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Elucidating the Physical and Mathematical Properties of the Prouhet-Thue-Morse Sequence in Quantum Computing

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ABSTRACT

This study explores the applications of the Prouhet-Thue-Morse (PTM) sequence in quantum computing, highlighting its mathematical elegance and practical relevance. We demonstrate the critical role of the PTM sequence in quantum error correction, in noise-resistant quantum memories, and in providing insights into quantum chaos. Notably, we demonstrate how the PTM sequence naturally appears in Ising X-X interacting systems, leading to a proposed robust encoding of quantum memories in such systems. Furthermore, connections to number theory, including the Riemann zeta function, bridge quantum computing with pure mathematics. Our findings emphasize the PTM sequence's importance in understanding the mathematical structure of quantum computing systems and the development of the full potential of quantum technologies and invite further interdisciplinary research.

1 | Introduction

The intersection of quantum computing and mathematics opens a fascinating frontier for research, with potential implications for both theoretical advancements and practical applications. Among mathematical constructs, the Prouhet-Thue-Morse (PTM) sequence stands out because of its surprisingly rich structure and multifaceted utility. Originally discovered in the 19th century [1] and renowned for its unique properties, the PTM sequence has garnered attention across various scientific disciplines [2].

Quantum computing, leveraging the principles of quantum mechanics, potentially offers unprecedented computational power and capabilities. However, this potential is often hindered by issues such as quantum error correction, noise resilience, and the intricate dynamics of quantum systems. This study seeks to bridge the gap between the abstract mathematics of the

PTM sequence and the practical challenges faced in quantum computing by demonstrating its natural emergence within quantum frameworks.

Intriguingly, we aim to show that it appears very naturally within the Hilbert spaces of quantum computing systems, suggesting profound links and practical applications that are yet to be fully explored. By investigating this manifestation of the PTM sequence, we aim to uncover how its unique properties can address critical issues such as error correction and noise resilience. We show how the PTM sequence's inherent symmetrical structure and fractal nature present opportunities for developing robust quantum error correction codes, thereby enhancing the reliability and efficiency of quantum computations. Additionally, we demonstrate how the PTM sequence appears naturally and easily in X-X Ising systems, further showcasing its significance. Furthermore, the sequence's integration into quantum memory systems and its role in understanding quantum chaos

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through constructs like the Walsh-Hadamard transform and the quantum baker's map illustrate its broader relevance.

Moreover, this study explores the deep connections between the PTM sequence and number theory, particularly its relation to the Riemann zeta function and, more generally, Dirichlet series. These connections not only enrich our understanding of the mathematical foundations underlying quantum computing but also pave the way for interdisciplinary research that could unlock new computational methods and insights.

In summary, this study aims to establish a comprehensive link between the Prouhet-Thue-Morse sequence and quantum computing, highlighting how the sequence naturally emerges within the computational space of quantum systems. By emphasizing the sequence's potential to enhance error correction, noise resilience, and our understanding of quantum dynamics, we endeavor to illustrate how bridging this gap can contribute to the evolution of quantum technologies and the realization of their full potential.

1.1 | The Prouhet-Thue-Morse Sequence

The Prouhet-Thue-Morse (PTM) sequence is a binary sequence with a rich mathematical structure and wide-ranging applications in physics, from quasicrystals to quantum mechanics [2, 3].

Definition. Mathematically, the sequence is defined by its n -th term t_n , which is the sum of the binary digits of $n \pmod 2$, formally given by:

$$t_n = \left(\sum_{k=0}^{\infty} \left\lfloor \frac{n}{2^k} \right\rfloor \right) \pmod 2. \tag{1}$$

Example.

$$\begin{aligned} (0)_2 = 0, & \quad t_0 = 0 \pmod 2 & = 0, \\ (1)_2 = 1, & \quad t_1 = 1 \pmod 2 & = 1, \\ (2)_2 = 10, & \quad t_2 = (1 + 0) \pmod 2 & = 1, \tag{2} \\ (3)_2 = 11, & \quad t_3 = (1 + 1) \pmod 2 & = 0, \\ (4)_2 = 100, & \quad t_4 = (1 + 0 + 0) \pmod 2 & = 1, \dots \end{aligned}$$

Historically, the sequence was first discovered by Eugène Prouhet in 1851 [1], later independently rediscovered by Axel Thue in 1906 [4], and again by Marston Morse in the 1920s [5]. Its discovery and subsequent investigations have revealed deep connections to various fields of mathematics and physics, illustrating the sequence's profound universality and versatility.

Beyond this initial, explicit, definition, the PTM sequence can also be constructed through iterative or recursive methods.

Definition. The iterative approach involves starting with the initial term 0 and then repeatedly appending the binary complement of the sequence so far. Formally, if T_i represents the sequence at the i -th iteration (with 2^i elements), then:

$$T_0 = \{0\}, \quad T_{i+1} = T_i \parallel \overline{T_i}, \tag{3}$$

where $\overline{T_i}$ denotes the binary complement of T_i , and \parallel denotes concatenation.

Example.

$$\begin{aligned} T_0 &= \{0\}, \\ T_1 &= \{0\} \parallel \overline{\{0\}} = \{0\} \parallel \{1\} = \{01\}, \\ T_2 &= \{01\} \parallel \overline{\{01\}} = \{01\} \parallel \{10\} = \{0110\}, \\ T_3 &= \{0110\} \parallel \overline{\{0110\}} = \{01101001\}, \dots \end{aligned} \tag{4}$$

Definition. Another method is through recursion, where the n -th term can be defined in relation to its predecessors, given by:

$$t_n = (t_{\lfloor n/2 \rfloor} + n) \pmod 2, \quad t_0 = 0. \tag{5}$$

These definitions reflect the self-similarity and fractal nature of the Prouhet-Thue-Morse sequence, emphasizing its deep mathematical properties and relevance to the study of non-periodic structures in physics [2, 3, 6-9]¹.

Definition. Another similar recursive formula is:

$$t_0 = 0, \quad t_{2n} = t_n, \quad t_{2n+1} = 1 - t_n \equiv \overline{t_n}. \tag{6}$$

Finally, an interesting property of the PTM sequence can be described as follows:

Definition. We define two sets based on the value of t_n within the first 2^N terms of the sequence:

- $E(N)$, the set of indices n for which $t_n = 0$, for all n in the range $0 \leq n \leq 2^N - 1$.
- $O(N)$, the set of indices n for which $t_n = 1$, for all n in the same range.

Property 1.1.1. For any natural number N and for all integers k such that $0 \leq k < N$:

$$\sum_{e \in E(N)} e^k = \sum_{o \in O(N)} o^k. \tag{7}$$

i.e., the sum of the k -th powers of the elements in $E(N)$ equals the sum of the k -th powers of the elements in $O(N)$.

Example. For $N = 3$, one has

$$\begin{aligned} T_3 &= \{0, 1, 1, 0, 0, 1, 0, 1\} \\ \text{index} & \quad 0, 1, 2, 3, 4, 5, 6, 7 \end{aligned} \tag{8}$$

$$E(3) = \{0, 3, 5, 6\}, \quad O(3) = \{1, 2, 4, 7\}. \tag{9}$$

$$\begin{aligned} 0^0 + 3^0 + 5^0 + 6^0 &= 1^0 + 2^0 + 4^0 + 7^0 = 4, \\ 0^1 + 3^1 + 5^1 + 6^1 &= 1^1 + 2^1 + 4^1 + 7^1 = 14, \\ 0^2 + 3^2 + 5^2 + 6^2 &= 1^2 + 2^2 + 4^2 + 7^2 = 70. \end{aligned} \tag{10}$$

While,

$$0^3 + 3^3 + 5^3 + 6^3 = 368 \neq 1^3 + 2^3 + 4^3 + 7^3 = 416. \quad (11)$$

Example. For $N = 4$, one has

$$T_4 = \{0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0\} \quad (12)$$

index 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15

$$E(4) = \{0, 3, 5, 6, 9, 10, 12, 15\}, \quad O(4) = \{1, 2, 4, 7, 8, 11, 13, 14\}. \quad (13)$$

$$\begin{aligned} 0^0 + 3^0 + 5^0 + 6^0 + 9^0 + 10^0 + 12^0 + 15^0 &= 8 \\ &= 1^0 + 2^0 + 4^0 + 7^0 + 8^0 + 11^0 + 13^0 + 14^0 = 8 \\ 0^1 + 3^1 + 5^1 + 6^1 + 9^1 + 10^1 + 12^1 + 15^1 &= 60 \\ &= 1^1 + 2^1 + 4^1 + 7^1 + 8^1 + 11^1 + 13^1 + 14^1 = 60 \\ 0^2 + 3^2 + 5^2 + 6^2 + 9^2 + 10^2 + 12^2 + 15^2 &= 620 \\ &= 1^2 + 2^2 + 4^2 + 7^2 + 8^2 + 11^2 + 13^2 + 14^2 = 620 \\ 0^3 + 3^3 + 5^3 + 6^3 + 9^3 + 10^3 + 12^3 + 15^3 &= 7200 \\ &= 1^3 + 2^3 + 4^3 + 7^3 + 8^3 + 11^3 + 13^3 + 14^3 = 7200. \end{aligned} \quad (14)$$

This relation demonstrates that the Thue-Morse sequence provides a solution to the Prouhet-Tarry-Escott problem (or multi-grades problem) for the given k [10].

1.2 | Definition of PTM (Logical) States

First, we define the notation $|k\rangle_2$, representing the N qubits state indexed by the base 2 notation of k with N digits, such that, for example, $|0\rangle_2 = \underbrace{|00\dots 0\rangle}_{N \text{ times}}$ or $|(2^N - 1)\rangle_2 = \underbrace{|11\dots 1\rangle}_{N \text{ times}}$.

Definition. We define the PTM (logical) states on a N qubits system, in the uncoupled basis, as:

$$|1_{\text{TM}}^{(N)}\rangle = \frac{1}{\sqrt{2^{N-1}}} \sum_{k=0}^{2^N-1} t_k |(k)_2\rangle = \frac{1}{\sqrt{2^{N-1}}} \sum_{o \in O(N)} |(o)_2\rangle, \quad (15)$$

and its complementary (using again $\bar{t} \equiv 1 - t$):

$$|0_{\text{TM}}^{(N)}\rangle = \frac{1}{\sqrt{2^{N-1}}} \sum_{k=0}^{2^N-1} \bar{t}_k |(k)_2\rangle = \frac{1}{\sqrt{2^{N-1}}} \sum_{e \in E(N)} |(e)_2\rangle. \quad (16)$$

Note the $1/\sqrt{2^{N-1}}$ prefactor, it arises simply because, by construction, in the first 2^N elements of the PTM sequence, exactly half are 1's. Note also that another useful property of the PTM sequence,

it being cube-free [2], appears at first glance to be irrelevant in the context of the odd and even sectors of the quantum computational space defined in the PTM states, while the same property proves to be crucial to quasicrystal applications for example [6].

If one notes that the PTM sequence t_i at index i is equal to 1 if and only if the binary representation of i has an odd amount of 1's, we get trivially the following property: $\forall N \in \mathbb{N}^*$,

$$|0_{\text{TM}}^{(N+1)}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |0_{\text{TM}}^{(N)}\rangle + |1\rangle \otimes |1_{\text{TM}}^{(N)}\rangle \right)$$

and

$$|1_{\text{TM}}^{(N+1)}\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |1_{\text{TM}}^{(N)}\rangle + |1\rangle \otimes |0_{\text{TM}}^{(N)}\rangle \right)$$

Here is an example of construction, the states contributing to $|0_{\text{TM}}^{(2)}\rangle$ (blue) and $|1_{\text{TM}}^{(2)}\rangle$ (red) are highlighted.

$$\begin{aligned} \boxed{N=2} \quad T_2 &= \{0 \quad 1 \quad 1 \quad 0\} \\ &\quad \quad \quad 0 \quad 1 \quad 2 \quad 3 \\ E(2) &= \{0 \quad \quad \quad 3\} \Rightarrow |0_{\text{TM}}^{(2)}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \\ (E(2))_2 &= \{00 \quad \quad \quad 11\} \\ O(2) &= \{ \quad 1 \quad 2 \quad \} \Rightarrow |1_{\text{TM}}^{(2)}\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle). \\ (O(2))_2 &= \{ \quad 01 \quad 10 \quad \} \end{aligned}$$

Moreover, by adding one more qubit, one can iterate the construction of, this time, $|0_{\text{TM}}^{(3)}\rangle$ and $|1_{\text{TM}}^{(3)}\rangle$. The components of the $|0_{\text{TM}}^{(2)}\rangle$ (resp. $|1_{\text{TM}}^{(2)}\rangle$) PTM state of the two qubit subsystem are underlined in blue (resp. red).

$$\begin{aligned} \boxed{N=3} \quad T_3 &= \{0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1\} \\ &\quad \quad \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \\ E(3) &= \{0 \quad \quad \quad 3 \quad \quad \quad 5 \quad 6 \quad \} \\ (E(3))_2 &= \{000 \quad \quad 011 \quad \quad 101 \quad 110 \quad \} \\ &\Rightarrow |0_{\text{TM}}^{(3)}\rangle = \frac{1}{2} (|000\rangle + |011\rangle + |101\rangle + |110\rangle) \\ &= \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |0_{\text{TM}}^{(2)}\rangle + |1\rangle \otimes |1_{\text{TM}}^{(2)}\rangle \right) \\ O(3) &= \{ \quad 1 \quad 2 \quad \quad 4 \quad \quad \quad 7 \} \\ (O(3))_2 &= \{ \quad 001 \quad 010 \quad \quad 100 \quad \quad \quad 111 \} \\ &\Rightarrow |1_{\text{TM}}^{(3)}\rangle = \frac{1}{2} (|001\rangle + |010\rangle + |100\rangle + |111\rangle) \\ &= \frac{1}{\sqrt{2}} \left(|0\rangle \otimes |1_{\text{TM}}^{(2)}\rangle + |1\rangle \otimes |0_{\text{TM}}^{(2)}\rangle \right) \end{aligned}$$

Definition. We define the qubit spin (Pauli) operators:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (17)$$

We will use $\sigma_i^{(k)}$ to indicate that the operator σ_i acts on the k -th qubit among N qubits, i.e. $\sigma_i^{(k)} = \underbrace{\mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2}_{k-1 \text{ times}} \otimes \sigma_i \otimes \underbrace{\mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2}_{N-k \text{ times}}$.

The PTM states present a certain number of interesting properties regarding spin operators $\sigma_{x,y,z}$, please refer to **A** for their proofs:

Property 1.2.1. $\forall N > 1, \langle 0_{\text{TM}}^{(N)} | \sum_{k=1}^N \sigma_{x,y,z}^{(k)} | 0_{\text{TM}}^{(N)} \rangle = 0$ and $\langle 1_{\text{TM}}^{(N)} | \sum_{k=1}^N \sigma_{x,y,z}^{(k)} | 1_{\text{TM}}^{(N)} \rangle = 0$.

Property 1.2.2. Let Z be an ensemble of $\sigma_z^{(k)}$ acting on M qubits ($M < N$)

$$\left\langle 0_{\text{TM}}^{(N)} \left| \prod_{k \in Z} \sigma_z^{(k)} \right| 0_{\text{TM}}^{(N)} \right\rangle = \left\langle 1_{\text{TM}}^{(N)} \left| \prod_{k \in Z} \sigma_z^{(k)} \right| 1_{\text{TM}}^{(N)} \right\rangle,$$

$$\text{and } \left\langle 1_{\text{TM}}^{(N)} \left| \prod_{k \in Z} \sigma_z^{(k)} \right| 0_{\text{TM}}^{(N)} \right\rangle = 0.$$

Property 1.2.3. Moreover $\forall j < N, i \in \{y, z\}$

$$\left\langle 0_{\text{TM}}^{(N)} \left| \left(\sum_{k=1}^N \sigma_i^{(k)} \right)^j \right| 0_{\text{TM}}^{(N)} \right\rangle = \left\langle 1_{\text{TM}}^{(N)} \left| \left(\sum_{k=1}^N \sigma_i^{(k)} \right)^j \right| 1_{\text{TM}}^{(N)} \right\rangle,$$

$$\text{and } \left\langle 1_{\text{TM}}^{(N)} \left| \left(\sum_{k=1}^N \sigma_i^{(k)} \right)^j \right| 0_{\text{TM}}^{(N)} \right\rangle = 0.$$

Property 1.2.4. $\forall k, j \leq N, \sigma_x^{(k)} \sigma_x^{(j)} | 0_{\text{TM}}^{(N)} \rangle = | 0_{\text{TM}}^{(N)} \rangle$ and $\sigma_x^{(k)} \sigma_x^{(j)} | 1_{\text{TM}}^{(N)} \rangle = | 1_{\text{TM}}^{(N)} \rangle$.

We will see in the following sections how these properties can be used for various applications.

1.3 | Relations Between the PTM States and the Hadamard Gate

Definition. We define the single qubit Hadamard gate, acting on the k -th qubit among N qubits:

$$H^{(k)} = \underbrace{\mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2}_{k-1 \text{ times}} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \underbrace{\mathbb{1}_2 \otimes \cdots \otimes \mathbb{1}_2}_{N-k \text{ times}}. \quad (18)$$

Property 1.3.1. The first interesting property of the Hadamard gate relating to these newly defined PTM states is that

$$\begin{aligned} \bigotimes_{k=1}^N H^{(k)} | (0)_2 \rangle &= \frac{1}{\sqrt{2}} \left(| 0_{\text{TM}}^{(N)} \rangle + | 1_{\text{TM}}^{(N)} \rangle \right) \\ \bigotimes_{k=1}^N H^{(k)} | (2^N - 1)_2 \rangle &= \frac{1}{\sqrt{2}} \left(| 0_{\text{TM}}^{(N)} \rangle - | 1_{\text{TM}}^{(N)} \rangle \right), \end{aligned} \quad (19)$$

allowing for a simple and natural passage from the PTM states to the uncoupled basis, namely

$$\begin{aligned} \bigotimes_{k=1}^N H^{(k)} | 0_{\text{TM}}^{(N)} \rangle &= \frac{1}{\sqrt{2}} (| (0)_2 \rangle + | (2^N - 1)_2 \rangle) \\ \bigotimes_{k=1}^N H^{(k)} | 1_{\text{TM}}^{(N)} \rangle &= \frac{1}{\sqrt{2}} (| (0)_2 \rangle - | (2^N - 1)_2 \rangle). \end{aligned} \quad (20)$$

An example of a circuit for encoding a state $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ into $|\psi_{\text{TM}}\rangle = \alpha |0_{\text{TM}}^{(N)}\rangle + \beta |1_{\text{TM}}^{(N)}\rangle$ is given in Figure 1.

1.4 | X-X Spin Chain and the PTM States

Property 1.2.4 can be generalized to verify that the PTM states are among the eigenstates of X - X rotations, and share the same eigenvalue:

$$\begin{aligned} \forall k, j \leq N, e^{i\theta \sigma_x^{(k)} \sigma_x^{(j)}} | 0_{\text{TM}}^{(N)} \rangle &= e^{i\theta} | 0_{\text{TM}}^{(N)} \rangle \\ \text{and } e^{i\theta \sigma_x^{(k)} \sigma_x^{(j)}} | 1_{\text{TM}}^{(N)} \rangle &= e^{i\theta} | 1_{\text{TM}}^{(N)} \rangle \end{aligned}$$

This means that X - X interactions only add a global phase in the Thue-Morse basis, making the encoding particularly useful to store memory qubits in X - X PTM spin lattices or chains.

1.5 | PTM States as Eigenstates of S_x

By defining $S_{x,z} = \sum_{k=1}^N \sigma_{x,z}^{(k)}$, we have that

$$S_z | (0)_2 \rangle = N | (0)_2 \rangle \quad \text{and} \quad S_z | (2^N - 1)_2 \rangle = -N | (2^N - 1)_2 \rangle. \quad (21)$$

Using Equation (19), the relation $H^{(k)} \sigma_z^{(k)} H^{(k)} = \sigma_x^{(k)}$ and consequently $\left(\bigotimes_{k=1}^N H^{(k)} \right) S_z \left(\bigotimes_{k=1}^N H^{(k)} \right) = S_x$, it is easy to see that

$$S_x | 0_{\text{TM}}^{(N)} \rangle = N | 1_{\text{TM}}^{(N)} \rangle \quad \text{and} \quad S_x | 1_{\text{TM}}^{(N)} \rangle = N | 0_{\text{TM}}^{(N)} \rangle. \quad (22)$$

The states $\frac{1}{\sqrt{2}} \left(| 0_{\text{TM}}^{(N)} \rangle \pm | 1_{\text{TM}}^{(N)} \rangle \right)$ are therefore eigenstates of S_x with eigenvalues $\pm N$.

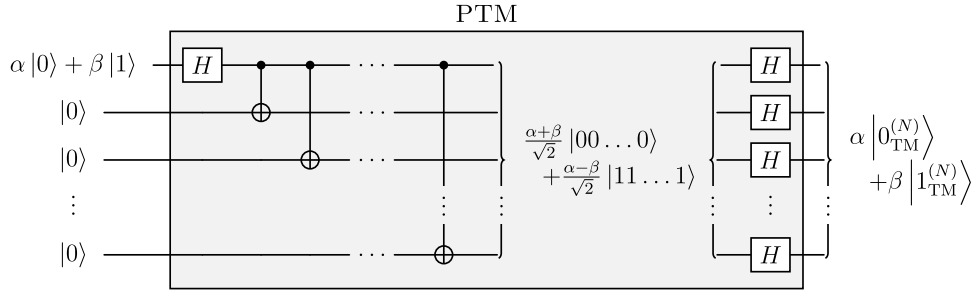


FIGURE 1 | Definition of the PTM Gate acting on N qubits.

2 | Appearance of the PTM Sequence in the Hilbert Spaces of Given Quantum Computing Platforms

2.1 | The PTM Sequence as the Indicator Function of the Purely Dephased X - X Ising Chain

Definition. We define an X - X Ising chain of decohering qubits as a system of N qubits evolving according to the equation:

$$\forall k \in [1, N], L_k = \underbrace{\mathbb{1}_2^{(1)} \otimes \dots \otimes \mathbb{1}_2^{(k-1)}}_{k-1} \otimes S_z^{(k)} \otimes \underbrace{\mathbb{1}_2^{(k+1)} \otimes \dots \otimes \mathbb{1}_2^{(N)}}_{N-k},$$

$$\partial_t \rho(t) = -\frac{i}{\hbar} [H(t), \rho(t)] + \sum_{k=1}^N \gamma_k \left(L_k \rho(t) L_k^\dagger - \frac{1}{2} \{ L_k^\dagger L_k, \rho(t) \} \right). \quad (23)$$

With $H(t) = g \sum_{k=1}^{N-1} S_x^{(k)} \otimes S_x^{(k+1)}$, and $\rho(t)$ the density matrix representing the state of the chain at time t . Here, L_k represents the Lindblad operator corresponding to the decoherence of the k -th qubit in the chain [11], with $S_z^{(k)} = \frac{1}{2} \sigma_z^{(k)}$, and γ_k is the decoherence rate associated with the k -th qubit.

If the system is initialized in the pure eigenstate $\rho(0) = |i_0\rangle \langle i_0|$, where i_0 belongs to either the set $E(N)$ or $O(N)$, then the final state will be a statistical mixture of states within the same set as the initial state. This implies that the PTM sequence acts as an indicator function, determining whether a state is present in the statistical mixture at the end of the evolution. In other words, $\forall t', \rho_{i,k}(t') \neq 0$ if $t_i = t_k = t_{i_0}$.

The linearity of time evolution allows us to extend this concept: if the initial state is a linear combination of eigenstates, all of which are elements of $E(N)$ (or $O(N)$), then the final state will also be a statistical mixture of states in $E(N)$ (or $O(N)$). Furthermore, if the initial state includes eigenstates from both sets, the final state will be a mixture of all eigenstates.

A special consideration applies to initial states of the form $|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} (|0\rangle_2 + |2^N - 1\rangle_2)$. By definition, $0 \in E(N)$, but $2^N - 1 \in E(N)$ if and only if N is even; otherwise, $2^N - 1 \in O(N)$. Therefore, an odd number of qubits initialized in a GHZ state will decohere into a statistical mixture of all eigenstates, whereas an even number of qubits will decohere into a statistical mixture of eigenstates only in E . See Figure 2.

2.2 | Initializing a Spin System in a Thue-Morse State

As we have seen in Section 1.5, the Thue-Morse states are mutually reciprocal under S_x , playing a role similar to $\frac{1}{\sqrt{2}} (|0\rangle_2 \pm |2^N - 1\rangle_2)$ for S_z , as hinted at by Equation (19).

By using the property that, for a single qubit, $\frac{\pi}{2T\sqrt{2}} (\sigma_z + \sigma_x)$ is the Hamiltonian that optimally generates the Hadamard gate in a time T up to a global phase [12], we get that $H^{(k)} \equiv \exp\left(\frac{-i\pi}{2\sqrt{2}} (\sigma_z^{(k)} + \sigma_x^{(k)})\right)$.

It is therefore clear that implementing the Hamiltonian $\frac{\pi}{2\sqrt{2}} (S_z + S_x)$ on a chain of spins in the $\frac{1}{\sqrt{2}} (|0\rangle_2 \pm |2^N - 1\rangle_2)$ state would allow to initialize a state in either $|0_{\text{TM}}^{(N)}\rangle$ or $|1_{\text{TM}}^{(N)}\rangle$, since, from Equation (19) we have that

$$\frac{1}{\sqrt{2}} \bigotimes_{k=1}^N H^{(k)} (|0\rangle_2 \pm |2^N - 1\rangle_2) = |0, 1_{\text{TM}}^{(N)}\rangle. \quad (24)$$

Examples of implementations for different N are shown on Figure 3.

3 | Uses of the PTM Sequence in Quantum Computing

3.1 | Quantum Error Correction (QEC)

Property 1.2.2 shows that the PTM states satisfy the Knill-Laflamme conditions [13], meaning up to $\frac{N-1}{2}$ single-qubit phase flip errors are detectable, and provided additional ancilla qubits, correctable.

Let us consider the traditional three qubit phase-flip error detection circuit depicted in Figure 4.

We notice, from Equation (19), that in essence by applying the Hadamard gates at the beginning, the state was mapped to the PTM basis as $|\phi_{\text{TM}}\rangle = \frac{\alpha+\beta}{\sqrt{2}} |0_{\text{TM}}^{(3)}\rangle + \frac{\alpha-\beta}{\sqrt{2}} |1_{\text{TM}}^{(3)}\rangle$ just before the error step. Therefore, to adapt this error correction code to the PTM basis, it is only necessary to map $|\phi\rangle$ to $|\phi_{\text{TM}}\rangle$ by

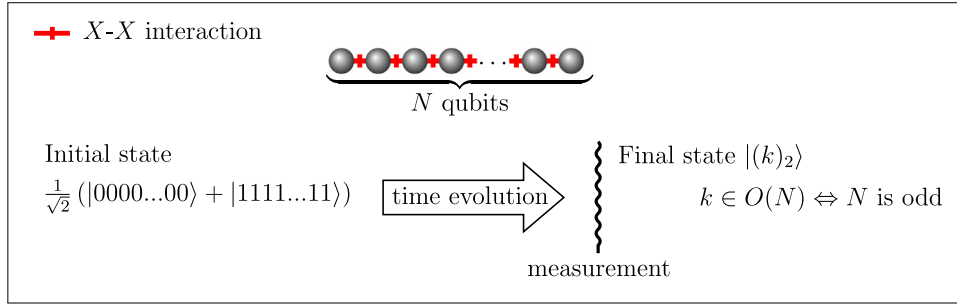


FIGURE 2 | Schematic representation of the interplay between the decoherence behavior of a X - X Ising chain initialized in the GHZ state, and the PTM sequence.

applying Hadamard gates at the end, as seen on Figure 5. This provides an error-correcting gate that is applied after the error has occurred, without the need for any state preparation prior to errors happening.

The same argument about error detection and correction can be applied to property 1.2.3 for errors due to a global magnetic field along the z or y direction.

Property 3.1.1. Property 1.2.3 can be generalized for d -level quantum bases of information, or qudits, if one defines $d = 2^N$ and the Thue-Morse states as:

$$|1_{\text{TM}}^{(N)}\rangle = \sqrt{\frac{2}{d}} \sum_{k=0}^{d-1} t_k |k\rangle, \quad |0_{\text{TM}}^{(N)}\rangle = \sqrt{\frac{2}{d}} \sum_{k=0}^{d-1} \bar{t}_k |k\rangle. \quad (25)$$

Qudits may present certain advantages over qubits, providing a higher density of information, and a reduced number of non-local gates. Moreover their higher error rates, due to the increased number of excited states, can be compensated in certain platforms with fast gate times and slow decoherence times [14, 15]. In this case, it can be proven that PTM states provide noise detection for J_z (spin- $\frac{d-1}{2}$ operator) up to order $\frac{1}{2} (\log_2(d) - 1)$, providing an encoding of error robust states conceptually similar to Chiesa et al. (2020) [16] all the while presenting the advantage of being encoded in a single physical system.

3.2 | Noise-Resistant Quantum Memories

Property 1.2.1. implies that the PTM states are robust to external magnetic fields along any spatial direction to first order.

Moreover, in the PTM logical basis, the different total spin operators take the following matrix forms:

$$\langle 0, 1_{\text{TM}}^{(N)} | S_x | 0, 1_{\text{TM}}^{(N)} \rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$S_z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (26)$$

This makes the PTM states particularly useful for storing a logical qubit in an environment with strong magnetic noise along the y or z -direction. This is not applicable in the x -direction, as only the states $|0_{\text{TM}}^{(N)}\rangle$ and $|1_{\text{TM}}^{(N)}\rangle$ are robust, but not superposition states.

3.3 | Link to Quantum Chaos

From the analysis presented in Equation (19), it becomes evident that the last row of the Walsh-Hadamard matrix corresponds to a modified PTM sequence, specifically represented as $(-1)^{t_n}$. This particular element and its connection to quantum chaos has been explored in depth in Refs. [17, 18], where the interplay between the PTM sequence's deterministic disorder property and the quantum Fourier transform is investigated. This synergy is central to the construction of an approximate multifractal eigenstate of the quantum baker's map, providing a novel approach to unraveling the complexity of quantum chaotic systems. Such findings not only highlight the multifaceted role of the PTM sequence within quantum chaos but also contribute significantly to our broader understanding of the dynamical properties of quantum systems.

Definition. An example of these interesting properties can be observed by defining the N -qubits Quantum Fourier Transform $QFT(N)$ gate, elementwise ($0 \leq j, k < N$)

$$(QFT(N))_{jk} = \frac{1}{\sqrt{2^N}} e^{i \frac{2\pi}{2^N} jk}. \quad (27)$$

This gate, that applies the discrete Fourier transform over the Hilbert space of the qubits, is essential for some quantum algorithms such as Shor's factorization algorithm [19].

Applying the QFT gate to a state in the PTM basis proves interesting, since

$$QFT(N) \left(\alpha |0_{\text{TM}}^{(N)}\rangle + \beta |1_{\text{TM}}^{(N)}\rangle \right) = \frac{\sqrt{2}}{2^N} \sum_{j=0}^{2^N-1} \sum_{k=0}^{2^N-1} \frac{1}{2} (\alpha + \beta + (-1)^{t_k} (\alpha - \beta)) e^{i \frac{2\pi}{2^N} jk} |j\rangle_2 \quad (28)$$

Therefore, by defining $F_j^{TM} \equiv \langle j | QFT(N) (\alpha |0_{\text{TM}}^{(N)}\rangle + \beta |1_{\text{TM}}^{(N)}\rangle) \rangle$ and since $\forall j \in \mathbb{N}^*, \sum_{k=0}^{2^N-1} e^{i \frac{2\pi}{2^N} jk} = 0$, we have

$$F_{j \neq 0}^{TM} = \frac{\alpha - \beta}{2^N \sqrt{2}} \sum_{k=0}^{2^N-1} (-1)^{t_k} e^{i \frac{2\pi}{2^N} jk}. \quad (29)$$

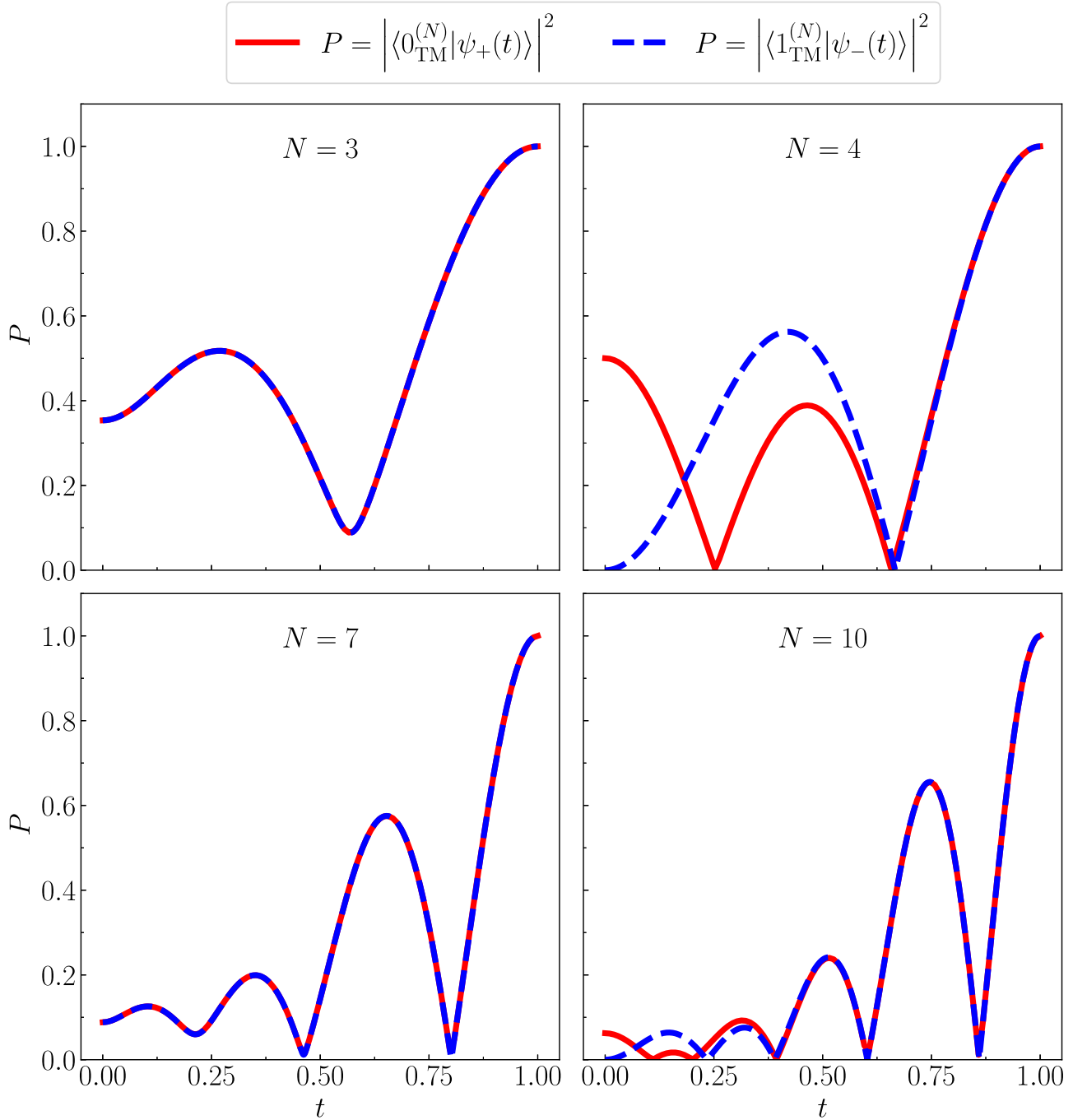


FIGURE 3 | Time-evolution of the populations P of different states initialized in $|\psi_+(0)\rangle = \frac{1}{\sqrt{2}}(|(0)_2\rangle + |(N)_2\rangle)$ (red) or $|\psi_-(0)\rangle = \frac{1}{\sqrt{2}}(|(0)_2\rangle - |(N)_2\rangle)$ (blue). With $|\psi_{\pm}(t)\rangle = \exp\left(\frac{-i\pi}{2\sqrt{2}}(S_z + S_x)t\right)|\psi_{\pm}(0)\rangle$. The final state corresponds to $|\psi_{\pm}(1)\rangle = \bigotimes_{k=1}^N H^{(k)}|\psi_{\pm}(0)\rangle$.

Moreover [10],

$$\forall x \in \mathbb{R}, \prod_{k=0}^{N-1} (1 - x^{2^k}) = \sum_{k=0}^{2^N-1} (-1)^k x^k, \quad (30)$$

hence,

$$F_{j \neq 0}^{TM} = \frac{\alpha - \beta}{2^N \sqrt{2}} \prod_{k=0}^{N-1} \left(1 - e^{i \frac{2\pi j}{2^{N-k}}}\right). \quad (31)$$

The specific case of $j = 0$, is given by

$$F_0^{TM} = \frac{\alpha + \beta}{\sqrt{2}}. \quad (32)$$

Further simplifying, we obtain, for $j \neq 0$

$$\left|F_j^{TM}\right|^2 = \frac{(\alpha - \beta)^2}{2} \prod_{k=0}^{N-1} \sin^2(\pi j 2^{k-N}). \quad (33)$$

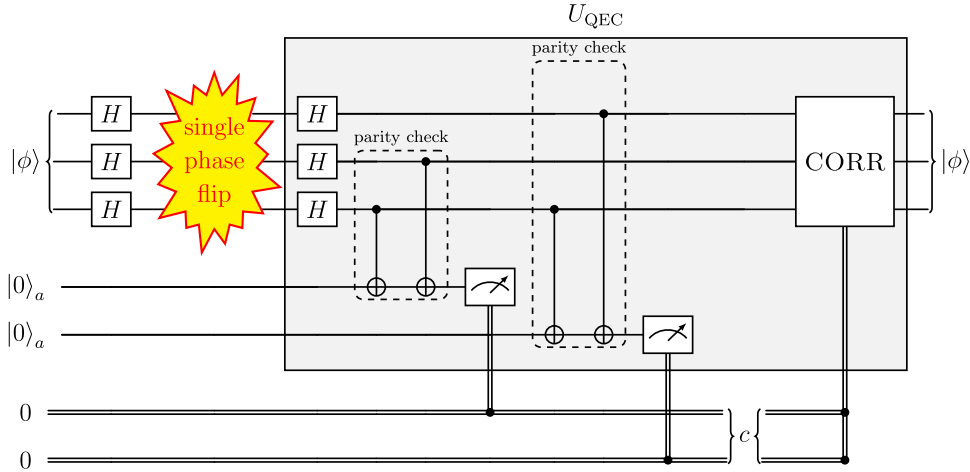


FIGURE 4 | Traditional (Shor's) three-qubit phase-flip error correction code. One can detect and correct 1 phase flip error using 2 ancilla qubits for parity checks, and with initial state $|\phi\rangle = \alpha|000\rangle + \beta|111\rangle$. CORR takes as an input the classical two-bits registry $c = (i)_2$, does nothing if $i = 0$, and flips qubit i otherwise, counting from top to bottom.

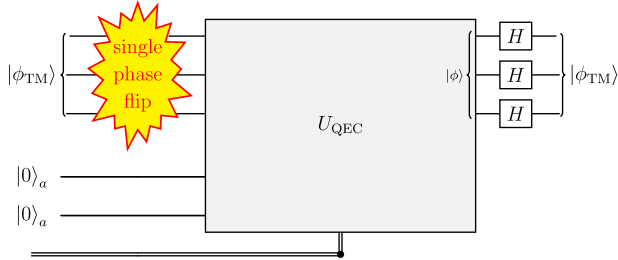


FIGURE 5 | Three-qubit single phase-flip error correction code for one logical PTM state.

For the case $\alpha = -\beta = \frac{1}{\sqrt{2}}$, $|F_j^{TM}\rangle^2 = \langle j | QFT(N) \otimes_{k=1}^N H^{(k)} | (2^N - 1)_2 \rangle$ are shown in Figure 6 for $N = 20$. One can notice self-similarity and infer the multifractal nature of the state in the Hilbert space.

It is these multifractal properties [20] and the recursive definition of the Thue-Morse states that have been used in to construct, for example, approximate eigenstates of the quantum baker map $B(N)$, such as in the works of Lakshminarayan et al. [17, 21, 22].

$$B(N) = QFT(N)^{-1} \begin{pmatrix} QFT(N-1) & 0 \\ 0 & QFT(N-1) \end{pmatrix}. \quad (34)$$

Moreover, considering the use of both the Hadamard gate $\otimes_{k=1}^N H^{(k)}$ and QFT gate $QFT(N)$ in Shor's factorization algorithm, it would be interesting to study further the appearance of quantum chaos that we showed to be arising from their interactions in such a well-known quantum algorithm [18, 22, 23].

3.4 | Link with Number Theory

There are several interesting formulae that use the PTM sequence [2, 24], including:

- Its product generating function formula

$$\forall x \in \mathbb{R}, \quad \prod_{i=0}^N (1 - x^{2^i}) = \sum_{j=0}^{2^{N+1}-1} (-1)^{t_j} x^j, \quad (35)$$

it can also be generalized for $N \rightarrow \infty$,

$$\forall x \in \mathbb{R}, \quad \prod_{i=0}^{\infty} (1 - x^{2^i}) = \sum_{j=0}^{\infty} (-1)^{t_j} x^j. \quad (36)$$

- $\prod_{n=0}^{\infty} \left(\frac{2n+1}{2n+2}\right)^{2(1-t_n)} \left(\frac{2n+3}{2n+2}\right) = \frac{\sqrt{2}}{\pi}$
- $\prod_{n=0}^{\infty} \left(\frac{2n+1}{2n+2}\right)^{2t_n} \left(\frac{2n+3}{2n+2}\right) = \frac{2\sqrt{2}}{\pi}$
- $\prod_{n=0}^{\infty} \left(\frac{2n+1}{2n+2}\right)^{(-1)^{t_n}} = \frac{\sqrt{2}}{2}$
- $\prod_{n=1}^{\infty} \left(\frac{2n}{2n+1}\right)^{(-1)^{t_n}} = 1.6281\dots$ converges toward a number whose algebraic nature remains unknown [2].
- $\tau = \sum_{i=0}^{\infty} \frac{t_i}{2^{i+1}} = 0.41245\dots$ is known as the PTM constant and has been proved to be a transcendental number.
- $\forall s \in \mathbb{C}$ s.t. $\text{Re}(s) > 1, \zeta(s) = \left(1 + \frac{1}{2^s}\right) \sum_{n \geq 1} \frac{t_{n-1}}{n^s}$ where ζ is the Riemann Zeta function.

In particular, the last formula presents an interesting linear combination of Dirichlet series. When linked to the work of Feiler and Schleich [25], it becomes of particular interest as it provides a link between quantum computing and number theory. Just as in the cited work, one can obtain the result of this sum of Dirichlet

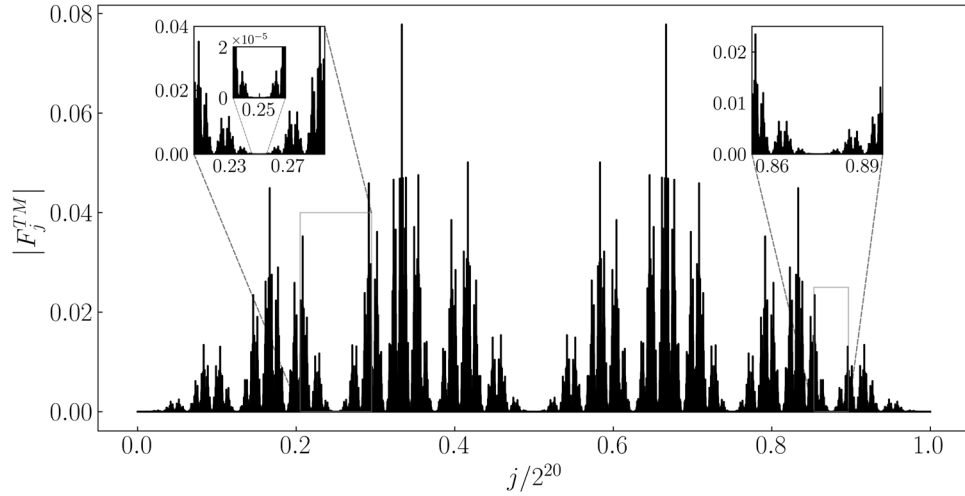


FIGURE 6 | Amplitudes of $F_j^{TM} = \frac{1}{\sqrt{2}} \left\langle j \left| QFT(20) \left(\left| 0_{TM}^{(20)} \right\rangle - \left| 1_{TM}^{(20)} \right\rangle \right) \right\rangle = \left\langle j \left| QFT(20) \otimes_{k=1}^{20} H^{(k)} \left| (2^{20} - 1)_2 \right\rangle \right\rangle$ for $j = 0$ to $2^{20} - 1$. Two zooms, around 0.2 and 0.88 show the self-similarity of the curve. An additional zoom around 0.25 further shows the self similar behavior.

series using the interference of two initial quantum states in a non-linear interaction with logarithmic energy spectrum. Let us assume a given system evolves under a non-linear oscillator Hamiltonian with a logarithmic energy spectrum: $H = \hbar\omega \ln(n+1) |n\rangle \langle n|$ [26]. Given a certain $s = \sigma + i\tau$ complex argument, one can consider two non-normalized states:

$$\begin{aligned} |\psi_1\rangle &= \sum_{n \geq 0} t_n \cdot (n+1)^{-\sigma/2} |n\rangle \\ |\psi_2\rangle &= \sum_{n \geq 0} t_{n+1} \cdot (n+1)^{-\sigma/2} |n\rangle. \end{aligned} \quad (37)$$

If one measures the conjugate auto-correlation probability amplitude $\langle \psi^* | |\psi(t)\rangle$ for $|\psi_1\rangle$ (resp. $|\psi_2\rangle$), evolved for time $t = \tau/\omega$, the result will be proportional to $\sum_{n \geq 1} \frac{t_{n-1}}{n^s}$ (resp. $\sum_{n \geq 1} \frac{t_n}{n^s}$).

Following the arguments of Feiler et al. [25], we remark that by evolving the initial state, normalized by \mathcal{N} ,

$$|\psi_0\rangle = \mathcal{N} \left(\left(1 + \frac{1}{2^s}\right) |\psi_1\rangle + \left(1 - \frac{1}{2^s}\right) |\psi_2\rangle \right), \quad (38)$$

for a time $t = \tau/g$ under an interaction with a logarithmic energy spectrum in the interaction picture [27], where the interaction Hamiltonian is given by $H_I = \hbar g \ln(\hat{n} + 1) \hat{\sigma}_z$, the measurement of the conjugate auto-correlation probability amplitude $\langle \psi_0^* | \psi_0(t = \tau/g) \rangle$ is expected to yield the exact value of the Riemann zeta function:

$$\zeta(s) = \mathcal{N}^2 \left[\left(1 + \frac{1}{2^s}\right) \sum_{n \geq 1} \frac{t_{n-1}}{n^s} + \left(1 - \frac{1}{2^s}\right) \sum_{n \geq 1} \frac{t_n}{n^s} \right].$$

This provides a method to determine the value of the Riemann Zeta function through quantum measurements.

4 | Conclusion

In this study, we have explored the many manifestations and uses of the Prouhet-Thue-Morse (PTM) sequence in quantum

computing, highlighting its intrinsic mathematical interest and practical significance. Through rigorous analysis, we have shown that the PTM sequence, beyond its mathematical appeal, plays a central role in quantum error correction, the design of noise-resistant quantum memories. We also elucidated its links with quantum chaos and number theory.

Our investigations have shown how logical states whose encoding is based on the PTM sequence exhibit remarkable properties, such as resilience to spin flip errors and low sensitivity to external magnetic fields, thus providing valuable insights for the design of fault-tolerant quantum computing architectures. Furthermore, the integration of the sequence into the Hilbert space of quantum computing platforms, particularly in the context of the XX-XX Ising chain and spin systems, demonstrates its potential to increase the efficiency and reliability of quantum information processing.

The connection between the PTM sequence and quantum chaos, through the Walsh-Hadamard transform and the quantum baker's map, enriches our understanding of the dynamical behaviors of quantum systems and provides a novel lens through which to view quantum chaos. Moreover, the intriguing links between the PTM sequence and number theory, exemplified by its relation to the Riemann zeta function and Dirichlet series, open new avenues for interdisciplinary research, bridging quantum computing, and mathematical number theory.

While the novelty of our results is tempered by the fundamental nature of the PTM sequence in both mathematics and physics, the implications of our findings for quantum computing are profound. The insights gained from this study not only contribute to the ongoing development of quantum computing technologies but also stimulate further research into the underlying mathematical structures that govern quantum mechanics.

As we continue to unravel the complexities of quantum systems and their computing capabilities, the PTM sequence stands as a testament to the deep connections between abstract mathematical concepts and practical quantum technologies. Our exploration

of these links also highlights the importance of interdisciplinary research, in this case, physics and mathematics, in order to reveal the full potential of quantum computing.

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Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

The codes and calculations done in this study are available from the authors upon reasonable request.

Conflicts of Interest

The authors declare no conflict of interest.

Endnotes

¹In some of these works, another property of the PTM sequence is used: it being cube-free, as defined in combinatorics; which is not considered in this work. The PTM sequence admits several equivalent definitions and is emphasized differently across communities. In some contexts, the terminology “Thue–Morse” or “Prouhet–Thue–Morse” is associated more specifically with one of its combinatorial, recursive, or number-theoretic characterizations. In the present work, we use the term PTM sequence for the binary sequence defined above and focus on the properties relevant to the quantum states introduced here.

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Appendix A: Proofs of Properties

A.1 | Proof of Property 1.2.1

$$\forall N > 1, \left\langle 0_{TM}^{(N)} \left| \sum_{k=1}^N \sigma_{x,y,z}^{(k)} \right| 0_{TM}^{(N)} \right\rangle = 0 \text{ and } \left\langle 1_{TM}^{(N)} \left| \sum_{k=1}^N \sigma_{x,y,z}^{(k)} \right| 1_{TM}^{(N)} \right\rangle = 0$$

Proof for σ_z : (by induction)

- One can easily check that for $N = 2$ it is true. Using $\sigma_z^{(1)} + \sigma_z^{(2)} = \sigma_z \otimes \mathbb{1}_2 + \mathbb{1}_2 \otimes \sigma_z$.

$$\begin{aligned} \left\langle 0_{TM}^{(2)} \left| \left(\sigma_z^{(1)} + \sigma_z^{(2)} \right) \right| 0_{TM}^{(2)} \right\rangle &= \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= 2 - 2 = 0, \end{aligned}$$

$$\left\langle 1_{TM}^{(2)} \left| \left(\sigma_z^{(1)} + \sigma_z^{(2)} \right) \right| 1_{TM}^{(2)} \right\rangle = \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = 0.$$

- Using the recursive properties of the Thue-Morse sequence,

$$\begin{aligned} \left\langle 0_{TM}^{(N)} \left| \sum_{k=1}^N \sigma_z^{(k)} \right| 0_{TM}^{(N)} \right\rangle &= \left\langle 0_{TM}^{(N-1)} \left| \left\langle 0 \left| \sum_{k=1}^N \sigma_z^{(k)} \right| 0 \right\rangle \right| 0_{TM}^{(N-1)} \right\rangle \\ &+ \left\langle 0_{TM}^{(N-1)} \left| \left\langle 0 \left| \sum_{k=1}^N \sigma_z^{(k)} \right| 1 \right\rangle \right| 1_{TM}^{(N-1)} \right\rangle \\ &+ \left\langle 1_{TM}^{(N-1)} \left| \left\langle 1 \left| \sum_{k=1}^N \sigma_z^{(k)} \right| 0 \right\rangle \right| 0_{TM}^{(N-1)} \right\rangle \\ &+ \left\langle 1_{TM}^{(N-1)} \left| \left\langle 1 \left| \sum_{k=1}^N \sigma_z^{(k)} \right| 1 \right\rangle \right| 1_{TM}^{(N-1)} \right\rangle \end{aligned}$$

and

$$\begin{aligned} \left\langle 1_{TM}^{(N)} \left| \sum_{k=1}^N \sigma_z^{(k)} \right| 1_{TM}^{(N)} \right\rangle &= \left\langle 1_{TM}^{(N-1)} \left| \left\langle 0 \left| \sum_{k=1}^N \sigma_z^{(k)} \right| 0 \right\rangle \right| 1_{TM}^{(N-1)} \right\rangle \\ &+ \left\langle 1_{TM}^{(N-1)} \left| \left\langle 0 \left| \sum_{k=1}^N \sigma_z^{(k)} \right| 1 \right\rangle \right| 0_{TM}^{(N-1)} \right\rangle \\ &+ \left\langle 0_{TM}^{(N-1)} \left| \left\langle 1 \left| \sum_{k=1}^N \sigma_z^{(k)} \right| 0 \right\rangle \right| 1_{TM}^{(N-1)} \right\rangle \\ &+ \left\langle 0_{TM}^{(N-1)} \left| \left\langle 1 \left| \sum_{k=1}^N \sigma_z^{(k)} \right| 1 \right\rangle \right| 0_{TM}^{(N-1)} \right\rangle. \end{aligned}$$

Assuming the property holds true for $N - 1$, and since $\sigma_z^{(k)}$ is diagonal, both equations reduce to $\sigma_z^{(1)}$ acting on the added qubit and are equal to:

$$\langle 0 | \sigma_z | 0 \rangle + \langle 1 | \sigma_z | 1 \rangle = 1 - 1 = 0$$

Proof for σ_x and σ_y : Since the effect of $\sigma_x^{(k)}$ is to flip the k^{th} qubit, $\sum_{k=1}^N \sigma_x^{(k)}$ maps $|0_{TM}^{(N)}\rangle \leftrightarrow |1_{TM}^{(N)}\rangle$, which are orthogonal to each other, hence the property is true.

For σ_y , it follows from $\sigma_y = -i\sigma_z\sigma_x$. Therefore, by defining $c_k := 0, 1$ as the state of the k^{th} qubit and $\bar{c}_k = 1 - c_k$

$$\begin{aligned} \left\langle 0, 1_{TM}^{(N)} \left| \sum_{k=1}^N \sigma_y^{(k)} \right| 0, 1_{TM}^{(N)} \right\rangle &= -i \left\langle 0, 1_{TM}^{(N)} \left| \sum_{k=1}^N \sigma_z^{(k)} \sigma_x^{(k)} \right| 0, 1_{TM}^{(N)} \right\rangle \\ &= -i \sum_{k=1}^N \left\langle 0, 1_{TM}^{(N)} \left| \sigma_z^{(k)} \sigma_x^{(k)} \right| 0, 1_{TM}^{(N)} \right\rangle \\ &= -i \sum_{k=1}^N \left\langle 0, 1_{TM}^{(N)} \left| \sigma_z^{(k)} \sigma_x^{(k)} \right| 0, 1_{TM}^{(N)} \right\rangle \\ &= -i \sum_{k=1}^N \langle c_k | \sigma_z | \bar{c}_k \rangle = 0. \end{aligned}$$

A.2 | Proof of Property 1.2.2

Let Z be an ensemble of $\sigma_z^{(k)}$ acting on M qubits ($M < N$)

$$\left\langle 0_{TM}^{(N)} \left| \prod_{k \in Z} \sigma_z^{(k)} \right| 0_{TM}^{(N)} \right\rangle = \left\langle 1_{TM}^{(N)} \left| \prod_{k \in Z} \sigma_z^{(k)} \right| 1_{TM}^{(N)} \right\rangle,$$

$$\text{and } \left\langle 1_{TM}^{(N)} \left| \prod_{k \in Z} \sigma_z^{(k)} \right| 0_{TM}^{(N)} \right\rangle = 0.$$

First, we recall that σ_z is involutory i.e. $(\sigma_z^{(k)})^2 = 1$ and having a $\sigma_z^{(k)}$ appear an odd amount of time is equivalent to it appearing only once. Therefore, we notice

$$\prod_{k \in Z} \sigma_z^{(k)} = \bigotimes_{k \in Z'} \sigma_z^{(k)}$$

with Z' being the ensemble of only one occurrence of the $\sigma^{(k)}$ s that appear an odd amount of times and not the others.

Induction:

- One can check that for $N = 2$, the equality does not hold for $Z' = (\sigma_z^{(1)}, \sigma_z^{(2)})$, but does for any subset.

$$\left\{ \begin{aligned} \left\langle 0_{TM}^{(2)} \left| \sigma_z^{(1)} \otimes \mathbb{1}_2 \right| 0_{TM}^{(2)} \right\rangle &= (1 \ 0 \ 0 \ 1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= 1 - 1 = 0 \\ \left\langle 1_{TM}^{(2)} \left| \sigma_z^{(1)} \otimes \mathbb{1}_2 \right| 1_{TM}^{(2)} \right\rangle &= (0 \ 1 \ 1 \ 0) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\ &= 1 - 1 = 0 \\ \left\langle 0_{TM}^{(2)} \left| \mathbb{1}_2 \otimes \sigma_z^{(2)} \right| 0_{TM}^{(2)} \right\rangle &= (1 \ 0 \ 0 \ 1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= 1 - 1 = 0 \\ \left\langle 1_{TM}^{(2)} \left| \mathbb{1}_2 \otimes \sigma_z^{(2)} \right| 1_{TM}^{(2)} \right\rangle &= (0 \ 1 \ 1 \ 0) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\ &= 1 - 1 = 0 \\ \left\langle 0_{TM}^{(2)} \left| \sigma_z^{(1)} \otimes \sigma_z^{(2)} \right| 0_{TM}^{(2)} \right\rangle &= (1 \ 0 \ 0 \ 1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ &= 1 + 1 = 2 \\ \left\langle 1_{TM}^{(2)} \left| \sigma_z^{(1)} \otimes \sigma_z^{(2)} \right| 1_{TM}^{(2)} \right\rangle &= (0 \ 1 \ 1 \ 0) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\ &= -1 - 1 = -2 \end{aligned} \right.$$

The orthogonality for $N = 2$ is conserved since we are dealing with diagonal operators.

- The orthogonality for any N is trivial since the matrices are diagonal. Let's look at $\left\langle 0_{TM}^{(N)} \left| \bigotimes_{k \in Z'} \sigma_z^{(k)} \right| 0_{TM}^{(N)} \right\rangle - \left\langle 1_{TM}^{(N)} \left| \bigotimes_{k \in Z'} \sigma_z^{(k)} \right| 1_{TM}^{(N)} \right\rangle$ and use the recursive definition of the states.

We also define $\epsilon^{(k)} = \begin{cases} \sigma_z^{(k)} & \text{if } \sigma_z^{(k)} \in Z' \\ \mathbb{1}_2 & \text{if not} \end{cases}$.

$$\begin{aligned} &\left\langle 0_{TM}^{(N)} \left| \bigotimes_{k \in Z'} \sigma_z^{(k)} \right| 0_{TM}^{(N)} \right\rangle - \left\langle 1_{TM}^{(N)} \left| \bigotimes_{k \in Z'} \sigma_z^{(k)} \right| 1_{TM}^{(N)} \right\rangle \\ &= \left\langle 0_{TM}^{(N-1)} \left| \left\langle 0 \left| \bigotimes_{k \in Z'} \sigma_z^{(k)} \right| 0 \right\rangle \right| 0_{TM}^{(N-1)} \right\rangle \\ &\quad + \left\langle 1_{TM}^{(N-1)} \left| \left\langle 1 \left| \bigotimes_{k \in Z'} \sigma_z^{(k)} \right| 1 \right\rangle \right| 1_{TM}^{(N-1)} \right\rangle \\ &\quad - \left\langle 1_{TM}^{(N-1)} \left| \left\langle 0 \left| \bigotimes_{k \in Z'} \sigma_z^{(k)} \right| 0 \right\rangle \right| 1_{TM}^{(N-1)} \right\rangle \\ &\quad - \left\langle 0_{TM}^{(N-1)} \left| \left\langle 1 \left| \bigotimes_{k \in Z'} \sigma_z^{(k)} \right| 1 \right\rangle \right| 0_{TM}^{(N-1)} \right\rangle \end{aligned}$$

$$\begin{aligned} &= \left\langle 0_{TM}^{(N-1)} \left| \bigotimes_{k \in Z' \setminus \sigma_z^{(N)}} \sigma_z^{(k)} \right| 0_{TM}^{(N-1)} \right\rangle \left\langle 0 \left| \epsilon^{(N)} \right| 0 \right\rangle \\ &\quad + \left\langle 1_{TM}^{(N-1)} \left| \bigotimes_{k \in Z' \setminus \sigma_z^{(N)}} \sigma_z^{(k)} \right| 1_{TM}^{(N-1)} \right\rangle \left\langle 1 \left| \epsilon^{(N)} \right| 1 \right\rangle \\ &\quad - \left\langle 1_{TM}^{(N-1)} \left| \bigotimes_{k \in Z' \setminus \sigma_z^{(N)}} \sigma_z^{(k)} \right| 1_{TM}^{(N-1)} \right\rangle \left\langle 0 \left| \epsilon^{(N)} \right| 0 \right\rangle \\ &\quad - \left\langle 0_{TM}^{(N-1)} \left| \bigotimes_{k \in Z' \setminus \sigma_z^{(N)}} \sigma_z^{(k)} \right| 0_{TM}^{(N-1)} \right\rangle \left\langle 1 \left| \epsilon^{(N)} \right| 1 \right\rangle \\ &= \left\langle 0_{TM}^{(N-1)} \left| \bigotimes_{k \in Z' \setminus \sigma_z^{(N)}} \sigma_z^{(k)} \right| 0_{TM}^{(N-1)} \right\rangle \\ &\quad - \left\langle 1_{TM}^{(N-1)} \left| \bigotimes_{k \in Z' \setminus \sigma_z^{(N)}} \sigma_z^{(k)} \right| 1_{TM}^{(N-1)} \right\rangle \left\langle 0 \left| \epsilon^{(N)} \right| 0 \right\rangle \\ &\quad + \left\langle 1_{TM}^{(N-1)} \left| \bigotimes_{k \in Z' \setminus \sigma_z^{(N)}} \sigma_z^{(k)} \right| 1_{TM}^{(N-1)} \right\rangle \\ &\quad - \left\langle 0_{TM}^{(N-1)} \left| \bigotimes_{k \in Z' \setminus \sigma_z^{(N)}} \sigma_z^{(k)} \right| 0_{TM}^{(N-1)} \right\rangle \left\langle 1 \left| \epsilon^{(N)} \right| 1 \right\rangle \end{aligned}$$

If we assume the property holds for $N - 1$, this expression equals to zero if

$-\sigma_z \in Z'$ and $Z' \setminus \sigma_z^{(N)}$ doesn't already contain all the other $N - 1$ $\sigma_z^{(k)}$ s.

Since $\sigma_z \in Z'$, this means $\epsilon^{(N)} = \sigma_z$ which implies

$\left\langle 0 \left| \epsilon^{(N)} \right| 0 \right\rangle = -\left\langle 1 \left| \epsilon^{(N)} \right| 1 \right\rangle$, so in order for the difference

to be zero, we need $\left\langle 0_{TM}^{(N-1)} \left| \bigotimes_{k \in Z' \setminus \sigma_z^{(N)}} \sigma_z^{(k)} \right| 0_{TM}^{(N-1)} \right\rangle -$

$\left\langle 1_{TM}^{(N-1)} \left| \bigotimes_{k \in Z' \setminus \sigma_z^{(N)}} \sigma_z^{(k)} \right| 1_{TM}^{(N-1)} \right\rangle = 0$, which does hold true

as long as $Z' \setminus \sigma_z^{(N)}$ doesn't already contain all the other $N - 1$ $\sigma_z^{(k)}$ s.

or

$-\sigma_z \notin Z'$ since $\epsilon^{(N)} = \mathbb{1}_2 \Rightarrow \left\langle 0 \left| \epsilon^{(N)} \right| 0 \right\rangle = \left\langle 1 \left| \epsilon^{(N)} \right| 1 \right\rangle$.

So, $\left\langle 0_{TM}^{(N)} \left| \bigotimes_{k \in Z'} \sigma_z^{(k)} \right| 0_{TM}^{(N)} \right\rangle - \left\langle 1_{TM}^{(N)} \left| \bigotimes_{k \in Z'} \sigma_z^{(k)} \right| 1_{TM}^{(N)} \right\rangle = 0$ iff E'

doesn't contain all $\sigma_z^{(k)}$ s. A similar argument can be made with $\sigma_y = -i\sigma_z\sigma_x$ using the diagonality of σ_z .

A.3 | Proof of Property 1.2.3

$\forall j < N, i \in \{y, z\}$

$$\left\langle 0_{TM}^{(N)} \left| \left(\sum_{k=1}^N \sigma_i^{(k)} \right)^j \right| 0_{TM}^{(N)} \right\rangle = \left\langle 1_{TM}^{(N)} \left| \left(\sum_{k=1}^N \sigma_i^{(k)} \right)^j \right| 1_{TM}^{(N)} \right\rangle,$$

$$\text{and } \left\langle 1_{TM}^{(N)} \left| \left(\sum_{k=1}^N \sigma_i^{(k)} \right)^j \right| 0_{TM}^{(N)} \right\rangle = 0.$$

For a given integer j ,

$$\left(\sum_{k=1}^N \sigma_{y,z}^{(k)}\right)^j = \sum_{j_1+j_2+\dots+j_N=j} \frac{j!}{j_1! j_2! \dots j_N!} \prod_{k=1}^N \left(\sigma_{y,z}^{(k)}\right)^{j_k},$$

which is then a state-independent weighted sum of products over ensembles E like in property 2. The ensembles E are then simply partitions of j of size N , and as long as $j < N$, it is not possible for E to contain all $\sigma_z^{(k)}$ s. If $j \geq N$, there exists ensembles E such that the equalities of property 1.2.2 do not hold anymore. Therefore property 1.2.3 is a consequence of property 1.2.2.

A.3.1 | Qudit Version for J_z

If one works with PTM logical states in a qudit of dimension $d = 2^N$.

$$\begin{aligned} |1_{\text{TM}}^{(N)}\rangle &= \sqrt{\frac{2}{d}} \sum_{k=0}^{d-1} t_k |k\rangle = \sqrt{\frac{2}{d}} \sum_{k \in O(N)} |k\rangle \\ \text{and } |0_{\text{TM}}^{(N)}\rangle &= \sqrt{\frac{2}{d}} \sum_{k=0}^{d-1} \bar{t}_k |k\rangle = \sqrt{\frac{2}{d}} \sum_{k \in E(N)} |k\rangle \end{aligned}$$

We have that

$$\begin{aligned} \langle 0_{\text{TM}}^{(N)} | J_z^j | 0_{\text{TM}}^{(N)} \rangle &= \sum_{k=0}^{d-1} \left(\frac{d-1-2k}{2}\right)^j \\ &= 2^{-j} \sum_{k \in E(N)} \sum_{l=0}^j \binom{j}{n} 2^l k^l (d-1)^{j-l} \\ &= 2^{-j} \sum_{l=0}^j \binom{j}{n} 2^l \left(\sum_{k \in E(N)} k^l\right) (d-1)^{j-l}, \end{aligned}$$

and similarly,

$$\langle 1_{\text{TM}}^{(N)} | J_z^j | 1_{\text{TM}}^{(N)} \rangle = 2^{-j} \sum_{l=0}^j \binom{j}{n} 2^l \left(\sum_{k \in O(N)} k^l\right) (d-1)^{j-l}.$$

Referring to the discussion about the Prouhet-Tarry-Escott problem, we know that $\sum_{k \in E(N)} k^l = \sum_{k \in O(N)} k^l$, and, therefore $\langle 0_{\text{TM}}^{(N)} | J_z^j | 0_{\text{TM}}^{(N)} \rangle = \langle 1_{\text{TM}}^{(N)} | J_z^j | 1_{\text{TM}}^{(N)} \rangle$. Moreover, J_z is diagonal therefore $\langle 1_{\text{TM}}^{(N)} | J_z^j | 0_{\text{TM}}^{(N)} \rangle = 0$.

A.4 | Proof of Property 1.2.4

$$\forall k, j \leq N, \quad \sigma_x^{(k)} \sigma_x^{(j)} |0_{\text{TM}}^{(N)}\rangle = |0_{\text{TM}}^{(N)}\rangle \text{ and } \sigma_x^{(k)} \sigma_x^{(j)} |1_{\text{TM}}^{(N)}\rangle = |1_{\text{TM}}^{(N)}\rangle$$

Since $\sigma_x^2 = 1$, if $k = j$, then the state remains unchanged. If exactly two different qubits are flipped without a relative phase (which is the action of two σ_x operators), the parity of the number of 1's stays unchanged.