Optimal Consumer Adoption of Energy Efficiency

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Abstract

Despite many decades of research, economically-grounded models that integrate energy consumption and energy-efficiency adoption decisions remain underdeveloped. As a result, critical questions such as the size of the energy-efficiency gap remain contested, and empirical estimates of phenomena like the rebound effect often rely on misspecified models. Moreover, policy prescriptions differ: some researchers advocate for targeted regulation to narrow the energy-efficiency gap or mitigate rebound, whereas others argue that such interventions would reduce welfare by restricting the choice set and suppressing legitimate consumer surplus.

The principal contribution of this dissertation is the development of a unified consumption–investment framework for analysing household energy consumption and energy-efficiency adoption. In contrast to accounting heuristics and static optimisation approaches, the proposed framework formulates adoption as a fully dynamic intertemporal choice problem. This formulation enables a consolidated treatment of preferences, wealth, uncertainty, and flexibility in decision-making, thereby providing a coherent foundation for studying consumer choice under uncertainty.

Four subsidiary contributions follow. First, it is demonstrated that rebound and backfire effects arise endogenously as path-dependent outcomes of the optimal strategies, ensuring consistency with welfare analysis. Second, the welfare implications of energy consumption and energy-efficiency adoption decisions are precisely characterised. Third, the resulting social costs are incorporated into a problem of optimal subsidy design which effectively targets agent characteristics such as income. Fourth, by embedding heterogeneous agents in the proposed framework, it is shown how macro-level technology diffusion patterns emerge endogenously from micro-level optimisation in a stochastic environment.

These contributions are operationalised through two decision models developed within the consumption–investment framework. The models demonstrate how consumption and technology adoption respond to changes in wealth, prices, and the stochastic environment, while also illustrating how the proposed definitions of rebound and backfire effects, welfare change, and optimal subsidy design operate in practice. Detailed case studies of energy retrofits of representative German single-family homes illustrate the plausibility of the models, providing concrete examples of how the framework can be applied to realistic adoption scenarios.

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Ad maiorem Dei gloriam

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A.B.

Contents

1 Introduction

This dissertation is about optimal household energy consumption and energy-efficiency adoption. To motivate the analysis, these decisions are considered in the context of the energy transition, alongside two longstanding puzzles in the academic literature: the energy-efficiency gap and the rebound effect. The research questions and scope are then delineated, the main contributions summarised, and an overview of the remainder of the thesis provided.

1.1 Motivation

Since the days of the Industrial Revolution, global primary energy consumption has surged, permeating nearly every aspect of modern life. It has been estimated the average person today has at their disposal nearly 700 times more useful energy than their ancestors at the beginning of the 19th century (Smil, 2022).[1] In the interim, the global primary-energy mix has come to be dominated by fossil fuels, whose abundance, high energy density, ease of storage and transport, and relatively low cost have underpinned their widespread adoption (Figure 1.1).

Three factors are especially pertinent to the discussion around present and future energy demand: the relationship between primary energy consumption and climate change, national security concerns related to energy trade, and scarcity of supply due to dwindling reserves (Cullen et al., 2011). Economically, each can be viewed as an externality generated by demand for useful energy. In order to be most effective, possible responses need to take into account the purpose served by energy consumption in the first place.[2] Broadly speaking, it is clear that energy is not demanded for its own sake, but for the direct services, such as a heated room or a lighted area, and indirect services, such as as embodied in food or consumer goods, that it provides. These desirable end-uses

[1] Useful energy is the final stage of the energy transformation chain that converts primary energy into secondary energy, then into final energy, and ultimately into useful energy delivered at the point of end-use (Weber et al., 2022, Ch. 2.4). For example, when a household gas boiler converts the chemical energy in natural gas into heat delivered to a room, or when an electric lightbulb converts electrical energy into illumination, the useful energy is the heat or light ultimately provided to the end-user.

[2] In this dissertation, the term "energy consumption" is used to denote the quantity of useful energy implicitly consumed.

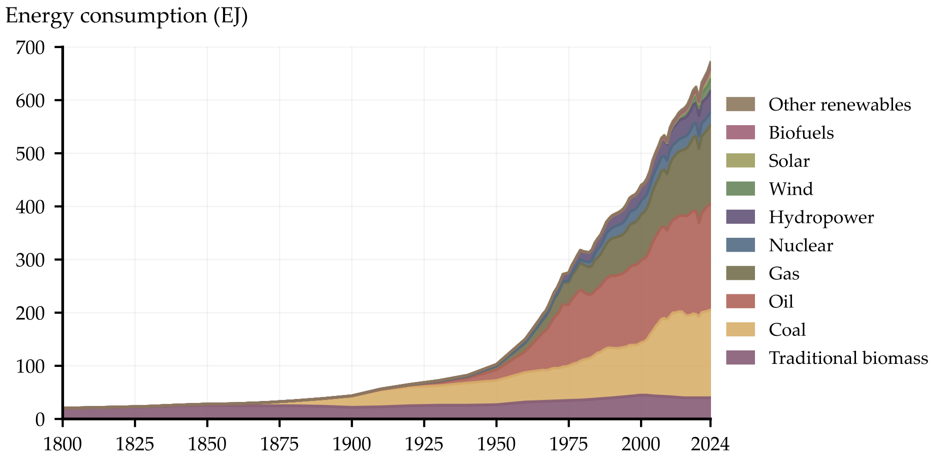

FIGURE 1.1. *Global primary energy consumption 1800–2024 in exa-joules. Own illustration based on data from Ritchie et al. (2020).*

are typically referred to as *energy services* (Fell, 2017).[3] With this in mind, consider a single household and a single energy service, such as thermal comfort; without loss of generality, restrict to externalities due to emissions.[4] Then the social cost generated by the agent's energy consumption might be disaggregated as follows (cf. Kaya & Yokoburi, 1997):

$$\text{CO}_2 \text{ emissions} = \text{Energy service} \times \frac{\text{Energy}}{\text{Energy service}} \times \frac{\text{CO}_2 \text{ emissions}}{\text{Energy}} . \tag{1.1}$$

The first factor is simply the quantity of the energy service demanded. The second factor is *energy intensity*, which is the inverse of *energy efficiency*, i.e. the ratio of useful outputs to energetic inputs (Saunders et al., 2021). Similarly, the third factor is *carbon intensity*; it is the inverse of *carbon efficiency*, and strategies which target its improvement fall broadly under the heading of "supply substitution" (Cullen et al., 2011). From the point of view

[3]With reference to Footnote 1, energy services are the outputs, often immaterial, derived from useful energy; for example, the heated room produced by the boiler and the illuminated area produced by the lightbulb. Although energy services are difficult to quantify precisely (cf. Cullen & Allwood, 2010), they are of primary economic interest, as it is these services that underpin consumer utility.

[4]We use the term "emissions" to refer broadly to greenhouse gas emissions rather than exclusively to carbon dioxide. This includes methane, nitrous oxide, and fluorinated gases, each weighted by its global warming potential as reported in standard inventories (Allen et al., 2016).

of developing mitigation strategies, it is important to recognise that at least two essential dimensions of the agent's decision-making are represented above:

$$CO_2 \text{ emissions} = \underbrace{\text{Energy service}}_{\text{Consumption decisions}} \times \underbrace{\frac{\text{Energy}}{\text{Energy service}} \times \frac{CO_2 \text{ emissions}}{\text{Energy}}}_{\text{Technology choice}} . \quad (1.2)$$

Technology choice is conceptualised here as covering both energy-efficiency decisions, carbon-efficiency decisions, or a combination of both. For instance, a consumer who heats their home using a conventional oil-fired boiler might choose to either upgrade the thermal performance of the dwelling envelope (pure energy-efficiency improvement), switch from an oil to a gas boiler (pure carbon-efficiency improvement), or undertake both improvements at the same time.[5]

Given that mitigation strategies must ultimately influence individual decisions, a natural set of questions follows. To what extent is household energy use and technology choice understood within a coherent economic framework? Can these two domains be meaningfully examined in isolation, or do behavioural and structural linkages imply joint determination? How do uncertainty, risk preferences, and expectations shape the timing and intensity of these choices? And to what degree might changes in one decision margin alter incentives on another, complicating simple engineering or accounting perspectives on policy effectiveness?

These questions are in fact intrinsically linked to two longstanding microeconomic puzzles: the energy-efficiency gap and the rebound effect. Briefly, the energy-efficiency gap refers to apparent consumer underinvestment in energy-efficiency improvements relative to the predictions of engineering or net-present-value calculations (Jaffe & Stavins, 1994a). On the other hand, the rebound effect captures the behavioural and economic responses that offset expected energy savings, such as increased utilisation of a more efficient service or reallocation of the budget to other energy-using activities (Sorrell & Dimitropoulos, 2008). Each of these phenomena has given rise to a rich theoretical

[5] Cullen & Allwood (2010) offer a heuristic that sharpens this distinction. Based on the energy-transformation chain described in Footnote 1, they propose conceptualising the technologies involved in the transformation as follows:

$$\underbrace{\text{Primary energy} \longrightarrow \cdots \longrightarrow}_{\text{Conversion device}} \underbrace{\text{Useful energy} \longrightarrow}_{\text{Passive system}} \text{Final service} . \quad (1.3)$$

At each stage of the transformation process, energy is converted into a more usable form, with attendant losses primarily in the form of low-grade heat. In this framework, oil and gas boilers exemplify conversion devices, whereas the dwelling envelope exemplifies the passive system that mediates the delivery of the final service. Technology choice is therefore construed as encompassing decisions about both the conversion device and the passive system.

and empirical literature, including many models of consumer choice, as reviewed in Chapter 2. Viewed together, they highlight the importance of modelling preferences, constraints, and incentives explicitly when analysing consumer decisions in energy use and technology adoption.

Decisions about energy consumption and technology adoption are not just of academic interest; they are central to contemporary policy debates on the energy transition. For instance, the International Energy Agency's *Net Zero by 2050* roadmap estimates that over half of the required emissions reductions by 2050 depend on end-user decisions (Cozzi et al., 2021, emphasis in original):

> *A transition of the scale and speed described by the net zero pathway cannot be achieved without sustained support and participation from citizens.* The changes will affect multiple aspects of people's lives—from transport, heating and cooking to urban planning and jobs. We estimate that around 55 % of the cumulative emissions reductions in the pathway are linked to consumer choices such as purchasing an EV, retrofitting a house with energy-efficient technologies or installing a heat pump.

Similar macro-level targets and pathways are ubiquitous and well-documented (DeAngelo et al., 2021). These analyses typically focus on aggregate outcomes, yet often lack an economically rigorous treatment of the decisions made by individual consumers (Allcott & Greenstone, 2012). Yet it is precisely these micro-level decisions, namely how households respond to prices, income, risk, and policy incentives, that ultimately determine realised energy use and technology uptake. Without a clear understanding of these micro-level determinants of energy consumption and technology adoption, it is difficult to anticipate how policy interventions translate into actual behaviour, whether intended efficiency gains are realised, or how the resulting benefits and costs are distributed across households. This gap motivates three key considerations.

First, the welfare implications of consumption and technology-choice decisions require careful assessment, particularly regarding how changes in energy efficiency, carbon intensity, or consumption levels affect social welfare, and how these effects vary across heterogeneous agents. Second, effective policy design depends on a detailed understanding of underlying consumer decisions. For instance, targeted subsidies intended to stimulate energy-efficiency adoption hinge on the behavioural responses of consumers (Allcott et al., 2015); only by modelling such responses explicitly can one evaluate alternative subsidy schemes and identify policies that maximise social welfare under realistic constraints. Third, heterogeneity in individual preferences, risk attitudes, expectations, and the timing of decisions shapes the aggregate patterns that emerge when these choices are combined. Joint decisions on technology investment and energy consumption can produce complex diffusion dynamics in a stochastic environment. By modelling these decisions explicitly, we can trace how individual behaviours translate into macro-level outcomes, including technology diffusion paths and aggregate energy demand.

1.2 Research questions & scope

The above considerations underline the importance of modelling consumer choices around consumption and technology adoption, both for improving theoretical understanding and for informing mitigation strategies. Before developing the analytical framework, it is useful clarify the conceptual perspective from which these decisions are studied. To this end, it is helpful to recall the following taxonomy from the decision sciences, whereby models are classified as being either normative, descriptive, or prescriptive (Baron, 2014). Broadly speaking, *normative models* describe how rational actors should behave in order to maximise a given objective, whereas *descriptive models* aim to describe how economic agents actually behave. The trio is completed by *prescriptive models*, which bridge the gap between normative and descriptive models by helping agents “do better”, that is, come closer to normative standards, given realistic cognitive and structural constraints (Baron, 2014, Luce & von Winterfeldt, 1994).

This thesis restricts focus to normative models because they provide axiomatic and fundamental representations of decision-making, characterising the behaviour of fully rational agents facing well-defined objectives and constraints. By specifying preferences, probabilities, and trade-offs in a formal, internally-consistent manner, normative models establish explicit and testable principles that form a logically coherent benchmark for rational choices. This benchmark is indispensable for welfare analysis, evaluating the efficiency of alternative policies, diagnosing sources of behavioural deviations, and providing the conceptual foundation upon which both descriptive and prescriptive approaches are ultimately built (Thaler, 2018).

However, despite many decades of research, fundamental models of energy consumption and technology adoption remain underdeveloped, as discussed in Chapter 2. As a result, critical questions such as the size of the energy-efficiency gap remain contested (Gillingham & Palmer, 2014), and empirical estimates of phenomena like the rebound effect often rely on misspecified models (Hunt & Ryan, 2015). Moreover, policy prescriptions differ: some researchers advocate for policies to narrow the energy-efficiency gap (Hirst & Brown, 1990) or mitigate rebound directly (Font Vivanco et al., 2016, van den Bergh, 2010), whereas others argue that such interventions would be welfare-reducing as they would suppress legitimate consumer surplus (Sutherland, 1996, Borenstein, 2015).

These gaps motivate the overarching research question:

> *How can consumer energy consumption and technology adoption be modelled within a unified, micro-founded framework?*

We additionally focus on the following five subsidiary questions. First, which dimensions of agent heterogeneity, such as preferences, wealth, or risk attitudes, are essential to represent in the model? Second, how can uncertainty and intertemporal dynamics in energy consumption and technology adoption be incorporated within a unified decision-making framework? Third, what are the welfare implications of agents' consumption and technology-choice decisions? Fourth, how can these insights inform the design of policies aimed at influencing consumer behaviour? And fifth, how do micro-level decisions regarding technology adoption and consumption translate into observable macro-level patterns of energy use and technology diffusion?

This dissertation focuses on space heating in residential buildings due to the scale and significance of the sector. This scope also facilitates comparison with the literature, where consumer decisions regarding energy use and improvements in building energy performance, commonly referred to as "retrofits", are studied extensively. As noted, one reason for this emphasis is the size of the sector: globally, roughly 5 % of greenhouse gas emissions can be traced back to direct on-site emissions from residential buildings due to space heating (Cabeza et al., 2022); if off-site generation of electricity and heat is included, the share of emissions rises to roughly 11 % (Ritchie, 2020). The other reason is that the building sector, despite having enormous potential for energy savings according to engineering estimates, has shown relatively slow progress in energy-efficiency adoption despite extensive policy interventions (Nejat et al., 2015). Germany, the focus of our case studies, aims to achieve overall climate neutrality by 2045, and has set the goal of reducing emissions in the building sector to 57 % of 2020 levels by 2030 (Bundestag, 2021).[6] The gap between these stated policy goals and actual retrofit rates is significant and well-documented. For instance, the German Energy Agency, a consultancy, estimates that the rate of deep retrofitting needs to roughly double, from the historic level of $1\,\%\,\mathrm{yr}^{-1}$ to around $1.9\,\%\,\mathrm{yr}^{-1}$, in order to achieve these goals (Jugel et al., 2021).

[6] Under Germany's Climate Protection Act (*Klimaschutzgesetz*), sectoral emission targets were originally legally binding, requiring immediate corrective action if exceeded. Following recent amendments, these targets remain codified and relevant for monitoring and policy evaluation, but the automatic enforcement mechanism has been largely removed; compliance is now assessed primarily on an aggregate basis (Flachsland et al., 2024).

1.3 *Outline of the dissertation*

The remainder of the dissertation is structured as follows. Chapter 2 surveys normative approaches to household energy consumption and energy-efficiency adoption. It reviews the cost-minimisation models of technology choice that dominate discussions of the energy-efficiency gap, and examines behavioural models of energy demand with particular attention to rebound effects. This motivates the need for intertemporal, utility-based reasoning and establishes consumption–investment models as a unified framework capable of capturing preferences, uncertainty, and decision flexibility. The chapter closes by surveying the current landscape of energy efficiency in the building sector.

Chapter 3 develops a model of consumption, investment, and energy-efficiency adoption under uncertainty. It derives closed-form results for core objects, including the adoption threshold and post-adoption strategies. It also advances welfare-consistent definitions of rebound and backfire effects, and demonstrates that efficiency improvements are generally welfare-enhancing. These welfare effects are integrated into a problem of optimal subsidy design. A detailed case study of a representative German single-family home illustrates the theoretical results.

Chapter 4 extends the model from Chapter 3 to incorporate dynamic energy prices. While some analytical characterisations remain available, an approximation scheme is introduced to support numerical implementation. A case study shows that the adoption boundary depends jointly on wealth and energy prices, and wealthier households invest at relatively lower energy prices. The chapter also studies the design of a subsidy program to minimise social cost.

Chapter 5 steps back to survey further extensions to the modelling framework. It evaluates alternative preference specifications and their suitability for dynamic consumption–investment problems with stochastic energy prices. The chapter also identifies conditions under which two-good utility functions remain tractable and economically coherent. Finally, it incorporates a bequest motive via a random-horizon formulation to bridge the gap to life-cycle models and concludes with brief remarks on additional technical and numerical extensions.

Chapter 6 concludes the dissertation by synthesising the main findings and drawing implications for energy-efficiency policy. It discusses limitations related to the modelling structure, solution tractability, and partial-equilibrium perspective. The chapter also offers an outlook on future work, including richer preference specifications, improved empirical grounding for baseline adoption rates, and the integration of general-equilibrium feedbacks into the consumption–investment framework.

2 Background & literature review

This chapter presents a review of normative approaches to household energy consumption and energy-efficiency adoption. Models of technology choice are considered first, with a focus on the cost-minimisation approaches that are dominant in the literature on the energy-efficiency gap. Models of household energy consumption are examined next, highlighting the treatment of the rebound effect and the need for intertemporal, utility-based reasoning. Building on this, consumption–investment models are motivated as a unified framework that incorporates preferences, uncertainty, and decision flexibility.

2.1 Technology choice & the energy-efficiency gap

The aim of this chapter is to highlight main aspects of the existing normative paradigms for technology choice and consumption decisions, the two dimensions of consumer choice motivated in Section 1.1. In so doing, we make a case for treating these decisions together in a dynamic model of intertemporal choice. We begin by considering technology choice, which is naturally related to the question of the energy efficiency gap.

Recall that the hypothesis of the energy-efficiency gap, which was formulated in the wake of the seminal work on consumer discount rates by Hausman (1979), posits that consumers systematically underinvest in cost-effective energy efficiency measures (Jaffe & Stavins, 1994b). In a recent review of the literature, Gerarden et al. (2017) identify three broad categories of explanations for this apparent underinvestment: market failures, behavioural explanations, and modelling flaws. Broadly speaking, market-failure explanations emphasise frictions that prevent private actors from fully capturing the benefits of energy-efficiency investments; these range from information asymmetries and principal–agent problems, to capital-market constraints and mispriced energy. Behavioural explanations, by contrast, highlight systematic patterns in decision-making that lead individuals to deviate from economically rational choices; examples of such explanations include limited attention, reliance on heuristics, short planning horizons, and biased beliefs. Our interest in normative models naturally aligns with the third category of explanations for the energy-efficiency gap, namely modelling flaws, which provides insight into why observed rates of energy-efficiency investment may not be as paradoxical as they first appear. The underlying motivation is well summarised by Gillingham & Palmer (2014):

> Many economists believe that consumer choices reveal more about the economics of energy efficiency improvements than do engineering calculations. If engineering estimates of the energy savings potential from seemingly cost-effective investments fail to include some costs or model the consumer's decision inappropriately, then the assessment of what is optimal from the consumer's perspective will be incorrect. Thus the engineering approach will result in the net benefits from energy efficiency investments being overstated, which means the [energy-efficiency] gap may be much smaller than estimated or there may be no gap at all.

This perspective connects directly to the normative–descriptive distinction discussed in Chapter 1.2. By focusing on models that provide a fundamental, theoretically grounded description of consumer technology choice, we hope to identify the minimal assumptions needed to explain observed adoption patterns.

As the above quotation suggests, technology adoption decisions are most commonly modelled using a cost-minimisation framework. It is useful to consider a simple example of this approach to highlight the aspects that economists typically take issue with. The following heuristic, adapted from Gerarden et al. (2017), illustrates the basic logic. The consumer's problem of technology choice is specified as

$$\underbrace{\min \text{Total cost}}_{\text{objective}} = \underbrace{K(E)}_{\text{equipment purchase cost}} + \underbrace{O(E, P_E) \times D(r, T)}_{\text{discounted operating costs}} + \text{other costs}\,, \tag{2.1}$$

where $O(E, P_E)$ denotes the annual operating cost, with E representing the agent's annual energy consumption and P_E the energy price, and $D(r, T)$ is the present-value factor, with r the discount rate and T the time horizon.[7] The term "other costs" refers to unobserved costs to technology adoption, such as subjective preferences, product heterogeneity, or opportunity costs (Gerarden et al., 2017). Although the identity is instructive for disaggregating some of the factors influencing technology adoption, it is clearly incapable of modelling consumption and adoption decisions in a unified way, since the rate of energy consumption is exogenously specified. Even more, the optimisation does not account in any fundamental way for the *reason* that the agent consumes energy in the first place, and thus cannot internally account for the relationship between the objective

[7] The factor $D(r, T)$ represents the sum of discounted future operating costs over the planning horizon T, typically calculated as $D(r, T) = \sum_{t=1}^{T}(1 + r)^{-t}$. Equation 2.1 assumes that annual operating costs $O(E, P_E)$ are constant over the horizon, so $D(r, T)$ converts the stream of identical future costs into their present value.

and the remaining terms in (2.1). For instance, is the agent minimising costs to free up resources for other goods, to reallocate expenditure to support future consumption, or perhaps from a bequest motive? The framework is silent on these underlying motives, and as such, cannot provide satisfying fundamental explanations for energy-efficiency adoption behaviour.[8,9]

These observations apply almost verbatim to a version of the min-cost model that instead emphasises monetary savings due to reduced energy consumption (cf. Allcott & Greenstone, 2012). We write this heuristically as

$$\underbrace{\max \text{Total savings}}_{\text{objective}} = \underbrace{(O_{\text{old}}(E, P_E) - O_{\text{new}}(E, P_E)) \times D(r, T)}_{\text{discounted energy savings}} - K(E) - \text{other costs}\ , \quad (2.2)$$

where $O_{\text{old}}(E, P_E)$ and $O_{\text{new}}(E, P_E)$ represent operating costs before and after the energy efficiency improvement respectively. As before, what is missing is a justification for why the agent maximises this particular objective. Even so, this formulation has stimulated an influential line of thought, beginning with Thompson (1997), who notes that different discount factors may be applied to the two cost streams $O_{\text{old}}(E, P_E)$ and $O_{\text{new}}(E, P_E)$. This re-framing of the energy efficiency investment as a choice between two *consumption* streams formed a crucial conceptual step toward the unified framework developed in this thesis.

It is fair to say that min-cost and max-savings models dominate baseline models in the literature (Häckel et al., 2017). Only a handful of normative alternatives exist. Most of these belong to the research area of real options, which primarily focuses on investment-timing flexibility, and could therefore benefit from a more explicit treatment of consumption choices. Detailed discussion is deferred to Section 2.3.

Lastly, although outside our scope, we mention that *descriptive* models of technology adoption, which seek to account for systematic deviations from the expected-utility benchmark, are numerous. Broadly speaking, these models rely on non-standard

[8] Reading between the lines, one infers that the framework implicitly assumes that energy consumption is a fixed yet necessary part of consumption spending which the agent would like to minimise, probably in order to free up resources for other goods. While defensible to some extent, this position cannot account for well-documented economic phenomena such as the responsiveness of energy consumption to changes in prices and income (Labandeira et al., 2017), or the rebound effect following an energy efficiency improvement (Schütt et al., 2024).

[9] It is noteworthy, and somewhat surprising, that even economists sometimes adopt a purely cost-minimisation approach, despite substantial evidence within the discipline, such as microeconomic analyses of the rebound effect, indicating that a notion of utility is essential for accurately modelling consumer behaviour. The intriguing questions this raises for the sociology and practice of science we leave to more qualified researchers.

preferences (e.g. self-control problems, reference-dependent utility), non-standard beliefs (e.g. biased expectations about energy savings), or non-standard decision making (e.g. limited attention, framing effects) to explain observed patterns of technology adoption (DellaVigna, 2009). Surveys of such approaches in the context of energy-efficiency decisions are provided by Gillingham & Palmer (2014) and Gillingham et al. (2009).

2.2 *Energy consumption & the rebound effect*

The next focus is on the complementary dimension of consumer choice: energy-consumption decisions. As in the previous case, there is an associated puzzle in the literature, the rebound effect, which refers to the idea that improvements in energy efficiency may induce behavioural responses that partially offset the expected energy savings. However, unlike in the previous case, the core conceptual issues have largely been resolved; the remaining challenges are primarily technical. Indeed, three recent articles, published almost simultaneously, advanced closely related frameworks that corrected earlier flaws in the literature and placed the analysis of rebound effects on sound normative footing (Borenstein, 2015, Chan & Gillingham, 2015, Hunt & Ryan, 2015).

To give a flavour of these arguments, and to contrast them with the considerations in Section 2.1, we present the simplest version of the standard model, in which there are only two consumption goods: the energy service s (e.g. thermal comfort) and the non-energy numeraire x. Using a standard microeconomic formulation, the agent's consumption problem in each period is formulated as

$$\max_{x,s} U(x,s) \qquad \text{subject to} \quad \begin{cases} s = \eta c \, , \\ x + pc = m \, , \end{cases} \tag{2.3}$$

where c is fuel consumption, η is the efficiency with which fuel is converted into service, p is the relative price of energy, and m is the budget. The direct rebound effect is then naturally defined as the elasticity $\partial_\eta s$, measuring how demand for the energy service responds to a change in efficiency. This framework can be enriched in several directions, such as multiple energy services, multiple fuels, or indirect rebound effects, but these extensions are not the focus here. What matters for our purposes is the *qualitative* contrast with min-cost or max-savings approaches: in the decision problem (2.3), the agent makes an explicit consumption trade-off grounded in the axiomatic framework of utility theory, rather than relying on an ad hoc optimisation of costs or savings. Consequently, the predictions regarding energy consumption are conceptually transparent, economically intuitive, and amenable to empirical testing (Schütt et al., 2024, Sorrell et al., 2009).

From our perspective, a set of closely related modelling limitations remains unresolved, and they directly motivate the need for a unified treatment of technology choice and energy consumption. The core of the issue emerges by noticing how the cost of the efficiency upgrade enters the household's decision problem. Chan & Gillingham (2015) abstract from costs altogether, whereas Hunt & Ryan (2015) fold the cost into the budget constraint without modelling its consequences for consumption choices. Only Borenstein (2015) accounts for the cost explicitly. Yet because his model collapses lifetime consumption into a single period, the intertemporal consequences of financing the upgrade, including how it affects consumption paths, saving decisions, precautionary motives, or future utility, are necessarily suppressed. This abstraction removes precisely those dynamic trade-offs that typically shape technology-adoption decisions.

Closely related is the absence of timing flexibility. In many real settings, households can postpone or accelerate an upgrade, creating an option value that depends on prices, income, and expectations (Dixit & Pindyck, 1994, Hassett & Metcalf, 1993). Static formulations cannot capture this dimension, nor can they represent the strategic interaction between consumption today and the possibility of investing tomorrow. A further limitation is the minimal treatment of uncertainty. Efficiency investments are often made under uncertain fuel prices, volatile usage needs, or evolving household circumstances, all of which have first-order consequences for optimal timing and for the welfare evaluation of rebound effects.

Taken together, these gaps suggest that while the microeconomic approach to modelling energy consumption is conceptually rigorous, its integration with technology choice remains incomplete. The result is that the existing normative models clarify the behavioural foundations of the rebound effect, yet do not fully exploit the power of the expected utility framework for analysing intertemporal choice, investment timing, or uncertainty.

2.3 *Towards a unified framework*

Addressing the limitations outlined above requires a coherent dynamic model that treats energy consumption and technology adoption jointly. We recap our requirements: the model must be *normative*, accounting for *intertemporal trade-offs* in *consumption* and *technology choice*, with the additional possibility of accounting for *risk and uncertainty*. We argue that consumption-investment models fit these criteria, thus offering the appropriate normative foundation.[10] Each of the emphasised ideas has been explored in the literature to some extent, although only a handful of articles have brought them together in a manner approximating our proposal for studying energy efficiency adoption. This section presents a narrative review of the literature centred around these themes, including relevant criticism and research gaps.

We begin by discussing *expected-utility maximisation*, the decision-theoretic foundation of the framework. While it remains the standard model of individual choice in economics, it has faced growing scrutiny in recent decades (Moscati, 2017). A full review of this debate lies beyond the scope of this work; for a comprehensive survey of expected utility theory and its alternatives, see Starmer (2000). We adopt the position that a decision model grounded in expected utility aligns with prevailing economic orthodoxy, including the view that the theory is best interpreted normatively (Thaler, 2018).[11] In the context of energy efficiency, expected utility theory predominantly shows up in real options models, which we discuss below. Apart from these analyses, only a few studies, such as Rockstuhl et al. (2021) and Häckel et al. (2017) seek to enrich min-cost models by incorporating elements of expected utility theory.

Considerations of *intertemporal choice* follow naturally from this foundation. Once consumption and investment decisions are framed within an expected-utility setting, the temporal structure of the problem becomes central: households choose not only how much to consume, but when to consume, and how current decisions shape future opportunities. These considerations have shaped the *life-cycle framework*, which is the standard way to model the intertemporal allocation of time, effort and money; the main insight of this framework is consumption smoothing, which is the tendency of households to allocate resources over time so as to stabilise marginal utility across periods (Browning & Crossley, 2001). *Consumption-investment models* form a particular subset of models within this larger framework. Beginning with the seminal work of Merton (1969), who was the first to incorporate uncertainty and portfolio selection into the life-cycle framework (Bodie, 2020), these models make the intertemporal trade-offs of consumption and investment explicit under uncertainty. Although widely used, these models face well-known empirical challenges. The equity premium and risk-free rate puzzles (Mehra & Prescott, 1985, Weil, 1989) show that their Euler-equation foundations struggle to

[10] We use the term *consumption–investment models* in a broad sense, encompassing both continuous consumption–savings decisions and discrete investment-timing decisions, the latter often formulated as optimal-stopping problems. A standard example involving both dimensions is a consumption–investment problem with flexible retirement timing (Jang et al., 2024, Perera, 2012, Jin Choi & Shim, 2006). This broad definition also covers the real-options literature, in which pure optimal-stopping problems dominate (cf. Dixit & Pindyck, 1994).

[11] However, see Tversky (1975) for a classic critique of the theory's normative status; Małecka (2019) provides a contemporary discussion.

match observed asset returns; excess smoothness and excess sensitivity of consumption (Flavin, 1981, Campbell & Mankiw, 1990) and the limited participation in risky assets (Haliassos & Bertaut, 1995) further highlight tensions between the theory and observed household behaviour. Considered together, these strands of evidence caution the use of these models as literal descriptions of household behaviour. Their reliance on frictionless markets and strong assumptions about information and preferences necessarily abstracts from many real-world complexities. Nonetheless, they remain indispensable as tractable, internally consistent tools for analysing intertemporal trade-offs.[12] To the best of our knowledge, this dissertation is the first to propose the use of these models for the study of energy-efficiency adoption. As is shown in the following chapters, their normative clarity, analytic structure, and ability to isolate the consequences of well-defined assumptions make them especially useful for studying energy-efficiency adoption when the aim is to model coherent patterns of optimal behaviour.

We make a brief detour to clarify the relationship between our proposal and a strand of the macrodynamics literature which studies energy efficiency adoption in a dynamic general equilibrium setting (Charlier et al., 2011, Pommeret & Schubert, 2009, Khan & Ravikumar, 2002). In these models, households act as producer–consumers, converting capital into consumption over time, and the planner uses Pigouvian taxation to decentralise the social optimum. In this respect, they share a resemblance to consumption–investment models because they place technology upgrading within an intertemporal wealth dynamic. The resemblance, however, is largely structural rather than substantive. These models are not intended as behavioural descriptions of household adoption decisions; they are macroeconomic efficiency analyses. Energy demand is usually fixed or modelled in reduced form, financial markets are highly stylised or absent, and individual-level uncertainty, borrowing constraints, precautionary motives, or heterogeneous consumption streams, i.e. the core ingredients of the consumption–investment perspective, are not modelled. The adoption choice is therefore embedded in an aggregate capital–technology dynamic rather than in a micro-founded consumption problem. For the purposes of a normative, household-level account of energy-efficiency adoption, these macrodynamic models are thus complementary but not foundational. They clarify how optimal policy interacts with technological diffusion in the aggregate economy, but they do not offer a microeconomic representation of the consumption trade-offs facing individual households. As such, we view these models as occupying a parallel tradition that shares

[12]The situation parallels the debate surrounding expected utility theory in behavioural economics: despite well-documented departures from its assumptions, the framework remains central because it provides a coherent and analytically disciplined benchmark (Thaler, 2018, Baron, 2014).

TABLE 2.1. *Key intrinsic and extrinsic risk factors in residential energy-efficiency investments. Adapted from Mills et al. (2006).*

Risk category	Intrinsic factors	Extrinsic factors
Economic	Misestimation of project scope Rework due to installation errors	Volatile energy prices Volatile labour & material costs Financing risk Subsidy uncertainty
Contextual	Occupant discomfort	Weather variability Policy instability Contractor unreliability
Technological	Infeasible measures Equipment failure Installation errors	Product unavailability Limited warranty coverage
Operational	Performance degradation Behavioural drift	Service provider unreliability
Measurement & verification	Metering errors Inaccurate baselines	Data privacy & access issues

certain dynamic elements but ultimately operates at a different level of analysis and cannot substitute for a first-principles consumption–investment framework.

This section closes by briefly surveying the literature on *risk and uncertainty* in energy-efficiency investments, since a central motivation for adopting consumption–investment frameworks is their capacity to treat these aspects explicitly. It is plain that energy-efficiency investments involve risk; what is less clear is which risk factors are relevant for decision-making in the real world, and of these, which ones are germane to the overarching goal of providing model-based evaluations of observed rates of energy-efficiency adoption. The taxonomy of Mills et al. (2006) provides a helpful organising principle: they propose labelling risk factors as “intrinsic” if they are measurable, verifiable, and controllable, or “extrinsic” if they are outside the agent’s direct control. Table 2.1 adapts their findings to the residential sector, highlighting that some risk factors are more consequential than others and that not all can be explicitly modelled. A synthesis of the risk- and mitigation factors considered in the energy literature is provided by Koutsandreas et al. (2022): what emerges is a clear focus on economic, contextual, and technological risks (with reference to Table 2.1). Assuming that this research focus reasonably reflects the risks relevant for modelling, the introduction of *decision flexibility* naturally points to the real-options framework, which has been widely applied in the energy literature. We list a few examples. Hassett & Metcalf (1992) consider energy-efficiency adoption in a model with stochastic energy prices and retrofit costs, and investment-timing flexibility; Menassa (2011) extend this work to involve complex options such as staging and abandonment; van Soest & Bulte (2001) and Ansar & Sparks (2009) present extensions incorporating uncertain technological progress; finally, Hagspiel et al. (2021) and Oliveira & Perkowski (2020) develop models of optimal investment timing under subsidy-retraction risk. In general, it is fair to say that real options models are flexible enough to incorporate a wide variety of risk factors, modelled as stochastic state variables, and related mitigation strategies, modelled as controls. Nevertheless, a common feature of these examples is that they all focus on optimal *investment timing*, while the energy-consumption decisions relevant for a fully integrated framework of consumer choice remain underexplored.

In sum, our examination of the literature makes a robust case for the core assumptions of our proposal as a normative framework for studying energy consumption and energy-efficiency adoption in a unified framework. The review highlights that while many individual elements, such as expected-utility maximisation, intertemporal trade-offs, uncertainty, and investment-timing flexibility, have been explored separately, there is a lack of models that systematically integrate them. The models developed in the following chapters are intended as a step in this direction.

2.4 Energy efficiency in the building sector

This background chapter concludes by providing additional context regarding energy efficiency in the European and German building sectors. The building sector is one of the most energy-intensive parts of advanced economies, responsible for roughly 40 % of final energy consumption and about 36 % of energy-related greenhouse gas emissions in the European Union (European Parliament, 2024). In Germany, building-related energy consumption accounted for nearly 35 % of total final energy use in 2023, with heating alone representing over 30 % of total consumption (Umweltbundesamt, 2025). These figures reflect the dominance of space heating and hot water demand in residential and non-residential buildings, and underscore the sector's significance for both climate mitigation and energy policy. The age profile of the building stock amplifies this difficulty: an estimated 85 to 95 % of current EU buildings were constructed before 2001, and a large share predates comprehensive insulation standards, leaving considerable potential for efficiency improvements (European Commission, 2020).

In response to these challenges, European climate and energy policy has framed the decarbonisation of buildings as a central objective. The European Green Deal has set a climate neutrality goal for 2050 (European Parliament, 2021), and the Renovation Wave strategy, communicated in 2020, aims to deliver substantial reductions in energy use and emissions from buildings (European Commission, 2020). Key targets include a

60 % reduction in building-sector greenhouse gas emissions, a 14 % reduction in final energy consumption, an 18 % reduction in energy demand for heating and cooling, and the doubling of the annual renovation rate to at least 2 % by 2030. Minimum energy performance standards for new and existing buildings, tighter requirements for heating system efficiency, and financial support mechanisms are all recommended to steer both renovation activity and technology uptake. Alongside these targets, the Renovation Wave strategy emphasises the multiple benefits of modernising the building stock, ranging from energy and cost savings to broader social, environmental, and economic impacts. Distributional considerations are also central, as the high upfront costs of deep renovation and advanced heating technologies can disproportionately burden low-income households. Energy poverty remains a significant challenge in the EU, with nearly 34 million Europeans unable to keep their dwellings adequately warm (European Commission, 2020).

Many of these strategic elements were subsequently translated into binding obligations through the recast Energy Performance of Buildings Directive (EPBD) (European Parliament, 2024). In particular, the directive introduces legally enforceable minimum energy performance standards, zero-emission requirements for new buildings, and mandatory national renovation trajectories, with particular focus on the worst-performing buildings. Its provisions cover four main areas. *Renovation* is addressed through national building renovation plans and trajectories. These introduce minimum energy performance standards for non-residential buildings, targeting the 16 % worst-performing by 2030 and the 26 % worst-performing by 2033, and set national trajectories to reduce the average primary energy use of residential buildings by 16 % by 2030 and 20 to 22 % by 2035. *Decarbonisation* requires new buildings to meet zero-emission standards, be solar-ready, and phase out fossil-fuel boilers, while whole-life carbon emissions for new buildings must also be considered. *Modernisation and digitalisation* measures include building automation and control, good indoor environmental quality, digital energy performance certificates, national building stock databases, and provisions for electric vehicle charging and bicycle parking. *Financing and technical assistance* ensure targeted financial support and safeguards for vulnerable households, and provide one-stop shops offering technical guidance to building owners.

Germany implements EU building energy requirements through the Building Energy Act (*Gebäudeenergiegesetz*, GEG), which consolidates earlier legislation on energy efficiency and renewable heating (Bundestag, 2020). The GEG sets legally binding energy performance standards for new and existing buildings and requires that at least 65 % of the energy for new or replacement heating systems comes from renewable sources, allowing a range of technologies such as heat pumps, solar thermal systems, biomass, and district heating. Functioning fossil-fuel systems may continue to operate until the end of their useful life, at which point replacement systems must comply with the renewable energy requirement. The law is complemented by federal funding programs to support the deployment of renewable heating technologies and building renovations, and includes safeguards to protect tenants from disproportionate cost burdens, for example through rent caps or targeted financial assistance. Together with efficiency standards for building envelopes and technical systems, the GEG provides a binding national framework to operationalise Germany's climate targets in the building sector while aligning with EU directives such as the EPBD.

Despite these ambitions, there remains a substantial gap between policy ambitions and realised action. On average, less than 1 % of the EU's building stock undergoes renovation each year, and many member states record rates below this average, despite the fact that energy-efficient retrofits could reduce total EU energy consumption by a significant margin (Esser et al., 2019). This inertia is compounded by the sheer scale of the existing stock: with tens of millions of residential units requiring deep renovation to meet near-zero energy targets, a doubling of current renovation activity still implies a sustained, decade-long mobilisation of capital, skills, and regulatory oversight (Hermelink & Bettgenhäuser, 2021). In Germany, despite national climate targets and programmes aimed at promoting retrofit activity, energy consumption for space heating has only modestly declined over the last decade, reflecting both slow renovation uptake and the persistent use of inefficient heating systems (Umweltbundesamt, 2025).

Even where efficiency measures are implemented, actual energy savings depend critically on occupant behaviour. Empirical studies document rebound effects in buildings, whereby improvements in thermal efficiency lower the effective cost of indoor heating and cooling, leading occupants to increase usage relative to pre-renovation patterns (Schütt et al., 2024). These behavioural responses can partially offset the engineering gains achieved through insulation and high-efficiency technologies, reducing the expected savings in energy consumption and emissions. The magnitude of these effects varies across contexts, but their presence complicates straightforward projections based solely on technical performance (Sorrell, 2015).

The electrification of heating is another core element of building decarbonisation. Heat pumps, which transfer ambient or ground heat into buildings with efficiencies several times greater than conventional gas or oil boilers, have been promoted as a key solution. Nevertheless, their uptake is hampered by high electricity prices, a shortage of installers, high upfront costs combined with limited financing, and in some cases, national and local policies that discourage heat pump deployment; for these reasons, annual heat pump installations across the EU fell from a level of 2.8 million units in 2022 to 2 million units in 2024 (European Commission, 2025). In Germany, overall heat-pump penetration remains low despite growing policy support; only 4 % of existing residential buildings had a heat pump installed as of 2022 (Wettengel, 2025). The performance of these systems in real-world conditions also highlights challenges related to sizing, installation quality, and integration with existing building fabric and control systems, suggesting that robust technical standards and post-installation verification are necessary to realise projected efficiency gains (Toleikyte et al., 2023).

Taken together, these empirical and policy developments illustrate the complexity of modernising the building sector. Significant potential exists in the building stock to reduce energy consumption and emissions while also realising broader social and economic benefits. Yet slow renovation rates, behavioural responses, technological integration issues, and uneven deployment of low-carbon heating temper the pace of change. The models developed in this dissertation aim to provide a normative basis for evaluating consumer incentives regarding energy consumption and technology choice in this sector, thereby offering a structured framework to interpret observed consumption and adoption patterns, assess policy effectiveness, and explore pathways for accelerating decarbonisation.

3 A basic model of consumption, investment, and energy-efficiency adoption[†]

This chapter develops a model of consumption, investment, and energy-efficiency adoption under uncertainty. It proposes new definitions of the rebound and backfire effects, and associated welfare change. These welfare considerations are then integrated into a model of optimal subsidy design. Macro-level energy consumption and technology diffusion across heterogenous agents are also formalised within the framework. Explicit results for core objects are derived, including the adoption threshold and post-adoption strategies. These are shown to depend on agent wealth, introducing a novel channel through which financial conditions influence technology-adoption decisions. An approximation scheme is proposed to estimate welfare implications explicitly, with adoption of energy efficiency shown to be welfare improving in the main. A detailed case study of a representative German single-family home illustrates the theoretical results. Numerical analysis indicates that the subsidy policy effectively steers aggregate energy consumption.

3.1 Introduction

This chapter initiates the formal mathematical analysis of the dissertation by developing a model of consumption, investment, and energy-efficiency adoption under uncertainty. Unlike static or heuristic approaches, the model treats technology adoption as a fully dynamic intertemporal choice problem, consolidating consumption, preferences, wealth, uncertainty, and flexibility in a single coherent framework.

The principal theoretical contributions are fourfold. First, rebound and backfire effects are shown to emerge endogenously as path-dependent outcomes of the optimal strategies, ensuring consistency with welfare analysis. Second, the welfare consequences of energy consumption and adoption decisions are explicitly characterised, revealing that energy-efficiency investments are generally welfare enhancing. Third, these insights are

[†]A version of this chapter has been submitted to a scientific journal for publication.

applied to the design of optimal subsidies, which can be targeted effectively to agent characteristics such as wealth or income. Fourth, by embedding heterogeneous agents in the model, the framework demonstrates how macro-level patterns of energy consumption and technology diffusion arise from micro-level optimisation in a stochastic environment. Such aggregate quantities can be used as normative baselines for benchmarking energy-efficiency adoption, helping to quantify the size and sources of the energy-efficiency gap.

In addition to the macro-dynamics literature surveyed in Chapter 2.3, the model relates to research on real options in energy-efficiency adoption in the building sector. Beyond the foundational work by Hassett & Metcalf (1993, 1992), we mention the studies by Tadeu et al. (2016), Lee et al. (2014) and Kumbaroğlu & Madlener (2012). It is worth noting that this literature generally adopts an "investment perspective", focusing on the minimisation of energy costs, whereas the present approach follows a theoretically grounded utility-maximisation perspective (cf. Chapter 2.1). The model also has connections to the literature on subsistence constraints in consumption-investment models (Jeon et al., 2020, Achury et al., 2012, Sethi et al., 1992). In particular, Achury et al. show that imposing a time-invariant subsistence requirement can reproduce key empirical regularities in household behaviour, including wealth-dependent saving rates, portfolio allocations, and effective risk aversion. These results support the proposed framework and motivate its extension to a setting with energy services

The remainder of the chapter is structured as follows. Chapter 3.2 develops the model, including the agent's decision problem, welfare implications, optimal subsidy design, and aggregate energy-efficiency uptake. Chapter 3.3 presents the solution, including explicit results for the agent's optimal strategies and approximate results for many of the welfare quantities. The model is applied to a case study in Chapter 3.4 following, which demonstrates the plausibility of the model and presents key comparative statics. Chapter 3.5 concludes.

3.2 *The model*

The model is built around the agent's decision problem of optimal consumption, investment, and energy-efficiency adoption, which is presented first. Chapter 3.2.2 then analyses the welfare implications of the agent's decisions, including a problem of optimal subsidy design to mitigate externalities from energy consumption. Lastly, Chapter 3.2.3 bridges to the macro-perspective, defining key aggregate quantities over a population of heterogenous agents.

3.2.1 The agent's decision problem

There are two consumption goods: the energy service "thermal comfort" s, and a perishable non-energy good x, the numeraire. The energy service is obtained from fuel consumption c according to $s = \eta c$, where $\eta > 0$ quantifies the total efficiency of the energy-conversion chain. The agent may, at any $\tau \in \mathcal{T}$ where $\mathcal{T}$ is the set of positive stopping times, choose to retrofit their dwelling at cost $K > 0$ to an efficiency-level $\widetilde{\eta} > \eta$. We specialise to the setup where K is large and assume the agent finances the investment through a loan. For simplicity, we assume that the loan is serviced indefinitely by a constant payment flow ρK, where $\rho > 0$ is the borrowing rate. To avoid confusion, the term "investment" is hereafter used exclusively to refer to the investment in the retrofit, with the term "allocation" being reserved for the agent's portfolio decisions.

The price of energy $P > 0$ is assumed constant, as is the rate of labour income $Y > 0$. There are two financial assets: a risk-free bond with price S_t^0 and dynamics

$$\mathrm{d}S_t^0 = \mu_R S_t^0 \,\mathrm{d}t \,, \tag{3.1}$$

where $\mu_R > 0$, and an index fund with price S_t whose dynamics follow

$$\mathrm{d}S_t = \mu_S S_t \,\mathrm{d}t + \sigma_S S_t \,\mathrm{d}B_t \,, \tag{3.2}$$

where $\mu_S > \mu_R$, $\sigma_S > 0$, and B_t is a standard Brownian motion. At any time t, the agent invests a share a_t of wealth in the index fund, with the remainder allocated to the bond. For realism, we assume that the borrowing rate ρ from above satisfies $\rho \geq \mu_R$. The market price of risk is denoted $\kappa := (\mu_S - \mu_R)/\sigma_S$.

For a given $\tau \in \mathcal{T}$, it follows that the agent's wealth is given by

$$\begin{aligned} \mathrm{d}W_t = {} & \frac{a_t W_t}{S_t} \,\mathrm{d}S_t + \frac{(1-a_t)W_t}{S_t^0} \,\mathrm{d}S_t^0 + (Y - x_t) \,\mathrm{d}t \\ & - \Big((s_t/\eta) P \,\mathbb{1}_{\{t<\tau\}} + ((s_t/\widetilde{\eta})P + \rho K) \mathbb{1}_{\{t \geq \tau\}} \Big) \,\mathrm{d}t \,, \quad t \geq 0 \,, \end{aligned} \tag{3.3}$$

which simplifies to

$$\begin{aligned} \mathrm{d}W_t = {} & (a_t \mu_S W_t + (1-a_t)\mu_R W_t) \,\mathrm{d}t + a_t \sigma_S W_t \,\mathrm{d}B_t + (Y - x_t) \,\mathrm{d}t \\ & - \Big((s_t/\eta) P \,\mathbb{1}_{\{t<\tau\}} + ((s_t/\widetilde{\eta})P + \rho K) \mathbb{1}_{\{t \geq \tau\}} \Big) \,\mathrm{d}t \,. \end{aligned} \tag{3.4}$$

Let $\{W_t^{w;\tau} \mid t \geq 0\}$ denote a solution to (3.4) for a given initial condition $W_0 = w$ and investment time τ. Assuming time-additive utility with exponential discounting, the agent's decision problem is to choose an allocation control, consumption controls, and an investment time τ to maximise the present value of utility up to a random time horizon $T > 0$. The random time horizon is intended to capture exogenous shocks to the consumption-investment process such as the purchase or sale of a house, a sudden change in employment status, or death. Thus, for a given initial wealth w, the agent's value function is written as

$$F(w) := \sup_{a,x,s,\tau} \mathbb{E}\left[\int_0^T \mathrm{e}^{-\delta t} U(x_t, s_t)\,\mathrm{d}t \;\middle|\; W_0 = w\right], \tag{3.5}$$

where $\delta > 0$ is the rate of time preference and $U\colon \mathbb{R}_+^2 \to \mathbb{R}$ is the utility function.

Following Merton (1971), assume that T follows an exponential distribution with cumulative distribution function $\mathcal{F}_T(t) = 1 - \mathrm{e}^{-\lambda t}$, where $\lambda > 0$ is the hazard rate; assume further that T is independent of the securities market and the controls. The expectation over the random time can then be rewritten as a weighted integral over all t as follows:

$$\mathbb{E}\left[\int_0^T \mathrm{e}^{-\delta t} U(x_t, s_t)\,\mathrm{d}t\right] = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\delta t} U(x_t, s_t)\mathbb{1}_{\{t<T\}}\,\mathrm{d}t\right] \tag{3.6}$$

$$= \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\delta t} U(x_t, s_t)(1 - \mathcal{F}_T(t))\,\mathrm{d}t\right] \tag{3.7}$$

$$= \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-(\delta+\lambda) t} U(x_t, s_t)\,\mathrm{d}t\right]. \tag{3.8}$$

We hence reduce to a problem of optimal control and stopping on an infinite horizon:

$$F(w) := \sup_{a,x,s,\tau} \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\widehat{\delta} t} U(x_t, s_t)\,\mathrm{d}t \;\middle|\; W_0 = w\right], \tag{3.9}$$

where $\widehat{\delta} := \delta + \lambda$ is the agent's effective discount rate. The remainder of this chapter assumes Stone-Geary preferences,

$$U(x, s) := \frac{\left((x - \underline{x})^{1-\beta}(s - \underline{s})^{\beta}\right)^{1-\gamma}}{1-\gamma}, \tag{3.10}$$

where $\underline{x} > 0$ and $\underline{s} > 0$ are the subsistence levels of the two goods, $\beta \in (0, 1)$ is the relative preference weight on the energy service, and $\gamma > 1$ is the coefficient of risk aversion (Neary, 1997). This specification retains the mathematical tractability of the Cobb-Douglas function while adding the realistic feature that a minimum level of consumption is required for survival. This turns out to be a necessary feature for problems of energy-efficiency adoption; the detailed discussion is deferred to Chapter 5.2.

3.2.2 *Welfare implications*

Having introduced the agent's decision problem, we turn to the welfare implications of their consumption and technology-adoption choices. In sequence, we develop definitions for the rebound and backfire effects and associated welfare implications, and conclude by formulating a problem of optimal subsidy design in a second-best world.

Classically, the rebound effect is studied in the context of a standard two-good utility-maximisation problem, where it is defined as the elasticity of energy-service demand with respect to efficiency (cf. Borenstein, 2015, Chan & Gillingham, 2015). Such an approach is not possible in the present model since the optimal consumption path is given by a stochastic process with a possible discontinuity when investment occurs. We therefore propose an alternative definition that aligns with the intuition underlying the rebound effect, which can be expressed as the following question: "How does energy-service demand after an energy-efficiency investment compare to a counterfactual scenario, where no investment occurs?" This type of counterfactual question is in line with standard welfare-economic reasoning, where behavioural responses are assessed by comparing actual outcomes with a clearly defined no-policy benchmark (cf. Varian, 1992, Ch. 10).

To this end, consider the no-investment limit $\tau = \infty$ in (3.4); denote this process by $\widehat{W}_t$. It follows the dynamics

$$\mathrm{d}\widehat{W}_t = (\widehat{a}_t \mu_S \widehat{W}_t + (1 - \widehat{a}_t)\mu_R \widehat{W}_t + Y - \widehat{x}_t - (\widehat{s}_t/\eta)P)\,\mathrm{d}t + \widehat{a}_t \sigma_S \widehat{W}_t\,\mathrm{d}B_t\,, \tag{3.11}$$

where the allocation and consumption controls also carry hats for clarity. We employ the standard notation $\{\widehat{W}_t^w \mid t \geq 0\}$ to denote a solution to (3.11) for a given initial condition $\widehat{W}_0 = w$. Thus, for a given initial wealth w, the counterfactual decision problem is given by

$$\widehat{F}(w) := \sup_{\widehat{a},\widehat{x},\widehat{s}} \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\widehat{\delta}t} U(\widehat{x}_t, \widehat{s}_t)\,\mathrm{d}t \;\middle|\; \widehat{W}_0 = w\right]. \tag{3.12}$$

Consequently, the difference in energy-service demand between the original decision problem (3.26) and the counterfactual is measured by the process

$$R_t := s_t^* - \widehat{s}_t^*\,, \tag{3.13}$$

where the asterisks indicate that the controls are optimal in their respective problems. It is then natural to say that *rebound occurs* if $R_t > 0$ for $t \geq \tau^*$, i.e. if energy-service demand after investment is greater than in the counterfactual. Note that any rebound observed in this context is "optimal" from the agent's perspective, with the definition naturally incorporating the utility-maximising levels of consumption, the optimal investment time, and the effects of the retrofit cost on the budget constraint.

The related notion of *backfire*, whereby net fuel savings vanish due to excessive rebound, can be similarly formalised. Defining the process

$$Q_t := \begin{cases} s_t^*/\eta - \widehat{s}_t^*/\eta\,, & t < \tau^*\,, \\ s_t^*/\widetilde{\eta} - \widehat{s}_t^*/\eta\,, & t \geq \tau^*\,, \end{cases} \tag{3.14}$$

we say that *backfire occurs* if $Q_t > 0$ for $t \geq \tau^*$. A key point, which is exploited in the sequel, is that it is possible to have rebound without backfire. Indeed, for $t \geq \tau^*$, $Q_t \leq 0$ is equivalent to imposing $s_t^* \leq \widetilde{\eta}\widehat{s}_t^*/\eta$, which allows for some level of rebound since $\widetilde{\eta} > \eta$. For instance, if $\eta = 0.5$ and $\widetilde{\eta} = 0.7$, we have $s_t^* \leq 1.4\,\widehat{s}_t^*$; hence, backfire occurs only if energy-service demand after the investment increases by more than 40 % relative to the counterfactual.

We make two additional remarks. Firstly, since R_t and Q_t are stochastic processes, our definitions of rebound and backfire are probabilistic. It is possible to remove the randomness by computing expected rebound (resp. expected backfire) at time t, $\mathbb{E}[R_t]$ (resp. $\mathbb{E}[Q_t]$). Secondly, the definitions of R_t and Q_t are time-dependent, so that changes in wealth affect the level of rebound and backfire. For instance, if the agent becomes wealthier in expectation as time passes, it is likely that the probability of rebound or backfire occurring increases in line with the increase in overall spending (cf. Proposition 3.6 below). To the best of our knowledge, this is the first formalisation in the literature of the time dependence of the rebound and backfire effects. The importance of these time effects becomes apparent when considering the welfare implications of the energy-efficiency investment, a task which we now take up.

The total welfare change due to the retrofit consists of two components: the change in the agent's welfare and the change in the social cost, each measured relative to the counterfactual. Consider first the agent's welfare change. Recall that the value function F and counterfactual value $\widehat{F}$ are the dynamic equivalents of indirect utility functions from microeconomics: they accept prices and wealth as arguments, and return the present value of lifetime utility (cf. Varian, 1992, Ch. 7.2). Consequently, the *expenditure function* $\widehat{F}^{-1}(u)$ encodes the amount of wealth required to attain a given level of lifetime utility in the counterfactual. The *equivalent variation* of the energy-efficiency investment is then given by

$$V_{\mathrm{ev}}(w) := \widehat{F}^{-1}(F(w)) - w\,. \tag{3.15}$$

This quantity gives the change in wealth which would be equivalent to the proposed change in utility due to the retrofit (cf. Varian, 1992, Ch. 10.1). Since it is measured in monetary terms, it can be directly compared to the social cost of the retrofit investment, which we now quantify.

All other things being equal, the social cost of the retrofit is driven by externalities from energy consumption.[13] Hence, the present value of social costs in the presence of a retrofit may be written as

$$\mathbb{E}\left[\int_0^{\tau^*} \mathrm{e}^{-\widehat{\epsilon}t}\,(\varpi_t^\pi\, s_t^*/\eta)\,\mathrm{d}t + \int_{\tau^*}^{\infty} \mathrm{e}^{-\widehat{\epsilon}t}\,(\varpi_t^\pi\, s_t^*/\widetilde{\eta})\,\mathrm{d}t\right], \tag{3.16}$$

where $\widehat{\epsilon} := \epsilon + \lambda > 0$ is the sum of the social discount rate $\epsilon > 0$ and hazard rate λ as in (3.26), and $\varpi_t > 0$ is the marginal social cost of energy consumption with initial condition $\varpi_0 = \pi > 0$. We assume that the marginal social cost ϖ_t follows a geometric Brownian motion, independent of the wealth process, with drift $\mu_\varpi > 0$ and volatility $\sigma_\varpi > 0$. In the counterfactual, the social cost is given by

$$\mathbb{E}\left[\int_0^{\infty} \mathrm{e}^{-\widehat{\epsilon}t}\,(\varpi_t^\pi\, \widehat{s}_t^*/\eta)\,\mathrm{d}t\right]. \tag{3.17}$$

Subtracting (3.17) from (3.16), we see that the change in social cost due to the energy-efficiency investment can be expressed succinctly in terms of the backfire measure as

$$V_{\mathrm{sc}}(w;\pi) := \mathbb{E}\left[\int_0^{\infty} \mathrm{e}^{-\widehat{\epsilon}t}\varpi_t^\pi Q_t^w\,\mathrm{d}t\right], \tag{3.18}$$

where it is made explicit that the backfire measure depends on initial wealth w. It follows that the total change in welfare due to the retrofit is given by netting out the social cost from the agent's equivalent variation, which yields the measure

$$V(w;\pi) := V_{\mathrm{ev}}(w) - V_{\mathrm{sc}}(w;\pi)\,. \tag{3.19}$$

The following intuitive result is obtained.

[13] In addition to direct externalities from energy consumption such as greenhouse gas emissions, Chan & Gillingham (2015) observe that there may also be externalities tied to the energy-*service* itself, e.g. congestion externalities from excessive driving. It is straightforward to extend the proposed framework to account for this, though we do not do so here since the energy service "thermal comfort" generates no material negative spillovers beyond energy use, and any incidental effects, e.g. neighbouring units benefiting from excess heat, are either positive or negligible relative to the externalities from fuel consumption.

Proposition 3.1. *In the absence of backfire, the energy-efficiency investment improves total welfare.*

The proof, along with the other mathematical proofs for this chapter, is provided in Appendix B.1. A direct implication of this result is that mitigation policies should target backfire rather than rebound. Indeed, as long as rebound remains below the level which leads to backfire, the agent's utility gains, which derive in part from rebound, can be preserved without driving social welfare below zero. For this reason, policies that aim to suppress rebound per se are at least partly misguided, as has been noted in the literature (Borenstein, 2015, Gillingham & Palmer, 2014).[14]

A natural extension of the above welfare considerations is the study of corrective policies for energy-consumption externalities. Since we are in a partial-equilibrium setup, a full characterisation of the design of optimal Pigouvian taxes lies outside our scope.[15] Given this limitation, we assume a second-best world in which corrective taxation is unavailable, and consider a social planner who addresses consumption externalities by subsidising the retrofit at rate $m \in \mathbb{R}$, reducing the agent's private cost to $(1-m)K$.[16] Consequently, the agent's investment time τ^*, as well as the allocation and consumption rules are altered. Assuming that the subsidy is paid when the agent invests, the social planner's objective of minimising social costs is written as

$$J(w;\pi) := \inf_m \mathbb{E}\left[V_{\mathrm{sc},m}(w;\pi) + \mathrm{e}^{-\epsilon\tau_m^*(w)}\Psi(mK) \right], \tag{3.20}$$

where

$$\Psi(x) := \xi_0 x + \frac{\xi_1}{2}x^2 \tag{3.21}$$

[14]With respect to Footnote 13, these conclusions are valid only if energy-service demand does not generate externalities. If this were not the case, the expression for V_{sc} in (3.18) would in fact contain a term including the rebound measure R_t, making rebound a legitimate policy target.

[15]Briefly, a Pigouvian tax alters the effective energy price faced by *all* agents in the economy. As a result, one must account not only for heterogeneous adoption responses, but also for the distributional consequences of higher energy prices, the reallocation of expenditure across consumption goods, and the potential general-equilibrium feedbacks on wages, capital returns, and output. In other words, the welfare accounting of a tax cannot be reduced to a localised transfer problem, but requires an explicit aggregation of utility losses and externality reductions across the entire population. This type of analysis is proper to a general-equilibrium framework, where heterogeneity across agents and market interactions can be made explicit.

[16]We underline that the typical role of subsidies is to correct for investment inefficiencies (e.g. behavioural distortions), which do not exist in our model since the agent is a rational expected-utility maximiser (cf. Allcott, 2016, Allcott & Greenstone, 2012). Note further that since Pigouvian taxes are absent, it is possible that the subsidy rate may be negative, i.e. a penalty may be imposed. This is justified in the event of consumption backfire; see Proposition 3.10 for an example.

is a convex cost function.[17] The parameter $\xi_0 \geq 1$ is commensurate with the marginal cost of public funds (Browning, 1976), and $\xi_1 > 0$ is an additional friction parameter to disincentivise large transfers. This is a standard bilevel or Stackelberg optimisation, since the planner anticipates the agent's response and chooses the subsidy accordingly (Colson et al., 2007). Notice that the optimal subsidy derived here is tailored to the characteristics of the individual agent; in particular, its level depends on the initial endowment w and income Y, dimensions which are essential to understanding free-riding behaviour (Rivers & Shiell, 2016, Nauleau, 2014).

3.2.3 *Modelling aggregate behaviour*

The model concludes by examining how aggregate energy-efficiency adoption and energy consumption evolve over time. The resulting macro-level curves may be used to benchmark the energy-efficiency gap, and for policy analysis (cf. Hassett & Metcalf, 1993). In the present case, the primary force shaping both curves is the interaction between agent heterogeneity and the stochastic environment.

Firstly, it is natural to assume that all agents face identical financial-market conditions, so that the risk-free rate μ_R and the risky-asset parameters μ_S and σ_S are taken to be common across agents. It is also natural to assume that the agents face the same price of energy P. The remaining parameters are then idiosyncratic. Firstly, there are the preference parameters β, γ, δ, and λ, and the subsistence consumption levels $\underline{x}$ and $\underline{s}$. Then there are the retrofit parameters ρ, K, η, and $\widetilde{\eta}$. Finally we have labour income Y and initial wealth w. By drawing N times from an assumed joint distribution for these parameters, we generate a population of representative agents

$$\mathcal{P} = \{(\beta^i, \gamma^i, \delta^i, \ldots, Y^i, w^i) \mid i = 1, \ldots, N\} . \tag{3.22}$$

Conditional on this population, the *share of adopters* at time t follows the stochastic process

$$S_t := \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{\tau^i(w^i) \leq t\}} , \tag{3.23}$$

where $\tau^i = \tau^i(w^i)$ is the optimal investment time of each agent. Intuitively, for a given realisation of the financial market, the process S_t experiences jumps of size $1/N$ each time an agent invests. Its expectation, $\mathbb{E}[S_t]$, is the main quantity of interest, denoting

[17] We write $V_{\text{sc},m}$ to indicate the influence of the subsidy on social cost through the channels of the agent's energy consumption and investment timing.

the expected uptake, or *diffusion*, of the energy efficiency measure. Similarly, aggregate energy consumption is given by

$$C_t := \sum_{i=1}^{N} \Big[(s_t/\eta) \mathbb{1}_{\{\tau^i > t\}} + (s_t/\widetilde{\eta}) \mathbb{1}_{\{\tau^i \le t\}} \Big] \,, \tag{3.24}$$

since the efficiency parameter is upgraded from η to $\widetilde{\eta}$ following the investment. Because the planner aims to minimise externalities from energy consumption, the quantity $\mathbb{E}[C_t]$ is in fact an indirect target of the subsidy policy considered in Chapter 3.2.2 above. By explicitly accounting for the heterogeneity underlying these dynamics, the subsidy allows the social planner to steer aggregate energy consumption more effectively by targeting energy-efficiency adoption where it generates the greatest marginal benefit.

3.3 The solution

Due to the assumptions of constant prices and wages, the agent's decision problem admits an almost complete closed-form solution, which is presented first. In Chapter 3.3.2, an approximation is then introduced to derive explicit solutions for the welfare aspects of the model. Finally, in Chapter 3.3.3, the aggregate quantities of interest are considered.

3.3.1 The agent's optimal strategies

We begin by simplifying the decision problem in (3.9). To this end, it proves helpful to have a separate notation for the agent's wealth in the case of immediate investment, i.e. $\tau = 0$. Denoting this process by $\widetilde{W}_t$, we write down its dynamics from (3.4) as

$$\mathrm{d}\widetilde{W}_t = \Big(\widetilde{a}_t \mu_S \widetilde{W}_t + (1-\widetilde{a}_t)\mu_R \widetilde{W}_t + \widetilde{Y} - \widetilde{x}_t - (\widetilde{s}_t/\widetilde{\eta})P \Big)\, \mathrm{d}t + \widetilde{a}_t \sigma_S \widetilde{W}_t \, \mathrm{d}B_t \,, \tag{3.25}$$

where $\widetilde{Y} := Y - \rho K$ is labour income net of the loan payment. For clarity, the allocation and consumption controls here have also been labelled with a tilde. Then, noticing the structure of the controlled dynamic in (3.4), we make use of the strong Markov property of geometric Brownian motion and the law of total expectation to rewrite (3.9) as a problem of optimal control and stopping on an infinite horizon:

$$F(w) = \sup_{a,x,s,\tau} \mathbb{E}\bigg[\int_0^{\tau} \mathrm{e}^{-\widehat{\delta} t} U(x_t, s_t)\, \mathrm{d}t + \mathrm{e}^{-\widehat{\delta}\tau} G(W_{\tau}^{w;\tau}) \,\bigg|\, W_0 = w \bigg] \,, \tag{3.26}$$

with

$$G(w) := \sup_{\widetilde{a},\widetilde{x},\widetilde{s}} \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\widehat{\delta}t} U(\widetilde{x}_t, \widetilde{s}_t)\,\mathrm{d}t \;\middle|\; \widetilde{W}_0 = w \right] \tag{3.27}$$

being the value function conditional on immediate investment. With a slight abuse of terminology, G is referred to as the *terminal gain* in the sequel.

The function G is a problem of optimal consumption and allocation with two goods and subsistence constraints over an infinite horizon. As such, it is an interesting and relevant extension to the literature on subsistence constraints discussed in Chapter 3.1. The typical first step in solving optimal control problems with labour income is to calculate human capital (cf. Bensoussan & Park, 2025, Kraft & Munk, 2011). In this instance, it is given by the present value of effective labour income net of subsistence consumption, i.e.

$$\widetilde{H} := \int_0^\infty \mathrm{e}^{-\mu_R t}(\widetilde{Y} - \underline{x} - (\underline{s}/\widetilde{\eta})P)\,\mathrm{d}t = \frac{1}{\mu_R}(\widetilde{Y} - \underline{x} - (\underline{s}/\widetilde{\eta})P)\,. \tag{3.28}$$

Since the agent is allowed to borrow against human capital, define the total money available for discretionary spending as

$$\widetilde{Z}_t := \widetilde{z}(\widetilde{W}_t) := \widetilde{W}_t + \widetilde{H}\,. \tag{3.29}$$

For ease of terminology, we refer to $\widetilde{Z}_t$ as “disposable capital” in the sequel. For economic realism, this quantity must be constrained to be positive; otherwise, the agent could borrow indefinitely. Consequently, for a given initial condition $w > -\widetilde{H}$, define the set of admissible controls as

$$\widetilde{\mathcal{A}}(w) = \left\{ (\widetilde{a}, \widetilde{x}, \widetilde{s}) \mid \widetilde{z}(\widetilde{W}_t^w) > 0 \;\forall\, t \geq 0 \right\}\,. \tag{3.30}$$

The following result is obtained.

Proposition 3.2. *Let $w > -\widetilde{H}$. The terminal gain in* (3.27) *is given by*

$$G(w) = \Gamma^{-\gamma} \frac{\widetilde{z}(w)^{1-\gamma}}{1-\gamma}\,, \tag{3.31}$$

where $\Gamma > 0$ is a constant defined in (B.9). *The optimal strategies are*

$$\widetilde{a}_t^* = \frac{\kappa}{\gamma\sigma_S} \frac{\widetilde{Z}_t^{\widetilde{z}(w)}}{\widetilde{W}_t^w}\,, \quad \widetilde{x}_t^* = \underline{x} + (1-\beta)\varphi \widetilde{Z}_t^{\widetilde{z}(w)}\,, \quad \widetilde{s}_t^* = \underline{s} + \beta\varphi \frac{\widetilde{Z}_t^{\widetilde{z}(w)}}{P/\widetilde{\eta}}\,, \tag{3.32}$$

where $\varphi > 0$ is a constant defined in (B.10).

A few remarks are in order. Notice first that in the limit $\widetilde{Z}_t \to 0$, allocation vanishes and consumption reduces to subsistence levels. On the other hand, allocation is always above the Merton level $\kappa/(\gamma\sigma_S)$, with this level being attained only asymptotically in the limit of large wealth, $\widetilde{W}_t \gg \widetilde{H}$. In the case of the consumption controls, optimal behaviour is straightforward to interpret: total expenditure on consumption above subsistence levels is given by $\varphi\widetilde{Z}_t$, with the sum being allotted to each good according to the preference weight β. Energy-service demand $\widetilde{s}_t^*$ is seen to depend on the so-called "implicit price of energy", $P/\widetilde{\eta}$, which reflects the price adjusted for conversion efficiency (cf. Chan & Gillingham, 2015). Due to the presence of the subsistence levels, price elasticity and income elasticity deviate from unity.

Note that in a model with exogenous labour income and no constraints or transaction costs on risky-asset allocation, it is possible for an impatient agent to borrow heavily against human capital, driving wealth into deeply negative regimes. To see this, consider that the optimally-controlled process $\widetilde{Z}_t$ follows a geometric Brownian motion

$$\mathrm{d}\widetilde{Z}_t = \left(\frac{\kappa^2 + \gamma\left(\kappa^2 - 2\widehat{\delta} + 2\mu_R\right)}{2\gamma^2} \right) \widetilde{Z}_t\,\mathrm{d}t + \frac{\kappa}{\gamma}\widetilde{Z}_t\,\mathrm{d}B_t \tag{3.33}$$

with solution

$$\widetilde{Z}_t = \widetilde{z}(w)\exp\left[\left(\mu_{\widetilde{Z}} - \tfrac{1}{2}\sigma_{\widetilde{Z}}^2\right)t + \sigma_{\widetilde{Z}}B_t\right], \tag{3.34}$$

where $\mu_{\widetilde{Z}}$ and $\sigma_{\widetilde{Z}}$ denote the drift and volatility respectively in (3.33). Hence, if $\mu_{\widetilde{Z}} - \sigma_{\widetilde{Z}}^2/2 < 0$, the process $\widetilde{Z}_t$ shrinks in expectation over time, so that as t becomes large, the agent exhausts human capital entirely. Such unrealistic behaviour can be avoided by requiring that the effective drift remain positive; straightforward computation shows that this requirement reduces to the condition

$$\widehat{\delta} < \frac{\kappa^2 + 2\mu_R}{2}\,. \tag{3.35}$$

This "patience condition" is taken as given in the following.

The agent's decision problem in full can now be taken up. As above, define the quantities

$$Z_t := z(W_t) := W_t + H\,, \quad \text{where} \quad H := \frac{1}{\mu_R}(Y - \underline{x} - (\underline{s}/\eta)P)\,. \tag{3.36}$$

Then for $w > -H$, define the set of admissible controls analogously to (3.30) as

$$\mathcal{A}(w) := \{(a, x, s, \tau) \mid z(W_t^{w;\tau}) > 0\ \forall\, t \geq 0\}\,. \tag{3.37}$$

Denote also the change in human capital due to adoption of the energy-efficiency measure as

$$\theta := \widetilde{H} - H = \frac{1}{\mu_R}\left((\underline{s}/\eta - \underline{s}/\widetilde{\eta})P - \rho K\right) . \tag{3.38}$$

Intuitively, the change simply equals the net present value of energy costs at subsistence levels. The quantity θ is hence referred to as the "subsistence net present value" in the sequel.[18] The solution to the agent's decision problem follows.

Theorem 3.3. *Let $w > -H - \theta$ be a given initial wealth. Define the threshold $w^* := \Lambda\theta - H$ for $\Lambda < 0$ a constant defined in* (B.37)*, and let $z^* = z(w^*)$. The following cases are obtained.*

(i) Suppose $\theta \geq 0$ or $w \geq w^*$. The value function of (3.26) *is then given by $F(w) = G(w)$, with $\tau^*(w) = 0$ being optimal in* (3.26)*. The optimal allocation and consumption strategies are given in* (3.32)*.*

(ii) Suppose $\theta < 0$ and $w < w^*$. The value function is then given by

$$F(w) = \inf_{\widehat{z}>0}\left[\widehat{f}(\widehat{z}) + \widehat{z}\,z(w)\right] , \tag{3.39}$$

where $\widehat{f}$ is defined in (B.31)*. The first hitting time*

$$\tau^*(w) = \inf\{t \geq 0 \mid W_t^w \geq w^*\} \tag{3.40}$$

is optimal in (3.26)*.*[19] *The remaining optimal strategies follow*

$$a_t^* = \begin{cases} -\dfrac{\kappa}{\sigma_S}\dfrac{\partial_w F(W_t^w)}{w\partial_w^2 F(W_t^w)} , & t < \tau^* , \\ \dfrac{\kappa}{\gamma\sigma_S}\dfrac{\widetilde{Z}_t^{z^*+\theta}}{\widetilde{W}_t^{w^*}} , & t \geq \tau^* , \end{cases} \tag{3.41}$$

and

$$x_t^* = \begin{cases} b_0(\partial_w F(W_t^w), (P/\eta)\partial_w F(W_t^w)) , & t < \tau^* , \\ \underline{x} + (1-\beta)\varphi\widetilde{Z}_t^{z^*+\theta} , & t \geq \tau^* , \end{cases} \tag{3.42}$$

[18] It bears emphasising that in contrast to a net present value analysis based on average demand (cf. Hassett & Metcalf, 1992), the above definition is based on the *subsistence* demand, where the agent has no more flexibility.

[19] We abuse notation slightly by using W_t^w to denote a solution to (3.4) with initial condition $W_0 = w$ for $t < \tau^*$.

and finally

$$s_t^* = \begin{cases} b_1(\partial_w F(W_t^w), (P/\eta)\partial_w F(W_t^w)) \,, & t < \tau^* \,, \\ \underline{s} + \beta\varphi \dfrac{\widetilde{Z}_t^{z^*+\theta}}{P/\widetilde{\eta}} \,, & t \geq \tau^* \,, \end{cases} \tag{3.43}$$

where b_0 and b_1 are deterministic functions defined in (B.7) *and* (B.8) *respectively.*

We make a few remarks. Firstly, the above result underscores the centrality of the subsistence requirement $\underline{s}$ in this model: since the agent cannot reduce demand below this level, immediate investment in the retrofit is optimal if the subsistence net present value θ is non-negative. Investment is also immediate if w exceeds the threshold level w^*, which is a linear function of θ. On the other hand, when $\theta < 0$ and $w < w^*$, there is an option value of waiting to invest (cf. McDonald & Siegel, 1986). As regards allocation and consumption, the post-investment strategies have been discussed in light of Proposition 3.2. On the other hand, in the case where waiting is optimal, the pre-investment strategies are available only in implicit form. However, they admit a natural closed-form approximation which is introduced in the sequel.

3.3.2 *Approximate welfare results*

Having thus solved the agent's decision problem, we consider now the welfare implications developed in Chapter 3.2.2. For ease of notation, the following results are stated directly in terms of disposable capital rather than wealth. We use the identity $w(z) := z - H$ as well as the investment threshold $z^* = z(w^*) > 0$ for disposable capital in the sequel. We begin by stating the solution to the counterfactual decision problem (3.12), which follows immediately from Proposition 3.2 by symmetry.

Corollary 3.4. *Let $z > 0$. The counterfactual value function in* (3.12) *is given by*

$$\widehat{F}(z) = \widehat{\Gamma}^{-\gamma} \frac{z^{1-\gamma}}{1-\gamma} \,, \tag{3.44}$$

where $\widehat{\Gamma} > 0$ is a constant identical to Γ of (B.9) *with $\widetilde{\eta}$ replaced by η. The optimal strategies are given by*

$$\widehat{a}_t^* = \frac{\kappa}{\gamma\sigma_S} \frac{\widehat{Z}_t^z}{\widehat{W}_t^{w(z)}} \,, \quad \widehat{x}_t^* := \underline{x} + (1-\beta)\varphi \widehat{Z}_t^z \,, \quad \widehat{s}_t^* := \underline{s} + \frac{\beta\varphi\widehat{Z}_t^z}{P/\eta} \,, \tag{3.45}$$

where $\widehat{Z}_t = z(\widehat{W}_t)$ and $\varphi > 0$ is a constant given in (B.10). *It follows that the dynamics of the optimally-controlled process $\widehat{Z}_t$ are identical to those of $\widetilde{Z}_t$ from* (3.33).

Intuitively, the "never-invest" optimal strategies are wholly analogous to the "immediate-invest" optimal strategies, except with a different efficiency parameter.

In order to facilitate closed-form results, we propose now an approximation to the implicit controls in Theorem 3.3. Recall that this is only necessary for the regime $t < \tau^*$ in the case where waiting is optimal, i.e. when $\theta < 0$ and $z < z^*$. A natural choice is to assume that the controls in this regime can be approximated by the counterfactual controls from Corollary 3.4 above. That is, the approximation assumes that the agent allocates wealth and consumes *as though* they were never going to invest, but then investment does in fact occur when the threshold z^* is attained. Appendix B.2 presents a formal argument and a numerical example which shows that as long as the derivatives of the option value of switching technologies are small enough, the approximation is a fair one. The following assumption is therefore introduced and immediately employed to examine the conditions that give rise to rebound and backfire.

Approximation 3.5. Suppose $\theta < 0$ and $z \in (0, z^*)$. Then define the first hitting time

$$\widehat{\tau}^*(z) := \inf\{t \geq 0 \mid \widehat{Z}_t^z > z^*\} . \tag{3.46}$$

Consequently, approximate the optimal controls from Theorem 3.3 as

$$a_t \approx \begin{cases} \dfrac{\kappa}{\gamma\sigma_S} \dfrac{\widehat{Z}_t^z}{\widehat{W}_t^{w(z)}} , & t < \widehat{\tau}^* , \\[2ex] \dfrac{\kappa}{\gamma\sigma_S} \dfrac{\widetilde{Z}_t^{z^*+\theta}}{\widetilde{W}_t^{w^*}} , & t \geq \widehat{\tau}^* , \end{cases} \tag{3.47}$$

and

$$x_t \approx \begin{cases} \underline{x} + (1-\beta)\varphi\widehat{Z}_t^z , & t < \widehat{\tau}^* , \\ \underline{x} + (1-\beta)\varphi\widetilde{Z}_t^{z^*+\theta} , & t \geq \widehat{\tau}^* , \end{cases} \qquad s_t \approx \begin{cases} \underline{s} + \dfrac{\beta\varphi\widehat{Z}_t^z}{P/\eta} , & t < \widehat{\tau}^* , \\[2ex] \underline{s} + \beta\varphi\dfrac{\widetilde{Z}_t^{z^*+\theta}}{P/\widetilde{\eta}} , & t \geq \widehat{\tau}^* . \end{cases} \tag{3.48}$$

Proposition 3.6. *Let $z > 0$.*

(i) Suppose $\theta > 0$. Then rebound occurs in expectation, i.e. $\mathbb{E}[R_t] > 0$. The probability of backfire occurring at time t is given by

$$\mathbb{P}(Q_t > 0) = 1 - \Phi\left(\frac{\log\kappa - \log\theta - (\mu_{\widetilde{Z}} - \frac{1}{2}\sigma_{\widetilde{Z}}^2)t}{\sigma_{\widetilde{Z}}\sqrt{t}}\right) , \tag{3.49}$$

where Φ is the cumulative distribution function of the standard normal distribution, and $\kappa > 0$ is a constant defined in (B.40).

(ii) *Suppose $\theta = 0$. Then rebound occurs in expectation and backfire does not occur in expectation, i.e.* $\mathbb{E}[Q_t] < 0$.

(iii) *Suppose $\theta < 0$ and $z \geq z^*$. Then rebound occurs in expectation if*

$$z > -\frac{\widetilde{\eta}\theta}{\widetilde{\eta} - \eta} . \tag{3.50}$$

Backfire does not occur in expectation.

(iv) *Suppose $\theta < 0$ and $z < z^*$, and assume Approximation 3.5. Then rebound occurs in expectation if*

$$\Lambda > -\frac{\widetilde{\eta}}{\widetilde{\eta} - \eta} , \tag{3.51}$$

where $\Lambda < 0$ is the constant from (B.37). *Backfire does not occur in expectation.*

In sum, if the subsistence net present value $\theta > 0$, rebound always occurs in expectation; moreover, the probability of backfire occurring increases as time passes. Conversely, if $\theta < 0$, rebound occurs under what turns out to be relatively mild conditions (cf. Chapter 3.4.1), whereas backfire never occurs. This general "no backfire" property for $\theta < 0$ will prove important in the sequel.

Having quantified the rebound and backfire measures, we are in a position to evaluate the net welfare implications of the energy-efficiency investment. We build towards the main result, Theorem 3.9 below, which gives the conditions under which the energy-efficiency investment is welfare improving. The agent's equivalent variation, V_{ev}, is computed first. No closed-form expression for V_{ev} is possible when waiting is optimal, since a closed-form solution for $F(z)$ is not available in this case. However, the following result is obtained in the case of immediate investment.

Lemma 3.7. *Suppose $\theta \geq 0$ or $z > z^*$. Then $V_{\text{ev}}(z) = (\widetilde{\eta}/\eta)^{\beta}(z + \theta) - z$.*

The monetary equivalent of the utility improvement due to the retrofit is hence given by an intuitive expression involving only the present level of disposable capital z, the ratio of the efficiency parameters $\widetilde{\eta}$ and η, the preference weight on the energy service β, and the improvement in human capital θ.

Consider now the present value of social costs, V_{sc}, as defined in (3.18), assuming additionally that $\widehat{\epsilon} - \mu_{\varpi} - \mu_{\widetilde{Z}} > 0$ to ensure convergence of the integral. The following results are obtained: the first gives an explicit formula for V_{sc}, and the second gives the condition under which the energy-efficiency investment improves welfare.

Lemma 3.8. *Let $z > 0$.*

(i) *Suppose* $\theta \geq 0$ *or* $z \geq z^*$. *Then* $V_{\mathrm{sc}}(z;\pi) = V_{\mathrm{sc}}(\pi) = \mathcal{I}(\pi)$, *where*

$$\mathcal{I}(\pi) = \left(\frac{(\widetilde{\eta}^{-1} - \eta^{-1})\underline{s}}{\widehat{\epsilon} - \mu_{\varpi}} + \frac{\beta\varphi\theta}{(\widehat{\epsilon} - \mu_{\varpi} - \mu_{\widetilde{Z}})P} \right) \pi \,. \tag{3.52}$$

(ii) *Suppose* $\theta < 0$ *and* $z < z^*$, *and assume Approximation 3.5. Then*

$$V_{\mathrm{sc}}(z;\pi) = \mathscr{L}(z;\widehat{\epsilon} - \mu_{\varpi})\mathcal{I}(\pi)\,, \tag{3.53}$$

where $\mathscr{L}(z;\varrho) = \mathbb{E}_z[\mathrm{e}^{-\varrho\widehat{\tau}^*}]$ *is the Laplace transform of* $\widehat{\tau}^*$, *given explicitly in* (B.49).

Theorem 3.9. *Let* $z > 0$.

(i) *Suppose* $\theta \geq 0$. *Then* $V(z;\pi) > 0$ *if and only if*

$$z < \frac{\mathcal{I}(\pi) - (\widetilde{\eta}/\eta)^{\beta}\,\theta}{(\widetilde{\eta}/\eta)^{\beta} - 1}\,. \tag{3.54}$$

(ii) *Suppose* $\theta < 0$ *and assume Approximation 3.5. Then* $V(z;\pi) > 0$.

Hence, the retrofit is generally welfare improving, except in the limiting case of very high wealth, where backfire eventually occurs as noted in Case (i) of Proposition 3.6.[20]

Lastly, this section considers the design of the corrective subsidy for energy-efficiency adoption defined in (3.20). Applying similar techniques as in Theorem 3.9, we are able to obtain an explicit solution in the case where the agent invests immediately, and to reduce to a deterministic optimisation if it is optimal for the agent to wait.

Proposition 3.10. *Let* $z > 0$.

(i) *Suppose* $\theta \geq 0$ *or* $z \geq z^*$. *The optimal subsidy rate in* (3.20) *is given by*

$$m^*(\pi) = -\frac{C_1(\pi) + \xi_0 K}{\xi_1 K^2} < 0\,, \tag{3.55}$$

where $C_1 > 0$ *is defined in* (B.54).

[20] We speculate that this exception arises because that unbounded energy demand is allowed in the model. If energy consumption were instead allowed to saturate, we believe it unlikely that backfire would occur, and the retrofit to be strictly welfare improving in all cases (Proposition 3.1).

(ii) *Suppose $\theta < 0$ and $z < z^*$, and assume Approximation 3.5. The optimal subsidy rate in* (3.20) *is given by*

$$m^*(z;\pi) = \underset{0\le m\le \overline{m}}{\arg\min} \Big[(D_0(z) + D_1(z)m)^{d_0}(C_0(\pi) + C_1(\pi)m) \\ + (D_0(z) + D_1(z)m)^{d_1}\Psi(mK) \Big] \,, \quad (3.56)$$

where $0 < \overline{m} < 1$ is defined in (B.56), *and the functions C_0, C_1, D_0, D_1, and constants d_0 and d_1, are defined in the proof.*

Hence, if $\theta \ge 0$ or $z \ge z^*$ the agent invests immediately anyway; thus, in light of Case (i) in Proposition 3.6, a negative subsidy is justified in order to mitigate externalities due to backfire. However, if $\theta < 0$, since welfare is guaranteed to improve by Theorem 3.9, the subsidy is positive to encourage earlier investment.

3.3.3 Aggregate quantities

Since an explicit solution for the agent's optimal investment time is obtained in Theorem 3.3, it is possible to simplify the expressions for the aggregate quantities defined in Chapter 3.2.3. Without risk of confusion, denote the optimal investment time by τ in this section. Then with $\mathcal{F}_\tau$ the cumulative distribution function of τ, define

$$\mathcal{G}(z^i;t) := \mathbb{1}_{\{z^i \ge z^*\}} + \mathbb{1}_{\{z^i < z^*\}} \mathcal{F}_{\tau^i(z^i)}(t) \,, \quad (3.57)$$

which gives the probability that agent i has invested at time t. Since the optimally-controlled state variable Z_t is a geometric Brownian motion, an explicit expression for $\mathcal{F}_\tau$ is available (Jeanblanc et al., 2009, Ch. 3.3.1), though we do not reproduce it here. It is then clear from (3.23) that the expected adoption share conditional on the population $\mathcal{P}$ is given by

$$\mathbb{E}\left[S_t \mid \mathcal{P}\right] = \frac{1}{N}\sum_{i=1}^{N} \mathcal{G}(z^i;t) \,, \quad (3.58)$$

which aggregates individual investment probabilities. Additionally, we note that the instantaneous rate of adoption, which is often of interest, is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\left[S_t \mid \mathcal{P}\right] = \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\{z^i < z^*\}} f_{\tau^i(z^i)}(t) \,, \quad (3.59)$$

where f_τ is the probability density function of τ. Similarly, expected aggregate energy consumption follows

$$\mathbb{E}[C_t \mid \mathcal{P}] = \sum_{i=1}^{N} \left[(s_t/\eta)(1 - \mathcal{G}(z^i;t)) + (s_t/\widetilde{\eta})\mathcal{G}(z^i;t) \right] . \tag{3.60}$$

Together, these quantities show how population-level trajectories are driven by the distribution of individual thresholds and the underlying financial uncertainty: agents above the wealth threshold adopt immediately, whereas those below it invest probabilistically over time, producing smooth aggregate dynamics in expectation.

3.4 Case study

The following sections present a detailed case study of an energy retrofit of a representative German single-family home, undertaken by an agent at the median of the wealth and income distributions. We analyse optimal strategies, welfare effects, optimal subsidy design, and aggregate quantities, concluding with comparative statics to assess parameter sensitivity. An excursion concerning optimal retrofit depth is presented in Appendix B.3.

The parameters required to specify the agent's decision problem are listed in Table 3.1. The risk-free rate μ_R, and the drift μ_S and volatility σ_S for the risky asset are in line with standard values (cf. Kraft & Munk, 2011). The labour income Y as well as the initial condition for the wealth diffusion w correspond to median values for a German homeowner with a mortgage (Bundesbank, 2023).[21] The agent's relative risk aversion γ and discount rate δ are also from Kraft & Munk (2011), with a hazard rate λ corresponding to a remaining life expectancy of 50 years. The subsistence level of the numeraire $\underline{x}$ is taken to coincide with the tax-free basic allowance (BMF, 2025). The relative weighting β, subsistence level $\underline{s}$, as well as the price of gas P were calibrated to obtain reasonable levels of energy-service demand. The efficiency parameters η, $\widetilde{\eta}$, borrowing rate ρ, and retrofit cost K are estimated from Galvin (2024, Case Study "EFH78"), corresponding to a typical German single-family home built during the period 1969–1978.

3.4.1 Optimal strategies

Firstly, due to the large cost of the retrofit, the subsistence net present value of the project is negative, namely $\theta = -16.2\,\text{k€}$. The corresponding investment threshold

[21] All monetary values are in 2021 €.

TABLE 3.1. *Parameter values for the case study in Chapter 3.4. Sources in main text.*

Parameter	Description	Value
	Financial assets	
μ_R	Drift, risk-free asset	$0.025\,\mathrm{yr}^{-1}$
μ_S	Drift, risky asset	$0.07\,\mathrm{yr}^{-1}$
σ_S	Volatility, risky asset	$0.2\,\mathrm{yr}^{-1}$
	Energy price, income, wealth	
P	Gas price	$0.21\,€\,\mathrm{kWh}^{-1}$
Y	Labour income	$47\,\mathrm{k€}\,\mathrm{yr}^{-1}$
w	Initial wealth	$45\,\mathrm{k€}$
	Preferences	
β	Weight, energy service	0.007
γ	Relative risk aversion	4
δ	Discount rate	$0.03\,\mathrm{yr}^{-1}$
λ	Hazard rate	$0.02\,\mathrm{yr}^{-1}$
	Subsistence consumption	
$\underline{x}$	Non-energy good	$12\,\mathrm{k€}\,\mathrm{yr}^{-1}$
$\underline{s}$	Indoor temperature	$15\,°\mathrm{C}$
	Retrofit parameters	
A	Dwelling area	$157\,\mathrm{m}^2$
η	Efficiency, existing state	$0.005\,°\mathrm{C}\,\mathrm{W}^{-1}$
$\widetilde{\eta}$	Efficiency, post-retrofit	$0.025\,°\mathrm{C}\,\mathrm{W}^{-1}$
ρ	Borrowing rate	$0.04\,\mathrm{yr}^{-1}$
K	Retrofit cost	$120\,\mathrm{k€}$

$w^* = 430\,\mathrm{k€}$, roughly 10 times the initial wealth level $w = 45\,\mathrm{k€}$. Figure 3.1 shows the probability density and cumulative distribution functions of the investment time τ^* for this initial wealth level, along with scaled multiples for comparison. As expected, the higher the initial wealth level, the earlier the expected investment time, and the higher the cumulative probability of investment.

The optimal allocation and consumption strategies from Theorem 3.3 are depicted in Figure 3.2 in a neighbourhood of the investment threshold. Comparing the pre- and post-investment regimes, a few patterns emerge. Allocation a closely aligns with the Merton portfolio allocation, with the agent engaging in limited borrowing against future labour income to increase exposure to the risky asset. Moreover, allocation decreases slightly following the investment, since capital is tied up in the illiquid retrofit investment. Non-energy consumption x also decreases after investment, since more of the budget is allocated to the energy service s, which is obtained with greater efficiency. Demand for the energy service s increases sharply following investment, whereas the corresponding

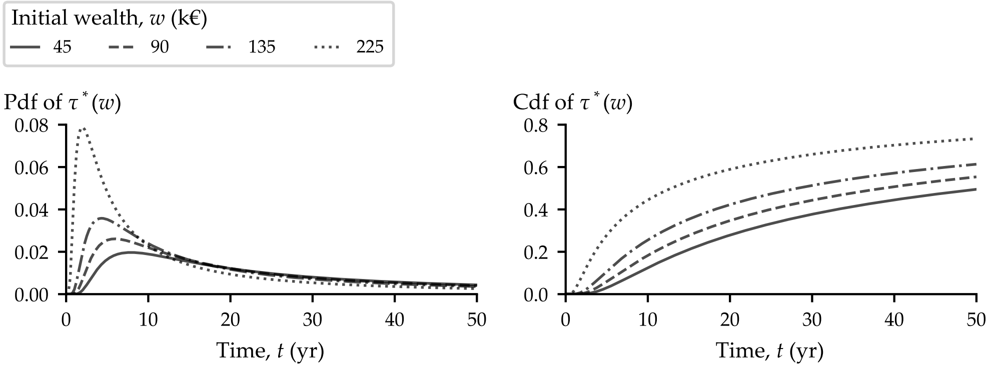

FIGURE 3.1. *Probability density function (Pdf) and cumulative distribution function (Cdf) of the optimal investment time τ^* for multiples of w from Table 3.1.*

fuel consumption $c = s/\eta$ (resp. $c = s/\widetilde{\eta}$ after investment) experiences a sharp decline due to the increased efficiency. That is, we have rebound without backfire, as established by Proposition 3.6.

The above observations can be appreciated in a different light in Figure 3.3, which depicts five optimally-controlled trajectories simulated over a 50 year horizon.[22] The upward jumps in energy-service demand depict clearly the moment of investment, with the corresponding downward jumps in the facing panel demonstrating how fuel consumption decreases for these same trajectories. The approximate rebound and backfire measures R_t and Q_t of Proposition 3.6 quantify this change directly, and are shown in the bottom row of Figure 3.4. For completeness, the top row of the same graphic presents the analogues of these measures for the allocation and non-energy consumption strategies by computing the difference to the counterfactual controls from (3.44). One sees clearly the effects discussed above: exposure to the risky asset increases prior to investment in order to build up capital stock, but decreases afterward as wealth becomes tied up in the retrofit project. On the other hand, consumption of the non-energy good declines relative to the counterfactual in trajectories where investment occurs, since the agent allocates comparatively more resources to the efficient energy service.

[22]We display trajectories instead of expectations since the demand jumps due to investments are masked completely if expectations are shown.

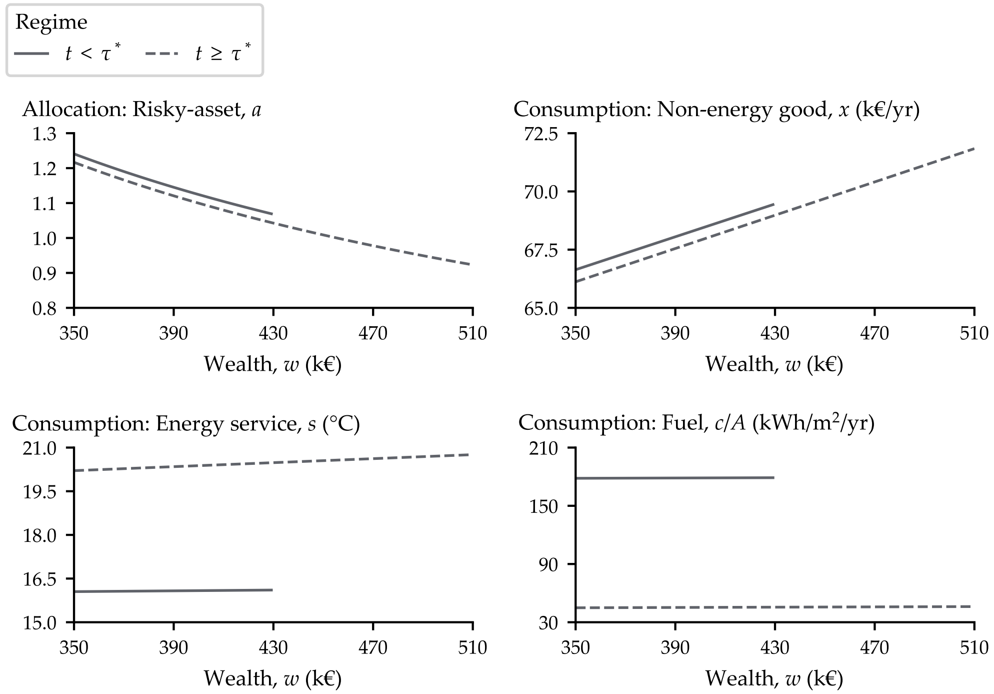

FIGURE 3.2. *Optimal strategies from Theorem 3.3 for the pre- and post-investment regimes in a neighbourhood of the investment threshold* $w^* = 430\,\text{k€}$. *For readability, fuel consumption c is normalised to the dwelling area A. Since the agent invests as soon as* $w > w^*$, *the regime* $t < \tau^*$ *cuts off at this point. On the other hand, since the agent's wealth may fall below the threshold after investment, the domain of the controls for the regime* $t \geq \tau^*$ *is* $w \in [-\widetilde{H}, \infty)$.

3.4.2 Welfare effects

We move on to the welfare implications of the agent's decisions. The additional parameters required for the analysis of the welfare aspects considered in Chapter 3.3.2 are listed in Table 3.2. The social discount rate ϵ is chosen slightly lower than the risk free rate μ_R as per Caplin & Leahy (2004), the drift in social cost μ_ϖ is estimated from the long term carbon price scenarios in Gerlagh & Liski (2017), and the marginal cost of public funds ξ_0 is from Kleven & Kreiner (2003).[23]

[23]The friction parameter ξ_1 was fixed by trial and error; see Chapter 3.4.4 following for the comparative statics.

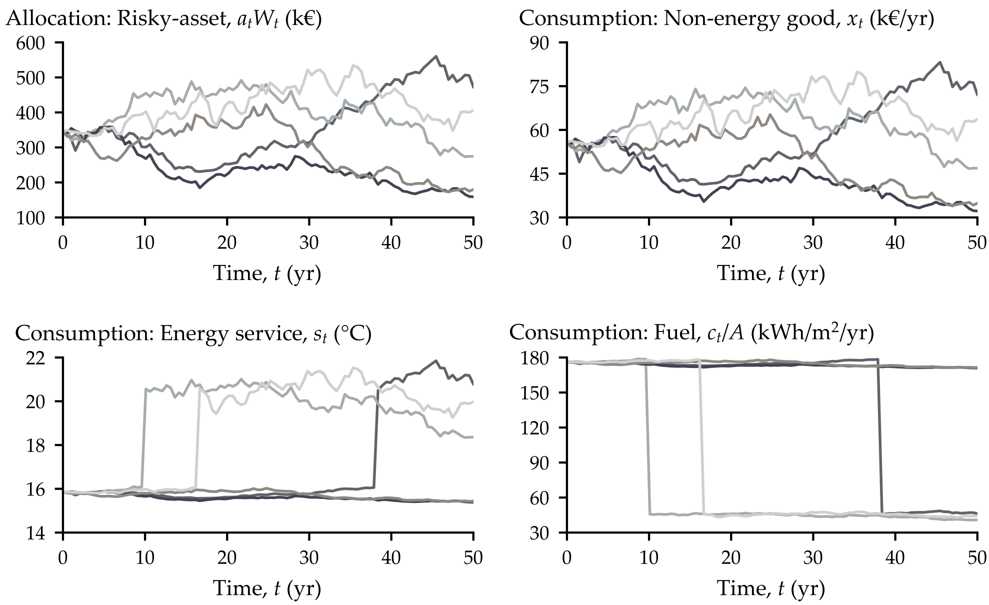

FIGURE 3.3. *Optimal strategies from Theorem 3.3 for five exemplary trajectories. Investment takes place in three of the trajectories, as indicated by the demand jumps in the bottom row. Fuel consumption c_t is normalised to the dwelling area A.*

Based on these parameter choices, Figure 3.5 shows the measure of welfare change V from Theorem 3.9, computed over a grid of carbon prices and initial wealth.[24] As expected, since $\theta < 0$, welfare change is positive everywhere and increasing in carbon prices and wealth. The optimal subsidy m^* from Proposition 3.10 is also shown in Figure 3.5; it is seen to be sharply increasing in carbon prices and slowly decreasing in wealth. The overall level of the subsidy is rather modest, attaining a maximum value of around 1.5 % of the total retrofit cost on the considered grid. For wealth levels exceeding w^*, the penalty rate from Proposition 3.10, which is anyway constant in w, is also roughly constant over the carbon price range 10 to 70 € tC^{-1}. It is given by $m^* = -1.76\,\%$ (not shown in Figure 3.5).

[24]Note that the social cost of energy consumption ϖ_t equals the product of the carbon price and the emissions factor for the fuel, in this case gas, which is 0.240×10^{-3} tC kWh^{-1} (Koffi et al., 2017). The range of carbon prices considered in Figure 3.5 corresponds roughly to the scenarios in the supplementary material of Gerlagh & Liski (2017), normalised to 2021 €.

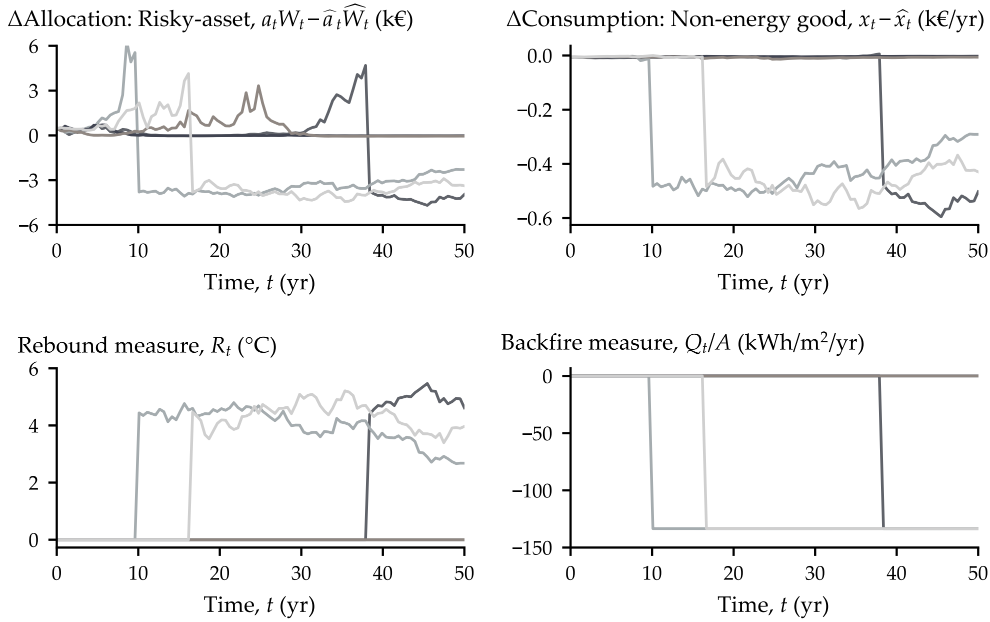

FIGURE 3.4. *Difference between the optimal strategies from Theorem 3.3 and the counterfactual strategies from Corollary 3.4 for the five trajectories shown in Figure 3.3. The backfire measure Q_t is normalised to the dwelling area A.*

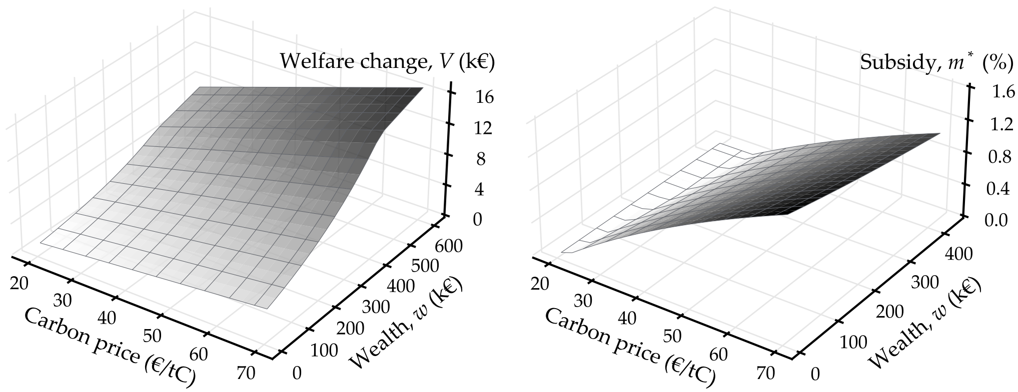

FIGURE 3.5. *On the left is the total welfare change V from Theorem 3.9, and on the right is the optimal subsidy level m^* from Proposition 3.10, each computed over a grid of carbon prices and wealth levels.*

TABLE 3.2. *Additional parameter values for the case study in Chapter 3.4. Sources in main text.*

Parameter	Description	Value
ϵ	Discount rate, social planner	$0.02\,\mathrm{yr}^{-1}$
μ_{ϖ}	Drift, marginal social cost	$0.013\,\mathrm{yr}^{-1}$
ξ_0	Marginal cost of public funds	2.12
ξ_1	Friction parameter, public funds	$1\,€^{-1}$

3.4.3 Aggregate behaviour

This section demonstrates the aggregate quantities of interest discussed in Chapter 3.3.3. Firstly, we restrict the physical scope of the study as follows: the single family home studied above is representative of a large cohort of the German building stock, approximately 1.5 million dwellings (Loga et al., 2015); assuming that around half of these are homeowners (cf. Destatis, 2025), we arrive at approximately 750 000 dwellings. The risk-free rate μ_R, the risky-asset parameters μ_S and σ_S, and energy price P are assumed common to all agents. For simplicity, the following calculation also ignores variations in the borrowing rate ρ, retrofit cost K, and the efficiency parameters η and $\widetilde{\eta}$, keeping them fixed at the respective levels in Table 3.1. The remaining parameters are idiosyncratic. The preference parameters β, γ, δ, and λ, along with the subsistence consumption levels $\underline{x}$ and $\underline{s}$, are assumed to follow independent uniform distributions centred on the values in Table 3.1, with a width of ±10 %. Then, a joint distribution for labour income Y and initial wealth w was calibrated to data from the Bundesbank (2023).[25] By drawing from these assumed distributions a population of representative agents as in (3.22) is generated.

Figure 3.6 shows the expected share of adoption and expected total energy consumption over this population, along with other possible trajectories. Immediate investment is optimal for roughly half of adopters, with uptake being rather gradual for the remaining share. Consequently, total energy consumption falls slowly over time as the share of adopters increases. Nevertheless, significant variation among the trajectories is observed. Figure 3.7 demonstrates the effect of the subsidy policy from Proposition 3.10 on adoption share and energy consumption; for realism, we ignore the penalty for immediate investment, namely Case (i) of Proposition 3.10, retaining only the positive subsidies which encourage investment, namely Case (ii). It is apparent that the main effect of the

[25] We summarise the procedure briefly. Median and mean values for German homeowner wealth and income determined log-normal marginals for Y and w. The empirical profile of mean wealth across income quantiles was then used to estimate a log–log relation $\log \mathbb{E}[w \mid Y] = \alpha + \beta \log Y$, which allowed us to recover the conditional variance via the law of total variance. For sampling, we draw $\log Y$ from its marginal distribution and $\log W$ conditionally on $\log Y$, yielding cross-sectional samples consistent with the reported moments.

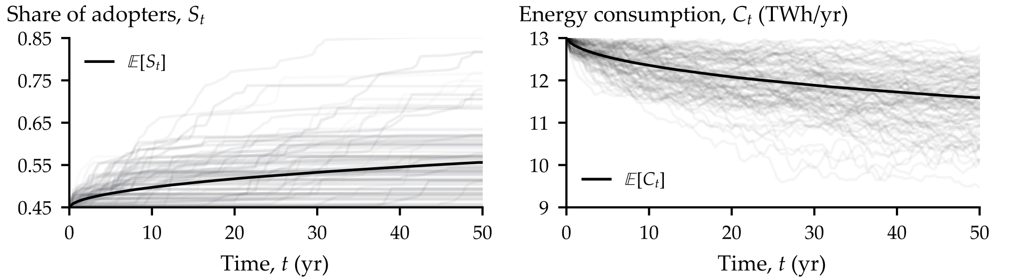

FIGURE 3.6. *On the left is the expected share of adopters* $\mathbb{E}[S_t]$ *from* (3.58)*, shown alongside other simulated trajectories. On the right is the expected aggregate energy consumption* $\mathbb{E}[C_t]$ *from* (3.60)*, computed for the same set of trajectories.*

policy is to increase the share of agents who adopt immediately, thus shifting the entire curve upward. The shift in the rate of adoption itself is slight.

3.4.4 Comparative statics

We take up the comparative statics of the case study. Since the model is complex, involving 15 parameters and several interdependencies, the following analysis will not be exhaustive. Instead, we focus firstly on the local effects of the model parameters on two fundamental outputs: the investment threshold w^* and optimal subsidy level

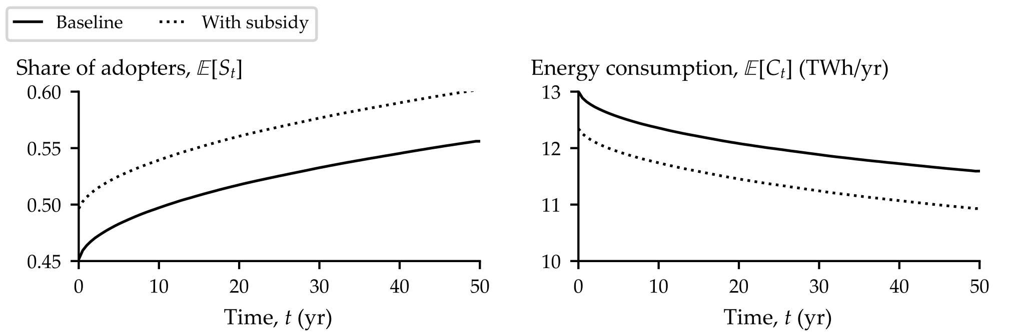

FIGURE 3.7. *On the left: the expected share of adopters* $\mathbb{E}[S_t]$ *for the baseline scenario from Figure 3.6 together with a second scenario where the optimal subsidy policy from Case (ii) of Proposition 3.10 is implemented. The panel on the right shows the analogous quantities for expected aggregate energy consumption* $\mathbb{E}[C_t]$.

m^*. Secondly, the effects of three key model parameters on the optimal strategies are examined. Finally, the effect of market volatility on technology diffusion is studied.

Table 3.3 lists the elasticities at w^* and m^* of the baseline parameters from Tables 3.1 and 3.2, assuming a carbon price of $45\,€\,\text{tC}^{-1}$.[26] Consider first the elasticities at w^*. Several are extremely large, indicating a certain level of model misspecification. These include the retrofit parameters η, $\widetilde{\eta}$, ρ, and K, the energy price P, and the subsistence level $\underline{s}$.[27] Each of these has an outsize effect on the retrofit threshold, and by extension on the other optimal strategies, indicating that an upper bound on energy-service consumption within the utility function would improve model realism. On the other hand, labour income Y and the preference weight β display large but plausible influences on w^*. The financial market parameters exhibit large to medium elasticities, with an increase in the risk-free rate μ_R increasing the attractiveness of the investment; conversely, if the risky asset becomes more attractive (higher μ_S or lower σ_S) the retrofit investment becomes relatively less attractive. An increase in the subsistence level of the non-energy good $\underline{x}$ is seen to drive up the investment threshold since comparatively more resources must be allocated to basic non-energy consumption. The remaining parameters manifest only a small local influence on w^*.

The optimal subsidy rate m^* is seen to be most sensitive to parameters that directly affect financing conditions and the effective return on energy efficiency. As above, the retrofit parameters, together with the gas price P and subsistence level $\underline{s}$, exhibit the largest effects; moreover, the direction of the effects is identical to the previous case, which is intuitive. For instance, a higher baseline efficiency η decreases the attractiveness of the retrofit for the agent, so the planner compensates by increasing m^*. As regards market conditions, the parameters μ_R, μ_S and σ_S exhibit moderate-to-large effects in the *opposite* direction as the effects on w^*. Hence, as financial markets become more rewarding, both the agent and the planner withdraw support for retrofit investment; the former because it is privately beneficial to do so, the latter because it is fiscally less efficient to subsidise. Conversely, as markets become riskier, or as the risk-free rate increases, both the agent and the planner shift toward the safer, socially productive retrofit. The subsidy level is seen to be progressive, with higher income and wealth levels associated with lower levels of subsidy, although the effect of the wealth level is comparatively small. Finally, among the social planner's parameters, the largest effects are attributed to the initial social cost π and marginal cost of public funds ζ_0.

[26] Since a closed-form expression for w^* is available, the elasticities are exact; the numerical calculations employed automatic differentiation (Maclaurin et al., 2015). On the other hand, since m^* does not admit an explicit solution, a finite-difference scheme was implemented.

[27] The elasticities of ρ and K are identical since only the combination ρK is present in the expression for w^*; the same is true for the combination $\underline{s}P$.

TABLE 3.3. *Local parameter elasticities for the case study in Chapter 3.4.*

Parameter	Description	Elasticity w^*	Elasticity m^*
	Financial assets		
μ_R	Drift, risk-free asset	-1.15	0.75
μ_S	Drift, risky asset	0.67	-3.87
σ_S	Volatility, risky asset	-0.43	2.45
	Energy price, income, wealth		
P	Gas price	-40.20	-4.12
Y	Labour income	-4.37	-2.29
w	Initial wealth		-0.06
	Agent preferences		
β	Weight, energy service	-3.72	-1.49
γ	Relative risk aversion	-0.44	1.25
δ	Discount rate, agent	-0.18	0.65
λ	Hazard rate	-0.12	-0.39
$\underline{x}$	Subsistence level, numeraire	1.12	0.59
$\underline{s}$	Subsistence level, energy service	-40.20	-2.54
	Retrofit parameters		
η	Efficiency, existing state	52.69	3.10
$\widetilde{\eta}$	Efficiency, post-retrofit	-12.49	-1.53
ρ	Borrowing rate	44.46	5.35
K	Retrofit cost	44.46	3.07
	Social planner parameters		
ϵ	Discount rate, planner		-0.82
π	Initial social cost		1.68
μ_π	Drift, social cost		0.70
ξ_0	Marginal cost of public funds		-1.23
ξ_1	Friction parameter, public funds		-0.45

Next, we examine the effects of changing three key parameters, namely the preference weight β, risk aversion γ, and risky-asset volatility σ_S, on the agent's allocation and consumption optimal strategies. For ease of comparison, we restrict attention to the regime $t > \tau^*$, since the controls are defined over the entire wealth domain (cf. Figure 3.2). The parameters were varied by $\pm 10\,\%$ relative to their baseline values in Table 3.3, with the resulting controls displayed in Figure 3.8.[28] The findings are intuitive and consistent with expectations. The preference weight β has almost no effect on either portfolio allocation or non-energy consumption; however, its influence on energy-service

[28]For σ_S, the $+10\,\%$ variation results in a mild violation of the patience condition (3.35).

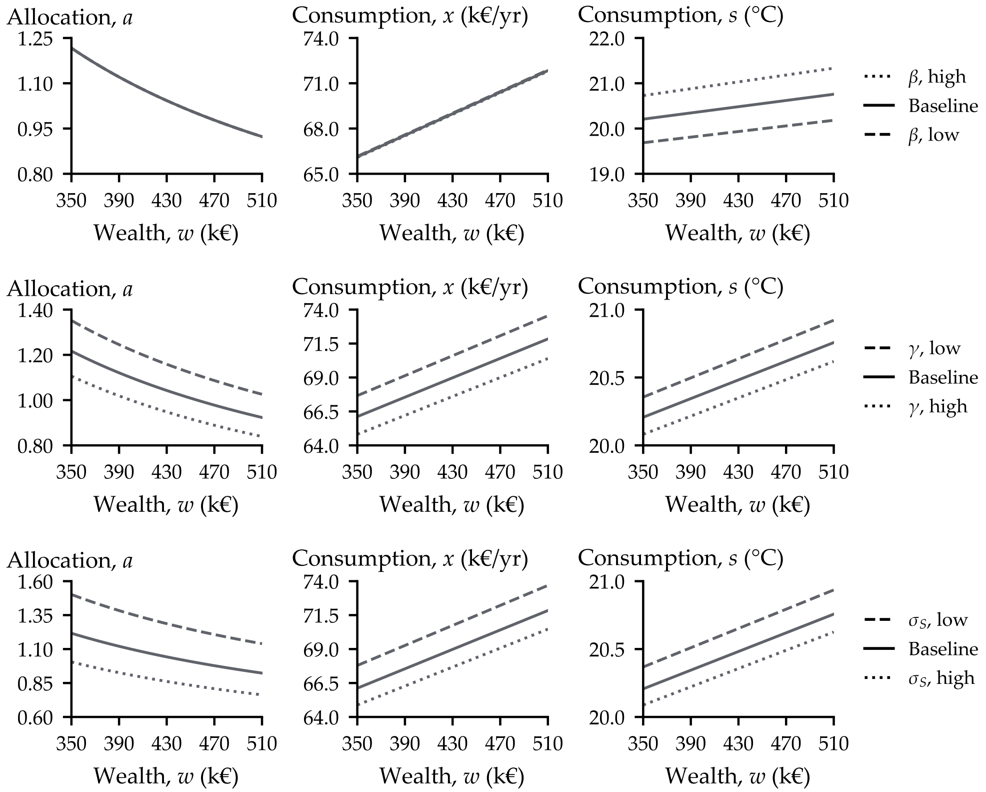

FIGURE 3.8. *Comparative statics for the model parameters β, γ, and σ_S. The baseline controls are identical to Figure 3.2. The "low" and "high" values for the parameters correspond to $\pm 10\,\%$ changes relative to the values in Table 3.3.*

consumption is strong and in the expected direction, with higher values associated with higher energy-service demand. The risk-aversion parameter γ significantly affects each of the optimal strategies: higher risk aversion is associated with lower investment in the risky asset and reduced consumption of both goods. The same holds for the risky-asset volatility σ_S, which influences portfolio allocation even more strongly than γ. The effects on consumption are also clear: greater market volatility leads to lower consumption as the agent responds through increased precautionary saving.

Finally, we examine the effect of market volatility on aggregate technology adoption and energy consumption. Using the same representative agents as in the simulations of Chapter 3.4.3, two volatility scenarios are examined, set at ±10 % of the values in Table 3.1. Figure 3.9 depicts the effects of these scenarios on expected adoption share

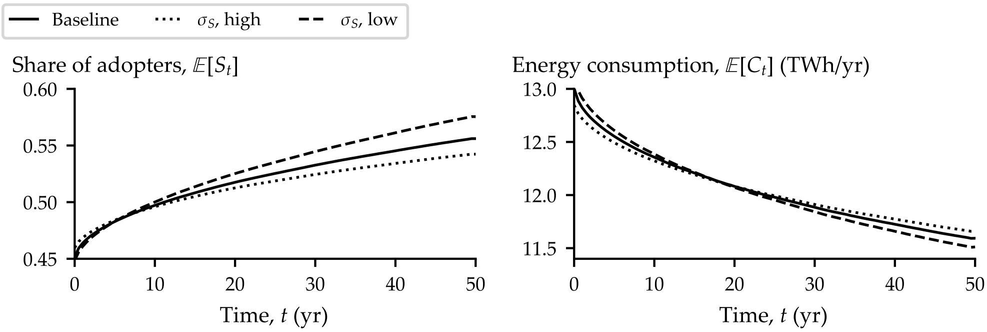

FIGURE 3.9. *Effect of the risky-asset volatility σ_S on expected adoption share $\mathbb{E}[S_t]$ and expected total energy consumption $\mathbb{E}[C_t]$; the baselines are from Figure 3.6.*

$\mathbb{E}[S_t]$ and expected aggregate energy consumption $\mathbb{E}[C_t]$. An interesting pattern emerges: although higher volatility lowers each agent's investment threshold (cf. Table 3.3) and slightly increases the initial share of early adopters relative to the baseline, the long-run effect on cumulative adoption moves in the *opposite* direction, namely, high volatility ultimately reduces adoption relative to the baseline, whereas low volatility leads to higher cumulative uptake. The underlying mechanism is as follows: higher volatility widens the dispersion of wealth paths, generating a small group of very early adopters, but an even larger group of agents whose wealth remains persistently below the investment region and therefore fails to reach the threshold within the finite horizon. Consequently, the aggregate adoption share, which depends on the entire distribution of stopping times rather than the marginal shift in individual thresholds, is systematically reduced.

Figure 3.10 repeats this exercise, but with the subsidy included (cf. Figure 3.7). Comparing the left panels of Figures 3.9 and 3.10, it is evident that the subsidy counteracts the wealth dispersion caused by increased market volatility by compressing the distribution of investment thresholds. This compression delays the "inflection point", where the ordering of the scenario curves reverses. The most striking effect, however, appears in the right panel, where the impact of volatility is completely reversed relative to Figure 3.9: the higher-volatility scenario consistently exhibits lower aggregate energy consumption. This makes sense in light of the policy's aim, which is to mitigate externalities from energy consumption, not simply to drive aggregate adoption; as such, the policy effectively targets energy-efficiency adoption where it generates the greatest marginal benefit.

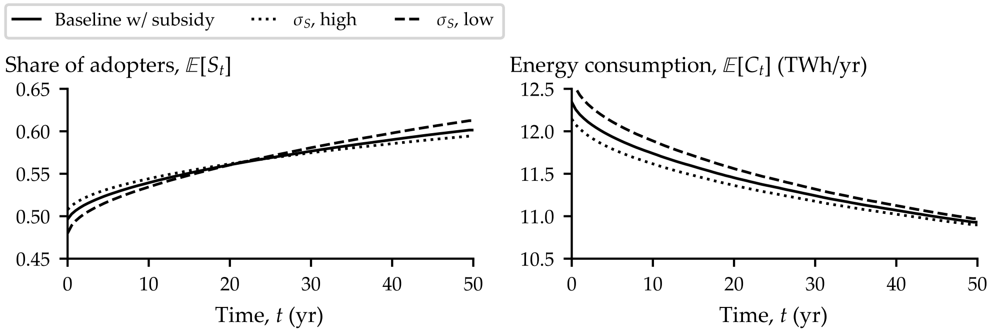

FIGURE 3.10. *Effect of the risky-asset volatility σ_S on expected adoption share $\mathbb{E}[S_t]$ and expected total energy consumption $\mathbb{E}[C_t]$ in the presence of the optimal subsidy from Case (ii) of Proposition 3.10.; the baselines are from Figure 3.7.*

3.5 Outlook

This chapter developed a model of consumption, investment, and energy-efficiency technology adoption under uncertainty. Despite its stylised nature, the model yielded substantial quantitative and qualitative insights, many in closed form. It demonstrated that the agent's optimal strategies, particularly the adoption of the energy-efficiency technology, are directly contingent on wealth, and that investment timing, energy demand, and portfolio choices co-evolve within a unified intertemporal structure. This joint evolution provided dynamic, internally consistent definitions of rebound and backfire effects, total welfare change, and optimal subsidy design, with several components characterised in closed form or through tractable approximations.

The analysis further showed that agent heterogeneity plays a central role: differences in wealth, preferences, income, and technology characteristics generate pronounced variation in adoption incentives, welfare outcomes, and responses to financial-market conditions. These heterogeneous responses aggregate in non-linear ways, such that macro-level energy use and technology diffusion emerge endogenously from micro-level optimisation. Comparative statics demonstrated how changes in energy prices, volatility, and financial parameters shift investment thresholds and alter aggregate adoption patterns.

These features have direct policy implications. Because investment incentives and welfare effects vary systematically across agents, effective subsidy design must account for heterogeneity rather than rely on uniform support. Moreover, since adoption thresholds respond sensitively to external conditions, particularly energy prices and macroeconomic volatility, subsidy schemes require regular recalibration to track underlying incentives and limit free-riding. The framework therefore provides a disciplined basis for identifying where subsidies have the highest marginal effect and for assessing how policies interact with non-linear aggregation effects that influence system-wide adoption.

The outlook for this research involves relaxing the model's more idealised assumptions. Since these simplifications were introduced to emphasise analytical tractability, it is likely that extensions will rely heavily on numerical methods. Three avenues appear particularly consequential. First, as the realism of theoretical and numerical results was dampened by the potential for unbounded energy consumption, constraining agent preferences is expected to improve alignment with observed behaviour; some concrete proposals are made in Chapter 5.2. Second, as human capital was shown to play a central role, incorporating uncertainty in labour income or introducing retirement is likely to generate quantitatively different outcomes. Third, as energy prices were found to exert a decisive influence on both consumption and investment decisions, accounting for price dynamics is essential for a comprehensive understanding of agent behaviour. Chapter 4 following takes up the task.

4 An extension to the basic model with dynamic energy prices

This chapter extends the basic model to incorporate dynamic energy prices. Some closed-form results are possible, with an approximation scheme introduced to support numerical implementation. A case study demonstrates that the resulting decision boundary depends on both wealth and energy prices, with investment in the efficiency measure becoming attractive at lower prices for wealthier agents. The design of a welfare-maximising subsidy is also studied. It is shown to be sensitive to the energy price and agent characteristics such as wealth.

4.1 Introduction

This chapter builds directly upon the previous one by introducing dynamics in the price of energy. As a consequence, the investment boundary becomes a function of energy price and wealth. We introduce the assumption of fixed energy demand in order to obtain some closed-form results which greatly aid the necessary numerical analysis. Economically speaking, this assumption allows us to study how an agent allocates wealth and consumes non-energy goods while maintaining a fixed indoor temperature. The assumption has the added benefit of sidestepping some of the issues with model realism encountered in the previous chapter, and is supported to some extent by empirical analyses which show that consumer energy demand is rather inelastic in the short term (Labandeira et al., 2017). Nevertheless, the assumption remains stylised, and realistic agent preferences are almost certain to lie between the specifications considered in the previous chapter and this one.

We also introduce the assumption of a fixed, finite time horizon at which the option to invest in the retrofit expires; economically, this corresponds to a date beyond which the agent is unwilling to take on the loan required to finance the retrofit. A similar setup is considered in Britto et al. (2024), although the agent's objective in that setting is pure wealth maximisation, with consumption and allocation controls assumed fixed. In contrast, we assume here that the agent's aim is to maximise utility from consumption subject to intertemporal budget constraints, similar to Chapter 3.

The remainder of the chapter is organised as follows. Chapter 4.2 develops the agent's investment problem and the problem of optimal subsidy design. Chapter 4.3 presents the solution, consisting of analytical results and a description of the numerical scheme used to approximate the remainder of the solution. Chapter 4.4 carries out a case study to demonstrate the applicability of the model. Chapter 4.5 concludes.

4.2 The model

4.2.1 The agent's decision problem

We introduce three changes to the setup from Chapter 3.2.1, leaving the rest intact. Firstly, rather than allowing flexibility in energy demand, we assume a fixed level of fuel consumption $C > 0$, which is reduced to a level $\widetilde{C} < C$ after the retrofit. The agent's preferences are hence defined over a single perishable consumption good x, which is also the numeraire. Secondly, we assume that the price of energy P_t increases at the fixed rate $\mu_P > 0$, so that

$$\mathrm{d}P_t = \mu_P P_t \,\mathrm{d}t \,. \tag{4.1}$$

It is economically reasonable to assume $\mu_P < \mu_R$, where the latter is the risk-free rate. We denote by $\{P_v^{t,p} \mid v \geq t\}$ a solution to (4.1) starting at time t with initial condition $P_t = p$. Finally, we introduce a fixed time horizon $0 < H < \infty$ at which the option to invest in the retrofit expires. Intuitively, since the large retrofit cost K is financed by a loan, H may be thought of as the last possible date at which the agent would consider taking on such a liability.

Hence, let $\mathcal{T}_{[0,H]}$ denote the set of stopping times in $[0, H]$. The set of admissible investment times is then given by

$$\mathcal{T} := \mathcal{T}_{[0,H]} \cup \{\infty\} \,, \tag{4.2}$$

where $\tau = \infty$ denotes the irreversible decision to never invest. From (3.4), we see that for a given $\tau \in \mathcal{T}$, wealth W_t evolves as

$$\begin{aligned} \mathrm{d}W_t = (a_t\mu_S W_t + (1-a_t)\mu_R W_t)\,\mathrm{d}t + a_t\sigma_S W_t\,\mathrm{d}B_t + (Y - x_t)\,\mathrm{d}t \\ - \left(CP_t \mathbb{1}_{\{t<\tau\}} + (\widetilde{C}P_t + \rho K)\mathbb{1}_{\{t\geq\tau\}}\right)\mathrm{d}t \,. \end{aligned} \tag{4.3}$$

For fixed t and a given investment time $\tau \geq t$, let $\{W_v^{t,(p,w),\tau} \mid v \geq t\}$ denote a solution to (4.3) starting at time t for given initial conditions $W_t = w$, $P_t = p$. Assuming time-additive utility with exponential discounting, the agent's decision problem is to choose an allocation control, a consumption control, and an investment time τ to maximise the present value of utility. This gives the value function at time t as

$$F_t(p,w) := \sup_{a,x,\tau} \mathbb{E}\left[\int_t^\infty e^{-\delta(v-t)} U(x_v)\, dv \;\middle|\; (P_t, W_t) = (p,w) \right], \quad t \in [0,H], \tag{4.4}$$

where $\delta > 0$ is the rate of time preference and $U\colon \mathbb{R}_+ \to \mathbb{R}$ is the utility function.[29] The remainder of this chapter assumes standard isoelastic preferences,

$$U(x) := \frac{x^{1-\gamma}}{1-\gamma}, \tag{4.5}$$

with $\gamma > 1$ the coefficient of risk aversion.

4.2.2 Optimal subsidy design

With this setup for the agent's decision problem, we turn to the welfare implications of the investment decision. In contrast with the consumption-investment model developed in Chapter 3, rebound is absent in the present case since energy demand is assumed rigid. By the same token, since backfire does not occur, the retrofit generates a social benefit due to reduced energy consumption (cf. Proposition 3.1). It is this quantity that is of interest to the social planner.

So for given initial conditions (p,w), denote the optimal investment time in (4.4) by $\tau^*(p,w)$; recall that it is a random variable, since it depends on the realised paths for P_t and W_t, the latter of which is stochastic. As in Chapter 3.2.2, assume that the social planner subsidises the cost of the retrofit at rate $m \in (0,1)$, reducing the agent's private cost to $(1-m)K$.[30] The natural policy instrument is a commitment to a fixed subsidy rate, and this is achieved by assuming an infinite planning horizon, as in (3.20). So let $\tau_m^*(p,w)$ be optimal in (4.4) for a given subsidy rate m. The social planner's value function is then given by

$$J(p,w;\pi) := \inf_{m\in[0,1]} \mathbb{E}\left[\int_{\tau_m^*(p,w)}^\infty e^{-\epsilon t}(\widetilde{C} - C)\varpi_t^\pi\, dt + e^{-\epsilon \tau_m^*(p,w)} \Psi(mK) \right], \tag{4.6}$$

where $\epsilon > 0$ is the social discount rate, ϖ_t^π is the marginal social cost of energy consumption with initial condition $\varpi_0 = \pi$, and Ψ is the convex cost function from (3.21).

[29] For ease of notation, the hazard rate considered in Section 3.9 is folded into the discount rate.

[30] In point of fact, $m < 0$ was allowed in Chapter 3.2.2 to mitigate externalities from backfire. Since demand effects are absent here, we restrict $m \in [0,1]$.

Similarly to Chapter 3.3.2, we assume that ϖ_t follows a geometric Brownian motion with drift μ_ϖ and volatility σ_ϖ, independent of W_t. Following arguments as in the proof of Proposition 3.10, it is easy to see that the above optimisation rewrites as

$$J(p,w;\pi) = \inf_{m\in[0,1]} \left[\mathscr{L}_m(p,w;\epsilon-\mu_\varpi)\frac{(\widetilde{C}-C)\pi}{\epsilon-\mu_\varpi} + \mathscr{L}_m(p,w;\epsilon)\Psi(mK) \right] , \tag{4.7}$$

where

$$\mathscr{L}_m(p,w;\varrho) = \mathbb{E}\left[\mathrm{e}^{-\varrho\tau_m^*(p,w)} \right] \tag{4.8}$$

is the Laplace transform of the distribution of the investment time.

4.3 *The solution*

The agent's decision problem, being one of optimal control and stopping, can be solved by dynamic programming (cf. Peskir & Shiryaev, 2006, Sec. 2.2). Due to the assumption of standard preferences, some closed-form results are possible. These facilitate the numerical analysis, which is necessary to approximate the remainder of the solution, including the optimal subsidy policy.

4.3.1 *Analytical results*

Similarly to Chapter 3.3.1, introduce the notation $\widetilde{W}_t$ to denote the wealth process W_t for $\tau = 0$, with $\widetilde{a}_t$ and $\widetilde{x}_t$ denoting the corresponding allocation and consumption controls respectively. And as in Chapter 3.2.2, introduce $\widehat{W}_t$ to denote the wealth process W_t for $\tau = \infty$, with $\widehat{a}_t$ and $\widehat{x}_t$ denoting the corresponding controls. It follows that the value of investing immediately is given by

$$G(p,w) = \sup_{\widetilde{a},\widetilde{x}} \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\delta t} U(\widetilde{x}_t)\,\mathrm{d}t \;\middle|\; (P_0,\widetilde{W}_0) = (p,w) \right] , \tag{4.9}$$

whereas the value of never investing is given by

$$\widehat{F}(p,w) := \sup_{\widehat{a},\widehat{x}} \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\delta t} U(\widehat{x}_t)\,\mathrm{d}t \;\middle|\; (P_0,\widehat{W}_0) = (p,w) \right] . \tag{4.10}$$

Notice now that at the decision horizon $t = H$, the agent is forced to choose between investing immediately and never investing, so that the value function is given by

$$F_H(p,w) = \max\{\widehat{F}(p,w), G(p,w)\} . \tag{4.11}$$

On the other hand, for $t < H$, the agent has the possibility to optimally choose the moment of investment while continuing to accrue utility from consumption. We build towards an explicit solution for (4.11) in Corollary 4.3 below; the value function for $t < H$ must be estimated numerically.

We take up first the value of immediate investment. Begin by defining human capital as the present value of labour income net of subsistence constraints:

$$\widetilde{h}(p) := \int_0^\infty \mathrm{e}^{-\mu_R t}(\widetilde{Y} - \widetilde{C}P_t^p)\,\mathrm{d}t = \frac{\widetilde{Y}}{\mu_R} - \frac{\widetilde{C}p}{\mu_R - \mu_P}\,, \tag{4.12}$$

where $\widetilde{Y} = Y - \rho K$ is labour income net of the loan repayment. Then with disposable capital defined as

$$\widetilde{Z}_t := \widetilde{z}(P_t, \widetilde{W}_t) := \widetilde{W}_t + \widetilde{h}(P_t)\,, \tag{4.13}$$

define the set of admissible controls analogously to (3.30). The following result is obtained.

Proposition 4.1. *Let $p > 0$ and $w \in \mathbb{R}$ be such that $\widetilde{z}(p,w) > 0$. The value of immediate investment in* (4.9) *is given by*

$$G(p,w) = \Gamma^{-\gamma}\frac{\widetilde{z}(p,w)^{1-\gamma}}{1-\gamma}\,, \tag{4.14}$$

where

$$\Gamma := \frac{\gamma\left(2\delta + \kappa^2 + 2(\gamma-1)\mu_R\right) - \kappa^2}{2\gamma^2} > 0\,. \tag{4.15}$$

Optimal allocation and consumption follow

$$\widetilde{a}_t = \frac{\kappa}{\gamma\sigma_S}\frac{\widetilde{Z}_t^{\widetilde{z}(w)}}{\widetilde{W}_t^w}\,, \qquad \widetilde{x}_t := \Gamma\,\widetilde{Z}_t^{\widetilde{z}(w)}\,. \tag{4.16}$$

The proof is analogous to that of Proposition 3.2 and is therefore omitted. We see that the allocation and consumption rules are also very similar to that case, so that the economic interpretation of the agent's consumption behaviour discussed in Chapter 3.3.1 holds. The main difference concerns energy demand, which is rigid in this model; consequently, the agent consumes the non-energy good as a fixed share Γ of disposable capital in each period. For economic realism, we continue to assume that the patience condition introduced in (3.35) holds.

The solution to the never-invest value function follows immediately from Proposition 4.1 due to symmetry. To this end, introduce human capital without the energy-efficiency improvement as

$$h(p) := \int_0^\infty \mathrm{e}^{-\mu_R t}(Y - CP_t^p)\,\mathrm{d}t = \frac{Y}{\mu_R} - \frac{Cp}{\mu_R - \mu_P}\,. \tag{4.17}$$

The corresponding measure of disposable capital is then given by

$$\widehat{Z}_t := z(P_t, \widehat{W}_t) := \widehat{W}_t + h(P_t)\,, \tag{4.18}$$

whence we have the following results.

Corollary 4.2. *Let $p > 0$ and $w \in \mathbb{R}$ be such that $z(p,w) > 0$. The never-invest value function in* (4.10) *is given by*

$$\widehat{F}(p,w) = \Gamma^{-\gamma}\frac{z(p,w)^{1-\gamma}}{1-\gamma}\,, \tag{4.19}$$

where $\Gamma > 0$ is from (4.15). *Optimal allocation and consumption follow*

$$\widehat{a}_t = \frac{\kappa}{\gamma\sigma_S}\frac{\widehat{Z}_t^{z(w)}}{\widehat{W}_t^w}\,, \qquad \widehat{x}_t := \Gamma\,\widehat{Z}_t^{z(w)}\,. \tag{4.20}$$

Corollary 4.3. *At the decision horizon $t = H$, the agent's value function satisfies*

$$F_H(p,w) = \begin{cases} \widehat{F}(p,w)\,, & p < b_H\,, \\ G(p,w)\,, & p \geq b_H\,, \end{cases} \tag{4.21}$$

where

$$b_H = \frac{(\mu_R - \mu_P)\rho K}{\mu_R(C - \widetilde{C})} \tag{4.22}$$

is the threshold energy price which makes $\widetilde{h}(p) > h(p)$.

Hence, since $\widehat{F}$ and G have identical functional forms and differ only in the size of human capital, which enters disposable capital linearly, the investment decision at the horizon $t = H$ is driven purely by the price of energy. If it is large enough to guarantee an improvement in human capital, the investment is undertaken; else, the agent never invests. Indeed, b_H is the standard net-present-value trigger, which is the optimal decision rule in the absence of investment-timing flexibility.

We state a final analytical result which gives an extremely useful insight into the agent's investment decision. Firstly, define the domain of the value function F_t as the subset where disposable capital is positive:

$$\mathcal{B} := \{(p, w) \in \mathbb{R}_+ \times \mathbb{R} \mid z(p, w) > 0\} . \tag{4.23}$$

Then, as is typical in optimal stopping problems, define the "continuation" and "stopping" regions at time t, denoted $\mathcal{C}_t$ and $\mathcal{S}_t$ respectively, as

$$\mathcal{C}_t := \{(p, w) \in \mathcal{B} \mid F_t(p, w) > G(p, w)\} , \tag{4.24}$$

$$\mathcal{S}_t := \{(p, w) \in \mathcal{B} \mid F_t(p, w) = G(p, w)\} . \tag{4.25}$$

The continuation region thus characterises states in which it is optimal to delay investment, while the stopping region identifies states in which immediate investment is optimal. Define also the constant

$$\overline{p} := \frac{\rho K}{C - \widetilde{C}} > b_H , \tag{4.26}$$

which is the energy price at which the per-period loan repayment is exactly offset by the contemporaneous monetary savings from reduced energy consumption. The following result is then obtained.

Corollary 4.4. *The stopping region satisfies* $\mathcal{S}_t \subseteq \{(p, w) \in \mathcal{B}_t \mid p \geq \overline{p}\}$ *for all* $t < H$.

The proof is by arguments that are similar to those found in the proof of Theorem 3.3. To wit, one evaluates the non-linear Hamilton-Jacobi-Bellman equation associated to the value function F_t at the terminal G, and checks for conditions which guarantee that the residual is non-negative. Economically, the result is noteworthy since it demonstrates the existence of an energy price level $\overline{p}$ below which investment is *never* optimal. For prices higher than $\overline{p}$, investment may be optimal; this is ultimately dictated by the location of the boundary dividing $\mathcal{C}_t$ and $\mathcal{S}_t$, which must be estimated numerically.

4.3.2 *Numerical approximation*

We take up the numerical scheme. Introduce first a discretisation of $[0, H]$ into N intervals $\{t_0, t_1, \ldots, t_N\}$ with $t_i = iH/N$ for $i = 0, 1, \ldots, N$ and $\Delta t := H/N$. Introduce also appropriate discretisations of the energy price P_t and the optimally-controlled "never invest" wealth process $\widehat{W}_t$ from Corollary 4.2. It follows from the dynamic programming principle that an approximation to the agent's value function in (4.4) is obtained by a backwards recursion of the Bellman equation as follows:

$$F_{t_N}(p,w) = F_H(p,w) \,, \tag{4.27}$$

$$F_{t_i}(p,w) = \max\{G(p,w), M_{t_i}(p,w)\} \,, \qquad i = N-1, N-2, \ldots, 0 \,, \tag{4.28}$$

where F_H is from Corollary 4.3, and where

$$M_{t_i}(p,w) := \sup_{a,x} \left[U(x(p,w))\Delta t + \mathrm{e}^{-\delta \Delta t} \mathbb{E}\left[F_{t_{i+1}}(P^{t_i,p}_{t_{i+1}}, \widehat{W}^{t_i,(p,w)}_{t_{i+1}}) \right] \right] \tag{4.29}$$

is the continuation value, that is, the value of waiting one period and optimally choosing later (cf. Peskir & Shiryaev, 2006, Sec. 1.2). Thus, at each $t < H$, the state space (p,w) splits into two regions: the "invest" region , where G is maximal, and the "wait" region, where M_t is maximal. At $t = H$, as we have already seen, the state space divides into an "invest" region where G is maximal, and a "never invest" region where $\widehat{F}$ is maximal. We therefore assume that the agent's optimal investment policy can be written as

$$\begin{cases} \text{invest if } P_t \geq b(t, W_t) \,, \\ \text{wait otherwise} \,, \end{cases} \tag{4.30}$$

where $b(t,W_t)\colon [0,H] \times \mathbb{R} \to \mathbb{R}_+$ is a time-dependent decision boundary satisfying $b(t,w) > \overline{p}$ for $t \in [0,H)$ due to Corollary 4.3, and $\lim_{t\to H} b(t,w) = b_H$ due to Corollary 4.4. Thus, the finite-time decision horizon introduces a discontinuity in the boundary, since it must "jump" from being above the level $\overline{p} > b_H$ for $t < H$ to attaining exactly the level b_H at $t = H$.[31] By the same token, it is clear that there is a significant value of waiting to invest, since energy prices higher than the net-present value threshold b_H are necessary for investment to be triggered before the option to wait expires.

We introduce finally an approximation to facilitate the numerical analysis, which is particularly challenging due to the presence of the controls a and x. The crux of the issue is that in simulation approaches to optimal control problems, the conditional expectation $\mathbb{E}[F_{t+1}]$ in (4.29) is typically estimated by a Monte Carlo simulation assuming random controls followed by a regression with a polynomial basis in the state variables (cf. Kharroubi et al., 2014). Therefore, the optimisation step in (4.29) is often numerically unstable

[31] To wit, if the finite-time decision horizon were removed, Corollary 4.4 would still hold, whereas Corollary 4.3 would no longer have force. Hence, we would have a stationary decision boundary $b(w)$ satisfying $b(w) > \overline{p}$.

or biased or both (Andréasson & Shevchenko, 2021). Moreover, this approach calls for repeated forward simulations with the optimal controls being re-estimated at each iteration, which is not only computationally prohibitive but also prone to a multiplication of errors (Shen, 2019). In order to avoid these difficulties, it is assumed that the agent allocates wealth and consumes according to the controls $\widehat{a}$ and $\widehat{x}$ associated with the never-invest case, as characterised in (4.20). This assumption parallels Approximation 3.5, which is discussed in detail in Chapter 3.3.2 and Appendix B.2: intuitively, the agent allocates wealth and consumes as though they were never going to invest, but then investment does in fact take place when the decision boundary is attained. We hence drop the supremum in (4.29) and approximate the continuation value as

$$M_{t_i}(p,w) \approx U(\widehat{x}(p,w))\Delta t + \mathrm{e}^{-\delta\Delta t}\mathbb{E}\left[F_{t_{i+1}}(P^{t_i,p}_{t_{i+1}}, \widehat{W}^{t_i,(p,w)}_{t_{i+1}})\right] . \tag{4.31}$$

The problem thus reduces to a pure optimal-stopping type, to which the Least Squares Monte Carlo (LSMC) scheme of Longstaff & Schwartz (2001) can be directly applied. We note that since the optimal stopping times $\tau^*(p,w)$ are the natural outputs of the LSMC algorithm, the Laplace transforms in the subsidy problem (4.7) are also straightforward to estimate.

4.4 Case study

To illustrate the above ideas, we consider in this section an agent with wealth and dwelling parameters as in Table 4.1. To facilitate comparison, many parameters are shared with the case study in Chapter 3.4 (cf. Table 3.1). The retrofit parameters are again from Galvin (2024, Case Study “EFH78”); however, instead of the estimated efficiencies reported in Table 3.1, we use here directly the fuel demand reported in the cited case study. The drift of the energy price μ_P is estimated from the real German consumer price index for gas during the period 1991–2024 (Destatis, 2023).[32] To place the following results in context, it is helpful to note that the average household gas price in Germany in 2024 was 12.5 cent/kWh (Bundesnetzagentur, 2025).

4.4.1 Optimal strategies

Following the assumption introduced in Chapter 4.3.2, the numerical solution to the agent’s investment problem is fully characterised by the time-dependent decision boundary which demarcates the agent’s action regions. This is estimated using the LSMC

[32] Only returns within 3 standard deviations of the mean were used for the estimation.

TABLE 4.1. *Parameters for the case study in Chapter 4.4. Sources in main text.*

Parameter	Description	Value
	Financial assets	
μ_R	Drift, risk-free asset	$0.025\,\text{yr}^{-1}$
μ_S	Drift, risky asset	$0.07\,\text{yr}^{-1}$
σ_S	Volatility, risky asset	$0.2\,\text{yr}^{-1}$
	Additional income parameters, wealth	
Y	Labour income	47 k€ yr^{-1}
C	Gas consumption, existing state	$30\,000\,\text{kWh}\,\text{yr}^{-1}$
μ_P	Drift, gas price	$0.011\,\text{yr}^{-1}$
w	Initial wealth	45 k€
	Retrofit parameters	
ρ	Borrowing rate	$0.04\,\text{yr}^{-1}$
K	Retrofit cost	120 k€
$\widetilde{C}$	Gas consumption, post-retrofit	$6000\,\text{kWh}\,\text{yr}^{-1}$
	Preferences	
γ	Risk aversion parameter	4
δ	Rate of time preference	$0.05\,\text{yr}^{-1}$
H	Decision horizon	10 yr

algorithm and shown in Figure 4.1 at times $t = 0$, $t = H/2$, and $t = H$. As discussed, the boundary $b(t, w)$ divides the wealth-price plane into "invest" and "wait" regions for $t < H$, and into "invest" and "never invest" regions for $t = H$. We see that the agent waits if energy prices and wealth levels are low, whereas they invest immediately if energy prices and wealth are large enough. It is clear that the price dynamics generate a significant value of waiting, as attested to by the gap between $b(t, w)$ for $t < H$ and $t = H$. Also apparent is the fact that the decision boundary shifts downwards in the (W_t, P_t)–plane towards the price level $\overline{p} = 20\,\text{cent}\,\text{kWh}^{-1}$ for $t < H$, finally attaining the level $b_H = 10\,\text{cent}\,\text{kWh}^{-1}$ at $t = H$ when the option to delay investment expires. The wealth dependence of the decision boundary for $t < H$ is clear, with the value of waiting being relatively reduced for higher wealth levels. This is due to the fact that, all other things being equal, higher wealth W_t lowers the marginal utility cost of future energy expenditures. With fixed energy use and deterministic energy prices, an increase in prices reduces non-energy consumption by a fixed amount, which entails a smaller utility loss for wealthier agents. Consequently, the option value of delaying investment is smaller, making earlier investment optimal even when energy prices are relatively high.

Based on this estimated decision boundary, Figure 4.2 shows the optimal allocation and consumption controls for five trajectories with the initial wealth level w from Table 3.1;

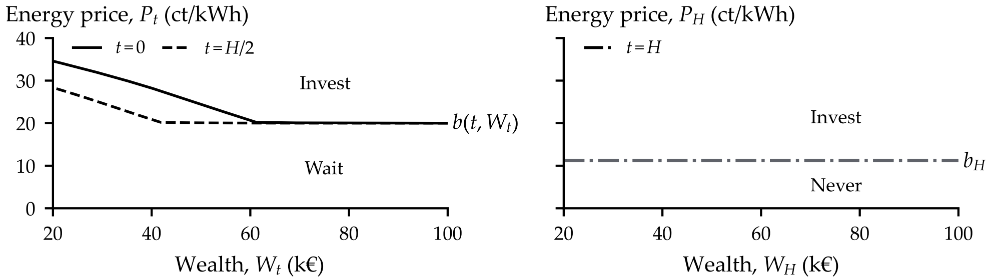

FIGURE 4.1. *The estimated decision boundary $b(t, w)$ for an agent with parameters as in Table 4.1. On the left: $b(t, w)$ at $t = 0$ and $t = H/2$. On the right: the terminal boundary b_H from Corollary 4.3.*

an initial energy price $p = 19$ cent/kWh is chosen to ensure that the decision boundary is attained within the simulated time frame. For clarity, as in Chapter 3.4.1, the difference to the counterfactual is displayed as well. The jumps in the difference plots indicate clearly the moment of investment, with the corresponding economic interpretation of the agent's behaviour being straightforward: since the retrofit investment relaxes the budget constraint, the agent allocates more to the risky asset, and increases utility by consuming greater amounts of the non-energy good. Comparison with Figures 3.3 and 3.4 shows that the allocation and consumption controls are of a similar order of magnitude to the basic model, although the strategies differ.

4.4.2 *Optimal subsidy design*

Given the agent's estimated decision boundary, we turn to the social planner's subsidy-design problem. To lay the groundwork, Figure 4.3 shows the effect of introducing a 30 % subsidy on the decision boundary b at time $t = 0$. As expected, since the retrofit cost K enters the key quantities b_H and $\overline{p}$, a reduction in K due to the subsidy fundamentally alters the geometry of the boundary, making investment optimal at comparatively lower energy prices. For this reason, it is clear that the subsidy policy directly influences the distribution of the optimal stopping times $\tau_m^*(p, w)$.

Therefore, assuming parameters for the social planner identical to Table 3.2, Figure 4.4 shows the optimal subsidy policy m^* from (4.7) for the low and high carbon price levels π considered in Chapter 3.4.2. Comparing with the agent's decision boundaries in Figure 4.1, we see that the level of subsidy is maximised in the "wait" region, close to the level of the terminal decision boundary b_H. As such, the policy has the clear intention of shifting b_H, so that at time $t = H$ at the latest, the agent has additional incentive to invest. On the other hand, if energy prices or wealth levels are too low, i.e. if the agent is

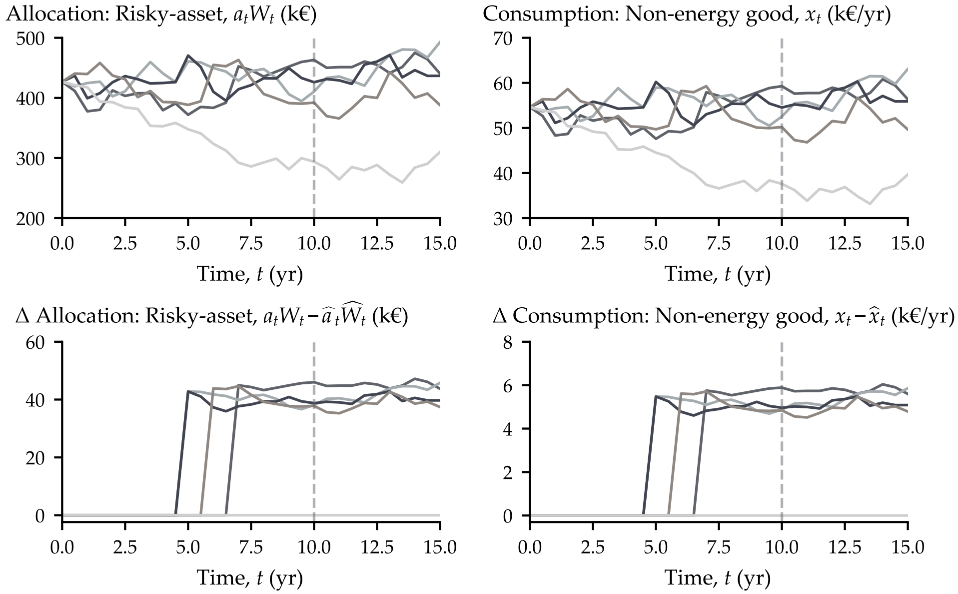

FIGURE 4.2. *Top row: optimal strategies from Corollary 4.2 (before investment) and Proposition 4.1 (after investment) for five exemplary trajectories, with the dashed vertical line at $t = H$ indicating the decision horizon. Bottom row: difference between these optimal strategies and the no-investment counterfactual (Corollary 4.2). Investment takes place in four of the five trajectories, as evidenced by the demand jumps in the bottom row. The low wealth trajectory, clearly visible in the upper panels, avoids investment.*

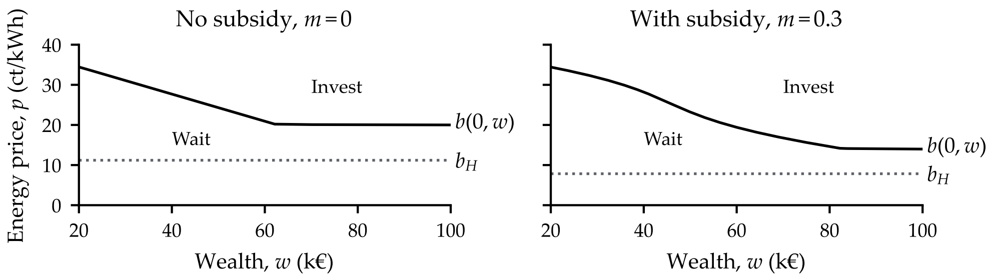

FIGURE 4.3. *Illustration of the effect of a* 30 % *subsidy on the decision boundary at $t = 0$ (right panel), as compared to the no-subsidy baseline (left panel).*

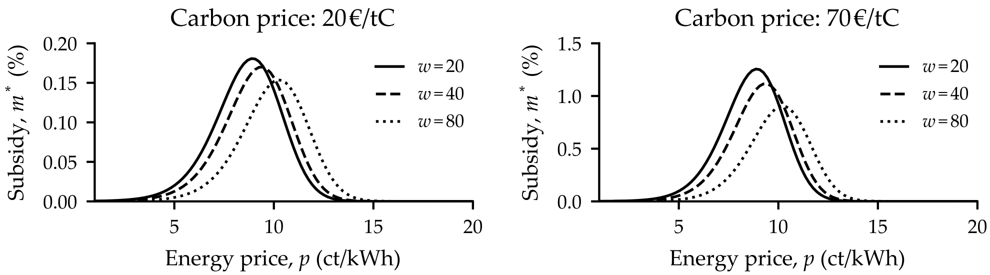

FIGURE 4.4. *Optimal subsidy policy $m^*(p, w)$ from* (4.6) *for two carbon price levels and three exemplary wealth levels.*

"too deep" in the "wait" region, no subsidy is offered, since the marginal effect on the distribution of investment times does not justify the cost of the policy. Comparing both panels of Figure 4.4, we see that as the carbon price increases, the optimal subsidy rises, reflecting the greater social value of low-carbon investment. Across wealth levels, the subsidy exhibits a nonlinear pattern: at low energy prices, poorer households require higher support because their limited wealth delays retrofit adoption and reduces the immediate social benefit. At higher energy prices, however, wealthier households may receive larger subsidies. This is due to the convex cost function: since the social planner optimises the subsidy for each household separately, and wealthier households are able to implement the retrofit sooner, the marginal impact of an additional unit of subsidy on adoption timing is larger for these households at higher prices. As a result, the social benefits of their investment are realised more quickly and fully, so it becomes optimal to assign them higher subsidies. The policy is therefore only partially progressive: it generally favours lower-wealth households at prices at our below the level b_H, but does not maintain strict progressivity across all price levels.

4.4.3 Comparative statics

This section performs a comparative statics analysis of the model, focusing on the parameters which typically drive non-linear change in consumption-investment setups, namely, risk aversion and market volatility. To this end, Figure 4.5 depicts the change in the investment boundary at time $t = 0$ due to ±40 % changes in the risk-aversion parameter γ and volatility parameter σ_S relative to their baseline values in Table 4.1. As expected, reductions in either risk aversion or market volatility strengthen the incentive to delay, shifting the investment threshold to higher energy prices. Conversely, greater risk aversion and heightened volatility erode the value of waiting, making investment

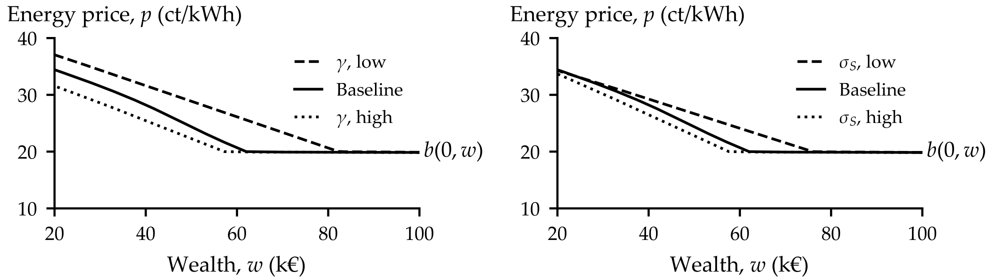

FIGURE 4.5. *Change in the decision boundary $b(0, w)$ for ±40 % changes in γ and σ_S relative to their baseline values in Table 4.1.*

optimal at lower prices. The consequence of this latter behaviour for the optimal subsidy policy is explored in Figure 4.6, which examines the effects of a 40 % increase in σ_S on the subsidy rate m^*. A sharp increase relative to the baseline in Figure 4.4 is observed, with the same partially-progressive behaviour in place. The mechanism is as follows: since higher market volatility induces earlier investment by households, private incentives to adopt are already strengthened. From the social planner's perspective, this raises the marginal effectiveness of subsidies: additional support builds on an existing willingness to invest, accelerating adoption and bringing forward emissions reductions. Accordingly, subsidies increase sharply, as they help translate heightened uncertainty into faster realisation of social benefits. This aspect is consistent with the comparative statics analysis in Chapter 3.4.4; by contrast, the policy in that case was strictly progressive due to demand flexibility, which constitutes a qualitative difference between the two setups.

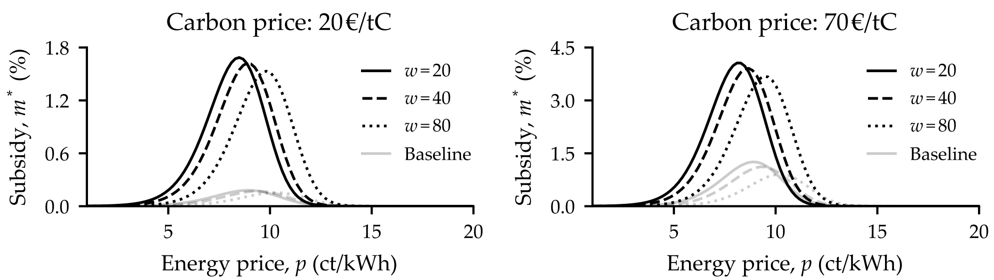

FIGURE 4.6. *Change in the optimal subsidy policy due to a 40 % increase in the market-volatility parameter σ_S relative to its baseline value in Table 4.1. The baseline scenarios (in light gray) are from Figure 4.4.*

4.5 Outlook

This chapter presented an extension of the model developed in Chapter 3 to incorporate dynamic energy prices. Since energy demand was assumed fixed, some of the unrealistic sensitivities encountered in the previous case were avoided. Analytical results were derived, and an approximation scheme proposed to facilitate numerical implementation. It was shown that the agent's decision boundary depended on energy prices and wealth in an economically plausible manner, with investment becoming attractive at relatively lower energy prices for wealthier households. The chapter also examined optimal subsidy design, showing that support was sharply targeted around the terminal decision boundary, with the explicit objective of nudging marginal agents into the investment region. The resulting policy was progressive in the main, with larger subsidies allocated at lower wealth levels; however, this progressivity diminished at higher energy prices, where wealthier households could receive larger subsidies due to the faster realisation of social benefits from earlier retrofit adoption.

Similarly to the basic model, the outlook for this research involves relaxing some of the model's more idealised assumptions and incorporating additional state variables. Two avenues are of particular interest. The first is the relaxation of the finite-horizon assumption, which was introduced to facilitate the transition to the LSMC method. This assumption had the effect of dampening the value of waiting for the agent, making the net-present value trigger b_H particularly consequential for both the agent and the social planner. A full analysis in an infinite-horizon setting is therefore valuable, as it would capture the true option value of waiting. Secondly, particular interest lies in the inclusion of uncertainty in energy prices, although it is likely that the solution in this case will need to be fully approximated numerically. The following chapter discusses these numerical methods, alongside more general considerations associated with extending the decision framework.

5 Further extensions to the basic model

This chapter surveys possible extensions to the modelling framework. It first reviews alternative preference specifications and assesses their suitability for dynamic consumption–investment problems with stochastic energy prices, highlighting conditions under which two-good utility functions remain well posed and economically interpretable. It then outlines how a bequest motive can be incorporated through a random-horizon formulation that preserves tractability while introducing life-cycle elements. The chapter concludes with a brief discussion of further technical and numerical extensions, providing a concise outlook on how the framework may be enriched in future work.

5.1 Introduction

This chapter outlines several directions in which the models of Chapters 3 and 4 may be extended. While those models provide tractable environments for analysing household energy consumption and retrofit behaviour, they rely on simplifying assumptions regarding preferences, investment flexibility, and energy-price dynamics, and they abstract from life-cycle considerations. The purpose here is therefore to review how the framework may be enriched without abandoning its core structure.

The discussion begins with agent preferences, focusing on conditions under which two-good utility functions generate well-posed dynamic programmes and economically meaningful behaviour when energy services are treated explicitly. Particular attention is given to utility functions that permit subsistence or saturation levels of energy services, and to the challenge of retaining tractability when energy prices are stochastic. The chapter then introduces a bequest motive through a random-horizon formulation that preserves stationarity. This provides a bridge to life-cycle modelling while avoiding the complexities associated with explicit human-capital dynamics. A subsequent section considers additional technical extensions, including flexible retrofit sizes and alternative investment dynamics, which broaden the empirical relevance of the framework.

The remainder of the chapter is structured as follows. Chapter 5.2.1 examines alternative preference specifications, and Chapter 5.2.2 introduces the bequest motive via a random stopping time. Chapter 5.3 outlines further technical extensions.

5.2 Agent preferences

5.2.1 The utility function

Both from a theoretical and practical perspective, the choice of agent preferences is fundamental to the specification of models developed within a consumption-investment framework. A key part of the research underlying this dissertation has therefore been to clarify which preference functions generate economically plausible behaviour when energy services and energy-efficiency investment enter the decision problem. These findings are presented here.

For concreteness, we specialise to the setup from Chapter 3, with the sole adjustment that the energy price is treated as a dynamic rather than fixed. Hence, retaining the notation from Chapter 3.2.1, we suppose that the wealth process follows

$$\begin{aligned} \mathrm{d}W_t = (a_t \mu_S W_t + (1-a_t)\mu_R W_t)\,\mathrm{d}t + a_t \sigma_S W_t\,\mathrm{d}B_t + (Y - x_t)\,\mathrm{d}t \\ - \Big(s_t P_t \mathbb{1}_{\{t<\tau\}} + (s_t P_t + \rho K)\mathbb{1}_{\{t\geq\tau\}} \Big)\,\mathrm{d}t\,, \end{aligned} \tag{5.1}$$

where the price of energy P_t follows a geometric Brownian motion,

$$\mathrm{d}P_t = \mu_P P_t\,\mathrm{d}t + \sigma_P P_t\,\mathrm{d}B_t^{(P)}\,, \tag{5.2}$$

with drift $\mu_P > 0$, volatility $\sigma_P > 0$, and $B_t^{(P)}$ a Brownian motion uncorrelated to the risky-asset innovations B_t. From (3.26), we write the agent's value function as

$$F(p,w) := \sup_{a,x,\tau} \mathbb{E}\left[\int_0^\tau \mathrm{e}^{-\widehat{\delta}t} U(x_t, s_t)\,\mathrm{d}t + \mathrm{e}^{-\widehat{\delta}\tau} G(P_\tau, W_\tau) \mid (P_0, W_0) = (p,w) \right], \tag{5.3}$$

where G is the value of immediate investment. Solving a concrete instance of model requires now a specification of the agent's preferences. This raises two issues. The first is general: which preference functions are mathematically admissible? The second is more specific: which preference functions are suitable for the task at hand? We take up these questions in turn.

The first question admits a definitive answer. The restriction arises from the requirement that that Legendre-Fenchel transform

$$\widetilde{U}(a,b) := \sup_{x,s>0} \left[U(x,s) - xa - sb \right]\,, \quad a,b > 0\,, \tag{5.4}$$

be well-defined and finite. This convex-analytic condition ensures that the associated Hamiltonian is proper and continuous, which is necessary to solve the dynamic programming equation associated to (5.3). We have the following result, which is closely related to the Inada conditions.

Theorem 5.1. *$\widetilde{U}(a,b)$ is well defined for all $a, b > 0$ if and only if*

$$\lim_{z\to\infty} \nabla U(z,z) = (0,0)\,, \tag{5.5}$$

where $\nabla U(z,z) = (\partial_x U(z,z), \partial_y U(z,z))$ is the gradient of U at (z,z).

The proof is presented in Appendix C.1. Economically, this condition on the preference function implies that as consumption and energy services become very large, the marginal utility of each must vanish, ensuring that the agent does not value additional consumption infinitely; in this way, they face a finite trade-off when allocating resources. Many classical utility functions satisfy (5.5) under certain restrictions. For instance, it is easy to check that the Cobb-Douglas utility function $u(x,s) = x^{\alpha}s^{\beta}$ satisfies the limit only if $\alpha + \beta < 1$. However, the case $\alpha + \beta = 1$ is more intuitive, and it is perhaps desirable to stay with this choice. A helpful strategy in this case is to compose with the isoelastic utility function, that is, to consider

$$U(x,s) = \frac{u(x,s)^{1-\gamma}}{1-\gamma}\,, \tag{5.6}$$

where u is the desired utility function and $\gamma > 1$ the risk-aversion parameter. A significant advantage of this approach is that is allows for a clean de-coupling of consumption preferences from risk-aversion. We have already encountered an example of this in the Stone-Geary function considered in Chapter 3. Further examples in the literature are found in Kraft & Munk (2011) and Cocco (2004), who use the above Cobb-Douglas specification, and Choi et al. (2008), who employ constant elasticity of substitution (CES) preferences. In this latter example, only implicit solutions are possible. This is a natural segue to the second question raised above, regarding the specific choice of preference function.

A central motivation here is the following. In decision problems with investment-timing flexibility and multiple state variable, such as (5.3), numerical methods are generally required to estimate the value function F and the associated optimal strategies (cf. Compernolle et al., 2021, Dammann & Ferrari, 2021). However, the computations simplify considerably if the terminal gain G in (5.3) can be expressed in closed form. Indeed, G solves an infinite-horizon pure optimal-control problem, which often admits closed-form solutions even with multiple state variables (cf. Kraft & Munk, 2011). Whether this is possible in a given setting depends crucially on the combination of the agent's preferences U and the dynamics of the state variables.

To our knowledge, no models in the literature obtain explicit solutions in a two-good setting without assuming Cobb–Douglas preferences and affine dynamics. Nevertheless, this pairing is unsuitable for our purposes. We present the economic mechanism here, deferring the formal argument to Appendix C. Essentially, Cobb–Douglas preferences imply unit price elasticity of demand, with the consequence that the marginal disutility of higher energy prices is constant. Thus, as P_t increases, the agent continuously reduces energy use, offsetting the utility loss by reallocating expenditure toward the non-energy good x. The energy-efficiency investment threshold therefore becomes a pure wealth effect, with investment occurring only when the marginal utility of wealth is sufficiently low.[33] The upshot is that the energy price therefore ceases to play a substantive role in the investment-timing decision.

This unsatisfactory mechanism ultimately motivated the Stone–Geary preferences assumed in Chapter 3. The key economic distinction is the presence of a subsistence level of energy services: near this lower bound, demand elasticities are non-unitary, so that higher energy prices generate a genuine incentive to invest in efficiency. As shown in Chapter 4, this restores the economically plausible channel through which energy prices influence investment behaviour. Nevertheless, because Stone–Geary preferences are non-homogeneous, the associated dynamic programming equation loses the scaling structure that delivers closed-form solutions under Cobb–Douglas preferences. As a result, the problem does not admit an explicit solution when P_t is stochastic, which is why Chapter 4 restricts attention to deterministic price paths.

A further shortcoming of Stone–Geary preferences, which we have encountered in Chapter 3, is that they permit unbounded consumption of the energy service. This complicates model calibration, as large energy-service consumption is difficult to justify empirically. For instance, for services such as thermal comfort, it is clear that an agent will not raise indoor temperatures without limit, regardless of wealth or energy prices. More generally, many energy services appear to have both a subsistence requirement and a saturation (or bliss) level beyond which marginal utility becomes zero or negative. A convenient specification that captures this behaviour is a quadratic utility function in the energy service s with lower bound $\underline{s}$. We write this as

$$u(s) := (s - \underline{s}) - \frac{1}{2\alpha}(s - \underline{s})^2 , \qquad s > \underline{s} , \tag{5.7}$$

[33] Compare the proof of Theorem 3.3.

where $\alpha > 0$ is a fixed parameter. This functional form attains a bliss level of utility at $\bar{s} := \underline{s} + \alpha$ and yields increasing utility over the subsistence range, while reaching a finite saturation point at $\bar{s}$. In doing so, it avoids the unrealistic unboundedness of the Stone–Geary specification. It is also particularly easy to parametrise: for instance, for the energy service thermal comfort, if we assume a subsistence level $\underline{s} = 15\,^{\circ}\mathrm{C}$ and a bliss level $\bar{s} = 23\,^{\circ}\mathrm{C}$, we have $\alpha = 7\,^{\circ}\mathrm{C}$. There exist models in the literature which solve consumption-investment problems assuming quadratic utility, albeit for a single good (Lim & Lee, 2018, Koo et al., 2015). Depending on the assumptions made for the dynamic quantities, explicit solutions are possible. We are not aware of literature which attempts to apply similar restrictions to problems involving two goods.

In sum, these observations highlight two promising directions for future research on agent preferences in consumption-investment models. First, preference functions should allow for non-unitary price elasticities, so that investment thresholds respond meaningfully to energy prices rather than being driven solely by wealth. Second, utility specifications should incorporate realistic subsistence and saturation levels for energy services, ensuring that consumption is bounded and marginal utility declines appropriately at high levels. Addressing these aspects would both improve the empirical plausibility of the models and enable more accurate analysis of the manner in which agent heterogeneity and stochastic prices influence consumption and adoption decisions, and consequently, social welfare. Whether such utility functions ultimately admit closed-form solutions under uncertain energy prices remains to be seen; some practical considerations for the use of numerical methods for this class of problems is presented in Britto & Oliveira (2024).

Finally, each of the decision models considered thus far assumed additive separability of utility with exponential discounting. An interesting avenue for extension is provided by Epstein–Zin preferences, which allow a separation of risk aversion from the elasticity of intertemporal substitution, thereby decoupling attitudes toward risk from pure time preferences. These preferences have proven useful in resolving empirical puzzles in macroeconomics and finance (Bansal & Yaron, 2004), and they have also been incorporated into consumption-investment frameworks (Kraft et al., 2016, Xing, 2016). In certain special cases, explicit solutions are available, suggesting that Epstein–Zin preferences could provide a tractable way to move beyond the restrictions of time-additive utility while capturing richer behavioural responses to uncertainty.

5.2.2 *The bequest motive*

We present now a technical extension to the model to highlight a method for introducing desirable aspects from finite-horizon consumption-investment models, such as reasonable levels of leverages against human capital, while retaining other advantageous properties of the infinite-horizon problems, most notably stationary optimal strategies, which are easier to study theoretically and empirically. The extension employs the approach of Merton (1971), which we have encountered in Chapter 3.2.1: by assuming a finite but random time horizon with an exponential distribution, the problem can be reduced to a standard infinite-horizon problem with an effectively higher discount rate.[34] This section discusses the application of this technique to include a bequest motive, thereby bridging the gap to life-cycle models. It turns out that the inclusion of a bequest motive encourages the agent to maintain a positive level of wealth, in contrast to the borrowing against human capital observed in Chapters 3 and 4.

Hence, given the wealth dynamic in (5.1), assume that the agent's decision problem is to select an allocation strategy, consumption strategies, and an investment time τ to maximise the present value of utility together with a bequest motive at a random horizon $T > 0$. The value function is hence given by

$$F(p,w) := \sup_{a,x,s,\tau} \mathbb{E}\left[\int_0^T \mathrm{e}^{-\delta t} U(x_t, s_t)\,\mathrm{d}t + \mathrm{e}^{-\delta T} V(W_T) \;\middle|\; (P_0, W_0) = (p,w) \right], \tag{5.8}$$

where $V\colon \mathbb{R} \to \mathbb{R}$ is the bequest motive. As in Chapter 3.2.1, suppose that the cumulative distribution function of T is exponential, namely $\mathcal{F}_T(t) = 1 - \mathrm{e}^{-\lambda t}$, where $\lambda > 0$ is the hazard rate. We have seen in (3.8) that this reduces the flow utility term to

$$\mathbb{E}\left[\int_0^T \mathrm{e}^{-\delta t} U(x_t, s_t)\,\mathrm{d}t \right] = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-(\delta+\lambda)t} U(x_t, s_t)\,\mathrm{d}t \right]. \tag{5.9}$$

On the other hand, the bequest motive now follows (cf. Kremer et al., 2013)

$$\begin{aligned} \mathbb{E}\left[\mathrm{e}^{-\delta T} V(W_T) \right] &= \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\delta t} V(W_t) \mathbb{1}_{\{T \in \mathrm{d}t\}} \right] && (5.10) \\ &= \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\delta t} V(W_t) \mathcal{F}_T'(t)\,\mathrm{d}t \right] && (5.11) \\ &= \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-(\delta+\lambda)t} \lambda V(W_t)\,\mathrm{d}t \right]. && (5.12) \end{aligned}$$

[34] The extension to non-exponential hazard distributions is straightforward; see Kremer et al. (2013). See also Rogers (2013, Ch. 2.11) for a variant involving non-exponential discounting.

Therefore, defining the preference function $u(x,s,w) := U(x,s) + \lambda V(w)$ and following arguments similar to (3.26), we see that (5.8) rewrites as the infinite-horizon decision problem

$$F(p,w) = \sup_{a,x,s,\tau} \mathbb{E}\left[\int_0^\tau \mathrm{e}^{-\widehat{\delta}t} u(x_t, s_t, W_t)\,\mathrm{d}t + \mathrm{e}^{-\widehat{\delta}\tau} G(P_\tau, W_\tau) \,\middle|\, (P_0, W_0) = (p,w) \right], \quad (5.13)$$

where $\widehat{\delta} = \delta + \lambda$ is the effective discount rate as before, and G is the value of immediate investment (with preference function u). Hence, in contrast to (5.3), the agent derives utility here from consumption *and* wealth, which should result in qualitatively different behaviour. For instance, Rogers (2013, Ch. 2.13) considers a Merton model with Cobb-Douglas preferences in consumption and wealth and shows that consumption is indeed reduced relative to the standard Merton benchmark. In sum, we view this extension as a promising way to enhance the realism of the baseline model while preserving its desirable stationary behaviour, and to explore how bequest motives can materially influence optimal consumption–investment decisions.

5.3 *Other technical extensions*

We conclude this chapter with a list of eight relevant extensions to the basic model beyond agent preferences. The aim is not to be exhaustive, but to illustrate how various aspects of energy-efficiency adoption can be incorporated into the proposed framework.

(i) A life-cycle model of energy-efficiency adoption. A life-cycle consumption-investment model is obtained from the basic model by relaxing the assumption of an infinite horizon. One considers instead a fixed finite horizon $T > 0$ and restricts the investment time τ to the stopping times which lie in $[0, T]$. In contrast to (5.3), where the "never invest" option is automatically included via the $\tau = \infty$ limit, care must be taken here to include an explicit comparison between investing and never investing before the horizon T is attained; we consider it natural in this setting to include a decision horizon $H < T$ to improve model realism (cf. Chapter 4). As noted, although the finite-horizon decision model is more realistic than its infinite-horizon counterpart, the outputs of the model are more challenging to work with due to the explicit time dependence of the optimal strategies.

(ii) Flexible investment size. This extension concerns flexibility that the agent might have in deciding the amount to invest in a retrofit, with larger investments leading to larger

improvements in efficiency (cf. Appendix B.3). For instance, one might assume that the improved efficiency parameter $\widetilde{\eta}$ depends on the retrofit cost K as

$$\widetilde{\eta}(K) = \eta + (\eta_{\max} - \eta)\frac{K}{\phi + K}\,, \tag{5.14}$$

where $\eta_{\max} > \eta$ and $\phi > 0$ are parameters that can be tuned to fit the particulars of the agent's dwelling. The function $\widetilde{\eta}(K)$ satisfies diminishing marginal utility of investment with maximum efficiency being attained only in the limit of very large K. The resulting interaction between wealth, investment timing, and investment size offers new possibilities for the design of subsidy schemes, as well as comparison with empirical observations.

(iii) Sequential efficiency improvements. A related extension concerns sequential investment in energy efficiency. As has been noted in the literature, it is often beneficial for households to focus on investments which deliver the greatest marginal improvement in efficiency; hence, low-cost yet highly-impactful upgrades such as attic insulation should be prioritised over more expensive retrofit measures (Galvin, 2010, Jakob, 2006). With this is mind, the agent's decision problem could be extended so that they have the option to invest in energy efficiency improvements in stages, with cheaper upgrades being given priority. This ties into the literature on sequential real options in energy (Penizzotto et al., 2024, Ma et al., 2022), which delivers insights into optimal decision-making in multi-stage projects.

(iv) Investment financing. The models considered in Chapters 3 and 4 each assumed that the retrofit was financed by an interest-only loan. This stylised assumption might be relaxed to consider loans which are repaid over a fixed duration.[35] On the other hand, many energy-efficiency investments might be conceivably modelled by a single lump-sum payment. We expect this change to result in minor deviations from the optimal investment-timing and consumption strategies in Chapters 3 and 4 since the relevant state variable for these strategies is disposable capital; however, a more pronounced difference in the allocation strategy is expected due to the step-change in wealth due to the investment. A related and empirically important extension is the inclusion of credit constraints, which have been shown empirically to affect energy-efficiency adoption (Berkouwer & Dean, 2022); such constraints can be modelled directly in a consumption–investment framework (cf. Vila & Zariphopoulou, 1997).

(v) Uncertain retrofit costs or subsidies. Generalising from the assumption of a constant retrofit cost K can be relevant for many applications. For instance, Galvin (2024) notes

[35]It is worth noting that this would induce an explicit time dependence in the model, breaking stationarity.

that retrofit costs in Germany increased 43 % between 2020 and 2024, and continue to trend upwards. Introducing dynamics in the retrofit costs could hence be a worthwhile model extension (cf. Hassett & Metcalf, 1992). A related issue concerns uncertainty in subsidies, which can have a large impact on optimal strategies; yet programs are often updated or scrapped (Rysanek & Choudhary, 2013). Extending the model to include subsidy-retraction risk would bring one in contact with an established literature on optimal investment under subsidy uncertainty (cf. Hagspiel et al., 2021, Oliveira & Perkowski, 2020).

(vi) Energy-carrier switch. An energy retrofit often involves switching energy carriers, for instance from an oil to gas boiler, or from a gas boiler to an electric heat pump.[36] In this case, the energy price in the wealth dynamic $\widetilde{W}_t$ must be modified so that the price of fuel for the new technology, say $\widetilde{P}_t$, is not identical to the price of fuel for the old technology, P_t. It follows that both P_t and $\widetilde{P}_t$ enter the optimal strategies as state variables; a relevant example is found in Britto et al. (2024). This extension is of particular relevance for studying adoption of heat pumps, a key technology for the energy transition (Hainsch et al., 2021). However, it is important to note that heat pumps follow different physics than conventional heating systems, which substantially complicates the analysis; see Appendix A.

(vii) Multiple energy carriers. Another extension comes from Chan & Gillingham (2015), who note that energy-services can often involve multiple related services and fuels. For instance, the energy service "thermal comfort", denoted s^h, is closely related to the service "warm water", denoted s^w. Suppose the agent can obtain either service from an existing gas boiler or from plug-in electric alternatives. The relationship between energy service and fuel consumption is then given by

$$s^h = \eta^{gh} c^{gh} + \eta^{eh} c^{eh} \,, \quad s^w = \eta^{gw} c^{gw} + \eta^{ew} c^{ew} \,, \tag{5.15}$$

where η and c are efficiency and fuel consumption as before, but now carrying superscripts to denote the relevant consumption-service channel, for instance, "gh" denotes "gas consumption for heating". Total expenditure on energy at time t is thus

$$P_t(c_t^{gh} + c_t^{gw}) + \widetilde{P}_t(c_t^{eh} + c_t^{ew}) \,, \tag{5.16}$$

and the wealth dynamic must be modified to take this into account. This framework, though much more complicated than the basic model, would yield valuable insights into

[36] A technology-adoption decision involving an energy-carrier switch is not necessarily an "energy-efficiency" decision, since the new technology may not improve primary-energy or useful-energy efficiency. We sidestep these subtleties here.

real-world technology choice and substitution, including hybrid heating technologies (Bennett et al., 2021, Heinen et al., 2016). As noted above, if a heat pump is involved, the problem complicates considerably due to the underlying physics.

(viii) Additional state variables. Finally, it may be of interest to model labour income or the risk-free rate as stochastic, or include additional risky assets. Although it is mathematically straightforward to incorporate these quantities into the wealth dynamic, each additional dynamic variable greatly increases the complexity and interpretability of the model. Numerical solutions also become difficult to obtain due to the curse of dimensionality (Bellman, 2010).

6 Discussion

This closing chapter synthesises the dissertation's main contributions and draws out their implications for energy-efficiency policy. It highlights the limitations of the modelling approach, clarifying how structural simplifications, computational constraints, and the partial-equilibrium setting shape the interpretation of the results. The chapter concludes with an outlook that identifies promising extensions, including richer preference specifications, improved grounding for baseline adoption rates, and the integration of general-equilibrium feedbacks into the consumption-investment framework.

6.1 Conclusions & policy implications

This dissertation has shown that dynamic models of optimal consumption and investment under uncertainty offer a coherent and flexible framework for analysing energy use and technology adoption. By situating investment decisions within a stochastic setting, such models accommodate interactions between economic conditions, household behaviour, and technology choice that static approaches typically abstract from. This perspective provides a structured method to examine how households weigh consumption, saving, and adoption decisions when future outcomes are uncertain.

It has been demonstrated how rebound and backfire effects arise endogenously as fully dynamic quantities: because energy demand, investment timing, and portfolio choices evolve jointly, rebound and backfire become path-dependent outcomes of the choice problem itself. This feature permits a rigorous and internally consistent definition of total welfare change, which is derived directly from the household's optimizing behaviour over time rather than inferred from heuristics or engineering estimates. The analysis further shows that welfare outcomes depend critically on agent heterogeneity. Differences in preferences, income, wealth, and the attributes of the energy-efficiency project generate pronounced variation in both investment incentives and welfare outcomes; this variation is a significant implication of the framework.

These findings have several consequences for policy design. First, because investment incentives and welfare effects depend on agent characteristics, subsidy schemes should be tailored to account for this heterogeneity. Second, because investment thresholds are sensitive to the external environment, particularly energy prices and macroeconomic conditions, subsidy programmes require routine recalibration as market conditions evolve. Regular adjustment has the potential not only to improve allocative efficiency but also to reduce free-riding, which often emerges when subsidies fail to track the underlying economic incentives. Third, although the models considered in this dissertation are stylised, the implied optimal subsidy rates are modest relative to those observed in practice (cf. KfW, 2025). Theoretically, high subsidy levels are typically rationalised by either very high implicit carbon prices faced by the social planner or by substantial behavioural frictions and deviations from fully rational decision-making (cf. Allcott, 2016). Against this backdrop, the models developed in this dissertation provide a novel and disciplined framework for reassessing the efficiency and justification of prevailing subsidy rates.

Finally, the case studies and comparative statics demonstrate that realistic macro-level patterns emerge from micro-level optimisation, even without imposing behavioural frictions. Non-linear aggregation effects play a central role. For example, it was shown that an increase in financial-market volatility could lead to a reduction in aggregate adoption even though the investment becomes more attractive for individual households, since higher volatility changes the distribution of triggering conditions across heterogeneous agents. Targeted subsidy policies can counteract, or in some cases reverse, these non-linearities by focusing support where marginal investment incentives are most sensitive to external conditions. This structure also provides a disciplined benchmark for evaluating energy-efficiency adoption. Because the underlying decision model is normative, it can be used to define a precise counterfactual against which observed adoption can be compared, thereby quantifying the energy-efficiency gap. Constructing such a benchmark requires careful specification of technologies, adopter characteristics, and preferences, but once established it provides a transparent measure of the extent to which observed patterns deviate from optimising behaviour.

Overall, the results suggest that the consumption-investment framework offers a coherent foundation for analysing energy use, technology adoption, and welfare change within a single intertemporal structure. By embedding heterogeneity, uncertainty, and investment timing in a unified framework, the approach yields internally consistent measures of rebound, welfare, and efficient adoption paths, as well as a disciplined benchmark against which observed behaviour can be assessed. This combination of normative clarity and empirical applicability indicates that the proposed framework can play a central role in advancing the economics of energy efficiency.

6.2 Limitations

The framework developed in this dissertation offers a transparent account of household adoption behaviour under uncertainty, but it does so by abstracting from several important dimensions of economic reality. These abstractions are standard in the literature and facilitate a focused examination of the core mechanisms, yet they also limit the scope and interpretability of the results. The principal limitations fall into three broad categories.

First, the modelling framework relies on strong structural simplifications. Markets are assumed to be complete and frictionless, stochastic processes are typically Gaussian and Markovian, and preferences and discounting follow tractable functional forms. These assumptions deliver analytical clarity and align the model with a well-understood class of optimal control problems, but they restrict its realism. Key behavioural features such as risk aversion, discount rates, and preference-separability across goods, are difficult to discern empirically, and the stochastic structure of financial markets, energy prices, or technological performance is only loosely approximated by commonly used diffusions. By the same token, the underlying physics and engineering aspects are treated in a somewhat stylised manner, even though a high degree of empirical precision is possible (cf. Appendix A). As a result, the quantitative implications should be interpreted as illustrative rather than fully structural.

A second limitation concerns tractability. Even under the highly stylised assumptions just described, the underlying optimisation problems are demanding; closed-form solutions are rare, and numerical techniques must approximate high-dimensional state spaces, non-linear controls, and stopping boundaries. Introducing more realistic features, such as more descriptive preferences and other behavioural effects, or non-Gaussian dynamics, quickly renders the problem computationally infeasible. This creates a tension between realism and feasibility: many extensions that would improve descriptive accuracy cannot be incorporated without losing the ability to solve the model.

Finally, the analysis is carried out in partial equilibrium, with most macroeconomic conditions held fixed. This isolates the mechanics of optimal consumption and adoption but omits important feedback channels. In reality, widespread increases in energy consumption or in the uptake of energy-efficient technologies affect energy prices, investment incentives for energy-producing firms, and the returns to capital and labour. Absent these interactions, a partial-equilibrium framework cannot speak to economy-wide welfare, or the design of optimal Pigouvian taxes. The resulting insights should therefore be viewed as conditional on an externally specified economic environment rather than as general-equilibrium statements.

6.3 Outlook

We identify several avenues for research in this domain. First, much work can be done to improve upon theoretical models that specify utility functions with energy services, with some indications given in Chapter 5.2. Several empirical avenues could inform and validate these models, namely, studies on the rebound effect and energy-price elasticity (Hunt & Ryan, 2015, Zarnikau, 2003); micro-datasets linking household characteristics, energy demand, and energy-efficiency adoption; and field experiments or pilot studies designed to elicit time-discounting and risk-preferences in the energy context (Bakaloglou & Belaid, 2022, Schleich et al., 2019).

We also see a gap regarding the use of micro-founded models to generate baselines for the study of the energy-efficiency gap. This dissertation focused on energy retrofits, but the proposed framework is general, requiring four steps. The first is to specify the particular energy-efficiency technology. Indeed, although LED lightbulbs, more fuel-efficient vehicles, and energetic retrofits are all examples of energy-efficiency technologies, they are rarely interchangeable within the class of decision models considered above. Restricting the scope at the outset to a particular technology and carefully considering its distinct characteristics is essential for specifying a meaningful model and evaluating its results. The second step identifies and selects potential adopters. Here, it is key to distinguish between heterogeneity of degree (e.g. risk aversion) and kind (e.g. landlords vs. tenants) since only in the latter case is a different model specification called for. Depending on the desired level of precision, it may be necessary to drill down to a "sub-technology" (e.g. the purchase of an efficient gas boiler as a specific example of an energy-efficiency retrofit) to clearly identify potential adopters and specify key model characteristics such as demand flexibility, investment flexibility, or financing needs. Once a technology has been selected and potential adopters identified, the third step is the specification of preferences; beyond the research gap indicated above, it is important to note that preferences may differ entirely across consumer groups. For example, in retrofit decisions, builders prioritize profit maximization, landlords focus on rent-driven wealth gains, and homeowners value consumption utility, so that both preferences and the optimisation objective are distinct across groups. Finally, the wealth dynamic must be specified. The inclusion of the energy price in the wealth dynamic is natural in this setting, but beyond this, several further modelling choices must be made: which consumption streams to include in the wealth dynamic, whether the agent allocates wealth to risky assets, and whether labour income is modelled explicitly, along with how retirement is treated if so. The nature of the energy-efficiency investment must also be specified, including whether it is fixed or flexible, and whether it is paid as a lump sum or financed over time. Each dynamic variable must also follow some specified, possibly stochastic, process. Once the decision model is operational, it can be used to study aggregate quantities, as we have seen in Chapter 3. With regard to using these models to generate baseline rates of energy efficiency, assumptions regarding the joint distribution of household characteristics and technologies can be informed by large-scale survey data, the availability of which is increasing (Belz et al., 2025, Frondel et al., 2025).

As for extensions beyond partial equilibrium, a natural direction concerns the evaluation of optimal corrective taxation for emissions. A tractable next step is to examine how an energy tax, coupled with a revenue-recycling scheme, affects energy consumption and technology adoption when households differ in wealth and thus in their sensitivity to operating costs. Because adoption in the present model is explicitly wealth-dependent, a uniform tax raises operating expenses unevenly across agents, potentially suppressing uptake among liquidity-constrained households even when the tax is welfare-improving on environmental grounds. Recycling revenues as lump-sum transfers or targeted subsidies can counteract this channel, but the distributional and efficiency implications depend on how transfers are allocated. A focused analysis of this tax–transfer mechanism would permit a clean quantification of how redistribution interacts with adoption dynamics and whether appropriately designed recycling can preserve the corrective intent of the tax while mitigating adverse effects on diffusion. Further general-equilibrium considerations of interest include how the tax–transfer mechanism interacts with endogenous wages, savings behaviour, and equilibrium energy prices, as well as potential feedbacks through capital accumulation or sectoral energy supply, thereby shaping both the aggregate diffusion path and the welfare distribution across heterogeneous households (cf. Pommeret & Schubert, 2009).

Lastly, computational methods for stochastic optimal control and stopping problems is an area of active research, with many recent promising advances. In particular, the advent of deep learning has brought with it new possibilities for solving consumption-investment models by approximating solutions to partial differential equations and free-boundary problems directly (Wang & Perdikaris, 2021, Wang & Zhang, 2020, Han et al., 2018), by using backward stochastic differential equation techniques (Gao et al., 2023, Han & Long, 2020), or by learning the optimal strategies outright (Li et al., 2024, Becker et al., 2021). The possibilities for tackling complex consumption–investment problems in multiple dimensions are therefore expanding, and with them the potential to deepen our economic understanding of optimal consumer adoption of energy efficiency. The time is ripe for fresh perspectives on an old debate.

A The physics of thermal comfort

This appendix examines the physical link between energy consumption and the energy service "thermal comfort". Here, "thermal comfort" is assumed synonymous with indoor air temperature. However, in reality thermal comfort has multiple factors, one of which is air temperature; other factors include the temperature of the radiative surfaces visible to a person's body (e.g. walls, floor, furniture, etc.), the presence of air drafts, the stratification of temperature within a room and between rooms, and the level of activity and amount of clothing worn by the occupant. We abstract from these considerations.

As described in Chapter 2, an economic agent consumes and pays for an amount of energy c while obtaining utility from a derived energy service s; hence c enters the wealth dynamic, whereas s enters the utility function. The most basic model for fuel-to-service conversion, which is also the standard model in the economics literature, assumes that $s = \eta c$, with $\eta > 0$ a conversion parameter (cf. Chan & Gillingham, 2015, Sorrell & Dimitropoulos, 2008). Although this identity makes intuitive sense, it is worth revisiting the underlying physics in order to make any simplifying assumptions explicit. We show that although the linear identity $s = \eta c$ is broadly correct, it makes strong assumptions on outdoor temperature. Moreover, heat pumps, which are a key technology for the energy transition, deviate sharply from the linear law. Finally, we discuss how to extend the model to include the effects of outdoor temperature, with important applications for demand flexibility.

Firstly, in the case of a conventional central heating system reliant on an oil or gas boiler, fuel is combusted in the boiler at efficiency $\eta_{\mathrm{b}} \in (0, 1)$ to produce hot water, which is then pumped and distributed through the dwelling at efficiency $\eta_{\mathrm{d}} \in (0, 1)$. If $c(t)$ denotes the rate of fuel consumption at time t (unit: W), then the delivered heating power $Q(t) = \eta_{\mathrm{b}}\eta_{\mathrm{d}}c(t)$. The link between delivered heat and indoor comfort follows from the heat balance equation (Rez, 2017, Ch. 2.1),

$$Q(t) = H(T_{\mathrm{in}}(t) - T_{\mathrm{out}}(t)) , \quad T_{\mathrm{in}}(t) > T_{\mathrm{out}}(t) , \tag{A.1}$$

where H is the thermal conductance of the dwelling envelope (W °C^{-1}), and $T_{\text{in}}(t)$ and $T_{\text{out}}(t)$ denote the indoor and outdoor temperatures respectively (°C).[37] Hence, associating the energy service "thermal comfort" with indoor temperature, $s := T_{\text{in}}$, under steady operating conditions we obtain the identity

$$s(t) = T_{\text{out}}(t) + \eta_{\text{b}}\eta_{\text{d}}c(t)/H\,. \tag{A.2}$$

Defining $\eta := \eta_{\text{b}}\eta_{\text{d}}/H$, we see that (A.2) reduces to $s = \eta c$ only if $T_{\text{out}}(t) \equiv 0\,°\text{C}$. Hence, the assumption of linear behaviour is reasonable, but it implicitly requires fixing outdoor temperature at a particular level and omits both seasonal patterns and stochastic fluctuations in temperature, all of which materially influence energy use. A simple improvement is to posit

$$s(t) = T_{\text{out}} + \eta c(t) \tag{A.3}$$

with constant outdoor temperature T_{out}, which accounts for the average winter temperature at the agent's location; in the context of the consumption-investment models developed in Chapters 3 and 4, the inclusion of a constant T_{out} would shift the agent's budget constraint, changing the decision problem slightly. Below, we consider a further extension which additionally accounts for seasonal and stochastic effects.

Aside from differences in terminology, the heat-balance relation (A.1) applies to all steady-state heating systems because it expresses an energy conservation law: the rate of useful heat delivery to the indoor air must equal the rate of heat loss through the building envelope. What differs across technologies is the *conversion pathway* by which primary energy input $c(t)$ is transformed into this useful heat flux $Q(t)$. As we have seen, in combustion-based systems such as oil or gas boilers and pellet stoves, the chemical energy of the fuel is converted directly into hot air or water at an overall efficiency $\eta_{\text{conv}} = \eta_{\text{b}}\eta_{\text{d}}$, so that $Q(t) = \eta_{\text{conv}}c(t)$. District-heating systems simply externalise this conversion: the building receives a pre-heated fluid whose enthalpy flow plays the same role as $Q(t)$. In electric resistance heaters, the conversion is effectively perfect, $\eta_{\text{conv}} \approx 1$, since electrical work is entirely dissipated as heat. Solar-thermal collectors and hybrid systems fit the same framework if one interprets $c(t)$ as the rate of absorbed solar energy rather than purchased fuel. In all cases, the indoor temperature dynamics remain governed by the same conductance relation (A.1); only the mapping between energy input and delivered heat changes through the technology-specific efficiency term.

[37]Here, H aggregates all of the thermally conducting surfaces of the dwelling, e.g. walls, roof, and windows. It is estimated relatively easily, (cf. Rez, 2017, Ch. 2.2). It is worth noting that in addition to the transmission losses in (A.1), the identity may be expanded to include additional terms such as ventilation losses, solar gains, etc.

On the other hand, heat pumps, which are instrumental for the electrification of the heating sector (Hainsch et al., 2021), follow a fundamentally different conversion pathway.[38] If $s = T_{\text{in}} > T_{\text{out}}$, the relation between delivered heat Q and instantaneous electricity consumption c is given by $Q = \text{COP}(s)c$, where $\text{COP}(s)$ is a non-linear function known as the *coefficient of performance*. We assume the model

$$\text{COP}(s) = \eta_{\text{hp}} \frac{s + \alpha}{s - T_{\text{out}}} , \tag{A.4}$$

where $\eta_{\text{hp}} \in (0, 1)$ is a device-specific "quality factor" measuring distance from the Carnot limit, and $\alpha = 273.15$ converts from Celsius to Kelvin. In practice, air-to-water heat pumps can be expected to have a quality factor η_{hp} of around 0.4, brine-to-water systems around 0.55, and groundwater systems around 0.5 (Bechem et al., 2015).[39] The following relations linking $c(t)$ and $s(t)$ are obtained:

$$c(t) = \frac{H(s(t) - T_{\text{out}}(t))^2}{\eta_{\text{hp}}(\alpha + s(t))} , \tag{A.5}$$

$$s(t) = T_{\text{out}}(t) + \frac{\eta_{\text{hp}} c(t) + \sqrt{(\eta_{\text{hp}} c(t))^2 + 4H(\alpha + T_{\text{out}}(t))\eta_{\text{hp}} c(t)}}{2H} . \tag{A.6}$$

Thus, in contrast to conventional heating systems, the relationship between electricity consumption and indoor comfort is intrinsically non-linear, with electricity consumption increasing sharply as the difference between indoor temperature rises. This difference should be taken into account when analysing the agent's optimal strategies in a consumption-investment model.[40]

Heat pumps can also operate in cooling mode and therefore serve as air conditioners, which broadens the scope of the model beyond space heating. In cooling operation the

[38] The following discussion assumes that the heat pump alone supplies the required heat flux. It is, however, possible to configure the dwelling so that only a portion of the heat load (e.g. 80 %) is provided by the heat pump, with the remainder supplied by a backup source such as resistance or fossil-fuel heating. This typically occurs at the coldest temperatures, when the heat pump's capacity and COP are reduced (Bennett et al., 2021, Heinen et al., 2016).

[39] Empirical estimations prefer to use polynomial approximations of $\text{COP}(s)$ over observed temperature ranges for increased accuracy, (cf. Ruhnau et al., 2019). However, as with all empirical models, care must be taken when extrapolating. By contrast, the simplified model (A.4) always obeys the laws of thermodynamics, although η_{hp} is here an idealised, operating-point dependent quality parameter.

[40] In particular, the calculation of the Legendre-Fenchel transform $\widetilde{U}$ from (5.4) becomes much more complicated. In the case of the linear relationship (A.3), the expression (5.4) reduces to $\widetilde{U}(a, b/\eta) - T_{\text{out}} b/\eta$, so that the convex dual is simply evaluated at the implicit price of energy b/η. The fixed offset $T_{\text{out}} b/\eta$ is then absorbed into the wealth drift in the HJB equation, cf. (B.3). This simplification is not obtained in the case of the heat-pump relation (A.5).

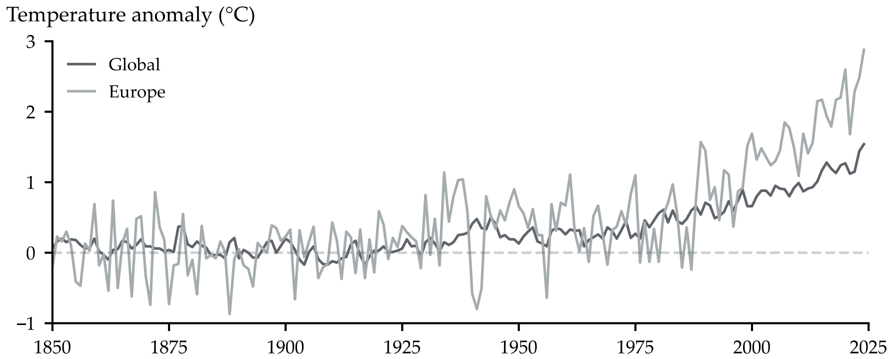

FIGURE A.1. *Global and European annual average near-surface temperature anomalies relative to the pre-industrial period 1850-1900. Own illustration based on data from Karl et al. (2015).*

device transfers heat from the indoor to the outdoor environment, and the same thermodynamic relations apply with the roles of hot and cold reservoirs reversed. Conventional air conditioners are, in essence, heat pumps configured exclusively for cooling, so their electricity demand can be represented within the same modelling framework. This symmetry implies that the analysis developed in this dissertation can be extended directly to future cooling demand, which is expected to grow in response to rising temperatures and increased adoption (Deroubaix et al., 2021).

Finally, we demonstrate how the above relations might be extended to include dynamics in the outdoor temperature. Particularly in the case of heat pumps, such an extension could deliver insight into demand flexibility as well as peak demand during extreme summer and winter temperatures (Ruhnau et al., 2019). To this end, consider the following model, which is standard in the literature on weather-derivative pricing (Benth & Šaltytė-Benth, 2005, Alaton et al., 2002):

$$T_{\text{out}}(t) = T_{\text{mean}} + T_{\text{seasonal}}(t) + X(t) \,, \tag{A.7}$$

$$T_{\text{seasonal}}(t) = A \sin\left(\frac{2\pi t}{365} + \phi\right) , \tag{A.8}$$

$$\mathrm{d}X(t) = (-\kappa X(t) + \mu)\,\mathrm{d}t + \sigma\,\mathrm{d}B(t) \,, \tag{A.9}$$

where T_{mean} is the reference average temperature at the agent's location, A and ϕ denote the amplitude and phase of the seasonal variation, and $X(t)$ represents a temperature anomaly. Here, $\kappa > 0$ is the mean-reversion speed, $\mu \in \mathbb{R}$ is a drift parameter, $\sigma > 0$ is a volatility parameter, and $B(t)$ is a standard Brownian motion.[41] Figure A.1 motivates this choice of stochastic process, showing why global and European temperature anomalies might be modelled as mean-reverting with drift. Finally, we note that if the consumption-investment model includes stochastic energy prices, one could introduce correlation between energy prices and weather, for which there is strong evidence (Mosquera-López et al., 2024, Su et al., 2023, Hulshof et al., 2016).

[41]The inclusion of a seasonal term introduces explicit time dependence in the agent's value function and the associated dynamic programming equation. Given that a deterministic seasonal component is of limited relevance for investment analysis, it can be omitted if technology adoption is the main modelling focus. However, it must be included if demand flexibility or peak demand is the focus.

B Supplementary material for Chapter 3

B.1 Mathematical proofs

Proof of Proposition 3.1. Notice that since τ^* is chosen optimally in (3.26), and since the choice set includes the counterfactual scenario $\tau = \infty$, the equivalent variation V_{ev} is necessarily non-negative. The claim follows since $V_{\text{sc}} \leq 0$ if $Q_t \leq 0$ for all t. $\square$

Proof of Proposition 3.2. The proof is standard. Firstly, the dynamic programming equation associated to (3.27) is

$$\widehat{\delta} G - \sup_{\widetilde{a},\widetilde{x},\widetilde{s}} \left[\mathcal{L}_w^{\widetilde{a},\widetilde{x},\widetilde{s}} G + U(\widetilde{x},\widetilde{s}) \right] = 0\,, \tag{B.1}$$

where

$$\mathcal{L}_w^{\widetilde{a},\widetilde{x},\widetilde{s}} G = (\widetilde{a}\kappa\sigma_S w + \mu_R w + \widetilde{Y} - \widetilde{x} - (\widetilde{s}/\widetilde{\eta})P)\partial_w G + \frac{1}{2}\widetilde{a}^2\sigma_S^2 w^2 \partial_w^2 G\,. \tag{B.2}$$

Assume $\partial_w G > 0$ and $\partial_w^2 G < 0$ and perform the optimisations over the controls to reduce (B.1) to the partial differential equation

$$\widehat{\delta} G - \left((\mu_R w + \widetilde{Y})\partial_w G - \frac{\kappa^2 (\partial_w G)^2}{2\sigma_S^2 \partial_w^2 G} + \widehat{U}(\partial_w G, (P/\widetilde{\eta})\partial_w G) \right) = 0\,, \tag{B.3}$$

where

$$\widetilde{a}^* = -\frac{\kappa}{\sigma_S}\frac{\partial_w G}{w \partial_w^2 G}\,, \tag{B.4}$$

and $\widehat{U}$ is the Legendre-Fenchel transform of U, defined as

$$\widehat{U}(\pi,\xi) = \sup_{b_0,b_1} \left[U(b_0,b_1) - b_0\pi - b_1\xi \right] \tag{B.5}$$

$$= -\underline{x}\pi - \underline{s}\xi + \frac{1}{\widehat{\gamma}} \left(\left(\frac{\pi}{1-\beta} \right)^{1-\beta} \left(\frac{\xi}{\beta} \right)^{\beta} \right)^{-\widehat{\gamma}} \tag{B.6}$$

with $\widehat{\gamma} := (1-\gamma)/\gamma$ and corresponding optimal controls

$$b_0(\pi,\xi) = \underline{x} + \left(\frac{\pi}{1-\beta}\right)^{-1} \left(\left(\frac{\pi}{1-\beta}\right)^{1-\beta} \left(\frac{\xi}{\beta}\right)^{\beta}\right)^{-\widehat{\gamma}}, \tag{B.7}$$

$$b_1(\pi,\xi) = \underline{s} + \left(\frac{\xi}{\beta}\right)^{-1} \left(\left(\frac{\pi}{1-\beta}\right)^{1-\beta} \left(\frac{\xi}{\beta}\right)^{\beta}\right)^{-\widehat{\gamma}}. \tag{B.8}$$

The second order condition guarantees that these controls are a local maximum if $0 < \beta < 1$ and $\gamma > 1$, which we have imposed.

At this juncture, given the homogeneity of the utility function, it is natural to guess a function of the form (3.31), substitute into (B.3), and solve for the constant of integration as

$$\Gamma = \left(\frac{(P/\widetilde{\eta})^{\beta}}{(1-\beta)^{1-\beta}\beta^{\beta}}\right)^{\widehat{\gamma}} \varphi > 0\,, \tag{B.9}$$

where

$$\varphi = \frac{\gamma\left(\kappa^2 + 2\widehat{\delta} - 2(\gamma-1)\mu_R\right) - \kappa^2}{2\gamma^2} > 0\,. \tag{B.10}$$

The controls (3.32) follow. That these are admissible can be seeing by noting that $\mathrm{d}\widetilde{Z}_t = \mathrm{d}\widetilde{W}_t$ and inserting the optimal controls to obtain the geometric Brownian motion (3.33), which is always positive for a positive initial condition. With $\mu_{\widetilde{Z}}$ and $\sigma_{\widetilde{Z}}$ denoting the drift of the process in (3.33), note that we have the identity

$$(1-\gamma)(\mu_{\widetilde{Z}} - \tfrac{1}{2}\gamma\sigma_{\widetilde{Z}}^2) = \widehat{\delta} - \varphi\,. \tag{B.11}$$

The transversality condition therefore computes as

$$\lim_{T\to\infty} \mathrm{e}^{-\widehat{\delta}T}\mathbb{E}\left[G(\widetilde{W}_T^w)\right] = \lim_{T\to\infty} \Gamma^{-\gamma}\frac{\widetilde{z}(w)^{1-\gamma}\mathrm{e}^{-\varphi T}}{1-\gamma} = 0\,. \tag{B.12}$$

Hence, the conditions for the verification theorem for this candidate solution (cf. Pham, 2009, Thm. 3.5.3) are satisfied. □

Proof of Theorem 3.3. The proof is in three steps. Firstly, the control-stopping problem is transformed into a pure stopping problem. In a following step, a characterisation of the stopping region is obtained, which yields the claimed stopping rule in Case (i). Finally, the decision problem is solved for Case (ii).

(a) From control-stopping to pure stopping. Firstly, note that at the moment of investment, we have the identity

$$\widetilde{Z}_\tau - Z_\tau = \theta\,. \tag{B.13}$$

It follows from Proposition 3.2 that the reward received upon stopping is given by

$$\Gamma^{-\gamma}\frac{(Z_\tau^{z(w)}+\theta)^{1-\gamma}}{1-\gamma}\,. \tag{B.14}$$

Hence, following a standard argument, associate the value function of (3.26) to the following ordinary differential equation in variational form:

$$\min\left[\widehat{\delta}F - \sup_{a,x,s}\left[\mathcal{L}_w^{a,x,s} + U(x,s)\right],\quad F(w) - \Gamma^{-\gamma}u(z(w)+\theta)\right] = 0\,, \tag{B.15}$$

where

$$u(x) := \frac{x^{1-\gamma}}{1-\gamma} \tag{B.16}$$

and

$$\mathcal{L}_w^{a,x,s}F = (a\kappa\sigma_S w + \mu_R w + Y - x - (s/\eta)P)\partial_w F + \frac{1}{2}a^2\sigma_S^2 w^2\partial_w^2 F \tag{B.17}$$

is the infinitesimal generator of wealth before investment. Now, as in the proof of Proposition 3.2, assume $\partial_w F > 0$ and $\partial_w^2 F < 0$ and perform the formal optimisations over the controls in (B.15). Then change variables by defining $f(z) := F(w(z))$ and $g(z) := \Gamma^{-\gamma}u(z+\theta)$ to transform (B.15) to

$$\min\left[\widehat{\delta}f - \left(\mu_R z\partial_z f - \frac{1}{2}\frac{\kappa^2(\partial_z f)^2}{\partial_z^2 f} + \widehat{U}(\partial_z f, (P/\eta)\partial_z f)\right),\quad f-g\right] = 0\,. \tag{B.18}$$

Define now the Legendre-Fenchel transform

$$\widehat{f}(\widehat{z}) = \sup_{z>0}\left[f(z) - z\widehat{z}\right]\,, \tag{B.19}$$

whence the identities

$$\partial_z f = \widehat{z}\,,\quad \partial_z^2 f = -(\partial_{\widehat{z}}^2\widehat{f})^{-1} \tag{B.20}$$

follow. Similarly define and compute

$$\widehat{g}(\widehat{z}) := \sup_{z>0}\left[g(z) - z\widehat{z}\right] = \Gamma^{-1}\widehat{u}(\widehat{z}) + \theta\widehat{z}\,, \tag{B.21}$$

where

$$\widehat{u}(\widehat{x}) = \sup_x \left[u(x) - x\widehat{x}\right] = \frac{\widehat{x}^{-\widehat{\gamma}}}{\widehat{\gamma}} \tag{B.22}$$

for $\widehat{\gamma} = (1-\gamma)/\gamma$. Insert these identities into (B.18) to obtain

$$\min\left[\widehat{\delta}\widehat{f} - \left((\widehat{\delta} - \mu_R)\widehat{z}\partial_{\widehat{z}}\widehat{f} + \frac{1}{2}\kappa^2\widehat{z}^2\partial^2_{\widehat{z}}\widehat{f} + \widehat{U}(\widehat{z}, (P/\eta)\widehat{z})\right), \quad \widehat{f} - \widehat{g}\right] = 0\,. \tag{B.23}$$

This is associated to the optimal stopping problem

$$\widehat{f}(\widehat{z}) = \sup_\tau \mathbb{E}\left[\int_0^\tau \mathrm{e}^{-\widehat{\delta}t}\widehat{U}(\widehat{Z}^{\widehat{z}}_t, (P/\eta)\widehat{Z}^{\widehat{z}}_t)\,\mathrm{d}t + \mathrm{e}^{-\widehat{\delta}\tau}\widehat{g}(\widehat{Z}_\tau)\right], \tag{B.24}$$

where the dual process $\widehat{Z}_t$ follows the geometric Brownian motion

$$\mathrm{d}\widehat{Z}_t = (\widehat{\delta} - \mu_R)\widehat{Z}_t\,\mathrm{d}t + \kappa\widehat{Z}_t\,\mathrm{d}B_t\,. \tag{B.25}$$

This decision problem is similar to the one studied by Rogers (2013, Ch. 2.16), whose method we follow in part (c) of the proof below.

(b) Characterisation of the stopping region, solution when $\theta \geq 0$. The continuation and stopping regions associated to (B.24) are given by

$$\mathcal{C} := \{\widehat{z} \in \mathbb{R}_+ \mid \widehat{f}(\widehat{z}) > \widehat{g}(\widehat{z})\}\,, \tag{B.26}$$

$$\mathcal{S} := \{\widehat{z} \in \mathbb{R}_+ \mid \widehat{f}(\widehat{z}) = \widehat{g}(\widehat{z})\}\,. \tag{B.27}$$

Further, with $\mathcal{L}_{\widehat{z}}$ the infinitesimal generator of the process $\widehat{Z}_t$, it is a standard result that

$$\mathcal{C} \supset \mathcal{D} := \{\widehat{z} \in \mathbb{R}_+ \mid \widehat{\delta}\widehat{g} - (\mathcal{L}_{\widehat{z}}\widehat{g} + \widehat{U}(\widehat{z}, (P/\eta)\widehat{z})) < 0\}\,, \tag{B.28}$$

$$\mathcal{S} \subset \mathcal{D}^c := \{\widehat{z} \in \mathbb{R}_+ \mid \widehat{\delta}\widehat{g} - (\mathcal{L}_{\widehat{z}}\widehat{g} + \widehat{U}(\widehat{z}, (P/\eta)\widehat{z})) \geq 0\}\,, \tag{B.29}$$

with $\tau^* = 0$ being optimal in the stopping problem if $\mathcal{D} = \emptyset$ (Øksendal & Sulem, 2019, Prop. 3.4). The condition defining the set $\mathcal{D}^c$ computes as

$$\gamma\varphi((\widehat{\eta}/\eta)^{-\beta\widehat{\gamma}} - 1)(z + \theta) + (\gamma - 1)\theta\mu_R \geq 0\,, \tag{B.30}$$

which is satisfied if $\theta \geq 0$. Hence, in this case $\mathcal{D} = \emptyset$ and $\tau^* = 0$ is optimal. It follows that $\widehat{f} = \widehat{g}$, which is true if and only if $F(w) = \Gamma^{-\gamma}u(z(w) + \theta)$. Thus, investing immediately is optimal, and Proposition 3.2 applies as claimed.

(c) Solution when $\theta < 0$. Suppose now that $\theta < 0$ so that we have to solve (B.24) in generality. This is in fact a standard optimal stopping problem in one dimension, albeit in dual space. Hence, the investment threshold is of the form $\widehat{z} \le \widehat{z}^*$, that is, the agent invests when the marginal utility of wealth is small enough. We thus conjecture a general solution to (B.23) of the form

$$\widehat{f}(\widehat{z}) = \begin{cases} \widehat{g}(\widehat{z})\,, & \widehat{z} \le \widehat{z}^*\,, \\ \Phi^{-1}\widehat{u}(\widehat{z}) + A_0(\widehat{z}/\widehat{z}^*)^{-a_0} + A_1(\widehat{z}/\widehat{z}^*)^{a_1}\,, & \widehat{z} \ge \widehat{z}^*\,, \end{cases} \tag{B.31}$$

where $\Phi > 0$ is identical to Γ of (B.9) with $\widetilde{\eta}$ replaced by η, A_0 and A_1 are constants of integration, and $-a_0 < 0 < 1 < a_1$ are the roots of the following quadratic equation in x:

$$\frac{1}{2}\kappa^2 x(x-1) + (\widehat{\delta} - \mu_R)x - \widehat{\delta} = 0\,. \tag{B.32}$$

The convexity of $\widehat{f}$ demands that we set A_1 to zero; it remains to solve for A_0 and $\widehat{z}^*$ from the smooth pasting conditions:

$$\widehat{f}(\widehat{z}^*) = \widehat{g}(\widehat{z}^*)\,, \quad \widehat{f}'(\widehat{z}^*) = \widehat{g}'(\widehat{z}^*)\,. \tag{B.33}$$

It follows that

$$A_0 = \left(- \frac{(a_0\gamma + \gamma - 1)(\Gamma - \Phi)^{\frac{\gamma}{\gamma-1}}((a_0+1)(\gamma-1)\Gamma\Phi)^{\frac{\gamma}{1-\gamma}}}{\theta} \right)^{\gamma-1}\,, \tag{B.34}$$

$$\widehat{z}^* = \left(- \frac{(a_0\gamma + \gamma - 1)(\Gamma - \Phi)}{(a_0+1)(\gamma-1)\Gamma\theta\Phi} \right)^{\gamma}\,, \tag{B.35}$$

completing the solution in dual space. The investment trigger in primal space follows by again invoking smooth pasting:

$$\widehat{z}^* = f'(z^*) = g'(z^*) = \Gamma^{-\gamma}(z^* + \theta)^{-\gamma}\,; \tag{B.36}$$

inserting the definitions of Γ and Φ gives the identity

$$z^* = \frac{1}{a_0\gamma + \gamma - 1}\left(\frac{(1+a_0)(\gamma-1)}{(\widetilde{\eta}/\eta)^{\beta\widehat{\gamma}} - 1} - a_0 \right)\theta =: \Lambda\theta\,, \tag{B.37}$$

where $\Lambda < 0$ so that $z^* > 0$. The claim follows. □

Proof of Proposition 3.6. Case (i). If $\theta > 0$, investment is immediate. It follows from the optimal controls in Proposition 3.2 and Corollary 3.4 that the rebound measure is given by the geometric Brownian motion

$$R_t = \frac{\beta\varphi}{P}\left(\widetilde{\eta}(z+\theta) - \eta z\right) \exp\left[(\mu_{\widetilde{Z}} - \tfrac{1}{2}\sigma^2_{\widetilde{Z}})t + \sigma_{\widetilde{Z}} B_t\right], \quad t \geq 0\,. \tag{B.38}$$

This quantity is strictly positive if $\theta > 0$. Hence $\mathbb{E}[R_t] > 0$ for all $t \geq 0$ as claimed. Similarly, the backfire measure is computed as

$$Q_t = (\widetilde{\eta}^{-1} - \eta^{-1})\underline{s} + \frac{\beta\varphi}{P}\theta \exp\left[(\mu_{\widetilde{Z}} - \tfrac{1}{2}\sigma^2_{\widetilde{Z}})t + \sigma_{\widetilde{Z}} B_t\right], \quad t \geq 0\,. \tag{B.39}$$

It follows that $Q_t \geq 0$ if

$$\vartheta^\theta_t \geq \frac{(\eta^{-1} - \widetilde{\eta}^{-1})\underline{s}P}{\beta\varphi} := \kappa\,, \tag{B.40}$$

where we denote

$$\vartheta^\theta_t := \theta \exp\left[(\mu_{\widetilde{Z}} - \tfrac{1}{2}\sigma^2_{\widetilde{Z}})t + \sigma_{\widetilde{Z}} B_t\right], \quad t \geq 0\,. \tag{B.41}$$

Since ϑ_t has a lognormal distribution at time t, the claim (3.49) follows.

Case (ii). If $\theta = 0$ exactly, from the expressions (B.38) and (B.39) above it is immediate that $\mathbb{E}[R_t] > 0$ and $\mathbb{E}[Q_t] < 0$.

Case (iii). If $\theta < 0$ and $z \geq z^*$, investment is again immediate, and the expression (B.38) for R_t holds. However, since $\theta < 0$, the requirement that $R_t > 0$ reduces to the requirement that the initial condition of the geometric Brownian motion in (B.38) be positive. The claim (3.50) follows. Similarly, since the expression (B.39) for Q_t holds in this case, It is immediate that $\mathbb{E}[Q_t] < 0$ if $\theta < 0$.

Case (iv). If $\theta < 0$ and $z < z^*$, waiting is optimal. So with $\widehat{\tau}^*$ the optimal investment time in Approximation 3.5, define the shifted Brownian motion

$$B^{\widehat{\tau}^*}_t := B_t - B_{\widehat{\tau}^*}\,, \quad t \geq \widehat{\tau}^*\,. \tag{B.42}$$

Using now the controls in Approximation 3.5, apply the definition of the rebound measure from (3.13) to obtain

$$R_t = \begin{cases} 0\,, & t < \widehat{\tau}^*\,, \\ \frac{\beta\varphi}{P}\left(\widetilde{\eta}(z^*+\theta) - \eta z^*\right) \exp\left[(\mu_{\widetilde{Z}} - \tfrac{1}{2}\sigma^2_{\widetilde{Z}})(t-\widehat{\tau}^*) + \sigma_{\widetilde{Z}} B^{\widehat{\tau}^*}_t\right], & t \geq \widehat{\tau}^*\,. \end{cases} \tag{B.43}$$

Similarly to Case (iii) and (3.50), demanding the non-negativity of R_t and inserting the identity $z^* = \Lambda\theta$ from (B.37) yields the claim. On the other hand, the backfire measure computes as

$$Q_t = \begin{cases} 0\,, & t < \widehat{\tau}^*\,, \\ (\widetilde{\eta}^{-1} - \eta^{-1})\underline{s} + \dfrac{\beta\varphi}{P}\vartheta_t^{\theta;\widehat{\tau}^*}\,, & t \geq \widehat{\tau}^*\,, \end{cases} \tag{B.44}$$

where $\vartheta_t^{\theta;\widehat{\tau}^*}$ is identical to (B.41) except with driver $B_t^{\widehat{\tau}^*}$ instead of B_t. As in Case (iii), the claim follows since $\theta < 0$. □

Proof of Lemma 3.8. We first define and compute the following integral using Fubini's theorem, utilising the fact that we have the product of independent geometric Brownian motions:

$$\mathcal{I}(\pi) := \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-\widehat{\epsilon}t}\varpi_t^\pi Q_t\,\mathrm{d}t\right] \tag{B.45}$$

$$= \left(\frac{(\widetilde{\eta}^{-1} - \eta^{-1})\underline{s}}{\widehat{\epsilon} - \mu_\varpi} + \frac{\beta\varphi\theta}{(\widehat{\epsilon} - \mu_\varpi - \mu_{\widetilde{Z}})P}\right)\pi\,, \tag{B.46}$$

where $\widehat{\epsilon} - \mu_\varpi - \mu_{\widetilde{Z}} > 0$ is required for convergence. Suppose now we are in Case (i). Then $\tau^*(z) = 0$, and it follows from (3.18) that $V_{\mathrm{sc}}(z;\pi) = \mathcal{I}(\pi)$.

For Case (ii), note firstly that due to Approximation 3.5, the integral in (3.18) in fact has the lower limit $t = \widehat{\tau}^*$ rather than $t = 0$, since $Q_t = 0$ for $t < \widehat{\tau}^*$ as shown in (B.44). Consequently, change the integration variable $t \mapsto t + \widehat{\tau}^*$ and apply the strong Markov property; it follows that

$$V_{\mathrm{sc}}(z;\pi) = \mathbb{E}\left[\mathrm{e}^{-\widehat{\epsilon}\widehat{\tau}^*(z)}\mathcal{I}(\varpi_{\widehat{\tau}^*(z)})\;\middle|\;(\widehat{Z}_0;\varpi_0) = (z;\pi)\right]\,. \tag{B.47}$$

Since $\mathcal{I}$ is linear in its argument and ϖ_t is a geometric Brownian motion, we have

$$\mathbb{E}\left[\mathrm{e}^{-\widehat{\epsilon}\widehat{\tau}^*(z)}\varpi_{\widehat{\tau}^*(z)}\right] = \pi\,\mathbb{E}\left[\mathrm{e}^{-(\widehat{\epsilon}-\mu_\varpi)\widehat{\tau}^*(z)}\right] =: \pi\,\mathscr{L}(z;\widehat{\epsilon} - \mu_\varpi)\,, \tag{B.48}$$

where $\mathscr{L}$ is the Laplace transform of the distribution of the investment time. This yields the claimed identity (3.53). Since $\widehat{Z}_t$ is a geometric Brownian motion, the Laplace transform is explicit (Jeanblanc et al., 2009, Ch. 3.3.2): for $z < z^*$ we have

$$\mathscr{L}(z;\varrho) = (z^*/z)^{a_0(\varrho)}\,, \tag{B.49}$$

where

$$a_0(\varrho) = \left(\nu_{\widetilde{Z}} - \sqrt{\nu_{\widetilde{Z}}^2 + 2\varrho}\right)/\sigma_{\widetilde{Z}}\,, \quad \text{for} \quad \nu_{\widetilde{Z}} := (\mu_{\widetilde{Z}} - \sigma_{\widetilde{Z}}^2/2)/\sigma_{\widetilde{Z}}\,. \tag{B.50}$$

The claim follows. □

Proof of Theorem 3.9. Firstly, from the proof of Proposition 3.6, note that if $\theta < 0$ we have $Q_t < 0$. Hence, the social cost $V_{\mathrm{sc}} < 0$ since the integral is over a strictly negative quantity. It follows from the positivity of V_{ev} that $V = V_{\mathrm{ev}} - V_{\mathrm{sc}}$ is strictly positive. Conversely, if $\theta \geq 0$, investment is immediate, and Lemma 3.7 as well as Case (i) of Lemma 3.8 apply. Inserting the identity from Lemma 3.7 with $\widehat{V}_{\mathrm{sc}} = \mathcal{I}(\pi)$ into (3.19) and solving for z such that $V(z) > 0$ yields the claim. □

Proof of Proposition 3.10. Firstly, make a calculation analogous to the proof of Lemma 3.8 to reduce the objective to the deterministic function

$$\begin{cases} \inf_m \left[\mathcal{I}(\pi; m) + \Psi(mK)\right] , & (\theta \geq 0) \vee (z \geq z^*) , \\ \inf_m \left[\mathscr{L}(z; \widehat{\epsilon} - \mu_{\varpi})\mathcal{I}(\pi; m) + \mathscr{L}(z; \widehat{\epsilon})\Psi(mK)\right] , & (\theta < 0) \wedge (z < z^*) . \end{cases} \tag{B.51}$$

To simplify notation, we group some constants. From (B.46) write $\mathcal{I}(\pi) = A_0 + A_1\theta$, where

$$A_0 := \frac{(\widetilde{\eta}^{-1} - \eta^{-1})\underline{s}\pi}{\widehat{\epsilon} - \mu_{\varpi}} , \quad A_1 := \frac{\beta\varphi\pi}{P(\widehat{\epsilon} - \mu_{\varpi} - \mu_{\widetilde{Z}})} , \tag{B.52}$$

and from (B.13) write $\theta = B_0 - B_1 K$, where

$$B_0 := (\eta^{-1} - \widetilde{\eta}^{-1})\underline{s}P/\mu_R , \quad B_1 := \rho/\mu_R . \tag{B.53}$$

Introduce now the subsidy by mapping $K \mapsto (1-m)K$ and propagate through the above identities. It follows that we can write $\mathcal{I}(\pi; m) = C_0 + C_1 m$ for constants

$$C_0 := A_0 + A_1\theta , \quad C_1 = A_1 B_1 K . \tag{B.54}$$

Hence, if $\theta \geq 0$ or $z \geq z^*$, (B.51) reduces to

$$\inf_m \left[(C_0 + C_1 m) + \Psi(mK)\right] . \tag{B.55}$$

Case (i) now follows by direct computation, noting that the second-order condition $\partial_m J = \xi_1 K^2 > 0$ guarantees an infimum.

For Case (ii), note that the subsistence net-present value in the presence of the subsidy, i.e. $B_0 - B_1(1-m)K$, has a root at

$$\overline{m} := 1 - \frac{B_0}{B_1 K} < 1 . \tag{B.56}$$

This corresponds to the maximum allowable subsidy, since the agent invests immediately when the subsistence net-present value is non-negative. Then from (B.49) and (B.37) we have the identity

$$\mathscr{L}(z;\varrho) = (D_0(z) + D_1(z)m)^{a_0(\varrho)} \,, \tag{B.57}$$

where

$$D_0(z) := \frac{\Lambda\theta}{z} \,, \quad D_1(z) := \frac{\Lambda B_1 K}{z} \,, \tag{B.58}$$

and $a_0(\varrho)$ is from (B.50). Hence, defining $d_0 := a_0(\widehat{\epsilon} - \mu_\varpi)$ and $d_1 := a_0(\widehat{\epsilon})$, the optimisation (B.51) reduces in the second case to (3.56). It follows that the optimal subsidy is given either by the interior solution or a boundary value; optimality must be verified by explicitly checking the value function J. □

B.2 Analysis of approximate controls

This section aims to formally treat Approximation 3.5. So let $w \in (-H, w^*)$ be a relevant initial condition for the wealth process W_t of (3.4), and let $\tau \in \mathcal{T}$ be a stopping time. With $\widehat{F}$ the counterfactual value function from Corollary 3.4 and G the terminal gain from Proposition 3.2, define

$$g(w) := \widehat{F}(w) - G(w) \,. \tag{B.59}$$

Then with $\widehat{a}$, $\widehat{x}$, and $\widehat{s}$ the optimal controls from Corollary 3.4, apply Dynkin's formula to $\widehat{F}$ to obtain

$$\mathbb{E}[\mathrm{e}^{-\widehat{\delta}\tau}\widehat{F}(W_\tau^{w;\tau})] = \widehat{F}(w) + \mathbb{E}\left[\int_0^\tau \mathrm{e}^{-\widehat{\delta}t}(-\widehat{\delta} + \mathcal{L}_w^{\widehat{a},\widehat{x},\widehat{s}})\widehat{F}(W_t^{w;\tau})\,\mathrm{d}t\right] \tag{B.60}$$

$$= \widehat{F}(w) + \mathbb{E}_w\left[\int_0^\tau \mathrm{e}^{-\widehat{\delta}t}(-U(\widehat{x}_t,\widehat{s}_t))\,\mathrm{d}t\right] \,, \tag{B.61}$$

where we use the fact that $\widehat{F}$ solves an HJB equation. Now, let $(a,x,s,\tau) \in \mathcal{A}(w)$ be arbitrary admissible controls in the agent's decision problem. Adding the present value of utility up to the time τ with respect to these controls to both sides of (B.61) and inserting the definition (B.59) yields

$$\mathbb{E}_w\left[\int_0^\tau \mathrm{e}^{-\widehat{\delta}t}U(x_t,s_t)\,\mathrm{d}t + \mathrm{e}^{-\widehat{\delta}\tau}(G(W_\tau^{w;\tau}) + g(W_\tau^{w;\tau}))\right] =$$

$$\widehat{F}(w) + \mathbb{E}_w\left[\int_0^\tau \mathrm{e}^{-\widehat{\delta}t}\,(U(x_t,s_t) - U(\widehat{x}_t,\widehat{s}_t))\,\mathrm{d}t\right] \,. \tag{B.62}$$

Taking now the supremum over a, x, s and τ, and applying the definition of the value function (3.26) thus gives the identity

$$F(w) = \widehat{F}(w) + \Omega(w) \; , \tag{B.63}$$

where

$$\Omega(w) := \sup_{a,x,s,\tau} \mathbb{E}_w \left[\int_0^\tau \mathrm{e}^{-\widehat{\delta} t} \left(U(x_t, s_t) - U(\widehat{x}_t, \widehat{s}_t) \right) \, \mathrm{d}t - \mathrm{e}^{-\widehat{\delta}\tau} g(W_\tau^{w;\tau}) \right] \tag{B.64}$$

is the *option value of switching technologies*. Thus, (B.63) states that the agent's value function equals the value of continuing forever with the present technology, $\widehat{F}$, plus the value of the option to switch. Note that since F is continuous with $F(w) = G(w)$ for $w \geq w^*$ (see Theorem 3.3), it must be the case that Ω is continuous as well, satisfying the boundary condition

$$\Omega(w) = G(w) - \widehat{F}(w) \; , \quad w \geq w^* \; . \tag{B.65}$$

The consequences of this for the optimal controls are straightforward. Consider that the implicit expressions for the pre-investment controls in Theorem 3.3 depend on the first and second derivatives of F. Therefore, since the derivative is a linear operator, as long as the derivatives of Ω are small relative to the derivatives of $\widehat{F}$, the counterfactual controls in (3.45) will provide a good approximation to the true controls. Intuitively, we expect the option value of switching $\Omega(w)$ to have relatively small derivatives for w far away from the investment threshold w^*; conversely, as w approaches the threshold, we expect the derivatives of $\Omega(w)$ to increase in relative size as the value of switching approaches the regime defined in (B.65). The upshot is that Approximation 3.5 is expected to be a fair approximation away from the threshold w^*, but to deteriorate in quality closer to the threshold.

Figure B.1 shows that this bears out for the case study in Chapter 3.4. It displays the relative errors between Approximation 3.5 and the numerically estimated true controls, as well as the relative error in the overall controlled wealth drift (cf. Equation (3.4))

$$a\mu_S w + (1-a)\mu_R w + Y - x - (s/\eta)P \; . \tag{B.66}$$

The graphic makes clear that the relative error in the consumption terms remains small right up to the threshold w^*; nevertheless, the cumulative effects of their deviations from the true controls results in a somewhat larger, though still manageable, error for the wealth drift as a whole. Overall, the approximation proves robust, supporting the qualitative conclusions of Chapter 3.3.2 and the numerical results in Chapter 3.4.

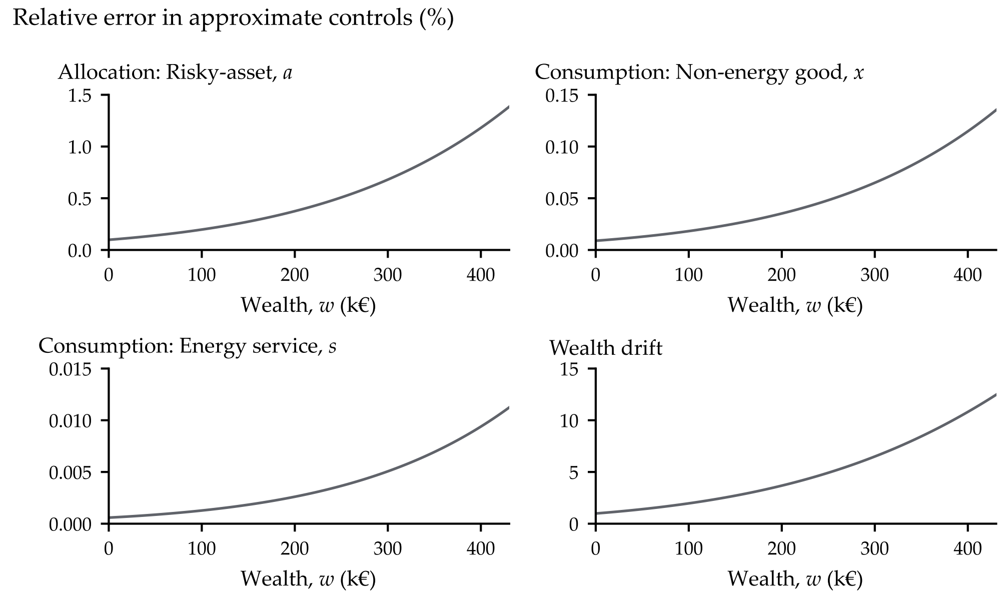

FIGURE B.1. *Relative error in the optimal strategies under Approximation 3.5, using parameters from the case study in Chapter 3.4. The errors are plotted over the range* $w \in [0, w^*]$*, where* $w^* = 430\,\text{k€}$ *is the investment threshold.*

B.3 Optimal retrofit depth

This section extends the case study in Chapter 3.4 to consider the interesting question of optimal retrofit depth. Namely, supposing that the post-retrofit efficiency $\widetilde{\eta}$ can be modelled as a function of cost K, we ask what level of efficiency improvement is optimal for the agent. To this end, Table B.1 lists additional parameters from Galvin (2024, Case Study "EFH78") corresponding to retrofits of progressively higher standards, here "Level 1", "Level 2", and "Level 3", with Level 1 being the focus of Chapter 3.4. The question of optimal retrofit depth is approached as follows: firstly, for each K and corresponding $\widetilde{\eta}(K)$ in Table B.1 the decision problem (3.9) is solved to obtain a value function $F(K)$; then, the K which maximises $F(K)$ is selected. This procedure is visually summarised in Figure B.2 for an agent with parameters as in Table 3.1. On the left, a standard logistic function approximating $\widetilde{\eta}(K)$ is fit to the available data points, and on the right, the corresponding value function $F(K)$ is displayed. In this instance, the optimal retrofit depth lies between Levels 1 and 2. For completeness, the welfare change V as a function of K is also shown; since backfire is absent here, social and private optimality coincide. We note in passing that a treatment of the optimal subsidy problem with flexible retrofit depth is beyond the scope of the model in Chapter 3. This is due to the fact that the bilevel optimisation (3.20) rests on the planner's knowledge of the agent's decision rules, which would have to be appropriately extended to account for flexibility in investment size.

TABLE B.1. *Parameters for the study of optimal retrofit depth in Appendix B.3, estimated from (Galvin, 2024). The breakdown of costs $K = K_{\text{min}} + K_{\text{ee}}$ is included since it is used for fitting the curve $\widetilde{\eta}(K_{\text{ee}})$ in Figure B.2. The so-called "anyway" costs K_{min} are the minimum necessary costs for renovation, with K_{ee} being the additional costs for the energy-efficiency measures.*

Parameter	Description (Unit)	Existing state	Level 1	Level 2	Level 3
η (resp. $\widetilde{\eta}$)	Efficiency (°C W^{-1})	0.005	0.025	0.030	0.039
K	Retrofit cost (k€)		120	125	137
K_{min}	Anyway costs (k€)		57	57	57
K_{ee}	Cost of energy-efficiency measure (k€)		63	68	80

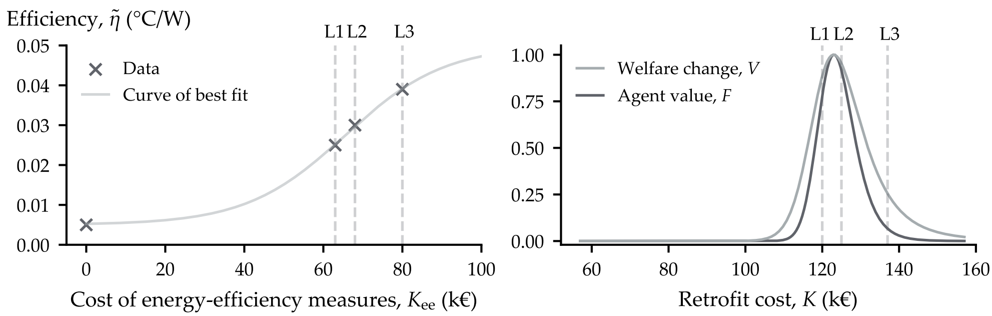

FIGURE B.2. *On the left: the data points from Table B.1, with Level 1 abbreviated as "L1", etc. together with the curve of best fit, assuming a standard logistic function. On the right: the corresponding value function F as well as the welfare change $\widehat{V}$ are displayed as functions of K, each normalised to a $[0, 1]$ scale.*

C Supplementary material for Chapter 5

C.1 Proof of Theorem 5.1

The proof is due to an anonymous user on the Economics Stack Exchange forum.[42] For the first direction the argument is by contrapositive. As U is a utility function, we have $\partial_x U > 0$ and $\partial_y U > 0$. If the limit (5.5) does not hold, there is an $\epsilon > 0$ and a sequence $z_t \to \infty$ such that for all t, either $U_x(z_t, z_t) \geq \epsilon$ or $U_y(z_t, z_t) \geq \epsilon$. Since U is concave, it is bounded from above by its Taylor expression; we thus have

$$U(0,0) \leq U(z_t, z_t) + U_x(z_t, z_t)(0 - z_t) + U_y(z_t, z_t)(0 - z_t) \,, \tag{C.1}$$

$$\Leftrightarrow \quad U(z_t, z_t) \geq U(0,0) + U_x(z_t, z_t) z_t + U_y(z_t, z_t) z_t \,, \tag{C.2}$$

$$\Rightarrow \quad U(z_t, z_t) \geq U(0,0) + \epsilon z_t \,. \tag{C.3}$$

Suppose $a = b = \epsilon/4 > 0$. Then,

$$U(z_t, z_t) - a z_t - b z_t \geq U(0,0) + \frac{\epsilon}{2} z_t \overset{t}{\to} \infty \,. \tag{C.4}$$

Hence $U(x, y) - ax - by$ is not bounded from above, so $\widetilde{U}(a, b)$ does not exist.

For the other direction, assume that the limit (5.5) holds. Then for given $a, b > 0$ there is a z large enough such that $\nabla U(z, z) < (a, b)$. Then from concavity,

$$U(x,y) - ax - by \leq U(z,z) + U_x(z,z)(x - z) + U_y(z,z)(y - z) - ax - by \,, \tag{C.5}$$

$$= U(z,z) - U_x(z,z)z - U_y(z,z)z + \underbrace{(U_x(z,z) - a)}_{\leq 0} x + \underbrace{(U_y(z,z) - b)}_{\leq 0} y \,, \tag{C.6}$$

$$\leq U(z,z) - U_x(z,z)z - U_y(z,z)z \,. \tag{C.7}$$

[42] The original response is found at the following link: https://economics.stackexchange.com/a/59862/23231.

As z was chosen independently of (x, y), we have that $U(x, y) - ax - by$ is bounded from above. Hence $\widetilde{U}(a, b)$ is well defined for all a, b. This completes the proof.

C.2 Energy consumption with Cobb-Douglas preferences

This appendix gives provides additional intuition for the claim that energy-efficiency adoption with Cobb-Douglas preferences leads to a price-independent decision boundary. With the setup from Chapter 5.2.1 and the decision problem (5.3), we specialise to the preference function

$$U(x, s) := \frac{(x^{1-\beta} s^{\beta})^{1-\gamma}}{1-\gamma}\,, \tag{C.8}$$

where $\beta \in [0, 1]$ is the preference weight on the energy service and $\gamma > 1$ is the coefficient of risk aversion.

We first solve for the terminal gain G in (5.3); the procedure is analogous to Chapters 3 and 4. Define human capital as

$$\widetilde{H} := \int_0^\infty \mathrm{e}^{-\mu_R t} \widetilde{Y} = \frac{\widetilde{Y}}{\mu_R}\,, \tag{C.9}$$

where $\widetilde{Y} = Y - \rho K$ is labour income net of the loan repayment. Then with $\widetilde{W}_t$ denoting the wealth process in the case of immediate investment, let $Z_t := \widetilde{z}(\widetilde{W}_t) := \widetilde{W}_t + \widetilde{H}$ denote disposable capital. As in Chapter 3.3.1, the allocation and consumption controls are admissible as long as disposable capital remains positive, which ensures finite borrowing. The following result is obtained.

Proposition C.1. *Let $w > -\widetilde{H}$, and let $\widehat{\delta} > 0$ be large enough to ensure that*

$$0 < \varphi := \frac{2\gamma\widehat{\delta} + (\gamma-1)\kappa^2 + (\gamma-1)\gamma\left(\beta(\beta(-\gamma) + \beta + 1)\sigma_P^2 - 2\beta\mu_P + 2\mu_R\right)}{2\gamma^2}\,. \tag{C.10}$$

Then the value of immediate investment G in (5.3) *is given by*

$$G(p, w) = \Gamma^{-\gamma} \frac{1}{1-\gamma} \left(\frac{\widetilde{z}(w)}{(p/\widetilde{\eta})^{\beta}}\right)^{1-\gamma}\,, \tag{C.11}$$

where

$$\Gamma := \left((1-\beta)^{1-\beta}\beta^{\beta}\right)^{-\widehat{\gamma}} \varphi > 0 \tag{C.12}$$

is a constant of integration and $\widehat{\gamma} = (1-\gamma)/\gamma$. *The optimal strategies follow*

$$\widetilde{a}_t^* = \frac{\kappa}{\gamma\sigma_S}\frac{\widetilde{Z}_t^{\widetilde{z}(w)}}{\widetilde{W}_t^w}\,,\quad \widetilde{x}_t^* = (1-\beta)\varphi\widetilde{Z}_t^{\widetilde{z}(w)}\,,\quad \widetilde{s}_t^* = \beta\varphi\frac{\widetilde{Z}_t^{\widetilde{z}(w)}}{P_t^p/\widetilde{\eta}}\,. \tag{C.13}$$

The proof is analogous to Proposition 3.2 and is therefore omitted. The interpretation of the controls is likewise analogous, except for the absence of subsistence constraints on consumption. This is mathematically convenient because it preserves the homogeneity of the preference function, which in turn allows us to consider stochastic energy prices; this contrasts with Proposition 4.1, where the energy price was held deterministic to obtain explicit solutions under subsistence constraints. Economically, however, the consumption controls in Proposition 4.1 are more realistic: the subsistence requirements ensure that the agent is not perfectly elastic in income and prices, and that non-energy consumption responds to energy prices. By contrast, in the present case the agent is perfectly elastic in income and prices, and non-energy consumption is unaffected by energy prices. This idealisation yields the behaviour described in Chapter 5.2.1, whereby the energy price does not affect the adoption threshold. We offer the following technical argument.

Let $\mathcal{L}_{p,w}$ denote the infinitesimal generator associated to the diffusion P_t and W_t, and let $\widehat{U}(a,b)$ denote the Legendre-Fenchel transform of U from (C.8). With $H = Y/\mu_R$ denoting human capital, the domain of value function $F(p,w)$ in (5.3) is given by $\mathcal{B} := \mathbb{R}_+ \times [-H,\infty)$. Recall now from the proof of Theorem 3.3 that the region where waiting to invest is optimal, denoted $\mathcal{C}$, and the region where immediate is optimal, denoted S, satisfy

$$\mathcal{C} \supseteq \mathcal{D} \;:= \Big\{(p,w)\in\mathcal{B} \mid \widehat{\delta}G - (\mathcal{L}_{p,w}G + \widehat{U}(\partial_w G, (p/\eta)\partial_w G)) < 0\Big\}\,, \tag{C.14}$$

$$\mathcal{S} \subseteq \mathcal{D}^c := \Big\{(p,w)\in\mathcal{B} \mid \widehat{\delta}G - (\mathcal{L}_{p,w}G + \widehat{U}(\partial_w G, (p/\eta)\partial_w G)) \geq 0\Big\}\,, \tag{C.15}$$

with immediate investment being optimal in (5.3) if $\mathcal{D} = \varnothing$. Defining the change in human capital due to the energy-efficiency investment as

$$\widetilde{H} - H = \frac{\rho K}{\mu_R} =: \theta\,, \tag{C.16}$$

and defining the constants

$$\overline{w} := \Lambda\theta - H, \quad \text{where} \quad \Lambda := 1 + \frac{\widehat{\gamma}\mu_R}{\varphi(1-(\widetilde{\eta}/\eta)^{-\beta\widehat{\gamma}})}\,, \tag{C.17}$$

straightforward computation gives that the sets $\mathcal{D}$ and $\mathcal{D}^c$ satisfy

$$\mathcal{D} = \{(p, w) \in \mathcal{B} \mid w < \overline{w}\} \ , \quad \mathcal{D}^c = \{(p, w) \in \mathcal{B} \mid w \geq \overline{w}\} \ . \tag{C.18}$$

Hence, if we picture the domain $\mathcal{B}$ with w on the horizontal axis and p on the vertical axis, the agent's decision boundary, which separates $\mathcal{C}$ and $\mathcal{S}$, must lie on or to the right of the vertical line $w = \overline{w}$.

We hypothesise, although we cannot prove, that the decision boundary is in fact commensurate with $w = \overline{w}$. That is, we conjecture that the first hitting time

$$\tau^*(p, w) = \tau^*(w) := \inf\{t \geq 0 \mid W_t^w \geq \overline{w}\} \tag{C.19}$$

is optimal in (5.3). The economic intuition is discussed above: since energy demand is perfectly elastic in prices, there is simply no price-driven incentive for the agent to improve energy-efficiency. We cannot show this explicitly because, with two state variables, the guess-and-verify approach used to obtain closed-form solutions in the one-dimensional case does not apply (cf. Compernolle et al., 2021, Dammann & Ferrari, 2021). Thus, numerical methods must be sought. Extensive numerical testing on our part indicated that the decision boundary is in fact degenerate in p, so that the investment decision reduces to a pure wealth effect: the agent invests when wealth is sufficiently high relative to the marginal utility of consumption (cf. the proof of Theorem 3.3). For this reason, the decision model (5.3) must move beyond Cobb-Douglas preferences, incorporating realistic features of energy-service demand such as a subsistence constraint, to generate economically meaningful results.

List of publications

Journal contributions

Britto, A. (2022). Letter to the editor: Discussion of proposed t-statistic in "ppcor: An R Package for a fast calculation to semi-partial correlation coefficients," CSAM 2015; 22:665-674. *Communications for Statistical Applications and Methods*, 29(3), 393–396. https://doi.org/10.29220/csam.2022.29.3.393

Britto, A., Dehler-Holland, J., & Fichtner, W. (2024). Wealth maximisation and residential energy-efficiency retrofits: Insights from a real options model. *Energy Economics*, 140, 108022. https://doi.org/10.1016/j.eneco.2024.108022

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Reijnders, A., Hamilton, J., Britto, A., Brubach, J.-B., Roy, P., Gibson, Q., Cava, R., & Burch, K. (2014). Optical investigation of the thermoelectric topological crystalline insulator $Pb_{0.77}Sn_{0.23}Se$. *Physical Review B*, 90(23), 235144. https://doi.org/10.1103/physrevb.90.235144

Conferences and preprints

Britto, A. (2021). A mathematical construction of an E6 grand unified theory. arXiv: 2102.13465. https://doi.org/10.48550/arxiv.2102.13465

Britto, A. & Dehler-Holland, J. (2021). Optimal investment in energy efficiency as a problem of growth-rate maximisation: Evidence and policy implications. *ECEEE 2021 Summer Study on Energy Efficiency: A New Reality*. https://doi.org/10.5445/IR/1000140253/pre

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