Why You Shouldn’t Use TDOA for Multilateration

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Input

Given

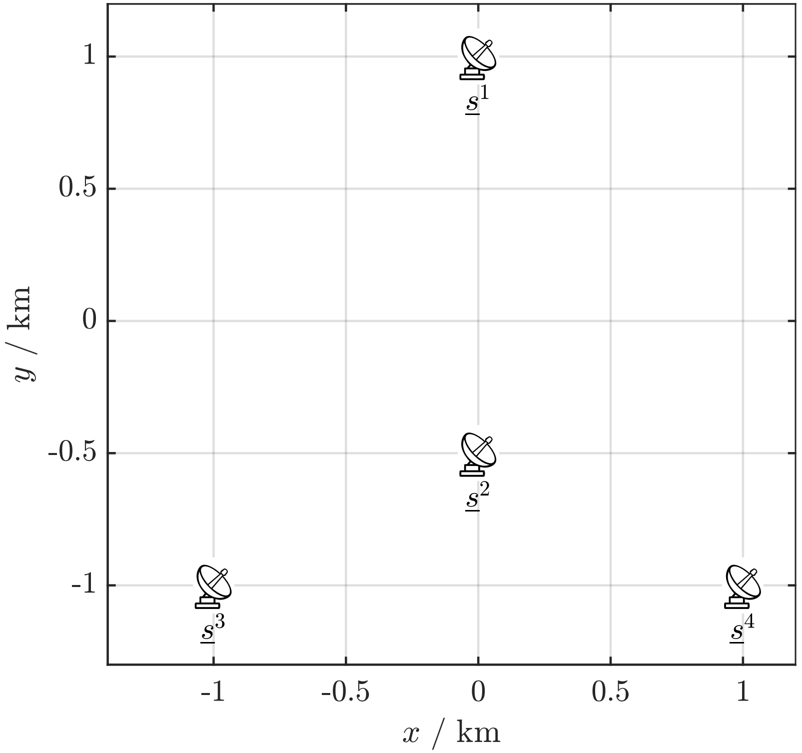
- Receiver positions
- Measured: TOAs

Multilateration

Output

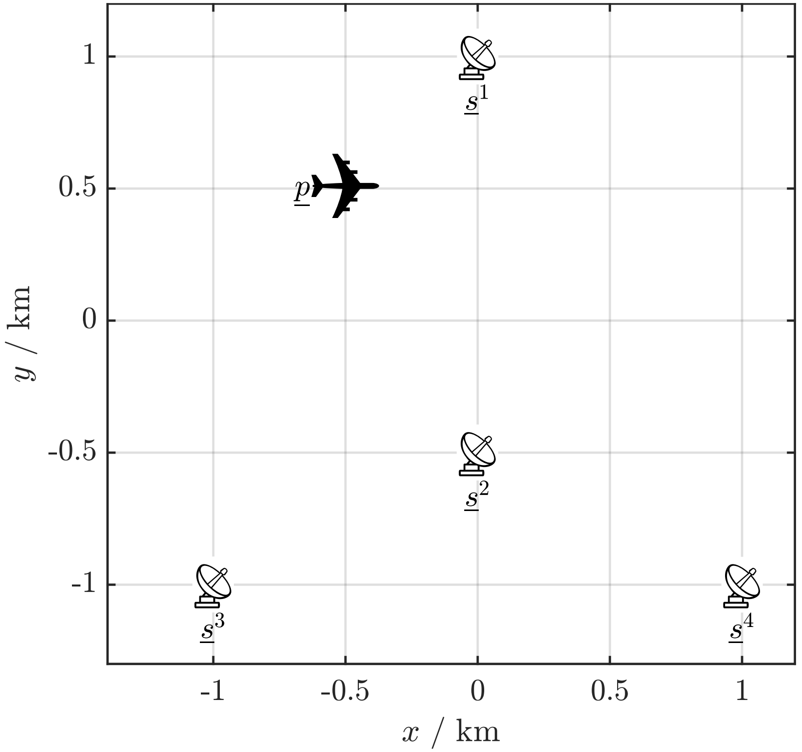
Estimate

- Aircraft Position
- TTT



Methods

- Closed-Form TTT Estimator
- Levenberg-Marquardt Minimizer



Multilateration

Input

Given

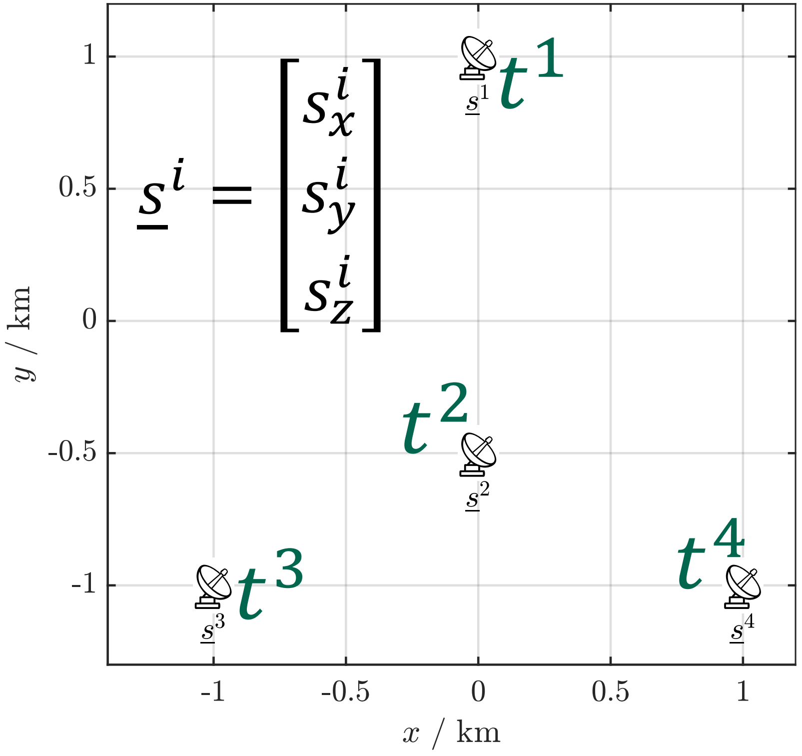
- Receiver positions $\underline{s}^i$
- Measured: ToAs t^i

Multilateration

Output

Estimate

- Aircraft Position $\underline{p}$
- TTT ^{p}t



Methods

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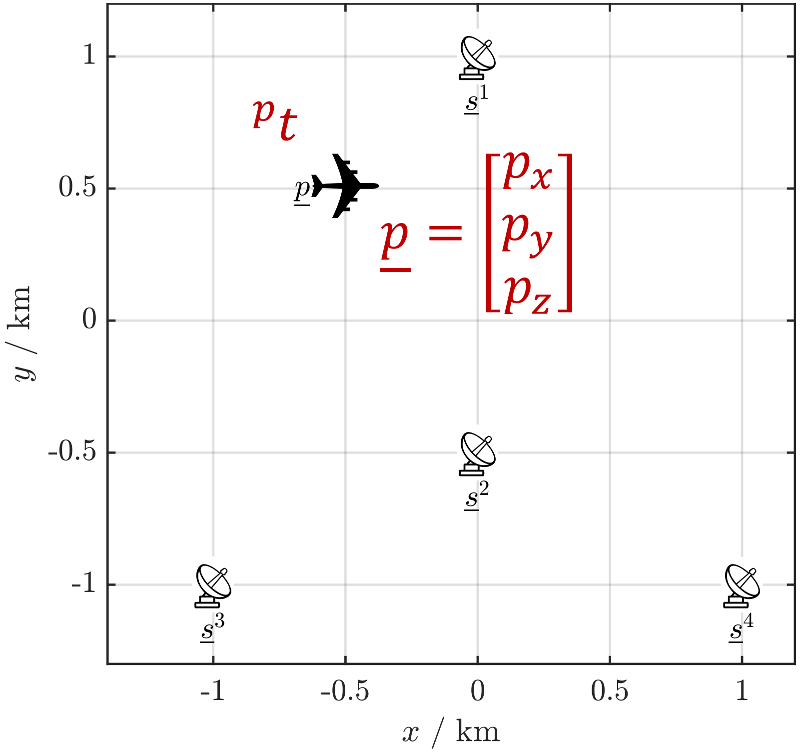

Maximum Likelihood (ML) Estimation

$$\underline{y} = \underline{h}(\underline{x}) + \underline{v}, \qquad \underline{v} \sim f_v(\underline{v})$$

$$\underline{\hat{x}}^{\mathrm{ML}} = \underset{\underline{x}}{\arg\max}\, f_v\left(\underline{y} - \underline{h}(\underline{x})\right)$$

- linear h, Gaussian f_v
  - LLS
- Linear Eqn System

- nonlinear h, Gaussian f_v
  - NLS
  - “Ausgleichsrechnung”,
    Carl Friedrich Gauss, 1800
- Gauß-Newton
- Levenberg-Marquardt
- Gradient Descent

- nonlinear h, non-Gaussian f_v
  - Nonlinear Optimization
- Newton’s method
- Gradient Descent

Weighted Linear Least Squares (WLLS)

- overdetermined probabilistic linear eqn system

$$\underline{y} = \mathbf{H}\underline{x} + \underline{v}, \qquad \underline{v} \in \mathcal{N}\left(\underline{v}; \underline{0}, \mathbf{C}_v\right)$$

- maximum likelihood

$$\hat{\underline{x}}^{\mathrm{ML}} = \arg\max_{\underline{x}} f_v\left(\underline{y} - \mathbf{H}\underline{x}\right)$$

- optimization problem
  - quadratic loss function

$$\hat{\underline{x}}^{\mathrm{ML}} = \arg\min_{\underline{x}} \left\| \mathbf{H}\underline{x} - \underline{y} \right\|^2_{\mathbf{C}_v^{-1}}$$

- analytic solution

$$\frac{\partial y}{\partial \underline{x}} \left(\underline{y} - \mathbf{H}\underline{x}\right)^\top \mathbf{C}_v^{-1} \left(\mathbf{H}\underline{x} - \underline{y}\right) = \underline{0}$$

- leads to linear eqn system
  - global optimum!

$$\mathbf{H}^\top \mathbf{C}_v^{-1} \mathbf{H}\underline{x} = \mathbf{H}^\top \mathbf{C}_v^{-1} \underline{y} \iff \mathbf{A}\underline{x} = \underline{b}$$

Maximum Likelihood (ML) Estimation

$$\underline{y} = \underline{h}(\underline{x}) + \underline{v}, \qquad \underline{v} \sim f_v(\underline{v})$$

$$\underline{\hat{x}}^{\mathrm{ML}} = \arg\max_{\underline{x}} f_v\left(\underline{y} - \underline{h}(\underline{x})\right)$$

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  - Nonlinear Optimization
- Newton’s method
- Gradient Descent

- Problem: $\underline{y} = \underline{h}(\underline{x}) + \underline{v}, \qquad \underline{v} \sim f_{\mathrm{SND}}(\underline{v})$
- ML/NLS Estimator: $\underline{\hat{x}} = \arg\min_{\underline{x}} \left\| \underline{h}(\underline{x}) - \underline{y} \right\|_2^2$
- Gauss-Newton: $\underline{x}_{k+1} = \underline{x}_k - a_k \cdot \left(\mathbf{H}_k^\top \mathbf{H}_k\right)^{-1} \mathbf{H}_k^\top \underbrace{\left(\underline{h}(\underline{x}) - \underline{y}\right)}_{\text{Output of objective function}}$
- Levenberg-Marq.: $\underline{x}_{k+1} = \underline{x}_k - a_k \cdot \left(\mathbf{H}_k^\top \mathbf{H}_k + \lambda \mathbf{I}\right)^{-1} \mathbf{H}_k^\top \overbrace{\left(\underline{h}(\underline{x}) - \underline{y}\right)}$

WNLS

- Problem: $\underline{y} = \underline{h}(\underline{x}) + \underline{v}, \qquad \underline{v} \sim f_v(\underline{v})$

- ML/NLS Estimator:
$$\hat{\underline{x}} = \arg\min_{\underline{x}} \left(\underline{h}(\underline{x}) - \underline{y}\right)^\top \mathbf{C}_v^{-1} \left(\underline{h}(\underline{x}) - \underline{y}\right)$$
$$= \arg\min_{\underline{x}} \left(\underline{h}(\underline{x}) - \underline{y}\right)^\top (\mathbf{R}\mathbf{R}^\top)^{-1} \left(\underline{h}(\underline{x}) - \underline{y}\right)$$
$$= \arg\min_{\underline{x}} \left(\mathbf{R}^{-1}\left(\underline{h}(\underline{x}) - \underline{y}\right)\right)^\top \left(\mathbf{R}^{-1}\left(\underline{h}(\underline{x}) - \underline{y}\right)\right)$$

- Gauss-Newton: $\underline{x}_{k+1} = \underline{x}_k - a_k \cdot \left(\mathbf{H}_k^\top \mathbf{H}_k\right)^{-1} \mathbf{H}_k^\top \left(\mathbf{R}^{-1}\left(\underline{h}(\underline{x}) - \underline{y}\right)\right)$

Primary Surveillance Radar (PSR)

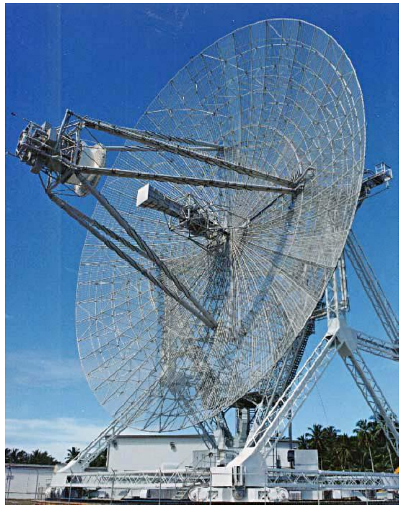
+ non-cooperative targets +

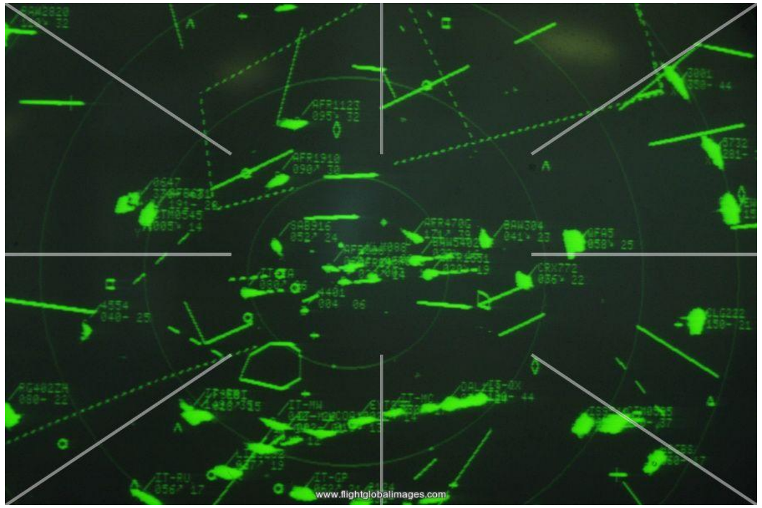
– $SNR \propto \frac{P_t}{R \cdot R \cdot R \cdot R}$ –

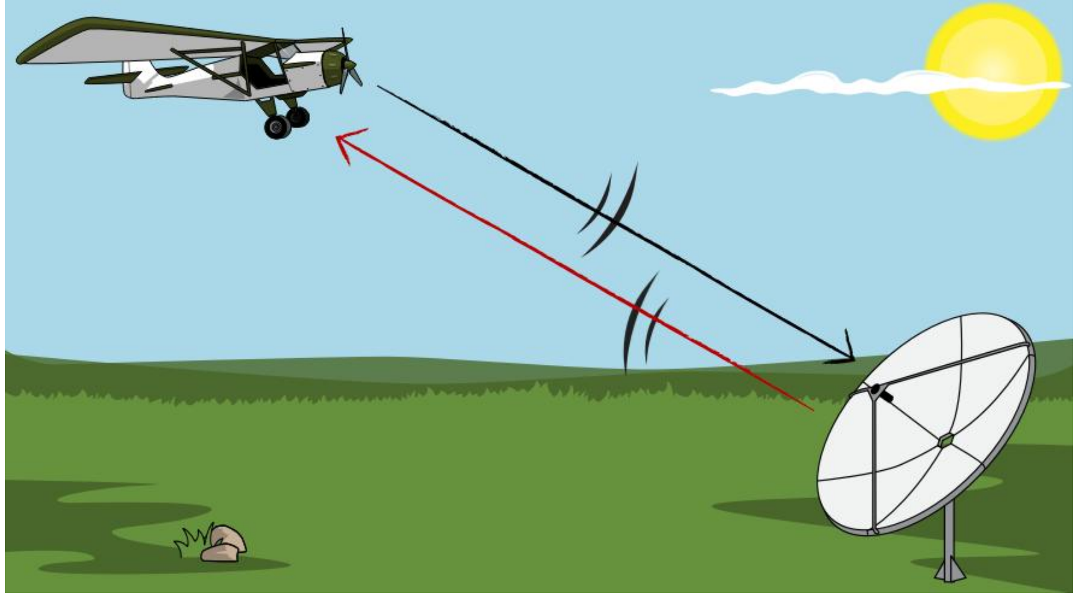
– powerful emissions –

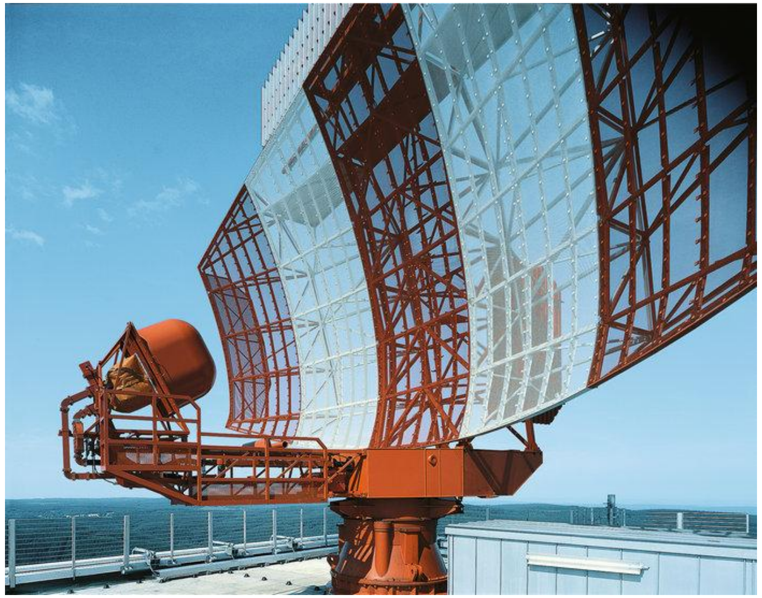
– expensive systems –

– low accuracy –









+ high localization accuracy +

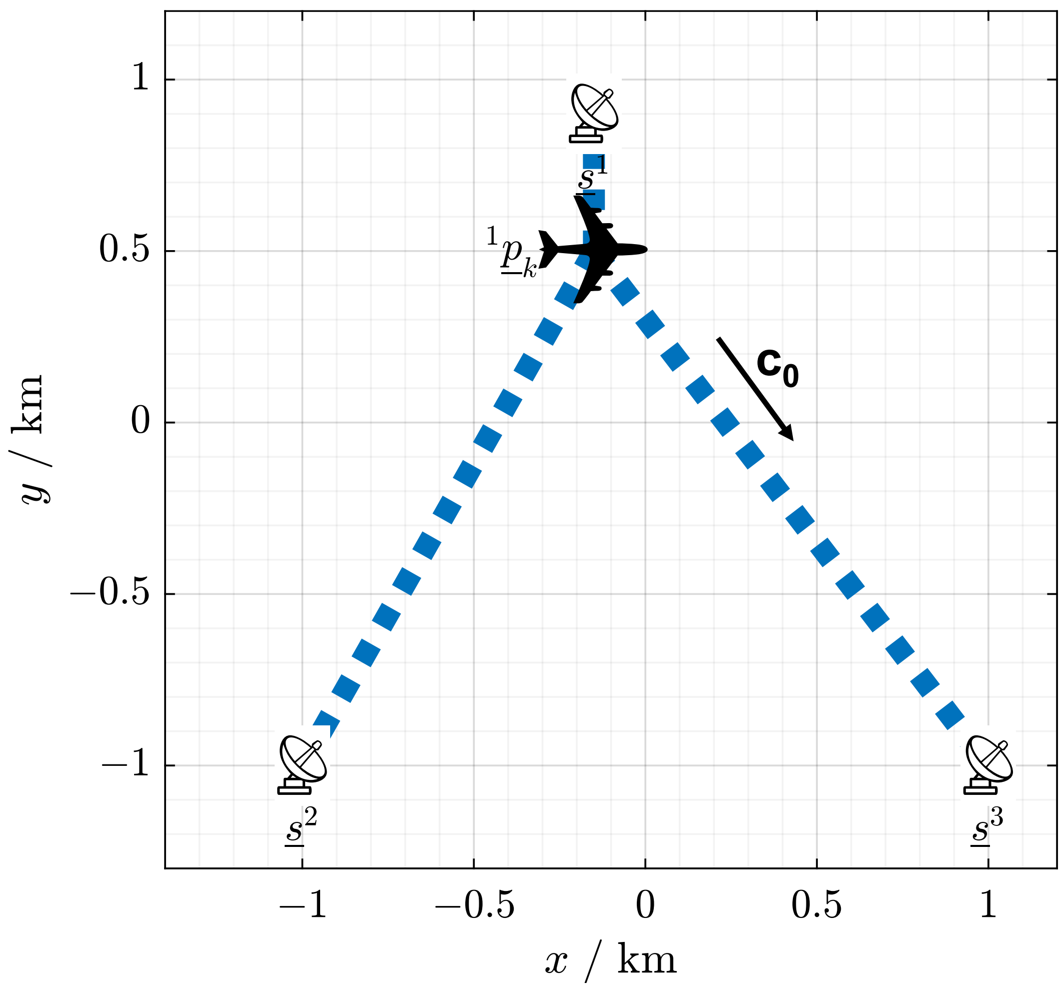
+ $SNR \propto \frac{P_t}{R \cdot R}$ +

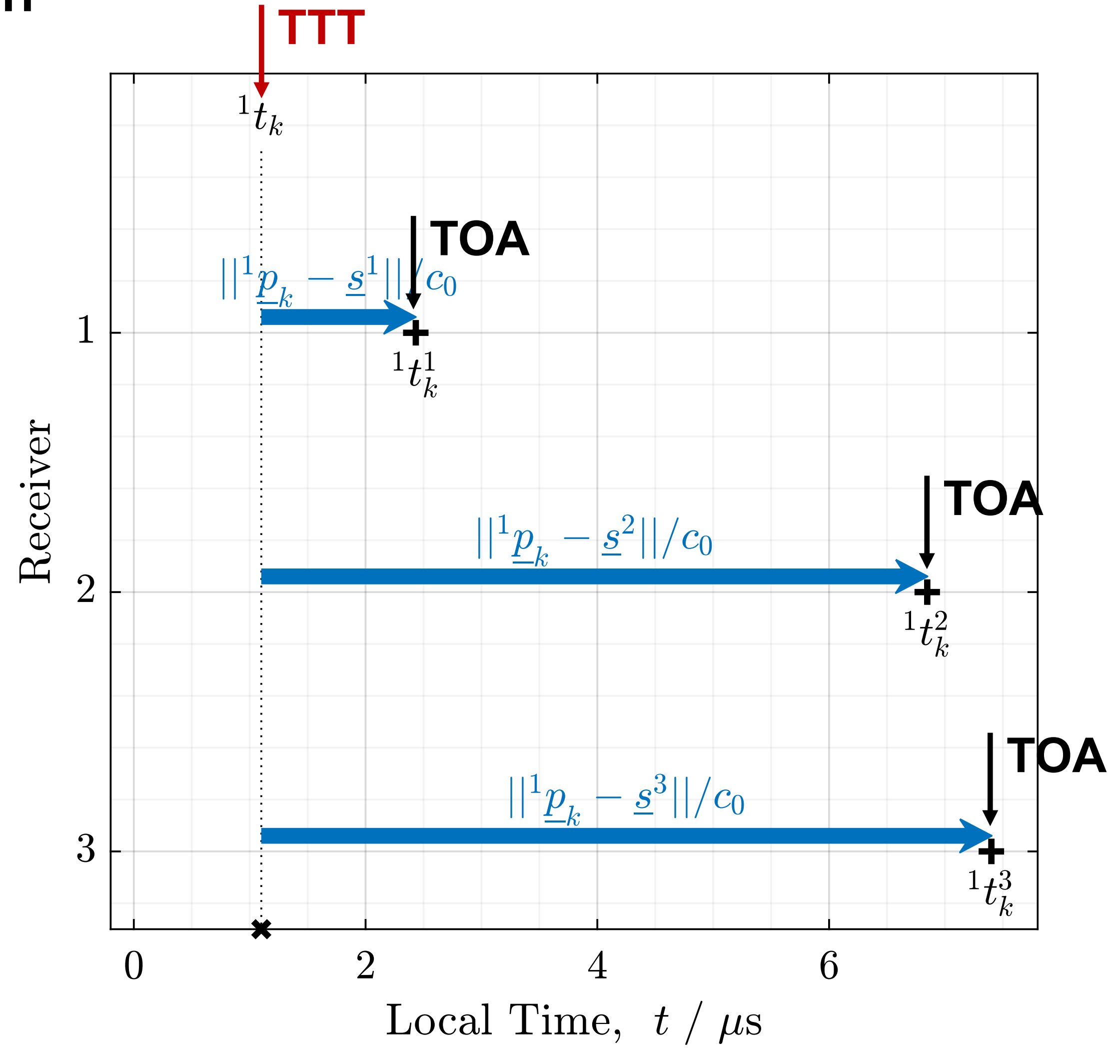
+ small transmitters & receivers +

– requires cooperative targets –

Multilateration

$$t^{i} = {}^{p}t + \left\| \underline{p} - \underline{s}^{i} \right\| / c_0 \,, \qquad i \in \{1, 2, 3\}$$





- solving for target pos
  - Nonlinear Least Squares

$$t^i = {}^p t + \left\| \underline{p} - \underline{s}^i \right\| / c_0 + v^i$$

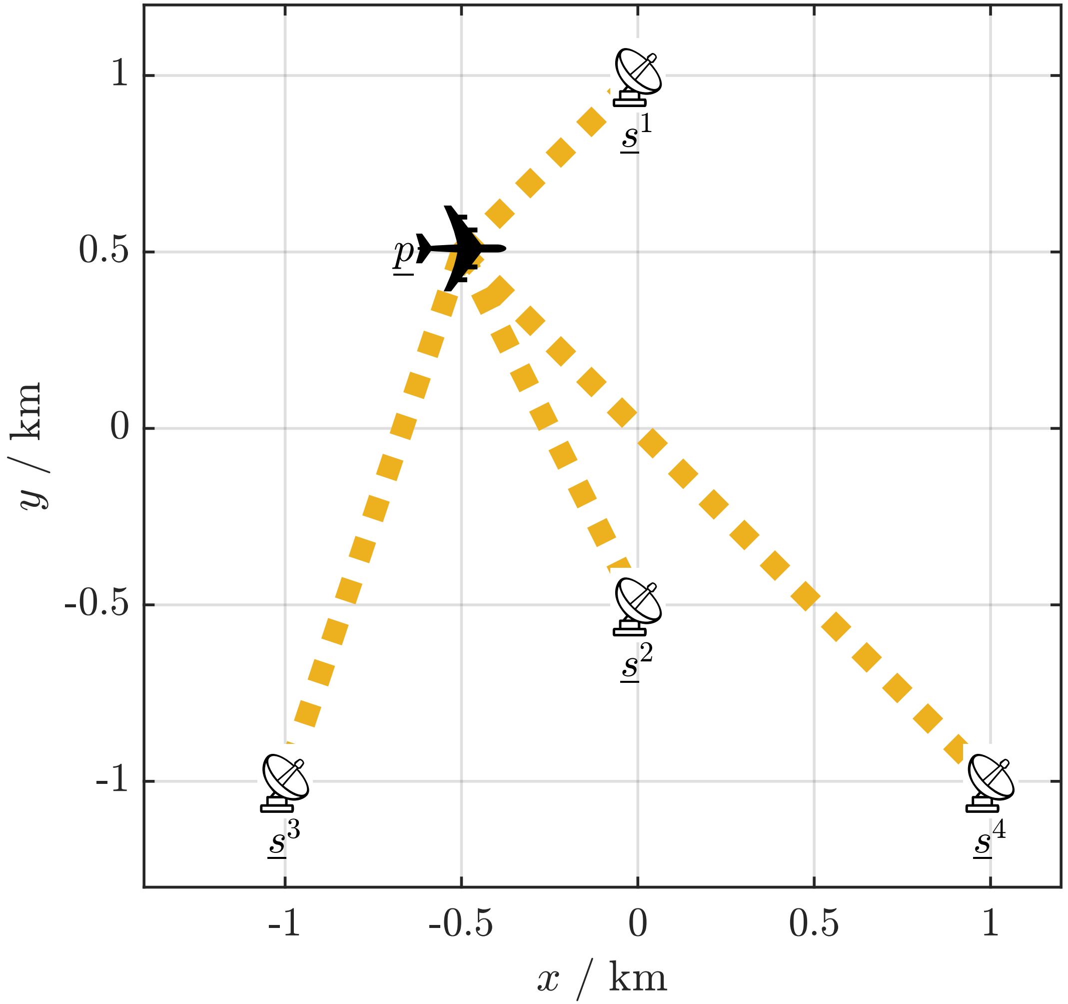
- dealing with TTT
  - TDOA: eliminate TTT
    - "star" variant
    - "successive" variant
    - "combinatoric" variant
  - solve for TTT
    - with NLS optimizer
    - exploit linearity

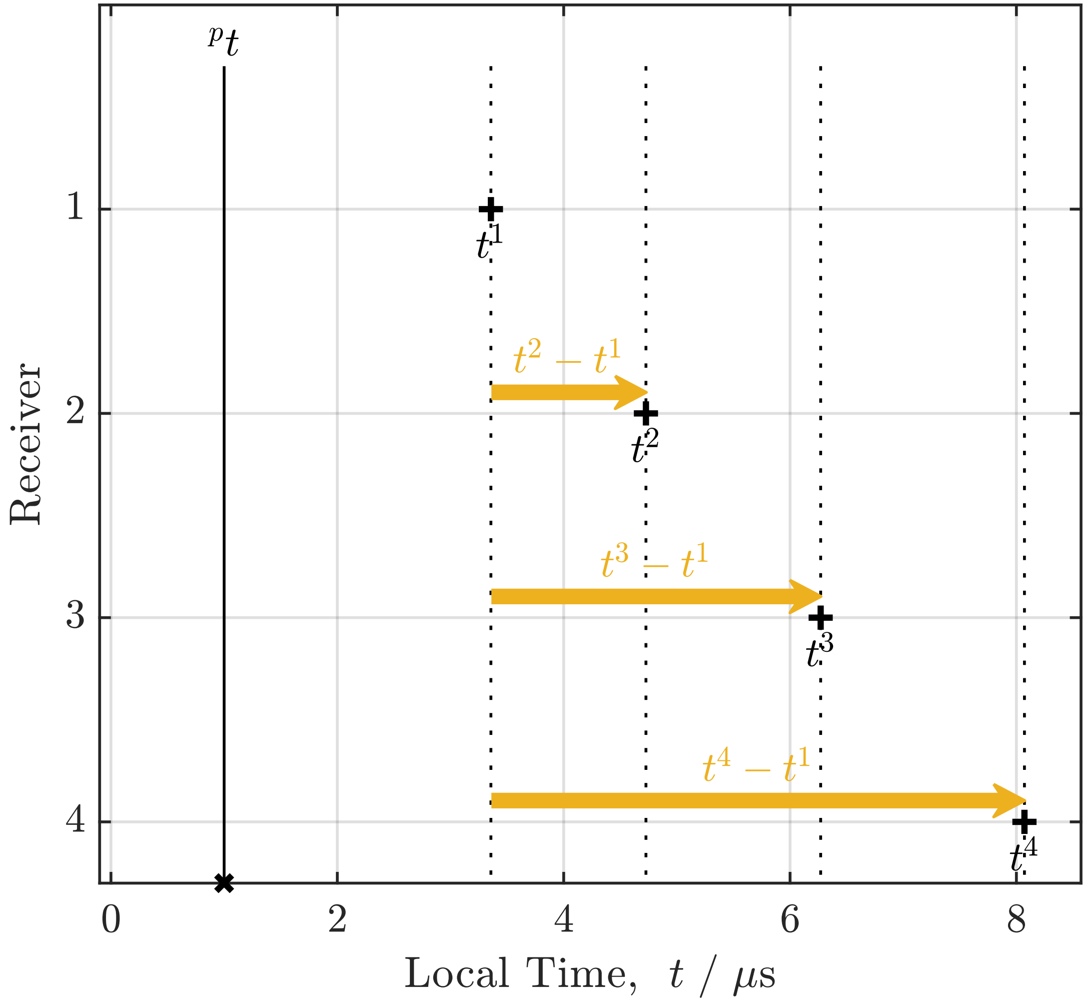
state of art

proposed

TDOA – "Star"

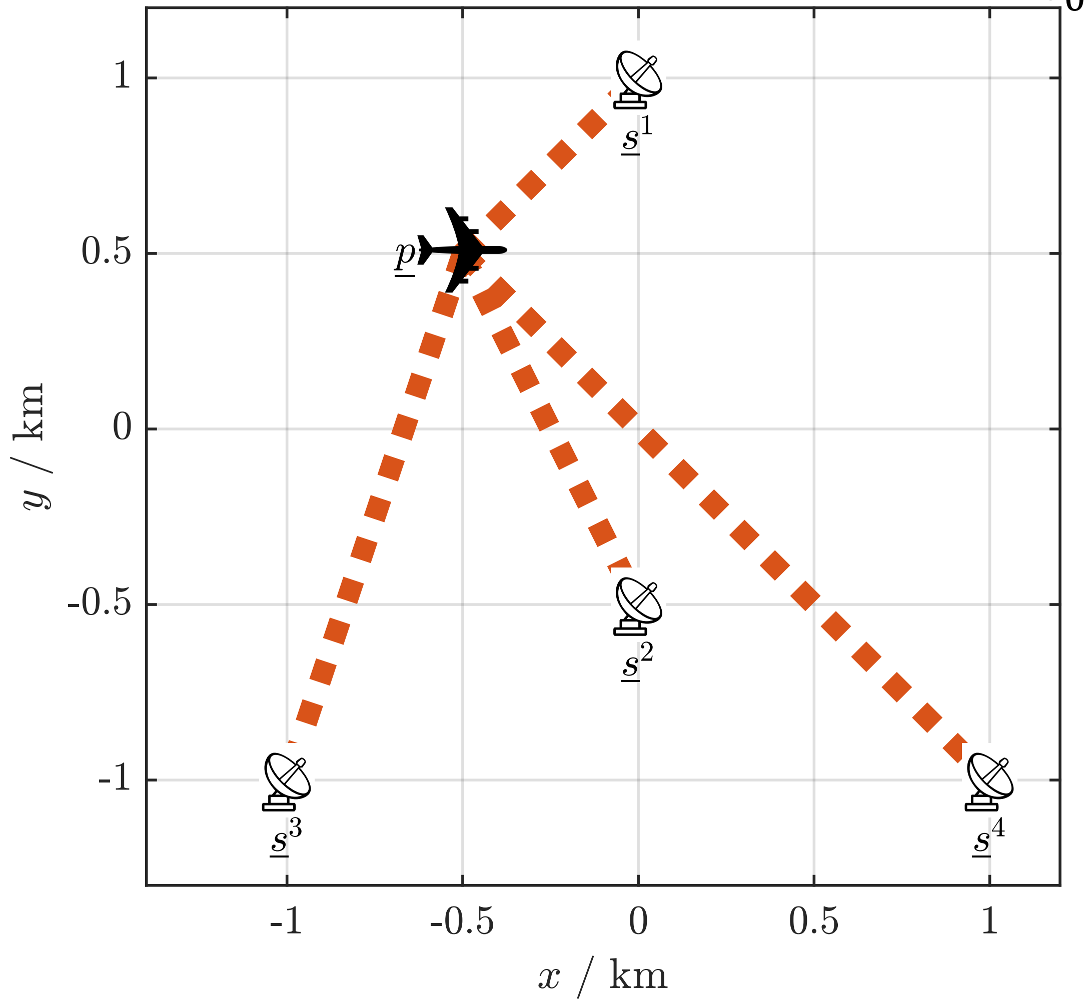
$$t^j - t^i = \frac{\left\| \underline{p} - \underline{s}^j \right\| - \left\| \underline{p} - \underline{s}^i \right\|}{c_0}, \qquad (i,j) \in \{(1,2),(1,3),(1,4)\}$$

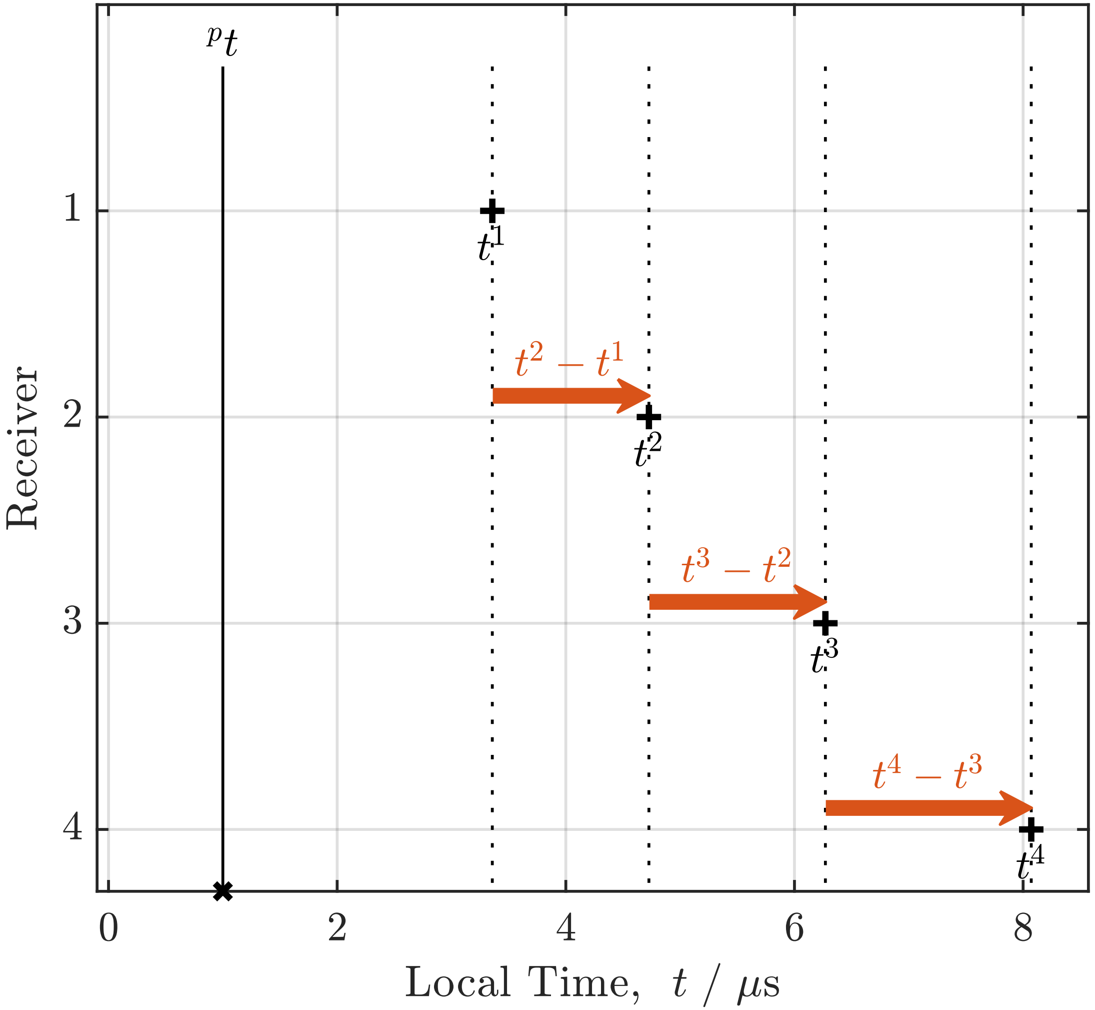




TDOA – "Consecutive"

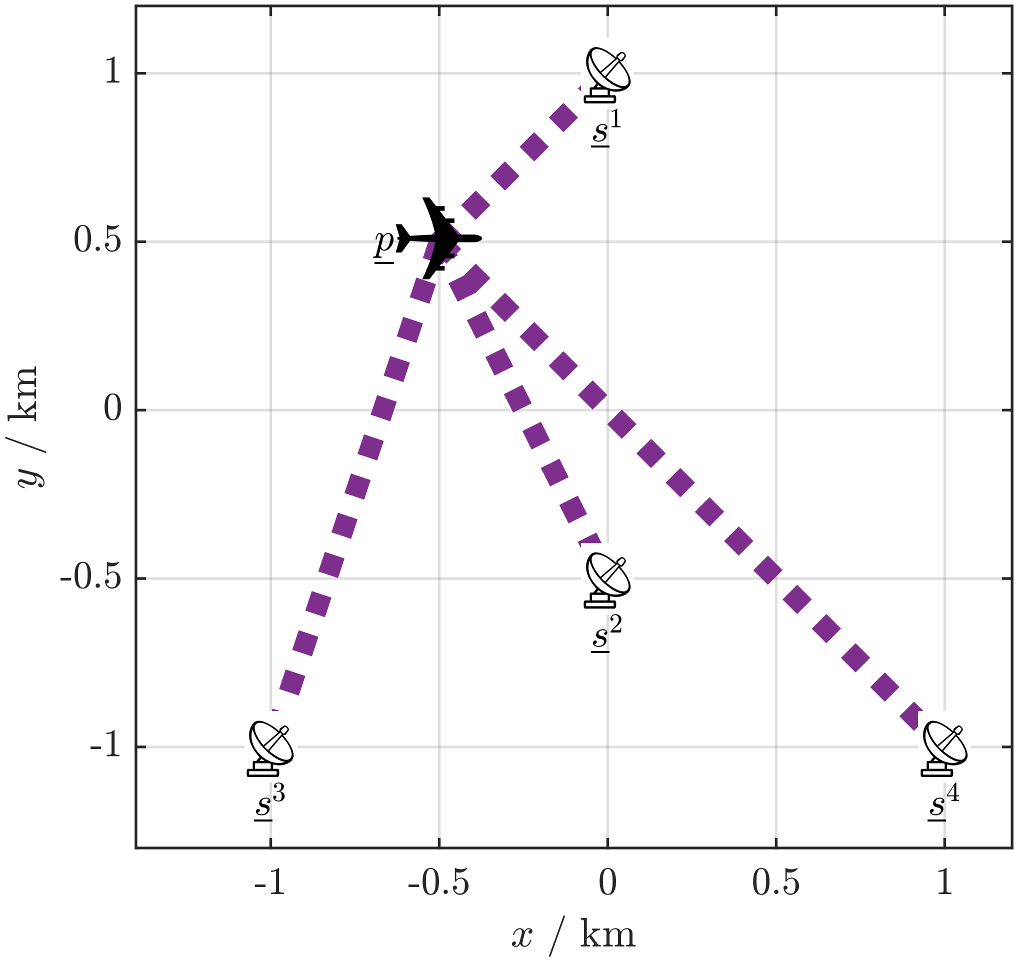
$$t^j - t^i = \frac{\left\| \underline{p} - \underline{s}^j \right\| - \left\| \underline{p} - \underline{s}^i \right\|}{c_0}, \qquad (i,j) \in \{(1,2),(2,3),(3,4)\}$$

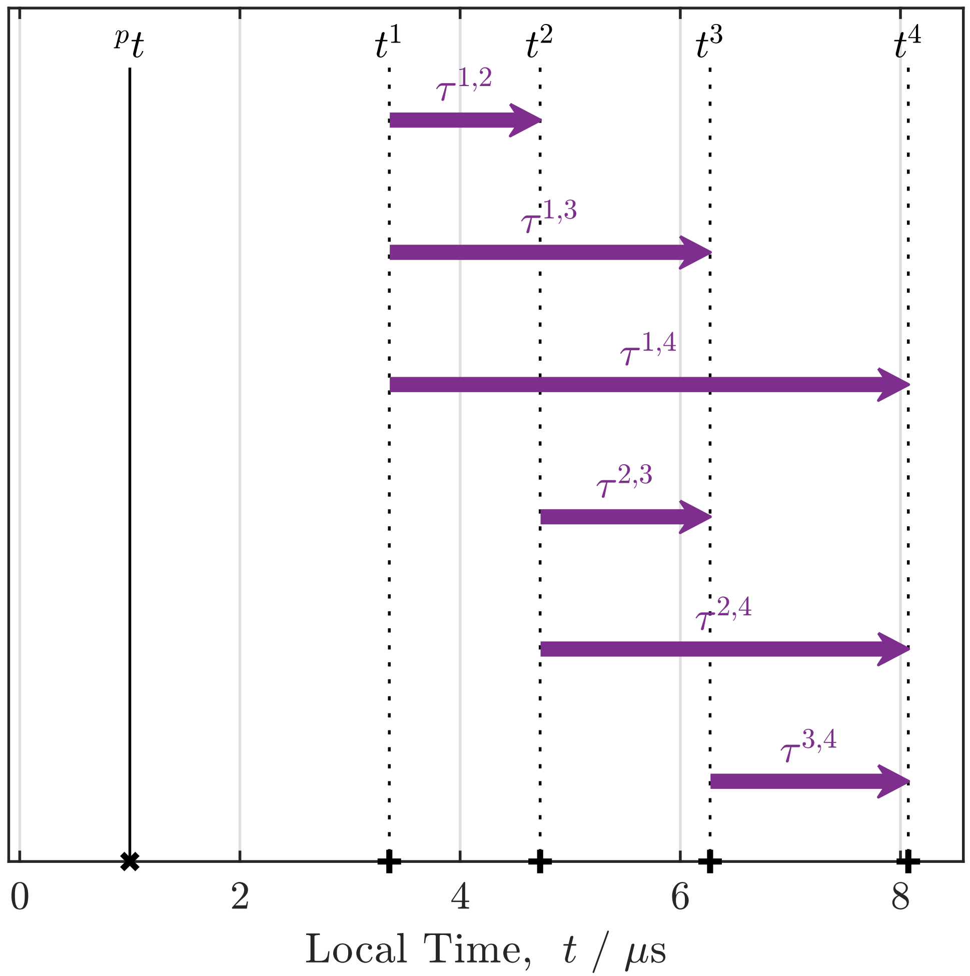




TDOA – "Combinatorial"

$$t^j - t^i = \frac{\left\| \underline{p} - \underline{s}^j \right\| - \left\| \underline{p} - \underline{s}^i \right\|}{c_0}, \qquad (i,j) \in \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$$





TDOA Measurement Equation

$$t^i = {}^{p}t + \left\| \underline{p} - \underline{s}^i \right\| / c_0 + v^i, \qquad v^i \sim \mathcal{N}(v^i, 0, \sigma_v^2)$$

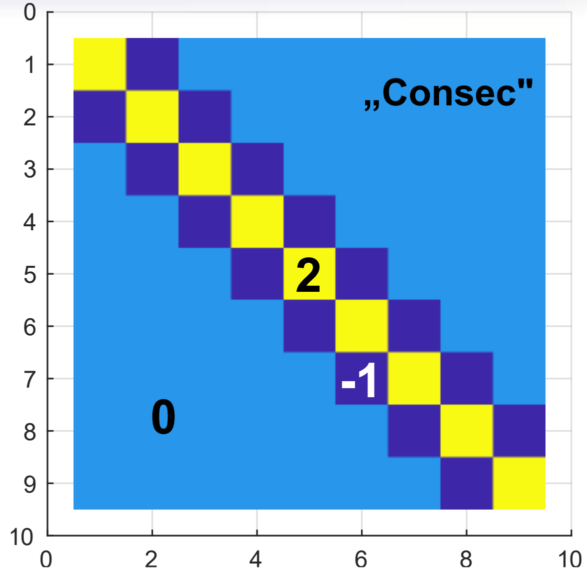
$$\underbrace{t^j - t^i}_{y^{ij}} = \underbrace{\frac{\left\| \underline{p} - \underline{s}^j \right\| - \left\| \underline{p} - \underline{s}^i \right\|}{c_0}}_{h^{ij}(\underline{x})} + \underbrace{v^j - v^i}_{v^{ij}}$$

$(i, j) \in \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$

$$\underline{y} = h(\underline{x}) + \underline{v}, \qquad \underline{v} \sim \mathcal{N}(\underline{v}, \underline{0}, \mathbf{C}_v)$$

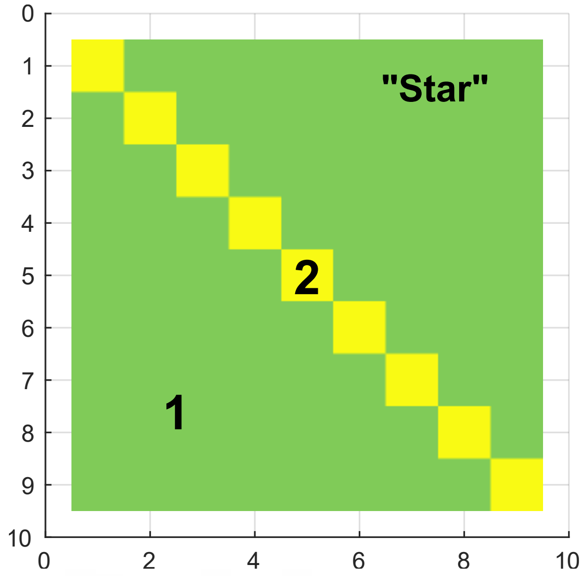
- "Consecutive" topology

$$\mathbf{C}_v = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$



- "Star" topology

$$\mathbf{C}_v = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$



TOA Exploiting Linearity

$$t^i = {}^p t + \left\| \underline{p} - \underline{s}^i \right\| / c_0 + v^i$$

- unknowns $\underline{x}$:
  - $\underline{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$
  - ${}^p t$
- one of the unknowns appears linearly only
- for $\underline{p}$ given: linear eqn for ${}^p t$

- hierarchical solver
  - outer: NLS solving for $\underline{p}$
    - inner: closed form LLS solver for ${}^p t$

Mixed NLS and LLS

- mixed linear+nonlinear measurement eqn
- $\underline{x}_{\mathrm{LN}} \in \mathbb{R}^n$ and $\underline{x}_{\mathrm{NL}} \in \mathbb{R}^m$ must be different variables
- given $\underline{x}_{\mathrm{NL}}$, we can solve for $\underline{x}_{\mathrm{LN}}$ in closed form / LLS
- $(n+m)$-dim NLS $\rightarrow m$-dim NLS!

$$\underline{y} = \mathbf{H}\underline{x}_{\mathrm{LN}} + h\left(\underline{x}_{\mathrm{NL}}\right) + \underline{v} \quad (1)$$

$$\underbrace{\underline{y} - h\left(\underline{x}_{\mathrm{NL}}\right)}_{\text{“new y”}} = \mathbf{H}\underline{x}_{\mathrm{LN}} + \underline{v}$$

LLS: $\mathbf{H}^\top \mathbf{C}_v^{-1} \mathbf{H} \cdot \underline{x}_{\mathrm{LN}} = \mathbf{H}^\top \mathbf{C}_v^{-1} \cdot \left[\underline{y} - h\left(\underline{x}_{\mathrm{NL}}\right)\right]$ $\quad | \text{ insert } \underline{x}_{\mathrm{LN}} \text{ in } (1)$

$$\underline{y} = \mathbf{H} \cdot \left(\mathbf{H}^\top \mathbf{C}_v^{-1} \mathbf{H}\right)^{-1} \mathbf{H}^\top \mathbf{C}_v^{-1} \left[\underline{y} - h\left(\underline{x}_{\mathrm{NL}}\right)\right] + h\left(\underline{x}_{\mathrm{NL}}\right) + \underline{v}$$

$$\underline{y} = \mathbf{H} \cdot (\mathbf{H}^\top \mathbf{C}_v^{-1} \mathbf{H})^{-1} \mathbf{H}^\top \mathbf{C}_v^{-1} \left[\underline{y} - h(\underline{x}_{\mathrm{NL}})\right] + h(\underline{x}_{\mathrm{NL}}) + \underline{v}$$

$$\underbrace{(\mathbf{I} - \mathbf{H} \cdot (\mathbf{H}^\top \mathbf{C}_v^{-1} \mathbf{H})^{-1} \mathbf{H}^\top \mathbf{C}_v^{-1})\underline{y}}_{\text{new "}y\text{"}} = \underbrace{(\mathbf{I} - \mathbf{H} \cdot (\mathbf{H}^\top \mathbf{C}_v^{-1} \mathbf{H})^{-1} \mathbf{H}^\top \mathbf{C}_v^{-1})h(\underline{x}_{\mathrm{NL}})}_{\text{new "}h(x)\text{"}} + \underline{v}$$

variant 1: new "y" new "$h(x)$" $\Rightarrow$ $y = h(x) + v$ NLS

$$\underline{y} = h(\underline{x}_{\mathrm{NL}}) + \underbrace{(\mathbf{I} - \mathbf{H} \cdot (\mathbf{H}^\top \mathbf{C}_v^{-1} \mathbf{H})^{-1} \mathbf{H}^\top \mathbf{C}_v^{-1})^{-1} \underline{v}}_{\text{new "}v\text{"}}$$

variant 2: new "v" $\Rightarrow$ $y = h(x) + v$ WNLS

… with $\mathbf{C}_v^{\mathrm{new}} = (\mathbf{I} - \mathbf{H} \cdot (\mathbf{H}^\top \mathbf{C}_v^{-1} \mathbf{H})^{-1} \mathbf{H}^\top \mathbf{C}_v^{-1})^{-1} \mathbf{C}_v^{\mathrm{orig}} [(\mathbf{I} - \mathbf{H} \cdot (\mathbf{H}^\top \mathbf{C}_v^{-1} \mathbf{H})^{-1} \mathbf{H}^\top \mathbf{C}_v^{-1})^{-1}]^\top$

$$t^i = {}^p t + \frac{1}{c} \left\| \underline{p} - \underline{s}^i \right\| + v^i$$

$$\underbrace{t^i - \frac{1}{c} \left\| \underline{p} - \underline{s}^i \right\|}_{\text{“new y”}} = {}^p t + v^i$$

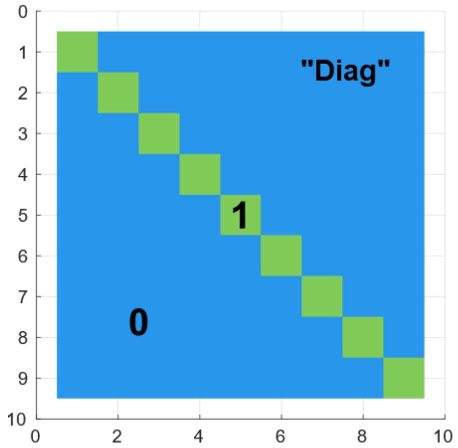
$$\Longrightarrow \quad {}^p\hat{t} = \frac{1}{N} \sum_{i=1}^{N} t^i - \frac{1}{c} \left\| \underline{p} - \underline{s}^i \right\|$$

LM Objective Function

$$t^i = {}^p t + \frac{1}{c}\left\|\underline{p} - \underline{s}^i\right\| + v^i$$

$$ {}^p\hat{t}\left(\underline{p}\right) = \frac{1}{N}\sum_{i=1}^{N} t^i - \frac{1}{c}\left\|\underline{p} - \underline{s}^i\right\|$$

$$\underline{\theta}_{\mathrm{LM}}\left(\underline{p}\right) = \begin{bmatrix} {}^p\hat{t}\left(\underline{p}\right) + \frac{1}{c}\left\|\underline{p} - \underline{s}^1\right\| - t^1 \\ {}^p\hat{t}\left(\underline{p}\right) + \frac{1}{c}\left\|\underline{p} - \underline{s}^2\right\| - t^2 \\ \vdots \\ {}^p\hat{t}\left(\underline{p}\right) + \frac{1}{c}\left\|\underline{p} - \underline{s}^N\right\| - t^N \end{bmatrix}$$



$$\hat{\underline{p}} = \arg\min_{\underline{p}} \left[\underline{\theta}_{\mathrm{LM}}\left(\underline{p}\right)\right]^{\top} \mathbf{C}_v^{-1} \left[\underline{\theta}_{\mathrm{LM}}\left(\underline{p}\right)\right]$$

Efficient MLAT Algorithms

- 4D LM
  - with TOAs
  - 264 ns

$$\theta_{\mathrm{LM},i}\left(\underline{p}, {}^{p}t\right) = {}^{p}t + \frac{1}{c}\left\|\underline{p} - \underline{s}^{i}\right\| - t^{i}$$

- 3D Weighted LM
  - with TDOAs
  - additional matrix mult.
  - 273 ns

$$\underline{\theta}_{\mathrm{LM}}\left(\underline{p}\right) = \mathbf{R}^{-1}\left(\underline{y} - \underline{h}\left(\underline{p}\right)\right)$$

- 3D LM
  - with TOAs
  - 200 ns

$$\theta_{\mathrm{LM},i}\left(\underline{p}\right) = {}^{p}\hat{t}\left(\underline{p}\right) + \frac{1}{c}\left\|\underline{p} - \underline{s}^{i}\right\| - t^{i}$$

LM objective function
- vector-valued
- in-place

```
function θ_TOA_LM!(θ, p)
    θ .= 1/c * [norm(p .- s[:,i]) for i in 1:N]
    tp = mean(ti .- θ)
    θ .+= tp .- ti
end
```

$$\frac{1}{c}\left\|\underline{p}-\underline{s}^i\right\|$$

$$^{p}\hat{t} = \frac{1}{N}\sum_{i=1}^{N} t^i - \frac{1}{c}\left\|\underline{p}-\underline{s}^i\right\|$$

$$\theta_{\mathrm{LM},i}\left(\underline{p}\right) = {}^{p}\hat{t}\left(\underline{p}\right) + \frac{1}{c}\left\|\underline{p}-\underline{s}^i\right\| - t^i$$

Conclusion

Disadvantages TDOA (state of art)

- Non-diag Cov
  - easy to forget / get wrong
    - Then non-efficient estimate
- Slower
  - more matrix multiplications

Advantages TOA (proposed)

- Only 3 variables
  - closed-form TTT
- No TTT initial guess
- Diagonal Cov

Thank you for your attention

Intelligent

Sensor-Actuator-Systems