

Deterministic Sampling with Separation of Variables in Spherical Coordinates

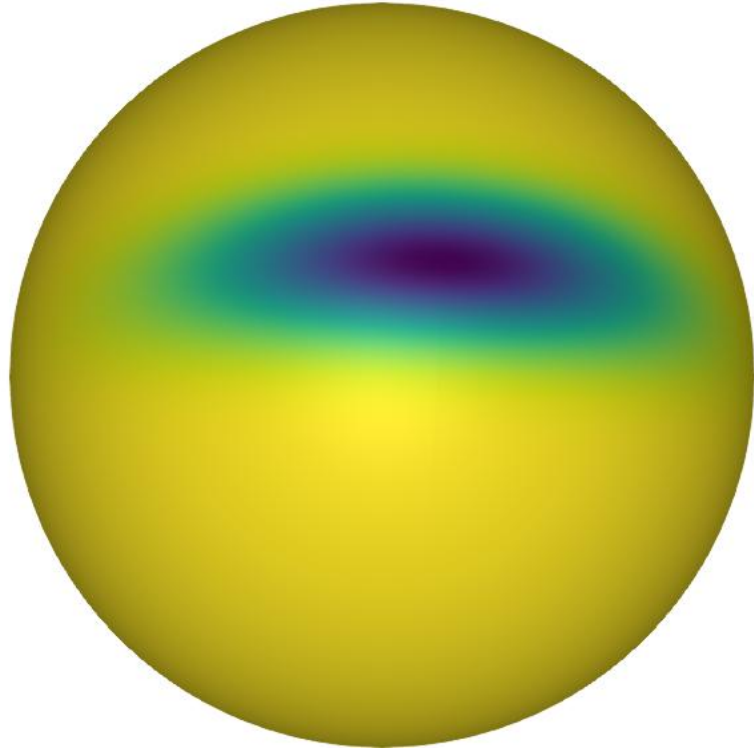
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FUSION 2025, Rio de Janeiro

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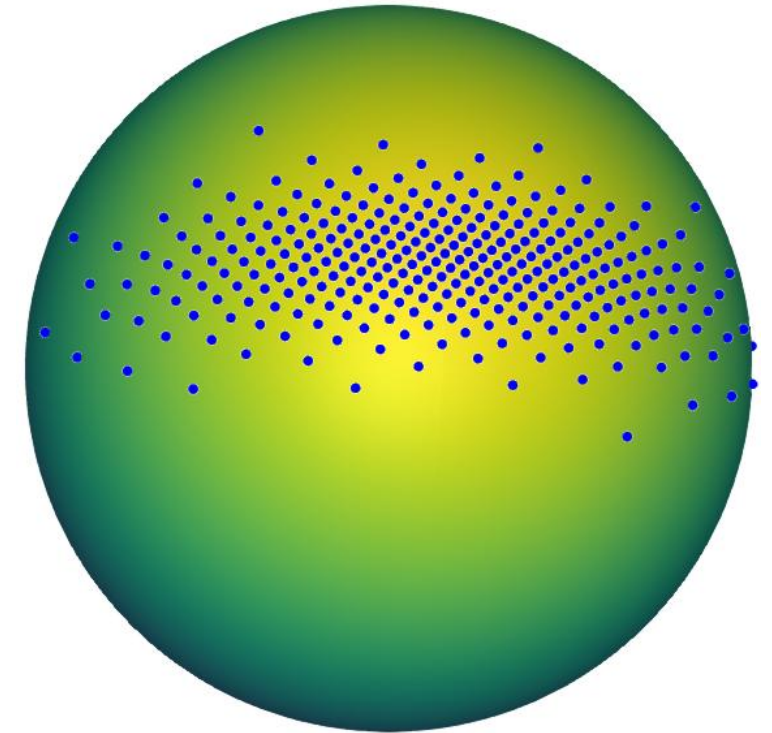
isas.iar.kit.edu

Spherically Separable Sampling

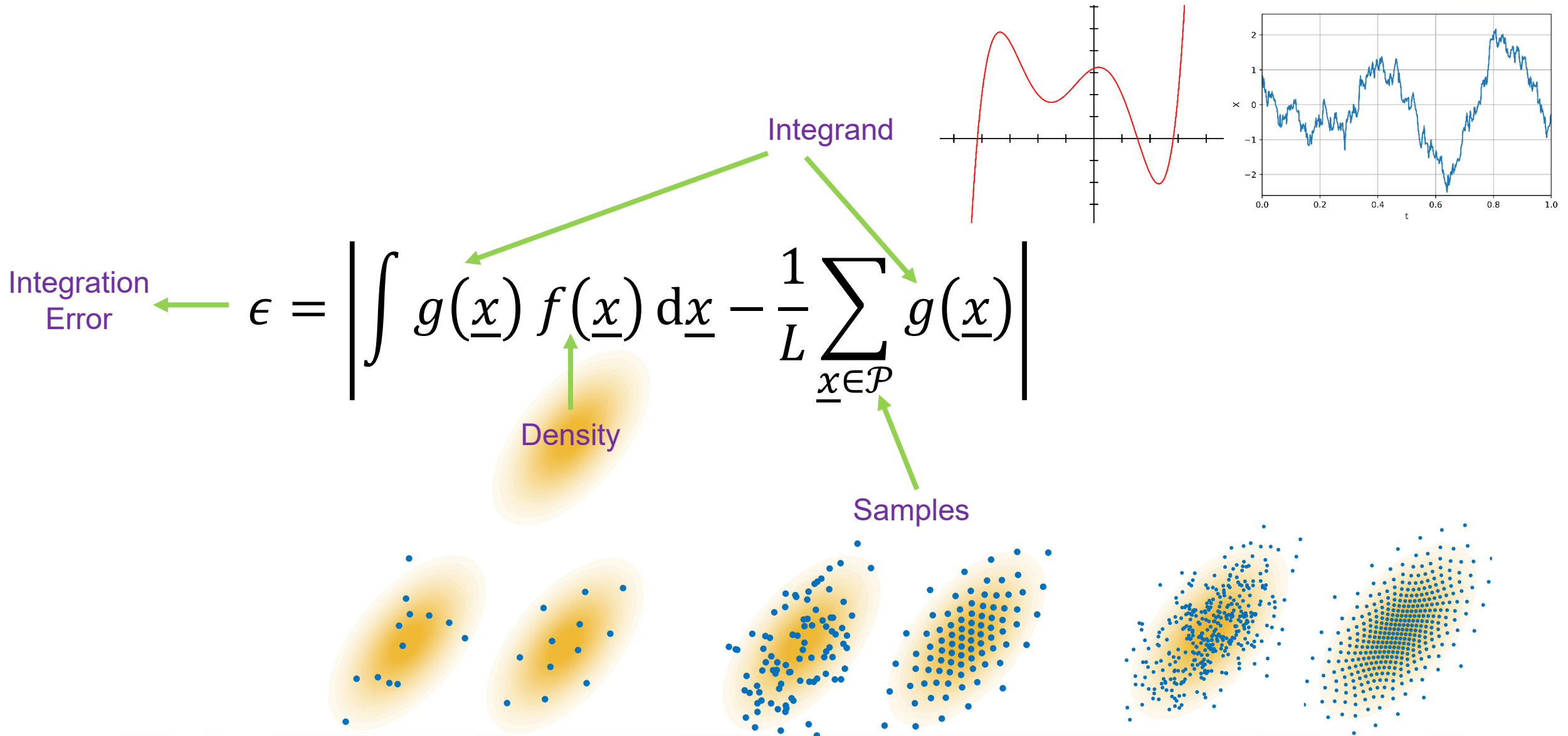


Methods

- Low-Discrepancy Uniform Samples
- Orthogonal Transformation

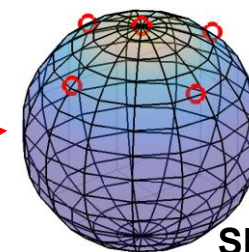


Numerical Integration Error

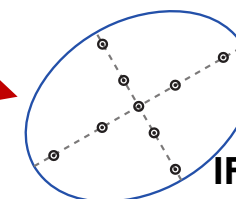


State of Art – Deterministic Sampling

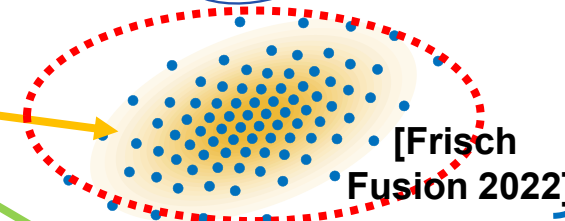
Dimension	Domain	Density	Method	#Samples
1	\mathbb{R}^d	Uniform	Random-iid	5
2	\mathbb{S}^d	Gaussian	Unscented	9
3	\mathbb{T}^d	Gaussian Mixture	Main Axes	30
4		Truncated Normal	Regular Grid	150
5		Wrapped normal	Low-Discrepancy + Orth. Trafo	300
6		Von Mises–Fisher	LCD	1000
7		Bingham	PSD	3000



[Kurz SPL 2016]



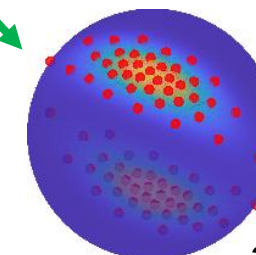
[Huber IFAC 2008]



[Frisch Fusion 2022]



[Hanebeck CISS 2020]



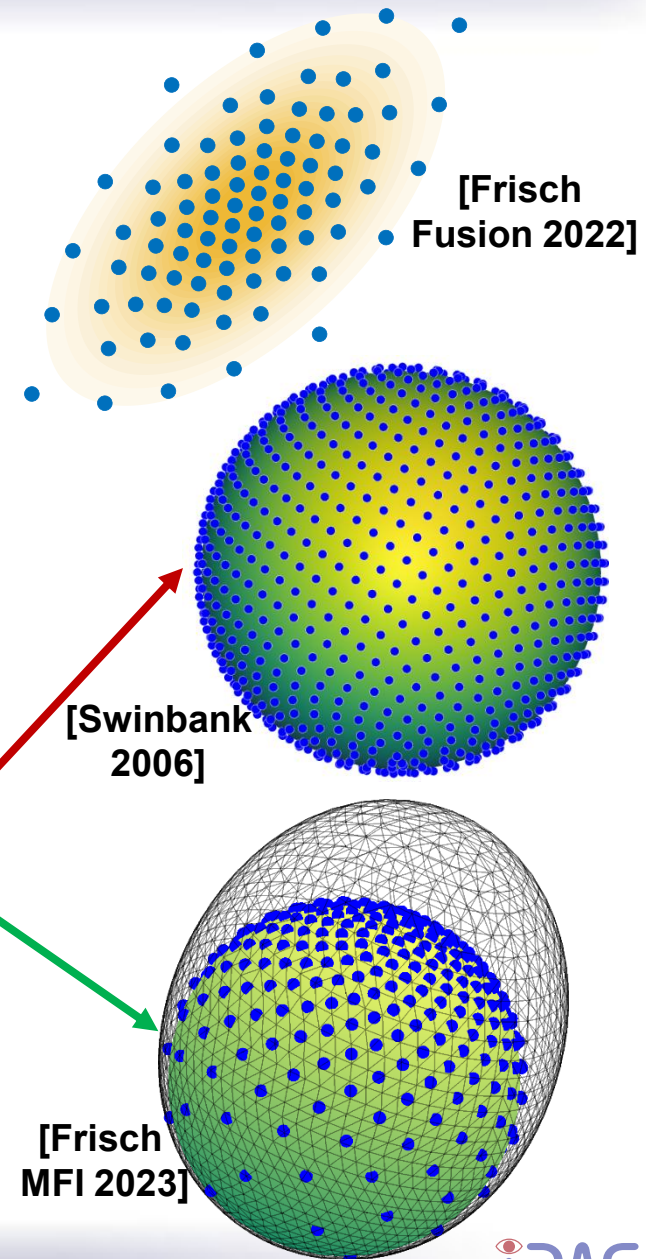
[Kailai Li arXiv 2021]

Fast

Slow

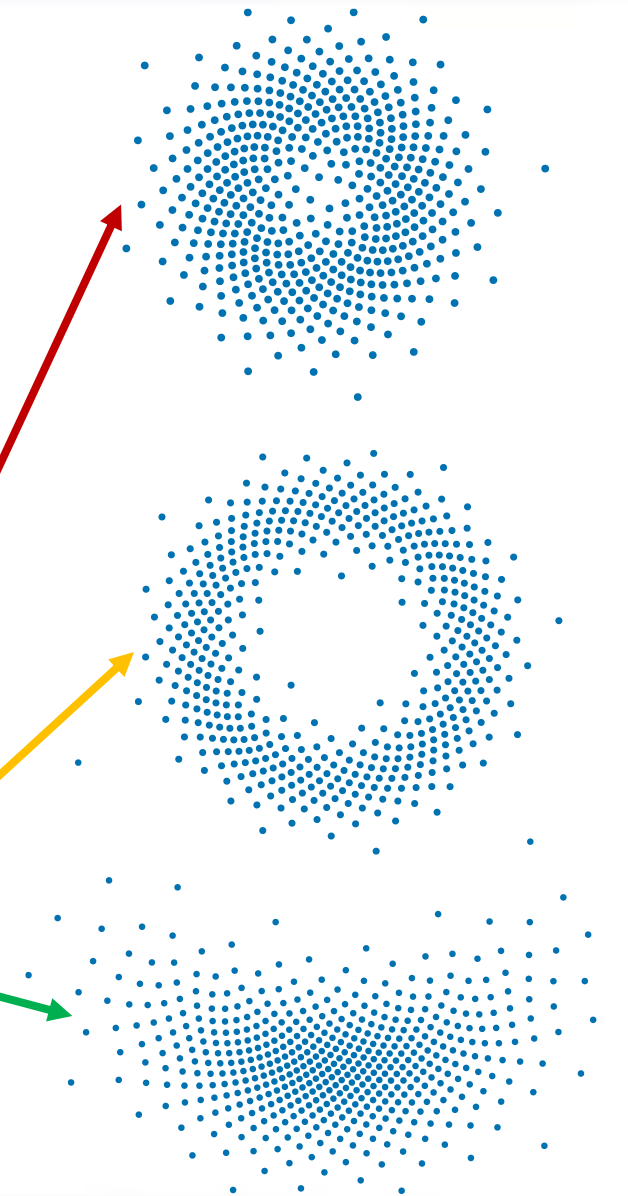
State of Art – Low-Discrepancy + Orthogonal Transformation Sampling

Dimension	Domain	Density	Method	#Samples
1	\mathbb{R}^d	Uniform	Random-iid	5
2	\mathbb{S}^d	Gaussian	Unscented	9
3	\mathbb{T}^d	Gaussian Mixture	Main Axes	30
4		Truncated Normal	Regular Grid	100
5		Wrapped normal	Low-Discrepancy + Orth. Trafo	300
6		Von Mises – Fisher	LCD	1000
7		Bingham-like	PCD	3000



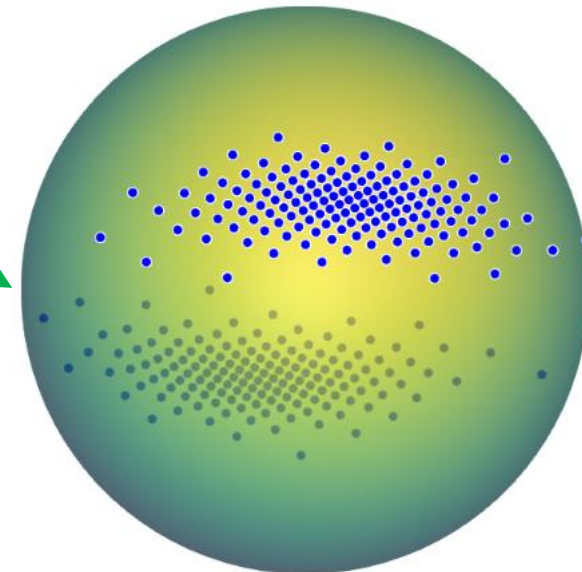
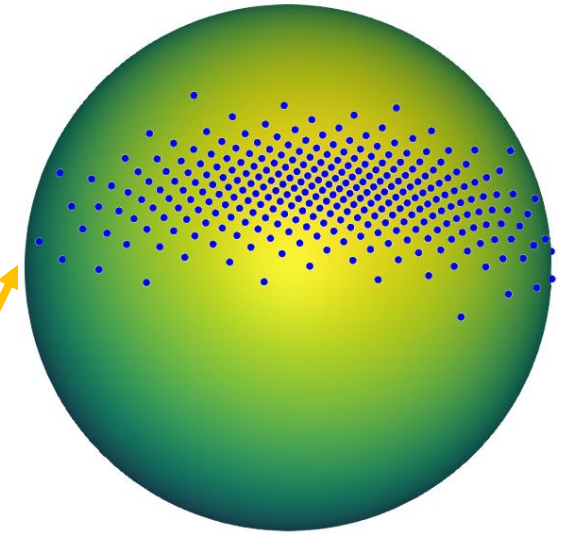
Proposed – \mathbb{R}^2

Dimension	Domain	Density	Method	#Samples
1	\mathbb{R}^d	Uniform	Random iid	5
2	\mathbb{S}^d	Gaussian	Unscented	9
3	\mathbb{T}^d	Gaussian Mixture	Main Axes	30
4		Truncated Normal	Regular Grid	100
5		Wrapped normal	Fibonacci / Low Discrepancy	300
6		Von Mises	LCD	500
7		Bingham	PCD	3000



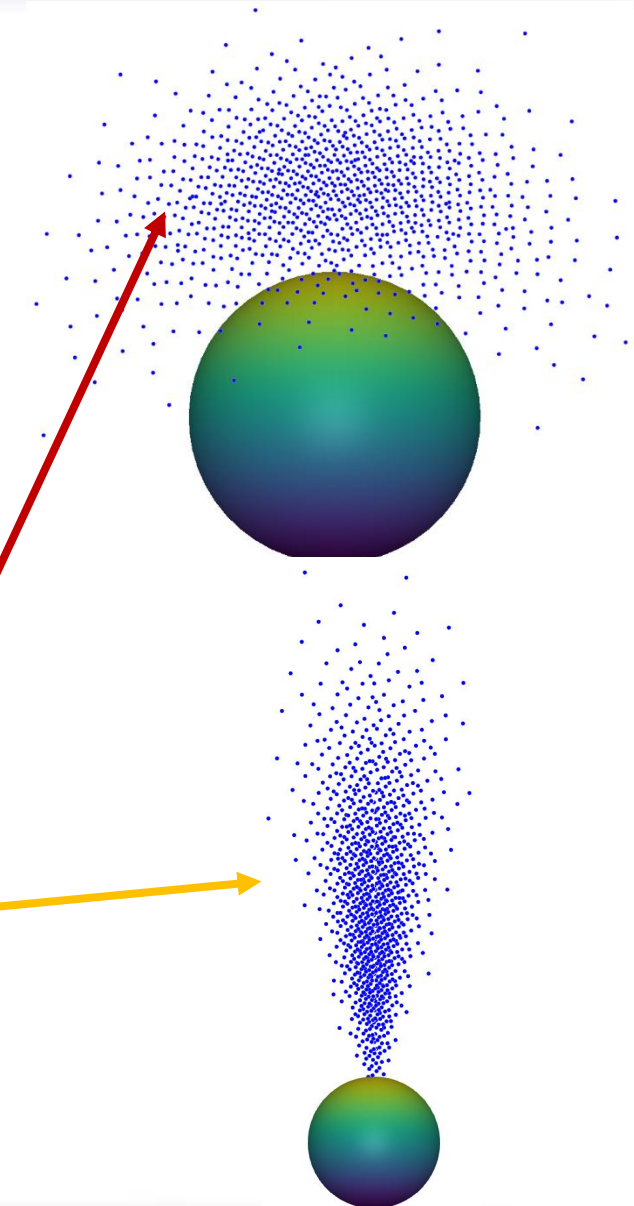
Proposed – \mathbb{S}^2

Dimension	Domain	Density	Method	#Samples
1	\mathbb{R}^d	Uniform	Random-iid	5
2	\mathbb{S}^d	Gaussian	Unscented	9
3	\mathbb{T}^d	Gaussian Mixture	Main Axes	30
4		Truncated Normal	Regular Grid	100
5		Wrapped normal	Low-Discrepancy + Orth. Trafo	300
6		Von Mises x von Mises	LCD	1000
7		Bingham-like	PCD	3000

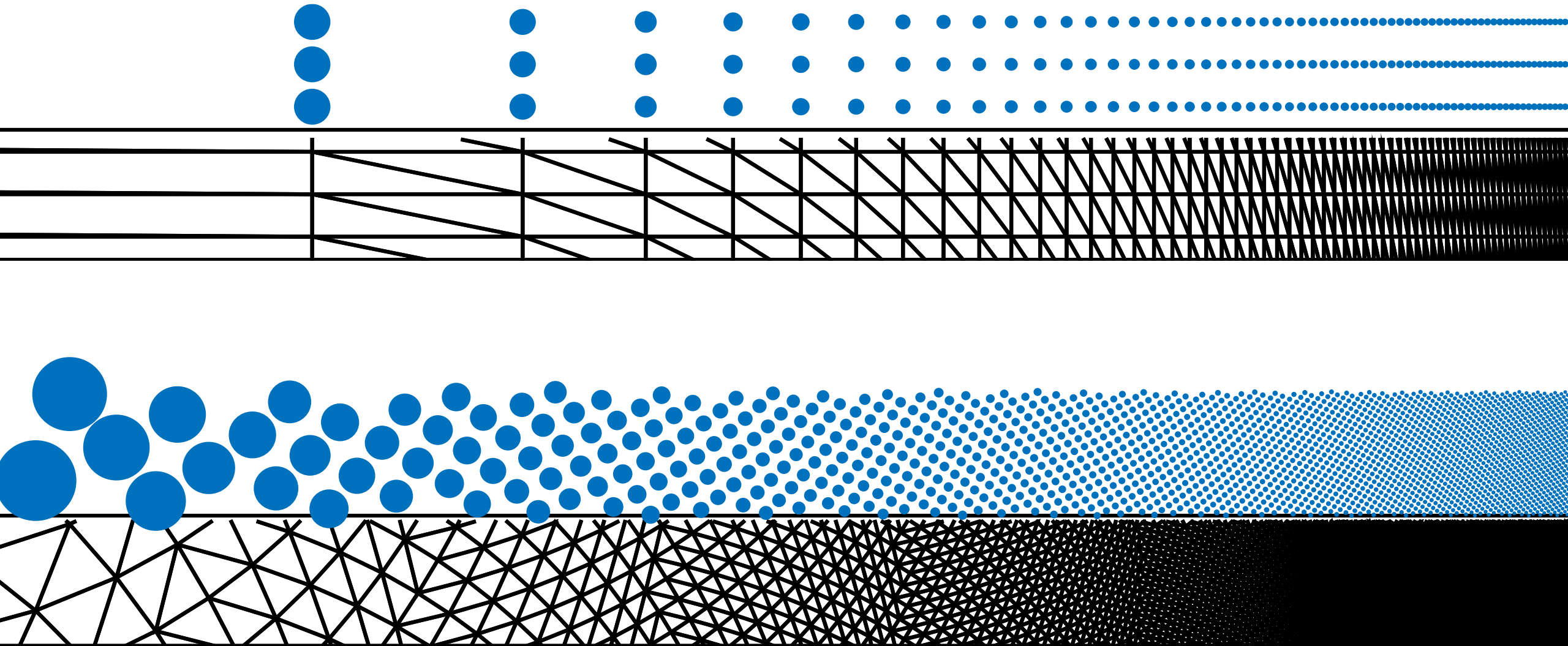


Proposed – \mathbb{R}^3

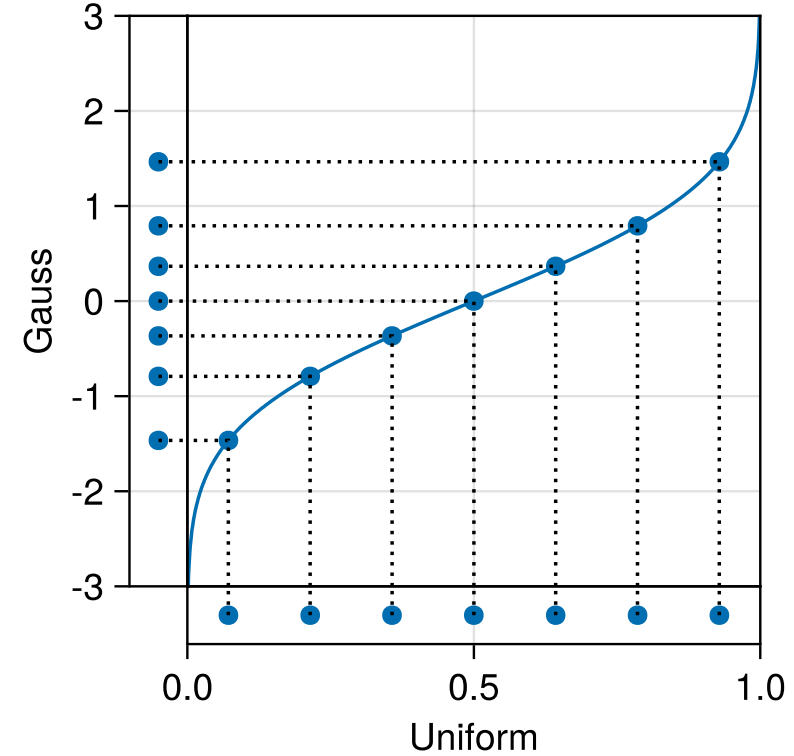
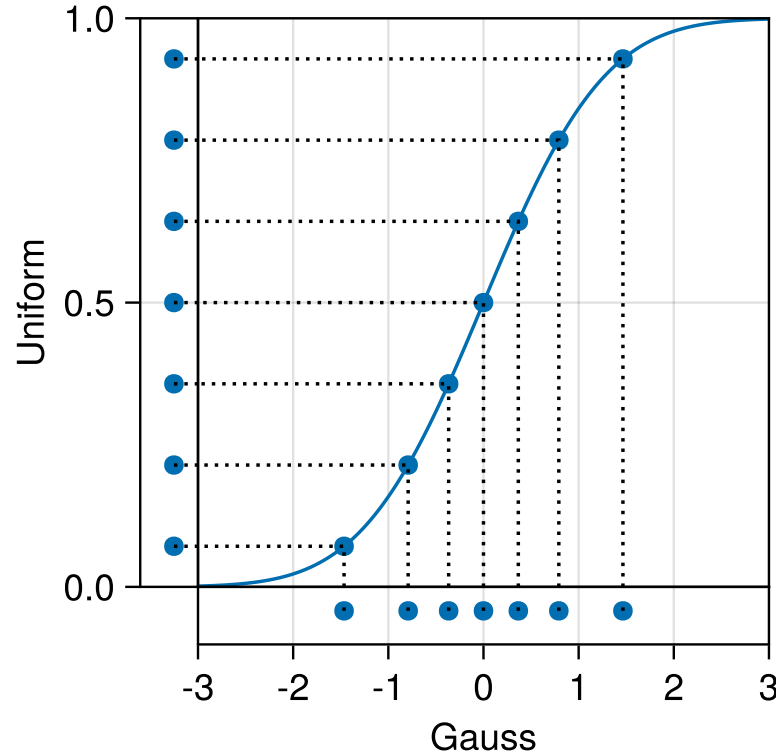
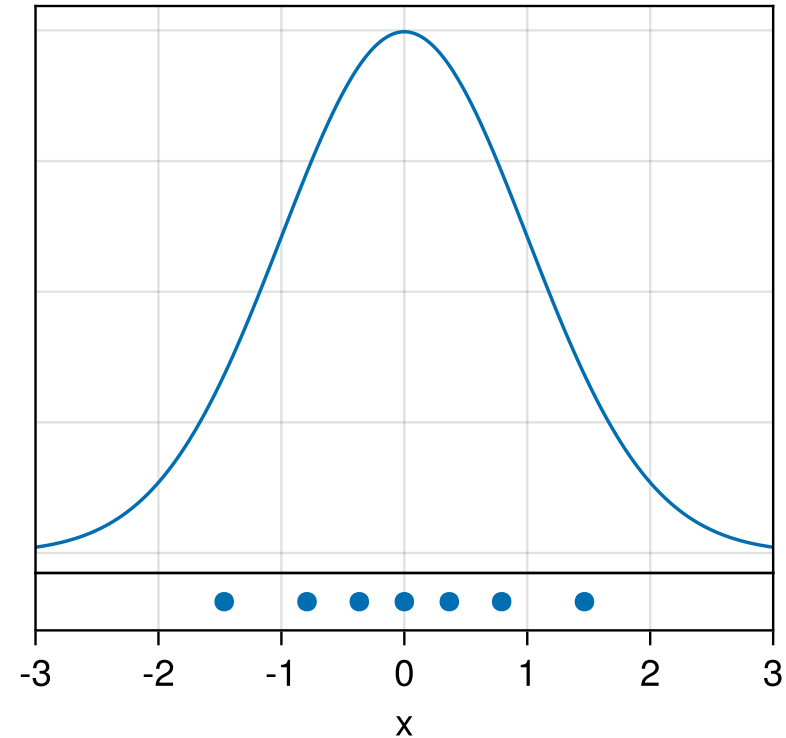
Dimension	Domain	Density	Method	#Samples
1	\mathbb{R}^d	Uniform	Random-iid	5
2	\mathbb{S}^d	Gaussian	Unscented	9
3	\mathbb{T}^d	Gaussian Mixture	Main Axes	30
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5		Wrapped normal	Fibonacci / Low-Discrepancy	300
6		Von Mises x von Mises	LCD	500
7		Bingham	PCD	3000



Anisotropic Rescaling



Inverse Transform Sampling – 1D Gauss

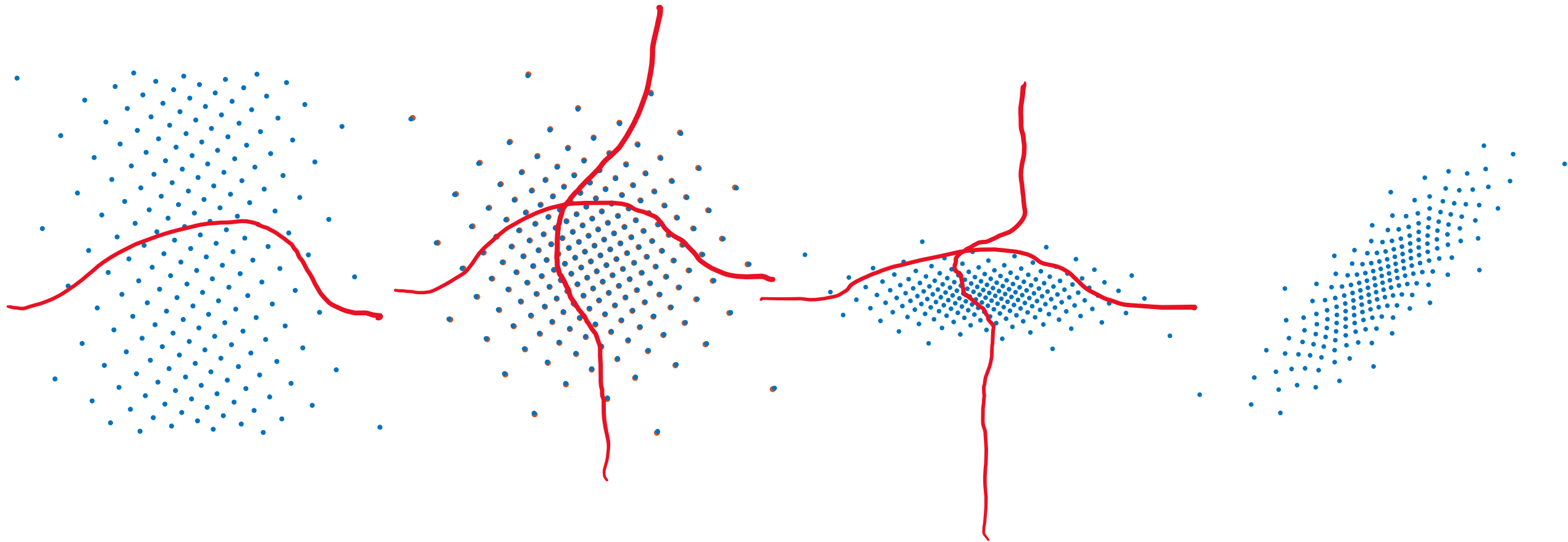


$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$F_{\theta}(\theta) = \frac{1}{2} \left(1 + \operatorname{erf} \left(\frac{x}{\sqrt{2}} \right) \right)$$

$$F_{\theta}^{-1}(p) = \sqrt{2} \operatorname{erf}^{-1}(2p - 1)$$

Gaussian Sampling




Gaussian Samples



- Gaussian

$$f(\underline{x}, \underline{m}, \sigma) = \exp \left\{ -\frac{(\underline{x} - \underline{m})^\top (\underline{x} - \underline{m})}{2 \sigma^2} \right\}$$

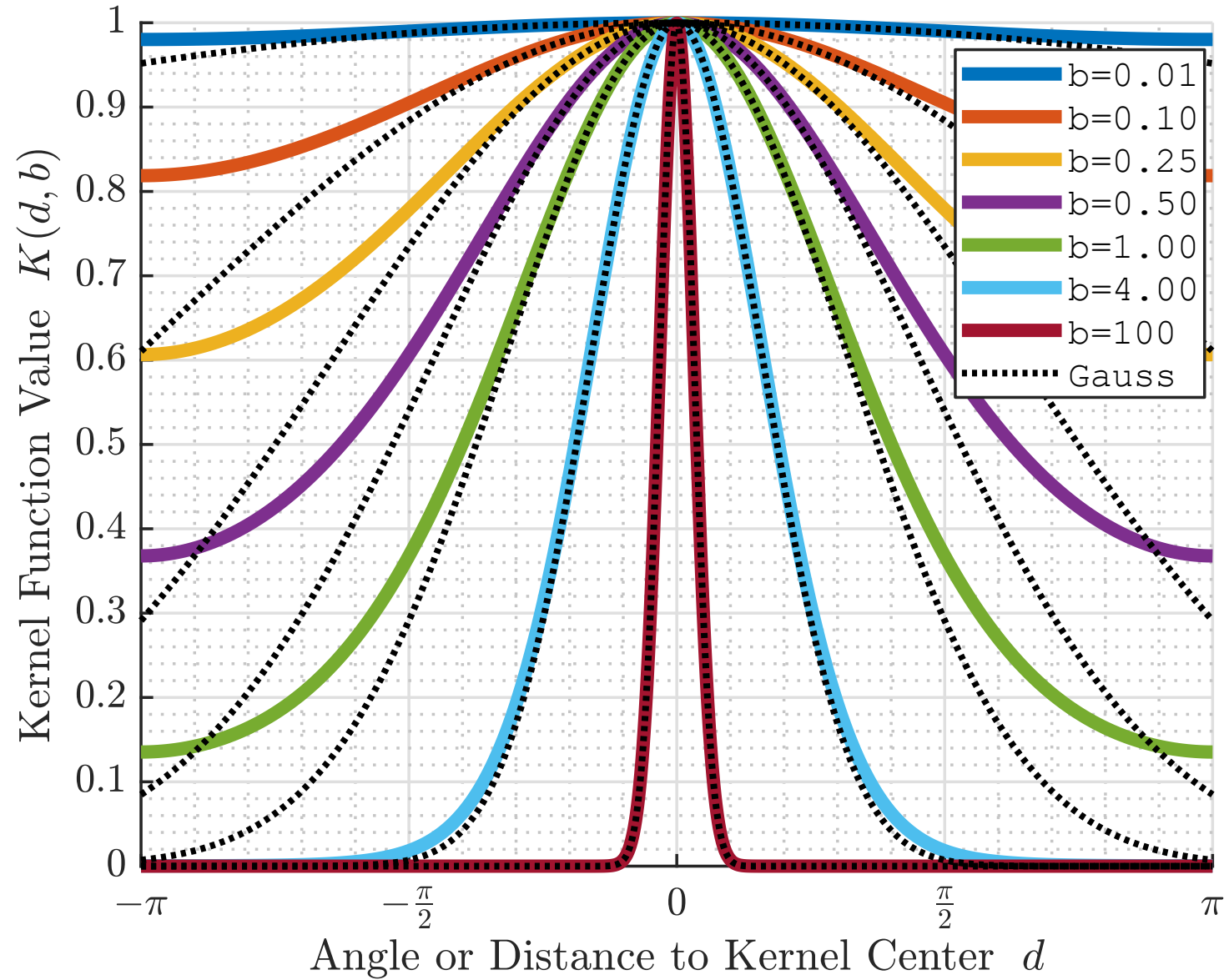
$$f(d, \sigma) = \exp \left\{ \frac{1}{\sigma^2} \cdot \left(-\frac{d^2}{2} \right) \right\}$$


- Von Mises–Fisher

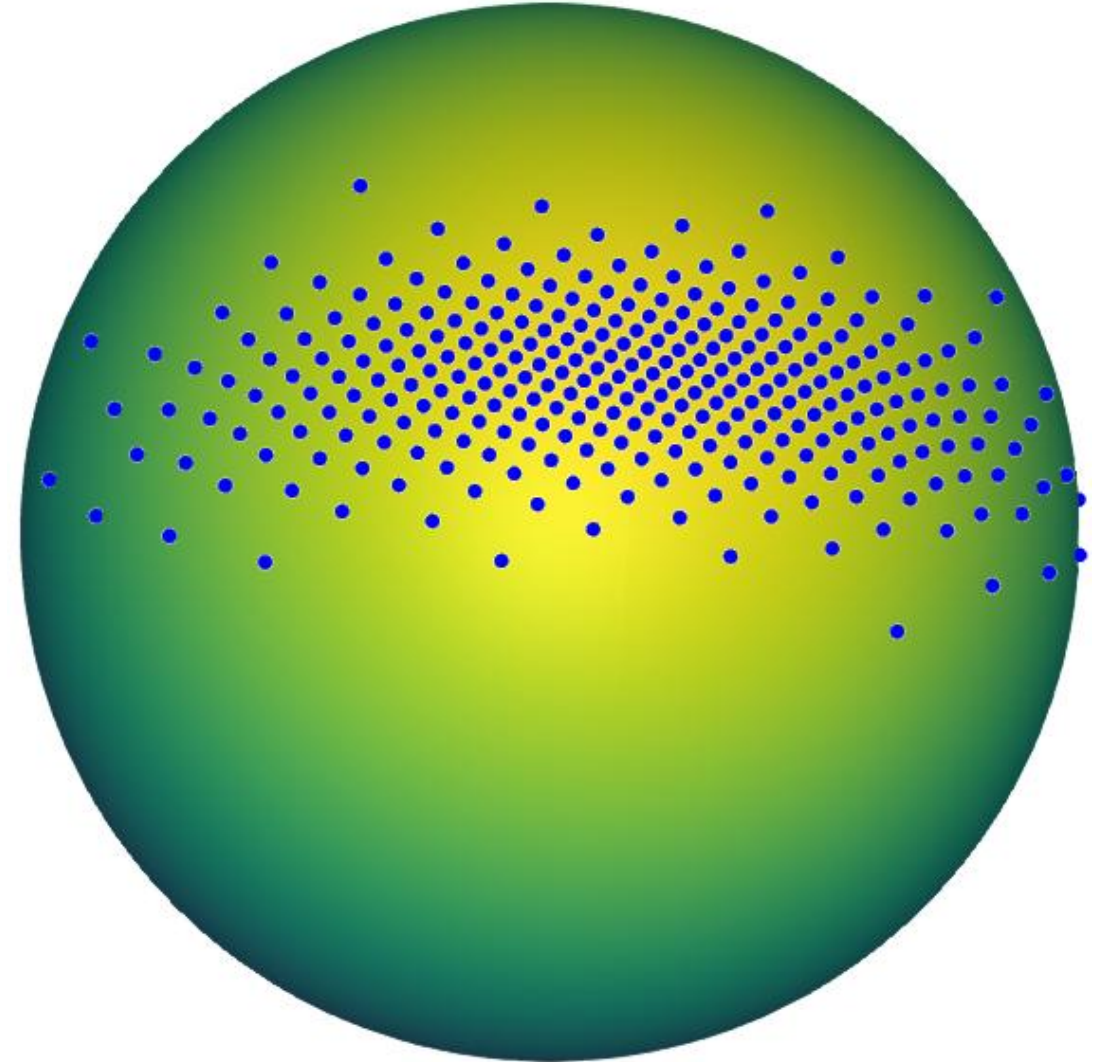
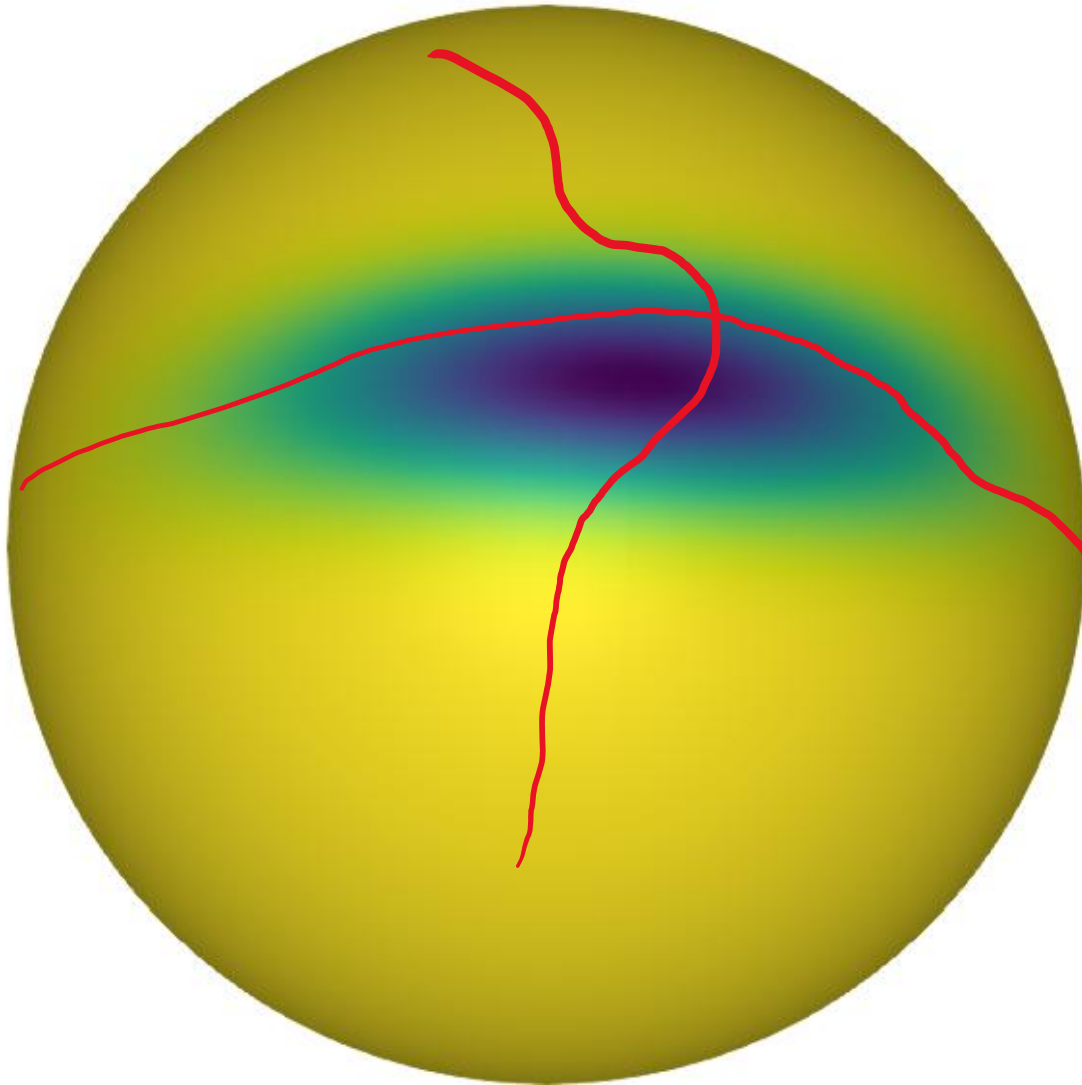
$$f(d, \kappa) = \exp\{\kappa \cdot \cos d\}$$


$$f(\underline{x}, \underline{m}, \kappa) = \exp\{\kappa \cdot \underline{x}^\top \underline{m}\}$$

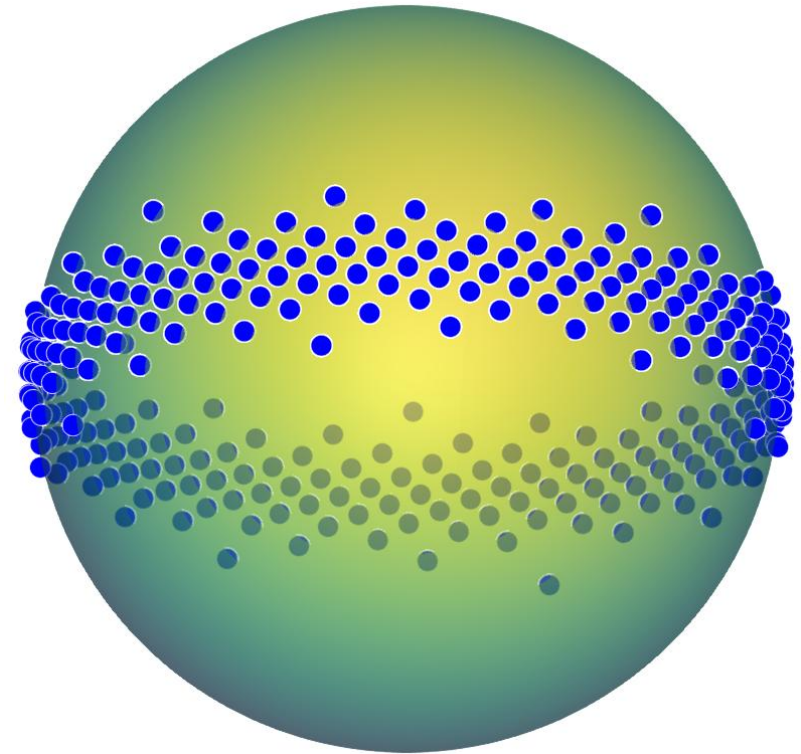
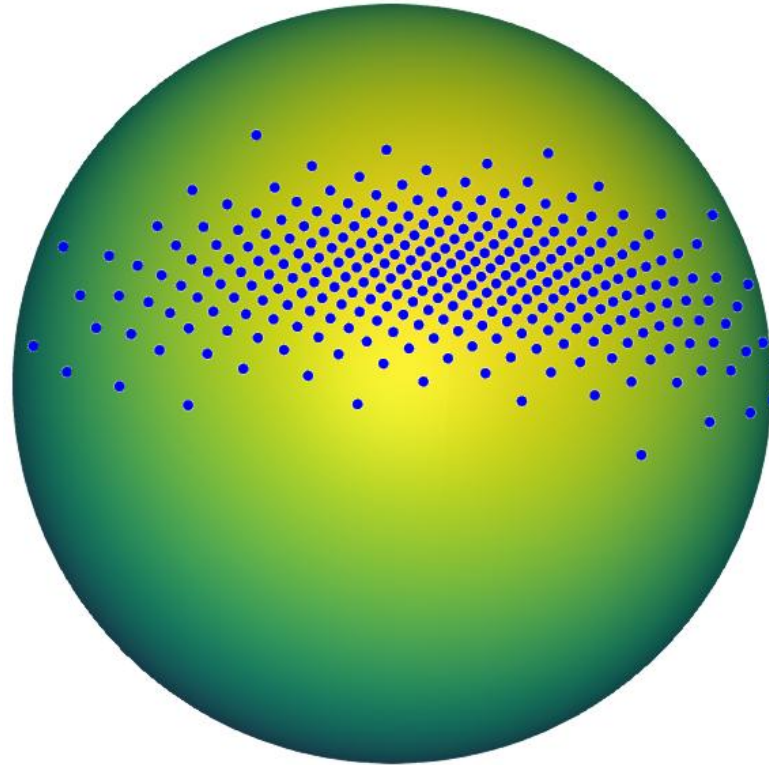
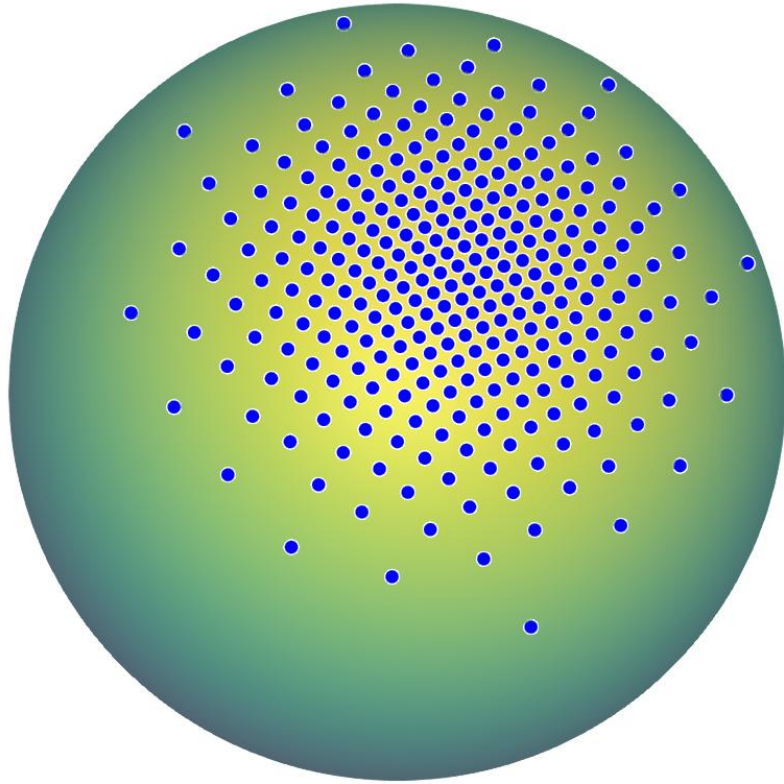
Gauss \leftrightarrow VM



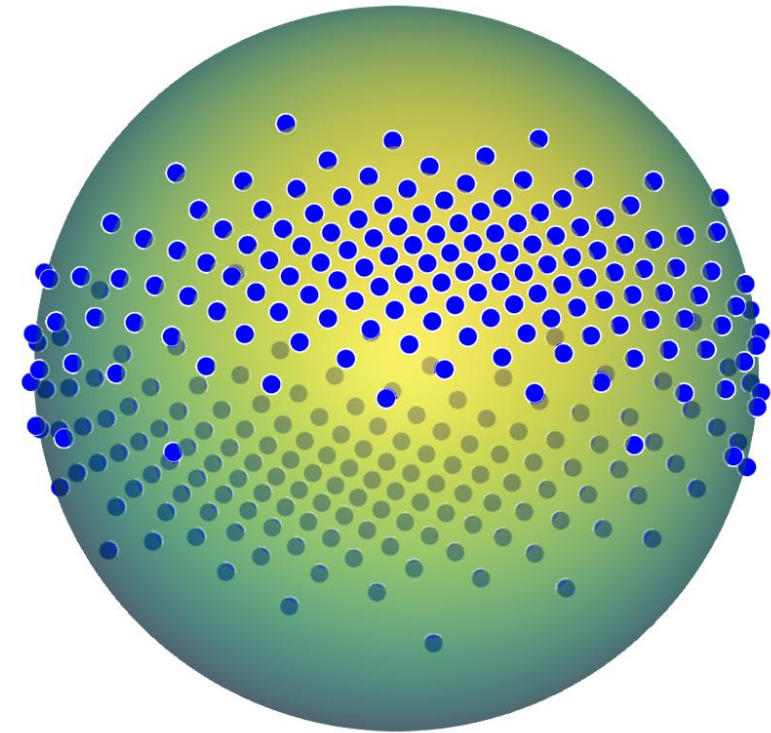
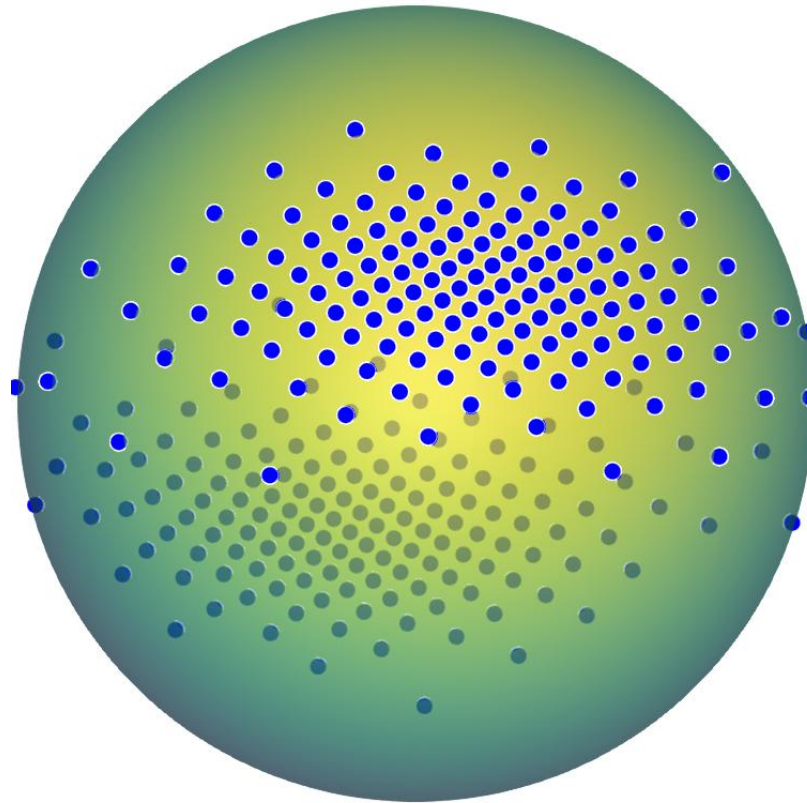
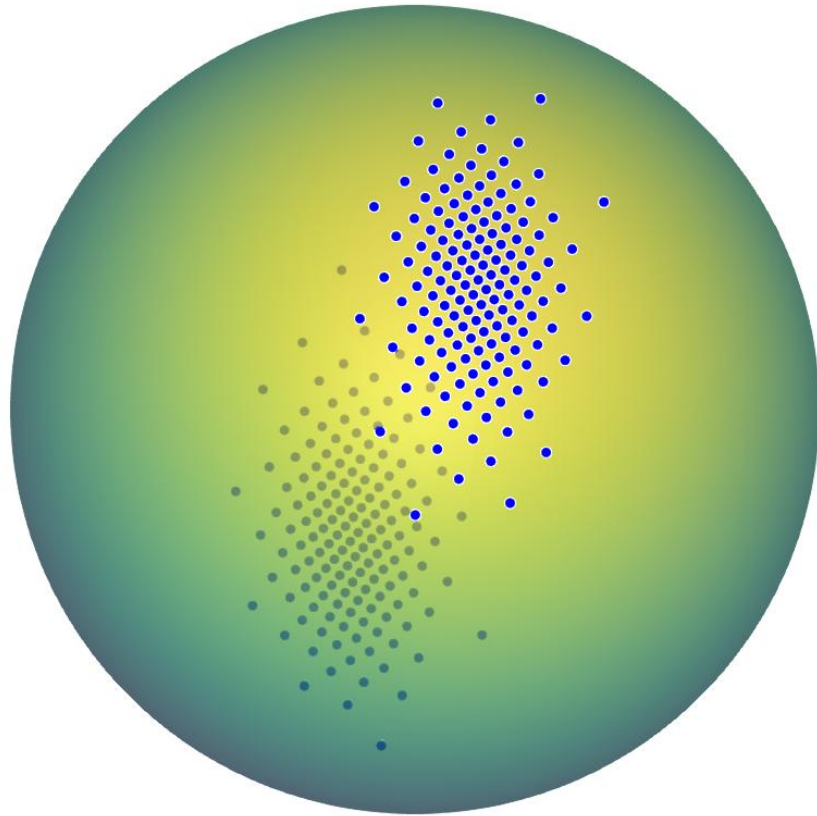
VM x VM Sampling



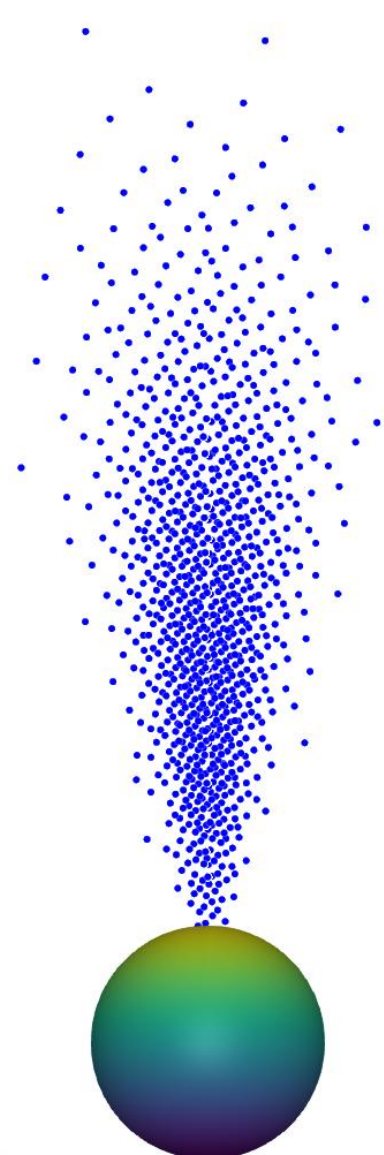
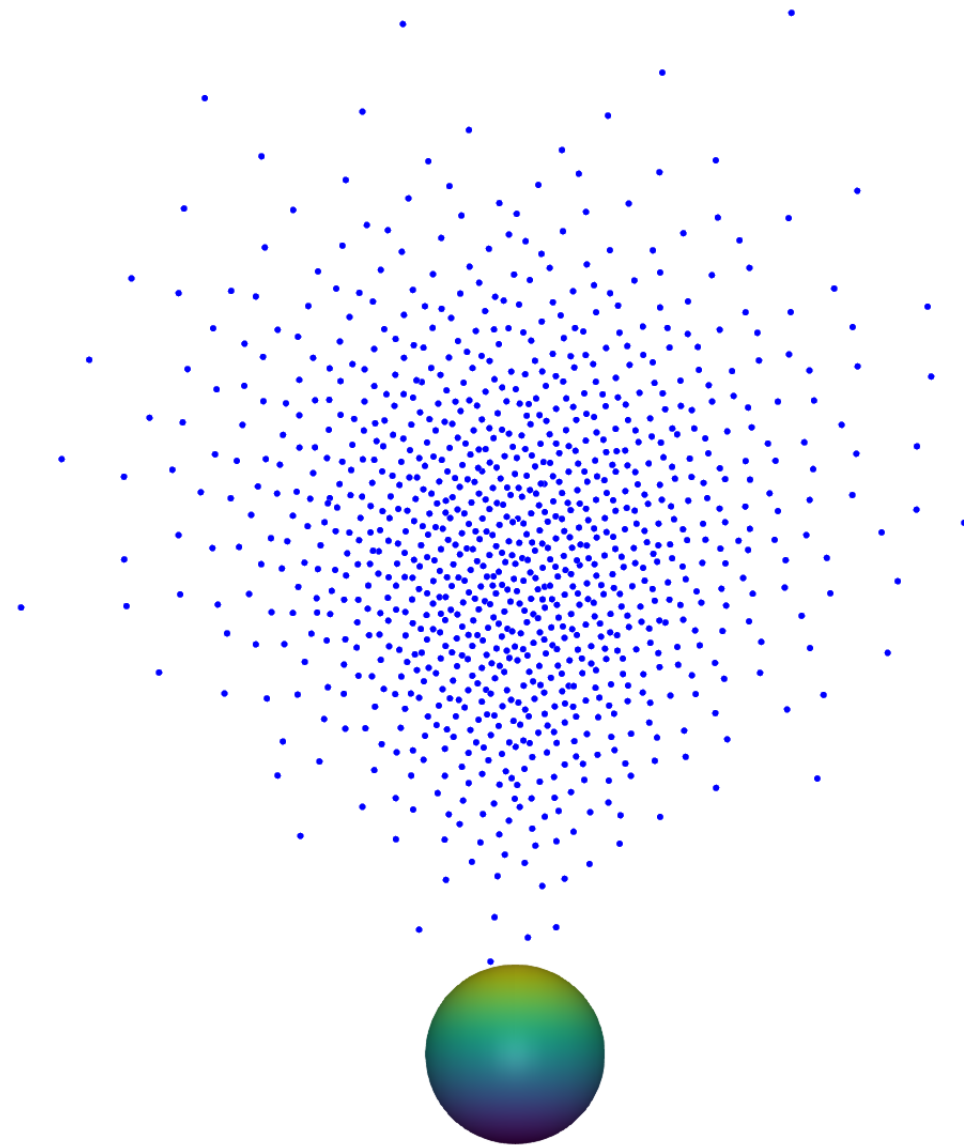
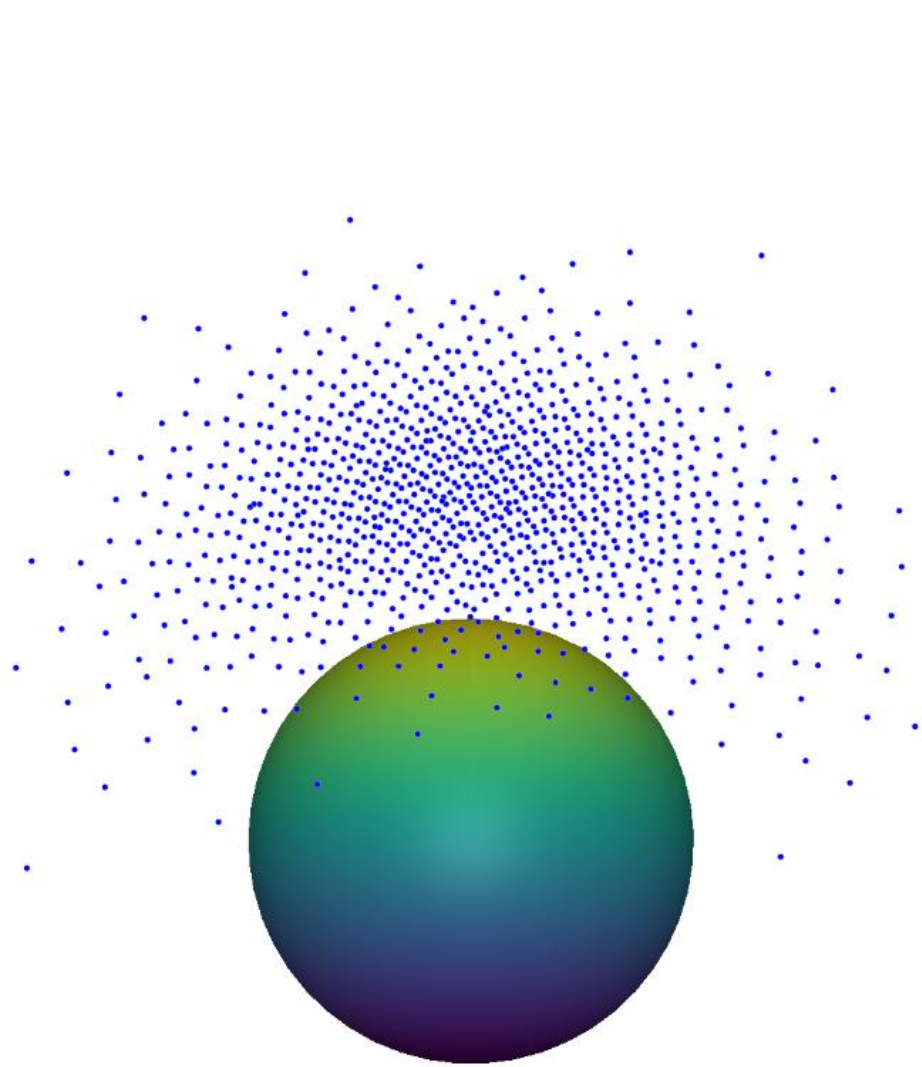
“Anisotropic VMF” Density



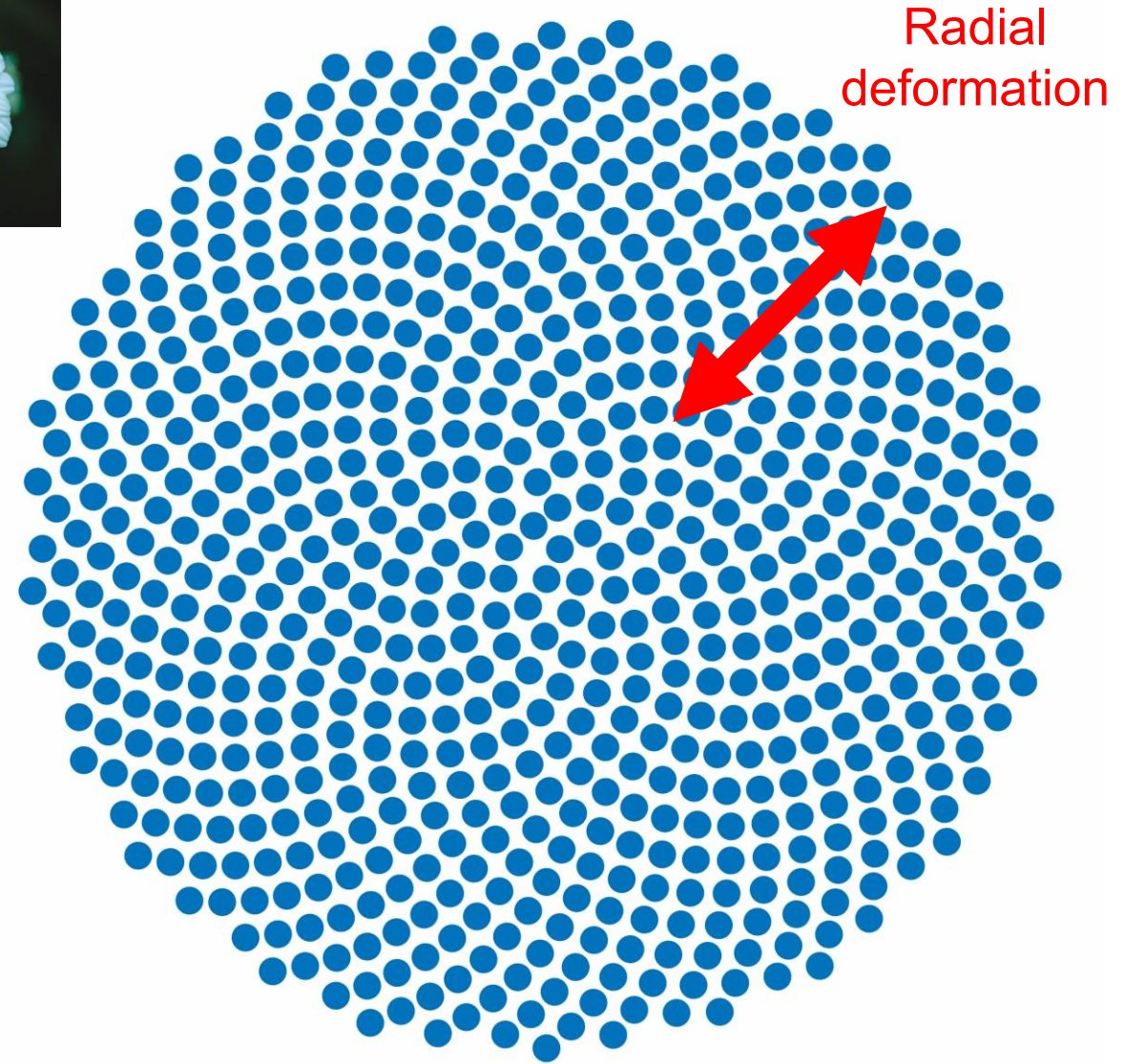
“Bingham-Like” Density



3D “Radar Tracking” Density

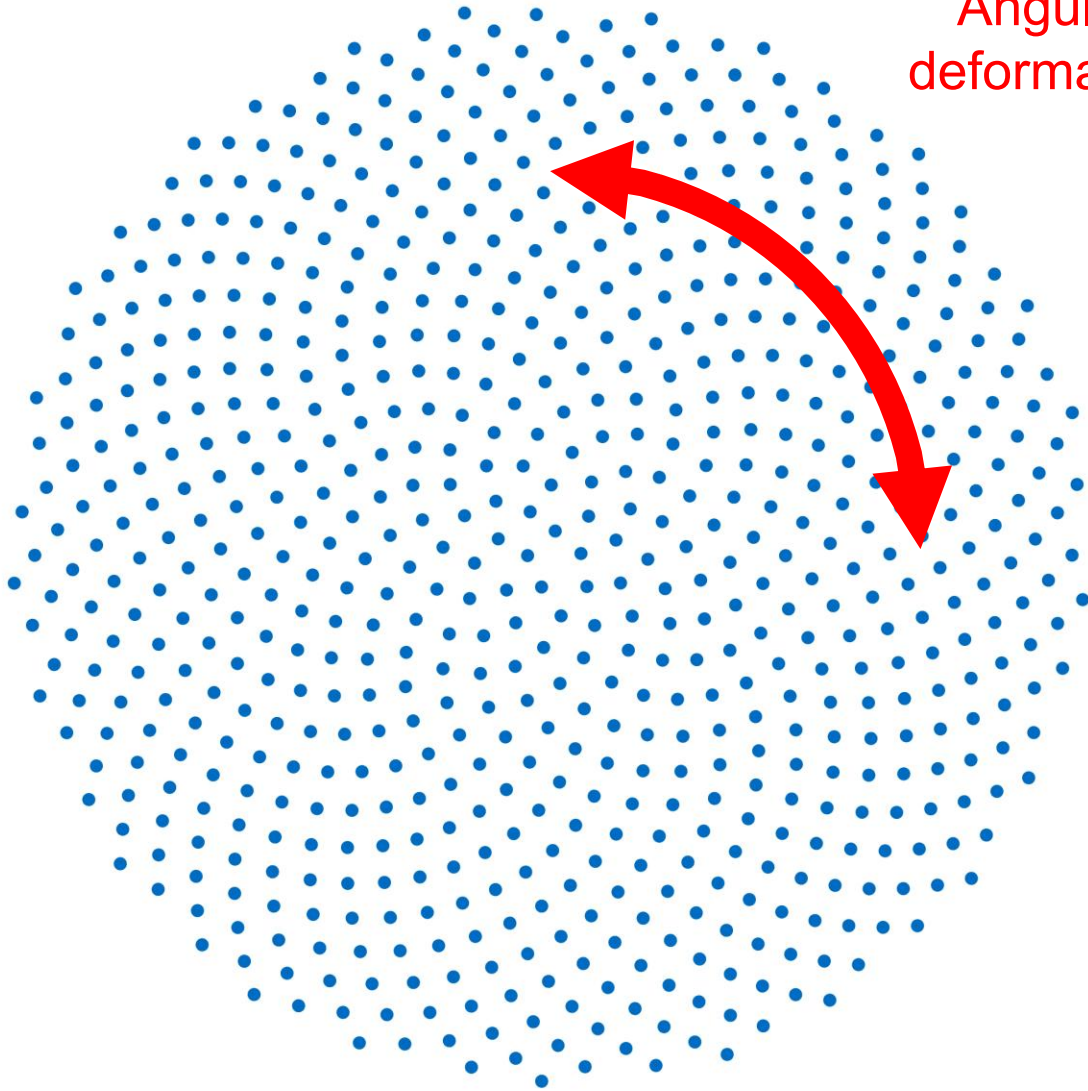


Polar Fibonacci Grid

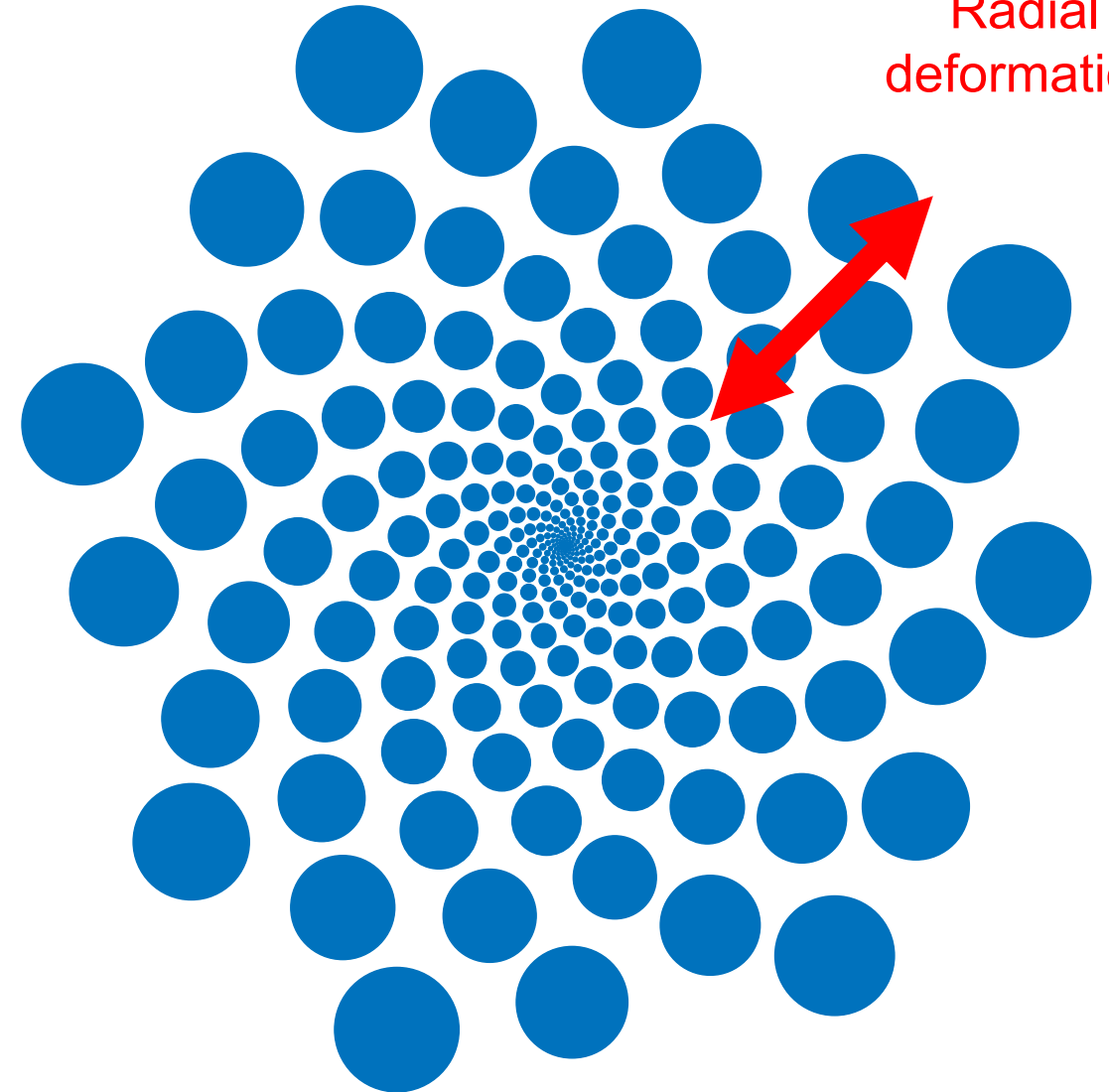


Polar Fibonacci Grid

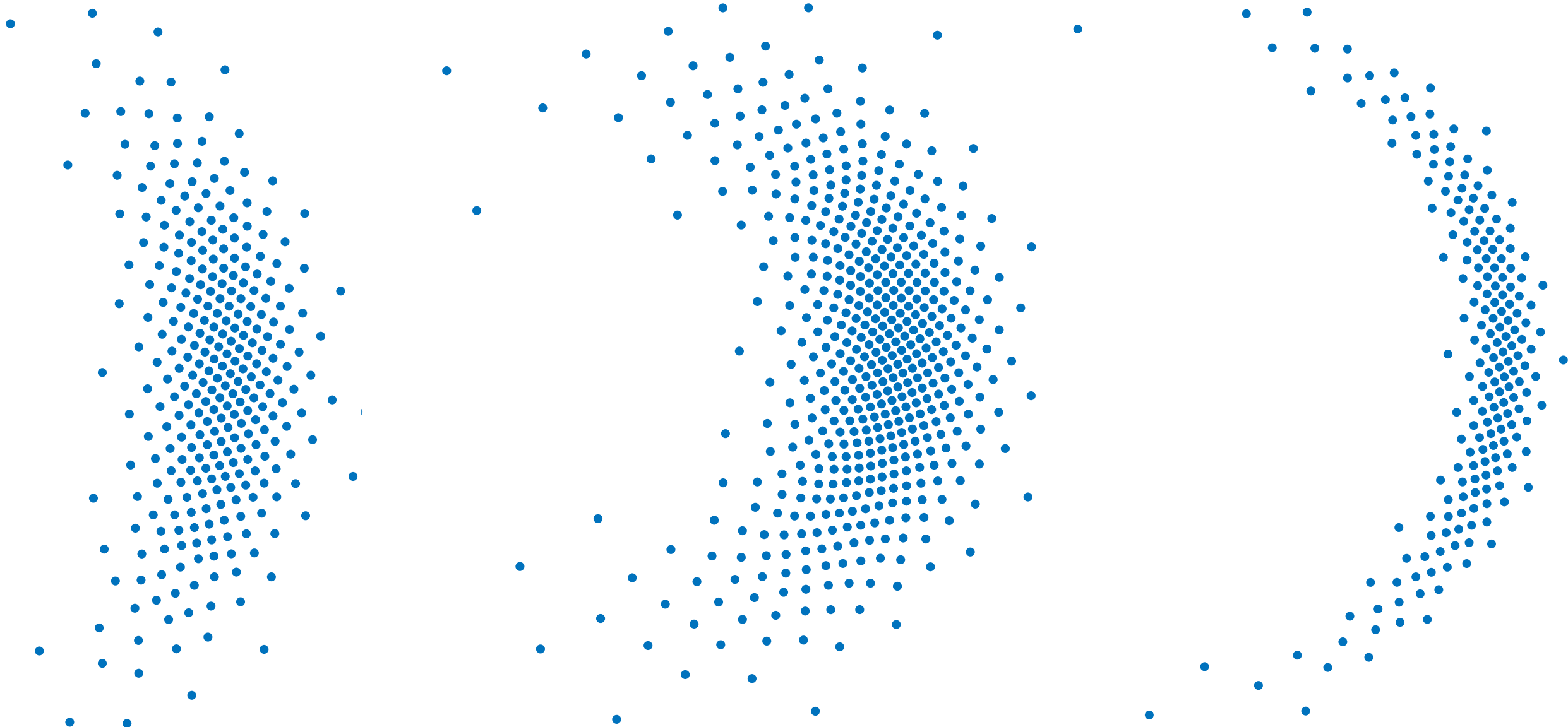
Angular
deformation



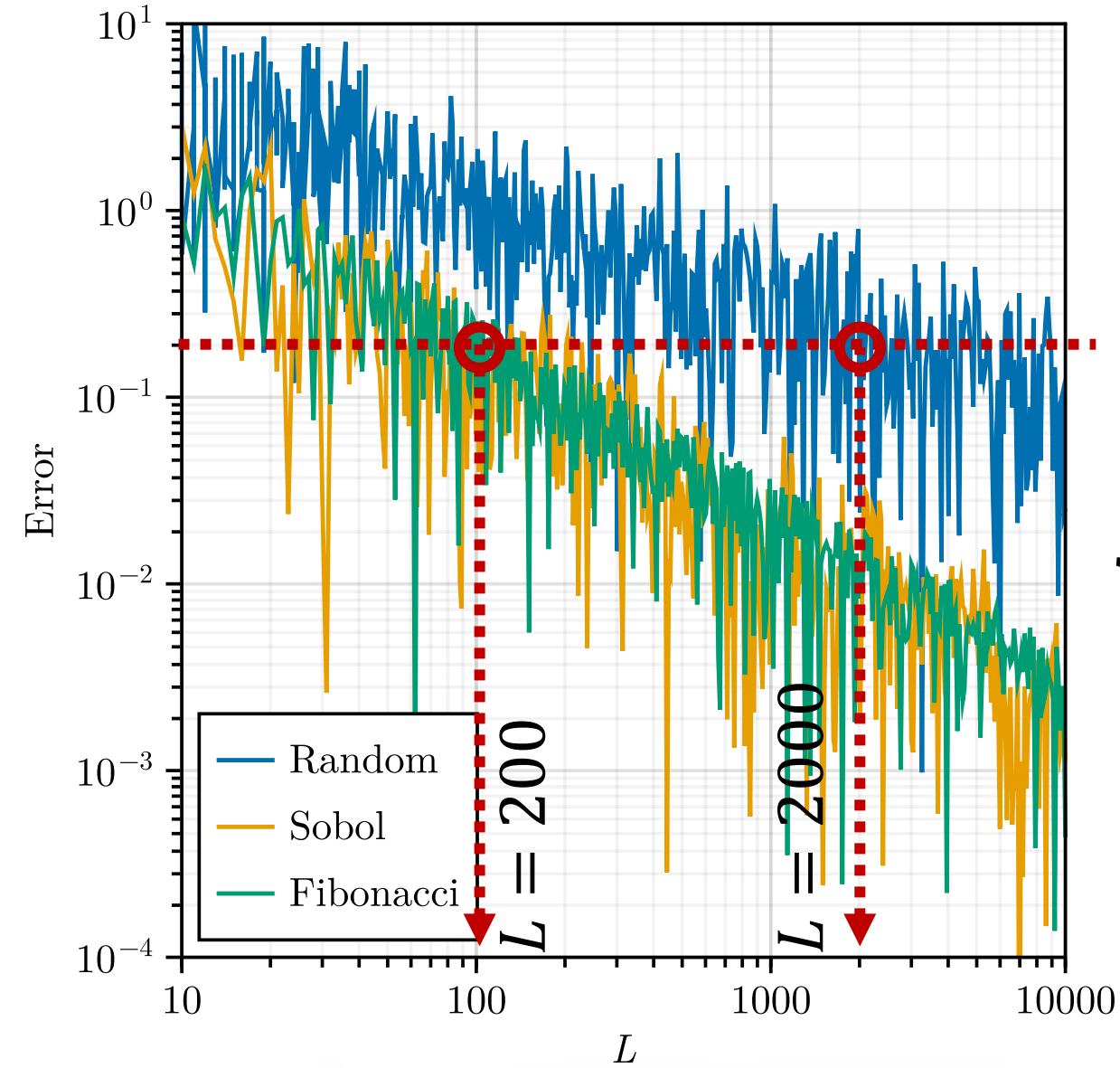
Radial
deformation



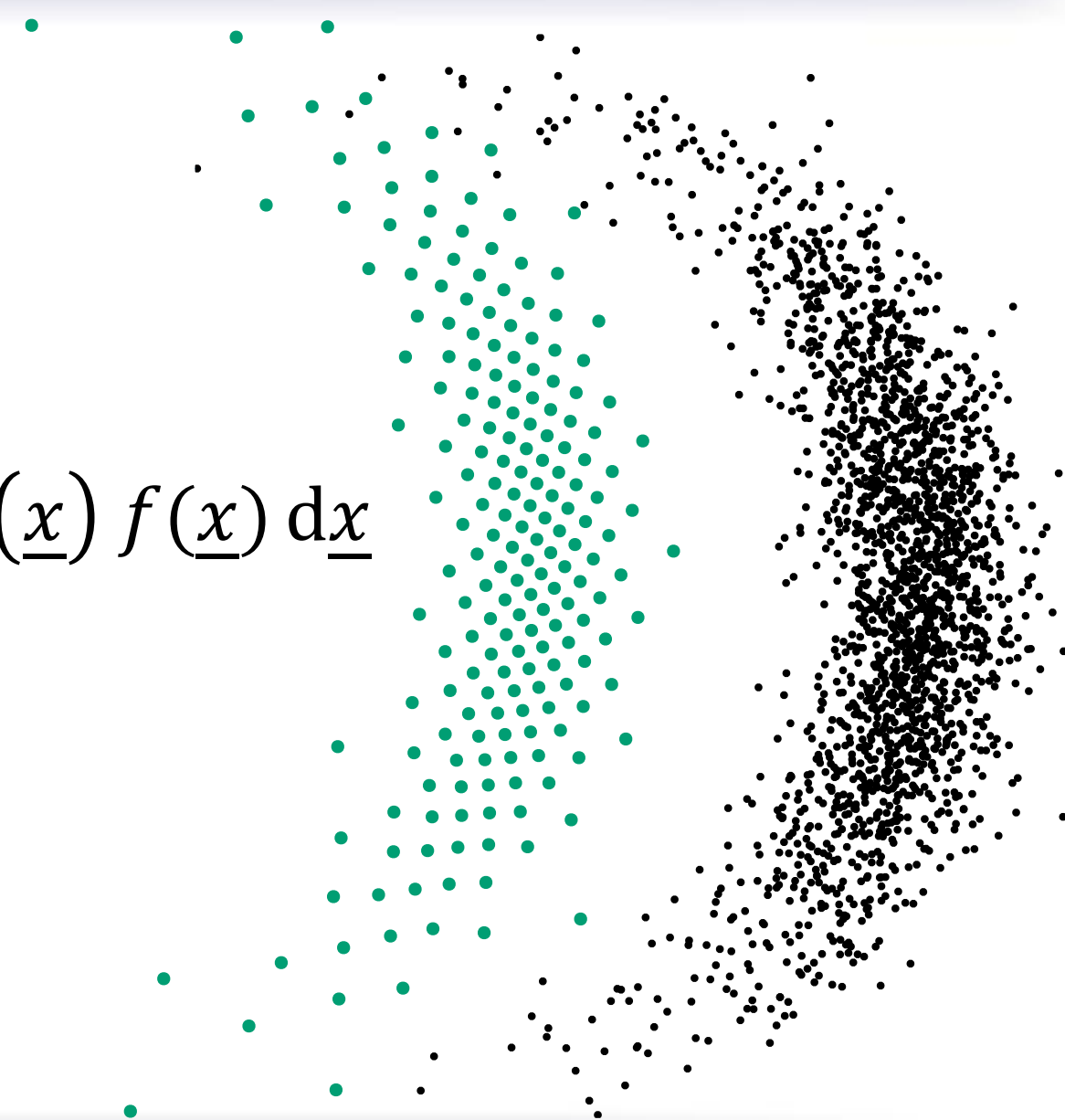
“Radar-Tracking” Density



Evaluation: Expectation Computation



$$\int g(\underline{x}) f(\underline{x}) d\underline{x}$$



Achieved

- Deterministic Sampling
 - homogeneous
 - superior convergence
- Various #Samples
- Various Densities
 - spherically separable

Applications

- Directional State Estimation
 - assumed density quaternion filtering
 - directional estimation
- Directional Control
 - directional sampling for MPC exploration

Thank you for your attention

Intelligent
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Sensor-Actuator-Systems