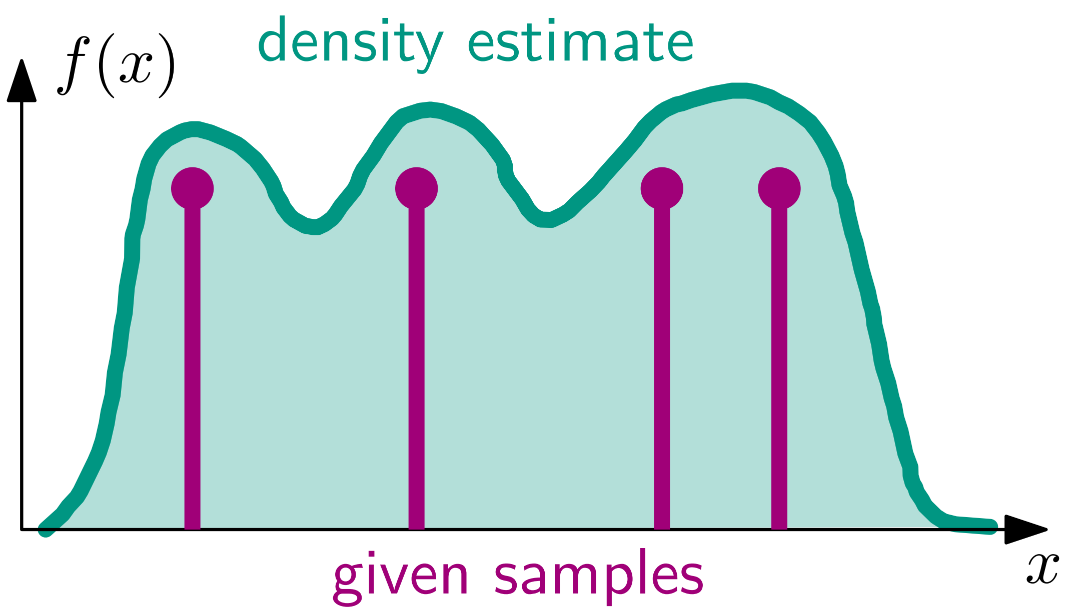

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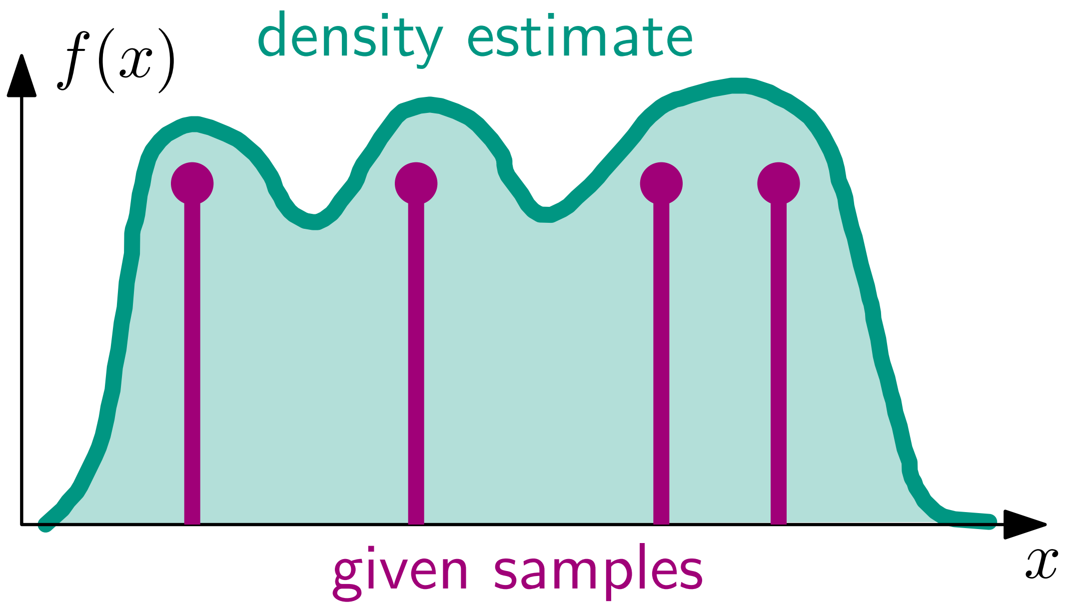

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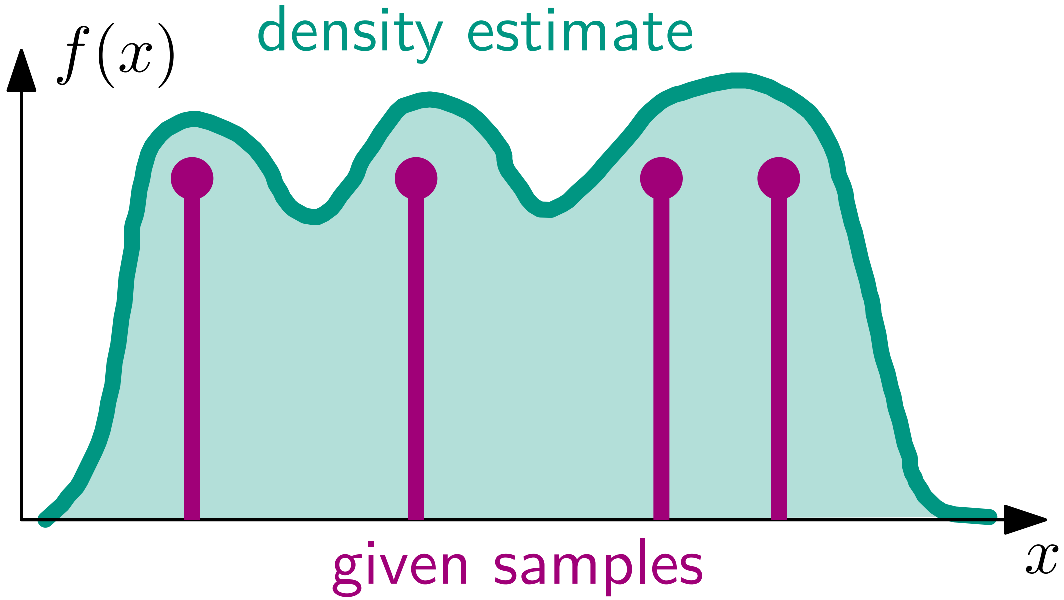

Form constraints

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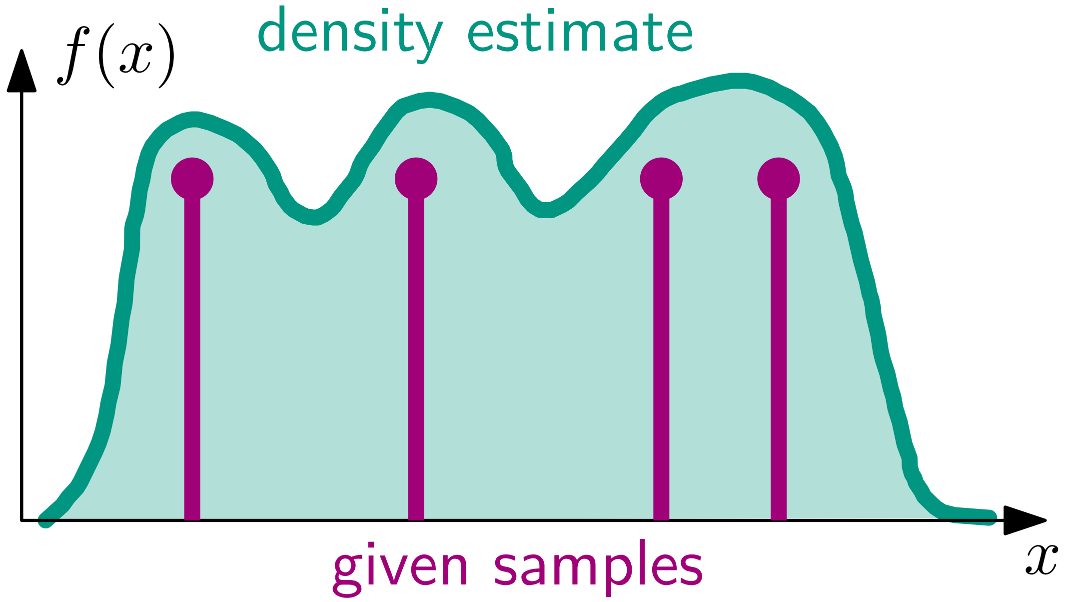

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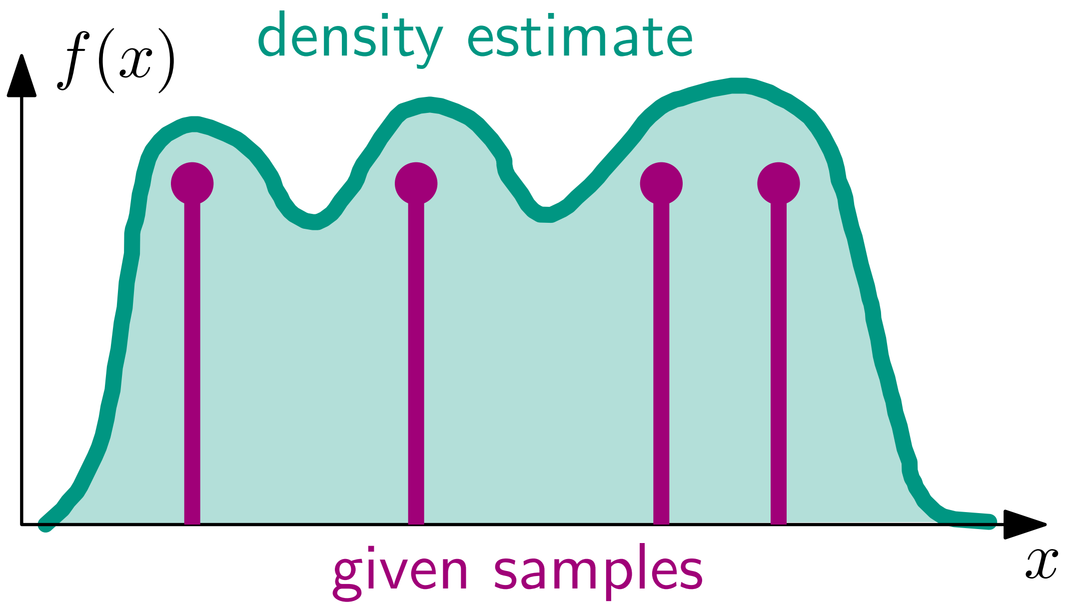

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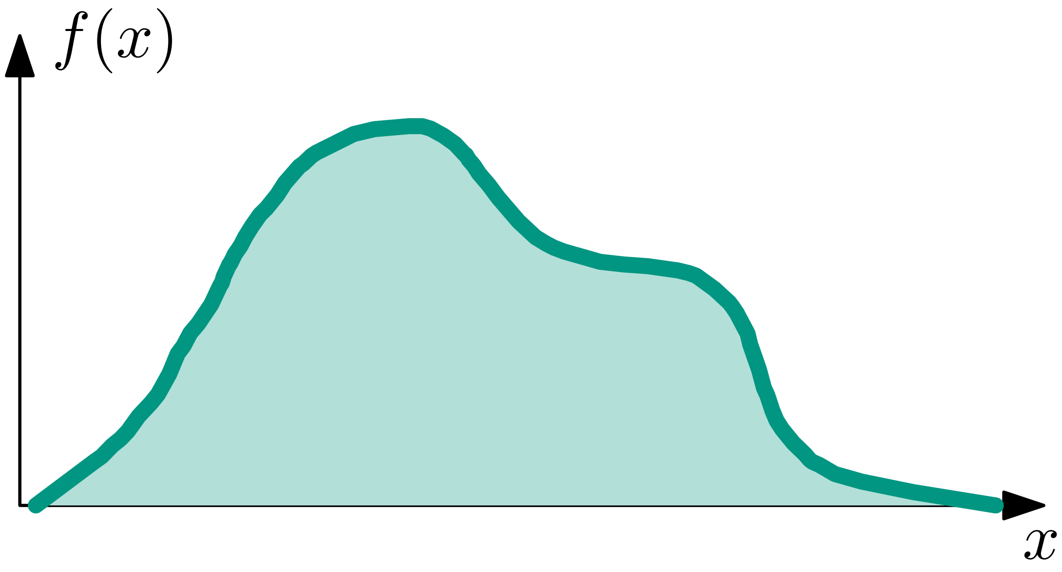

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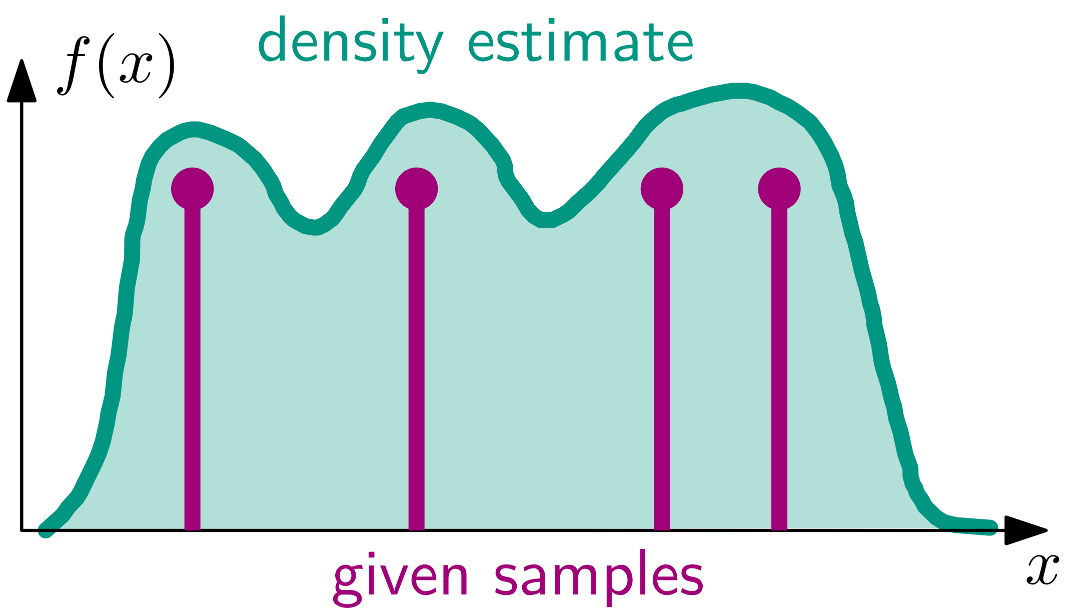

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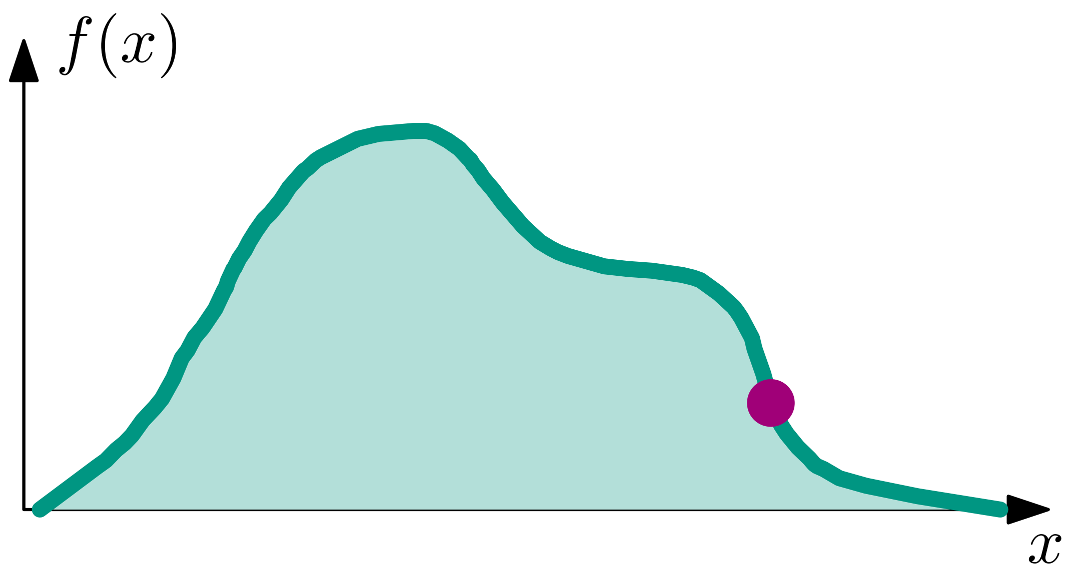

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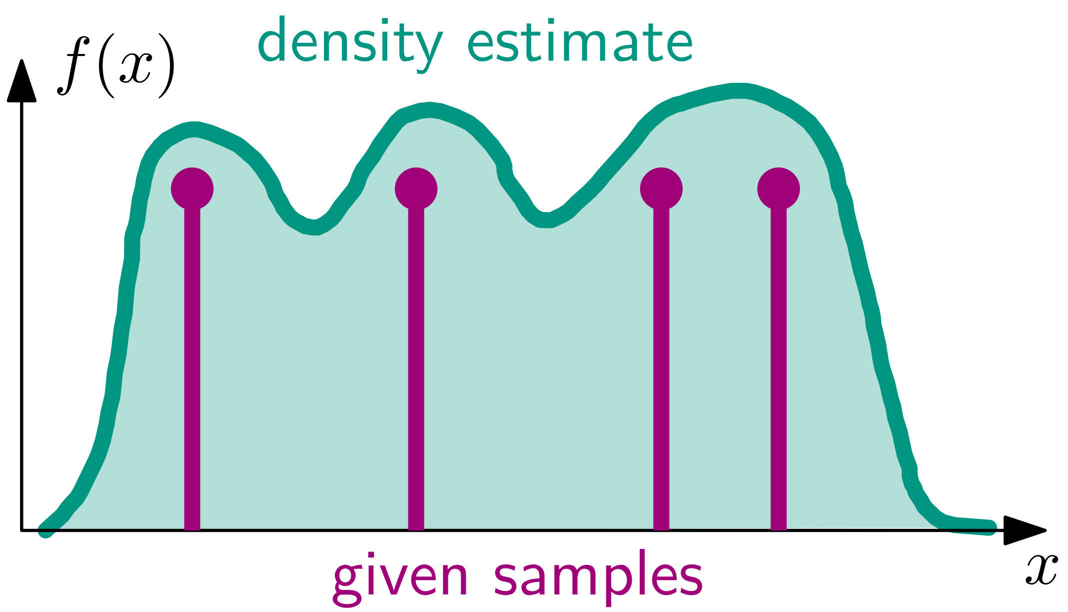

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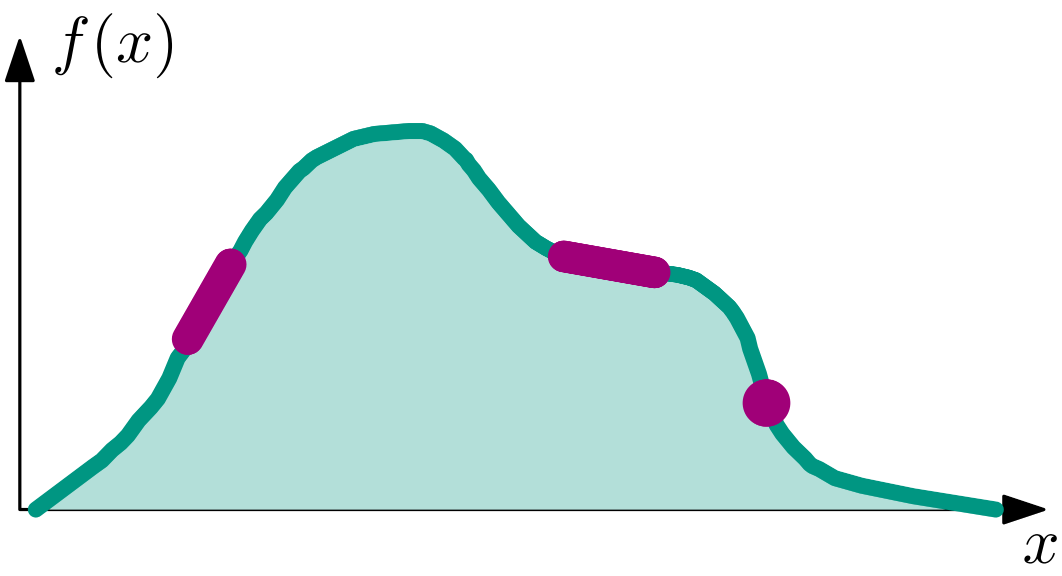

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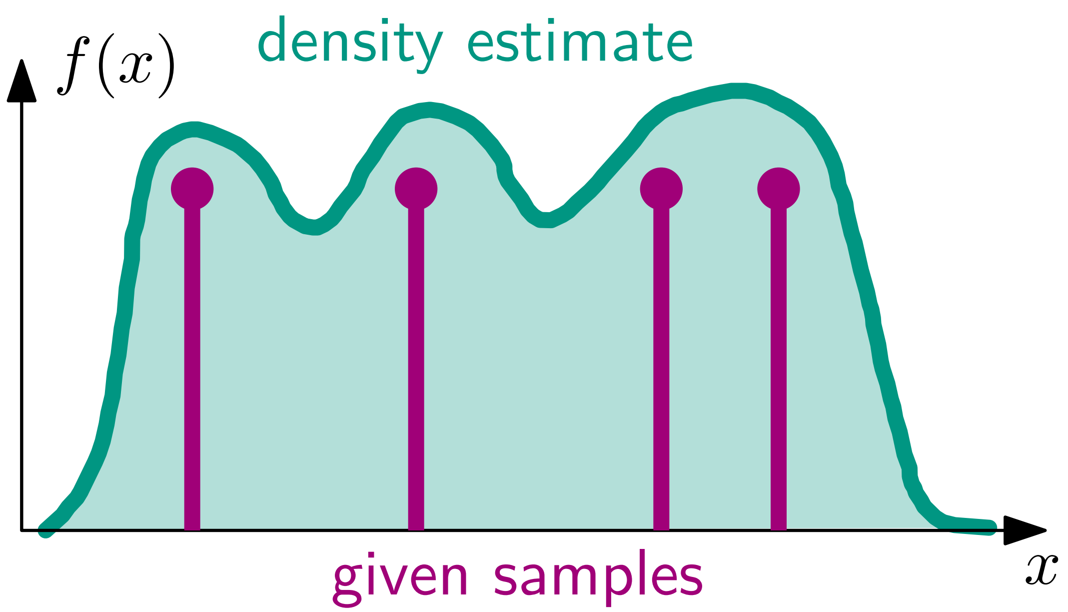

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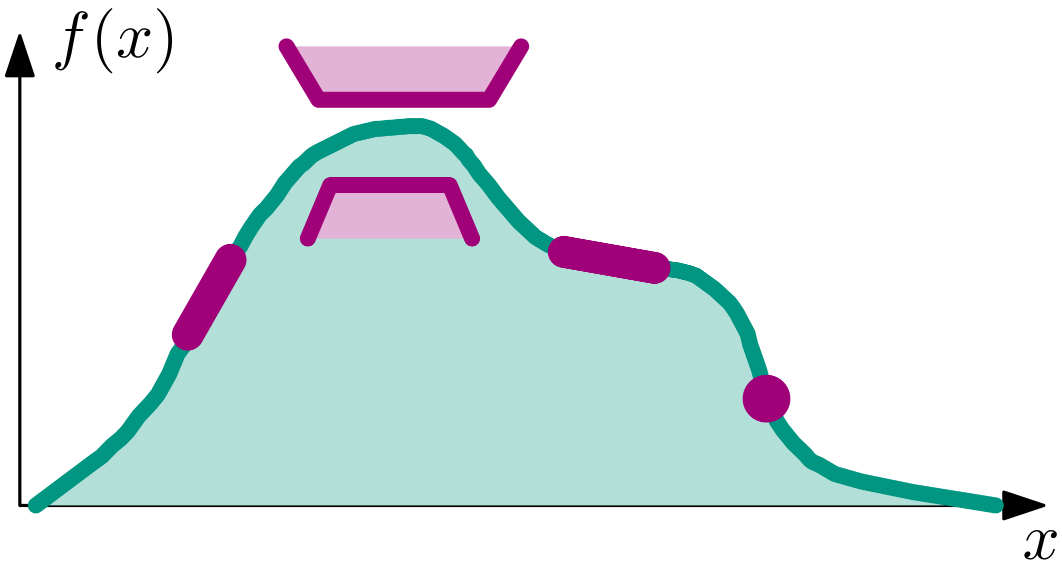

Context & Motivation (4)

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More typical use cases:

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Context & Motivation (4)

More typical use cases:

- Increasing number of components of mixture density

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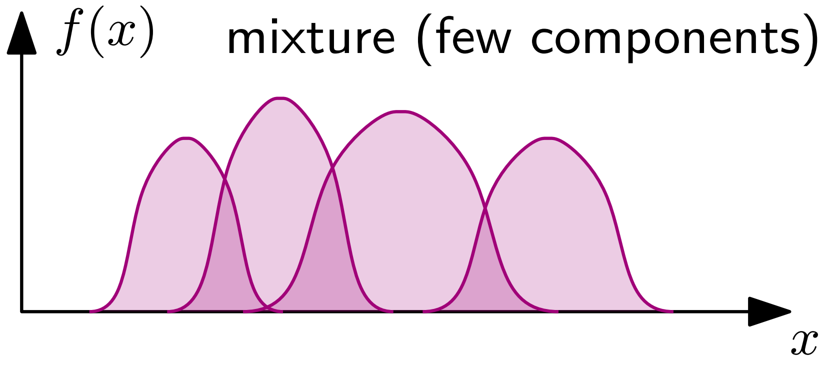

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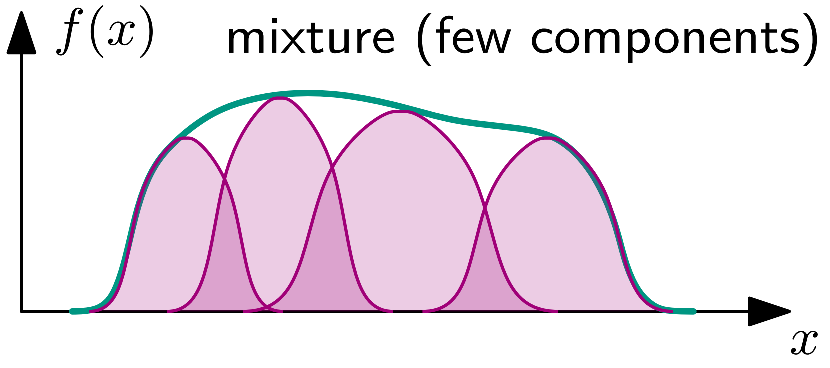

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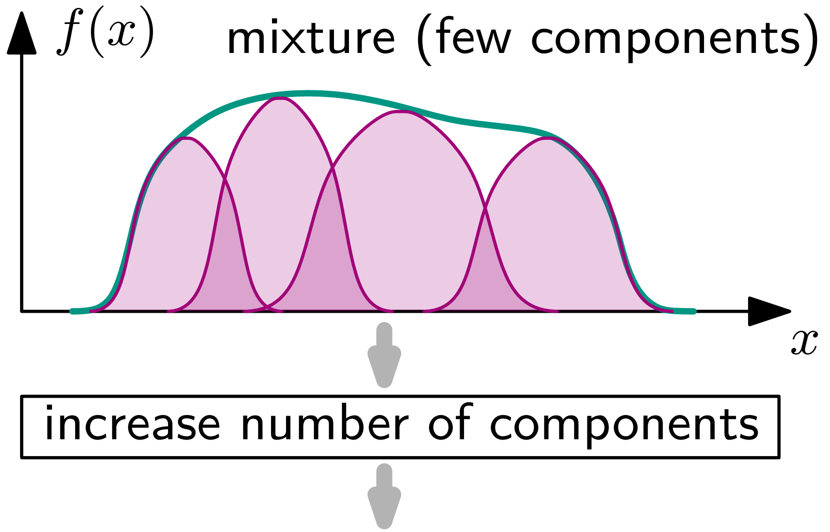

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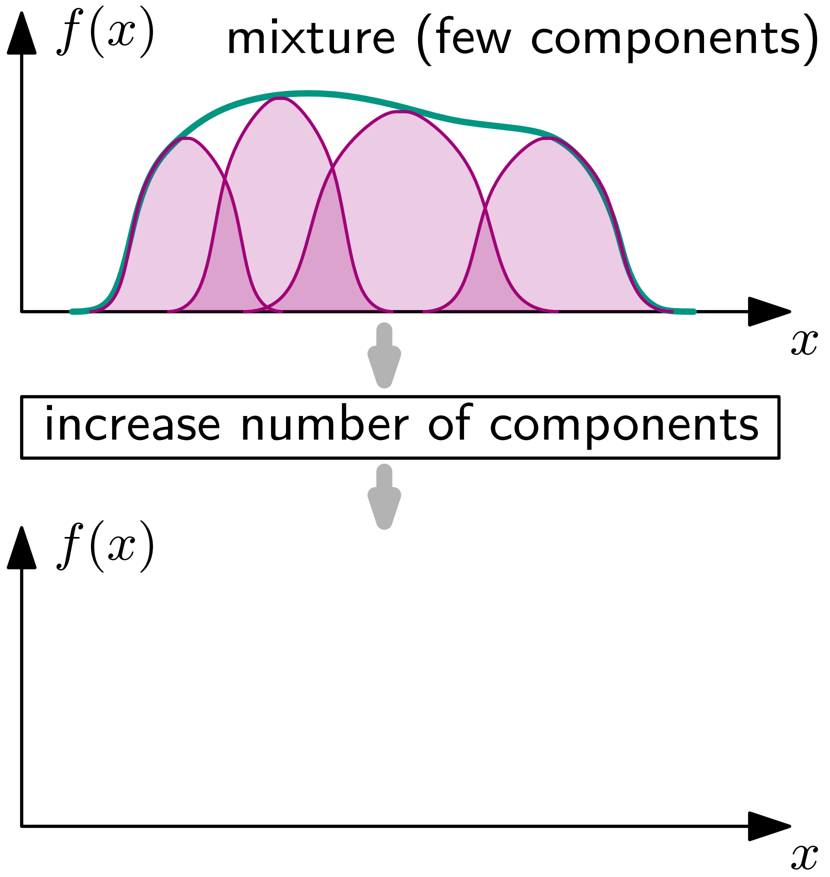

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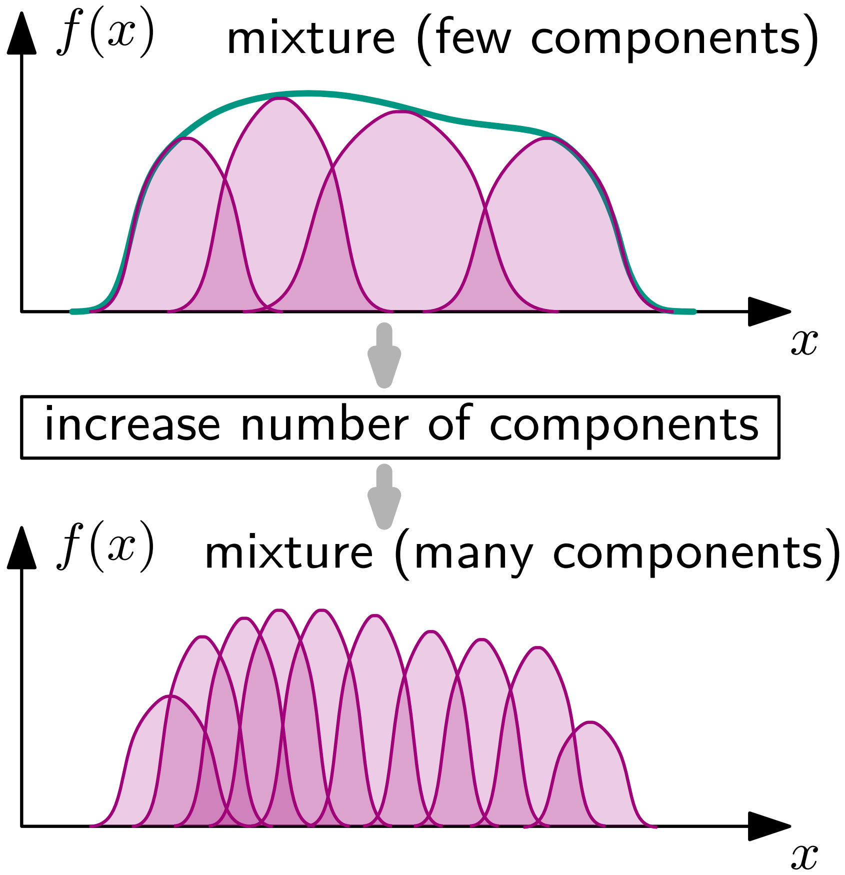

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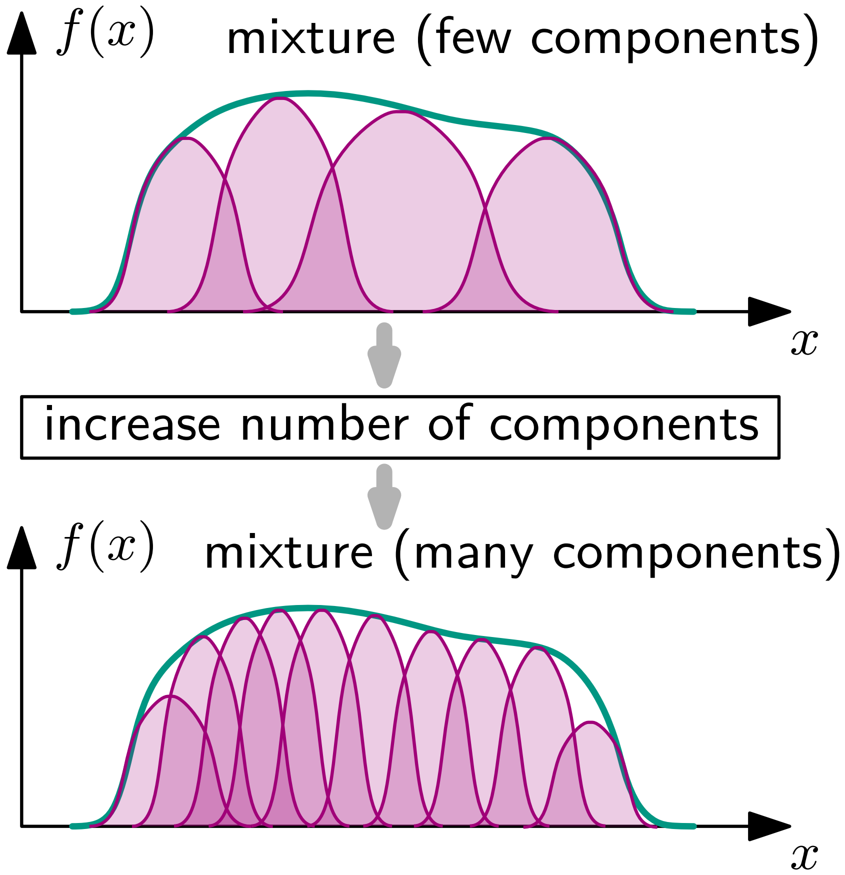

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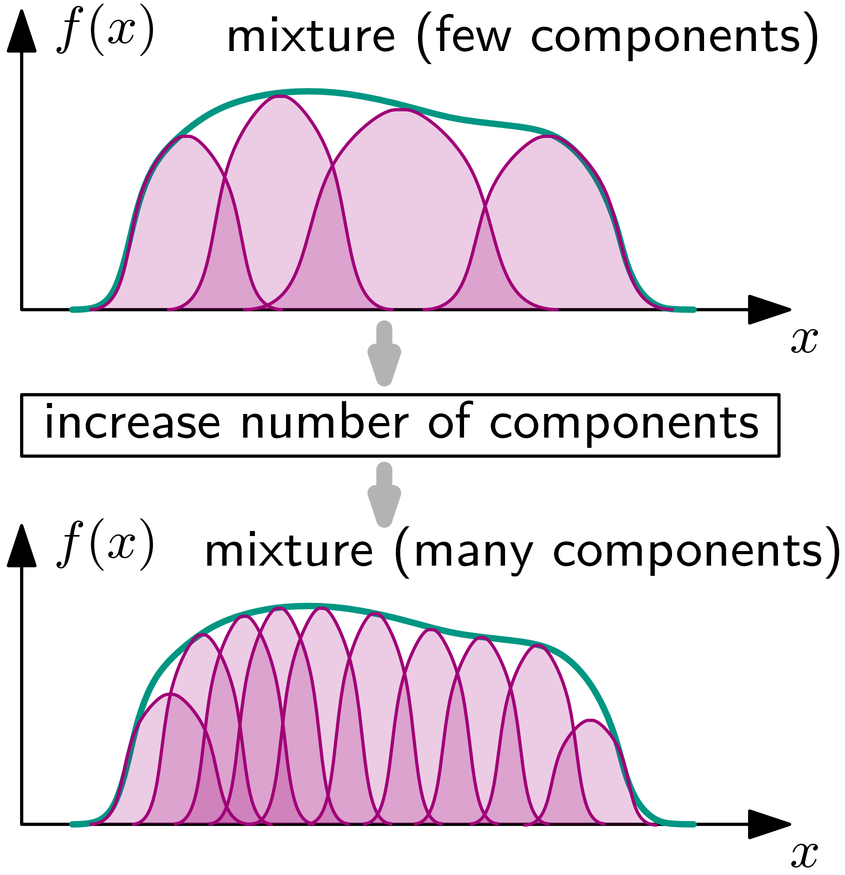
Context & Motivation (4)

More typical use cases:
- Increasing number of components of mixture density
- Estimate continuous pdf from set of given moments



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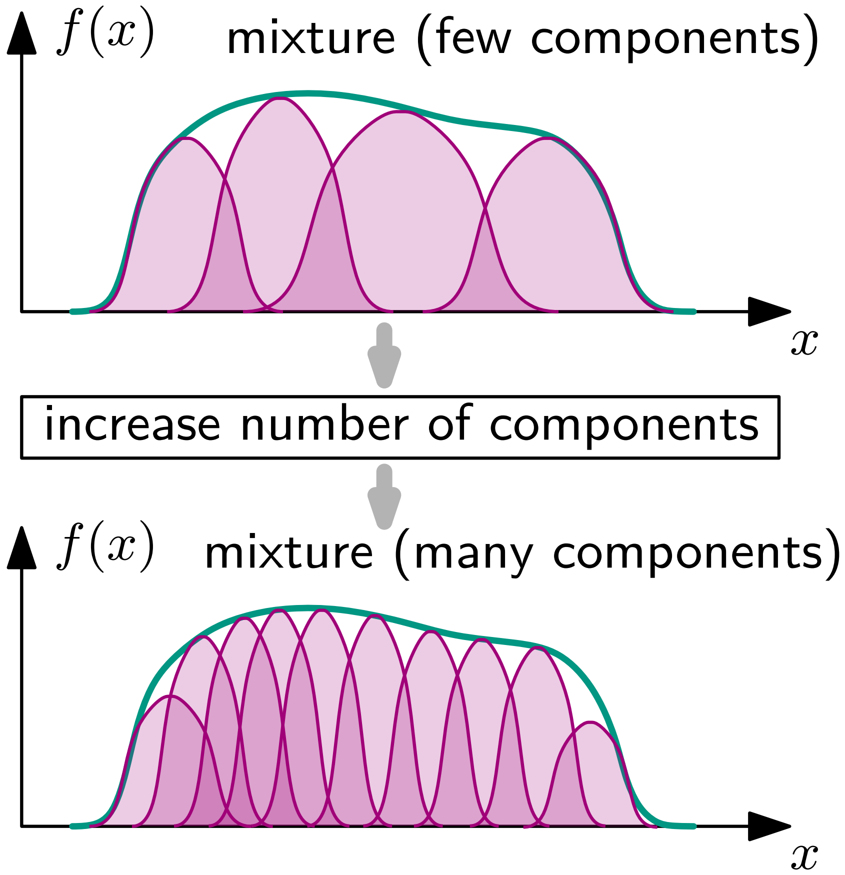

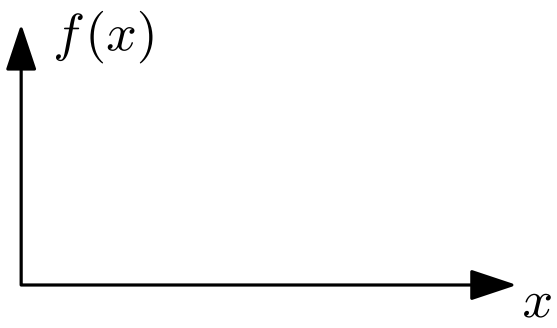

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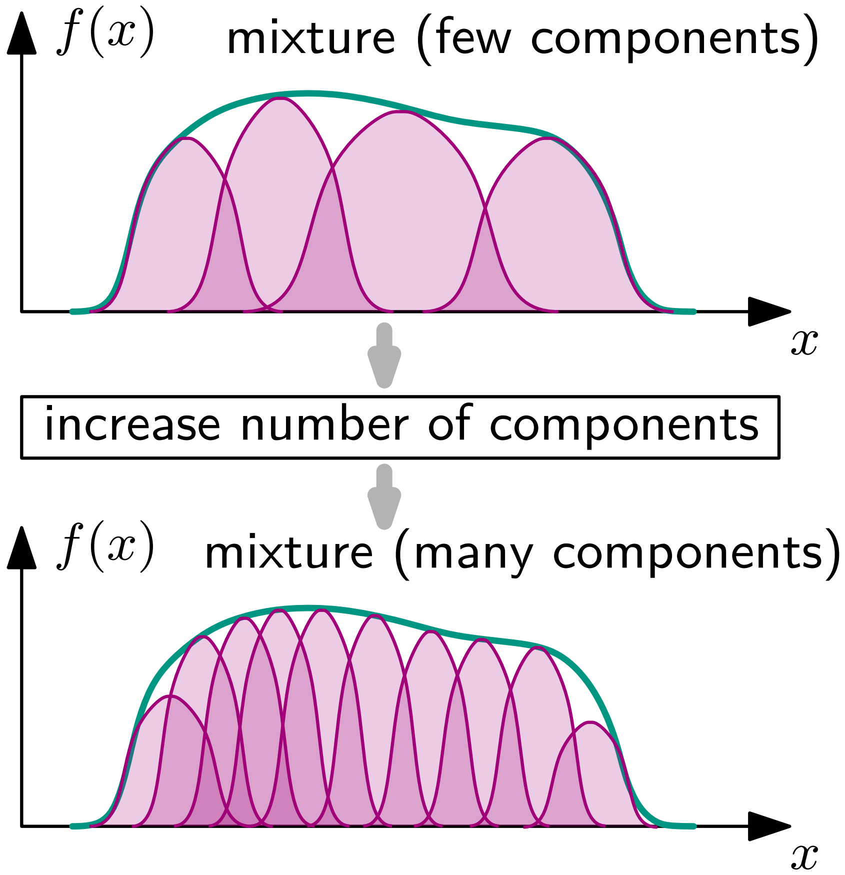

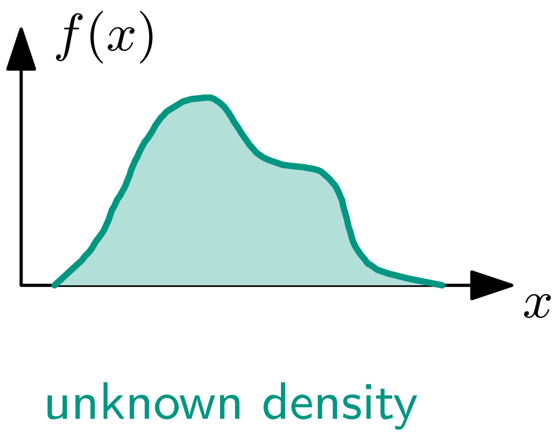

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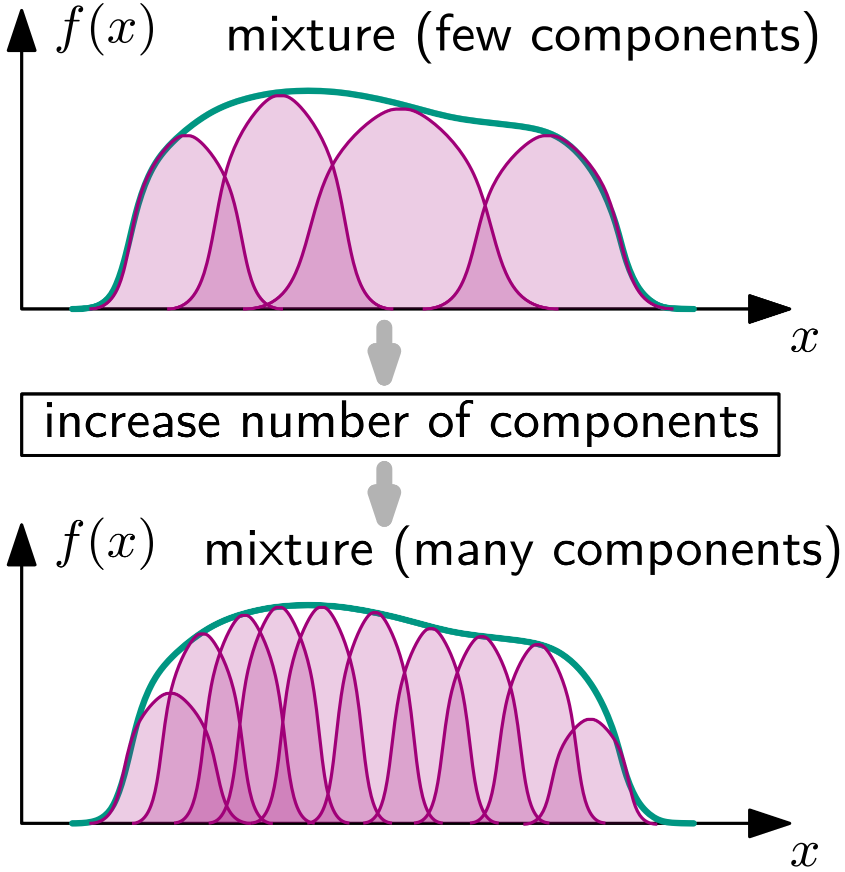
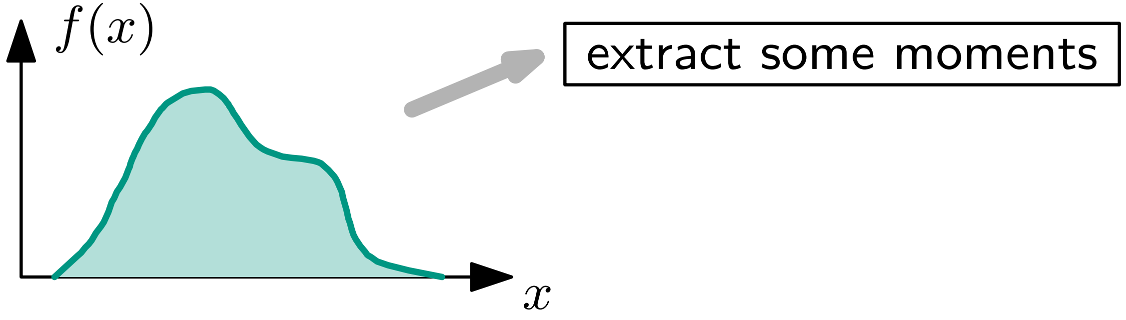


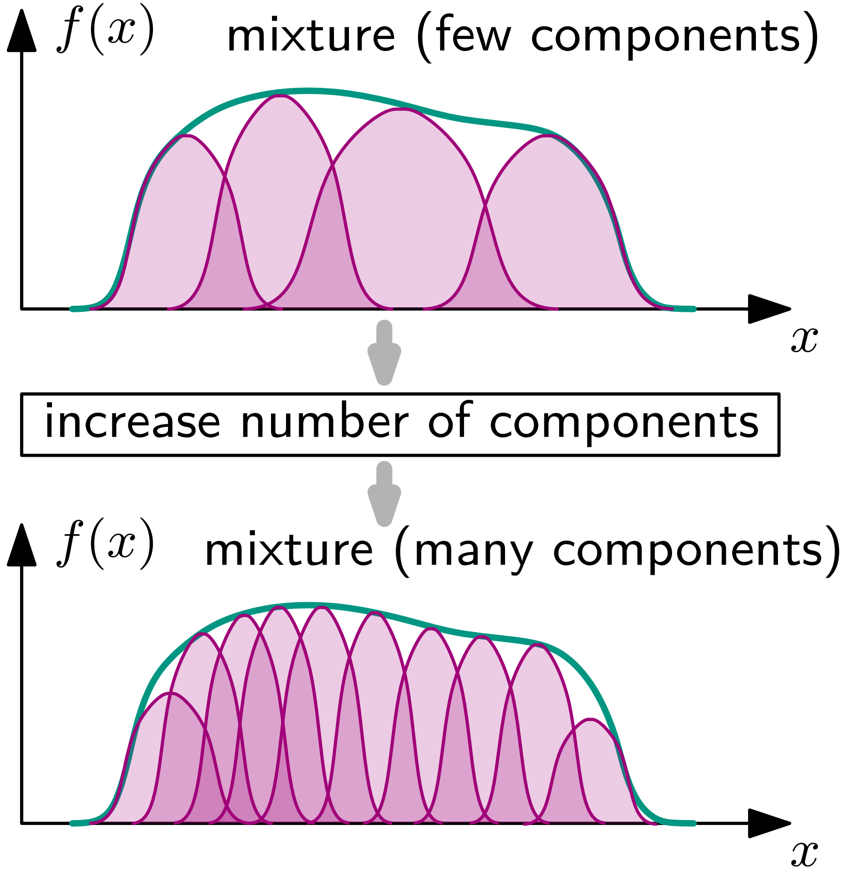


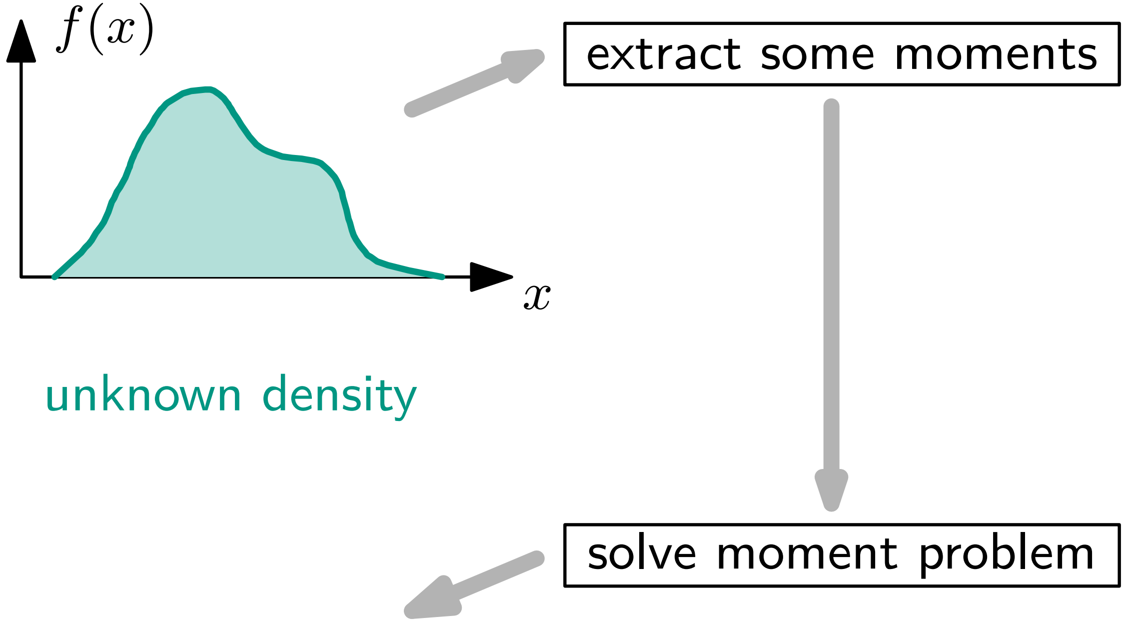
unknown density

Context & Motivation (4)

More typical use cases:

- Increasing number of components of mixture density
- Estimate continuous pdf from set of given moments





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Context & Motivation (4)

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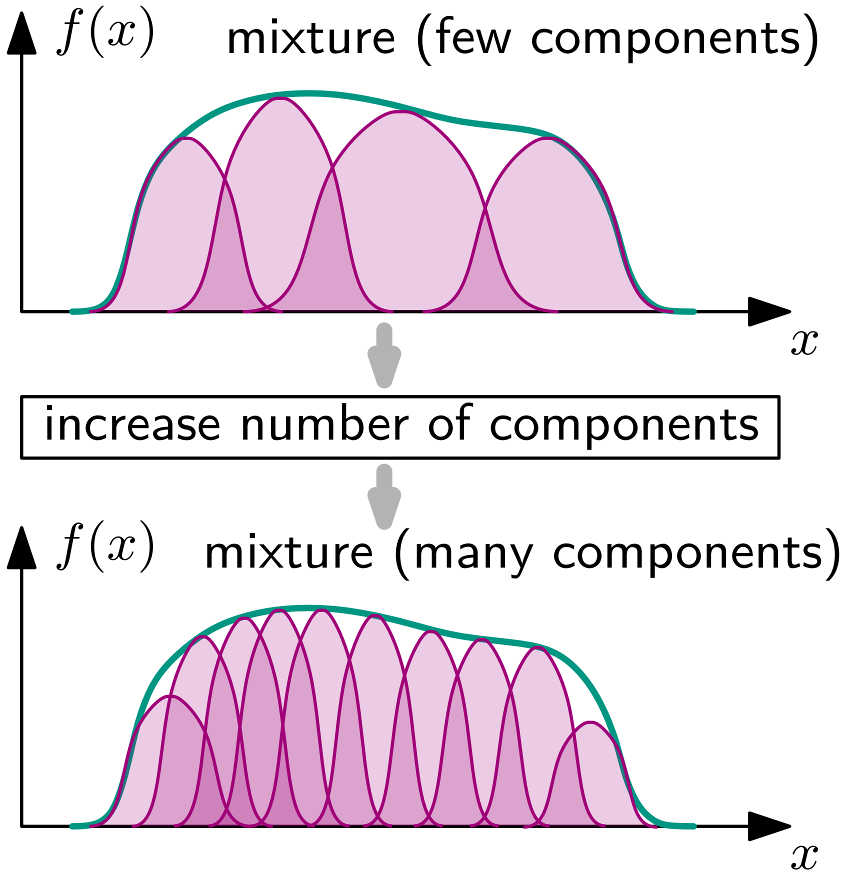

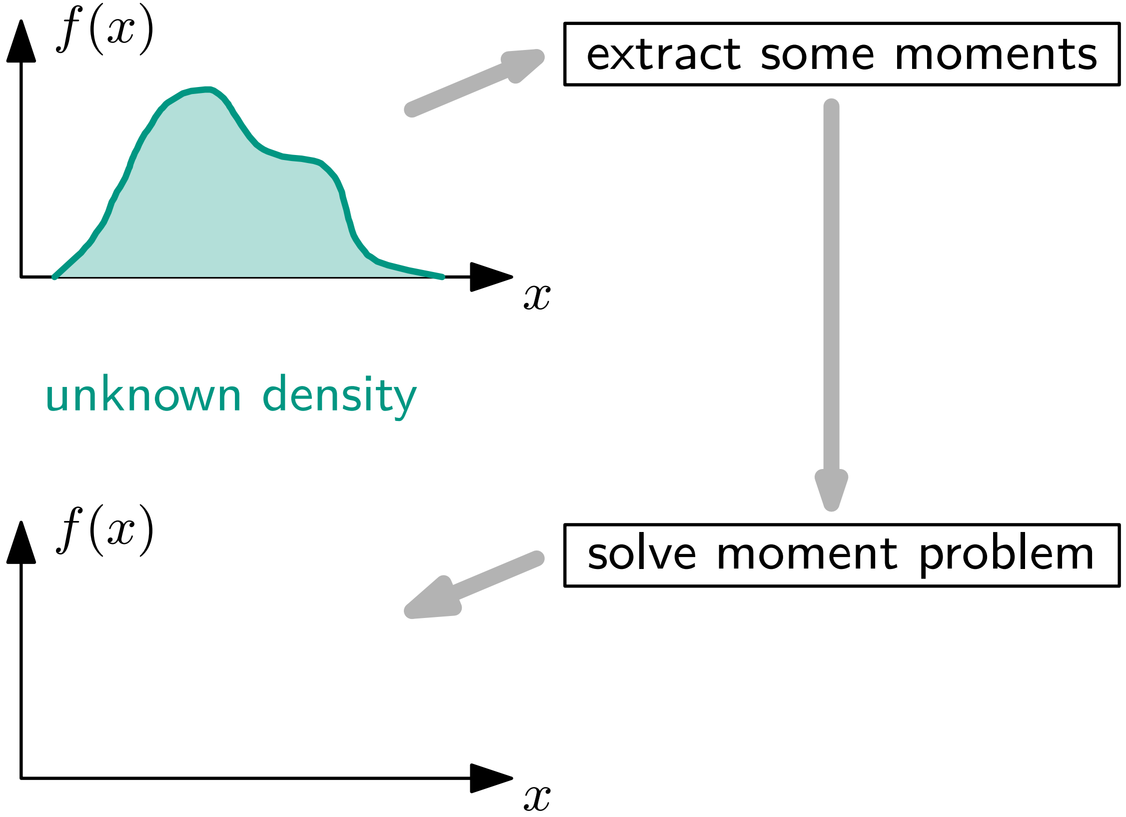

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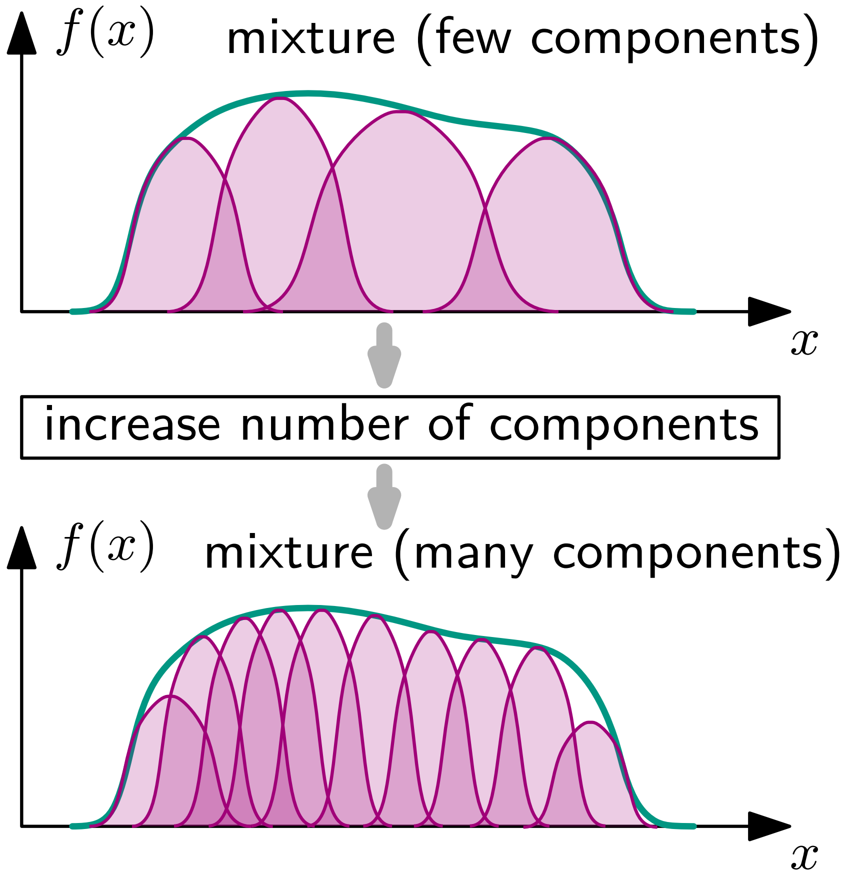

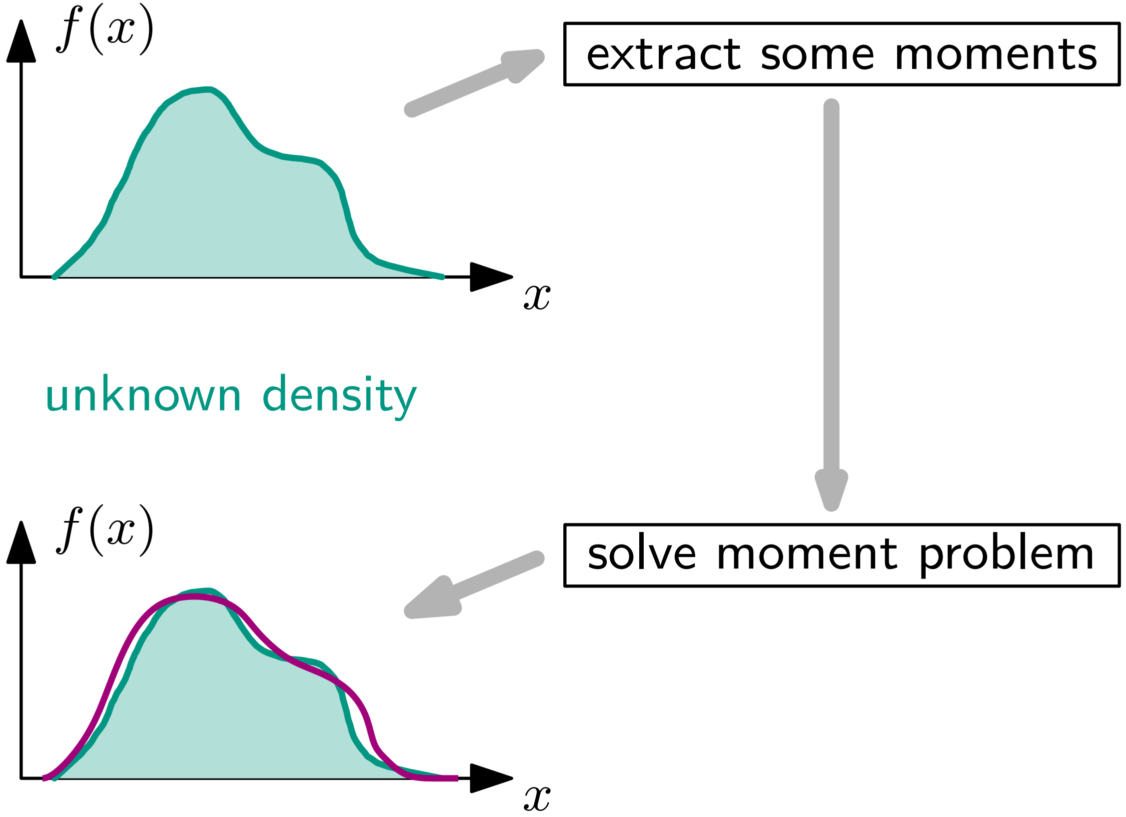

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Context & Motivation (5)

General setup:

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Context & Motivation (5)

Karlsruhe Institute of Technology

General setup:

- (Virtual) underlying true density $\tilde{f}(\underline{x})$

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Context & Motivation (5)

Karlsruhe Institute of Technology

General setup:

- (Virtual) underlying true density $\tilde{f}(\underline{x})$
- Discrete set of specifications S_i

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Context & Motivation (5)

General setup:

- (Virtual) underlying true density $\tilde{f}(\underline{x})$
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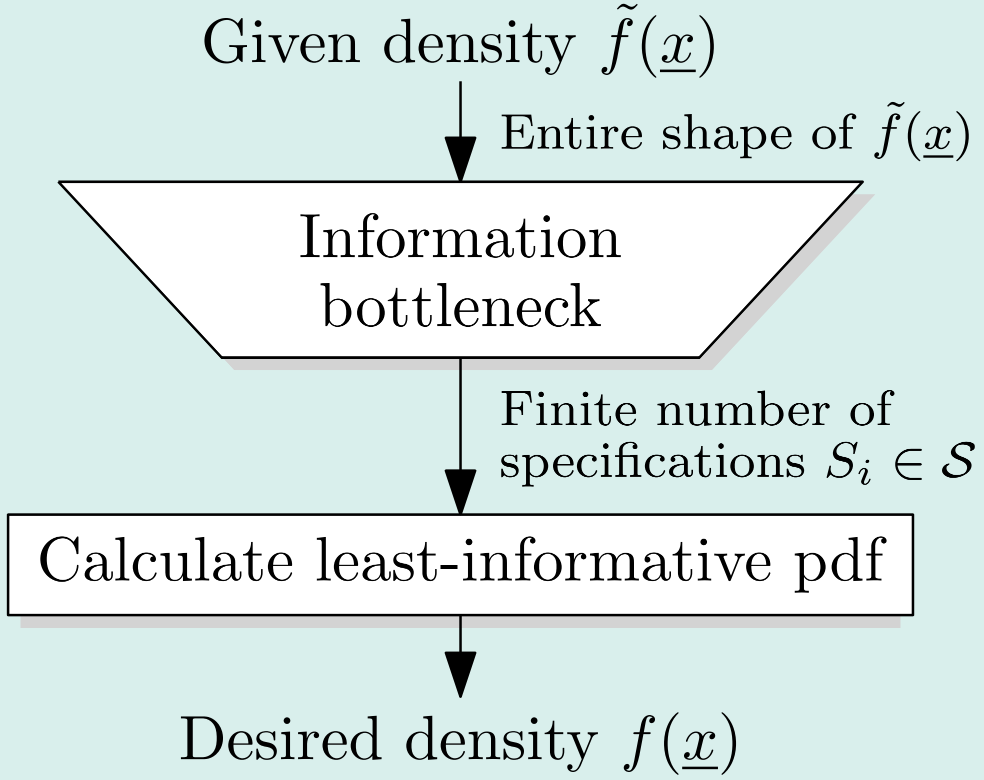

END