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Mathematical Analysis of Regularity Propagation in Variable-Viscosity Fluid Models

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Abstract

In this thesis we investigate several variable-viscosity fluid models, including incompressible viscous fluids with usual or odd viscosity and compressible viscous fluids. We mostly focus on the regularity propagation of sharp interfaces between two immiscible, viscous fluids. This thesis is divided into three parts.

The first part (Chapter 2) investigates the existence of weak solutions to the two-dimensional inhomogeneous incompressible Navier-Stokes equations with variable, odd viscosity. *Odd* or *Hall* viscosity is the anti-symmetric part of the viscosity tensor, and it is present in fluids with broken microscopic time-reversal symmetry and broken parity. We prove the existence of weak solutions in both the evolutionary and stationary cases. Furthermore, we study the limit of the weak solutions as the odd viscosity coefficient converges to a constant. Lastly, examples of stationary solutions for parallel, concentric and radial flows, are considered.

The second part (Chapter 3) addresses the two-dimensional incompressible Navier-Stokes equations with freely transported viscosity coefficient. Under a suitable smallness assumption on the initial velocity, a global-in-time well-posedness result is established which allows for *discontinuous* density and viscosity coefficients without size restriction on the jumps. As an application, the global-in-time well-posedness of the two-dimensional inhomogeneous incompressible Navier-Stokes equations with density-dependent viscosity, and the local-in-time well-posedness of the two-dimensional incompressible Boussinesq equations with temperature-dependent viscosity is proven. Both results allow for discontinuous, largely varying viscosity. The regularity of an interface of discontinuity and the tangential regularity of the density/temperature along this interface is also shown to persist over time.

The third part (Chapter 4) concerns the two-dimensional compressible Navier-Stokes equations with density-dependent viscosity coefficients. Under a suitable smallness assumption on the initial velocity and initial energy of the system, a global-in-time well-posedness result is proven in a framework allowing for discontinuous density and viscosity coefficients without size restriction on the jumps. We also show the persistence of regularity of the discontinuity curve as well as the tangential regularity of the density over time.

Keywords: Fluid mechanics, Navier-Stokes equations, Boussinesq equations, variable viscosity, odd viscosity, density-patch problem, two-phase flow, tangential regularity

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Contents

CHAPTER 1

Introduction

In this thesis we investigate various models describing the motion of variable-viscosity fluids. Fluids with variable viscosity are of significant physical relevance and appear in a variety of circumstances including geophysics and biological systems. Mathematically the variable viscosity results in a stronger nonlinear coupling in the system of fluid motion. The fluids we study here include incompressible viscous fluids with usual or odd viscosity and some compressible viscous fluids. We particularly focus on the regularity propagation of sharp interfaces between immiscible fluids in mixtures of viscous fluids. The fluid density or temperature can exhibit discontinuities across this interface, and we aim to analyze the propagation of regularity of the interface over time, as well as the tangential regularity of the density and temperature along this interface.

This introduction is divided into three parts. Section 1.1 gives an overview of the fluid models which will be studied in this thesis. A review of the existing mathematical literature on these models is given in Section 1.2. The main results of this thesis are presented in Section 1.3.

1.1. Presentation of the fluid models

We consider a fluid in the whole plane $\mathbb{R}^2$. When the fluid moves, there is internal resistance to flow. This internal resistance, or friction, is measured by *viscosity*. It is caused by forces appearing between neighboring fluid particles that tend to make their velocities equal. The forces between fluid particles are described mathematically by a tensor $T \in \mathbb{R}^{2\times 2}$ called the *stress tensor*. It is composed of two parts: viscous stresses and hydrostatic stresses. These are usually of the form [158]

$$T = \sigma - \pi \mathrm{Id},$$

where the tensor $\sigma \in \mathbb{R}^{2\times 2}$ is called the *viscous stress tensor* and $\pi \in \mathbb{R}$ is the pressure of the fluid. Above, Id denotes the identity matrix in $\mathbb{R}^{2\times 2}$.

We assume the absence of external forces. By the momentum conservation law, the fluid motion is described by the Navier-Stokes equation [219]

$$\rho(\partial_t u + u \cdot \nabla u) - \operatorname{div} T = 0. \tag{1.1.1}$$

Here, the time and space variables are denoted by $t \in [0, \infty)$ and $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, respectively, and $\rho = \rho(t, x) \in [0, \infty)$ is the fluid density and $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u(t, x) \in \mathbb{R}^2$ the fluid velocity.

Since internal friction can only occur when the fluid is in motion, the viscous stress tensor must be a function of the velocity gradient: $\sigma = \sigma(\nabla u)$. In this thesis we always consider fluids for which the viscous stress tensor σ is linearly proportional to the velocity gradient ∇u

[1]. For such fluids the viscous stress tensor has the general form [16]

$$\sigma_{ij} = \eta_{ijkl} \cdot \frac{1}{2}(\partial_{x_k} u_l + \partial_{x_l} u_k),$$

for some coefficients $\eta_{ijkl} \in \mathbb{R}$. The tensor η is called the *viscosity tensor*, and it can be decomposed into a symmetric part and an antisymmetric part:

$$\eta_{ijkl} = \eta^S_{ijkl} + \eta^A_{ijkl}, \quad \text{with} \quad \eta^S_{ijkl} = \eta^S_{klij}, \; \eta^A_{ijkl} = -\eta^A_{klij}.$$

The symmetric part η^S is often called *even viscosity* and the antisymmetric part η^A is referred to as *odd viscosity*. For systems with symmetry under both parity and time-reversal, which are satisfied by conventional fluids at thermal equilibrium, Onsager reciprocal relation [197] demands that odd viscosity vanishes: $\eta^A = 0$.

For an isotropic fluid in two space dimensions, the symmetric part η^S is characterized by two scalar viscosity coefficients μ and λ, and the antisymmetric part η^A by one, μ_o, which leads to the following form of the viscous stress tensor [16]:

$$\sigma = \mu S u + \lambda(\operatorname{div} u)\mathrm{Id} + \mu_o S_o u. \tag{1.1.2}$$

Here we denote

$$Su = \nabla u + (\nabla u)^T = \begin{pmatrix} 2\partial_{x_1} u_1 & \partial_{x_1} u_2 + \partial_{x_2} u_1 \\ \partial_{x_1} u_2 + \partial_{x_2} u_1 & 2\partial_{x_2} u_2 \end{pmatrix}, \tag{1.1.3}$$

$$S_o u = \nabla u^\perp + \nabla^\perp u = \begin{pmatrix} -(\partial_{x_1} u_2 + \partial_{x_2} u_1) & \partial_{x_1} u_1 - \partial_{x_2} u_2 \\ \partial_{x_1} u_1 - \partial_{x_2} u_2 & \partial_{x_1} u_2 + \partial_{x_2} u_1 \end{pmatrix}. \tag{1.1.4}$$

The coefficients μ, λ are the so-called Lamé viscosity coefficients and μ_o the odd viscosity coefficient. With this form (1.1.2) of the viscous stress tensor, the Navier-Stokes equation (1.1.1) becomes

$$\rho(\partial_t u + u \cdot \nabla u) - \operatorname{div}(\mu S u + \mu_o S_o u) - \nabla(\lambda \operatorname{div} u) + \nabla \pi = 0. \tag{1.1.5}$$

Generally, viscosity can depend on the values of other state variables of the fluid, such as the fluid temperature ϑ, density ρ or concentration c [92, 175]. In particular, viscosity can be variable when these state variables are non-constant. In the following we present several fluid models in which the viscosity is variable. It is the aim of this thesis to study the regularity propagation of geometric structures such as sharp interfaces in these models.

1.1.1. Incompressible fluid models

1.1.1.1. The classical incompressible fluid model

In the case of many fluids, including most liquids, compressibility effects are negligible, so that these fluids can often be considered incompressible [32, 232]. Mathematically, incompressibility is expressed as

$$\operatorname{div} u = 0. \tag{1.1.6}$$

[1]In general, the viscous stress tensor can depend nonlinearly on the velocity gradient. In that case the fluid is called *non-Newtonian*. Examples of non-Newtonian fluids include *shear-thinning* fluids such as polymer solutions, blood and mayonnaise (viscosity decreases with increasing velocity gradient), and *shear-thickening* fluids such as suspensions (viscosity increases with increasing velocity gradient) [144, 236].

Furthermore, for homogeneous fluids we take the fluid density and temperature to be constant: $\rho = \vartheta = 1$, and assume that time-reversal and parity hold. Then odd viscosity must vanish by Onsager reciprocal relation, and hence by virtue of (1.1.6) only the Lamé viscosity coefficient μ plays a role in the viscous stress tensor (1.1.2). Under the assumption $\rho = \vartheta = 1$ this viscosity coefficient becomes a positive constant $\mu = \nu \equiv const$.

Combining the Navier-Stokes equations (1.1.5) and the incompressibility condition (1.1.6) yields the two-dimensional classical incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla \pi = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ \operatorname{div} u = 0. \end{cases} \tag{NS}$$

These equations were first proposed by C. L. M. H. Navier in 1822 (see also [194]) and were later justified by G. G. Stokes in 1834 [227]. They are regarded as a fundamental basis of fluid mechanics.

1.1.1.2. Incompressible fluid models with variable density

Here we assume the fluid to be incompressible such that (1.1.6) holds. Furthermore, we take the fluid temperature to be constant, but take density fluctuations into account. The evolution equation of the density, which is derived from the mass conservation law [158], reads

$$\partial_t \rho + \operatorname{div}(\rho u) = 0. \tag{1.1.7}$$

We make the further assumption that time-reversal and parity hold, so that odd viscosity must vanish by Onsager reciprocal relation. Hence, by (1.1.2), the fluid is characterized by one viscosity coefficient, μ, which is assumed to depend on the density ρ. To stress the density-dependence, *we write* $\mu(\rho)$ *instead of* μ, and we regard μ as a *given* function

$$\mu : [0,\infty) \to \mathbb{R}, \quad 0 < \mu_* \le \mu \le \mu^*, \tag{1.1.8}$$

for some positive constants μ_*, μ^*.

Combining the Navier-Stokes equation (1.1.5), the incompressibility condition (1.1.6), and the mass conservation law (1.1.7) yields the two-dimensional inhomogeneous incompressible Navier-Stokes equations with density-dependent viscosity[2]

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ \rho(\partial_t u + u \cdot \nabla u) - \operatorname{div}(\mu(\rho) S u) + \nabla \pi = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{INS}$$

Here we have used the incompressibility condition (1.1.6) to rewrite the density equation (1.1.7) as $(\text{INS})_1$. The above system of equations is usually used to model the motion of most conventional liquids, as they are considered incompressible and have no odd viscosity.

[2]In this model, the density ρ can also be replaced by the concentration c. Indeed, concentration-dependent viscosity plays an important role in many fluids such as certain nanofluids [13, 239], salt crystallization [153] or complex (dusty) plasmas [148, 212]. A well-known example of concentration-dependent viscosity is Einstein's law for dilute particle suspensions $\mu(c) = \tilde{\mu}(1 + 2.5c)$ [85, 86], where $\tilde{\mu}$ is some constant reference viscosity (see also the extensions of Einstein's viscosity law [211, 233] and references therein).

1.1.1.3. Incompressible fluid models with odd viscosity

We now consider incompressible fluids for which time-reversal and parity are broken, so that the viscosity tensor has a non-vanishing odd part $\mu_o \not\equiv 0$. Hence, the fluid is characterized by two viscosity coefficients, one for the symmetric part of the viscosity tensor, μ, and one for the antisymmetric part, μ_o. Both viscosity coefficients are assumed to depend on the density ρ. For μ we again assume (1.1.8), and μ_o is given by

$$\mu_o : [0, \infty) \to \mathbb{R}, \quad -\mu^* \leq \mu_o \leq \mu^*.$$

Combining the Navier-Stokes equation (1.1.5), the incompressibility condition (1.1.6), and the mass conservation law (1.1.7) yields the following two-dimensional inhomogeneous incompressible Navier-Stokes equations with density-dependent, odd viscosity:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^2, \\ \rho(\partial_t u + u \cdot \nabla u) - \operatorname{div}(\mu(\rho) S u) - \operatorname{div}(\mu_o(\rho) S_o u) + \nabla \pi = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{oddINS}$$

We will also study the stationary counterpart to (oddINS):

$$\begin{cases} \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho) S u) - \operatorname{div}(\mu_o(\rho) S_o u) + \nabla \pi = 0, \\ \operatorname{div}(\rho u) = 0, \quad \operatorname{div} u = 0. \end{cases} \tag{oddINS'}$$

Examples for fluids with non-vanishing odd viscosity include polyatomic gases [151], chiral active fluids [19, 225, 226], magnetized plasmas [204] and fluids of vortices [241].

1.1.1.4. Incompressible fluid models with variable temperature: Boussinesq equations

In many fluids, viscosity is quite sensitive to temperature changes. For example, the viscosity of water at $0°C$ is $1.8 \times 10^{-3} \frac{Ns}{m^2}$, whereas at $20°C$ it is $1.0 \times 10^{-3} \frac{Ns}{m^2}$ [144]. The next model describes such a fluid whose temperature is variable and whose viscosity is temperature-dependent; that is, the viscosity is given by $\mu(\vartheta)$, where $\vartheta \in \mathbb{R}$ denotes the fluid temperature, and μ is a given function

$$\mu : \mathbb{R} \to \mathbb{R}, \quad 0 < \mu_* \leq \mu \leq \mu^*.$$

Common viscosity laws are [203]

- Arrhenius Law or Andrade's Law $\mu(\vartheta) = C_1 \exp(\frac{C_2}{C_3 + \vartheta})$ for liquids,
- Sutherland's Law $\mu(\vartheta) = \mu(\tilde{\vartheta})(\frac{\vartheta}{\tilde{\vartheta}})^{\frac{3}{2}} \frac{\tilde{\vartheta} + C_4}{\vartheta + C_5}$ for gases,

where $C_1, ..., C_5$ are constants and $\tilde{\vartheta}$ is a reference temperature (valid for Kelvin degrees).

According to the energy conservation law, the evolution equation for the fluid temperature in the case of very weak heat conduction reads as [200]

$$\partial_t \vartheta + u \cdot \nabla \vartheta = 0. \tag{1.1.9}$$

Instead of the Navier-Stokes equations (1.1.5), we consider here the *Boussinesq approximation*. It neglects density fluctuations in the computation of momentum changes from acceleration, but takes them into account when they give rise to buoyancy effects, i.e. a vertical force caused by the higher-density (colder) fluid to descend and the lower-density (hotter) fluid to rise [24]. This buoyancy effect appears as a forcing term ϑe_2 in the momentum equation, where $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

With the incompressibility condition (1.1.6) and the temperature equation (1.1.9), the Boussinesq equations read as [78, 109]

$$\begin{cases} \partial_t \vartheta + u \cdot \nabla \vartheta = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ \partial_t u + u \cdot \nabla u - \operatorname{div}(\mu(\vartheta) S u) + \nabla \pi = \vartheta e_2, \\ \operatorname{div} u = 0. \end{cases} \tag{B}$$

The Boussinesq equations commonly appear in geophysics (see e.g. the books [107, 184, 200]), describing for instance large scale atmospheric and oceanic flows, as well as in the study of Rayleigh-Benard convection [49].

1.1.2. Compressible fluid models

The last fluid model we present here is the compressible counterpart of (INS). For compressible Newtonian fluids without odd viscosity and negligible temperature variations, the viscous stress tensor (1.1.2) becomes $\sigma = \mu(\rho) S u + \lambda(\rho)(\operatorname{div} u)\mathrm{Id}$, where the function μ satisfies (1.1.8), and λ is such that

$$\lambda : [0,\infty) \to \mathbb{R}, \quad 0 < \nu_* \le \nu := 2\mu + \lambda \le \nu^*,$$

for some positive constants ν_*, ν^*. The viscosity coefficient $\nu(\rho) = 2\mu(\rho) + \lambda(\rho)$ is often referred to as the *bulk viscosity.*

In stark contract to incompressible fluids, for compressible fluids the pressure is a function of the density, i.e. it is given by $P(\rho)$ for some function

$$P : [0,\infty) \to \mathbb{R}, \quad P' > 0.$$

Typical pressure laws include [92]

- $P(\rho) = a\rho^\gamma$, with constants $a > 0$, $\gamma \in (1, \frac{5}{3}]$ (polytropic gases)
- $P(\rho) = R\vartheta_0\rho$, with constants $R > 0, \vartheta_0 \in \mathbb{R}$ (isothermal ideal gases)

Combining the Navier-Stokes equations (1.1.5) (replacing π by $P(\rho)$) and the mass conservation law (1.1.7) yields the two-dimensional compressible Navier-Stokes equations with density-dependent viscosity

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ \rho(\partial_t u + u \cdot \nabla u) - \operatorname{div}(\mu(\rho) S u) - \nabla(\lambda(\rho) \operatorname{div} u) + \nabla P(\rho) = 0. \end{cases} \tag{CNS}$$

The above equations are most commonly used to describe the motion of gases or that of liquids at very high speed [93, 175, 191].

1.2. Review of known results

In this section we review some of the mathematical literature on the equations (INS), (oddINS), (oddINS'), (B) and (CNS). Here and throughout the remainder of this thesis we adopt the following notation:

- *(Inequalities).* The expression $A \lesssim B$ means that $A \leq CB$ for some constant $C > 0$.
- *(Vectors and matrices).* For two vectors $v, w \in \mathbb{R}^2$, the expression $v \otimes w$ stands for the matrix with entries $(v \otimes w)_{ij} = v_i w_j$, $i, j = 1, 2$.
  For two matrices $A, B \in \mathbb{R}^{2\times 2}$ the Frobenius inner product is $A : B = \sum_{i,j=1}^{2} A_{ij} B_{ij}$.
  For a vector $v \in \mathbb{R}^2$ we write $v^\perp = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix}$ for a rotation in the plane by ninety degrees.
- *(Function spaces).* Let $d \in \mathbb{N}$.
  For $p \in [1, \infty]$ and $s \in \mathbb{R}$ the spaces $L^p(\mathbb{R}^d)$ and $W^{s,p}(\mathbb{R}^d)$ ($\dot{W}^{s,p}(\mathbb{R}^d)$) denote the usual Lebesgue and (homogeneous) Sobolev-Slobodetskiĭ spaces, respectively, with norms $\|\cdot\|_{L^p} = \|\cdot\|_{L^p(\mathbb{R}^d)}$ and $\|\cdot\|_{W^{2,p}} = \|\cdot\|_{W^{2,p}(\mathbb{R}^d)}$ ($\|\cdot\|_{\dot{W}^{2,p}} = \|\cdot\|_{\dot{W}^{2,p}(\mathbb{R}^d)}$). We furthermore write $H^s(\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d)$ ($\dot{H}^s(\mathbb{R}^d) = \dot{W}^{s,2}(\mathbb{R}^d)$).
  We denote $L^p(\mathbb{R}^d; \mathbb{R}^n)$ simply by $L^p(\mathbb{R}^d)$ for $n \in \mathbb{N}$, if the dimension n is clear from the context, with norm $\|\cdot\|_{L^p} = \|\cdot\|_{L^p(\mathbb{R}^d;\mathbb{R}^n)}$, and similarly for other function spaces.
  The notation $\langle \cdot, \cdot \rangle$ indicates the $L^2(\mathbb{R}^d)$-inner product.
  For times $t > 0$ and exponents $p, q \in [1, \infty]$ we denote $L^p_t L^q = L^p([0, t]; L^q(\mathbb{R}^d))$ and $L^p L^q = L^p([0, \infty); L^q(\mathbb{R}^d))$.
  The set of functions which are continuous, bounded maps from the interval $I \subset \mathbb{R}$ to $L^p(\mathbb{R}^d)$ is denoted by $\mathcal{C}_b(I; L^p(\mathbb{R}^d))$.
- *(Operators).* Derivatives with respect to time and space are denoted by ∂_t and $\partial_i = \partial_{x_i}$, $i = 1, 2$, respectively. We set $\nabla = \begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix}$, $\operatorname{div} = \nabla\cdot$, $\nabla^\perp = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix}$, $\operatorname{curl} = \nabla^\perp\cdot$, and $\Delta = \operatorname{div}\nabla$. For a vector field $v = (v_j)_{j=1,2}$ we set $\nabla v = (\partial_j v_i)_{i,j=1,2}$, $Dv = (\partial_i v_j)_{i,j=1,2}$.
  The material derivative associated to the velocity vector field u is denoted by $\dot{f} \equiv \frac{D}{Dt} f \equiv \partial_t f + u \cdot \nabla f$ for functions f.
  For a regular vector field X we write $\partial_X := X \cdot \nabla$ for the directional derivative along X. For $g \in L^\infty(\mathbb{R}^2, \mathbb{R})$ it is understood in the weak sense through $\partial_X g = \operatorname{div}(Xg) - g \operatorname{div} X$.
  The Riesz transform on $\mathbb{R}^2$ is denoted by $\mathcal{R} = \begin{pmatrix} \mathcal{R}_1 \\ \mathcal{R}_2 \end{pmatrix} = \frac{\frac{1}{i}\nabla}{\sqrt{-\Delta}}$. Here, $i \in \mathbb{C}$ with $i^2 = -1$.
  For any $p \in (1, \infty)$, the Riesz transform $\mathcal{R}$ is a bounded linear operator on $L^p(\mathbb{R}^2)$.
  The operator $(-\Delta)^{-1}$ usually appears together with second-order derivatives $\partial_j \partial_k$, such that the total application $-(-\Delta)^{-1}\partial_j\partial_k$ is understood as the composition of Riesz operators $\mathcal{R}_j\mathcal{R}_k$. The function of the form $(-\Delta)^{-1}\partial_j f$ with $f \in L^2(\mathbb{R}^2)$ is understood as a linear operator which applies on functions of the form $\partial_k g$ with $g \in L^2(\mathbb{R}^2)$ via $\langle (-\Delta)^{-1}\partial_j f, \partial_k g \rangle_{\dot{H}^1(\mathbb{R}^2), \dot{H}^{-1}(\mathbb{R}^2)} = \langle \mathcal{R}_j \mathcal{R}_k f, g \rangle$.
  The operator $\mathbb{P} : L^2(\mathbb{R}^2; \mathbb{R}^2) \to L^2(\mathbb{R}^2; \mathbb{R}^2)$ represents the Leray projector, which projects a vector field onto its divergence-free part: $\operatorname{div}\mathbb{P} = 0$ (see [96]).
  The commutator of two operators A and B is defined as $[A, B] := AB - BA$.

1.2.1. The incompressible Navier-Stokes equations

1.2.1.1. The classical incompressible Navier-Stokes equations

The incompressible Navier-Stokes equations have a long history reaching back to the 19th century. For constant density and viscosity $\rho = 1$, $\mu(\rho) = \nu \equiv$ *const* the equations (INS) turn into the classical two-dimensional Navier-Stokes equations (NS):

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla \pi = 0, & (t, x) \in (0, \infty) \times \mathbb{R}^2, \\ \operatorname{div} u = 0. \end{cases} \tag{NS}$$

The literature on (NS) is vast. In the seminal work [163] J. Leray proved the global-in-time existence of finite-energy weak solutions (u, π) in the space $L^\infty([0,\infty); L^2(\mathbb{R}^2;\mathbb{R}^2)) \cap L^2([0,\infty); \dot{H}^1(\mathbb{R}^2;\mathbb{R}^2)) \times L^2_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^2)$. By the celebrated work of O. A. Ladyzhenskaya [156], it is well-known that Leray's weak solution in dimension *two* are unique in the energy space $L^2(\mathbb{R}^2;\mathbb{R}^2)$ and the Cauchy problem for the classical Navier-Stokes equations (NS) is well-posed globally in time. In dimension *three*, the uniqueness and regularity of Leray's weak solutions are extensively studied. The global-in-time existence and uniqueness of strong solutions with *small* initial data has been shown [161, 234]; however, the global-in-time well-posedness for arbitrarily *large* data in three dimensions remains open and is famously known as the Millennium Problem for the Navier-Stokes equations [143]. We refer to the book [161] for a review on the progress towards an answer to this problem.

1.2.1.2. The incompressible Navier-Stokes equations with variable density

For *variable density and viscosity*, the fluid motion evolves according to the two-dimensional inhomogeneous incompressible Navier-Stokes equations with density-dependent viscosity (INS):

$$\begin{cases} \partial_t \rho + u\cdot\nabla\rho = 0, \quad (t,x)\in(0,\infty)\times\mathbb{R}^2, \\ \rho(\partial_t u + u\cdot\nabla u) - \operatorname{div}(\mu(\rho)Su) + \nabla\pi = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{INS}$$

This system comes with the following basic features:

- *(Energy balance).* Sufficiently smooth solutions $(\rho, u, \nabla\pi)$ to (INS) satisfy the energy balance

$$\int_{\mathbb{R}^2} \rho(t,x)|u(t,x)|^2\,dx + \int_0^t\int_{\mathbb{R}^2} \mu(\rho(t',x))|Su(t',x)|^2\,dx\,dt' = \int_{\mathbb{R}^2}\rho(0,x)|u(0,x)|^2\,dx, \tag{1.2.1}$$

  for all $t > 0$;
- *(Density bounds).* For sufficiently smooth solutions $(\rho, u, \nabla\pi)$ and $0 \le m \le M$ the Lebesgue measure of the sets

$$\{x\in\mathbb{R}^2 : m \le \rho(t,x) \le M\}$$

  is time-independent; in particular

$$\inf_{x\in\mathbb{R}^2}\rho(t,x) = \inf_{x\in\mathbb{R}^2}\rho(0,x), \quad \sup_{x\in\mathbb{R}^2}\rho(t,x) = \sup_{x\in\mathbb{R}^2}\rho(0,x),$$

  for all $t > 0$;
- *(Scaling).* If (ρ, u, π) is a solution to (INS) on $[0,T]\times\mathbb{R}^2$ for some $T > 0$, then the rescaled triplet $(\rho,u,\pi)_\lambda$ defined by

$$(\rho,u,\pi)_\lambda(t,x) = (\rho(\lambda^2 t,\lambda x), \lambda u(\lambda^2 t,\lambda x), \lambda^2\pi(\lambda^2 t,\lambda x)), \quad \lambda > 0, \tag{1.2.2}$$

  is a solution of (INS) on $[0,\lambda^{-2}T]\times\mathbb{R}^2$.

The system (INS) has been widely explored by numerous mathematicians. The existence of global-in-time finite energy weak solutions was established by P.-L. Lions [175]. See also the earlier works by J. Simon [222] and A. V. Kazhikov [149] for the constant viscosity case.

P.-L. Lions' existence proof in [175] is based on the energy balance (1.2.1). His weak solutions satisfy the energy *in*equality, i.e. (1.2.1) with $=$ replaced by $\leq$, and the density property above.

The uniqueness and regularity of P.-L. Lions' weak solutions are still big open questions. The regularity was addressed by B. Desjardins [75], who showed that, under the additional assumption that the viscosity variation is small $\|\mu(\rho)-1\|_{L^\infty(\mathbb{T}^2)} \ll 1$, and the initial velocity belongs to $H^1(\mathbb{T}^2)$, Lions' weak solution satisfies $u \in L^\infty([0,T];H^1(\mathbb{T}^2))$, where $\mathbb{T}^2$ stands for the two-dimensional torus. Moreover, with additional regularity assumptions on the initial data, he proved that $u \in L^2([0,t_0];H^2(\mathbb{T}^2))$ for some time t_0. However, this result still does not give an answer to the uniqueness and regularity question of P.-L. Lions' weak solutions.

For *constant viscosity* the existence and uniqueness of strong solutions in the case of smooth initial data and density bounded away from zero was proved by O. A. Ladyzhenskaya and V. A. Solonnikov [157]. More precisely, the authors achieved global well-posedness in dimension two, local well-posedness in dimension three, and global well-posedness in dimension three provided the initial velocity is suitably small. Recently, global-in-time well-posedness results in the more general case with *discontinuous* density and in the presence of vacuum (i.e. allowing the density to be zero) are now known to hold true, thanks to the remarkable contributions by R. Danchin and P. B. Mucha [57, 59, 60]: in [57], the authors basically proved global well-posedness of (INS) allowing density discontinuities across a C^1-interface with a sufficiently small jump. The smallness condition of the jump was removed in [60], where the presence of vacuum was allowed. See also an earlier result by M. Paicu, Z. Zhang and P. Zhang [199] with only bounded density and without vacuum.

For *variable viscosity* the local-in-time well-posedness for smooth initial data was proved by Y. Cho and H. Kim [46], see also the book [14]. Under *small variation assumptions*, either with small density variation [110, 137, 179] or small viscosity variation [11, 101, 136, 142, 198] global-in-time well-posedness results have been achieved in two spatial dimensions. In the three-dimensional case, global well-posedness was proved under smallness assumptions on the initial velocity and some additional regularity assumptions on the initial density, see e.g. [10, 119, 140, 248].

Motivated by the scaling (1.2.2), a number of works have been dedicated to the study of the system in critical functional spaces which are invariant under the same scaling, see for example [51, 62, 138] for the system with constant viscosity, and [7, 9] for variable viscosity.

1.2.1.3. The stationary Navier-Stokes equations

The stationary counterpart of (INS) reads as

$$\begin{cases} \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho) S u) + \nabla \pi = 0, \quad x \in \Omega, \\ \operatorname{div}(\rho u) = 0, \quad \operatorname{div} u = 0. \end{cases} \tag{INS'}$$

The equations are posed on a domain $\Omega \subset \mathbb{R}^2$ which can either be a simply connected bounded domain, an exterior domain, or the whole plane. If Ω has a boundary $\partial\Omega$, then the system (INS') is supplemented with the boundary condition

$$u|_{\partial\Omega} = g, \tag{1.2.3}$$

where the function $g : \partial\Omega \to \mathbb{R}^2$ is given and satisfies, in view of $\operatorname{div} u = 0$, the following *zero flux condition*

$$\int_{\partial\Omega} g \cdot n d\sigma = 0, \tag{1.2.4}$$

with the outer unit normal n of the boundary $\partial\Omega$.

The *homogeneous* system (INS')-(1.2.3), i.e. when the density is constant $\rho = 1$, has been extensively studied. J. Leray [162] proved the existence of weak solutions $u \in H^1(\Omega; \mathbb{R}^2)$ in the case when Ω is a simply connected, bounded domain. Similar existence results were obtained in [94, 97, 154] with some relaxations on the zero flux condition (1.2.4), such as *small* zero flux or zero *total* flux for multi-connected domains. Weak solutions $u \in \dot{H}^1(\Omega; \mathbb{R}^2)$ in the exterior domain case were also constructed by J. Leray [162], and in the whole plane case by J. Guillod and P. Wittwer [111]. See also the books [96, 156] for further details on the homogeneous system.

In the *inhomogeneous* case with *constant* viscosity, i.e. with ρ variable and $\mu \equiv const$, the existence of weak solutions of (INS')-(1.2.3) has been proven by N. N. Frolov [95]. He constructed solutions of the form

$$(\rho, u) = (\eta(\phi), \nabla^\perp \phi), \tag{1.2.5}$$

for some suitably chosen functions $\eta \in L^\infty(\mathbb{R}; [0, \infty))$ and ϕ. The function ϕ is usually referred to as the *stream function*, and with the density and velocity of the form (1.2.5), the conditions $(\text{INS'})_2$ are automatically satisfied, at least on a formal level:

$$\begin{aligned} &\operatorname{div} u = \operatorname{div} \nabla^\perp \phi = 0, \\ &\operatorname{div}(\rho u) = u \cdot \nabla \rho = \eta'(\phi) \nabla^\perp \phi \cdot \nabla \phi = 0. \end{aligned}$$

In the *variable viscosity* case, weak solutions to (INS')-(1.2.3) were constructed by Z. He and X. Liao [123]. These solutions are of Frolov's form (1.2.5). The key observation is the following non-linear equation for the stream function, which is obtained by an application of the curl operator $\nabla^\perp \cdot$ to the momentum equation $(\text{INS'})_1$:

$$\mathcal{L}_\mu \phi = \nabla^\perp \cdot \operatorname{div}(\rho \nabla^\perp \phi \otimes \nabla^\perp \phi), \tag{1.2.6}$$

with the fourth-order elliptic operator

$$\mathcal{L}_\mu = (\partial_{22} - \partial_{11}) \mu(\rho) (\partial_{22} - \partial_{11}) + (2\partial_{12}) \mu(\rho) (2\partial_{12}). \tag{1.2.7}$$

Observe that for constant viscosity $\mu \equiv \nu > 0$, this operator coincides with the Bilaplacian, $\mathcal{L}_\nu = \nu \Delta^2$. The non-linear fourth-order elliptic equation (1.2.6) (supplemented with suitable boundary conditions) is then solved by a fixed point argument.

The work [123] also analyzes the elliptic operator $\mathcal{L}_\mu$ in (1.2.7) for *discontinuous* viscosity coefficients. It is shown that for any $p > 2$ there exists a bounded measurable (highly oscillating) function $\bar{\mu} : \mathbb{R}^2 \to \{\frac{1}{K}, K\}$ taking only two possible values with $K = \frac{2}{p-2} + 1 > 1$, such that there exist solutions ϕ to the homogeneous elliptic equation $\mathcal{L}_{\bar{\mu}} \phi = 0$ with

$$\nabla u = \nabla \nabla^\perp \phi \notin L^p_{\text{loc}}(\mathbb{R}^2).$$

In particular, the boundedness of the viscosity coefficient alone is not sufficient to guarantee the Lipschitz continuity of the velocity vector field $\nabla u \in L^\infty(\mathbb{R}^2)$.

1.2.1.4. The density-patch problem

The *density-patch problem*, posed by P.-L. Lions in 1996 in [175], considers the following situation: suppose the incompressible Navier-Stokes equations (INS) are supplemented with initial density $\rho_0(x) = 1_{D_0}(x)$ for some bounded, simply connected domain $D_0 \subset \mathbb{R}^2$. By P.-L. Lions' work [175], there exists a weak solution such that $\rho(t,x) = 1_{D_t}(x)$ for all times $t \geq 0$, with a bounded, simply connected domain $D_t \subset \mathbb{R}^2$ whose measure is preserved: $|D_t| = |D_0|$. The question of the density-patch problem is *whether the boundary regularity of the initial domain is propagated over time?*

If the fluid velocity is sufficiently regular, such as $\nabla u \in L^1_{\text{loc}}([0,\infty); L^\infty(\mathbb{R}^2))$, then the density patch at time t is given by

$$D_t = \mathcal{X}(t, D_0),$$

where $\mathcal{X} : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ is the flow map associated to the velocity field:

$$\begin{cases} \frac{d}{dt}\mathcal{X}(t,x) = u(t,\mathcal{X}(t,x)), \quad (t,x) \in [0,\infty) \times \mathbb{R}^2, \\ \mathcal{X}(0,x) = x, \quad x \in \mathbb{R}^2. \end{cases} \tag{1.2.8}$$

However, the weak solutions provided by P.-L. Lions [175] are in general not regular enough to define the flow map (1.2.8). Indeed, the Lipschitz regularity

$$\nabla u \in L^1_{\text{loc}}([0,\infty); L^\infty(\mathbb{R}^2)) \tag{1.2.9}$$

appears to be necessary to propagate boundary regularity, as even the slightly less regular case $\nabla u \in L^1_{\text{loc}}([0,\infty); BMO(\mathbb{R}^2))$ can produce density patches with time-decreasing regularity index: $\partial D_t \in \mathcal{C}^{\alpha \exp(-Ct)}$ for initially Hölder continuous interfaces $\partial D_0 \in \mathcal{C}^\alpha$ $(\alpha \in (0,1))$ [18, 58, 129]. See also the examples in [48] for a loss of regularity in transport equations with non-Lipschitzean vector fields. On the other hand, the strong solutions for (INS) found in e.g. [14, 46, 157] are too regular to allow for *discontinuous* densities such as indicator functions. A good approach is to construct solutions in a regularity class that is in-between the weak solution framework of P.-L. Lions [175] and the strong solution framework.

Observe that although a density of patch-type is discontinuous across the boundary of the patch, it is smooth in the tangential direction of the boundary. Indeed, if the vector field $\tau = \tau(t,x) \in \mathbb{R}^2$ is tangential to the boundary ∂D_t, then one has

$$\partial_{\tau(t,\cdot)} 1_{D_t} = 0 \tag{1.2.10}$$

in the distribution sense. In particular, densities of patch-type $\rho(t,x) = \rho^+ 1_{D_t}(x) + \rho^- 1_{D_t^c}(x)$, with constants $\rho^+, \rho^- \geq 0$, are smooth in tangential direction:

$$\partial_{\tau(t,\cdot)} \rho(t,\cdot) = 0. \tag{1.2.11}$$

The evolution equation of the tangential vector field τ is derived from the transport equation for the zero level set function of ∂D_t to be

$$\begin{cases} \partial_t \tau + u \cdot \nabla \tau = \partial_\tau u, \\ \tau|_{t=0} = \tau_0. \end{cases} \tag{τ}$$

Repeatedly applying the tangential derivative ∂_τ to the first equation $(\tau)_1$ and noticing the vanishing of the commutator $[\partial_\tau, \partial_t + u \cdot \nabla] = 0$, implies that

$$(\partial_t + u \cdot \nabla u)\partial_\tau^k \tau = \partial_\tau^{k+1} u, \quad \forall k \in \mathbb{N}.$$

Hence, propagating boundary regularity of ∂D_t requires suitable tangential regularity estimates for the velocity. The tangential regularity of the density (1.2.11) plays a crucial role in establishing these estimates.

In the *constant viscosity* case, this strategy was successfully implemented by X. Liao and P. Zhang [166, 167], who proved the persistence of $W^{k+2,p}$ boundary regularity ($p \in (2,4), k \in \mathbb{N}$) of density patches of the form $\rho_0(x) = \rho^+ 1_{D_0}(x) + \rho^- 1_{D_0^c}(x)$ with positive constants $\rho^+, \rho^- > 0$. A similar result holds in three dimensions with small density jump and small initial velocity [165]. Using different methods, the $\mathcal{C}^{k,\alpha}$-regularity ($k \in \{1,2\}, \alpha \in (0,1)$) was shown to be propagated over time in [65] with a small density jump, and in [100] for positive density without size restriction on the jump. In the presence of vacuum, the persistence of $\mathcal{C}^{1,\alpha}$-regularity ($\alpha \in (0,1)$ in dimension two, $\alpha \in (0, \frac{1}{2})$ in dimension three) was shown by R. Danchin and P. B. Mucha [60] and C. Prange and J. Tan [207].

For *variable viscosity*, the propagation of $H^{\frac{5}{2}}(\mathbb{R}^2)$-regularity and $\mathcal{C}^{1,\alpha}(\mathbb{R}^2)$-regularity ($\alpha \in (0,1)$) was proved by M. Paicu and P. Zhang [198] and F. Gancedo and E. Garcia-Juarez [101], respectively, under a *small viscosity jump* assumption. In three space dimensions a result similar to the one in [198] holds under an additional smallness assumption on the initial velocity (see [198, Remark 1.1 (1)]). A key ingredient in the analysis of the above results is the following decomposition of the divergence of the viscous stress tensor:

$$\operatorname{div}(\mu(\rho)Su) = \Delta u + \operatorname{div}((\mu(\rho) - 1)Su), \tag{1.2.12}$$

where the second term on the right hand side is treated as a perturbation if the viscosity variation is small $|\mu(\rho) - 1| \ll 1$. However, to the best of our knowledge the density-patch problem remains open for general viscosity coefficients, which might exhibit *large* jumps.

1.2.1.5. The incompressible Boussinesq equations

The Boussinesq equations (B) have the more general form

$$\begin{cases} \partial_t \vartheta + u \cdot \nabla \vartheta - \operatorname{div}(\kappa(\vartheta)\nabla\vartheta) = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ \partial_t u + u \cdot \nabla u - \operatorname{div}(\mu(\vartheta)Su) + \nabla \pi = \vartheta e_2, \\ \operatorname{div} u = 0. \end{cases} \tag{1.2.13}$$

with the heat conduction coefficient $\kappa(\vartheta)$, where κ is some given, smooth function. It is common to take [203]

- constant heat conductivity law $\kappa(\vartheta) = c_1$ for liquids,
- $\kappa(\vartheta) = c_2 \mu(\vartheta)$ for gases,

with constants $c_1, c_2 \in \mathbb{R}$.

The classical constant coefficient scenario $\kappa(\vartheta) \equiv \tilde{\kappa}$, $\mu(\vartheta) \equiv \tilde{\mu}$, with positive constants $\tilde{\kappa}, \tilde{\mu} > 0$, is known to be globally well-posed, see e.g. [34]. In case of strong heat conduction $\kappa(\vartheta) \geq \kappa_* > 0$, the diffusion term $\operatorname{div}(\kappa(\vartheta)\nabla\vartheta)$ regularizes the temperature ϑ over time, leading to a smooth viscosity coefficient $\mu(\vartheta)$. Consequently, the viscosity term can be rewritten as

$$\operatorname{div}(\mu(\vartheta)Su) = \mu(\vartheta)\Delta u + \nabla\mu(\vartheta) \cdot Su, \tag{1.2.14}$$

where $\nabla\mu(\vartheta) \cdot Su$ is considered as a lower-order term with respect to u. This formulation results in global-in-time well-posedness results, see [121, 122, 180, 238] and references therein.

In the past few years much effort has been made to extend these results to the partial/no dissipation cases

(i) $\kappa(\vartheta) = 0, \mu(\vartheta) \geq \mu_* > 0$ (dissipation but no thermal diffusion),
(ii) $\kappa(\vartheta) \geq \kappa_* > 0, \mu(\vartheta) = 0$ (thermal diffusion but no dissipation),
(iii) $\kappa(\vartheta) = \mu(\vartheta) = 0$ (inviscid Boussinesq equations).

For *constant viscosity and heat conductivity*, the Boussinesq equations (1.2.13) are globally well-posed in the cases (i) and (ii) for regular initial data [35, 132], and locally well-posed in the case (iii), see e.g. [36, 38, 54, 84]. In this thesis we focus on the first of the above cases, (i), which corresponds to the Boussinesq system (B). Similar to the density-patch problem for the Navier-Stokes equations one can consider here the *temperature-patch problem* for the Boussinesq equations without thermal diffusion, where the initial temperature is the indicator function of some bounded, simply connected domain $D_0 \subset \mathbb{R}^2$:

$$\vartheta_0(x) = 1_{D_0}(x),$$

and one can ask for the regularity propagation of the boundary ∂D_0 over time. If the viscosity is *constant*, the propagation of $\mathcal{C}^{1,\alpha}$ boundary regularity of the temperature patch ($\alpha \in (0,1)$) was proven in [64, 99]. However, to the best of our knowledge, the temperature-patch problem has not yet been addressed in the literature for *variable* viscosity coefficients.

For results on more general Boussinesq systems such as the anisotropic Boussinesq equations and the fractional Boussinesq equations, we refer to the review notes [243].

1.2.2. The compressible Navier-Stokes equations

Recall the two-dimensional compressible Navier-Stokes equations with density-dependent viscosity coefficients (CNS):

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ \rho(\partial_t u + u \cdot \nabla u) - \operatorname{div}(\mu(\rho) S u) - \nabla(\lambda(\rho) \operatorname{div} u) + \nabla P(\rho) = 0. \end{cases} \tag{CNS}$$

The local-in-time well-posedness for smooth initial data away from vacuum was proved by J. Nash [193] and A. Tani [231]. See also [146, 147, 224] for the constant viscosity case. The global-in-time well-posedness goes back to A. Matsumura and T. Nishida [186] provided the initial data is small in $H^3(\mathbb{R}^3)$ for constant viscosity and small in $H^4(\mathbb{R}^3)$ for variable viscosity. The regularity assumptions of [186] on the initial data were later relaxed to smallness in the critical Besov space [39, 43, 117] in the constant viscosity case. An interesting result was achieved by A. Kazhikhov and A. Vaigant [235], who proved the global existence and uniqueness of strong solutions of (CNS) on the square $(0,1)^2 \subset \mathbb{R}^2$, for the pressure law $P(\rho) = a\rho^\gamma$ $(a, \gamma \geq 0)$, $\mu > 0$ a constant and λ as a function of the density: $\lambda(\rho) = b\rho^\beta$ for $b > 0, \beta \geq 3$, notably without any smallness assumption.

It is well-known that smooth solutions to the Navier–Stokes equations (CNS) satisfy the following energy balance

$$\begin{aligned} \int_{\mathbb{R}^2} \Big[\rho \frac{|u|^2}{2} + H(\rho)\Big](t,x)dx + \int_0^t \int_{\mathbb{R}^2} \Big[\frac{\mu(\rho)}{2}|Su|^2 + \lambda(\rho)(\operatorname{div} u)^2\Big](t',x)dxdt' \\ = \int_{\mathbb{R}^2} \Big[\rho \frac{|u|^2}{2} + H(\rho)\Big](0,x)dx, \end{aligned} \tag{1.2.15}$$

for all $t > 0$, where $H(\rho)$ is the classical potential energy

$$H(\rho) = \rho \int_{\tilde{\rho}}^{\rho} \frac{P(s) - P(\tilde{\rho})}{s^2} ds,$$

and $\tilde{\rho}$ is some constant reference density. The existence of global-in-time finite-energy weak solutions to the system (CNS) similar to P.-L. Lion's weak solutions [175] for the incompressible model (INS), is still a widely open problem. For *constant viscosity*, global weak solutions of (CNS) were obtained by P.-L. Lions [176] and E. Feireisl, A. Novotný and H. Petzeltová [93] with the pressure law $P(\rho) = a\rho^\gamma$ ($a > 0$, $\gamma > \frac{d}{2}$). The construction is based on the energy balance (1.2.15), and the weak solutions obtained in [93, 176] are known to satisfy the energy *in*equality ((1.2.15) with $=$ replaced by $\leq$). For a certain class of density-dependent viscosity coefficients and with some Sobolev regularity assumption on the density, the existence of weak solutions in the spirit of P.-L. Lions [175] was established in [28]. In [27, 31] the existence of global-in-time weak solutions was proven for more general stress tensors than [93, 176], which include some anisotropic fluids. However, the existence of weak solutions to (CNS) for general density-dependent viscosity coefficients similar to the incompressible weak solutions [175] remains unresolved. It is also still completely open whether the global weak solutions of (CNS) with constant viscosity obtained in [93, 176] are unique; see [105] for some weak-strong uniqueness results.

Since the mid-90s there has been growing interest in solutions that emanate from *discontinuous* initial densities, and tracking these discontinuities over time. For *constant viscosity*, remarkable contributions in this direction were made by D. Hoff [125, 127, 128], who considered initial densities which are piecewise Hölder continuous on both sides of some suitable curve. In two space dimensions he proved the propagation of $\mathcal{C}^{1,\alpha}$-regularity of the curve ($\alpha \in (0,1)$), and that the density also remains $\mathcal{C}^\alpha$ on both sides of the transported curve. Moreover, its jump decays exponentially in time. The key concepts in D. Hoff's analysis [125, 127, 128] can be summarized as follows:

- *(Energy functionals).* In his pioneer work [127], D. Hoff introduced the following energy functionals

$$A_1(t) = \sup_{[0,t]} \sigma \|\nabla u\|_{L^2}^2 + \int_0^t \sigma \|\sqrt{\rho}\dot{u}\|_{L^2}^2 dt', \quad A_2(t) = \sup_{[0,t]} \sigma^2 \|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \int_0^t \sigma^2 \|\nabla \dot{u}\|_{L^2}^2 dt', \tag{1.2.16}$$

  with the material derivative of the velocity $\dot{u} = (\partial_t + u \cdot \nabla)u$ and the time weight $\sigma(t) = \min\{1, t\}$. Notice that these functionals appear naturally by taking the L^2-inner product between the momentum equation $(\text{CNS})_2$ and $\dot{u}$ resp. $\ddot{u} = (\partial_t + u \cdot \nabla)\dot{u}$. He establishes bounds for the energy functionals (1.2.16) provided that the L^2-norm of the initial velocity is small and the density is bounded away from zero and from above, along with some technical assumptions. With the boundedness of these energy functionals one in particular has $\rho\dot{u} \in L^p(\mathbb{R}^2)$ for any $p \in [2, \infty)$.
- *(Vorticity and effective flux).* He observed that the vorticity and effective flux

$$\omega = \nabla^\perp \cdot u = -\partial_2 u_1 + \partial_1 u_2, \quad F = (2\mu + \lambda)\text{div}\, u - P(\rho) + \tilde{P},$$

  satisfy the elliptic equations

$$\mu\Delta\omega = \nabla^\perp \cdot (\rho\dot{u}), \quad \Delta F = \text{div}\,(\rho\dot{u}), \tag{1.2.17}$$

  which are obtain after an application of the curl operator $\nabla^\perp \cdot$ resp. the divergence div to the momentum equation $(\text{CNS})_2$.

The effective flux was first identified in [130], and it plays a crucial role in the mathematical analysis of the compressible Navier–Stokes model with constant viscosity coefficients. For instance, it serves as a key tool in the study of the propagation of density oscillations (see [218]) and in the construction of finite-energy weak solutions in [93, 175]. In the framework of D. Hoff [127], one has $\rho\dot{u} \in L^p(\mathbb{R}^2)$ for any $p \in [2, \infty)$ by use of the energy functionals (1.2.16), which implies, thanks to (1.2.17), that both F and ω are Hölder continuous for positive times, even if the initial data are discontinuous.

- *(Velocity gradient decomposition).* Propagating the piece-wise Hölder continuity of the density and the regularity of the curve of discontinuity requires the Lipschitz regularity of the velocity field (1.2.9). To this end, D. Hoff decomposes the velocity gradient into two parts; a regular part, which is associated to the effective flux and vorticity, and another part, which is associated to the pressure $P(\rho) - \tilde{P}$, and which is less regular. Denoting by $\mathbb{P}$ the Leray projector, the velocity gradient can be expressed as

$$\begin{aligned}\nabla u &= \nabla \mathbb{P} u + \nabla(\mathrm{Id} - \mathbb{P})u \\ &= \nabla \mathbb{P} u - \nabla(-\Delta)^{-1}\nabla \mathrm{div}\, u \\ &= \nabla \mathbb{P} u - \frac{1}{2\mu + \lambda}\nabla(-\Delta)^{-1}\nabla F - \frac{1}{2\mu+\lambda}\nabla(-\Delta)^{-1}\nabla(P(\rho) - \tilde{P}). \end{aligned} \tag{1.2.18}$$

The first two terms in this decomposition are Hölder continuous since from (1.2.17) it follows that both $\nabla^2 \mathbb{P} u$ and ∇F belong to $L^p(\mathbb{R}^2)$, $p \in [2, \infty)$. The third term is less regular. To prove its boundedness, D. Hoff first propagates the piecewise Hölder regularity of the density with the help of the regularity of the effective flux F and the following reformulation of the mass equation $(\mathrm{CNS})_1$:

$$\partial_t \log \rho + u \cdot \nabla \log \rho + \frac{P(\rho) - \tilde{P}}{2\mu + \lambda} = -\frac{F}{2\mu + \lambda},$$

where the pressure term on the left hand side acts as a damping term. Then the piecewise Hölder regularity of the density provides the piecewise Hölder regularity of the third term in (1.2.18).

The proof of uniqueness of D. Hoff's solutions constructed in [125, 127, 128] was given by R. Danchin, F. Fanelli, and M. Paicu [55]. The small density jump was removed recently by X. Liao and S. M. Zodji [169], who proved the $W^{2,p}$-regularity propagation ($p \in (2, \infty)$ in dimension two, $p \in (3, 6)$ in dimension three) of the curve provided the bulk viscosity is sufficiently large, $\nu \gg 1$, and if $d = 3$, additionally the initial energy is small.

In the *variable viscosity* scenario, the global well-posedness with piecewise Hölder continuous density was proved recently by S. M. Zodji [252] with *small viscosity variation* $|\mu(\rho) - 1| \ll 1$. The result basically relies on a combination of the decompositions (1.2.12) and (1.2.18) of the divergence of the viscous stress tensor and velocity gradient, respectively. However, the *large* viscosity variation case is still unresolved.

1.2.3. Related fluid models

In this subsection we present two further models which play an important role in fluid mechanics. These will not be studied in this thesis, but they are related to the models from Section 1.1 in that they both deal with sharp interfaces.

1.2.3.1. The two-phase Navier-Stokes equations

Free boundary problems arise in numerous physical, chemical and biological processes, such as the melting/solidifying of substances or the mixing of fluids. They usually consist of (a system of) partial differential equations posed on a domain which is one of the unknowns of the problem and which may evolve over time. The density-patch problem for (INS), (CNS) and temperature-patch problem for (B) can be considered free boundary problems due to the time-evolving interface between the patch and the outer fluid.

Here we briefly discuss the two-phase Navier-Stokes equations, which is one type of free boundary problem. Other types of intensively studied free boundary problems include for example the free boundary problem for the Navier-Stokes equations [112, 113, 221], water waves [159, 244] and vortex-sheets [20].

The two-phase Navier-Stokes equations describe the motion of two immiscible fluids and the interface between them, such as a drop of one fluid inside another, or two fluids on top of each other. Let us consider two fluids occupying the regions Ω_t^+ and Ω_t^-, respectively, at time $t \geq 0$, which are separated by the interface Γ_t such that $\mathbb{R}^2 = \Omega_t^+ \cup \Omega_t^- \cup \Gamma_t$. The fluids have positive, constant density and viscosity ρ^+, ν^+ and ρ^-, ν^-, respectively. As before, let $u = u(t,x) \in \mathbb{R}^2$ and $\pi = \pi(t,x) \in \mathbb{R}$ denote the fluid velocity and pressure inside the fluid domain. Then the fluid moves in accordance to the following two-phase Navier-Stokes equations [209]

$$\begin{cases} \partial_t u + u \cdot \nabla u - \nu^\pm \Delta u + \frac{1}{\rho^\pm} \nabla \pi = 0 & \text{in } \Omega_t^\pm,\ t > 0, \\ \operatorname{div} u = 0 & \text{in } \Omega_t^\pm,\ t > 0, \\ [\![u]\!] = 0 & \text{on } \Gamma_t,\ t > 0, \\ [\![T^\pm n]\!] = \sigma H n & \text{on } \Gamma_t,\ t > 0, \\ V = u \cdot n & \text{on } \Gamma_t,\ t > 0. \end{cases} \tag{1.2.19}$$

In the above, V, n, H denote the normal velocity, outer unit normal, and mean curvature of the boundary Γ_t, respectively, and $\sigma \geq 0$ is the surface tension coefficient. The stress tensor is $T^\pm = \nu^\pm \rho^\pm S u - \pi \mathrm{Id}$, and $[\![\cdot]\!]$ denotes the jump of a quantity across the free interface Γ_t. The dynamical boundary condition $(1.2.19)_4$ states that passing through the interface produces a discontinuity in the normal stress proportional to the mean curvature of the boundary. The fifth line $(1.2.19)_5$ expresses that the free boundary Γ_t consists of the same fluid particles for all times.

Observe that if surface tension is absent, $\sigma = 0$, then solutions of the density-patch problem for the Navier-Stokes equations (INS) are solutions of (1.2.19). Indeed, if $(\rho, u, \nabla\pi)$ solves (INS) with initial density $\rho_0 = \rho^+ 1_{\Omega_0^+} + \rho^- 1_{\Omega_0^-}$, then it in particular verifies $(\text{INS})_2$ in $\Omega_t^+ \cup \Omega_t^-$, and hence $(1.2.19)_1$ with $\nu^+ = \frac{\mu(\rho^+)}{\rho^+}$, $\nu^- = \frac{\mu(\rho^-)}{\rho^-}$, provided that both the vectors u and $T^\pm n$ are continuous across the freely transported interface Γ_t (as long as Γ_t remains well-defined).

The two-phase Navier-Stokes equations (1.2.19) have been thoroughly studied since the 1980s in various configurations of Ω_t^+ and Ω_t^-; see the books [73, 209] for a comprehensive overview. In the presence of surface tension, $\sigma > 0$, well-posedness results have been achieved locally in time [70, 71, 208] and globally in time under some smallness assumptions [72, 209, 223, 229]. See [5] for the global-in-time existence of varifold solutions with rather general initial data. Without surface tension, $\sigma = 0$, the existence of global-in-time weak solutions was shown in [196, 228] and the global-in-time well-posedness holds under some smallness assumption [68, 69, 213].

One of the main challenges in solving (1.2.19) is the fact that the interface is *unknown* and *changing over time*. The usual strategy is therefore to transform the equations to a domain with a fixed interface. The transformations most commonly used in the literature are the Langrangian transform (1.2.8), or some geometric map such as a "flattening function" (in the case of an infinite ocean of finite depth) or the so-called Hanzawa transform (see e.g. [209]). One then obtains a quasilinear parabolic system posed on a fixed domain with nonlinear boundary conditions, which is solved via fixed point iteration.

1.2.3.2. The vortex-patch problem for the Euler equations

If internal friction in the fluid is negligible, it is common to consider the fluid *inviscid*. The fluid motion is then described by the Euler equations, which were derived by L. Euler in 1757 [87]. For an incompressible fluid in two space dimensions these equations read as [158]

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla \pi = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ \operatorname{div} u = 0. \end{cases} \tag{1.2.20}$$

There is a large body of literature dedicated to the above equations. The local-in-time well-posedness goes back to L. Lichtenstein [170–173]. The global existence and uniqueness of solutions with bounded vorticity

$$\omega = \nabla^\perp \cdot u = -\partial_2 u_1 + \partial_1 u_2$$

is a classical result by V. I. Yudovich [247]. In fact, in two space dimensions, the vorticity is freely transported by the fluid velocity

$$\partial_t \omega + u \cdot \nabla \omega = 0,$$

so that in particular (by incompressibility) the vorticity remains bounded for all times if it is initially bounded:

$$\|\omega(t)\|_{L^\infty} \le \|\omega_0\|_{L^\infty}, \quad t \ge 0.$$

This, together with the *Biot-Savart law*

$$u(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(t,y)dy, \quad (t,x) \in [0,\infty) \times \mathbb{R}^2,$$

which allows the velocity to be reconstructed from its vorticity, implies the log-Lipschitz continuity of the velocity, and hence, the existence of a unique, continuous flow map $\mathcal{X}$ as defined in (1.2.8) (see [18]).

The *vortex-patch problem* (see A. Majda [185]) considers the following scenario: suppose the initial vorticity is the indicator function $\omega_0 = 1_{D_0}$ of some bounded, simply connected domain $D_0 \subset \mathbb{R}^2$. By the preceding arguments, the vorticity is at all times an indicator function

$$\omega(t) = 1_{D_t}, \quad \text{with } D_t = \mathcal{X}(t, D_0).$$

The question of the vortex-patch problem is *whether the boundary regularity of the initial domain ∂D_0 is propagated over time?*

J.-Y. Chemin's celebrated works [40, 41] confirm this regularity propagation, by use of tangential regularity methods and a nondegenerate family of vector fields. The regularity propagation requires an a priori Lipschitz-bound on the velocity (1.2.9): $\nabla u \in L^1_{\mathrm{loc}}([0,\infty); L^\infty(\mathbb{R}^2))$. Indeed, as discussed above for density patches, the violation of this Lipschitz regularity can result in a loss of boundary regularity for the vortex patch. By the Biot-Savart law, the velocity gradient can be expressed in terms of the double Riesz transform of the vorticity

$$\nabla u = \mathcal{R}\mathcal{R}^{\perp}\omega, \quad \text{with } \mathcal{R} = \frac{\frac{1}{i}\nabla}{\sqrt{-\Delta}}.$$

However, this does not immediately yield an L^{∞}-bound on the velocity gradient, since the double Riesz transform is not bounded in $L^{\infty}(\mathbb{R}^2)$, but maps the space $L^{\infty}(\mathbb{R}^2)$ into $BMO(\mathbb{R}^2)$. In [41, 42] J.-Y. Chemin shows that tangential regularity (1.2.10) can help to establish such an L^{∞}-bound for the double Riesz-transform, by proving the following Lipschitz estimate for the velocity field with a logarithm growth in the tangential regularity of ω with respect to the vector field τ:

$$\|\nabla u\|_{L^\infty} \lesssim \|\omega\|_{L^p} + \|\omega\|_{L^\infty} \log\Big(e + \|\frac{1}{|\tau|}\|_{L^\infty} \frac{\|\omega\|_{L^\infty}\|\tau\|_{\mathcal{C}^\alpha} + \|\mathrm{div}\,(\tau\omega)\|_{\mathcal{C}^{\alpha-1}}}{\|\omega\|_{L^\infty}}\Big) \tag{1.2.21}$$

for $p \in [1, \infty)$ and $\alpha \in (0, 1)$. This inequality comes essentially from the analysis of the elliptic equation

$$\Delta\phi = \omega, \quad \text{where } u = \nabla^{\perp}\phi,$$

which, together with the tangential regularity of ω in τ-direction, allows one to recover the boundedness of $\partial_{ij}\phi$ in *all* directions $i, j \in \{1, 2\}$. See [18] for more details.

A more geometric viewpoint of the vortex-patch problem was employed in A. L. Bertozzi and P. Constantin's work [22]. A thorough review of results on the two-dimensional vortex-patch problem can be found in [103]. See also [98] for the problem in three space dimensions and [88] for the inhomogeneous case.

1.3. Main results

We now present the main results of this thesis. This section and the remaining chapters are structured as follows:

- Subsection 1.3.1 and Chapter 2 focus on the inhomogeneous incompressible Navier-Stokes equations with odd viscosity in the evolutionary (oddINS) and stationary (oddINS') cases. The global-in-time existence of finite-energy weak solutions in the spirit of P.-L. Lions [175] and Z. He and X. Liao [123] is established. We also investigate the limit of these weak solutions as the odd viscosity tends to some constant $\mu_o \to c$, and we consider examples of stationary parallel, concentric and radial flows.
- Subsection 1.3.2 and Chapter 3 are devoted to the inhomogeneous incompressible Navier-Stokes equations with density-dependent viscosity (INS). We establish a global-in-time well-posedness result that allows for discontinuous viscosity coefficients without size restriction on the jumps. This well-posedness result can be applied to the density-patch problem to prove the regularity propagation of sharp interfaces between immiscible fluids.

  Furthermore, a local-in-time well-posedness result for the two-dimensional Boussinesq equations with temperature-dependent viscosity (B) is also presented and proved, with possibly large viscosity jumps. A lower bound on the existence time is established. As an application of this result, one can prove the persistence of regularity of temperature patches locally in time.
- Subsection 1.3.3 and Chapter 4 address the compressible Navier-Stokes equations with density-dependent viscosity (CNS). We show that these equations are globally well-posed allowing for discontinuous viscosity coefficients with possibly large jumps. The regularity propagation of density-patches follows as a consequence of this result.

1.3.1. Existence of weak solutions to the incompressible Navier-Stokes equations with variable, odd viscosity

We consider the two-dimensional inhomogeneous incompressible Navier-Stokes equations with odd viscosity (oddINS) under the influence of an external force f:

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \quad (t,x) \in (0,\infty) \times \Omega, \\ \rho(\partial_t u + u \cdot \nabla u) - \operatorname{div}(\mu(\rho) S u) - \operatorname{div}(\mu_o(\rho) S_o u) + \nabla \pi = \rho f, \\ \operatorname{div} u = 0, \end{cases} \tag{oddINS}$$

and its stationary counterpart to (oddINS):

$$\begin{cases} \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho) S u) - \operatorname{div}(\mu_o(\rho) S_o u) + \nabla \pi = f, \quad x \in \Omega \\ \operatorname{div}(\rho u) = 0, \quad \operatorname{div} u = 0, \end{cases} \tag{oddINS'}$$

with given functions

$$\mu, \mu_o \in \mathcal{C}([0,\infty); \mathbb{R}), \quad \mu_* \le \mu \le \mu^*, \quad -\mu^* \le \mu_o \le \mu^*, \tag{1.3.1}$$

for some positive constants $0 < \mu_* \le \mu^*$.

Despite the large number of works dedicated to the incompressible inhomogeneous Navier-Stokes equations (INS), the presence of *odd viscosity* in these equations (i.e. (oddINS)) has received little attention from mathematicians until now. In particular, the existence of finite energy weak solutions in the spirit of P.-L. Lions [175] are yet unknown. Our first objective is to establish the existence of such weak solutions, both in the evolutionary (oddINS) and the stationary (oddINS') cases. The first result of this thesis can be summarized as follows.

Theorem 1. 1. *(Evolutionary case). Let $\Omega \subset \mathbb{R}^2$ be either a bounded, connected Lipschitz domain, a rectangle $(0, L_1) \times (0 \times L_2)$, where $L_1, L_2 > 0$, or the whole plane.*

a) *(Existence). Given an external force $f \in L^2((0,\infty) \times \Omega; \mathbb{R}^2)$, and initial density and velocity $(\rho_0, u_0) \in L^\infty(\mathbb{R}^2; [\rho_*, \rho^*]) \times L^2(\mathbb{R}^2; \mathbb{R}^2)$, with positive constants $0 < \rho_* \le \rho^*$, there exists at least one weak solution $(\rho, u, \nabla \pi)$ of (oddINS). This weak solution satisfies the energy inequality*

$$\int_\Omega \rho |u|^2 \, dx + \int_0^t \int_\Omega \mu(\rho) |\nabla u + \nabla^T u|^2 \, dx dt' \le \int_\Omega \rho_0 |u_0|^2 \, dx + 2 \int_0^t \int_\Omega \rho f \cdot u \, dx dt', \tag{1.3.2}$$

for almost every $t > 0$.

b) *(Convergence to (INS)). Let a sequence $(\mu_o^\epsilon)_{\epsilon \in (0,1)}$ of functions in $\mathcal{C}(\mathbb{R}; [-\mu^*, \mu^*])$ be given, such that*

$$\|\mu_o^\epsilon - \nu_o\|_{\mathcal{C}([\rho_*, \rho^*])} \to 0, \quad \epsilon \to 0, \tag{1.3.3}$$

for some constant $\nu_o \in [-\mu^*, \mu^*]$. For each $\epsilon \in (0,1)$ let $(\rho^\epsilon, u^\epsilon)$ denote a weak solution of (oddINS) with odd viscosity coefficient μ_o^ϵ which satisfies the energy inequality (1.3.2). Then there exists a function pair (ρ, u) such that up to a subsequence

$$\begin{aligned} &\rho^\epsilon \to \rho, \quad \text{in } \mathcal{C}([0,T]; L^p(\Omega \cap B_R(0))), \quad \forall p \in [1,\infty), \forall T, R > 0, \\ &u^\epsilon \rightharpoonup^* u, \quad \text{in } L^\infty((0,T); L^2(\Omega)) \cap L^2((0,T); H^1(\Omega)), \quad \forall T > 0, \end{aligned}$$

as $\epsilon \to 0$, and (ρ, u) is a weak solution of (INS).

2. *(Stationary case).*
   a) *(Existence). Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected $\mathcal{C}^{1,1}$ domain. Given an external force $f \in H^{-1}(\Omega;\mathbb{R}^2)$ and boundary value $g \in H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^2)$ satisfying the zero flux condition* (1.2.4)*: $\int_{\partial\Omega} g \cdot n \, d\sigma = 0$, there exists at least one weak solution $(\rho, u) \in L^\infty(\Omega;[0,\infty)) \times H^1(\Omega;\mathbb{R}^2)$ of* (oddINS')-(1.2.3) *which is of Frolov's form* (1.2.5).
   If Ω is the exterior domain of a bounded simply connected $\mathcal{C}^1$ domain in $\mathbb{R}^2$, or $\Omega = \mathbb{R}^2$, then there exists a weak solution $(\rho, u) \in L^\infty(\Omega;[0,\infty)) \times (D^1(\Omega))^2$ of (oddINS') *(and* (1.2.3) *if $\Omega \neq \mathbb{R}^2$), where*

$$D^1(\Omega) = \dot{H}^1(\Omega) \cap \Big(\cap_{n\in\mathbb{N}} H^1(\Omega \cap B_n)\Big).$$

   b) *(Convergence to* (INS')*). Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected $\mathcal{C}^{1,1}$ domain. Suppose the sequence $(\mu_o^\epsilon)_{\epsilon\in(0,1)}$ is given as in 1b) above, and $(\rho^\epsilon, u^\epsilon)$ is any weak solution of* (oddINS') *of Frolov's form, i.e. $(\rho^\epsilon, u^\epsilon) = (\eta^\epsilon(\phi^\epsilon), \nabla^\perp\phi^\epsilon)$ for some functions $\phi^\epsilon \in H^2(\Omega)$ and $\eta^\epsilon \in L^\infty(\mathbb{R};[0,\rho^*])$. Then there exists a function pair (ρ, u) such that up to a subsequence*

$$\rho^\epsilon \rightharpoonup \rho, \quad \text{in } L^p(\Omega), \quad \forall p \in (1,\infty),$$
$$u^\epsilon \rightharpoonup u, \quad \text{in } H^1(\Omega),$$

   as $\epsilon \to 0$, and (ρ, u) is a weak solution of (INS').

The definitions of weak solutions and the precise statements are given below in Chapter 2 in Definitions 2.2.1, 2.2.5 and Theorems 2.2.2, 2.2.6, respectively. The existence proofs rely on the observation that odd viscosity does not affect the energy balance (1.2.1) for smooth solutions due to the cancellation $S_o u : Su = 0$.

1.3.2. Regularity propagation for the incompressible Navier-Stokes equations with variable viscosity

1.3.2.1. Variable-density case

Recall the two-dimensional inhomogeneous incompressible Navier-Stokes equations with density-dependent viscosity (INS):

$$\begin{cases} \partial_t\rho + u \cdot \nabla\rho = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ \rho(\partial_t u + u \cdot \nabla u) - \operatorname{div}(\mu(\rho) Su) + \nabla\pi = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{INS}$$

Motivated by P.-L. Lions' density-patch problem, we are going to study the well-posedness of (INS) in a functional framework that allows for *discontinuous* densities and viscosities. Here, we assume the initial density ρ_0 to satisfy

$$\begin{aligned} &\partial_{\tau_0}\rho_0 \in L^{p_0}(\mathbb{R}^2) \ \text{ for some suitable } p_0 > 2, \\ \text{with} \quad &\tau_0 \in L^\infty \cap \dot{W}^{1,p_0}(\mathbb{R}^2;\mathbb{R}^2) \ \text{ being a non-degenerate vector field.} \end{aligned} \tag{1.3.4}$$

This assumption is inspired by the following considerations:

- *(Tangential regularity of density patches).* Assumption (1.3.4) is clearly satisfied for densities of patch-type, keeping in mind the tangential regularity (1.2.11), if the initial boundary ∂D_0 has W^{2,p_0}-regularity and the vector field τ_0 is chosen to be the tangent vector of ∂D_0.
- *(Failure of Lipschitz regularity).* In general, some regularity of the viscosity $\mu(\rho)$ is required to guarantee the Lipschitz regularity (1.2.9), as the boundedness of the viscosity $0 < \mu_* \le \mu(\rho) \le \mu^*$ alone is not sufficient; see the examples for the stationary Navier-Stokes system (INS') in [123]. The tangential regularity of the density (1.3.4) gives the initial viscosity the same regularity $\partial_{\tau_0}\mu(\rho_0) = \mu'(\rho_0)\partial_{\tau_0}\rho_0 \in L^{p_0}(\mathbb{R}^2)$, if $\|\mu'\|_{L^\infty([0,\infty))} < \infty$, which will help us achieve the a priori Lipschitz estimate (1.2.9).

Recall the evolution equation of the tangent vector field (τ):

$$\begin{cases} \partial_t \tau + u \cdot \nabla \tau = \partial_\tau u, \\ \tau|_{t=0} = \tau_0. \end{cases} \tag{τ}$$

The second main result of this thesis can be summarized as follows (see Theorem 3.1.9 in Chapter 3). Its proof can be found in Chapter 3.

Theorem 2. *Given positive upper and lower viscosity bounds $0 < \mu_* \le \mu^*$ there exists $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0]$ the following holds true.*
Let $\rho_0 \in L^\infty(\mathbb{R}^2; [\rho_*, \rho^*])$, $0 < \rho_* \le \rho^*$, be an initial density satisfying $\rho_0 - \tilde\rho \in L^2(\mathbb{R}^2)$ for some $\tilde\rho > 0$. Assume the dependence of the viscosity coefficient on the density function ρ to be $\mu(\rho)$ for some $\mu \in W^{1,\infty}([\rho_*, \rho^*]; [\mu_*, \mu^*])$. Let $u_0 \in H^1 \cap \dot H^{-1}(\mathbb{R}^2; \mathbb{R}^2)$ be divergence-free and $\tau_0 \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$ such that $|\tau_0|^{-1} \in L^\infty(\mathbb{R}^2)$ and $(\nabla\tau_0, \partial_{\tau_0}\rho_0) \in L^{2+\epsilon}(\mathbb{R}^2; \mathbb{R}^{2\times 2+1})$ in the sense of distributions. If (with $\bar\tau_0 = \frac{\tau_0}{|\tau_0|}$)

$$e^{c_1(\|u_0\|^2_{L^2(\mathbb{R}^2)} + \|\rho_0 - 1\|^2_{L^2(\mathbb{R}^2)}\|\nabla u_0\|^2_{L^2(\mathbb{R}^2)})\exp(c_1\|u_0\|^2_{L^2(\mathbb{R}^2)})} \cdot \Big(\|u_0\|_{L^2(\mathbb{R}^2)} + \|\rho_0 - 1\|_{L^2(\mathbb{R}^2)}\|\nabla u_0\|_{L^2(\mathbb{R}^2)}\Big)^{\frac{\epsilon}{2}}$$
$$\cdot\Big(\|u_0\|_{\dot H^{-1}(\mathbb{R}^2)} + \|\rho_0 - 1\|_{L^2(\mathbb{R}^2)}\|u_0\|_{L^2(\mathbb{R}^2)}\Big) \cdot \Big(\|\nabla u_0\|_{L^2(\mathbb{R}^2)} + \|(\nabla\bar\tau_0, \partial_{\bar\tau_0}\mu_0)\|^{\frac{2+\epsilon}{\epsilon}}_{L^{2+\epsilon}(\mathbb{R}^2)}\Big) \le c_2, \tag{1.3.5}$$

where c_1, c_2 are positive constants depending only on $\rho_*, \rho^*, \mu_*, \mu^*$, ϵ and $\|\mu'\|_{L^\infty([\rho_*,\rho^*])}$, then the system (INS)-(τ) *supplemented with the initial data (ρ_0, u_0, τ_0) has a unique global-in-time solution $(\rho, u, \nabla\pi, \tau)$ such that*

$$\begin{aligned} &\rho \in L^\infty((0,\infty)\times\mathbb{R}^2; [\rho_*, \rho^*]), \quad \rho - \tilde\rho \in \mathcal{C}_b([0,\infty); L^q(\mathbb{R}^2)), \ \forall q \in [2,\infty), \\ &u \in \mathcal{C}_b([0,\infty); L^2(\mathbb{R}^2;\mathbb{R}^2)) \cap L^2((0,\infty); \dot H^1(\mathbb{R}^2;\mathbb{R}^2)), \\ &\nabla u \in \mathcal{C}_b([0,\infty); L^2(\mathbb{R}^2;\mathbb{R}^{2\times2})) \cap L^1((0,\infty); L^{2+\epsilon}\cap L^\infty(\mathbb{R}^2;\mathbb{R}^{2\times2})), \\ &\tau \in \mathcal{C}_b([0,\infty); L^\infty \cap \dot W^{1,2+\epsilon}(\mathbb{R}^2;\mathbb{R}^2)), \quad |\tau|^{-1} \in L^\infty((0,\infty)\times\mathbb{R}^2), \\ &\partial_\tau\rho \in L^\infty((0,\infty); L^{2+\epsilon}(\mathbb{R}^2)), \quad \partial_\tau\nabla u, \nabla\partial_\tau u \in L^1((0,\infty); L^{2+\epsilon}(\mathbb{R}^{2\times2})). \end{aligned}$$

In particular, if the initial density is of the patch-type

$$\rho_0(x) = \rho_0^+(x)1_{D_0}(x) + \rho_0^-(x)1_{D_0^c}(x),$$

for some bounded, simply connected domain $D_0 \subset \mathbb{R}^2$ with $W^{2,2+\epsilon}$-boundary, and functions $\rho_0^+ \in W^{1,2+\epsilon}(\overline{D_0})$, $\rho_0^- - \tilde\rho \in L^2 \cap W^{1,2+\epsilon}(\overline{D_0}^c)$, then there exists a nondegenerate vector field $\tau_0 \in L^\infty \cap \dot W^{1,2+\epsilon}(\mathbb{R}^2;\mathbb{R}^2)$ tangential to the boundary ∂D_0. If the smallness assumption (1.3.5) *is satisfied, then the unique solution above preserves the patch structure for all times $t > 0$,*

$$\rho(t,x) = \rho^+(t,x)1_{D_t}(x) + \rho^-(t,x)1_{D_t^c}(x),$$

for some bounded, simply connected domain $D_t \subset \mathbb{R}^2$ *with* $W^{2,2+\epsilon}$*-boundary, and functions* $\rho^+(t,\cdot) \in W^{1,2+\epsilon}(\overline{D_t})$, $\rho^-(t,\cdot) - \tilde{\rho} \in L^2 \cap W^{1,2+\epsilon}(\overline{D_t}^c)$*. Thus, the density-patch-type problem in the absence of vacuum for the density-dependent incompressible Navier-Stokes equations* (INS) *is uniquely globally-in-time solvable under the smallness assumption* (1.3.5)*. This solution solves also the two-phase Navier-Stokes equations* (1.2.19) *without surface tension (*$\sigma = 0$*) with* $\Omega_t^+ = D_t$, $\Omega_t^- = \overline{D_t}^c$ *and the interface* $\Gamma_t = \partial D_t$*.*

1.3.2.2. Variable-temperature case

Recall the two-dimensional Boussinesq equations without thermal conduction and with temperature-dependent viscosity (B):

$$\begin{cases} \partial_t \vartheta + u \cdot \nabla \vartheta = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ \partial_t u + u \cdot \nabla u - \operatorname{div}(\mu(\vartheta) S u) + \nabla \pi = \vartheta e_2, \\ \operatorname{div} u = 0. \end{cases} \tag{B}$$

To the best of our knowledge, the temperature-patch problem has not yet been addressed in the literature for *variable* viscosity coefficients. As a step towards an answer to this problem we establish the following local-in-time well-posedness result for (B), which is the third main result of this thesis. In the following we give a condensed version of this result and refer to Theorem 3.1.7 in Chapter 3 for the full statement.

Theorem 3. *Given positive upper and lower viscosity bounds* $0 < \mu_* \leq \mu^*$ *there exists* ϵ_0 *such that for* $\epsilon \in (0, \epsilon_0]$ *the following holds true.*
Let $u_0 \in H^1(\mathbb{R}^2;\mathbb{R}^2)$ *be a divergence-free vector field and* $\vartheta_0 \in L^1 \cap L^r(\mathbb{R}^2)$ *for some* $r \in [2+\epsilon, \infty]$*. Assume the dependence of the viscosity coefficient on the temperature function* ϑ *to be* $\mu(\vartheta)$ *for some* $\mu \in \mathcal{C}(\mathbb{R}; [\mu_*, \mu^*])$*. Let* $\tau_0 \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ *be a vector field such that* $|\tau_0|^{-1} \in L^\infty(\mathbb{R}^2)$ *and* $(\nabla \tau_0, \partial_{\tau_0}\mu(\vartheta_0)) \in L^{2+\epsilon}(\mathbb{R}^2;\mathbb{R}^{2\times 2+1})$*.*
Then the system (B)-(τ) *supplemented with the initial data* $(\vartheta_0, u_0, \tau_0)$ *has a unique solution* $(\vartheta, u, \nabla\pi, \tau)$ *on the time interval* $[0,T]$*, with the existence time* $T > 0$ *bounded from below as follows (with* $\bar{\tau}_0 = \frac{\tau_0}{|\tau_0|}$*)*

$$\begin{aligned} \max_{q \in \{1, 2+\epsilon\}} &\Big(\|u_0\|_{L^2(\mathbb{R}^2)} + T^{\frac{3}{2}-\frac{1}{q}} \|\vartheta_0\|_{L^q(\mathbb{R}^2)} \Big)^{\frac{2\epsilon}{(2+\epsilon)^2}} \Big(T^{\frac{1}{2}} \|\nabla u_0\|_{L^2(\mathbb{R}^2)} + T^{\frac{\epsilon}{2+\epsilon}} \|(\nabla \bar{\tau}_0, \partial_{\bar{\tau}_0}\mu_0)\|_{L^{2+\epsilon}(\mathbb{R}^2)} \\ &+ T^{\frac{3}{2}-\frac{1}{q}} \|\vartheta_0\|_{L^q(\mathbb{R}^2)} \big(1 + \|u_0\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} + (T^{\frac{3}{2}-\frac{1}{q}} \|\vartheta_0\|_{L^q(\mathbb{R}^2)})^{\frac{1}{2}}\big)\Big)^{\frac{\epsilon^2+2\epsilon+4}{(2+\epsilon)^2}} \leq c_3, \end{aligned} \tag{1.3.6}$$

where $c_3 > 0$ *depends only on* μ_*, μ^* *and* ϵ*. The solution satisfies*

$$\begin{aligned} &\vartheta \in \mathcal{C}([0,T]; \cap_{1 \leq \tilde{r} \leq r, \tilde{r} < \infty} L^{\tilde{r}}(\mathbb{R}^2)) \cap L^\infty((0,T); L^1 \cap L^r(\mathbb{R}^2)) \\ &u \in \mathcal{C}([0,T]; L^2(\mathbb{R}^2;\mathbb{R}^2)) \cap L^2((0,T); \dot{H}^1(\mathbb{R}^2;\mathbb{R}^2)), \\ &\nabla u \in \mathcal{C}([0,T]; L^2(\mathbb{R}^2;\mathbb{R}^{2\times 2})) \cap L^1((0,T); L^\infty(\mathbb{R}^2;\mathbb{R}^{2\times 2})), \\ &\tau \in L^\infty((0,T); L^\infty \cap \dot{W}^{1,2+\epsilon}(\mathbb{R}^2;\mathbb{R}^2)), \quad |\tau|^{-1} \in L^\infty((0,T)\times\mathbb{R}^2), \\ &\partial_\tau \mu(\vartheta) \in L^\infty((0,T); L^{2+\epsilon}(\mathbb{R}^2)) \text{ in the distribution sense.} \end{aligned}$$

In particular, if the initial temperature is a temperature patch

$$\vartheta_0(x) = 1_{D_0}(x),$$

for some bounded, simply connected domain $D_0 \subset \mathbb{R}^2$ with $W^{2,2+\epsilon}$-boundary, if the vector field τ_0 from the above assumption is tangential to the boundary ∂D_0 and $\mu \in W^{1,\infty}(\mathbb{R};[\mu_*,\mu^*])$, then the unique solution above satisfies for all times $t \in [0,T]$,

$$\vartheta(t,x) = 1_{D_t}(x)$$

for some bounded, simply connected domain $D_t \subset \mathbb{R}^2$ with $W^{2,2+\epsilon}$-boundary.

The proof of Theorem 3 is given in Chapter 3.

Let us comment on the finite time T. The fact that Theorem 3 gives only *local-in-time* solutions of the Boussinesq equations essentially comes from the buoyancy forcing term on the right hand side of the momentum equation $(\mathrm{B})_2$. The latter causes the $L^2(\mathbb{R}^2)$-norm of the velocity vector field to grow in time, even for constant diffusion coefficients and smooth and fast decaying small initial data; see the lower growth bounds in [26, 150].

1.3.3. Regularity propagation for the compressible Navier-Stokes equations with variable viscosity

Recall the two-dimensional compressible Navier-Stokes equations with density-dependent viscosity coefficients (CNS):

$$\begin{cases}\partial_t\rho + \mathrm{div}\,(\rho u) = 0, \quad (t,x) \in (0,\infty)\times\mathbb{R}^2,\\ \rho(\partial_t u + u\cdot\nabla u) - \mathrm{div}\,(\mu(\rho)Su) - \nabla(\lambda(\rho)\mathrm{div}\,u) + \nabla P(\rho) = 0.\end{cases} \tag{CNS}$$

Our main goal for (CNS) is to remove the size restriction on the viscosity variation assumed in the existing literature. We address this in the fourth main result of this thesis, which can be summarized as follows (see Theorem 4.1.1 in Chapter 4).

Theorem 4. *Given positive upper and lower viscosity and pressure bounds $0 < \mu_* \leq \mu^*$, $0 < \nu_* \leq \nu^*$, $\pi_* > 0$ there exists ϵ_0 such that for $\epsilon \in (0,\epsilon_0]$ the following holds true.*
Let $\rho_0 \in L^\infty(\mathbb{R}^2;[\rho_*,\rho^*])$, $0 < \rho_* \leq \rho^*$, be an initial density satisfying $\rho_0 - \tilde\rho \in L^2(\mathbb{R}^2)$ for some $\tilde\rho > 0$. Assume the dependence of the viscosity coefficients and pressure on the density function ρ to be $\mu(\rho), \lambda(\rho)$ and $P(\rho)$ for some $\mu, \lambda, P \in W^{2,\infty}(\frac{1}{4}\rho_*, 4\rho^*)$ such that

$$\mu_* \leq \mu|_{(\frac14\rho_*,4\rho^*)} \leq \mu^*, \quad \nu_* - 2\mu_* \leq \lambda|_{(\frac14\rho_*,4\rho^*)} \leq \nu^* - 2\mu^*,$$
$$\inf_{s\in(\frac14\rho_*,4\rho^*)} sP'(s) \geq \pi_*, \qquad \inf_{s\in(\frac14\rho_*,4\rho^*)} P'(s) =: \tilde\pi_* > 0.$$

Let $u_0 \in H^1(\mathbb{R}^2;\mathbb{R}^2)$ and $\tau_0 \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ such that $|\tau_0|^{-1} \in L^\infty(\mathbb{R}^2)$ and $(\nabla\tau_0, \partial_{\tau_0}\rho_0) \in L^{2+\epsilon}(\mathbb{R}^2;\mathbb{R}^{2\times2+1})$ in the sense of distribution. If $(\rho_0 - \tilde\rho, \rho_0 u_0) \in \dot H^{-2\delta}(\mathbb{R}^2;\mathbb{R}^{1+2})$ for some $\delta \in (\frac12 - \frac14\frac{\epsilon}{2+\epsilon}, \frac12)$, and (with $\bar\tau_0 = \frac{\tau_0}{|\tau_0|}$)

$$(\|(\rho_0-\tilde\rho,\rho_0u_0)\|_{L^2\cap\dot H^{-2\delta}(\mathbb{R}^2)} + \|\nabla u_0\|_{L^2(\mathbb{R}^2)})(1 + \|(\nabla\bar\tau_0, \partial_{\bar\tau_0}\rho_0)\|_{L^{2+\epsilon}(\mathbb{R}^2)}) \leq c_4, \tag{1.3.7}$$

where c_4 is a positive constant depending only on $\rho_*,\rho^*,\mu_*,\mu^*,\nu_*,\nu^*,\pi_*,\tilde\pi_*$, $\|(\mu,\lambda,P)\|_{W^{2,\infty}(\frac14\rho_*,4\rho^*)}$, ϵ,δ, then the system (CNS)-(τ) supplemented with the initial data (ρ_0,u_0,τ_0) has a unique global-in-time solution (ρ,u,τ) such that

$$\rho \in L^\infty([0,\infty)\times\mathbb{R}^2;[\frac{1}{4}\rho_*,4\rho^*]), \quad \rho - \tilde\rho \in \mathcal{C}_b([0,\infty);L^2(\mathbb{R}^2)),$$

$$
\begin{aligned}
&u \in \mathcal{C}_b([0,\infty); L^2(\mathbb{R}^2;\mathbb{R}^2)) \cap L^2([0,\infty); \dot{H}^1(\mathbb{R}^2;\mathbb{R}^2)),\\
&\nabla u \in L^\infty((0,\infty); L^2(\mathbb{R}^2;\mathbb{R}^{2\times 2})) \cap L^1((0,\infty); L^{2+\epsilon} \cap L^\infty(\mathbb{R}^2;\mathbb{R}^{2\times 2})),\\
&t^{\frac{3}{4}}\nabla u \in L^\infty((0,\infty); L^\infty(\mathbb{R}^2;\mathbb{R}^{2\times 2})), \quad t^{\frac{1}{2}}\nabla u \in L^2((0,\infty); L^\infty(\mathbb{R}^2;\mathbb{R}^{2\times 2})),\\
&\tau \in \mathcal{C}_b([0,\infty);\ (L^\infty \cap \dot{W}^{1,2+\epsilon})(\mathbb{R}^2;\mathbb{R}^2)), \quad |\tau|^{-1} \in L^\infty((0,\infty)\times\mathbb{R}^2),\\
&\partial_\tau \rho \in (L^1 \cap L^\infty)((0,\infty); L^{2+\epsilon}(\mathbb{R}^2)), \quad \partial_\tau \nabla u,\ \nabla \partial_\tau u \in L^1((0,\infty); L^{2+\epsilon}(\mathbb{R}^2;\mathbb{R}^{2\times 2})).
\end{aligned}
$$

In particular, if the initial density is of the patch-type

$$
\rho_0(x) = \rho_0^+(x) 1_{D_0}(x) + \rho_0^-(x) 1_{D_0^c}(x),
$$

for some bounded, simply connected domain $D_0 \subset \mathbb{R}^2$ *with* $W^{2,2+\epsilon}$*-boundary, and functions* $\rho_0^+ \in W^{1,2+\epsilon}(\overline{D_0})$, $\rho_0^- - \tilde{\rho} \in L^2 \cap W^{1,2+\epsilon}(\overline{D_0}^c)$*, then there exists a nondegenerate vector field* $\tau_0 \in L^\infty \cap \dot{W}^{1,2+\epsilon}(\mathbb{R}^2;\mathbb{R}^2)$ *tangential to the boundary* ∂D_0*. If the smallness assumption* (1.3.7) *is satisfied, then the unique solution above preserves the patch structure for all times* $t > 0$*,*

$$
\rho(t,x) = \rho^+(t,x) 1_{D_t}(x) + \rho^-(t,x) 1_{D_t^c}(x),
$$

for some bounded, simply connected domain $D_t \subset \mathbb{R}^2$ *with* $W^{2,2+\epsilon}$*-boundary, and functions* $\rho^+(t,\cdot) \in W^{1,2+\epsilon}(\overline{D_t})$, $\rho^-(t,\cdot) - \tilde{\rho} \in L^2 \cap W^{1,2+\epsilon}(\overline{D_t}^c)$*. Thus, the density-patch-type problem in the absence of vacuum for the compressible Navier-Stokes equations* (CNS) *is uniquely globally-in-time solvable under the smallness assumption* (1.3.7)*.*

Theorem 4 is proved in Chapter 4. It is essentially the compressible counterpart of Theorem 2. The proof requires additional arguments to handle the time-decay of the divergence of the velocity, $\operatorname{div} u$, and the tangential derivative of the density, $\partial_\tau \rho$, for both of which the density-dependence of the pressure $P(\rho)$ plays an essential role.

CHAPTER 2

THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS WITH VARIABLE, ODD VISCOSITY

The results presented in this chapter are based on the work [250].

2.1. Introduction

This chapter is devoted to the two-dimensional inhomogeneous incompressible Navier-Stokes equations with odd viscosity (oddINS) and (oddINS').

Odd or *Hall* viscosity is the anti-symmetric part of the viscosity tensor (in the sense of (2.1.4) below), and it is present in fluids with broken microscopic time-reversal symmetry and broken parity, see [16, 17]. We are going to discuss the physical model in Subsection 2.1.3 below. Recall that this system reads as

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho) S u) - \operatorname{div}(\mu_o(\rho) S_o u) + \nabla \pi = \rho f, \\ \operatorname{div} u = 0, \end{cases} \tag{oddINS}$$

where the scalar density function ρ, the velocity vector field $u = (u_1, u_2)^T$, and the scalar pressure π are unknown and depend on the time and space variables $(t, x) \in [0, \infty) \times \Omega$. The domain Ω is either the whole space $\mathbb{R}^2$, or a bounded, open connected Lipschitz domain in $\mathbb{R}^2$. The variable shear and odd viscosity coefficients depend on the density function, $\mu(\rho), \mu_o(\rho)$, for some given functions

$$\mu \in \mathcal{C}(\mathbb{R}; [\mu_*, \mu^*]), \quad \mu_o \in \mathcal{C}(\mathbb{R}; [-\mu^*, \mu^*]), \tag{2.1.1}$$

with two positive constants $\mu^*, \mu_* > 0$. The external force $f \in (L^2((0, \infty) \times \Omega))^2$ is given. We denote

$$u \otimes u = (u_i u_j)_{1 \le i,j \le 2}, \quad \nabla u = (\partial_j u_i)_{1 \le i,j \le 2}, \quad \nabla^T u = (\partial_i u_j)_{1 \le i,j \le 2},$$
$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \nabla = \begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix}, \quad \nabla^\perp = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix}, \quad u^\perp = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix},$$

so that the strain tensors read as

$$S u = \nabla u + \nabla^T u = \begin{pmatrix} 2\partial_1 u_1 & \partial_2 u_1 + \partial_1 u_2 \\ \partial_2 u_1 + \partial_1 u_2 & 2\partial_2 u_2 \end{pmatrix},$$
$$S_o u = \nabla u^\perp + \nabla^\perp u = \begin{pmatrix} -(\partial_1 u_2 + \partial_2 u_1) & \partial_1 u_1 - \partial_2 u_2 \\ \partial_1 u_1 - \partial_2 u_2 & (\partial_1 u_2 + \partial_2 u_1) \end{pmatrix}.$$

Equations $(\text{oddINS})_1$ and $(\text{oddINS})_2$ stand for the conservation of mass and the conservation of momentum, respectively, and $(\text{oddINS})_3$ expresses the incompressibility of the fluid.

The physics literature regarding odd viscosity in fluid dynamics is vast and has been an active area of research during the last thirty years. Despite of that, the notion of odd viscosity has as of yet received little attention from mathematicians.

There is a large body of mathematical literature concerning the Navier-Stokes equations (oddINS) *without* odd viscosity, i.e. the system (INS):

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho) S u) + \nabla \pi = \rho f, \\ \operatorname{div} u = 0. \end{cases} \tag{INS}$$

In particular, the existence of weak solutions of (INS) is well known in both the evolutionary and stationary cases [123, 175] (for the definition of weak solution we refer to Definitions 2.2.1 and 2.3.3 below). It is natural to ask whether similar existence results hold if the odd viscosity is taken into account. In this chapter we give a positive answer to this question. More precisely, we establish the following facts:

1. Weak solutions to (oddINS) exist in both the evolutionary and stationary cases (see Theorems 2.2.2 and 2.2.6).
2. As the odd viscosity tends to some constant: $\mu_o(\rho) \to \nu_o \equiv const$, any weak solution of (oddINS) which satisfies an energy inequality, converges (in a suitable sense) to a solution of the Naiver-Stokes equations without odd viscosity (INS) (see Corollaries 2.2.4 and 2.2.8).
3. For vanishing shear viscosity $\mu(\rho) \equiv 0$, there do not in general exist stationary weak solutions of (oddINS) with $u \in H^1_{\text{loc}}$ (see Example 2.3.8).

This chapter is structured as follows. We begin in Section 2.1 with a review of known mathematical results concerning the existence theory of weak solutions to (INS) for the evolutionary and the stationary cases. The main existence result of weak solutions to the evolutionary and the stationary Navier-Stokes equations (oddINS) and (oddINS'), respectively, is stated in Section 2.2. The proofs are performed in Section 2.3. Subsection 2.3.1 is concerned with the existence proof in the evolutionary case and Subsection 2.3.2 with that of the stationary case. In Subsection 2.3.2 we also consider examples of stationary parallel, concentric and radial flows.

2.1.1. The evolutionary Navier-Stokes equations

There has been extensive reseach on the Navier-Stokes equations (INS) without odd viscosity. When $\rho \equiv 1$ (and hence $\mu(\rho)$ is a positive constant) the system turns into the classical homogeneous incompressible Navier-Stokes equation, which has been intensively studied. J. Leray [162] proved global-in-time existence of finite energy weak solutions in dimension $d = 2, 3$. For general ρ and $\mu(\rho)$ the existence of weak solutions was shown in P.-L. Lions' book [175]. There, P.-L. Lions proved the existence of weak solutions (ρ, u) of (INS) on various domains in the sense of Definition 2.2.1 below (with $\mu_o(\rho) \equiv 0$).

Here we are going to show that P.-L. Lion's results hold true even with the additional odd viscosity term. The main observation is that the *a priori* energy balance, which is satisfied by smooth enough solutions, is preserved under odd viscosity. More precisely, the cancellation of the Frobenius inner product

$$\begin{aligned} (\nabla u^\perp + \nabla^\perp u) : (\nabla u + \nabla^T u) &= \sum_{i,j=1}^{2} (\nabla u^\perp + \nabla^\perp u)_{ij} (\nabla u + \nabla^T u)_{ij} \\ &= 2(\partial_2 u_2 - \partial_1 u_1)(\partial_1 u_2 + \partial_2 u_1) + 2(\partial_1 u_1 - \partial_2 u_2)(\partial_2 u_1 + \partial_1 u_2) \\ &= 0 \end{aligned} \tag{2.1.2}$$

holds and hence, a priori, the energy equality holds when we test $(\text{oddINS})_2$ by u:

$$\int_\Omega \rho|u|^2\,dx + \int_0^t\int_\Omega \mu(\rho)|Su|^2\,dxdt' = \int_\Omega \rho_0|u_0|^2\,dx + 2\int_0^t\int_\Omega \rho f\cdot u\,dxdt', \quad \forall t>0.$$

The energy *in*equality is known to hold for P.-L. Lions' weak solutions (i.e. if $\mu_o(\rho)\equiv 0$). The weak solutions we obtain for (oddINS) satisfy the same energy inequality (see (2.2.6) below). This entails in particular that odd viscosity does not affect the energy of the fluid.

2.1.2. The stationary Navier-Stokes equations

The stationary counterpart of the equations (oddINS) reads as follows:

$$\begin{cases} \operatorname{div}(\rho u\otimes u) - \operatorname{div}(\mu(\rho)Su) - \operatorname{div}(\mu_o(\rho)S_o u) + \nabla\pi = f, \\ \operatorname{div} u = 0,\ \operatorname{div}(\rho u) = 0, \end{cases} \tag{oddINS'}$$

For $\mu_o(\rho)\equiv 0$ previous results on (oddINS') include the following: If the flow is homogeneous, i.e. $\rho\equiv 1$ (and hence $\mu(\rho)$ is a positive constant), then the system has been extensively studied and J. Leray [162] proved the existence of weak solutions $u\in(H^1(\Omega))^2$ on a simply connected domain Ω in $\mathbb{R}^2$. If the flow is inhomogeneous but $\mu(\rho)>0$ is a constant, then N. N. Frolov [95] proved the existence and regularity of weak solutions of the form

$$(\rho,u) = (\eta(\phi),\nabla^\perp\phi), \tag{2.1.3}$$

where ϕ denotes the stream function of u, and η is a Hölder continuous function. This Ansatz ensures that the divergence-free conditions $(\text{oddINS'})_2$ are automatically satisfied: Formally we have

$$\begin{aligned} \operatorname{div}(u) &= \operatorname{div}(\nabla^\perp\phi) = 0, \\ \operatorname{div}(\rho u) &= \operatorname{div}(\eta(\phi)\nabla^\perp\phi) = \eta'(\phi)\nabla\phi\cdot\nabla^\perp\phi = 0, \end{aligned}$$

and if $\phi\in H^2_{\text{loc}}$, $(\text{oddINS'})_2$ holds not only formally, but also in the sense of distributions.

The Hölder continuity of η in [95] was relaxed to continuity in [214]. The existence result of weak solutions with *variable* viscosity $\mu(\rho)$ was established by Z. He and X. Liao in [123]. The weak solutions there are also of Frolov's form (2.1.3).

Assuming the form (2.1.3) not only ensures that $(\text{oddINS'})_2$ is satisfied, but it also has the advantage that the original problem (oddINS') for (ρ,u) reduces to solving a fourth order elliptic equation for ϕ. More precisely, applying the two-dimensional curl operator $\nabla^\perp\cdot$ to the momentum equation $(\text{oddINS'})_1$ yields

$$\mathcal{L}\phi = -\nabla^\perp\cdot f + \nabla^\perp\cdot\operatorname{div}(\eta(\phi)\nabla^\perp\phi\otimes\nabla^\perp\phi),$$

where the fourth order elliptic operator $\mathcal{L}$ is computed below in Section 2.3.2. It then remains to study the existence of weak solutions to this equation. The elliptic equation for the stream function has also been used in [123] with a different fourth order elliptic operator. The key observation is that the equation stays elliptic in the presence of odd viscosity.

We are going to explain in this article how the proof from [123] can be modified to show a similar existence result in the presence of non-vanishing odd viscosity, $\mu_o(\rho)\not\equiv 0$, in the momentum equation. In analogy to [123] we also consider examples of flows under certain symmetry assumptions on the density, and analyse the effects of the presence of odd viscosity on the velocity vector field.

In what follows, we give a brief explanation of the odd viscosity term in $(\text{oddINS})_2$, and give an overview of previous physical results on this model. To the best of our knowledge there is much less mathematical research on hydrodynamic equations with odd viscosity, which is why this introduction focuses on the physical results.

2.1.3. Odd viscosity: Properties and previous results

Viscosity of a fluid typically measures the resistance of the fluid to velocity gradients. It is expressed mathematically by a tensor η_{ijkl} that acts as a coefficient of proportionality between viscous stress σ_{ij} and strain rate $\partial_k u_l$, namely $\sigma_{ij} = \eta_{ijkl}\partial_k u_l$ [158]. Symmetry under both parity and time-reversal are conditions which are satisfied by conventional fluids at thermal equilibrium, and in that case Onsager reciprocal relation [197] demands that η_{ijkl} must be symmetric in the sense that $\eta_{ijkl} = \eta_{klij}$. Fluids in which time-reversal and parity are broken exhibit non-dissipative viscosity that is odd under each of these symmetries, which leads to the presence of an antisymmetric part in the viscosity tensor:

$$\eta_{ijkl} = \eta^S_{ijkl} + \eta^A_{ijkl}, \quad \text{with} \quad \eta^S_{ijkl} = \eta^S_{klij}, \quad \eta^A_{ijkl} = -\eta^A_{klij}. \tag{2.1.4}$$

In a two-dimensional incompressible isotropic fluid with broken parity and broken time-reversal, even and odd viscosity are specified by a single scalar $\mu(\rho)$ and $\mu_o(\rho)$, respectively, and hence the viscous stress tensor is characterized by two viscosity coefficients, one for the even part and one for the odd part [16, 158]

$$\sigma = \mu(\rho)(\nabla u + \nabla^T u) + \mu_o(\rho)(\nabla u^\perp + \nabla^\perp u),$$

where $\mu(\rho)$ is the kinematic shear (or “even") viscosity, and $\mu_o(\rho)$ the kinematic odd viscosity. When time-reversal and parity are broken, the viscosity tensor can have a non-vanishing odd part $\mu_o(\rho) \not\equiv 0$, while it must vanish if at least one of these symmetries holds.

The odd viscosity is quite different from the conventional shear viscosity, and significant odd viscosity may lead to counterintuitive properties of the fluid, which were illustrated with examples in [16]. For instance, odd viscosity can produce forces perpendicular to the direction of the fluid flow. An important difference between the two viscosity coefficients is that the shear viscosity $\mu(\rho)$ is associated with dissipation, while the odd viscosity $\mu_o(\rho)$ is of non-dissipative nature.

The interest in odd viscosity was motivated by the seminal paper by J. E. Avron, R. Seiler and P. G. Zograf [17], where it was shown that in general, quantum Hall states have non vanishing odd viscosity. Since then the role of odd viscosity in the context of quantum Hall fluids has been an active area of research, see [2, 25, 115, 133, 134, 210] and references therein. In classical fluids, odd viscosity can appear in various systems including polyatomic gases [151], chiral active fluids [19, 225, 226], magnetized plasmas [204] and fluids of vortices [241]. In three dimensions, odd terms in the viscosity tensor have been known for some time in hydrodynamic theories of superfluid He-3A [237].

In the following we will review some results on the constant and variable viscosity cases.

2.1.3.1. Constant Viscosity

When the density ρ is positive and constant, say $\rho \equiv 1$, then by (2.1.1), $\mu(\rho) \equiv \nu > 0$ and $\mu_o(\rho) \equiv \nu_o \in \mathbb{R} \setminus \{0\}$ are constants, and the system (oddINS) turns into the homogeneous incompressible Navier-Stokes equations with odd viscosity

$$\partial_t u + u \cdot \nabla u - \nu \Delta u - \nu_o \Delta u^\perp + \nabla \pi = 0, \quad \operatorname{div} u = 0. \tag{2.1.5}$$

Much research has been done on related free-surface problems [1, 4, 77, 189].

The authors in [108] considered equation (2.1.5) with $\mu(\rho) = 0$ and a free-surface boundary, under the additional assumption that the fluid is irrotational: $\nabla^\perp \cdot u = 0$. They developed asymptotic models of surface waves and proved well-posedness of these models in an analytic function space and also in a Sobolev setting.

In [16] J. E. Avron studied the odd viscosity effects in classical two-dimensional hydrodynamics by examining solutions to the wave equation as well as the homogeneous Navier-Stokes equations where the stress tensor is dominated by odd viscosity. These effects were shown to be subtle in the case when the fluid is incompressible, and they are most prominent in compressible flows. It has also been shown that if the fluid is almost incompressible, the odd viscosity effects are most visible at the dynamical boundary subject to no-stress or free-surface boundary conditions [1, 102]. Further investigations on observable effects of odd viscosity in classical two-dimensional incompressible hydrodynamics were conducted in [19, 160, 181].

One important feature of the homogeneous incompressible Navier-Stokes equations with odd viscosity (2.1.5) is that odd viscosity effects are absent when the fluid is spread on the entire plane or confined in rigid domains with no-slip boundary conditions. Indeed, due to the incompressibility of the fluid, the odd part of the viscosity tensor can be written as a gradient of the vorticity of the flow: $\Delta u^\perp = -\nabla \omega$, where $\omega = \nabla^\perp \cdot u$, so that taking the two-dimensional curl yields the equation for the vorticity

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega,$$

which is independent of ν_o. Hence, vorticity is generated only by the symmetric part of the viscosity tensor, and as a consequence, the effects of odd viscosity on the flow of an incompressible fluid can come only through boundary conditions. For example, the signature of $\mu_o(\rho)$ is present in surface waves and in the interface between two fluids governed by kinematic and no-stress boundary conditions, which explicitly depends on the odd viscosity [102].

It is easy to see that for the same reason the systems (oddINS) and (INS) are equivalent if $\mu_o(\rho)$ is a constant. Nevertheless, for density-dependent viscosity coefficient $\mu_o(\rho)$ as in (oddINS), it is in general not possible to write the odd viscosity term as a gradient, and $\mu_o(\rho)$ might influence the flow.

2.1.3.2. Variable Viscosity

In [3] the authors considered compressible fluids satisfying the strictly non-dissipative ($\mu(\rho) \equiv 0$) equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\nu_o \rho S_o u) + \nabla(p(\rho)) = -\rho(B u^\perp + E), \end{cases}$$

where B and $E = (E_1, E_2)$ are electromagnetic fields and ν_o is a real non-zero constant, i.e., equations $(\text{oddINS})_{1,2}$ with $\mu(\rho) \equiv 0$ and $\mu_o(\rho) = \nu_o \rho$ and an external force $f = Bu^\perp + E$ depending on u. These equations were paired with free-surface boundary conditions. The authors studied questions related to the variational formalism for the free-surface dynamics, linear surface waves and the incompressible limit.

The incompressible counterpart of the system in [3] was treated in [89, 90] with the same choice for $\mu_o(\rho)$ and no external force, and they proved local-in-time existence and uniqueness in the Sobolev setting $H^s(\mathbb{R}^2)$ for $s > 2$, provided, among other assumptions, that the initial data satisfies $\rho_0 - 1 \in H^{s+1}(\mathbb{R}^2)$ and $u_0 \in (H^s(\mathbb{R}^2))^2$. These high regularity assumptions on the initial data were crucial for the authors to be able to exploit an underlying hyperbolic structure of $(\text{oddINS})_2$ when $\mu(\rho) \equiv 0$. More precisely, working with a new set of variables

$$\omega = \nabla^\perp \cdot u, \quad \eta = \nabla^\perp \cdot (\rho u), \quad \theta = \eta - \Delta \rho,$$

allows one to resort to general theory of transport equations since the quantities ω and θ both satisfy simple transport equations transported by $H^s(\mathbb{R}^2)$ vector fields. It was also explained in [89] about the challenges of obtaining a priori estimates for higher order derivatives when the shear viscosity term vanishes, due to the odd viscosity leading to a loss of derivatives.

In a recent paper [90] the system with $\mu(\rho) \equiv 0$ and $\mu_o(\rho) = \nu_o \rho$ was reformulated into a system bearing strong similarities to the ideal magnetohydrodynamics equations. By use of this formulation the authors proved local well-posedness in a Besov space setting and provided a lower bound on the lifespan of the solution.

Here we will take $\mu(\rho) \geq \mu_* > 0$, and the presence of the shear viscosity term in $(\text{oddINS})_2$ is crucial to obtain higher order estimates for the velocity u, if the density is only bounded, without any regularity assumptions.

2.2. Main results

In this section we give the definition of weak solutions of the systems (oddINS) and (oddINS'), and we state the main results of this chapter.

2.2.1. The evolutionary system

We first fix a domain $\Omega \subset \mathbb{R}^2$ as in one of the following three cases.

1. *(Dirichlet case)*: Ω is an open, bounded, connected domain in $\mathbb{R}^2$ with a Lipschitz boundary $\partial\Omega$. We look for solutions satisfying the non-slip boundary condition $u = 0$ on $\partial\Omega$.
2. *(Periodic case)*: Ω is a rectangle $(0, L_1) \times (0 \times L_2)$, where $L_1, L_2 > 0$. We look for solutions (ρ, u) which are periodic in x_i of period L_i, $i = 1, 2$, respectively.
3. *($\mathbb{R}^2$ case)*: Ω is the whole plane $\mathbb{R}^2$. We look for solutions $u \in L^\infty_{\text{loc}}(0, \infty; (L^2(\mathbb{R}^2))^2)$.

We pose the following assumptions on the initial data: Let the initial density have a positive upper and lower bound

$$\rho_0 \in L^\infty(\Omega), \quad 0 < \rho_* \leq \rho_0 \leq \rho^* \text{ a.e. in } \Omega, \tag{2.2.1}$$

and the initial velocity vector field

$$u_0 \in (L^2(\Omega))^2. \tag{2.2.2}$$

Let $f \in (L^2((0,\infty)\times\Omega))^2$ be an external force. In the periodic case we consider ρ_0, u_0 and f as functions defined on $\mathbb{R}^2$ by extending them periodically.

We define weak solutions for the evolutionary Navier-Stokes equations (oddINS) in the following way.

Definition 2.2.1 (Weak solutions of the evolutionary system). *We call a pair (ρ, u) a weak solution of* (oddINS) *with initial data given by* (2.2.1) *and* (2.2.2)*, if one of the following is satisfied for all $T > 0$.*

1. *(Dirichlet case): $\rho \in L^\infty((0,\infty)\times\Omega)\cap \mathcal{C}_b([0,\infty); L^p(\Omega))$, for any $p \in [1,\infty)$ and $u \in L^2(0,T;(H_0^1(\Omega))^2)$;*
2. *(Periodic case): $\rho \in L^\infty((0,\infty)\times\Omega)\cap \mathcal{C}_b([0,\infty); L^p(\Omega))$ for any $p \in [1,\infty)$, and $u \in L^2(0,T;(H^1(\Omega))^2)$, ρ and u periodic in x_i of period L_i, $i = 1,2$;*
3. *($\mathbb{R}^2$ case): $\rho \in L^\infty((0,\infty)\times\mathbb{R}^2)\cap \mathcal{C}_b([0,\infty); L^p_{\mathrm{loc}}(\mathbb{R}^2))$ for any $p \in [1,\infty)$, and $u \in L^2(0,T;(H^1(\mathbb{R}^2))^2)$;*

and additionally there holds

$$\operatorname{div} u = 0, \quad \textit{in the sense of distribution,} \tag{2.2.3}$$

$$-\int_\Omega \rho_0\psi(0)\,dx - \int_0^\infty\int_\Omega \rho\partial_t\psi\,dxdt = \int_0^\infty\int_\Omega \rho u\cdot\nabla\psi\,dxdt, \tag{2.2.4}$$

$$\begin{aligned} -\int_\Omega \rho_0 u_0\cdot\varphi(0)\,dx + \int_0^\infty\int_\Omega & -\rho u\cdot\partial_t\varphi - \rho(u\otimes u):\nabla\varphi \\ &+ \Big(\frac{\mu(\rho)}{2}Su + \frac{\mu_o(\rho)}{2}S_o u\Big) : S\varphi\,dxdt = \int_0^\infty\int_\Omega \rho f\cdot\varphi\,dxdt, \end{aligned} \tag{2.2.5}$$

for all $\psi \in \mathcal{C}_c^\infty([0,\infty)\times\Omega)$, $\varphi \in (\mathcal{C}_c^\infty([0,\infty)\times\Omega))^2$, with $\operatorname{div}\varphi = 0$*, in the Dirichlet and whole plane case, or for all $\psi \in \mathcal{C}_c^\infty([0,\infty);\mathcal{C}^\infty(\Omega))$, $\varphi \in (\mathcal{C}_c^\infty([0,\infty);\mathcal{C}^\infty(\Omega)))^2$, with* $\operatorname{div}\varphi = 0$*, and ψ, φ being periodic in x_i of period L_i, $i = 1,2$, in the periodic case. Here the Frobenius inner product is defined by $A : B = \sum_{i,j=1}^2 a_{ij}b_{ij}$ for matrices $A = (a_{ij})_{i,j=1,2}, B = (b_{ij})_{i,j=1,2} \in \mathbb{R}^{2\times 2}$.*

Our main result concerning the existence of weak solutions to the Navier-Stokes system (oddINS) reads as follows.

Theorem 2.2.2 (Existence of weak solutions of the evolutionary system). *Let Ω be as in one of the three cases above, and let ρ_0 and u_0 be given by* (2.2.1) *and* (2.2.2)*, respectively. Then there exists at least one weak solution (ρ, u) to the system* (oddINS) *with initial data ρ_0 and u_0. Furthermore, this weak solution satisfies the energy inequality*

$$\int_\Omega \rho|u|^2\,dx + \int_0^t\int_\Omega \mu(\rho)|Su|^2\,dxdt' \le \int_\Omega \rho_0|u_0|^2\,dx + 2\int_0^t\int_\Omega \rho f\cdot u\,dxdt', \quad \forall t > 0. \tag{2.2.6}$$

We follow a standard procedure to prove Theorem 2.2.2 in Section 2.3.1. For the Dirichlet and periodic cases we use a Galerkin method as in [92, Section 7], and the whole plane case will be a consequence of the Dirichlet case for the derivation of which we follow the lines of [175]. Here we have to deal with additional second-order derivative terms of the velocity in the momentum equation due to the presence of odd viscosity.

Let us make a few comments on the weak solutions obtained in Theorem 2.2.2.

Remark 2.2.3. As in [175], the results can be improved in the following sense:

- The assumption $\rho_* > 0$ in (2.2.1) can be relaxed to $\rho_* = 0$, where vacuum in the fluid is allowed. This however requires additional assumptions on the initial data, and in Definition 2.2.1 the condition $u \in L^2(0,T;(H^1(\Omega))^2)$ is replaced by $\sqrt{\rho}u \in L^\infty(0,T;(L^2(\Omega))^2)$ and $\nabla u \in L^2((0,T)\times\Omega))^{2\times 2}$, $T > 0$.
- The solutions given in Theorem 2.2.2 are continuous in the sense that

$$\rho u, \sqrt{\rho}u, u \in \mathcal{C}([0,T];L^2_w(\Omega)), \quad \forall T > 0,$$

  where $L^2_w(\Omega)$ denotes the space $L^2(\Omega)$ endowed with the weak topology.

The energy inequality (2.2.6) entails the following property of our weak solutions: let $\mu \in \mathcal{C}(\mathbb{R};[\mu_*,\mu^*])$ be fixed. We are interested in what happens to a weak solution in the limit $\mu_o \to \nu_o$ for some constant ν_o, and are wondering whether the system (oddINS) converges to (INS) in some sense. The following corollary formulates this question more rigorously and gives a confirming answer.

Corollary 2.2.4. *Let a sequence* $(\mu_o^\epsilon)_{\epsilon\in(0,1)}$ *of functions in* $\mathcal{C}(\mathbb{R};[-\mu^*,\mu^*])$ *be given, such that*

$$\|\mu_o^\epsilon - \nu_o\|_{\mathcal{C}([\rho_*,\rho^*])} \to 0, \quad \epsilon \to 0, \tag{2.2.7}$$

for some constant $\nu_o \in [-\mu^*,\mu^*]$*. For each* $\epsilon \in (0,1)$ *let* $(\rho^\epsilon, u^\epsilon)$ *denote a weak solution of* (oddINS) *with odd viscosity coefficient* μ_o^ϵ *which satisfies the energy inequality* (2.2.6)*. Then there exists a function pair* (ρ, u) *such that up to a subsequence*

$$\begin{aligned} \rho^\epsilon &\to \rho, \quad && \text{in } \mathcal{C}([0,T];L^p(\Omega\cap B_R)), \quad \forall p \in [1,\infty), \forall T, R > 0, \\ u^\epsilon &\rightharpoonup^* u, \quad && \text{in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \quad \forall T > 0, \end{aligned} \tag{2.2.8}$$

as $\epsilon \to 0$*, and* (ρ, u) *is a weak solution of* (INS)*, i.e.* (ρ, u) *satisfies the conditions in Definition 2.2.1 with* $\mu_o = 0$*.*

Corollary 2.2.4 is proved in Subsection 2.3.1.4 below. The convergence “(oddINS)→(INS)” as $\mu_o \to \nu_o$ is compatible with the observation mentioned in the introduction, namely that for constant odd viscosity coefficient the odd viscosity term can be absorbed into the pressure and therefore does not affect the fluid flow (if the boundary conditions do not depend on μ_o).

2.2.2. The stationary system

We next consider the stationary system (oddINS’) on a bounded simply connected $\mathcal{C}^{1,1}$ domain Ω in $\mathbb{R}^2$. We fix an external force $f \in (H^{-1}(\Omega))^2$, and the boundary value $g \in (H^{\frac{1}{2}}(\partial\Omega))^2$. In order for the boundary value condition $u|_\Omega = g$ to be compatible with the divergence-free condition for u, we need to postulate the zero flux condition

$$\int_{\partial\Omega} g \cdot n \, d\sigma = 0, \tag{2.2.9}$$

where $n = (n_1, n_2)$ denotes the outer normal vector to the boundary $\partial\Omega$.

Firstly, we give the definition of a weak solution to the boundary value problem of the stationary Navier-Stokes equations (oddINS’).

Definition 2.2.5 (Weak solutions of the stationary system). *We call a pair* $(\rho, u) \in L^\infty(\Omega; [0,\infty)) \times (H^1(\Omega))^2$ *a weak solution to* (oddINS') *with boundary value* $g \in (H^{\frac{1}{2}}(\partial\Omega))^2$ *and external force* $f \in (H^{-1}(\Omega))^2$*, if*

$$\operatorname{div} u = 0, \quad \operatorname{div}(\rho u) = 0, \qquad \textit{in the sense of distribution}, \tag{2.2.10}$$

$$u|_{\partial\Omega} = g \qquad \textit{in the trace sense}, \tag{2.2.11}$$

and the integral identity

$$\int_\Omega \Big(\frac{\mu(\rho)}{2} Su + \frac{\mu_o(\rho)}{2} S_o u \Big) : S\varphi \, dx = \int_\Omega \rho(u \otimes u) : \nabla\varphi \, dx + \langle f, \varphi \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \tag{2.2.12}$$

holds for any $\varphi \in (H_0^1(\Omega))^2$ *with* $\operatorname{div}\varphi = 0$.

The existence and regularity result of weak solutions to the boundary value problem (oddINS') reads as follows.

Theorem 2.2.6 (Existence and regularity of weak solutions of the stationary system). *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded simply connected* $\mathcal{C}^{1,1}$ *domain, and let the external force* $f \in (H^{-1}(\Omega))^2$*, and the boundary value* $g \in (H^{\frac{1}{2}}(\partial\Omega))^2$ *be given and satisfy the zero-flux condition* (2.2.9).

1. *There exists at least one weak solution* $(\rho, u) \in L^\infty(\Omega; [0,\infty)) \times (H^1(\Omega))^2$ *of the stationary Navier-Stokes equations* (oddINS') *which is of Frolov's form.*
2. *Let* $k \in \mathbb{N}$ *and let* Ω *be a bounded simply connected* $\mathcal{C}^{k+1,1}$ *domain. If* $\mu_e \in \mathcal{C}^k(\mathbb{R}; [\mu_*, \mu^*])$*,* $\mu_o \in \mathcal{C}^k(\mathbb{R}; [-\mu^*, \mu^*])$*,* $\eta \in \mathcal{C}^k(\mathbb{R}; [0, \rho^*])$*, and the external force* $f \in (H^{k-1}(\Omega))^2$ *and the boundary value* $g \in (H^{k+\frac{1}{2}}(\partial\Omega))^2$*, then the weak solution* (ρ, u) *from (1) satisfies* $\rho \in H^k(\Omega)$ *and* $u \in (H^{k+1}(\Omega))^2$.

We follow the arguments of [123] to prove Theorem 2.2.6 in Section 2.3.2. Since the presence of odd viscosity in the equation does not change much in the proof from [123], we are only going to explain the main steps. The idea is to look for solutions which are of Frolov's form, i.e. $(\rho, u) = (\eta(\phi), \nabla^\perp\phi)$, for the stream function ϕ (determined by (2.3.10)), and some function $\eta \in L^\infty(\mathbb{R}; [0,\infty))$.

Remark 2.2.7. Similarly to [123] one can show the existence of weak solutions on an exterior domain or on the whole plane $\mathbb{R}^2$. More precisely, if Ω is the exterior domain of a bounded simply connected $\mathcal{C}^1$ set in $\mathbb{R}^2$, or $\Omega = \mathbb{R}^2$, then there exists a function pair $(\rho, u) \in L^\infty(\Omega; [0,\infty)) \times (D^1(\Omega))^2$ satisfying (2.2.10), (2.2.11) (if $\Omega \neq \mathbb{R}^2$) and (2.2.12). Here, the space $D^1(\Omega)$ is defined by

$$D^1(\Omega) = \dot{H}^1(\Omega) \cap \Big(\cap_{n \in \mathbb{N}} H^1(\Omega \cap B_n) \Big).$$

Moreover, one can prove a statement analogous to Corollary 2.2.4.

Corollary 2.2.8. *Suppose the sequence* $(\mu_o^\epsilon)_{\epsilon \in (0,1)}$ *is given as in Corollary 2.2.4, and* $(\rho^\epsilon, u^\epsilon)$ *is any weak solution of* (oddINS') *of Frolov's form, i.e.,* $(\rho^\epsilon, u^\epsilon) = (\eta^\epsilon(\phi^\epsilon), \nabla^\perp\phi^\epsilon)$ *for some* *function $\phi^\epsilon \in H^2(\Omega)$, and $\eta^\epsilon \in L^\infty(\mathbb{R};[0,\rho^*])$. Then there exists a function pair (ρ,u) such that up to a subsequence*

$$\begin{aligned} \rho^\epsilon &\rightharpoonup \rho, \quad \textit{in } L^p(\Omega), \quad \forall p\in(1,\infty),\\ u^\epsilon &\rightharpoonup u, \quad \textit{in } H^1(\Omega), \end{aligned}$$

as $\epsilon\to 0$, and (ρ,u) is a weak solution of the stationary system (oddINS') *with $\mu_o(\rho)\equiv 0$:*

$$\begin{cases} \operatorname{div}(\rho u\otimes u) - \operatorname{div}(\mu(\rho)(\nabla u + \nabla^T u)) + \nabla\pi = f,\\ \operatorname{div} u = 0,\ \operatorname{div}(\rho u) = 0,\\ u|_{\partial\Omega} = g, \end{cases} \tag{2.2.13}$$

i.e. (ρ,u) satisfies the conditions in Definition 2.2.5 with $\mu_o(\rho)\equiv 0$.

The main step of the proof is to show that the sequence of stream functions $(\phi^\epsilon)_{\epsilon\in(0,1)}$ is bounded in $H^2(\Omega)$, by adjusting the arguments from [123, Section 2.1]. This then yields the uniform boundedness of $(u^\epsilon)_{\epsilon\in(0,1)}$ in $H^1(\Omega)$, from which the assertion follows by standard compactness arguments.

2.3. Proofs

2.3.1. The evolutionary system

In this section we prove the existence of weak solutions to the evolutionary system. We begin by regularising the given data in Subsection 2.3.1.1. Afterwards we prove the existence and weak convergence of a sequence of solutions to the regularised system. To do so, we treat the Dirichlet and periodic cases (Subsection 2.3.1.2) separately from the whole plane case (Subsection 2.3.1.3).

2.3.1.1. Regularisation of the data

In this subsection we regularise the initial data and external force as in [175].

Let $\alpha\in\mathcal{C}_c^\infty(\mathbb{R}^2)$ and $\beta\in\mathcal{C}_c^\infty(\mathbb{R})$ be smooth, compactly supported functions satisfying

$$\begin{aligned} &\operatorname{supp}\alpha\subset B_1(0)\subset\mathbb{R}^2, \quad 0\le\alpha\le 1, \quad \int_{\mathbb{R}^2}\alpha\,dx = 1,\\ &\operatorname{supp}\beta\subset(-1,1)\subset\mathbb{R}, \quad 0\le\beta\le 1, \quad \int_{\mathbb{R}}\beta dt = 1. \end{aligned}$$

Let $\epsilon\in(0,1)$. We set $\alpha_\epsilon = \epsilon^{-2}\alpha(\epsilon^{-1}\cdot)$, $\beta_\epsilon = \epsilon^{-1}\beta(\epsilon^{-1}\cdot)$. We choose approximate functions $\mu^\epsilon,\mu_o^\epsilon\in\mathcal{C}^\infty([0,\infty))$ with $\mu^\epsilon\ge\frac{\mu_*}{2}$, and

$$\|(\mu^\epsilon-\mu,\mu_o^\epsilon-\mu_o)\|_{\mathcal{C}([\rho_*,\rho^*])}\le\epsilon,.$$

Furthermore, in the Dirichlet case we set $\Omega_\delta = \{x\in\Omega : \operatorname{dist}(x,\partial\Omega)>\delta\}$, $\delta>0$. Then we define

$$\begin{aligned} f^\epsilon &= (1_{[\epsilon,\infty)}(f1_{\Omega_{2\epsilon}}) *_x \alpha_\epsilon) *_t \beta_\epsilon, \quad \rho_0^\epsilon = (\rho_0 1_\Omega + 1_{\mathbb{R}^2\setminus\Omega}) *_x \alpha_\epsilon, \quad \text{in the Dirichlet case,}\\ f^\epsilon &= (1_{[\epsilon,\infty)} f *_x \alpha_\epsilon) *_t \beta_\epsilon, \quad \rho_0^\epsilon = \rho_0 *_x \alpha_\epsilon, \quad \text{in the periodic and whole plane case,} \end{aligned}$$

and u_0^ϵ in the following way: Let $\overline{m}_0^\epsilon = \rho_0^\epsilon((u_0 1_{\Omega_{2\epsilon}}) * \alpha_\epsilon)$, and let $q_0^\epsilon \in \mathcal{C}^\infty(\Omega)$ be a solution of

$$\begin{aligned}
&\operatorname{div}\big(\frac{1}{\rho_0^\epsilon}(\nabla q_0^\epsilon - \overline{m}_0^\epsilon)\big) = 0,\\
&\frac{\partial q_0^\epsilon}{\partial n} = 0 \quad \text{on } \partial\Omega, \quad \text{in the Dirichlet and periodic case,}\\
&q_0^\epsilon \in H^1(\mathbb{R}^2), \quad \text{in the whole plane case,}
\end{aligned}$$

where n denotes the unit outward normal to $\partial\Omega$. Then we have $\overline{m}_0^\epsilon = \rho_0^\epsilon \overline{u}_0^\epsilon + \nabla q_0^\epsilon$ for some $\overline{u}_0^\epsilon \in \mathcal{C}^\infty(\Omega)$ satisfying

$$\begin{aligned}
&\operatorname{div} \overline{u}_0^\epsilon = 0,\\
&\overline{u}_0^\epsilon \cdot n = 0, \quad \text{on } \partial\Omega, \quad \text{in the Dirichlet and periodic case,}\\
&\overline{u}_0^\epsilon \in L^2(\mathbb{R}^2), \quad \text{in the whole plane case.}
\end{aligned}$$

Finally we choose

$$u_0^\epsilon \in \mathcal{C}_c^\infty(\Omega), \quad \operatorname{div} u_0^\epsilon = 0, \quad \|u_0^\epsilon - \overline{u}_0^\epsilon\|_{L^2(\Omega)} \le \epsilon.$$

With the above definitions there holds

$$\begin{aligned}
&f^\epsilon \in \mathcal{C}^\infty([0,\infty)\times\Omega), \quad \rho_0^\epsilon \in \mathcal{C}^\infty(\Omega), \quad u_0^\epsilon \in \mathcal{C}_c^\infty(\Omega),\\
&f^\epsilon \to f, \quad \text{in } (L^2((0,T)\times\Omega))^2, \quad \forall T > 0,\\
&\rho_0^\epsilon \to \rho_0, \quad \text{in } L^p(\Omega \cap B_R), \quad \forall p \in [1,\infty),\ R > 0,\\
&m_0^\epsilon := \overline{m}_0^\epsilon + \rho_0^\epsilon(u_0^\epsilon - \overline{u}_0^\epsilon) \to m_0 := \rho_0 u_0, \quad \text{in } L^2(\Omega),
\end{aligned}$$

as $\epsilon \to 0$.

Since we use a fixed point argument to prove the existence of smooth solutions to the regularised system, we recall here the Schauder fixed point theorem (see e.g. [106]).

Theorem 2.3.1 (Schauder fixed point theorem). *Let Y be a closed, convex set in a Banach space X, and let F be a continuous mapping of Y into itself such that $F(Y)$ is relatively compact. Then F has a fixed point, i.e., there exists an element $x \in Y$ such that $F(x) = x$.*

In the following we distinguish between the case that the domain Ω is bounded (Dirichlet and periodic case) and the case that Ω is unbounded (whole plane case).

2.3.1.2. Dirichlet and Periodic case

We use the Faedo-Galerkin approximations to construct a sequence of smooth solutions $(\rho^{\epsilon,n}, u^{\epsilon,n})_{\epsilon\in(0,1),n\in\mathbb{N}}$ as in [92]. We start by proving their existence using the Schauder fixed point theorem.

Existence of regularised solutions Let the set of divergence-free test functions on Ω be denoted by

$$\mathcal{D}_\sigma := \{\varphi \in (\mathcal{C}_c^\infty(\Omega))^2 : \operatorname{div}\varphi = 0\}, \quad \text{or}$$

$$:= \{\varphi \in (\mathcal{C}^\infty(\Omega))^2 : \operatorname{div}\varphi = 0,\ \ \varphi \text{ is periodic in } x_i \text{ of period } L_i,\ i = 1,2\},$$

in the Dirichlet or periodic case, respectively. Furthermore, let

$$\begin{aligned} L^2_\sigma &:= \{\varphi \in (L^2(\Omega))^2 : \operatorname{div}\varphi = 0\}, \quad \text{ or} \\ &:= \{\varphi \in (L^2(\Omega))^2 : \operatorname{div}\varphi = 0,\ \ \varphi \text{ is periodic in } x_i \text{ of period } L_i,\ i = 1,2\}, \\ V_\sigma &:= \{\varphi \in (H^1_0(\Omega))^2 : \operatorname{div}\varphi = 0\}, \quad \text{ or} \\ &:= \{\varphi \in (H^1(\Omega))^2 : \operatorname{div}\varphi = 0,\ \ \varphi \text{ is periodic in } x_i \text{ of period } L_i,\ i = 1,2\}, \end{aligned}$$

in the Dirichlet or periodic case, respectively. Since $\mathcal{D}_\sigma$ is dense in the Hilbert spaces (L^2_σ, $\|\cdot\|_{L^2}$) and (V_σ, $\|\cdot\|_{H^1}$), we can choose a countable system of functions $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{D}_\sigma$ which form an orthonormal basis in L^2_σ and are dense in V_σ. For $n \in \mathbb{N}$ let

$$X_n := \operatorname{span}\{\varphi_1, ..., \varphi_n\}.$$

It is easy to see that $\overline{X_n}^{\|\cdot\|_{L^2}} = X_n$, so that $(X_n, \|\cdot\|_{X_n})$ is a Hilbert space endowed with the norm $\|\cdot\|_{X_n} := \|\cdot\|_{L^2}$.

Notice that for $p \in [1,\infty]$ there exists a constant $C = C(n, \{\|\varphi_j\|_{W^{1,p}}\}_{j=1,\dots,n})$ such that

$$\|\varphi\|_{W^{1,p}} \le C\|\varphi\|_{X_n}, \quad \forall\varphi \in X_n. \tag{2.3.1}$$

We fix $T > 0$ and $n \in \mathbb{N}$. Our first goal is to find for each $\epsilon \in (0,1)$ a function pair $(\rho^{\epsilon,n}, u^{\epsilon,n}) \in \mathcal{C}([0,T]\times\Omega)\times\mathcal{C}([0,T];X_n)$ which satisfies (2.2.4) and (2.2.5) with test-functions $\phi \in \mathcal{C}^\infty_c([0,T)\times\Omega)$ and $\varphi \in \mathcal{C}^\infty_c([0,T);X_n)$, respectively, and initial data replaced by ρ^ϵ_0, u^ϵ_0, μ^ϵ, μ^ϵ_o, and f^ϵ.

Let us first introduce some notation. For a function $u \in \mathcal{C}([0,T];X_n)$ we denote by $\rho[u]$ the unique solution of the transport equation

$$\begin{cases} \partial_t\rho + u\cdot\nabla\rho = 0, & \text{in } (0,T)\times\Omega, \\ \rho(0,\cdot) = \rho^\epsilon_0. \end{cases}$$

By the smoothness of u in space we know that $\rho[u], \partial_t\rho[u] \in \mathcal{C}([0,T];\mathcal{C}^k(\Omega))$ for any $k \in \mathbb{N}_0$, with

$$\begin{aligned} &\|\rho[u](t)\|_{L^p} \le \|\rho^\epsilon_0\|_{L^p}\exp\Big(C\int_0^t \|\nabla u(t')\|_{L^\infty}dt'\Big), \\ &\|\rho[u](t)\|_{L^\infty} \le \|\rho^\epsilon_0\|_{L^\infty}, \end{aligned}$$

for $t \in [0,\infty)$ and $p \in [1,\infty)$. Clearly, $\rho[u]$ satisfies the weak formulation (2.2.4) with velocity field u and initial data ρ^ϵ_0.

In order to reformulate equation (2.2.5) as a fixed point problem we define for a function $\rho : \Omega \to [\rho_*, \rho^*]$ the operator

$$M_\rho : X_n \to X_n^*, \quad M_\rho(\varphi) = \Big(X_n \ni \tilde\varphi \mapsto \int_\Omega \rho\varphi\cdot\tilde\varphi\,dx\Big)$$

where X_n^* denotes the dual space of X_n. M_ρ is linear, continuous and bijective with inverse given by

$$M_\rho^{-1} : X_n^* \to X_n, \quad M_\rho^{-1}\varphi^* = \sum_{j=1}^n \varphi^*(\frac{1}{\rho}\varphi_j)\varphi_j,$$

and its operator norm satisfies the bound $\|M_\rho^{-1}\|_{X_n^*\to X_n} \le \frac{n}{\rho_*}$.

Lemma 2.3.2. *For $\epsilon \in (0,1)$ and sufficiently large $n \in \mathbb{N}$ there exists $u^{\epsilon,n} \in \mathcal{C}([0,T];X_n) \cap \mathcal{C}^1((0,T];X_n)$ such that*

$$\begin{aligned}
&\int_\Omega \rho^{\epsilon,n}(t) u^{\epsilon,n}(t) \cdot \varphi(t)\, dx - \int_\Omega \rho_0^\epsilon u_0^\epsilon \cdot \varphi(0)\, dx - \int_0^t \int_\Omega \rho^{\epsilon,n} u^{\epsilon,n} \cdot \partial_t \varphi \, dx dt' \\
&= \int_0^t \int_\Omega \rho^{\epsilon,n}(u^{\epsilon,n} \otimes u^{\epsilon,n}) : \nabla \varphi - \frac{\mu^{\epsilon,n}}{2} S u^{\epsilon,n} : S\varphi - \frac{\mu_o^{\epsilon,n}}{2} S_o u^{\epsilon,n} : S\varphi + \rho^{\epsilon,n} f^\epsilon \cdot \varphi \, dx dt'
\end{aligned} \tag{2.3.2}$$

for any function $\varphi \in \mathcal{C}^1([0,T];X_n)$ and time $t \in [0,T]$, where $\rho^{\epsilon,n} = \rho[u^{\epsilon,n}]$, and $\mu^{\epsilon,n} = \mu^\epsilon(\rho^{\epsilon,n})$, $\mu_o^{\epsilon,n} = \mu_o^\epsilon(\rho^{\epsilon,n})$. Moreover, $(\rho^{\epsilon,n}, u^{\epsilon,n})$ satisfies the energy equality

$$\begin{aligned}
&\frac{1}{2} \int_\Omega \rho^{\epsilon,n}(t) |u^{\epsilon,n}(t)|^2 \, dx + \int_0^t \int_\Omega \frac{\mu^{\epsilon,n}}{2} |S u^{\epsilon,n}|^2 \, dx dt' \\
&= \frac{1}{2} \int_\Omega \rho^{\epsilon,n}(0) |u^{\epsilon,n}(0)|^2 \, dx + \int_0^t \int_\Omega \rho^{\epsilon,n} f^\epsilon \cdot u^{\epsilon,n} \, dx dt'
\end{aligned} \tag{2.3.3}$$

for any $t \in [0,T]$.

Proof. We first reduce the problem of showing (2.3.2) to a simpler (but equivalent) problem: We claim that it suffices to look for $u^{\epsilon,n}$ satisfying

$$\begin{aligned}
&\int_\Omega \rho^{\epsilon,n}(t) u^{\epsilon,n}(t) \cdot \tilde\varphi - \rho_0^\epsilon u_0^\epsilon \cdot \tilde\varphi \, dx \\
&= \int_0^t \int_\Omega \rho^{\epsilon,n}(u^{\epsilon,n} \otimes u^{\epsilon,n}) : \nabla \tilde\varphi - \frac{\mu^{\epsilon,n}}{2} S u^{\epsilon,n} : S\tilde\varphi - \frac{\mu_o^{\epsilon,n}}{2} S_o u^{\epsilon,n} : S\tilde\varphi + \rho^{\epsilon,n} f^\epsilon \cdot \tilde\varphi \, dx dt'
\end{aligned} \tag{2.3.4}$$

for any $\tilde\varphi \in X_n$ and $t \in [0,T]$. Indeed, assume that (2.3.4) holds and take $\varphi \in \mathcal{C}^1([0,T];X_n)$. Differentiating (2.3.4) with respect to time and then testing with $\tilde\varphi = \varphi(t) \in X_n$ for fixed $t \in [0,T]$ implies

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega \rho^{\epsilon,n} u^{\epsilon,n} \cdot \varphi \, dx - \int_\Omega \rho^{\epsilon,n} u^{\epsilon,n} \cdot \partial_t \varphi \, dx \\
&= \int_\Omega \rho^{\epsilon,n}(u^{\epsilon,n} \otimes u^{\epsilon,n}) : \nabla \varphi - \frac{\mu^{\epsilon,n}}{2} S u^{\epsilon,n} : S\varphi - \frac{\mu_o^{\epsilon,n}}{2} S_o u^{\epsilon,n} : S\varphi + \rho^{\epsilon,n} f^\epsilon \cdot \varphi \, dx,
\end{aligned}$$

which yields the claimed equality (2.3.2) after integration in t.

In order to prove the existence of a solution to (2.3.4), we use the Schauder fixed point Theorem 2.3.1. We begin by reformulating this equation into a fixed point problem. One can rewrite the equation as

$$u^{\epsilon,n}(t) = M^{-1}_{\rho[u^{\epsilon,n}](t)} \Big((\rho_0^\epsilon u_0^\epsilon)^* + \int_0^t A(u^{\epsilon,n})(t') dt' \Big), \quad \forall t \in [0,T],$$

where the element $(\rho_0^\epsilon u_0^\epsilon)^* \in X_n^*$ and the operator $A : \mathcal{C}([0,T];X_n) \to L^\infty_T(X_n^*)$ are defined as

$$\begin{aligned}
(\rho_0^\epsilon u_0^\epsilon)^* &= \Big(X_n \ni \tilde\varphi \mapsto \int_\Omega \rho_0^\epsilon u_0^\epsilon \cdot \tilde\varphi \, dx \Big), \\
A(u) &= \Big(X_n \ni \tilde\varphi \mapsto \int_\Omega \rho[u](u \otimes u) : \nabla \tilde\varphi - \frac{\mu^\epsilon(\rho[u])}{2} S u : S\tilde\varphi \\
&\qquad - \frac{\mu_o^\epsilon(\rho[u])}{2} S_o u : S\tilde\varphi + \rho[u] f^\epsilon \cdot \tilde\varphi \, dx \Big),
\end{aligned}$$

for $u \in \mathcal{C}([0,T];X_n)$.

We define the set $E_T = \{u \in \mathcal{C}([0,T]; X_n) : \|u - u_0^\epsilon\|_{L_T^\infty(X_n)} \le 1\}$, which is clearly convex in $\mathcal{C}([0,T]; X_n)$ (and non-empty for n sufficiently large), and the operator

$$F : E_T \to \mathcal{C}([0,T]; X_n), \quad F(u)(t) := M_{\rho[u](t)}^{-1}\Big((\rho_0^\epsilon u_0^\epsilon)^* + \int_0^t A(u)(t')\,dt'\Big).$$

We first show that F has a fixed point provided $T = T(n)$ is sufficiently small.

1. Continuity. In order to verify that F is continuous, let $u, v \in \mathcal{C}([0,T]; X_n)$ and $t \in [0,T]$. Writing

$$\begin{aligned} F(u)(t) - F(v)(t) &= M_{\rho[u](t)}^{-1}\Big(\int_0^t A(u)(t') - A(v)(t')dt'\Big) \\ &\quad + (M_{\rho[u](t)}^{-1} - M_{\rho[v](t)}^{-1})\Big((\rho_0^\epsilon u_0^{\epsilon,n})^* + \int_0^t A(v)(t')\,dt'\Big), \end{aligned}$$

and using

$$\begin{aligned} \|M_{\rho[u](t)}^{-1} - M_{\rho[v](t)}^{-1}\|_{X_n^* \to X_n} &= \|M_{\rho[v](t)}^{-1}(M_{\rho[v](t)} - M_{\rho[u](t)})M_{\rho[u](t)}^{-1}\|_{X_n^* \to X_n} \\ &\le C\|\rho[u](t) - \rho[v](t)\|_{L^\infty} \\ &\le C\|u - v\|_{L_T^\infty(X_n)} \exp(C\|v\|_{L_T^\infty(X_n)}) \end{aligned} \tag{2.3.5}$$

it is easy to see that

$$\begin{aligned} \|F(u) - F(v)\|_{L_T^\infty(X_n)} &\le C\|u - v\|_{L_T^\infty(X_n)} \exp(C\|v\|_{L_T^\infty(X_n)})(1 + \|u\|_{L_T^\infty(X_n)} \\ &\quad + \|v\|_{L_T^\infty(X_n)} + \|v\|_{L_T^\infty(X_n)}^2 + \|u_0^\epsilon\|_{L^2} + \|f^\epsilon\|_{L_T^2(L^2)}) \end{aligned}$$

for some constant $C = C(n,T)$, which shows that F is locally Lipschitz continuous and in particular continuous.

2. Invariance of E_T under F. Next we verify that F maps the ball E_T into itself provided the time T is sufficiently small. Firstly, let $R > 0$ such that $\|u\|_{L_T^\infty(X_n)} \le R$ for any $u \in E_T$. For $t \in (0,T]$ we have

$$\begin{aligned} \|F(u)(t) - u_0^\epsilon\|_{X_n} &\le C\|(\rho_0^\epsilon u_0^\epsilon)^* + \int_0^t A(u)(t')dt' - M_{\rho[u](t)}u_0^\epsilon\|_{X_n^*} \\ &\le C\|\rho_0^\epsilon - \rho[u](t)\|_{L^\infty}\|u_0^\epsilon\|_{L^2} + C\int_0^t \|u\|_{X_n}^2 + \|u\|_{X_n} + \|f^\epsilon\|_{L^2}dt', \end{aligned}$$

so that

$$\|F(u) - u_0^\epsilon\|_{L_T^\infty(X_n)} \le C\|\rho_0^\epsilon - \rho[u]\|_{L_T^\infty(L^\infty)}\|u_0^\epsilon\|_{L^2} + CT(R^2 + R + \|f^\epsilon\|_{L_T^\infty(L^2)})$$

for a constant $C = C(n)$ independent of T. Hence we can choose $T(n) \in (0,T]$ sufficiently small such that F maps $E_{T(n)}$ into itself.

3. Relative compactness. It is left to show that the image $F(E_{T(n)})$ is relatively compact in the set $E_{T(n)}$. To do so we use the version of the Arzela-Ascoli Theorem from [190, Theorem 47.1]: If $F(E_{T(n)})$ is equicontinuous, and the sets $F_t = \{v(t) : v \in F(E_{T(n)})\}$ are relatively compact in X_n for all $t \in [0, T(n)]$, then it follows that $F(E_{T(n)})$ is relatively compact in $E_{T(n)}$. Notice that the latter condition is obvious here since $F_t \subset \{\varphi \in X_n : \|\varphi - u_0^\epsilon\|_{X_n} \le 1\}$ is a bounded set in the finite-dimensional vector space X_n. In order to show that $F(E_{T(n)})$ is equicontinuous, let $u \in E_{T(n)}$ and $t_1, t_2 \in [0, T(n)]$. Writing

$$F(u)(t_1) - F(u)(t_2) = (M_{\rho[u](t_1)}^{-1} - M_{\rho[u](t_2)}^{-1})\Big((\rho_0^\epsilon u_0^\epsilon)^* + \int_0^{t_1} A(u)(t')\,dt'\Big)$$

$$+ M^{-1}_{\rho[u](t_2)}\Big(\int_{t_2}^{t_1} A(u)(t')\,dt'\Big),$$

and using similar estimates as before, we deduce

$$\|F(u)(t_1) - F(u)(t_2)\|_{X_n} \leq C|t_1 - t_2| + C|t_1 - t_2|^{\frac{1}{2}}$$

for some constant $C > 0$ independent of t_1, t_2 and u, which yields equicontinuity.

4. Conclusions. By the Schauder fixed point Theorem 2.3.1 there exists a fixed point $u^{\epsilon,n} \in \mathcal{C}([0,T(n)];X_n)$, i.e., $F(u^{\epsilon,n}) = u^{\epsilon,n}$. It is straightforward to verify that $F(u)$ belongs to $\mathcal{C}^1((0,T];X_n)$ for any $u \in \mathcal{C}([0,T];X_n)$, which implies that the fixed point is continuously differentiable in time: $u^{\epsilon,n} \in \mathcal{C}^1((0,T(n)];X_n)$.

Differentiating (2.3.4) with respect to t, choosing $\varphi = u^{\epsilon,n}(t)$, and integrating by parts (which is justified by the smoothness of $\rho^{\epsilon,n}$ and $u^{\epsilon,n}$ in space, and where the boundary values in the periodic case vanish due to the periodicity) yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^{\epsilon,n}|u^{\epsilon,n}|^2\,dx + \frac{\mu_*}{2}\int_\Omega |Su^{\epsilon,n}|^2\,dx \leq \sqrt{\rho^*}\|f^\epsilon\|_{L^2(\Omega)}\|\sqrt{\rho^{\epsilon,n}}u^{\epsilon,n}\|_{L^2(\Omega)}$$

for every $t \in [0,T(n)]$. Here we used the cancellation (2.1.2). By Gronwall's inequality it follows that,

$$\begin{aligned}\|(\sqrt{\rho^{\epsilon,n}}u^{\epsilon,n})(t)\|^2_{L^2(\Omega)} &\leq \|\sqrt{\rho_0^\epsilon}u_0^\epsilon\|^2_{L^2(\Omega)} + 2\rho^*\|f^\epsilon\|^2_{L^2((0,t)\times\Omega)}e^{2t}\\ &\leq \rho^*\|u_0\|^2_{L^2(\Omega)} + 2\rho^*\|f\|^2_{L^2((0,t)\times\Omega)}e^{2t}\end{aligned}$$

for some constant $C > 0$ independent of ϵ and n. Consequently, $u^{\epsilon,n}$ is uniformly bounded in $L^\infty_{T(n)}(L^2) \cap L^2_{T(n)}(H^1)$, with a bound independent of ϵ, n and $T(n)$. This allows us to iterate the above procedure to obtain a solution which is defined for all times up to T.

Finally, the energy equality (2.3.3) follows from choosing $\varphi = u^{\epsilon,n}$ in (2.3.2). □

The limit $n \to \infty$ We next show that for fixed $\epsilon \in (0,1)$ the sequence of weak solutions $(\rho^{\epsilon,n}, u^{\epsilon,n})_{n\in\mathbb{N}}$ constructed in Lemma 2.3.2 converges weakly up to a subsequence to some function pair $(\rho^\epsilon, u^\epsilon)$ satisfying

$$\begin{aligned}&\int_\Omega \rho^\epsilon(t)u^\epsilon(t)\cdot\varphi(t)\,dx - \int_\Omega \rho_0^\epsilon u_0^\epsilon\cdot\varphi(0)\,dx - \int_0^t\int_\Omega \rho^\epsilon u^\epsilon\cdot\partial_t\varphi\,dxdt'\\ &= \int_0^t\int_\Omega \rho^\epsilon(u^\epsilon\otimes u^\epsilon):\nabla\varphi - \frac{\mu^\epsilon(\rho^\epsilon)}{2}Su^\epsilon : S\varphi - \frac{\mu_o^\epsilon(\rho^\epsilon)}{2}S_o u^\epsilon : S\varphi + \rho^\epsilon f^\epsilon\cdot\varphi\,dxdt'\end{aligned} \tag{2.3.6}$$

for any function $\varphi \in \mathcal{C}^1([0,T];V_\sigma)$ and time $t \in [0,T]$.

Since $(u^{\epsilon,n})_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty_T(L^2)\cap L^2_T(H^1)$, there exists $u^\epsilon \in L^\infty_T(L^2)\cap L^2_T(H^1)$ ($u^\epsilon \in L^\infty_T(L^2)\cap L^2_T(H^1_0)$ in the Dirichlet case) such that up to a subsequence

$$u^{\epsilon,n} \rightharpoonup^* u^\epsilon, \quad \text{in } L^\infty_T(L^2), \quad \nabla u^{\epsilon,n} \rightharpoonup \nabla u^\epsilon, \quad \text{in } L^2_T(L^2).$$

Moreover, [175, Theorem 2.4] yields the existence of $\rho^\epsilon \in L^\infty_T(L^\infty)$ such that up to a subsequence

$$\rho^{\epsilon,n} \to \rho^\epsilon, \quad \text{in } \mathcal{C}([0,T];L^p), \quad \forall p \in [1,\infty)$$

with ρ^ϵ being the solution to

$$\partial_t \rho^\epsilon + \operatorname{div}(\rho^\epsilon u^\epsilon) = 0, \quad \rho^\epsilon(0,\cdot) = \rho_0$$

in the sense of distributions, i.e. (2.2.4) holds with ρ^ϵ. Consequently $\mu^{\epsilon,n} \to \mu^\epsilon(\rho^\epsilon)$, $\mu_o^{\epsilon,n} \to \mu_o^\epsilon(\rho^\epsilon)$ in $\mathcal{C}([0,T];L^p)$ for $p \in [1,\infty)$. Since $\rho^{\epsilon,n}u^{\epsilon,n} \to \rho^\epsilon u^\epsilon$ in $L^2_T(L^r)$ for all $r \in [1,\infty)$ and $u^{\epsilon,n} \rightharpoonup^* u^\epsilon$ in $L^\infty_T(L^2)$ it follows that $\rho^{\epsilon,n}u^{\epsilon,n} \otimes u^{\epsilon,n} \rightharpoonup \rho^\epsilon u^\epsilon \otimes u^\epsilon$ in $L^2_T(L^p)$ for any $p \in [1,2)$. All of this together ensures the convergence of the integrals in (2.3.2) for fixed $\varphi \in \mathcal{C}^1([0,T];X_{n_0})$ for any $n_0 \in \mathbb{N}$. Since $\operatorname{span}\{\varphi_n\}_{n\in\mathbb{N}}$ is dense in V_σ, the integral identity also holds for any function $\varphi \in \mathcal{C}^1([0,T];V_\sigma)$.

Observe that $u^{\epsilon,n}(0)$ is given by the projection in L^2 of u_0^ϵ onto X_n. This implies that $u^{\epsilon,n}(0) \to u_0^\epsilon$ in L^2. With the same arguments as in [175] we can therefore take the limit $n \to \infty$ in the energy equality (2.3.3) to obtain the energy *in*equality for $(\rho^\epsilon, u^\epsilon)$

$$\frac{1}{2}\int_\Omega \rho^\epsilon(t)|u^\epsilon(t)|^2\,dx + \int_0^t \int_\Omega \frac{\mu^\epsilon(\rho^\epsilon)}{2}|Su^\epsilon|^2\,dxdt' \le \frac{1}{2}\int_\Omega \rho^\epsilon(0)|u^\epsilon(0)|^2\,dx + \int_0^t\int_\Omega \rho^\epsilon f^\epsilon \cdot u^\epsilon\,dxdt' \tag{2.3.7}$$

for all $t \in [0,T]$.

The limit $\epsilon \to 0$ Analogous arguments as before yield the existence of a pair of functions $(\rho,u) \in \mathcal{C}([0,T];L^p) \times (L^\infty_T(L^2) \cap L^2_T(H^1))$ ($u \in L^\infty_T(L^2)\cap L^2_T(H^1_0)$ in the Dirichlet case) which is the weak limit of a subsequence of $(\rho^\epsilon,u^\epsilon)_{\epsilon\in(0,1)}$ and satisfies (2.2.4) and

$$\begin{aligned}&\int_\Omega \rho(t)u(t)\cdot\varphi(t)\,dx - \int_\Omega \rho_0 u_0\cdot\varphi(0)\,dx - \int_0^t\int_\Omega \rho u\cdot\partial_t\varphi\,dxdt' \\ &= \int_0^t\int_\Omega \rho(u\otimes u):\nabla\varphi - \frac{\mu(\rho)}{2}Su:S\varphi - \frac{\mu_o(\rho)}{2}S_o u:S\varphi + \rho f\cdot\varphi\,dxdt'\end{aligned}$$

for any $\varphi \in \mathcal{C}^1([0,T];V_\sigma)$ for all $t \in [0,T]$. Since $T > 0$ was arbitrary we can find a weak solution (ρ,u) which is defined for all times. This entails the weak formulations (2.2.4) and (2.2.5), and the existence part of Theorem 2.2.2 for the Dirichlet and periodic case is proven.

By taking the limit $\epsilon \to 0$ in (2.3.7) we obtain the energy inequality for (ρ,u).

2.3.1.3. Whole plane case

We now turn to the existence proof for the whole plane case $\Omega = \mathbb{R}^2$. Similarly to [175], we derive the result for the whole plane case from the Dirichlet case.

Applying the Dirichlet case with Ω chosen as the ball $B_R := B_R(0)$ of radius $R > 0$, and initial data

$$\rho_0^R := \rho_0 1_{B_R}, \quad u_0^R := u_0 1_{B_R}, \quad f^R := f 1_{B_R},$$

yields a sequence $(\rho^R,u^R)_{R>0}$ of weak solutions on B_R with

$$\begin{aligned}&\rho^R \in L^\infty((0,\infty)\times B_R)\cap\mathcal{C}_b([0,\infty);L^p(B_R)), \quad \forall p\in[1,\infty),\\ &u^R \in L^2(0,T;(H^1_0(B_R))^2), \quad \forall T>0.\end{aligned}$$

In the following we are going to show that up to a subsequence there holds

$$\rho^R \to \rho \ \text{ in } \mathcal{C}([0,T];L^p(B_M)), \qquad u^R \rightharpoonup u \ \text{ in } L^2(0,T;(H^1(B_M))^2)$$

as $R \to \infty$, for any $T, M > 0$ and $p \in [1, \infty)$, where (ρ, u) is a weak solution of (oddINS) on the whole plane $\mathbb{R}^2$.

Indeed, by Remark 2.2.3 the energy inequality (2.2.6) holds for the weak solutions (ρ^R, u^R), which entails that the bound

$$\|u^R\|_{L^2(0,T;H^1(\mathbb{R}^2))} \leq C_T, \qquad \|\sqrt{\rho^R} u^R\|_{L^\infty(0,T;L^2(\mathbb{R}^2))} \leq C_T$$

holds uniformly in R, for all $T > 0$. Here we view ρ^R and u^R as functions defined on $\mathbb{R}^2$ by extending them by zero onto $\mathbb{R}^2 \setminus B_R$. This implies the existence of a function $u \in L^2_{\mathrm{loc}}(0, \infty; (H^1(\mathbb{R}^2))^2)$ such that

$$u^R \rightharpoonup^* u \ \text{ in } L^\infty(0, T; (L^2(\mathbb{R}^2))^2), \quad \nabla u^R \rightharpoonup \nabla u \ \text{ in } L^2(0, T; (L^2(\mathbb{R}^2))^4), \quad \forall T > 0,$$

as $R \to \infty$, up to a subsequence. By [175, Theorem 2.5] there exists a function $\rho \in L^\infty((0, \infty) \times \mathbb{R}^2)$ satisfying the transport equation $(\text{oddINS})_1$ in the weak sense with initial datum ρ_0 and velocity vector field u, and

$$\rho^R \to \rho \ \text{ in } \mathcal{C}([0, T]; L^p(B_M)), \quad \forall p \in [1, \infty), \ T, M > 0$$

as $R \to \infty$, up to a subsequence. Hence also $\mu(\rho^R) \to \mu(\rho)$, $\mu_o(\rho^R) \to \mu_o(\rho)$ in $\mathcal{C}([0, T]; L^p(B_M))$ for any $p \in [1, \infty)$, $T, M > 0$. Observe that also $\rho^R f^R \to \rho f$ in $L^2(0, T; (L^2(\mathbb{R}^2))^2)$ as $R \to \infty$.

Notice that $\rho^R u^R \otimes u^R$ is bounded in $L^2(0, T; (L^{\frac{4}{3}}(\mathbb{R}^2))^4)$, and $\mu(\rho^R) S u^R$, $\mu_o(\rho^R) S_o u^R$ and $\rho^R f^R$ are bounded in $L^2((0, T) \times \mathbb{R}^2)$. Using the weak formulation (2.2.5) we thus have for every $\varphi \in L^2(0, T; (H^2(\mathbb{R}^2))^2)$ with $\operatorname{div} \varphi = 0$,

$$|\langle \partial_t(\rho^R u^R), \varphi \rangle| \leq C \|\varphi\|_{L^2(0,T;H^2(\mathbb{R}^2))}$$

for some constant $C > 0$ independent of R. Since additionally $\rho^R |u^R|^2$ is bounded in $L^\infty(0, T; (L^1(\mathbb{R}^2))^2)$, it follows by [175, Theorem 2.5] that

$$\sqrt{\rho^R} u^R \to \sqrt{\rho} u \ \text{ in } L^p(0, T; (L^r(B_M))^2), \quad \forall p \in (2, \infty), \ r \in [1, \frac{2p}{p-2}),$$

for any $M > 0$. This ensures the convergence of the integrals in the weak formulation as $R \to \infty$, so that (ρ, u) satisfies (2.2.5) with $\Omega = \mathbb{R}^2$.

2.3.1.4. Proof of Corollary 2.2.4

Let $(\mu_o^\epsilon)_{\epsilon \in (0,1)}$ denote the sequence from Corollary 2.2.4, and let $(\rho^\epsilon, u^\epsilon)_{\epsilon \in (0,1)}$ be the corresponding sequence of weak solutions of (oddINS) constructed in Theorem 2.2.2. The energy inequality (2.2.6) entails that $(u^\epsilon)_{\epsilon \in (0,1)}$ is uniformly bounded in $L^\infty(0, T; (L^2(\Omega))^2) \cap L^2(0, T; (\dot{H}^1(\Omega))^2)$ for every $T > 0$.

Similar arguments as in Subsection 2.3.1.3 yield the existence of a function pair (ρ, u) with

$$\rho^\epsilon \to \rho \text{ in } \mathcal{C}([0, T]; L^p(B_M)), \quad u^\epsilon \rightharpoonup u \text{ in } L^2(0, T; (H^1(B_M))^2)$$

as $\epsilon \to 0$, for any $T, M > 0$ and $p \in [1, \infty)$, where (ρ, u) is a weak solution of (oddINS) with shear and odd viscosity coefficients $\mu(\rho)$ and c_0, respectively. To prove that (ρ, u) is in fact a weak solution of (INS) it remains to verify the cancellation

$$\int_\Omega (\nabla u^\perp + \nabla^\perp u) : (\nabla \varphi + \nabla^T \varphi) \, dx = 0 \tag{2.3.8}$$

for all φ as in (2.2.5). Indeed, using integration by parts twice and that u and φ are both divergence-free the left hand side can be written as

$$\begin{aligned}
&\int_\Omega 2(\partial_1 u_1 - \partial_2 u_2)(\partial_1\varphi_2 + \partial_2\varphi_1) - 2(\partial_1 u_2 + \partial_2 u_1)(\partial_1\varphi_1 - \partial_2\varphi_2)\, dx \\
&= \int_\Omega -2u_1(\partial_{11}\varphi_2 + \partial_{12}\varphi_1) + 2u_2(\partial_{12}\varphi_2 + \partial_{22}\varphi_1) \\
&\quad + 2u_2(\partial_{11}\varphi_1 - \partial_{12}\varphi_2) + 2u_1(\partial_{12}\varphi_1 - \partial_{22}\varphi_2)\, dx \\
&= \int_\Omega -2u_1\partial_{11}\varphi_2 + 2u_2\partial_{22}\varphi_1 + 2u_2\partial_{11}\varphi_1 - 2u_1\partial_{22}\varphi_2\, dx \\
&= \int_\Omega 2\partial_1 u_1\partial_1\varphi_2 - 2\partial_2 u_2\partial_2\varphi_1 + 2\partial_2 u_2\partial_1\varphi_2 - 2\partial_1 u_1\partial_2\varphi_1\, dx \\
&= \int_\Omega 2\partial_1 u_1(\partial_1\varphi_2 + \partial_2\varphi_1) - 2\partial_1 u_1(\partial_1\varphi_2 + \partial_2\varphi_1)\, dx \\
&= 0.
\end{aligned}$$

This implies that the odd viscosity terms in the weak formulation (2.2.5) vanish and hence (ρ, u) is a weak solution of (INS).

2.3.2. The stationary system

In this section we consider the stationary Navier-Stokes equation (oddINS'). We first explain the main steps of the existence proof of Theorem 2.2.6, and then study examples of the flow under certain symmetry assumptions on the density.

2.3.2.1. Proof of Theorem 2.2.6

In this paragraph we explain the strategy of the proof of Theorem 2.2.6. Since most arguments coincide with the ones in the proof of [123, Theorem 1.5], we omit the details of the proof and only describe the main ideas. We look for weak solutions which are of Frolov's form

$$(\rho, u) = (\eta(\phi), \nabla^\perp\phi), \tag{2.3.9}$$

for the stream function ϕ, and some given function $\eta \in L^\infty(\mathbb{R}; [0, \infty))$.

The existence proof of Theorem 2.2.6 is carried out in two steps: The first step is to formulate the boundary value problem for the stream function and to show the existence of a weak solution to this problem. In a second step one goes back to the original equation and shows that any pair which is of Frolov's form (2.3.9) is indeed a weak solution to (oddINS'). In the presence of odd viscosity the main changes are in step one since the boundary value problem for the stream function is modified. However, since the problem stays elliptic (due to $\mu(\rho) \geq \mu_* > 0$) the arguments from [123] still work.

The regularity results of Theorem 2.2.6 are proven using the elliptic equation (2.3.10) for ϕ, with the same arguments as in [123], which we omit here.

Step 1: We begin by formulating the boundary value problem for the stream function. Firstly, we transform the equation for u into a fourth order elliptic equation for the stream function ϕ by applying the two-dimensional curl operator $\nabla^\perp\cdot$ to the momentum equation $(\text{oddINS'})_1$. A straightforward calculation yields

$$\nabla^\perp \cdot \operatorname{div}(\mu(\rho)Su) = \mathcal{L}_\mu\phi, \quad \nabla^\perp \cdot \operatorname{div}(\mu_o(\rho)S_o u) = \mathcal{J}_{\mu_o}\phi,$$

where the operators $\mathcal{L}_\mu$ and $\mathcal{J}_{\mu_o}$ are defined as

$$\begin{aligned}\mathcal{L}_\mu &= (\partial_{22}-\partial_{11})(\mu(\rho)(\partial_{22}-\partial_{11})) + (2\partial_{12})(\mu(\rho)(2\partial_{12})),\\ \mathcal{J}_{\mu_o} &= (\partial_{22}-\partial_{11})(\mu_o(\rho)(2\partial_{12})) - (2\partial_{12})(\mu_o(\rho)(\partial_{22}-\partial_{11})).\end{aligned}$$

The momentum equation therefore becomes

$$\mathcal{L}\phi = -\nabla^\perp\cdot f + \nabla^\perp\cdot \operatorname{div}(\eta(\phi)\nabla^\perp\phi\otimes\nabla^\perp\phi),$$

where $\mathcal{L} = \mathcal{L}_\mu + \mathcal{J}_{\mu_o}$. Observe that $\mathcal{L}$ is an elliptic operator. Here we call an operator $\tilde{\mathcal{L}} = \sum_{|\alpha|,|\beta|\le 2} D^\alpha(a_{\alpha\beta}D^\beta)$ elliptic if there exists some $\delta>0$ such that

$$\delta|\xi|^2 \le \sum_{|\alpha|=|\beta|=2} \operatorname{Re}\Big(a_{\alpha\beta}(x)\xi_\beta\xi_\alpha\Big) \le \delta^{-1}|\xi|^2$$

for almost every $x\in\Omega$ and every $\xi = (\xi_\alpha)_{|\alpha|=2}$, $\xi_\alpha\in\mathbb{R}$. We write

$$\begin{aligned}\mathcal{L}_\mu &= \partial_{11}(\mu(\rho)\partial_{11}) + \partial_{22}(\mu(\rho)\partial_{22}) - \partial_{11}((\mu(\rho)-\frac{\mu_*}{2})\partial_{22}) - \partial_{22}((\mu(\rho)-\frac{\mu_*}{2})\partial_{11})\\ &\quad + 2\partial_{12}((\mu(\rho)-\frac{\mu_*}{2})\partial_{12}) + 2\partial_{21}(\mu(\rho)\partial_{21}) =: \sum_{|\alpha|=|\beta|=2} D^\alpha(a^e_{\alpha\beta}D^\beta),\\ \mathcal{J}_{\mu_o} &= \partial_{22}(\mu_o(\rho)\partial_{12}) + \partial_{22}(\mu_o(\rho)\partial_{21}) - \partial_{12}(\mu_o(\rho)\partial_{22}) - \partial_{21}(\mu_o(\rho)\partial_{22}) - \partial_{11}(\mu_o(\rho)\partial_{12})\\ &\quad - \partial_{11}(\mu_o(\rho)\partial_{21}) + \partial_{12}(\mu_o(\rho)\partial_{11}) + \partial_{21}(\mu_o(\rho)\partial_{11}) =: \sum_{|\alpha|=|\beta|=2} D^\alpha(a^o_{\alpha\beta}D^\beta),\end{aligned}$$

so that for $\xi = (\xi_\alpha)_{|\alpha|=2}$, $\xi_\alpha\in\mathbb{R}$, there holds

$$\frac{\mu_*}{2}|\xi|^2 \le \sum_{|\alpha|=|\beta|=2} a^e_{\alpha\beta}(x)\xi_\alpha\xi_\beta \le 2\mu^*|\xi|^2, \qquad \sum_{|\alpha|=|\beta|=2} a^o_{\alpha\beta}(x)\xi_\alpha\xi_\beta = 0$$

for almost every $x\in\Omega$. Hence, the operator $\mathcal{L}$ with coefficients $a_{\alpha\beta} = a^e_{\alpha\beta} + a^o_{\alpha\beta}$ for $|\alpha|,|\beta|\le 2$, is elliptic.

Let $\phi_0 \in H^{\frac{3}{2}}(\partial\Omega)$ and $\phi_1\in H^{\frac12}(\partial\Omega)$ be given functions. Pairing the equation for ϕ with boundary conditions we obtain the following boundary value problem for the stream function

$$\begin{cases}\mathcal{L}\phi = -\nabla^\perp\cdot f + \nabla^\perp\cdot\operatorname{div}(\eta(\phi)\nabla^\perp\phi\otimes\nabla^\perp\phi),\\ \phi|_{\partial\Omega} = \phi_0,\ \frac{\partial\phi}{\partial n}|_{\partial\Omega} = \phi_1.\end{cases} \tag{2.3.10}$$

We define weak solutions of (2.3.10) as follows.

Definition 2.3.3 (Weak solutions of the elliptic equation). *Let $\Omega\subset\mathbb{R}^2$ be a bounded simply connected $\mathcal{C}^{1,1}$ domain, let $\eta\in L^\infty(\mathbb{R};[0,\infty))$, and $\mu_e\in\mathcal{C}(\mathbb{R};[\mu_*,\mu^*])$, $\mu_o\in\mathcal{C}(\mathbb{R};[-\mu^*,\mu^*])$, $\mu_*,\mu^*>0$. Moreover, let $\phi_0\in H^{\frac32}(\partial\Omega)$, $\phi_1\in H^{\frac12}(\partial\Omega)$, and $f\in(H^{-1}(\mathbb{R}^2))^2$. We call $\phi\in H^2(\Omega)$ a weak solution to* (2.3.10) *if*

$$\phi|_{\partial\Omega} = \phi_0, \quad \frac{\partial\phi}{\partial n}|_{\partial\Omega} = \phi_1, \quad \textit{in the trace sense,}$$

and

$$\begin{aligned}&\int_\Omega \mu(\rho)((\partial_{22}\phi-\partial_{11}\phi)(\partial_{22}\psi-\partial_{11}\psi) + (2\partial_{12}\phi)(2\partial_{12}\psi))\,dx\\ &+\int_\Omega \mu_o(\rho)((2\partial_{12}\phi)(\partial_{22}\psi-\partial_{11}\psi) - (\partial_{22}\phi-\partial_{11}\phi)(2\partial_{12}\psi))\,dx\\ &= \int_\Omega \rho(\nabla^\perp\phi\otimes\nabla^\perp\phi):\nabla\nabla^\perp\psi\,dx + \langle f,\nabla^\perp\psi\rangle_{H^{-1}(\Omega)\times H^1_0(\Omega)}\end{aligned} \tag{2.3.11}$$

for any $\psi\in H^2_0(\Omega)$, where $\rho = \eta(\phi)$.

We have the following result concerning the existence of weak solutions to the elliptic equation (2.3.10).

Lemma 2.3.4. *Let $\eta \in L^\infty(\mathbb{R};[0,\infty))$, $\mu_e \in \mathcal{C}(\mathbb{R};[\mu_*,\mu^*])$, $\mu_o \in \mathcal{C}(\mathbb{R};[-\mu^*,\mu^*])$ and $f \in (H^{-1}(\Omega))^2$ be given. Then for any functions $\phi_0 \in H^{\frac{3}{2}}(\partial\Omega)$, $\phi_1 \in H^{\frac{1}{2}}(\partial\Omega)$ there exists a weak solution $\phi \in H^2(\Omega)$ to the boundary value problem* (2.3.10)*.*

We are not going to give a proof of the preceding lemma here, but one can for example follow the lines of [123, Section 2]; see also [162] for the classical stationary Navier-Stokes equations.

Step 2: We now go back to the original equation. Given a boundary value g, we show that for suitably chosen boundary values ϕ_0 and ϕ_1, a pair of Frolov's form (2.3.9) is a weak solution of (oddINS') provided ϕ is a weak solution of (2.3.10).

Notice that if $\phi \in H^2_{\text{loc}}(\Omega)$ satisfies (2.3.11), then $(\rho, u) = (\eta(\phi), \nabla^\perp\phi)$ satisfies (2.2.12). Moreover, $\operatorname{div}(\rho u) = 0$ holds in the distribution sense. Therefore, we have indeed proved that one can obtain a weak solution (ρ, u) of (oddINS') from a weak solution ϕ of (2.3.10) in the following way.

Lemma 2.3.5. *Let $\eta \in L^\infty(\mathbb{R};[0,\infty))$, $\mu_e \in \mathcal{C}(\mathbb{R};[\mu_*,\mu^*])$, $\mu_o \in \mathcal{C}(\mathbb{R};[-\mu^*,\mu^*])$ and $f \in (H^{-1}(\Omega))^2$ be given. Let $g \in (H^{\frac{1}{2}}(\partial\Omega))^2$ satisfy the zero-flux condition* (2.2.9) *and let $C_0 \in \mathbb{R}$. Moreover, let $\phi_0 \in H^{\frac{3}{2}}(\partial\Omega)$ and $\phi_1 \in H^{\frac{1}{2}}(\partial\Omega)$ be defined by*

$$\phi_0(\gamma(s)) = -\int_0^s g\cdot n\,d\theta + C_0, \quad \phi_1(\gamma(s)) = (u_0\cdot n^\perp)(\gamma(s)),$$

for $s \in [0,2\pi)$, where $\gamma : [0,2\pi) \to \partial\Omega$ is a parametrisation of the boundary $\partial\Omega$. If $\phi \in H^2(\Omega)$ is a weak solution of (2.3.10)*, then the pair*

$$(\rho, u) = (\eta(\phi), \nabla^\perp\phi)$$

is a weak solution of (oddINS')*.*

The regularity results in Theorem 2.2.6 (2) follow from successively applying the differential operator ∂_j, $j = 1, 2$, to the elliptic equation $(2.3.10)_1$ and using elliptic theory from [79] to deduce bounds on $\|\nabla^{k+2}\phi\|_{L^2(\Omega)}$. We omit the details and instead refer to [123, Paragraph 1.3.2].

2.3.2.2. Proof of Corollary 2.2.8

Let $(\phi^\epsilon)_{\epsilon\in(0,1)}$ denote the weak solutions of (2.3.10) corresponding to the odd viscosity coefficients $(\mu_o^\epsilon)_{\epsilon\in(0,1)}$. If we can show that the sequence of stream functions are uniformly bounded in $H^2(\Omega)$, then the sequence $(u^\epsilon)_{\epsilon\in(0,1)}$ is uniformly bounded in $(H^1(\Omega))^2$, and the claim follows by compactness arguments and the cancellation (2.3.8).

To show the uniform boundedness of $(\phi^\epsilon)_{\epsilon\in(0,1)}$ in $H^2(\Omega)$ one can follow the lines of [123, Paragraph 1.3.1], which we are only giving a rough sketch of here.

By the inverse trace theorem and Whitney's extension theorem ϕ_0 can be extended onto $\mathbb{R}^2$ (still denoted by ϕ_0) such that $\phi_0 \in H^2(\mathbb{R}^2)$ and $\frac{\partial\phi_0}{\partial n}|_{\partial\Omega} = \phi_1$. Next we let $\delta \in (0,1)$

and $\zeta^\delta \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ be a smooth cut-off function satisfying $\zeta^\delta = 1$ near the boundary $\partial\Omega$, and $\zeta^\delta(x) = 0$ for $\operatorname{dist}(x, \partial\Omega) \geq \delta$. Let $\phi_0^\delta := \phi_0\zeta^\delta$. Notice that $\tilde{\phi}^{\epsilon,\delta} := \phi^\epsilon - \phi_0^\delta \in H_0^2(\Omega)$, so that $\|\tilde{\phi}^{\epsilon,\delta}\|_{H^2(\Omega)} \sim \|\Delta\tilde{\phi}^{\epsilon,\delta}\|_{L^2(\Omega)}$.

The idea is to prove the uniform boundedness via a contradiction argument. Supposing that a subsequence $(\phi^{\epsilon_n})_{n\in\mathbb{N}}$ satisfies $\|\phi^{\epsilon_n}\|_{H^2(\Omega)} \to \infty$ as $n \to \infty$, we can test (2.3.11) by $\tilde{\phi}^{\epsilon,\delta}$ to derive an inequality of the form

$$1 \leq C \int_\Omega \eta(\phi^{\epsilon_n})(\nabla^\perp g_n \otimes \nabla^\perp \phi_0^\delta) : \nabla\nabla^\perp g_n \, dx + \frac{C}{\|\tilde{\phi}^{\epsilon_n,\delta}\|_{H^2(\Omega)}},$$

where $g_n = (\|\tilde{\phi}^{\epsilon_n,\delta}\|_{H^2(\Omega)})^{-1}\tilde{\phi}^{\epsilon_n,\delta}$. Using the uniform boundedness of the sequence $(g_n)_{n\in\mathbb{N}}$ in $H_0^2(\Omega)$ and compactness arguments one can show that the right hand side of the inequality converges to zero as $n \to \infty$ and $\delta \to 0$, which is a contradiction.

2.3.2.3. Examples of parallel, concentric and radial flows

In this paragraph we look at more concrete examples of solutions (ρ, u) of the stationary Navier-Stokes system (oddINS') with external force $f = 0$, under some symmetry assumptions on the density function, namely parallel, concentric and radial flows. The examples we consider are adapted from the case $\mu_o(\rho) \equiv 0$ in [123].

In the following we consider a weak solution (ρ, u) of (oddINS') on $\mathbb{R}^2$. We write $\partial_j = \partial_{x_j}$, $j = 1, 2$, and make use of polar coordinates

$$(x_1, x_2) = (r\cos\theta, r\sin\theta),$$

so that

$$\nabla_x = e_r\partial_r + \frac{e_\theta}{r}\partial_\theta, \qquad \text{where} \qquad e_r = \begin{pmatrix} \frac{x_1}{r} \\ \frac{x_2}{r} \end{pmatrix}, \quad e_\theta = \begin{pmatrix} \frac{x_2}{r} \\ -\frac{x_1}{r} \end{pmatrix}.$$

Example 2.3.6 (Parallel flow). Let $\rho = \rho(x_2)$ in $\mathbb{R}^2$ with $\partial_2\rho \neq 0$. The divergence-free condition $\operatorname{div} u = 0$ and $\operatorname{div}(\rho u) = 0$ imply that u is of the form

$$u = u_1(x_2)e_1,$$

where u_1 is a function only depending on the variable x_2, and $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the first standard basis vector in $\mathbb{R}^2$. Consequently, we see that

$$\begin{aligned} &\operatorname{div}(\rho u \otimes u) = 0, \\ &\operatorname{div}(\mu(\rho)(\nabla u + \nabla^T u)) = \partial_2(\mu(\rho)\partial_2 u_1)e_1, \\ &\operatorname{div}(\mu_o(\rho)(\nabla u^\perp + \nabla^\perp u)) = \partial_2(\mu_o(\rho)\partial_2 u_1)e_2, \end{aligned}$$

where $e_2 = (0\ 1)^T$. Hence, the system (oddINS') reads

$$\begin{pmatrix} -\partial_2(\mu(\rho)\partial_2 u_1) + \partial_1\pi \\ -\partial_2(\mu_o(\rho)\partial_2 u_1) + \partial_2\pi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.3.12}$$

The first line implies that π is of the form $\pi(x_1, x_2) = x_1\alpha(x_2) + \beta(x_1)$ for some functions α, β, whereas the second line implies that $\partial_2\pi$ must be independent of x_1. Therefore, $\alpha \equiv C$ for some constant $C \in \mathbb{R}$, which yields the two equations

$$\partial_2(\mu(\rho)\partial_2 u_1) = C, \quad \partial_2(\mu_o(\rho)\partial_2 u_1) = 0. \tag{2.3.13}$$

Example 2.3.7 (Concentric flow). Let $\rho = \rho(r)$ in $\mathbb{R}^2$ with $\partial_r \rho \neq 0$. The divergence-free condition $\operatorname{div} u = 0$ and $\operatorname{div}(\rho u) = 0$ imply that u is of the form

$$u = rg(r)e_\theta$$

for some function g only depending on r. It is straightforward to compute

$$\begin{aligned}
\operatorname{div}(\rho u \otimes u) &= -r\rho g^2 e_r,\\
\operatorname{div}(\mu(\rho)(\nabla u + \nabla^T u)) &= \frac{\partial_r(r^3\mu(\rho)\partial_r g)}{r^2} e_\theta,\\
\operatorname{div}(\mu_o(\rho)(\nabla u^\perp + \nabla^\perp u)) &= \frac{\partial_r(\mu_o(\rho) r^3 \partial_r g)}{r^2} e_r,
\end{aligned}$$

and thus the system (oddINS') reads

$$(-r\rho g^2 - \frac{\partial_r(\mu_o(\rho)r^3\partial_r g)}{r^2} + \partial_r \pi)e_r + (-\frac{\partial_r(r^3\mu(\rho)\partial_r g)}{r^2} - \frac{1}{r}\partial_\theta \pi)e_\theta = 0. \tag{2.3.14}$$

By the linear independence of e_r and e_θ, each of the two bracket terms must be zero. The equation in e_θ direction yields $\pi(r,\theta) = \tilde{\alpha}(r)\theta + \tilde{\beta}(r)$ for some functions $\tilde{\alpha}, \tilde{\beta}$ depending only on r, and substituting $\partial_r \pi$ into the equation in e_r direction yields

$$\partial_r(r^3\mu(\rho)\partial_r g) = Cr \tag{2.3.15}$$

for some constant $C \in \mathbb{R}$.

Example 2.3.8 (Radial flow). Let $\rho = \rho(\theta)$ in $\mathbb{R}^2$ with $\partial_\theta \rho \neq 0$. The divergence-free condition $\operatorname{div} u = 0$ and $\operatorname{div}(\rho u) = 0$ imply that u is of the form

$$u = \frac{h(\theta)}{r} e_r$$

for some function h depending only on θ. Consequently there holds

$$\begin{aligned}
\operatorname{div}(\rho u \otimes u) &= -\rho \frac{h^2}{r^3} e_r,\\
\operatorname{div}(\mu(\rho)(\nabla u + \nabla^T u)) &= \frac{\partial_\theta(\mu(\rho)\partial_\theta h)}{r^3} e_r - 2\frac{\partial_\theta(\mu(\rho)h)}{r^3} e_\theta,\\
\operatorname{div}(\mu_o(\rho)(\nabla u^\perp + \nabla^\perp u)) &= -2\frac{\partial_\theta(\mu_o(\rho)h)}{r^3} e_r - \frac{\partial_\theta(\mu_o(\rho)\partial_\theta h)}{r^3} e_\theta,
\end{aligned}$$

and thus the system (oddINS') reads

$$(-\rho\frac{h^2}{r^3} - \frac{\partial_\theta(\mu(\rho)\partial_\theta h)}{r^3} + 2\frac{\partial_\theta(\mu_o(\rho)h)}{r^3} + \partial_r\pi)e_r + (2\frac{\partial_\theta(\mu(\rho)h)}{r^3} + \frac{\partial_\theta(\mu_o(\rho)\partial_\theta h)}{r^3} - \frac{1}{r}\partial_\theta\pi)e_\theta = 0.$$

The equation in e_θ direction implies that

$$\pi(r,\theta) = 2\frac{\mu(\rho)h}{r^2} + \frac{\mu_o(\rho)\partial_\theta h}{r^2} + \hat{\alpha}(r)$$

for some function $\hat{\alpha}$ depending only on r, and substituting $\partial_r \pi$ into the equation in e_r direction yields

$$\rho h^2 + \partial_\theta(\mu(\rho)\partial_\theta h) - 2\partial_\theta(\mu_o(\rho)h) + 4\mu(\rho)h + 2\mu_o(\rho)\partial_\theta h = C \tag{2.3.16}$$

for some constant $C \in \mathbb{R}$.

If we take $\mu(\rho) \equiv 0$, equation (2.3.16) becomes

$$C = \rho h^2 - 2\partial_\theta(\mu_o(\rho)h) + 2\mu_o(\rho)\partial_\theta h. \tag{2.3.17}$$

This means that for a general domain Ω and general boundary value we can not expect $u \in H^1_{\mathrm{loc}}(\Omega)$ if $\mu_o(\rho)$ has a jump. Indeed, let μ_o and ρ be given by

$$\mu_o = 1_{[0,\pi)} + 2 \cdot 1_{[\pi,2\pi)}, \quad \rho = \mu_o^{-1}(1)1_{[0,\pi)} + \mu_o^{-1}(2)1_{[\pi,2\pi)},$$

and assume there exists a solution $u \in H^1_{\mathrm{loc}}(\Omega)$, i.e. $h \in H^1_{\mathrm{loc}}([0, 2\pi))$. We test (2.3.17) with $\varphi \in \mathcal{C}_c^\infty((0, 2\pi))$ to derive

$$C \int_0^{2\pi} \varphi \, d\theta = \mu_o^{-1}(1) \int_0^\pi h^2\varphi \, d\theta + \mu_o^{-1}(2) \int_\pi^{2\pi} h^2\varphi \, d\theta - 2h(\pi)\varphi(\pi).$$

If $h(\pi) \neq 0$ this implies

$$\delta_\pi = \frac{\mu_o^{-1}(1)}{2h(\pi)} h^2 1_{[0,\pi)} + \frac{\mu_o^{-1}(2)}{2h(\pi)} h^2 1_{[\pi,2\pi)} - \frac{C}{2h(\pi)},$$

where δ_π denotes the Dirac function. This is a contradiction since the right hand side is contained in $L^1_{\mathrm{loc}}((0, 2\pi))$, whilst the left hand side is not. If $h(\pi) = 0$, then

$$\mu_o^{-1}(1)h^2 1_{[0,\pi)} + \mu_o^{-1}(2)h^2 1_{[\pi,2\pi)} = C \quad \text{in } \mathcal{D}'((0, 2\pi)),$$

which implies that h^2 must have a jump if $C \neq 0$, or $h = 0$ if $C = 0$. In the former case we obtain a contradiction since h can not have a jump if it is contained in $H^1_{\mathrm{loc}}([0, 2\pi))$. In the latter case u can not satisfy the boundary condition for non-zero boundary value.

Consequently, it appears to be necessary to impose some regularity assumptions on $\mu_o(\rho)$ in order to obtain the existence of solutions of (oddINS) and (oddINS') if the shear viscosity in the momentum equation vanishes.

CHAPTER 3

The incompressible Navier-Stokes equations with variable viscosity

The results presented in this chapter are based on the joint work [168] with Xian Liao.

3.1. Introduction

This chapter is devoted to the global-in-time well-posedness of the Cauchy problem for systems of equations that describe the evolution of incompressible fluids with general variable viscosity coefficients in two spatial dimensions. The study of fluids with variable viscosity coefficients is of significant physical relevance, cf. [174, Section 6], and has attracted considerable interest in the mathematical community, cf. the books [14, 92, 175, 176, 183]. We focus primarily on fluids whose viscosity coefficients may be *discontinuous, potentially exhibiting large jumps,* as commonly arise in applications such as particle suspensions or mixtures of multiple immiscible fluids with different viscosities.

A variety of systems have been proposed to model the dynamics of variable-viscosity fluids. In this chapter, we investigate the following three models:

1. The incompressible Navier-Stokes equations with freely transported viscosity coefficient (see (μINS) below).

   Although this system is not widely studied, we believe it captures the essential dynamics and the main difficulties arising in the analysis of incompressible fluids with general viscosity coefficients.
2. The Boussinesq equations without heat conduction (see (B) below).

   This model can be viewed as a variant of the previous one, incorporating buoyancy effects.
3. The density-dependent incompressible Navier-Stokes equations (see (INS) below).

   This model is well known, and the analysis of the previous systems can be applied with appropriate adaptations.

This chapter is structured as follows. In Subsection 3.1.1 we present the above mentioned fluid models and review the relevant literature, focusing on the typical approaches for handling variable viscosity coefficients when they are smooth, of small variation, or piecewise constant. We then introduce our new approach in Subsection 3.1.2, which is designed to treat general variable viscosity coefficients that may exhibit large discontinuities. The main results for these three models are presented in Subsection 3.1.3 in Theorems 3.1.3, 3.1.7 (cf. Theorem 3) and 3.1.9 (cf. Theorem 2), respectively. Their proof ideas are explained in Section 3.2. In Section 3.3 we first establish the a priori estimates mentioned in Section 3.2 step by step, and afterwards we prove the main results. Appendix 3.A is devoted to the proof of our key Lemma 3.1.1 - 2. Some commutator estimates are proved in Appendix 3.B. In Appendix 3.C we show the decay estimates for the fluid velocity. Finally, we construct a nondegenerate tangent vector field for patches in Appendix 3.D.

3.1.1. Presentation of three fluid models

3.1.1.1. The incompressible Navier-Stokes equations with freely transported viscosity coefficient

As a model example we consider *constant-density fluids with largely varying viscosity coefficients*, such as those occurring in the mixing of two rivers with different temperatures. The motion can be described by the following incompressible Navier-Stokes equations with freely transported, variable viscosity coefficient

$$\begin{cases} \partial_t \mu + u \cdot \nabla \mu = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ \partial_t u + u \cdot \nabla u - \operatorname{div}(\mu S u) + \nabla \pi = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{μINS}$$

Here, $t \in [0,\infty)$ and $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ denote the time and space variables, respectively. The unknowns of the equations are the velocity vector field $u = u(t,x) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2$, the viscosity coefficient $\mu = \mu(t,x) \in (0,\infty)$ and the gradient of the pressure $\nabla \pi = \nabla \pi(t,x) = \begin{pmatrix} \partial_1 \pi \\ \partial_2 \pi \end{pmatrix} \in \mathbb{R}^2$, which is the Lagrangian multiplier associated to the divergence-free condition on the velocity $(\mu\text{INS})_3$. The matrix $Su \in \mathbb{R}^{2\times 2}$ denotes twice the symmetric part of the velocity gradient:

$$(Su)_{ij} = 2 \cdot \frac{1}{2}(\partial_i u_j + \partial_j u_i), \quad i,j = 1,2.$$

We aim to investigate the *nonlinear interplay* between the unknown viscosity coefficient μ, which satisfies the free transport equation $(\mu\text{INS})_1$, and the velocity field u, which satisfies the incompressible Navier-Stokes equations $(\mu\text{INS})_2$ with this varying viscosity coefficient μ. Central to our study is a detailed analysis of the divergence of the viscous stress tensor:

$$\operatorname{div}(\mu S u), \tag{3.1.1}$$

in the case where μ is a highly variable positive function (see Subsection 3.1.2 below).

Constant-viscosity case: Classical Navier-Stokes equations. If $\mu = \nu > 0$ is a positive constant, then the divergence-free condition $\operatorname{div} u = 0$ simplifies the above viscosity term (3.1.1) into

$$\operatorname{div}(\mu S u) = \nu \Delta u, \tag{3.1.2}$$

a diffusion term that plays an important role in the classical Navier-Stokes solution theory in J. Leray's pioneer work [163]. It is well-known, following the celebrated work of O. A. Ladyzhenskaya [156], that in space dimension *two*, J. Leray's weak solutions in the energy space $L^2(\mathbb{R}^2;\mathbb{R}^2)$ are unique and the Cauchy problem for the classical Navier-Stokes equations (i.e. the system (μINS) with $\mu = \nu > 0$) is well-posed globally in time. In *three* spatial dimensions, the uniqueness and the regularity of Leray's weak solutions are extensively studied, and at the same time, it has been shown that strong solutions with *small* initial data exist uniquely for all time; see the recent monographs [161, 234] and references therein. The global-in-time well-posedness problem for arbitrarily *large* initial data in three dimensions remains open and is famously known as the Millennium Problem for the Navier-Stokes equations [143].

Below, we review the standard approaches for handling the variable viscosity term in (3.1.1) across various fluid models, focusing in particular on the cases of smooth viscosity (see (3.1.4)), small viscosity variation (see (3.1.6)), and piecewise-constant viscosity (see (3.1.9)).

3.1.1.2. The Boussinesq equations without heat condution

Smooth-viscosity case: Boussinesq equations with heat conduction. We begin with the two-dimensional Boussinesq equations with heat conduction

$$\begin{cases} \partial_t \vartheta + u \cdot \nabla \vartheta - \operatorname{div}(\kappa \nabla \vartheta) = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ \partial_t u + u \cdot \nabla u - \operatorname{div}(\mu S u) + \nabla \pi = \vartheta e_2, \\ \operatorname{div} u = 0. \end{cases} \tag{3.1.3}$$

Here, the unknowns are the temperature $\vartheta \in \mathbb{R}$, the velocity field $u \in \mathbb{R}^2$ and the pressure $\pi \in \mathbb{R}$. The heat conduction coefficient and the viscosity coefficient

$$\kappa = \kappa_{\text{tem}}(\vartheta), \quad \mu = \mu_{\text{tem}}(\vartheta)$$

are both smooth functions[1] of the unknown temperature ϑ. The buoyancy force term ϑe_2 in $(3.1.3)_2$ accounts for the gravitational effects. The Boussinesq equations (3.1.3) have been known as one of the most important models in geophysical fluid dynamics [107].

Variable viscosity coefficients have been successfully incorporated in the presence of strong heat conduction $\kappa(t,x) \geq \kappa_* > 0$. In this case, the diffusion term $\operatorname{div}(\kappa \nabla \vartheta)$ regularizes the temperature ϑ over time, leading to a smooth viscosity coefficient $\mu = \mu_{\text{tem}}(\vartheta)$. Consequently, the viscosity term (3.1.1) can be rewritten as

$$\operatorname{div}(\mu S u) = \mu \Delta u + \nabla \mu \cdot S u, \tag{3.1.4}$$

where $\nabla \mu \cdot S u$ is considered as a lower-order term with respect to u. This formulation can be used to establish global-in-time well-posedness results, as discussed in [121, 122, 180, 238] and references therein. The classical constant coefficient scenario has been extensively studied in the literature, see the review notes [243] for more general results.

In the case of very weak heat conduction with $\kappa = 0$ which we consider in this chapter, the temperature ϑ satisfies the free transport equation, transforming (3.1.3) into

$$\begin{cases} \partial_t \vartheta + u \cdot \nabla \vartheta = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ \partial_t u + u \cdot \nabla u - \operatorname{div}(\mu S u) + \nabla \pi = \vartheta e_2, \\ \operatorname{div} u = 0. \end{cases} \tag{B}$$

This motivates our consideration of (μINS), which is derived from (B) by neglecting the buoyancy effect ϑe_2 on the right hand side of $(\mathrm{B})_2$. Specifically, multiplying $(\mathrm{B})_1$ by $\mu'_{\text{tem}}(\vartheta)$ (formally) yields the free transport equation of μ in $(\mu\mathrm{INS})_1$.

Recently there has been notable progress in the mathematical analysis of (μINS) and (B), cf. [8, 12, 195], under either the smoothness assumption $\nabla \mu_0 \in L^p$ or small variation assumption (see (3.1.5) below). *It remains an open problem whether global-in-time well-posedness results still hold in the presence of large variation in the initial data.* Our primary global-in-time well-posedness result for the system (μINS), under a scaling-invariant smallness assumption, is presented in Theorem 3.1.3 below. Notably, this result permits large jumps in the viscosity coefficient. As a consequence, we establish a lower bound on the existence time of solutions to (B), expressed in terms of the initial data, in Theorem 3.1.7 below.

[1] It is common to adapt cf. [203, Part I] the constant heat conductivity law $\kappa_{\text{tem}} = C_1$ and exponential viscosity law $\mu_{\text{tem}}(\vartheta) = C_2 \exp(C_3/(C_4 + \vartheta))$ for liquids, while $\kappa_{\text{tem}}(\vartheta) = C_5 \mu(\vartheta)$ and Sutherland's Law $\mu_{\text{tem}}(\vartheta) = \underline{\mu}(\frac{\vartheta}{\underline{\vartheta}})^{\frac{3}{2}} \frac{\vartheta + C_6}{\vartheta + C_7}$ for gases, where C_j, $j = 1, \cdots, 7$ are constants and $\vartheta_0, \underline{\mu} = \mu_{\text{tem}}(\underline{\vartheta})$ are reference temperature and viscosity coefficient. In particular, Andrade's Law: $\mu_{\text{tem}}(\vartheta) = C_2 \exp(C_3/\vartheta)$ with $C_2 = e^{-12.9896}, C_3 = 1780.622, C_4 = 0$ gives good accurate values in the range of $[10 - 100°]$ for waters, and Sutherland's Law $\mu_{\text{tem}}(\vartheta) = \underline{\mu}(\frac{\vartheta}{\underline{\vartheta}})^{\frac{3}{2}} \frac{\vartheta + C_6}{\vartheta + C_7}$ with $\underline{\vartheta} = 273\,K$, $\underline{\mu} = 1.716 \times 10^{-5}$, $C_6 = C_7 = 110.5\,K$ is good approximation for air close to the reference temperature $273\,K$.

3.1.1.3. The inhomogeneous incompressible Navier-Stokes equations with density-dependent viscosity

Variable viscosity coefficients have also been investigated recently in the context of density-dependent incompressible fluids with freely transported density function, described by the system

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ \rho(\partial_t u + u \cdot \nabla u) - \operatorname{div}(\mu S u) + \nabla \pi = 0, \\ \operatorname{div} u = 0. \end{cases} \tag{INS}$$

Here $\rho = \rho(t,x) \geq 0$ is the unknown density function, and the viscosity coefficient μ is a given smooth function of ρ as

$$\mu = \mu_{\mathrm{den}}(\rho) : [0,\infty) \to (0,\infty).$$

The three equations in (INS) represent the mass conservation law, the momentum conservation law, and the incompressibility condition, respectively. Formally, the system (μINS) can be seen as the density-dependent incompressible Navier-Stokes equations (INS) with the density dependence in the transport term in the momentum equation $(\mathrm{INS})_2$ being neglected. Specifically, similarly as above, multiplying $(\mathrm{INS})_1$ by $\mu'_{\mathrm{den}}(\rho)$ gives $(\mu\mathrm{INS})_1$, while $(\mathrm{INS})_2$ simplifies to $(\mu\mathrm{INS})_2$ by replacing $\rho(\partial_t u + u \cdot \nabla u)$ by $(\partial_t u + u \cdot \nabla u)$ (similar as in the Boussinesq-approximation).

The system (INS) has been widely explored by numerous mathematicians. P.-L. Lions establishes the existence of global-in-time weak solutions in [175], which improves an earlier work [222] for the constant viscosity case. In the case of constant viscosity $\mu = \nu > 0$, the existence and uniqueness of strong solutions of (INS) in the case of smooth initial data (ρ_0, u_0) are demonstrated by O. A. Ladyzhenskaya and V. A. Solonnikov [157]. Motivated by the natural scaling of (INS), a number of works have been dedicated to the study of the system in critical functional spaces which are invariant under the same scaling, see for example [7, 9, 51, 138] and references therein. Recently, the global-in-time well-posedness results in the more general case with discontinuous densities in the presence of vaccuum are now known to hold true, thanks to the remarkable contributions by R. Danchin and P. B. Mucha [57, 59, 60].

Small-variation case: Density-dependent incompressible Navier-Stokes equations. For general viscosity $\mu = \mu_{\mathrm{den}}(\rho)$, local-in-time well-posedness for smooth initial data for (INS) was established in Y. Cho and H. Kim [46], see also the book [14]. Under *small variation* assumptions, either with small density variation [110, 137, 179] or small viscosity variation [11, 101, 136, 142, 198], global-in-time well-posedness results have been achieved in two spatial dimensions. An earlier work by Desjardins [75] addresses the regularity of P.-L. Lions' weak solutions. For the three spatial dimensional case, see [10, 119, 140, 248] and references therein.

In the case where μ is close to a positive constant $\nu > 0$:

$$\|\mu - \nu\|_{L^\infty(\mathbb{R}^2)} \ll 1, \tag{3.1.5}$$

a key ingredient in the analysis is the following decomposition of the viscosity term (3.1.1):

$$\operatorname{div}(\mu S u) = \nu \Delta u + \operatorname{div}((\mu - \nu) S u), \tag{3.1.6}$$

where $\operatorname{div}((\mu - \nu) S u)$ is considered as a perturbation term.

However, this decomposition does not apply when μ varies significantly. *It remains open whether the global-in-time wellposedness of* (INS) *holds in two space dimensions with large* *initial data.* We establish in Theorem 3.1.9 below the global-in-time wellposedness of (INS), assuming some smallness condition while allowing for large variations in the density and viscosity.

Piecewise-constant case: Free interface problem. When describing the time evolution of two immiscible fluids, which are separated by a free interface, one considers the following two-phase Navier-Stokes equations

$$\begin{cases} \rho(\partial_t u + u \cdot \nabla u) - \operatorname{div}(\mu S u) + \nabla \pi = 0, \quad \operatorname{div} u = 0 & \text{in } \Omega_t^- \cup \Omega_t^+, \\ [\![u]\!] = 0, \quad [\![T(u,\pi)n]\!] = \sigma H n, \quad U = u \cdot n & \text{on } \Gamma_t. \end{cases} \tag{3.1.7}$$

Here, two fluids occupy the domains Ω_t^+, Ω_t^- respectively, with Γ_t as the separating interface. The vector $n = n(t,x)$ denotes the outward unit normal to Ω_t^+, and $[\![\cdot]\!]$ represents the jump of a function across the interface Γ_t in the direction of n. The functions $H = H(t,x)$ and $U = U(t,x)$ denote the curvature and the normal velocity of Γ_t with respect to n, respectively, and $\sigma \geq 0$ is the surface tension coefficient. The total stress tensor $T(u,\pi)$ is defined by

$$T(u,\pi) = \mu S u - \pi \mathrm{Id}, \text{ with } \mathrm{Id} \in \mathbb{R}^{2\times 2} \text{ denoting the unit matrix.}$$

In the case where two different fluids having positive constant densities ρ^+, ρ^- and positive constant viscosity coefficients $\mu^+ = \rho^+\nu^+, \mu^- = \rho^-\nu^-$, the momentum equation in $(3.1.7)_1$ reads as

$$\partial_t u + u \cdot \nabla u - \nu^\pm \Delta u + \frac{1}{\rho^\pm}\nabla \pi = 0 \text{ in } \Omega_t^- \cup \Omega_t^+. \tag{3.1.8}$$

In this scenario, the viscosity term (3.1.1) simplifies to

$$\operatorname{div}(\mu S u) = \mu^\pm \Delta u \quad \text{in } \Omega_t^- \cup \Omega_t^+, \tag{3.1.9}$$

which reduces the problem (3.1.7) to solving the Navier-Stokes equations with a *constant* viscosity coefficient within each domain. The main challenge then lies in determining the free interface Γ_t.

Notice that in the absence of surface tension ($\sigma = 0$), if $(\rho, u, \nabla\pi)$ solves the density-dependent incompressible Navier-Stokes equations (INS) with the initial density $\rho_0 = \rho^+ 1_{\Omega_0^+} + \rho^- 1_{\Omega_0^-}$, then it also satisfies (3.1.7) and hence (3.1.8), provided that both the vectors u and $T(u,\pi)n$ are continuous across the freely transported interface Γ_t (as long as Γ_t remains well-defined). Similarly, in the case of constant density function $\rho^\pm = 1$, if $(\mu, u, \nabla\pi)$ solves (μINS) with the initial viscosity $\mu_0 = \mu^+ 1_{\Omega_0^+} + \mu_- 1_{\Omega_0^-}$ and both u and $T(u,\pi)n$ are continuous across the well-defined free-transported interface Γ_t, then it satisfies (3.1.7), which in this context becomes

$$\begin{cases} \partial_t u + u \cdot \nabla u - \operatorname{div}(\mu S u) + \nabla \pi = 0, \quad \operatorname{div} u = 0 & \text{in } \Omega_t^- \cup \Omega_t^+, \\ [\![u]\!] = [\![T(u,\pi)n]\!] = 0, \quad U = u \cdot n & \text{on } \Gamma_t. \end{cases} \tag{3.1.10}$$

The models (3.1.7) and (3.1.10) are known as sharp interface models. For discussions on the sharp interface limit of Navier-Stokes/Allen-Cahn or Navier-Stokes/Cahn-Hilliard equations, see [6, 177] and the references therein.

The two-phase Navier-Stokes equations (3.1.7) with piecewise-constant densities and viscosity coefficients (3.1.8) have been thoroughly studied since the 1980s in various configurations of Ω_t^- and Ω_t^+; see the books [73, 209] for a comprehensive overview. In the presence of surface tension ($\sigma > 0$), local-in-time existence and uniqueness results are provided in e.g. [71, 208] and global-in-time well-posedness is proved in [72, 223]. See also [5] for the global-in-time existence of varifold solutions with rather general initial data. When the surface tension is absent ($\sigma = 0$), global-in-time well-posedness has been obtained in e.g. [68, 69, 213]. *However, it remains unclear whether $\rho^{\pm}, \mu^{\pm}$ can be taken as largely variable smooth functions within their respective domains $\Omega_t^{\pm}$.* In Corollary 3.1.5 - 2 and Theorem 3.1.9 below we address this question for the systems (3.1.10) and (3.1.7) (with $\sigma = 0$), respectively.

The literature also contains extensive discussions on the regularity of solutions for other evolutionary models with variable viscosity coefficients beyond those presented above. This includes for instance compressible models [125, 186, 252], zero Mach-number systems and combustion models [56], and MHD equations with density-dependent viscosity [141]. However, to our knowledge, at least one of the decompositions

(3.1.4) (regular case), (3.1.6) (perturbative case), or (3.1.9) (piecewise-constant case)

for the viscosity term (3.1.1) has been applied in the regularity theory. In this chapter, we aim to address more general variable viscosity coefficients, with our first key observation being the following global-in-space decomposition.

3.1.2. A hidden fourth-order elliptic operator in the viscous stress tensor

In the present chapter, building on insights from the previous work [123] by Z. He and the first author for the stationary Navier-Stokes equations with variable viscosity coefficient, we decompose the divergence of the viscous stress tensor (3.1.1) straightforwardly into a divergence-free component and a curl-free component. This approach allows us to handle more general variable viscosity coefficients effectively.

In this section, time dependence is largely disregarded.

Lemma 3.1.1 (Helmholtz decomposition, "global good unknown" a, elliptic operator $\mathcal{L}_\mu$, and $L^{2+\epsilon}(\mathbb{R}^2)$-estimate). *Let $u = \nabla^\perp \phi$ with $\nabla^\perp = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix}$, $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that* $\operatorname{div} u = 0$. *Let $\mu \in L^\infty(\mathbb{R}^2)$.*

1. *(Decomposition of* $\operatorname{div}(\mu S u)$ *and "global good unknown" a). The following (formal) decomposition holds*
$$\operatorname{div}(\mu S u) = \nabla^\perp a + \nabla b \tag{3.1.11}$$
where
$$\Delta a = \mathcal{L}_\mu \phi, \ \text{with } \mathcal{L}_\mu = (\partial_{22} - \partial_{11})\mu(\partial_{22} - \partial_{11}) + (2\partial_{12})\mu(2\partial_{12}), \tag{3.1.12}$$
$$\Delta b = \mathcal{J}_\mu \phi, \ \text{with } \mathcal{J}_\mu = (\partial_{22} - \partial_{11})\mu(2\partial_{12}) - (2\partial_{12})\mu(\partial_{22} - \partial_{11}). \tag{3.1.13}$$
Here μ is understood as the multiplication operator by the function μ.
Furthermore, if $\nabla u \in L^2(\mathbb{R}^2; \mathbb{R}^{2\times 2})$, then the following holds in the $L^2(\mathbb{R}^2)$-functional setting (where the Fourier transform applies).

- *$a, b \in L^2(\mathbb{R}^2)$ can be determined from $\mu S u$ as follows:*

$$\begin{aligned} a &= -(-\Delta)^{-1}\nabla^\perp \cdot \operatorname{div}(\mu S u) = -(-\Delta)^{-1}(\nabla^\perp \otimes \nabla) : (\mu S u) \\ &= (\mathcal{R}^\perp \otimes \mathcal{R}) : (\mu S u), \end{aligned} \tag{3.1.14}$$

$$\begin{aligned} b &= -(-\Delta)^{-1}\nabla \cdot \operatorname{div}(\mu S u) = -(-\Delta)^{-1}(\nabla \otimes \nabla) : (\mu S u) \\ &= (\mathcal{R} \otimes \mathcal{R}) : (\mu S u), \end{aligned} \tag{3.1.15}$$

where $\mathcal{R} = \frac{\frac{1}{i}\nabla}{\sqrt{-\Delta}}$ and $\mathcal{R}^\perp = \frac{\frac{1}{i}\nabla^\perp}{\sqrt{-\Delta}}$ are the Riesz operators.

- *If we introduce the scalar fluid vorticity $\omega = \nabla^\perp \cdot u = \Delta\phi$, then a, b can be respresented in terms of μ, ω and Riesz operators as follows:*

$$a = \mathcal{R}_\mu \omega, \quad \mathcal{R}_\mu := (\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\mu(\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1) + (2\mathcal{R}_1\mathcal{R}_2)\mu(2\mathcal{R}_1\mathcal{R}_2), \tag{3.1.16}$$

$$b = \mathcal{Q}_\mu \omega, \quad \mathcal{Q}_\mu := (\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\mu(2\mathcal{R}_1\mathcal{R}_2) - (2\mathcal{R}_1\mathcal{R}_2)\mu(\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1). \tag{3.1.17}$$

2. *(Ellipticity of $\mathcal{L}_\mu$ and $L^{2+\epsilon}(\mathbb{R}^2)$-estimates). Let $0 < \mu_* \le \mu^*$ be two positive constants and $\mu \in L^\infty(\mathbb{R}^2; [\mu_*, \mu^*])$. Then the operator $\mathcal{L}_\mu$ given in* (3.1.12) *above is a fourth-order elliptic operator.*

   Furthermore, there exists an $\epsilon_0 > 0$ depending only on μ_*, μ^*, such that the operator $\mathcal{R}_\mu$ in (3.1.16) defines an isomorphism on $L^{2+\epsilon}(\mathbb{R}^2)$, for all $\epsilon \in [0, \epsilon_0]$. Correspondingly there exists a positive constant $C \ge 1$ depending only on μ_*, μ^* such that

$$C^{-1}\|a\|_{L^{2+\epsilon}(\mathbb{R}^2)} \le \|\nabla u\|_{L^{2+\epsilon}(\mathbb{R}^2)} \le C\|a\|_{L^{2+\epsilon}(\mathbb{R}^2)}, \quad \forall \epsilon \in [0, \epsilon_0]. \tag{3.1.18}$$

Proof. The decomposition (3.1.11) can be checked (see also [123]) by straightforward computations:

$$\nabla^\perp \cdot \operatorname{div}(\mu S \nabla^\perp \phi) = \mathcal{L}_\mu \phi \quad \text{and} \quad \nabla \cdot \operatorname{div}(\mu S \nabla^\perp \phi) = \mathcal{J}_\mu \phi. \tag{3.1.19}$$

(3.1.11) is equivalent to (3.1.14)-(3.1.15). The relations (3.1.16)-(3.1.17) between a, b and ω follow from (3.1.12)-(3.1.13) directly. This completes the proof of Lemma 3.1.1 - 1.

The ellipticity of $\mathcal{L}_\mu$ follows immediately from the following reformulation of $\mathcal{L}_\mu$ (see also [123])

$$\begin{aligned} \mathcal{L}_\mu &= \partial_{11}(\mu\partial_{11}) + \partial_{22}(\mu\partial_{22}) - \partial_{11}\Big((\mu - \frac{\mu_*}{2})\partial_{22}\Big) - \partial_{22}\Big((\mu - \frac{\mu_*}{2})\partial_{11}\Big) + \partial_{12}\Big((4\mu - \mu_*)\partial_{12}\Big) \\ &=: \sum_{|\alpha|=|\beta|=2} D^\alpha(l^\mu_{\alpha\beta} D^\beta), \end{aligned}$$

where

$$\frac{\mu_*}{2}|\xi|^2 \le \sum_{|\alpha|=|\beta|=2} l^\mu_{\alpha\beta}\xi_\alpha\xi_\beta \le 2\mu^*|\xi|^2, \quad \forall \xi = (\xi_\alpha)_{|\alpha|=2} \in \mathbb{R}^3. \tag{3.1.20}$$

The proof of the invertibility of $\mathcal{R}_\mu$ in $L^{2+\epsilon}(\mathbb{R}^2)$, $\epsilon \in [0, \epsilon_0]$, is postponed to Appendix 3.A. It is strongly related to the ellipticity of the operator $\mathcal{L}_\mu$.

The estimate (3.1.18) follows from the relations $a = \mathcal{R}_\mu\omega = \mathcal{R}_\mu\nabla^\perp \cdot u$ and $\nabla u = \mathcal{R}\mathcal{R}^\perp\omega = \mathcal{R}\mathcal{R}^\perp\mathcal{R}_\mu^{-1}a$ together with the $L^{2+\epsilon}(\mathbb{R}^2)$-boundedness of the Riesz-transforms $\mathcal{R}, \mathcal{R}^\perp$ and $\mathcal{R}_\mu, \mathcal{R}_\mu^{-1}$. □

Remark 3.1.2. 1. (Constant viscosity case). Note that if $\mu = \nu$ is a positive constant, then $\mathcal{L}_\mu = \nu\Delta^2$ is a biharmonic operator, while $a = \nu\omega$ and $b = 0$ by (3.1.16) and (3.1.17), respectively.

2. (Time-evolutionary vorticity equation). With the decomposition (3.1.11) the velocity equation $(\mu\text{INS})_2$ becomes

$$\partial_t u + u \cdot \nabla u - \nabla^\perp a + \nabla(\pi - b) = 0.$$

We apply $\nabla^\perp \cdot$ to it and derive the equation for the vorticity

$$\partial_t \omega + u \cdot \nabla\omega - \Delta a = 0, \tag{3.1.21}$$

where $u = \nabla^\perp \Delta^{-1}\omega$ is given by the Biot-Savart law. With μ freely transported by the velocity field u as in $(\mu\text{INS})_1$, $a = \mathcal{R}_\mu \omega$ is given by applying nonlocal Riesz operators $\mathcal{R}$ composed with the local multiplication operator by μ to ω. This "nonlocal" vorticity equation (3.1.21) is essence of the system (μINS).

The challenge then lies in deriving bounds for ω or ∇u from a. In this chapter we first establish the time-weighted H^1-energy estimates for a, which imply (by interpolation) $L^{2+\epsilon}$-estimate for a hence for ∇u by (3.1.18). In order to achieve the crucial L^∞-estimate of ∇u, we have to further make use of the "localized version" of a (see its definition in (3.1.31) below).

3. (Straightforward $L^2(\mathbb{R}^2)$-equivalence between ∇u and a). If $\mu \in L^\infty(\mathbb{R}^2; [\mu_*, \mu^*])$, then it is straightforward to derive the equivalence of the L^2-norms of ω and a

$$\mu_* \|\omega\|_{L^2(\mathbb{R}^2)} \le \|a\|_{L^2(\mathbb{R}^2)} \le 8\mu^* \|\omega\|_{L^2(\mathbb{R}^2)}, \tag{3.1.22}$$

which is (3.1.18) in the case $\epsilon = 0$. Indeed, on one side, using that the operator norm of the Riesz operators on $L^2(\mathbb{R}^2)$ is 1, we have

$$\|a\|_{L^2(\mathbb{R}^2)} \le 8\mu^* \|\omega\|_{L^2(\mathbb{R}^2)}. \tag{3.1.23}$$

On the other side, by the fact that $\text{Id} = \mathcal{R}_1\mathcal{R}_1 + \mathcal{R}_2\mathcal{R}_2$ and $(\mathcal{R}_1\mathcal{R}_1 + \mathcal{R}_2\mathcal{R}_2)^2 = (\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)^2 + (2\mathcal{R}_1\mathcal{R}_2)^2$ (understood as operators defined on $L^2(\mathbb{R}^2)$) and the symmetry of the double Riesz transform on $L^2(\mathbb{R}^2)$, we derive

$$\begin{aligned}
&\mu_* \|\omega\|^2_{L^2(\mathbb{R}^2)} = \mu_* \Big\langle \omega, (\mathcal{R}_1\mathcal{R}_1 + \mathcal{R}_2\mathcal{R}_2)^2 \omega \Big\rangle_{L^2(\mathbb{R}^2)} \\
&= \mu_* \Big\langle \omega, \big((\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)^2 + (2\mathcal{R}_1\mathcal{R}_2)^2\big)\omega \Big\rangle_{L^2(\mathbb{R}^2)} \\
&= \mu_* \Big\langle (\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\omega, (\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\omega \Big\rangle_{L^2(\mathbb{R}^2)} + \mu_* \Big\langle (2\mathcal{R}_1\mathcal{R}_2)\omega, (2\mathcal{R}_1\mathcal{R}_2)\omega \Big\rangle_{L^2(\mathbb{R}^2)} \\
&\le \Big\langle \mu(\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\omega, (\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\omega \Big\rangle_{L^2(\mathbb{R}^2)} + \Big\langle \mu(2\mathcal{R}_1\mathcal{R}_2)\omega, (2\mathcal{R}_1\mathcal{R}_2)\omega \Big\rangle_{L^2(\mathbb{R}^2)} \\
&= \Big\langle (\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\mu(\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\omega, \omega \Big\rangle_{L^2(\mathbb{R}^2)} + \Big\langle (2\mathcal{R}_1\mathcal{R}_2)\mu(2\mathcal{R}_1\mathcal{R}_2)\omega, \omega \Big\rangle_{L^2(\mathbb{R}^2)} \\
&\overset{(3.1.16)}{=} \langle a, \omega \rangle_{L^2(\mathbb{R}^2)},
\end{aligned}$$

which, together with the Cauchy-Schwarz inequality, implies that

$$\|\omega\|_{L^2(\mathbb{R}^2)} \le \frac{1}{\mu_*} \|a\|_{L^2(\mathbb{R}^2)}. \tag{3.1.24}$$

4. (Size restriction on ϵ). The upper bound ϵ_0 cannot be arbitrarily large. Indeed, for any $p > 2$, there exists a bounded measurable (highly oscillating) function $\tilde{\mu}$ taking only two possible values, $\tilde{\mu} \in \{\frac{1}{K}, K\}$ with $K = \frac{2}{p-2} + 1 > 1$, such that there exist solutions to the *homogeneous* elliptic equation $L_{\tilde{\mu}}\phi = 0$ with

$$\nabla u = \nabla\nabla^{\perp}\phi \notin L^p_{\mathrm{loc}}(\mathbb{R}^2), \tag{3.1.25}$$

see [123]. In particular, this case corresponds to $\Delta a = 0$ while $\nabla u \notin L^p_{\mathrm{loc}}(\mathbb{R}^2)$ by (3.1.12), that is, a can not control ∇u in $L^p(\mathbb{R}^2)$ (locally). This can be viewed as a generalization of the failure of $L^p(\mathbb{R}^2)$-estimate associated to the second-order elliptic operator $\mathrm{div}\,(\mu\nabla)$ studied in [15], to the fourth-order elliptic operator $\mathcal{L}_\mu$ here.

3.1.3. Main results

We first note that, as pointed out in Remark 3.1.2 – 4 above, if the viscosity coefficient μ is only assumed to be bounded from above and below, one cannot, in general, expect even $\nabla u \in L^p_{\mathrm{loc}}(\mathbb{R}^2)$, let alone the crucial Lipschitz regularity $\nabla u \in L^\infty(\mathbb{R}^2)$. This motivates us to incorporate tangential regularity along a vector field $\tau = \tau(t,x) \in \mathbb{R}^2$ which is transported by the Navier–Stokes flow. Such additional tangential regularity – for instance $\partial_\tau \nabla u \in L^{2+\epsilon}(\mathbb{R}^2)$ – is expected to help recover control of the velocity gradient, in the spirit of the Sobolev embedding $W^{1,2+\epsilon}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$.

We are thus led to consider the corresponding coupled system, consisting of the Navier–Stokes or Boussinesq equations together with the following evolution equation for the tangential vector field

$$\partial_t\tau + u\cdot\nabla\tau = \tau\cdot\nabla u. \tag{τ}$$

In other words, the tangential derivative $\partial_\tau := \tau\cdot\nabla$ commutes with the material derivative $\frac{D}{Dt} := \partial_t + u\cdot\nabla$, since

$$[\frac{D}{Dt}, \partial_\tau] = (\frac{D}{Dt}\tau - \partial_\tau u)\cdot\nabla = (\partial_t\tau + u\cdot\nabla\tau - \tau\cdot\nabla u)\cdot\nabla = 0.$$

Combined with the free transport equation $(\mu\mathrm{INS})_1$: $\frac{D}{Dt}\mu = 0$, this implies that the tangential derivative $\partial_\tau\mu$ is also transported by the flow:

$$\partial_\tau\frac{D}{Dt}\mu = 0 \Leftrightarrow \frac{D}{Dt}(\partial_\tau\mu) = 0. \tag{3.1.26}$$

In particular, the $L^p(\mathbb{R}^2)$-norm of $\partial_\tau\mu$ is preserved by the flow a priori for any $p \in [1,\infty]$.

However, the tangential regularity of μ with respect to the vector field τ involves not only $\|\partial_\tau\mu(t)\|_{L^p(\mathbb{R}^2)}$, but also the regularity of the vector field τ itself: $\|\nabla\tau(t)\|_{L^p(\mathbb{R}^2)}$ (see e.g. [41]). This quantity arises naturally when estimating commutators of the form $[\partial_\tau, \nabla] = \nabla\tau\cdot\nabla$. Unfortunately, the norm $\|\nabla\tau(t)\|_{L^p(\mathbb{R}^2)}$ grows exponentially in time with a bound of the form $\exp(C\|\nabla u\|_{L^1([0,t];L^\infty(\mathbb{R}^2))})$. Consequently, a smallness condition on the initial data is required in order to control the tangential regularity together with the Lipschitz norm of the velocity field globally-in-time.

3.1.3.1. The incompressible Navier-Stokes equations with freely transported viscosity coefficient

Our main result for the system (μINS) reads as follows.

Theorem 3.1.3 (Global-in-time well-posedness of (μINS)**).** *Let $0 < \mu_* \leq 1 \leq \mu^*$ be two positive constants and let $\epsilon_0 = \epsilon_0(\mu_*, \mu^*)$ be given by Lemma 3.1.1 - 2.*

Let $\mu_0 \in L^\infty(\mathbb{R}^2; [\mu_*, \mu^*])$ be an initial viscosity function satisfying $\mu_0 - 1 \in L^2(\mathbb{R}^2)$, and let $u_0 \in H^1 \cap \dot{H}^{-1}(\mathbb{R}^2; \mathbb{R}^2)$ be a divergence-free vector field. Furthermore, let $\tau_0 \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$ be a nondegenerate vector field such that $|\tau_0|^{-1} \in L^\infty(\mathbb{R}^2)$ and $(\nabla \tau_0, \partial_{\tau_0}\mu_0) \in L^{2+\epsilon}(\mathbb{R}^2; \mathbb{R}^{2\times 2+1})$ for some $\epsilon \in (0, \epsilon_0]$ in the sense of distributions. If the following smallness condition is fulfilled

$$e^{c\|u_0\|^4_{L^2(\mathbb{R}^2)}} \|u_0\|^{\frac{\epsilon}{2}}_{L^2(\mathbb{R}^2)} \cdot \Big(\|u_0\|_{\dot{H}^{-1}(\mathbb{R}^2)} + \|\mu_0 - 1\|_{L^2(\mathbb{R}^2)} \|u_0\|_{L^2(\mathbb{R}^2)} \Big) \cdot \Big(\|\nabla u_0\|_{L^2(\mathbb{R}^2)} + \|(\nabla \bar{\tau}_0, \partial_{\bar{\tau}_0}\mu_0)\|^{\frac{2+\epsilon}{\epsilon}}_{L^{2+\epsilon}(\mathbb{R}^2)} \Big) \leq c_0, \tag{3.1.27}$$

where $\bar{\tau}_0 = \frac{\tau_0}{|\tau_0|}$ and c, c_0 are positive constants depending only on μ_*, μ^*, ϵ, then the system (μINS)-(τ) *supplemented with the initial data (μ_0, u_0, τ_0) has a unique global-in-time solution $(\mu, u, \nabla\pi, \tau)$ satisfying*

$$\begin{aligned}
&\mu \in L^\infty((0,\infty) \times \mathbb{R}^2; [\mu_*, \mu^*]), \quad \mu - 1 \in \mathcal{C}_b([0,\infty); L^q(\mathbb{R}^2)),\ \forall q \in [2,\infty),\\
&u \in \mathcal{C}_b([0,\infty); L^2(\mathbb{R}^2;\mathbb{R}^2)) \cap L^2((0,\infty); \dot{H}^1(\mathbb{R}^2;\mathbb{R}^2)),\\
&\nabla u \in \mathcal{C}_b([0,\infty); L^2(\mathbb{R}^2;\mathbb{R}^{2\times 2})) \cap L^1((0,\infty); L^\infty(\mathbb{R}^2;\mathbb{R}^{2\times 2})),\\
&\nabla(\pi - b) \in L^2((0,\infty); L^2(\mathbb{R}^2;\mathbb{R}^2)),\\
&\tau \in L^\infty((0,\infty); L^\infty \cap \dot{W}^{1,2+\epsilon}(\mathbb{R}^2;\mathbb{R}^2)), \quad \frac{1}{|\tau|} \in L^\infty((0,\infty)\times\mathbb{R}^2),\\
&\partial_\tau \mu \in L^\infty([0,\infty); L^{2+\epsilon}(\mathbb{R}^2)) \text{ in the distribution sense,}
\end{aligned} \tag{3.1.28}$$

with b defined in (3.1.15)*. Furthermore, it holds*

- *(Time-weighted) $H^1(\mathbb{R}^2)$-energy estimates for the "global good unknown" a defined in* (3.1.14)*:*

$$\begin{aligned}
&a \in \mathcal{C}_b([0,\infty); L^2(\mathbb{R}^2)) \cap L^2((0,\infty); \dot{H}^1(\mathbb{R}^2)),\\
&t^{\frac{1}{2}}\nabla a \in L^\infty((0,\infty); L^2(\mathbb{R}^2;\mathbb{R}^2)) \cap L^2((0,\infty); \dot{H}^1(\mathbb{R}^2;\mathbb{R}^2));
\end{aligned} \tag{3.1.29}$$

- *$W^{1,2+\epsilon}(\mathbb{R}^2)$-boundedness:*

$$a, \alpha, \partial_\tau u \in L^1((0,\infty); W^{1,2+\epsilon}(\mathbb{R}^2)); \tag{3.1.30}$$

  here, α denotes the "local good unknown", which is defined by

$$\alpha = (\bar{\tau} \otimes n) : (\mu S u), \tag{3.1.31}$$

  with the (unit) tangential and normal vectors $\bar{\tau} = \frac{\tau}{|\tau|}$ and $n = -\frac{\tau^\perp}{|\tau|}$, respectively.

- *$H^1(\mathbb{R}^2)$-boundedness for the material derivative $\frac{D}{Dt}u = \partial_t u + u \cdot \nabla u$ and the divergence of the total stress tensor $T(u,\pi) = \mu S u - \pi Id$*

$$\frac{D}{Dt}u = \operatorname{div} T(u,\pi) \in L^2((0,\infty); L^2(\mathbb{R}^2;\mathbb{R}^2)), \tag{3.1.32}$$

$$t^{\frac{1}{2}}\frac{D}{Dt}u = t^{\frac{1}{2}}\operatorname{div} T(u,\pi) \in L^2((0,\infty); \dot{H}^1(\mathbb{R}^2;\mathbb{R}^2)). \tag{3.1.33}$$

The proof of Theorem 3.1.3 is found in Subsection 3.3.4. The proof ideas for the global-in-time a priori estimates are discussed in Section 3.2 below. Let us now make a few comments on the results in Theorem 3.1.3.

Remark 3.1.4. (i) (Jump of $\partial_n u$ in case of jumping μ). We establish the following expression for the normal derivative of the velocity $\partial_n u$ by use of α, μ, $\bar{\tau}$, n and the tangential derivative $\partial_{\bar{\tau}} u$ (see (3.2.13) below)

$$\partial_n u = \partial_n \nabla^\perp \phi = \frac{\alpha}{\mu}\bar{\tau} - 2(n \cdot \partial_{\bar{\tau}} u)\bar{\tau} + (\partial_{\bar{\tau}} u)^\perp. \tag{3.1.34}$$

The regularity of τ in (3.1.28) and the regularity of $\alpha, \partial_{\bar{\tau}} u$ in (3.1.30) imply that $\partial_n u$ has a jump exactly when μ has a jump.

It is this jump that makes it particularly challenging to show the boundedness of the velocity gradient in the case of a variable viscosity coefficient. Here, we are able to prove this thanks to the regularity of the local good unknown α.

(ii) (Key quantity for the Lipschitz estimate: "Local good unknown" α). A new key ingredient in establishing the L^∞-estimate for ∇u in terms of the H^1-energy estimates for a in (3.1.29), is the successful *localization* of the global good unknown a into its localized version $\alpha = (\bar{\tau} \otimes n) : (\mu S u)$, such that

- α equals to a, up to tangential regularity terms (see (3.2.14) below);
- the normal derivative $\partial_n u$ is expressed by α and tangential regularity terms pointwisely (see (3.1.34)).

Finally, the velocity gradient is bounded by a up to tangential regularity terms as follows (see also (3.3.30) below):

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \lesssim \|a\|_{L^{2+\epsilon}(\mathbb{R}^2)}^{\frac{\epsilon}{2+\epsilon}} \Big(\|\nabla a\|_{L^{2+\epsilon}(\mathbb{R}^2)} + \|(\nabla \bar{\tau}, \partial_{\bar{\tau}} \mu)\|_{L^{2+\epsilon}} \|(\nabla u, a)\|_{L^\infty} \Big)^{\frac{2}{2+\epsilon}}. \tag{3.1.35}$$

Let us take a closer look at the role of α in the two-phase problem (3.1.7). Observe that if we multiply the jump condition $\sigma H n = [\![T(u,\pi)n]\!]$ on the interface Γ_t by the continuous tangent vector $\bar{\tau}$ we derive that

$$0 = \bar{\tau} \cdot \sigma H n = [\![\bar{\tau} \cdot (T(u,\pi)n)]\!] = [\![\bar{\tau} \cdot (\mu S u\, n)]\!] = [\![\alpha]\!],$$

where we used the definition $T(u,\pi) = \mu S u - \pi \mathrm{Id}$. Thus, α is continuous, which is consistent with our analysis.

Notice that

$$\alpha = \bar{\tau} \cdot (T(u,\pi)n)$$

is the tangential component of the Cauchy stress vector $T(u,\pi)n$, which is referred to as *shear stress* in physical contexts. The idea of multiplication by the tangent vector has appeared e.g. in Nalimov's formulation of the one-dimensional water waves problem [192]. We believe that our analysis of α in the variable viscosity setting is novel, and that the localization approach provides valuable insights into the free interface problem.

(iii) (Assumptions revisited: No smallness condition on the initial viscosity variation). The low frequency control by $\|u_0\|_{\dot{H}^{-1}}$ and $\|\mu_0 - 1\|_{L^2}\|u_0\|_{L^2}$ provides sufficient time decay (see Proposition 3.3.3 below), while the high frequency control by $\|\nabla u_0\|_{L^2}$ and $\|(\nabla \bar{\tau}_0, \partial_{\bar{\tau}_0} \mu_0)\|_{L^{2+\epsilon}}$ provides sufficient regularity (see Proposition 3.3.4). The combination of these bounds on the left hand side in (3.1.27), which is invariant under the scaling

$$(\mu_{0,\lambda}, u_{0,\lambda}, \bar{\tau}_{0,\lambda})(x) = (\mu_0, \lambda^{-1} u_0, \bar{\tau}_0)(\lambda^{-1} x), \quad \lambda > 0,$$

controls the critical norm $\|\nabla u\|_{L^1_t L^\infty_x}$ (see Proposition 3.3.8). In particular, the smallness condition (3.1.27) permits arbitrarily large initial norms $\|\mu_0-1\|_{L^2}$ and $\|(\nabla\bar{\tau}_0, \partial_{\bar{\tau}_0}\mu_0)\|_{L^{2+\epsilon}}$, as long as the norm $\|u_0\|_{L^2}$ is sufficiently small.

(iv) (Tangential regularity). The tangential regularity assumption seems necessary, as we expect finite-time formation of singularity if no regularity assumptions are imposed on the significantly varying viscosity coefficient (see (3.1.25)).

Theorem 3.1.3 establishes the propagation of $W^{1,2+\epsilon}(\mathbb{R}^2)$-tangential regularity for $\epsilon \in (0,\epsilon_0]$. The main difficulty in preserving arbitrary $W^{1,p}(\mathbb{R}^2)$, $p \in (1,\infty)$-tangential regularity lies in showing that $\mathcal{R}_\mu$ is an isomorphism in $L^p(\mathbb{R}^2)$. This property, however, depends in a nontrivial way on the $W^{1,p}$-tangential regularity one seeks to propagate.

As direct applications of Theorem 3.1.3, we address both the smooth-viscosity case and the viscosity-patch problem. The proof is also found in Subsection 3.3.4. The construction of a nondegenerate tangent vector field is found in Appendix 3.D.

Corollary 3.1.5. *The following results hold for two distinct cases of the viscosity coefficient.*

1. *(Smooth-viscosity case for* (μINS)*). Assume the hypotheses of Theorem 3.1.3, and in addition that $\mu_0 \in \dot{W}^{1,q}(\mathbb{R}^2)$ for some $q \in [2,\infty]$. Then the solution provided by Theorem 3.1.3 additionally satisfies $\mu \in L^\infty([0,\infty); \dot{W}^{1,q}(\mathbb{R}^2))$ and*

$$t^{\frac{1}{q}}\nabla u \in L^2((0,\infty); \dot{H}^1(\mathbb{R}^2;\mathbb{R}^{2\times 2})). \tag{3.1.36}$$

In particular, the smallness condition (3.1.27) *can be replaced by*

$$e^{c\|u_0\|^4_{L^2(\mathbb{R}^2)}}\|u_0\|^{\frac{\epsilon}{2}}_{L^2(\mathbb{R}^2)} \cdot \left(\|u_0\|_{\dot{H}^{-1}(\mathbb{R}^2)} + \|\mu_0 - 1\|_{L^2(\mathbb{R}^2)}\|u_0\|_{L^2(\mathbb{R}^2)}\right) \cdot\left(\|\nabla u_0\|_{L^2(\mathbb{R}^2)} + \|\partial_1\mu_0\|^{\frac{2+\epsilon}{\epsilon}}_{L^{2+\epsilon}(\mathbb{R}^2)}\right) \le c_0$$

which corresponds to the condition (3.1.27) *with the constant vector field $\bar{\tau}_0 = \begin{pmatrix}1\\0\end{pmatrix}$.*

2. *(Viscosity patch-type problem for* (μINS) *and the two-phase problem* (3.1.10)*). Let $0 < \mu_* \le 1 \le \mu^*$ be two positive constants and let $\epsilon_0 = \epsilon_0(\mu_*,\mu^*)$ be given by Lemma 3.1.1 - 2.*

Let the initial viscosity be of the form

$$\mu_0(x) = \mu_0^+(x)1_D(x) + \mu_0^-(x)1_{D^c}(x), \text{ such that } \mu_0 \in [\mu_*,\mu^*], \tag{3.1.37}$$

where $D \subset \mathbb{R}^2$ is a bounded, simply connected domain of class $W^{2,2+\epsilon}(\mathbb{R}^2)$, $\epsilon \in (0,\epsilon_0]$, and $\mu_0^+ \in W^{1,2+\epsilon}(\overline{D})$ is a positive continuous bounded function defined on $\overline{D}$ while $\mu_0^- - 1 \in L^2 \cap W^{1,2+\epsilon}(\overline{D}^c)$ is a continuous bounded function defined on $\overline{D}^c$. Let $\tau_0 \in L^\infty \cap \dot{W}^{1,2+\epsilon}(\mathbb{R}^2;\mathbb{R}^2)$, with $\frac{1}{|\tau_0|} \in L^\infty(\mathbb{R}^2)$, be a nondegenerate vector field which is tangent to the boundary ∂D.

Let $u_0 \in H^1 \cap \dot{H}^{-1}(\mathbb{R}^2;\mathbb{R}^2)$ be divergence-free. If (μ_0, u_0, τ_0) satisfies the hypotheses of Theorem 3.1.3, then the system (μINS) supplemented with the initial data (μ_0, u_0) has a unique global-in-time solution $(\mu, u, \nabla\pi)$ which satisfies the estimates in Theorem 3.1.3. Furthermore, for all times $t > 0$, the viscosity retains its patch structure

$$\mu(t,\cdot) = \mu^+(t,\cdot)1_{D_t}(x) + \mu^-(t,\cdot)1_{D_t^c}(x),$$

where $D_t \subset \mathbb{R}^2$ is a bounded, simply connected domain of class $W^{2,2+\epsilon}(\mathbb{R}^2)$, and $\mu^+(t,\cdot) \in W^{1,2+\epsilon}(\overline{D_t})$, $\mu^-(t,\cdot) - 1 \in L^2 \cap W^{1,2+\epsilon}(\overline{D_t}^c)$. This solution solves the two-phase Navier-Stokes equations with constant density (3.1.10), with $\Omega_t^+ = D_t$, $\Omega_t^- = \overline{D_t}^c$ and the interface $\Gamma_t = \partial D_t$.

Remark 3.1.6 (Viscosity layer problem: Arbitrarily many and arbitrarily closely located concentric shell layers). We can straightforwardly generalize the result for the viscosity patch-type problem (3.1.37) to the N-viscosity layer problem with the initial viscosity

$$\mu_0(x) = \sum_{j=1}^{N} \eta_{j,0}(x) 1_{D^{(j)}}(x) + 1_{(\cup_{j=1}^N D^{(j)})^c}(x). \tag{3.1.38}$$

Here, $\eta_{j,0} \in W^{1,2+\epsilon}(\overline{D^{(j)}}; [\mu_*, \mu^*])$ are continuous bounded functions and $D^{(j)} \subset \mathbb{R}^2$, $j = 1, ..., N$, are bounded, simply connected domains of class $W^{2,2+\epsilon}(\mathbb{R}^2)$ which are mutually non-intersecting: $\partial D^{(j)} \cap \partial D^{(i)} = \emptyset$ for $i \neq j$, where $\epsilon \in (0, \epsilon_0]$, with $\epsilon_0 = \epsilon_0(\mu_*, \mu^*)$ given by Lemma 3.1.1 - 2. Thus, either all the domains $D^{(j)}$, $j = 1, \cdots, N$ are disdjoint, or $D^{(i)} \subset D^{(j)}$ for some $i \neq j$.

In the special case where

$$\eta_{j,0} \text{ are positive constants and } D^{(j)} = B_{r^{(j)}}(0), \tag{3.1.39}$$

with $r^{(1)} < \cdots < r^{(N)}$, the smallness assumption (3.1.27) is satisfied (see Appendix 3.D) if

$$\begin{aligned} e^{c\|u_0\|^4_{L^2(\mathbb{R}^2)}} \|u_0\|^{\frac{\epsilon}{2}}_{L^2(\mathbb{R}^2)} \cdot \big(\|u_0\|_{\dot{H}^{-1}(\mathbb{R}^2)} + \|\mu_0 - 1\|_{L^2(\mathbb{R}^2)} \|u_0\|_{L^2(\mathbb{R}^2)}\big) \\ \cdot \big(\|\nabla u_0\|_{L^2(\mathbb{R}^2)} + \frac{1}{r^{(1)}}\big) \leq \tilde{c}_0, \end{aligned} \tag{3.1.40}$$

that is, the smallness of u_0, depends (only) on μ_*, μ^*, $\|\mu_0 - 1\|_{L^2}$ and $\frac{1}{r^{(1)}}$, but not on $r^{(j)} - r^{(i)}$ or N. Hence, Theorem 3.1.3 can handle arbitrarily many concentric discs and the boundaries $\partial D^{(j)}$, $j = 1, ..., N$ can be arbitrarily close.

The smallness condition (3.1.40) is the smallness condition (3.1.27) for the viscosity patch-type problem (3.1.37) when $\mu_0^+ > 0$ is a positive constant, $\mu_0^- = 1$ and $D = B_{r^{(1)}}(0)$.

3.1.3.2. The Boussinesq equations without heat condution

The proof strategy of Theorem 3.1.3 can be adapted to obtain the following local-in-time well-posedness result for the Boussinesq equations without heat conduction (B). For the proof, see Subsection 3.3.5.

Theorem 3.1.7 (Lower bound for the existence time of solutions to the Boussinesq equations without heat conduction (B)**).** *Let $0 < \mu_* \leq 1 \leq \mu^*$ be two positive constants and let $\epsilon_0 = \epsilon_0(\mu_*, \mu^*)$ be given by Lemma 3.1.1 - 2.*

Let $u_0 \in H^1(\mathbb{R}^2; \mathbb{R}^2)$ be a divergence-free vector field and $\vartheta_0 \in L^1 \cap L^r(\mathbb{R}^2)$ for some $r \in (2, \infty]$. Assume the dependence of the viscosity coefficient μ on the temperature function ϑ to be $\mu = \mu_{\text{tem}}(\vartheta)$ for some $\mu_{\text{tem}} \in \mathcal{C}(\mathbb{R}; [\mu_*, \mu^*])$. Let $\tau_0 \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$ be a vector field such that $|\tau_0|^{-1} \in L^\infty(\mathbb{R}^2)$ and $(\nabla \tau_0, \partial_{\tau_0} \mu_0) \in L^{2+\epsilon}(\mathbb{R}^2; \mathbb{R}^{2\times 2+1})$, for some $\epsilon \in (0, \min\{\epsilon_0, r-2\}]$.

Then the system (B)-(τ) *supplemented with the initial data* $(\vartheta_0, u_0, \tau_0)$ *has a unique solution* $(\vartheta, u, \nabla\pi, \tau)$ *on the time interval* $[0,T]$, *with the existence time* $T>0$ *bounded from below as follows*

$$\max_{q\in\{1,2+\epsilon\}}\Big(\|u_0\|_{L^2(\mathbb{R}^2)}+T^{\frac32-\frac1q}\|\vartheta_0\|_{L^q(\mathbb{R}^2)}\Big)^{\frac{2\epsilon}{(2+\epsilon)^2}}\Big(T^{\frac12}\|\nabla u_0\|_{L^2(\mathbb{R}^2)}+T^{\frac{\epsilon}{2+\epsilon}}\|(\nabla\bar\tau_0,\partial_{\bar\tau_0}\mu_0)\|_{L^{2+\epsilon}(\mathbb{R}^2)}$$
$$+T^{\frac32-\frac1q}\|\vartheta_0\|_{L^q(\mathbb{R}^2)}\big(1+\|u_0\|^{\frac12}_{L^2(\mathbb{R}^2)}+(T^{\frac32-\frac1q}\|\vartheta_0\|_{L^q(\mathbb{R}^2)})^{\frac12}\big)\Big)^{\frac{\epsilon^2+2\epsilon+4}{(2+\epsilon)^2}}\le c_1, \tag{3.1.41}$$

where $c_1>0$ *depends only on* μ_*, μ^*, ϵ. *The solution satisfies* $\vartheta\in\mathcal{C}([0,T];\cap_{1\le\tilde r\le r,\tilde r<\infty}L^{\tilde r}(\mathbb{R}^2))\cap L^\infty([0,T];L^1\cap L^r(\mathbb{R}^2))$ *and all the estimates in* (3.1.28) *on* $[0,T]$, *except the property for* $\mu-1$.

Furthermore, for the quantity $a_\vartheta = a-\mathcal{R}_{-1}\vartheta$, *with* a *defined in* (3.1.14) *and* $\mathcal{R}_{-1}=\partial_1(-\Delta)^{-1}$, *we have the (time weighted)* $H^1(\mathbb{R}^2)$*-energy estimates*

$$a_\vartheta\in\mathcal{C}([0,T];L^2(\mathbb{R}^2))\cap L^2([0,T];L^2(\mathbb{R}^2;\mathbb{R}^2)),$$
$$t^{\frac12}\nabla a_\vartheta\in L^\infty([0,T];L^2(\mathbb{R}^2;\mathbb{R}^2))\cap L^2([0,T];\dot H^1(\mathbb{R}^2;\mathbb{R}^2)).$$

Moreover, $a, \alpha, \partial_\tau u\in L^1([0,T];W^{1,2+\epsilon}(\mathbb{R}^2))$ *and* $\frac{D}{Dt}u=\operatorname{div}T(u,\pi)+\vartheta e_2\in L^1([0,T];L^{2+\epsilon}(\mathbb{R}^2))$, *with the same notations* $\alpha, \frac{D}{Dt}u, T(u,\pi)$ *as in Theorem 3.1.3.*

Remark 3.1.8. The main observation enabling us to apply the methods for the system (μINS) to the Boussinesq system (B) (and also the density-dependent case (INS), see Theorem 3.1.9 below) is the validity of the robust estimate (3.1.35) for the velocity gradient. As long as higher-order energy estimates can be established to deduce the $\|a\|_{W^{1,2+\epsilon}(\mathbb{R}^2)}$-estimate, the $\|\nabla u\|_{L^\infty(\mathbb{R}^2)}$-estimate follows. For the Boussinesq equations the $H^1(\mathbb{R}^2)$-energy estimates hold for the quantity a_ϑ, which basically coincides with $a=(\mathcal{R}^\perp\otimes\mathcal{R}):(\mu Su)$ but corrected by $\mathcal{R}_{-1}\vartheta$ due to the additional buoyancy force ϑe_2 in (B). As there is no regularity assumption on ϑ, we do not have $H^1(\mathbb{R}^2)$-energy estimates for a in this case.

The bound (3.1.41) is inspired by the invariance of the quantities

$$t^{\frac32-\frac1q}\|\vartheta_0\|_{L^q(\mathbb{R}^2)},\quad \|u_0\|_{L^2(\mathbb{R}^2)},\quad t^{\frac12}\Big(\|\nabla u_0\|_{L^2(\mathbb{R}^2)}+\|(\nabla\bar\tau_0,\partial_{\bar\tau_0}\mu_0)\|^{\frac{2+\epsilon}{\epsilon}}_{L^{2+\epsilon}(\mathbb{R}^2)}\Big),$$

under the scaling

$$(\vartheta_\lambda,u_\lambda)(t,x)=(\lambda^{-3}\vartheta,\lambda^{-1}u)(\lambda^{-2}t,\lambda^{-1}x),\quad \lambda>0.$$

3.1.3.3. The inhomogeneous incompressible Navier-Stokes equations with density-dependent viscosity

Our main result for system (INS) reads as follows.

Theorem 3.1.9 (Global-in-time well-posedness of the inhomogeneous incompressible Navier-Stokes equations (INS)**).** *Let* $0<\mu_*\le 1\le\mu^*$ *be two positive constants and let* $\epsilon_0=\epsilon_0(\mu_*,\mu^*)$ *be given by Lemma 3.1.1 - 2.*

Let $\rho_0\in L^\infty(\mathbb{R}^2;[\rho_*,\rho^*])$, $0<\rho_*\le\rho^*$, *be an initial density satisfying* $\rho_0-1\in L^2(\mathbb{R}^2)$. *Assume the dependence of the viscosity coefficient* μ *on the density function* ρ *to be* $\mu=\mu_{\mathrm{den}}(\rho)$

for some $\mu_{\text{den}} \in W^{1,\infty}([\rho_*, \rho^*]; [\mu_*, \mu^*])$. *Let* $u_0 \in H^1 \cap \dot{H}^{-1}(\mathbb{R}^2; \mathbb{R}^2)$ *be divergence-free and* $\tau_0 \in L^\infty(\mathbb{R}^2; \mathbb{R}^2)$ *such that* $|\tau_0|^{-1} \in L^\infty(\mathbb{R}^2)$ *and* $(\nabla \tau_0, \partial_{\tau_0} \rho_0) \in L^{2+\epsilon}(\mathbb{R}^2; \mathbb{R}^{2\times 2+1})$ *for some* $\epsilon \in (0, \epsilon_0]$ *in the sense of distributions. If*

$$e^{c_2(\|u_0\|^2_{L^2(\mathbb{R}^2)} + \|\rho_0 - 1\|^2_{L^2(\mathbb{R}^2)} \|\nabla u_0\|^2_{L^2(\mathbb{R}^2)}) \exp(c_2 \|u_0\|^2_{L^2(\mathbb{R}^2)})} \cdot \Big(\|u_0\|_{L^2(\mathbb{R}^2)} + \|\rho_0 - 1\|_{L^2(\mathbb{R}^2)} \|\nabla u_0\|_{L^2(\mathbb{R}^2)} \Big)^{\frac{\epsilon}{2}}$$
$$\cdot \Big(\|u_0\|_{\dot{H}^{-1}(\mathbb{R}^2)} + \|\rho_0 - 1\|_{L^2(\mathbb{R}^2)} \|u_0\|_{L^2(\mathbb{R}^2)} \Big) \cdot \Big(\|\nabla u_0\|_{L^2(\mathbb{R}^2)} + \|(\nabla \bar{\tau}_0, \partial_{\bar{\tau}_0} \mu_0)\|^{\frac{2+\epsilon}{\epsilon}}_{L^{2+\epsilon}(\mathbb{R}^2)} \Big) \leq c_3, \tag{3.1.42}$$

where c_2, c_3 *are positive constants depending only on* $\rho_*, \rho^*, \mu_*, \mu^*$, ϵ *and* $\|\mu'_{\text{den}}\|_{L^\infty([\rho_*, \rho^*])}$, *then the system* (INS)-(τ) *supplemented with the initial data* (ρ_0, u_0, τ_0) *has a unique global-in-time solution* $(\rho, u, \nabla\pi, \tau)$ *such that* (3.1.28) *holds, with* μ *replaced by* ρ. *Furthermore, we have the (time weighted)* $L^2(\mathbb{R}^2)$*-energy estimates*

$$a \in \mathcal{C}_b([0, \infty); L^2(\mathbb{R}^2)) \cap L^2((0, \infty); \dot{H}^1(\mathbb{R}^2)),$$
$$t^{\frac{1}{2}} \frac{D}{Dt} u \in L^\infty((0, \infty); L^2(\mathbb{R}^2; \mathbb{R}^2)) \cap L^2((0, \infty); \dot{H}^1(\mathbb{R}^2; \mathbb{R}^2)),$$

$a, \alpha, \partial_\tau u \in L^1((0, \infty); W^{1,2+\epsilon}(\mathbb{R}^2))$ *and* $\rho \frac{D}{Dt} u = \operatorname{div} T(u, \pi) \in L^1((0, \infty); L^{2+\epsilon}(\mathbb{R}^2))$, *with the same notations* $a, \alpha, \frac{D}{Dt} u, T(u, \pi)$ *as given in Theorem 3.1.3.*

In particular, if the initial density is of the patch-type

$$\rho_0(x) = \rho_0^+(x) 1_D(x) + \rho_0^-(x) 1_{D^c}(x),$$

for some bounded, simply connected domain $D \subset \mathbb{R}^2$ *with* $W^{2,2+\epsilon}$*-boundary, and functions* $\rho_0^+ \in W^{1,2+\epsilon}(\overline{D})$, $\rho_0^- - 1 \in L^2 \cap W^{1,2+\epsilon}(\overline{D}^c)$, *then there exists a nondegenerate vector field* $\tau_0 \in L^\infty \cap \dot{W}^{1,2+\epsilon}(\mathbb{R}^2)$ *tangential to the boundary* ∂D. *If the smallness assumption* (3.1.42) *is satisfied, then the unique solution above preserves the patch structure for all times* $t > 0$,

$$\rho(t, x) = \rho^+(t, x) 1_{D_t}(x) + \rho^-(t, x) 1_{D_t^c}(x),$$

for some bounded, simply connected domain $D_t \subset \mathbb{R}^2$ *with* $W^{2,2+\epsilon}$*-boundary, and functions* $\rho^+(t, \cdot) \in W^{1,2+\epsilon}(\overline{D_t})$, $\rho^-(t, \cdot) - 1 \in L^2 \cap W^{1,2+\epsilon}(\overline{D_t}^c)$. *Thus, the density-patch-type problem in the absence of vacuum for the density-dependent incompressible Navier-Stokes equations* (INS) *is uniquely globally-in-time solvable under the smallness assumption* (3.1.42). *This solution solves also the two-phase Navier-Stokes equations* (3.1.7) *without surface tension* ($\sigma = 0$) *with* $\Omega_t^+ = D_t$, $\Omega_t^- = \overline{D_t}^c$ *and the interface* $\Gamma_t = \partial D_t$.

Remark 3.1.10. 1. As before, the smallness condition (3.1.42) allows large variations in the initial density and viscosity, provided that the initial velocity remains sufficiently small. The left hand side of (3.1.42) is invariant under the scaling

$$(\rho_{0,\lambda}, u_{0,\lambda})(x) = (\rho_0, \lambda^{-1} u_0)(\lambda^{-1} x), \quad \lambda > 0.$$

The results for the density-patch-type problem in Theorem 3.1.9 can be generalized to the density layer problem as in Remark 3.1.6.

2. For the density-dependent case (INS) the $L^2(\mathbb{R}^2)$-energy estimates hold for the material derivative $\frac{D}{Dt} u = (\partial_t + u \cdot \nabla) u$. As there is no regularity assumption on ρ, we do not have $H^1(\mathbb{R}^2)$-energy estimates for a, which is related to $\frac{D}{Dt} u$ by $\nabla^\perp a = \mathbb{P}(\rho \frac{D}{Dt} u)$, with $\mathbb{P}$ denoting the Leray-Helmholtz projection on the divergence-free vector fields.

To conclude this subsection, we briefly revisit some studies on the vortex patch problem and the density patch problem in fluid mechanics.

- Vortex-patch problem for the (classical) incompressible Euler equations with the initial vorticity $\omega_0 = 1_{D_0}$.

  J.-Y. Chemin's celebrated works [40, 41] confirm the regularity propagation of the domain boundary ∂D_0 for all time, by use of a nondegenerate family of vector fields. See also A. L. Bertozzi and P. Constantin's work [22] from a more geometric viewpoint and P. Serfati's work [217]. Their strategy was also used recently to solve the regularity propagation of temperature-fronts for the Boussinesq equations (3.1.3) in [37]. A thorough review of results on the two-dimensional vortex-patch problem can be found in [103]. See also [98] for the problem in three space dimensions and [88] for the inhomogeneous case.

- Density-patch problem for the inhomogeneous Navier-Stokes equations with the initial density $\rho_0 = 1_{D_0}$.

  In the case of constant viscosity coefficient $\mu = \nu > 0$ and in the absence of vacuum with $\rho_0 = \rho^+ 1_{D_0} + 1_{D_0^c}$, $\rho^+ > 0$, it was proven by the first author and P. Zhang [166, 167] that the $W^{k+2,p}$-regularity of the interface ∂D_0 is propagated throughout time for $k \in \mathbb{N}$, $p \in (2,4)$. A similar result was obtained by F. Gancedo and E. Garcia-Juarez in [100] using bootstrapping arguments. The density-patch problem in a bounded domain was solved by R. Danchin and P. B. Mucha in [60]. Specifically, they showed that the $\mathcal{C}^{1,\alpha}$-regularity of the fluid-vacuum interface is preserved over time ($\alpha \in (0,1)$ in dimension two and $\alpha \in (0, \frac{1}{2})$ in dimension three). Very recently, an analogous result for the density-patch problem in $\mathbb{R}^2$ was obtained by T. Hao et al. [116]. See also the earlier works [65, 167] for a small density jump and [165] for the three-dimensional case.

  If μ is variable but close to a positive constant as in (3.1.5) and the density is bounded away from zero, then global-in-time results were successfully obtained: M. Paicu and P. Zhang [198] proved the propagation of $H^{\frac{5}{2}}$-regularity, and F. Gancedo and E. Garcia-Juarez [101] the propagation of $\mathcal{C}^{1,\alpha}$-regularity, $\alpha \in (0,1)$, both in two space dimensions.

  The results for two-phase Navier-Stokes equations (3.1.7)-(3.1.8) in [68, 69, 213], with (slightly perturbed) piecewise-constant viscosity coefficients and without surface tension $\sigma = 0$, can also be included in this category.

To the best of the authors' knowledge, the density patch problem for (INS) with general variable viscosity which might have *large jumps*, as stated in Theorem 3.1.9, was not addressed in the literature before.

3.2. Outline of the proof strategy

Theorems 3.1.3, 3.1.7 and 3.1.9 are proved by deriving suitable a priori estimates, constructing an approximate solution sequence and then showing its convergence to some limit function which solves the corresponding system. Here we outline the main ideas for establishing the global-in-time a priori estimates for system (μINS), which are proved in three steps:

- Step I. $L^2(\mathbb{R}^2)$-energy estimates for u and $H^1(\mathbb{R}^2)$-energy estimates for a in terms of $\nabla u \in L^\infty(\mathbb{R}^2)$;
- Step II. Time-independent Lipschitz estimate for u in terms of $a \in W^{1,2+\epsilon}(\mathbb{R}^2)$ and $\nabla \bar{\tau}, \partial_{\bar{\tau}}\mu, \nabla \partial_{\bar{\tau}} u \in L^{2+\epsilon}(\mathbb{R}^2)$;
- Step III. $L^1_t \mathrm{Lip}(\mathbb{R}^2)$-bound for u and the conclusion of $H^1(\mathbb{R}^2)$-energy estimates for a.

In the following we explain the main ideas of each step.

3.2.1. Step I. Energy estimates

Smooth solutions of the density-dependent Navier-Stokes equations (INS) in d space dimensions, $d \geq 2$, come with the following energy balance

$$\int_{\mathbb{R}^d} \rho |u|^2 dx + \int_0^t \int_{\mathbb{R}^d} \mu |Su|^2 dx dt' = \int_{\mathbb{R}^d} \rho_0 |u_0|^2 dx. \tag{3.2.1}$$

Based on this energy balance, P.-L. Lions [175] proved the global in time existence of weak solutions to (INS) with finite energy in any space dimension $d \geq 2$. The uniqueness and regularity of such weak solutions are still open questions even in two space dimensions. Under the additional assumption that the viscosity jump is sufficiently small (3.1.5) and the initial velocity belongs to $H^1(\mathbb{T}^2)$, B. Desjardins [75] proved that the global weak solution $(\rho, u, \nabla\pi)$ of [175] on the two-dimensional torus $\mathbb{T}^2$ satisfies $u \in L^\infty_{\mathrm{loc}}([0,\infty); H^1(\mathbb{T}^2))$. With additional regularity assumptions on the initial data he could also establish $u \in L^2([0,T_*]; H^2(\mathbb{T}^2))$ for some short time T_*. However, these regularity results still do not give an answer to the uniqueness and regularity question.

In the same spirit, for the Navier-Stokes equations with freely transported viscosity coefficient (μINS) we aim to establish

- an energy balance similar to (3.2.1) as well as a time weighted energy-balance version $\|(u, t'^{(\frac{1}{2})-} u)\|_{L^\infty_t L^2 \cap L^2_t \dot{H}^1}$, by use of the initial data $u_0 \in L^2 \cap \dot{H}^{-1}(\mathbb{R}^2)$, $\mu_0 - 1 \in L^2(\mathbb{R}^2)$;
- an L^2-estimate as well as its time weighted version for $\|(a, t'^{\frac{1}{2}} a, t'^{1-} a)\|_{L^\infty_t L^2 \cap L^2_t \dot{H}^1}$ in terms of $V(t) := \exp(C \|\nabla u\|_{L^1_t L^\infty})$ and the initial data $u_0 \in H^1 \cap \dot{H}^{-1}(\mathbb{R}^2)$, based on the vorticity equation (3.1.21);
- a time-weighted $\dot{H}^1$-estimate for $\|t'^{\frac{1}{2}} \nabla a\|_{L^\infty_t L^2 \cap L^2_t \dot{H}^1}$ in terms of $V(t)$, $\|t'^{\frac{1}{2}} \nabla u\|_{L^2_t L^\infty}$ and the initial data $u_0 \in \dot{H}^1(\mathbb{R}^2)$, based on the vorticity equation (3.1.21).

The time-weighted estimate $\|t'^{(\frac{1}{2})-} u\|_{L^\infty_t L^2 \cap L^2_t \dot{H}^1}$ has been established for the density-dependent Navier-Stokes equations (INS) in e.g. [11, 242]; see also [10] for the three-dimensional case. Roughly speaking, the strong decay assumption in the low frequency part $u_0 \in \dot{H}^{-1}(\mathbb{R}^2)$ implies stronger decay in time of the solution u. A similar consideration applies to the time-weighted estimates for a. Compared with the derivation of classical energy estimates for u, due to the non-local representation of $a = \mathcal{R}_\mu \omega$ (recalling (3.1.16)) in terms of ω, we have to make use of commutator estimates for the Riesz transform, as well as the commutation relation $[\mu, \frac{D}{Dt}] = 0$, that is the transport equation $\frac{D}{Dt}\mu = 0$, when deriving energy estimates for a. Notice that in the energy estimates for a we simply use the Lipschitz norm of the velocity field $\|\nabla u\|_{L^1_t L^\infty}$ and $\|t'^{\frac{1}{2}} \nabla u\|_{L^2_t L^\infty}$, instead of the classical $\|\nabla u\|_{L^4_t L^4}$-norm (see e.g. [198]). Indeed, although a priori the initial lower and upper bounds μ_*, μ^* for μ_0 are transported by the free transport equation $(\mu\mathrm{INS})_1$,

$$\mu_* \leq \mu(t,x) \leq \mu^*,$$

we can not, in light of (3.1.25), control $\|\omega\|_{L^4(\mathbb{R}^2)}$ or $\|\nabla u\|_{L^4(\mathbb{R}^2)}$ by $\|a\|_{H^1(\mathbb{R}^2)}$ with only positive bounded μ while with no regularity or small variation assumptions on μ. See more discussions in Step II below.

The energy estimates for a are not yet closed, and we discuss in Step II the (time-independent) Lipschitz estimate for u in terms of $\|a\|_{W^{1,2+\epsilon}(\mathbb{R}^2)}$ and the tangential regularity. Finally, a bootstrap argument concludes the global-in-time estimates in Step III.

3.2.2. Step II. The time-independent Lipschitz estimate

It is well-known that for evolution equations arising in fluid mechanics, the $L^1_t\mathrm{Lip}(\mathbb{R}^2)$-regularity of the fluid velocity is crucial for regularity theory. In order to obtain such an estimate we begin with establishing a *time-independent* Lipschitz estimate for the velocity vector field, which is a key step.

The main obstacle to derive the desired Lipschitz estimate is that one can not bound $\|\nabla u\|_{L^\infty(\mathbb{R}^2)}$ by $\|a\|_{H^2(\mathbb{R}^2)}$ (from the energy estimates in Step I) directly, and even worse, we can not control $\|\nabla u\|_{L^4(\mathbb{R}^2)}$ a priori by $\|a\|_{H^1(\mathbb{R}^2)}$ or $\|a\|_{L^4(\mathbb{R}^2)}$, provided with the a priori bound $\mu_* \leq \mu(t,x) \leq \mu^*$, as mentioned above.

Recall that the velocity gradient $\nabla u = \nabla\nabla^\perp\phi$ is related to a by (3.1.12):

$$\mathcal{L}_\mu\phi = \Delta a, \text{ with } \mathcal{L}_\mu = (\partial_{22}-\partial_{11})\mu(\partial_{22}-\partial_{11}) + (2\partial_{12})\mu(2\partial_{12}), \tag{3.2.2}$$

where $\mathcal{L}_\mu$ is a fourth-order elliptic operator. Given the failure of the $L^p(\mathbb{R}^2)$-estimate (3.1.25), we impose a tangential regularity assumption on the initial viscosity μ_0 with respect to some nondegenerate regular vector field τ_0, aiming to obtain the Lipschitz estimate for the velocity by exploiting ellipticity and tangential regularity. Note that the discontinuity of μ in the normal direction $\tau_0^\perp$ is allowed.

In the past twenty years significant developments have been made in the study of elliptic and parabolic systems with rough coefficients, see e.g. the book [155]. H. Dong and D. Kim established in [80] L^p-estimates for solutions of higher order elliptic and parabolic systems with so-called variably partially BMO coefficients, which in particular includes discontinuous coefficients which may have jumps in one direction and are continuous in the other directions. Roughly speaking, this means that for every localized cylinder there exists a local coordinate system such that the coefficients $\mu(y', y_d)$ are BMO with respect to the first $d-1$ components $y' \in \mathbb{R}^{d-1}$, while only measurable and bounded in the last component $y_d \in \mathbb{R}$. This partial regularity in the coefficients implies then the regularity of the solution in y', and finally the ellipticity (or parabolicity) of the equation allows one to recover the desired regularity of the solution in y_d as well.

Observe that functions with tangential regularity, e.g. the initial data μ_0 given in Theorem 3.1.3, fall into Dong-Kim's coefficient category. Indeed, for the *stationary* Navier-stokes equation, it was shown by use of Dong-Kim's results in [123] that on a bounded $C^{1,1}$-domain $\Omega \subset \mathbb{R}^2$, given a weak solution $(\rho, u) \in L^\infty(\Omega; [0,\infty)) \times H^1(\Omega)$ satisfying appropriate boundary conditions and provided the coefficient μ has tangential regularity, we have

$$\nabla u \in L^p(\Omega) \quad \text{for any } p \in (1,\infty),$$

(note that $p=\infty$ can not be achieved by Dong-Kim's results). Unfortunately, Dong-Kim's estimates for $\mathcal{L}_\mu\phi = \Delta a$ can not give the explicit dependence on the tangential regularity of the coefficient μ, which is extremely important for us since the tangential regularity also evolves in time and should be tracked precisely. We follow the essential idea to separate the "good" and "bad" directions, but in a more transparent way, below.

Lemma 3.2.1 (Decomposition of $\mathcal{L}_\mu$ in tangential and normal directions in terms of "local good unknown" α). *Let* $\tau = \begin{pmatrix}\tau_1\\ \tau_2\end{pmatrix}(x)$ *be a regular nondegenerate vector field such that*

$$\tau \in L^\infty(\mathbb{R}^2;\mathbb{R}^2), \quad \nabla\tau \in L^p(\mathbb{R}^2;\mathbb{R}^{2\times 2}), \text{ for some } p \in (2,\infty), \quad \frac{1}{|\tau|} \in L^\infty(\mathbb{R}^2). \tag{3.2.3}$$

We introduce correspondingly

- *The unit tangential and normal vectors*

$$\overline{\tau} = \begin{pmatrix} \overline{\tau}_1 \\ \overline{\tau}_2 \end{pmatrix} = \frac{\tau}{|\tau|} = \begin{pmatrix} \frac{\tau_1}{|\tau|} \\ \frac{\tau_2}{|\tau|} \end{pmatrix}, \quad n = -\overline{\tau}^\perp = -\frac{\tau^\perp}{|\tau|} = \begin{pmatrix} \frac{\tau_2}{|\tau|} \\ -\frac{\tau_1}{|\tau|} \end{pmatrix} = \begin{pmatrix} \overline{\tau}_2 \\ -\overline{\tau}_1 \end{pmatrix}, \tag{3.2.4}$$

and their tensor products

$$\overline{\tau} \otimes \overline{\tau} = \begin{pmatrix} \overline{\tau}_1^2 & \overline{\tau}_1\overline{\tau}_2 \\ \overline{\tau}_1\overline{\tau}_2 & \overline{\tau}_2^2 \end{pmatrix}, \; n \otimes n = \begin{pmatrix} \overline{\tau}_2^2 & -\overline{\tau}_1\overline{\tau}_2 \\ -\overline{\tau}_1\overline{\tau}_2 & \overline{\tau}_1^2 \end{pmatrix}, \; \overline{\tau} \otimes n = (n \otimes \overline{\tau})^T = \begin{pmatrix} \overline{\tau}_1\overline{\tau}_2 & -\overline{\tau}_1^2 \\ \overline{\tau}_2^2 & -\overline{\tau}_1\overline{\tau}_2 \end{pmatrix}.$$

- *The associated directional differential operators*

$$\partial_{\overline{\tau}} = \overline{\tau} \cdot \nabla, \quad \partial_n = n \cdot \nabla, \tag{3.2.5}$$

and their adjoint operators

$$\partial_{\overline{\tau}}^* = -\mathrm{div}\, \overline{\tau}, \quad \partial_n^* = -\mathrm{div}\, n, \tag{3.2.6}$$

where the operator $\mathrm{div}\, v$ *is understood as* $\mathrm{div}\, v(f) = \mathrm{div}\,(vf) = \sum_{j=1}^2 \partial_j(v_j f)$, *for* $v = \overline{\tau}$, n.

Then the following formulas hold

1. a) $\nabla = \overline{\tau}\partial_{\overline{\tau}} + n\partial_n = -\partial_{\overline{\tau}}^* \overline{\tau} - \partial_n^* n$ *and* $\nabla^\perp = -n\partial_{\overline{\tau}} + \overline{\tau}\partial_n = \partial_{\overline{\tau}}^* n - \partial_n^* \overline{\tau}$. *More precisely,*

$$\begin{aligned} \partial_1 &= \overline{\tau}_1\partial_{\overline{\tau}} + \overline{\tau}_2\partial_n &&= -\partial_{\overline{\tau}}^*(\overline{\tau}_1 \cdot) - \partial_n^*(\overline{\tau}_2 \cdot), \\ \partial_2 &= \overline{\tau}_2\partial_{\overline{\tau}} - \overline{\tau}_1\partial_n &&= -\partial_{\overline{\tau}}^*(\overline{\tau}_2 \cdot) + \partial_n^*(\overline{\tau}_1 \cdot), \end{aligned} \tag{3.2.7}$$

   b) $\Delta = \nabla \cdot \nabla = -\partial_{\overline{\tau}}^* \partial_{\overline{\tau}} - \partial_n^* \partial_n$ *and* $n\Delta = -\partial_{\overline{\tau}}\nabla^\perp + \partial_n \nabla$,

   c) $\nabla^\perp \otimes \nabla = \partial_{\overline{\tau}}^*(n \otimes \overline{\tau})\partial_{\overline{\tau}} + \partial_{\overline{\tau}}^*(n \otimes n)\partial_n - \partial_n^*(\overline{\tau} \otimes \overline{\tau})\partial_{\overline{\tau}} - \partial_n^*(\overline{\tau} \otimes n)\partial_n$.

2. *Let* $\mu \in L^\infty(\mathbb{R}^2)$, *and recall the definition of the operator* $\mathcal{L}_\mu$ *in Lemma 3.1.1*

$$\begin{aligned} &\mathcal{L}_\mu \phi = (\nabla^\perp \otimes \nabla) : (\mu S \nabla^\perp \phi), \\ \textit{with} \quad &S\nabla^\perp \phi = \nabla\nabla^\perp\phi + (\nabla\nabla^\perp\phi)^T = \begin{pmatrix} -2\partial_{12}\phi & (\partial_{11} - \partial_{22})\phi \\ (\partial_{11} - \partial_{22})\phi & 2\partial_{12}\phi \end{pmatrix}. \end{aligned}$$

   - *We can reformulate the operator* $\mathcal{L}_\mu$ *as follows*

$$\begin{aligned} &\mathcal{L}_\mu \phi \\ &= -\partial_{\overline{\tau}}^*\Big((\overline{\tau}_2^2 - \overline{\tau}_1^2)\partial_{\overline{\tau}}\omega_1\Big) + \partial_{\overline{\tau}}^*\Big(2\overline{\tau}_1\overline{\tau}_2\partial_n\omega_1\Big) + \partial_n^*\Big(2\overline{\tau}_1\overline{\tau}_2\partial_{\overline{\tau}}\omega_1\Big) + \partial_n^*\Big((\overline{\tau}_2^2 - \overline{\tau}_1^2)\partial_n\omega_1\Big) \\ &\quad - \partial_{\overline{\tau}}^*\Big(2\overline{\tau}_1\overline{\tau}_2\partial_{\overline{\tau}}\omega_2\Big) - \partial_{\overline{\tau}}^*\Big((\overline{\tau}_2^2 - \overline{\tau}_1^2)\partial_n\omega_2\Big) - \partial_n^*\Big((\overline{\tau}_2^2 - \overline{\tau}_1^2)\partial_{\overline{\tau}}\omega_2\Big) + \partial_n^*\Big(2\overline{\tau}_1\overline{\tau}_2\partial_n\omega_2\Big), \end{aligned} \tag{3.2.8}$$

     where we denote

$$\omega_1 = \mu(\partial_{22} - \partial_{11})\phi, \quad \omega_2 = \mu 2\partial_{12}\phi, \qquad \textit{such that} \quad \mu S\nabla^\perp\phi = \begin{pmatrix} -\omega_2 & -\omega_1 \\ -\omega_1 & \omega_2 \end{pmatrix}.$$

- *We can furthermore decompose* $\mathcal{L}_\mu\phi$ *into*

$$\mathcal{L}_\mu\phi = -\,\partial_n^*\partial_n\alpha + \mathcal{L}_\mu^\tau\phi, \tag{3.2.9}$$

where α *is the local good unknown defined by* (3.1.31)*:*

$$\alpha = (\bar\tau\otimes n) : (\mu S\nabla^\perp\phi), \tag{3.2.10}$$

and

$$\begin{aligned}
\mathcal{L}_\mu^\tau\phi =& -\,\partial_{\bar\tau}^*\Big((\bar\tau_2^2-\bar\tau_1^2)\partial_{\bar\tau}\omega_1 + 2\bar\tau_1\bar\tau_2\partial_{\bar\tau}\omega_2\Big) + 2\partial_n^*\Big(2\bar\tau_1\bar\tau_2\partial_{\bar\tau}\omega_1 - (\bar\tau_2^2-\bar\tau_1^2)\partial_{\bar\tau}\omega_2\Big)\\
&-\partial_1\Big(\partial_2(2\bar\tau_1\bar\tau_2)\omega_1 - \partial_2(\bar\tau_2^2-\bar\tau_1^2)\omega_2\Big) + \partial_2\Big(\partial_1(2\bar\tau_1\bar\tau_2)\omega_1 - \partial_1(\bar\tau_2^2-\bar\tau_1^2)\omega_2\Big)\\
&-\partial_n^*\Big(\partial_n(\bar\tau_2^2-\bar\tau_1^2)\omega_1 + \partial_n(2\bar\tau_1\bar\tau_2)\omega_2\Big)\\
=&\nabla\cdot\Big(\big(\bar\tau(\bar\tau_2^2-\bar\tau_1^2) - 2n(2\bar\tau_1\bar\tau_2)\big)\partial_{\bar\tau}\omega_1 + \big(\bar\tau 2\bar\tau_1\bar\tau_2 - 2n(\bar\tau_2^2-\bar\tau_1^2)\big)\partial_{\bar\tau}\omega_2\Big)\\
&+\nabla^\perp\cdot\Big(-\omega_1\nabla(2\bar\tau_1\bar\tau_2) + \omega_2\nabla(\bar\tau_2^2-\bar\tau_1^2)\Big)\\
&+\nabla\cdot\Big(\omega_1\,\partial_n(\bar\tau_2^2-\bar\tau_1^2)n + \omega_2\,\partial_n(2\bar\tau_1\bar\tau_2)n\Big).
\end{aligned}$$

Here, α *can be equivalently written as*

$$\begin{aligned}
\alpha &= -(\bar\tau_2^2-\bar\tau_1^2)\omega_1 - 2\bar\tau_1\bar\tau_2\omega_2\\
&= -(\bar\tau_2^2-\bar\tau_1^2)\mu(\partial_{22}-\partial_{11})\phi - 2\bar\tau_1\bar\tau_2\mu(2\partial_{12})\phi,
\end{aligned} \tag{3.2.11}$$

or

$$\alpha = \mu\Delta\phi - 2\mu(\bar\tau\cdot\partial_{\bar\tau}\nabla\phi), \tag{3.2.12}$$

which implies the relation between $\partial_n\nabla^\perp\phi$ *and* α *below (if* $\mu\neq 0$*)*

$$\partial_n\nabla^\perp\phi = \frac{\alpha}{\mu}\bar\tau + 2(\bar\tau\cdot\partial_{\bar\tau}\nabla\phi)\bar\tau - \partial_{\bar\tau}\nabla\phi. \tag{3.2.13}$$

- *Recall the definition of* a *in* (3.1.12)*:* $\Delta a = \mathcal{L}_\mu\phi$*. Then we have the following relation*

$$\begin{aligned}
&\nabla(a-\alpha)\\
&=\mathcal{R}\mathcal{R}\cdot\Big(-\bar\tau\partial_{\bar\tau}\alpha + \big(\bar\tau(\bar\tau_2^2-\bar\tau_1^2) - 2n(2\bar\tau_2\bar\tau_2)\big)\partial_{\bar\tau}\omega_1 + \big(2\bar\tau_2\bar\tau_2 - 2n(\bar\tau_2^2-\bar\tau_1^2)\big)\partial_{\bar\tau}\omega_2\Big)\\
&\quad+\mathcal{R}\mathcal{R}^\perp\cdot\Big(-\omega_1\nabla(2\bar\tau_2\bar\tau_2) + \omega_2\nabla(\bar\tau_2^2-\bar\tau_1^2)\Big)\\
&\quad+\mathcal{R}\mathcal{R}\cdot\Big(\omega_1\,\partial_n(\bar\tau_2^2-\bar\tau_1^2)n + \omega_2\,\partial_n(2\bar\tau_2\bar\tau_2)n\Big),
\end{aligned} \tag{3.2.14}$$

where $\mathcal{R} = \frac{\frac{1}{i}\nabla}{\sqrt{-\Delta}}$ *denotes the Riesz operator. Here the equality can be understood in* $L^p(\mathbb{R}^2)$ *if* $a,\alpha\in W^{1,p}(\mathbb{R}^2)$*,* $\partial_{\bar\tau}\mu, \partial_{\bar\tau}\nabla^2\phi, \nabla\bar\tau \in L^p(\mathbb{R}^2)$ *and* $\mu, \nabla^2\phi \in L^\infty(\mathbb{R}^2)$*.*

Proof. The formulas (1a)-(1c) in the first statement follow from straightforward calculations.

The formula (3.2.8) follows from (1c) directly.

The equivalence of the definitions (3.2.10), (3.2.11) and (3.2.12) follows by direct computation. The relation (3.2.13) follows from (3.2.12) and (1b).

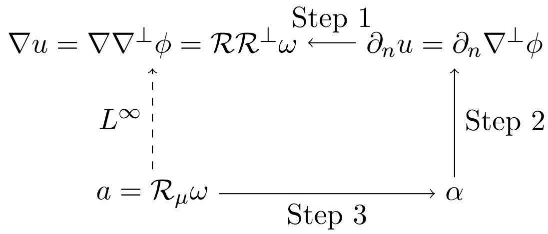

Figure 3.1.: Idea of the proof of Proposition 3.3.5.

We derive (3.2.9) from (3.2.8), by applying the following commutator identities (with appropriately chosen f, g) to (3.2.8):

$$\partial_{\bar\tau}^*(f\partial_n g) - \partial_n^*(f\partial_{\bar\tau} g) = \partial_1(f\partial_2 g) - \partial_2(f\partial_1 g) = -\partial_1((\partial_2 f)g) + \partial_2((\partial_1 f)g),$$
$$\partial_n^*(f\partial_n g) - \partial_n^*\partial_n(fg) = -\partial_n^*((\partial_n f)g).$$

Finally, (1b) and (3.2.9) imply $\Delta a = \mathcal{L}_\mu\phi = \Delta\alpha + \partial_{\bar\tau}^*\partial_{\bar\tau}\alpha + \mathcal{L}_\mu^\tau\phi$ and hence (3.2.14) follows. This completes the proof of Lemma 3.2.1. □

Making use of Lemma 3.2.1 we can derive the following $L^\infty(\mathbb{R}^2)$-bound for ∇u (see Proposition 3.3.5 below):

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \le C(p)\|\omega\|_{L^p(\mathbb{R}^2)}^{1-\frac{2}{p}}\Big(\|\nabla a\|_{L^p(\mathbb{R}^2)} + \|(\nabla\bar\tau, \partial_{\bar\tau}\mu)\|_{L^p(\mathbb{R}^2)}\|\nabla u\|_{L^\infty(\mathbb{R}^2)} + \|\partial_{\bar\tau}\omega\|_{L^p(\mathbb{R}^2)}\Big)^{\frac{2}{p}}. \tag{3.2.15}$$

To prove (3.2.15) we write $u = \nabla^\perp\phi$ and start with the bound for the "good" direction in terms of the tangential regularity:

$$\|\partial_{\bar\tau}\nabla^2\phi\|_{L^p} + \|\nabla\partial_{\bar\tau}\nabla\phi\|_{L^p} + \|\partial_{\bar\tau}\nabla u\|_{L^p} + \|\nabla\partial_{\bar\tau}u\|_{L^p} \lesssim \|\nabla\bar\tau\|_{L^\infty}\|\nabla u\|_{L^\infty} + \|\partial_{\bar\tau}\omega\|_{L^p},$$

by use of commutator estimates. Now, with the relations (3.2.13), (3.2.14), we can derive the Lipschitz estimate for the velocity $u = \nabla^\perp\phi$ following the steps illustrated in Figure 3.1 (it is not possible to control ∇u by a in L^∞ directly):

- Step 1. It remains to control $\|\partial_n u\|_{L^\infty}$, since the L^∞-bound for the "good" direction $\partial_{\bar\tau}u$ follows from interpolating between $\|\partial_{\bar\tau}u\|_{L^p} \lesssim \|\omega\|_{L^p}$ and $\|\nabla\partial_{\bar\tau}u\|_{L^p}$, which is controlled by the righthand side of (3.2.15).
- Step 2. It remains to control $\|\alpha\|_{L^\infty}$, in view of the expression (3.2.13).
- Step 3. The control on $\|\alpha\|_{L^\infty}$ follows from $\nabla a \in L^p$ and the tangential regularity by (3.2.14).

We later take $p = 2+\epsilon$ with $\epsilon \in (0, \epsilon_0]$ (see Corollary 3.3.6 below), since we have to estimate the $L^{2+\epsilon}(\mathbb{R}^2)$-norm of $\omega, \partial_{\bar\tau}\omega$ in (3.2.15) in terms of $a, \partial_{\bar\tau}a$, respectively, where the boundedness of $\mathcal{R}_\mu^{-1}$ in $L^{2+\epsilon}(\mathbb{R}^2)$ is used.

We remark that although one can simply perform Young's inequality in (3.2.15) to get a uniform bound for $\|\nabla u\|_{L^\infty(\mathbb{R}^2)}$, we don't do so since $\|\nabla\bar\tau(t)\|_{L^p(\mathbb{R}^2)}$ grows exponentially in (the time integration of) $\|\nabla u\|_{L^\infty(\mathbb{R}^2)}$: We take the spatial derivative to the τ-equation (τ) and test it by $|\nabla\tau|^{p-2}\nabla\tau$, to derive the following bound for $\nabla\tau$

$$\|\nabla\tau\|_{L^\infty([0,t);L^p(\mathbb{R}^2))} \le \Big(\|\nabla\tau_0\|_{L^p(\mathbb{R}^2)} + \int_0^t \|\nabla\partial_\tau u\|_{L^p(\mathbb{R}^2)}dt'\Big)\exp(\|\nabla u\|_{L^1([0,t];L^\infty(\mathbb{R}^2))}). \tag{3.2.16}$$

We use the smallness assumption (3.1.27) and a bootstrap argument in Step III to close the estimate.

3.2.3. Step III. The $L^1_t\mathrm{Lip}(\mathbb{R}^2)$-estimate

After establishing the time-independent Lipschitz estimate for the velocity (3.2.15), we conclude the uniform-in-time bound for $\|\nabla u\|_{L^1_t L^\infty_x}$ by a bootstrap argument.

Recall

- the time-weighted energy estimates for u and a from Step I, which imply the estimates for $\|a\|_{L^1_t W^{1,2+\epsilon}}$ and $\|t'^{\frac{1}{2}} a\|_{L^2_t W^{1,2+\epsilon}}$ in terms of $\|\nabla u\|_{L^1_t L^\infty}$ and $\|t'^{\frac{1}{2}} \nabla u\|_{L^2_t L^\infty}$;
- the time-independent Lipschitz estimate (3.2.15) with $p = 2+\epsilon$ from Step II;
- the $L^{2+\epsilon}$-estimate for $\nabla\tau(t,\cdot)$ in (3.2.16), which depends exponentially on $\|\nabla u\|_{L^1_t L^\infty}$.

In order to close the estimate for the scaling-invariant quantity $\|\nabla u\|_{L^1_t L^\infty}$, we make use of the scaling-invariant smallness condition (3.1.27). However, since the norms $\|u_0\|_{L^2}$ and $\|u_0\|_{\dot{H}^{-1}}$, which appear both in the time-weighted estimate for $\|(t')^{(\frac{1}{2})-} u\|_{L^\infty_t L^2}$ (see (3.3.9) below), do not share the same scaling, it turns out to be more convenient to consider directly the rescaled solution (see (3.3.35)-(3.3.37) below). See Subsection 3.3.3 for more details.

Notation

Recall the notation conventions introduced at the beginning of Section 1.2 in Chapter 1. Additionally, we denote the exponential growth in the time integration of the velocity gradient by

$$V(t) := \exp\Big(C\|\nabla u\|_{L^1_t L^\infty}\Big) \quad \text{and} \quad \tilde{V}(t) := V(t)\exp\Big(C\|t'^{\frac{1}{2}}\nabla u\|_{L^2_t L^\infty}\Big). \tag{3.2.17}$$

Here and in what follows C denotes some positive constant which may depend only on μ_*, μ^*, ϵ and may vary from line to line. Lastly, we denote $\langle t\rangle = e + t$ for times $t \in [0,\infty)$.

3.3. Proofs

The goal of this section is to prove Theorem 3.1.3, Corollary 3.1.5 and Theorems 3.1.7 and 3.1.9. To this end, we first establish a priori estimates in a series of propositions in Subsections 3.3.1, 3.3.2 and 3.3.3, which correspond to the estimates stated in Steps I, II, III in Section 3.2 respectively. Theorem 3.1.3 and its Corollary 3.1.5 are proved in Section 3.3.4. The proofs of Theorems 3.1.7 and 3.1.9 are found in Section 3.3.5.

We are going to use frequently the following well-known interpolation inequalities, see e.g. [18].

Lemma 3.3.1. *If $g \in H^1(\mathbb{R}^2) \cap W^{1,r}(\mathbb{R}^2)$, $r \in (2,\infty)$, then*

$$\|g\|_{L^r(\mathbb{R}^2)} \lesssim_r \|g\|_{L^2(\mathbb{R}^2)}^{\frac{2}{r}} \|\nabla g\|_{L^2(\mathbb{R}^2)}^{1-\frac{2}{r}}, \tag{3.3.1}$$

$$\|g\|_{L^\infty(\mathbb{R}^2)} \lesssim_r \|g\|_{L^r(\mathbb{R}^2)}^{1-\frac{2}{r}} \|\nabla g\|_{L^r(\mathbb{R}^2)}^{\frac{2}{r}}. \tag{3.3.2}$$

Let us recall some classical commutator estimates for the Riesz transform.

Lemma 3.3.2. *Let* $\mathcal{R} = \frac{\frac{1}{i}\nabla}{\sqrt{-\Delta}}$ *denote the Riesz transform on* $\mathbb{R}^2$.

1. *For* $p, p_1 \in (1,\infty)$ *and* $p_2 \in [1,\infty]$ *satisfying* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, *we have the following commutator estimate*

$$\|[\mathcal{R}^2, \partial_X]g\|_{L^p} \lesssim_{p,p_1,p_2} \|\nabla X\|_{L^{p_2}} \|g\|_{L^{p_1}}, \tag{3.3.3}$$

where $g \in L^{p_1}(\mathbb{R}^2)$ *and* $X \in C_c^1(\mathbb{R}^2;\mathbb{R}^2)$.

2. *For* $p \in (2,\infty)$, *we have the following commutator estimate*

$$\|\partial_X \mathcal{R}^2 g\|_{L^p} \lesssim_p \|\partial_X g\|_{L^p} + \|\nabla X\|_{L^p}\|\mathcal{R}^2 g\|_{L^\infty}, \tag{3.3.4}$$

$$\|\partial_X \mathcal{R}^2 g - \mathcal{R}^2 \mathrm{div}\,(Xg)\|_{L^p} \lesssim_p \|\nabla X\|_{L^p}\|\mathcal{R}^2 g\|_{L^\infty}, \tag{3.3.5}$$

for any $g \in C_c^1(\mathbb{R}^2)$ *and* $X \in C_c^1(\mathbb{R}^2;\mathbb{R}^2)$. *Furthermore, for* $\mu \in L^\infty(\mathbb{R}^2)$ *with* $\|\mu\|_{L^\infty} \le \mu^*$ *and* $\partial_X \mu \in L^q(\mathbb{R}^2)$, $q \in [p,\infty]$, *we have*

$$\|[\mathcal{R}_\mu, \partial_X]g\|_{L^p} \lesssim_{p,\mu^*} \|(\nabla X, \partial_X \mu)\|_{L^p}\|(\mathcal{R}^2 g, \mathcal{R}_\mu g)\|_{L^\infty}, \ \text{if } q = p, \tag{3.3.6}$$

$$\|[\mathcal{R}_\mu, \partial_X]g\|_{L^p} \lesssim_{p,q,\mu^*} \|\nabla X\|_{L^p}\|(\mathcal{R}^2 g, \mathcal{R}_\mu g)\|_{L^\infty} + \|\partial_X \mu\|_{L^q}\|g\|_{L^{\frac{qp}{q-p}}}, \ \text{if } q \in (p,\infty], \tag{3.3.7}$$

where $\mathcal{R}_\mu = (\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\mu(\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1) + (2\mathcal{R}_1\mathcal{R}_2)\mu(2\mathcal{R}_1\mathcal{R}_2)$ *is defined in* (3.1.16).

The proof of the first estimate (3.3.3) can be found in A. P. Calderón's article [33, Theorem 1]. The proof of the second statement is very much in the spirit of [198, Lemma 5.1] and [55, Lemma 2.10], and is postponed in Appendix 3.B.

3.3.1. Step I. Energy estimates

We start with some basic energy estimates for (μINS). These have already been established for the density-dependent Navier-Stokes equations (INS) in e.g. [11, 242]; see also [10] for the three-dimensional case. Using the same ideas we prove the following estimates for our system (μINS) in Appendix 3.C, with particular emphasis on their explicit dependence on the initial data.

Proposition 3.3.3 (Energy estimates for u). *Let* (μ, u) *be a sufficiently smooth and decaying (at infinity) solution of (μINS) on some time interval* $[0, T^*)$ *with* $\mu_0 - 1 \in L^2(\mathbb{R}^2)$ *and* $u_0 \in L^2(\mathbb{R}^2) \cap \dot{H}^{-2\delta}(\mathbb{R}^2)$ *for some* $\delta \in (0, \frac{1}{2})$. *Then the following energy estimates hold for* $t \in [0, T^*)$:

$$\|u\|_{L_t^\infty L^2} + \|\nabla u\|_{L_t^2 L^2} \le C(\mu_*)\|u_0\|_{L^2}, \tag{3.3.8}$$

$$\begin{aligned}\|\langle t\rangle^{\delta_-} u\|_{L^2} + \|\langle t'\rangle^{\delta_-}\nabla u\|_{L_t^2 L^2} \le{}& C(\mu_*, \delta, \delta - \delta_-)(\|u_0\|_{L^2 \cap \dot{H}^{-2\delta}} + \|\mu_0 - 1\|_{L^2}\|u_0\|_{L^2}) \\ &\times e^{C(\|u_0\|^2_{L^2\cap\dot{H}^{-2\delta}} + \|\mu_0-1\|^2_{L^2}\|u_0\|^2_{L^2} + \|u_0\|^4_{L^2})},\end{aligned} \tag{3.3.9}$$

where $\delta_- > 0$ *stands for any positive number strictly smaller than* δ.

We now turn to establishing energy estimates for the *global good unknown* a. Recall the vorticity reformulation (3.1.21) of the velocity equation $(\mu\mathrm{INS})_2$ as

$$\begin{cases} \partial_t\omega + u\cdot\nabla\omega - \Delta a = 0, \quad (t,x)\in(0,\infty)\times\mathbb{R}^2, \\ u = \nabla^\perp\Delta^{-1}\omega, \quad a = \mathcal{R}_\mu\omega. \end{cases} \tag{3.3.10}$$

If μ is smooth, then the vorticity equation (3.3.10) is parabolic. However, for more general (discontinuous) viscosities, it is not clear whether the equation has a parabolic character. This is largely because of the non-local operator $\mathcal{R}_\mu$, which itself is composed of local and non-local operators. Nevertheless, we have the following (time-weighted) energy estimates for the vorticity equation (3.3.10).

Proposition 3.3.4 (H^1-energy estimates for a). *Let (μ,u) be a sufficiently regular solution of $(\mu\mathrm{INS})$ on some time interval $[0,T^*)$, where $\mu\in L^\infty([0,\infty)\times\mathbb{R}^2;[\mu_*,\mu^*])$ with $0<\mu_*\le\mu^*$. Then for all times $t\in[0,T^*)$,*

$$\|a\|^2_{L^\infty_t L^2} + \|\nabla a\|^2_{L^2_t L^2} \le C(\mu_*,\mu^*)\|\omega_0\|^2_{L^2}V(t), \tag{3.3.11}$$

$$\|t'^{\frac12}a\|^2_{L^\infty_t L^2} + \|t'^{\frac12}\nabla a\|^2_{L^2_t L^2} \le C(\mu_*,\mu^*)\|u_0\|^2_{L^2}V(t), \tag{3.3.12}$$

$$\|t'^{\frac12}\nabla a\|^2_{L^\infty_t L^2} + \|t'^{\frac12}\Delta a\|^2_{L^2_t L^2} \le C(\mu_*,\mu^*)(\|\nabla a\|^2_{L^2_t L^2} + \|t'^{\frac12}\nabla u\|^2_{L^2_t L^\infty}\|a\|^2_{L^\infty_t L^2})V(t), \tag{3.3.13}$$

where $V(t)=\exp(\int_0^t C\|\nabla u\|_{L^\infty}dt')$ denotes the exponential growth in the time integration of the velocity gradient. Moreover, if we additionally assume the hypotheses of Proposition 3.3.3, then

$$\|t'^{\frac12+\delta_-}a\|^2_{L^\infty_t L^2} + \|t'^{\frac12+\delta_-}\nabla a\|^2_{L^2_t L^2} \le C(\mu_*,\mu^*)\|\langle t'\rangle^{\delta_-}\nabla u\|^2_{L^2_t L^2}\,V(t). \tag{3.3.14}$$

Proof. It is convenient to use the vorticity equation (3.3.10) instead of the velocity equation $(\mu\mathrm{INS})_2$.

- **Proof of (3.3.11):** We rewrite the ω-equation (3.3.10) as

$$\dot\omega - \Delta a = 0, \ \text{ with } \dot\omega := \frac{D}{Dt}\omega = (\partial_t + u\cdot\nabla)\omega, \tag{3.3.15}$$

and take the $L^2(\mathbb{R}^2)$-inner product between the above equation and a to obtain

$$\int_{\mathbb{R}^2}\dot\omega\mathcal{R}_\mu\omega dx + \int_{\mathbb{R}^2}|\nabla a|^2dx = 0.$$

The self-adjointness of the double Riesz transform yields (recalling the transport equation $\frac{D}{Dt}\mu=\dot\mu=0$ and the divergence free condition $\operatorname{div}u=0$ in $(\mu\mathrm{INS})$)

$$\begin{aligned}
&\int_{\mathbb{R}^2}\dot\omega\mathcal{R}_\mu\omega dx\\
&= \int_{\mathbb{R}^2}\Big((\mathcal{R}_2\mathcal{R}_2-\mathcal{R}_1\mathcal{R}_1)\dot\omega\cdot\mu(\mathcal{R}_2\mathcal{R}_2-\mathcal{R}_1\mathcal{R}_1)\omega + (2\mathcal{R}_1\mathcal{R}_2)\dot\omega\cdot\mu(2\mathcal{R}_1\mathcal{R}_2)\omega\Big)dx\\
&= \frac12\frac{d}{dt}\int_{\mathbb{R}^2}\mu\Big(((\mathcal{R}_2\mathcal{R}_2-\mathcal{R}_1\mathcal{R}_1)\omega)^2 + ((2R_1R_2)\omega)^2\Big)dx\\
&\quad + \int_{\mathbb{R}^2}\mu\Big([(\mathcal{R}_2\mathcal{R}_2-\mathcal{R}_1\mathcal{R}_1),u\cdot\nabla]\omega\cdot(\mathcal{R}_2\mathcal{R}_2-\mathcal{R}_1\mathcal{R}_1)\omega\\
&\qquad\qquad + [(2\mathcal{R}_1\mathcal{R}_2),u\cdot\nabla]\omega\cdot(2\mathcal{R}_1\mathcal{R}_2)\omega\Big)dx.
\end{aligned}$$

Thus,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\mu\Big(((\mathcal{R}_2\mathcal{R}_2-\mathcal{R}_1\mathcal{R}_1)\omega)^2+((2\mathcal{R}_1\mathcal{R}_2)\omega)^2\Big)dx+\int_{\mathbb{R}^2}|\nabla a|^2dx\\
&=-\int_{\mathbb{R}^2}\mu\Big([\mathcal{R}_2\mathcal{R}_2-\mathcal{R}_1\mathcal{R}_1,u\cdot\nabla]\omega\,\cdot\,(\mathcal{R}_2\mathcal{R}_2-\mathcal{R}_1\mathcal{R}_1)\omega\\
&\qquad\qquad+[2\mathcal{R}_1\mathcal{R}_2,u\cdot\nabla]\omega\,\cdot\,(2\mathcal{R}_1\mathcal{R}_2)\omega\Big)dx.
\end{aligned}
\tag{3.3.16}
$$

Recall (the proof of) (3.1.24) for the first integral on the left hand side

$$
\int_{\mathbb{R}^2}\mu\Big(((\mathcal{R}_2\mathcal{R}_2-\mathcal{R}_1\mathcal{R}_1)\omega)^2+((2\mathcal{R}_1\mathcal{R}_2)\omega)^2\Big)dx=\langle a,\omega\rangle_{L^2(\mathbb{R}^2)}\geq\mu_*\|\omega\|_{L^2}^2. \tag{3.3.17}
$$

The integral on the righthand side of (3.3.16) can be bounded with the help of the commutator estimate from Lemma 3.3.2 by $C\mu^*\|\nabla u\|_{L^\infty}\|\omega\|_{L^2}^2$, and thus integrating the result over $[0,t]$ yields

$$
\frac{\mu_*}{2}\|\omega(t)\|_{L^2}^2+\int_0^t\|\nabla a\|_{L^2}^2dt'\leq\frac{\mu^*}{2}\|\omega_0\|_{L^2}^2+C\mu^*\int_0^t\|\nabla u\|_{L^\infty}\|\omega\|_{L^2}^2dt'.
$$

An application of Grönwall's inequality and the bound (3.1.23) imply the estimate (3.3.11).

- **Proof of (3.3.12):** We multiply (3.3.16) by t to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(t\int_{\mathbb{R}^2}\mu\Big(((\mathcal{R}_2\mathcal{R}_2-\mathcal{R}_1\mathcal{R}_1)\omega)^2+((2\mathcal{R}_1\mathcal{R}_2)\omega)^2\Big)dx\Big)+t\int_{\mathbb{R}^2}|\nabla a|^2dx\\
&=\frac{1}{2}\int_{\mathbb{R}^2}\mu\Big(((\mathcal{R}_2\mathcal{R}_2-\mathcal{R}_1\mathcal{R}_1)\omega)^2+((2\mathcal{R}_1\mathcal{R}_2)\omega)^2\Big)dx\\
&\quad-t\int_{\mathbb{R}^2}\mu\Big([\mathcal{R}_2\mathcal{R}_2-\mathcal{R}_1\mathcal{R}_1,u\cdot\nabla]\omega\,\cdot\,(\mathcal{R}_2\mathcal{R}_2-\mathcal{R}_1\mathcal{R}_1)\omega\\
&\qquad\qquad+[2\mathcal{R}_1\mathcal{R}_2,u\cdot\nabla]\omega\,\cdot\,(2\mathcal{R}_1\mathcal{R}_2)\omega\Big)dx,
\end{aligned}
$$

where integration over $[0,t]$ together with the commutator estimate (3.3.3) implies

$$
\frac{\mu_*}{2}t\|\omega\|_{L^2}^2+\int_0^t t'\|\nabla a\|_{L^2}^2dt'\lesssim_{\mu^*}\int_0^t\|\omega\|_{L^2}^2dt'+\int_0^t\|\nabla u\|_{L^\infty}\|t'^{\frac{1}{2}}\omega\|_{L^2}^2dt'.
$$

Thus, (3.3.12) follows from Grönwall's inequality, (3.1.23) and (3.3.8).

- **Proof of (3.3.13):** For the higher order estimates we apply $\mathcal{R}_\mu$ to the vorticity equation (3.3.15) to get

$$
\mathcal{R}_\mu\dot{\omega}-\mathcal{R}_\mu\Delta a=0,
$$

and take the L^2 inner product with $\dot{\omega}$ to derive

$$
\int_{\mathbb{R}^2}\mathcal{R}_\mu\dot{\omega}\,\dot{\omega}dx-\int_{\mathbb{R}^2}\mathcal{R}_\mu\Delta a\,\dot{\omega}dx=0. \tag{3.3.18}
$$

We use integration by parts to calculate

$$
\begin{aligned}
\int_{\mathbb{R}^2}\mathcal{R}_\mu\dot{\omega}\,\dot{\omega}dx&=\int_{\mathbb{R}^2}\mu\Big(((\mathcal{R}_2\mathcal{R}_2-\mathcal{R}_1\mathcal{R}_1)\dot{\omega})^2+((2\mathcal{R}_1\mathcal{R}_2)\dot{\omega})^2\Big)dx,\\
-\int_{\mathbb{R}^2}\mathcal{R}_\mu\Delta a\,\dot{\omega}dx&=-\int_{\mathbb{R}^2}(\Delta a)(\frac{D}{Dt}\mathcal{R}_\mu\omega)dx-\int_{\mathbb{R}^2}(\Delta a)\,[\mathcal{R}_\mu,\frac{D}{Dt}]\omega dx
\end{aligned}
$$

$$=: I_1 + I_2.$$

As $\mathcal{R}_\mu \omega = a$ and $[\nabla, \frac{D}{Dt}] = [\nabla, u \cdot \nabla]$, integration by parts gives

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^2} \nabla a \cdot \frac{D}{Dt} \nabla a dx + \int_{\mathbb{R}^2} \nabla a \cdot [\nabla, u \cdot \nabla] a dx \\ &= \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^2} |\nabla a|^2 dx + \int_{\mathbb{R}^2} \nabla a \cdot \nabla u \cdot \nabla a dx. \end{aligned}$$

Furthermore, since $\frac{D}{Dt}\mu = 0$, the commutator in the second integral I_2 reads

$$\begin{aligned} [\mathcal{R}_\mu, \frac{D}{Dt}] &= (\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\mu[\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1, u \cdot \nabla] + (2\mathcal{R}_1\mathcal{R}_2)\mu[2\mathcal{R}_1\mathcal{R}_2, u \cdot \nabla] \\ &\quad + [\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1, u \cdot \nabla]\mu(\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1) + [2\mathcal{R}_1\mathcal{R}_2, u \cdot \nabla]\mu(2\mathcal{R}_1\mathcal{R}_2). \end{aligned}$$

All of these identities we apply to (3.3.18) and use (3.3.17) with ω replaced by $\dot{\omega} = \Delta a$, to arrive at

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla a\|_{L^2}^2 + \mu_*\|\Delta a\|_{L^2}^2 \le -\int_{\mathbb{R}^2} \nabla a \cdot \nabla u \cdot \nabla a dx \\ &- \int_{\mathbb{R}^2} (\Delta a)\Big((\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\big(\mu[\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1, u \cdot \nabla]\omega\big) + (2\mathcal{R}_1\mathcal{R}_2)\big(\mu[2\mathcal{R}_1\mathcal{R}_2, u \cdot \nabla]\omega\big) \\ &\quad + [\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1, u \cdot \nabla]\big(\mu(\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\omega\big) + [2R_1R_2, u \cdot \nabla]\big(\mu(2\mathcal{R}_1\mathcal{R}_2)\omega\big)\Big) dx. \end{aligned} \tag{3.3.19}$$

The last two integrals on the right hand side are bounded by $C\mu^*\|\Delta a\|_{L^2}\|\nabla u\|_{L^\infty}\|\omega\|_{L^2}$ due to the commutator estimate (3.3.3), and the first integral satisfies

$$\Big| - \int_{\mathbb{R}^2} \nabla a \cdot \nabla u \cdot \nabla a dx \Big| \le \|\nabla u\|_{L^\infty}\|\nabla a\|_{L^2}^2.$$

Consequently,

$$\frac{1}{2}\frac{d}{dt}\|\nabla a\|_{L^2}^2 + \mu_*\|\Delta a\|_{L^2}^2 \lesssim \mu^*\|\Delta a\|_{L^2}\|\nabla u\|_{L^\infty}\|\omega\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla a\|_{L^2}^2. \tag{3.3.20}$$

We multiply (3.3.20) by t to obtain

$$\frac{1}{2}\frac{d}{dt}(t\|\nabla a\|_{L^2}^2) + \mu_* t\|\Delta a\|_{L^2}^2 \lesssim \|\nabla a\|_{L^2}^2 + \mu^* t\|\Delta a\|_{L^2}\|\nabla u\|_{L^\infty}\|\omega\|_{L^2} + \|\nabla u\|_{L^\infty} t\|\nabla a\|_{L^2}^2.$$

This implies

$$\frac{1}{2}\frac{d}{dt}(t\|\nabla a\|_{L^2}^2) + \frac{\mu_*}{2} t\|\Delta a\|_{L^2}^2 \lesssim \|\nabla a\|_{L^2}^2 + \frac{(\mu^*)^2}{\mu_*} t\|\nabla u\|_{L^\infty}^2\|\omega\|_{L^2}^2 + \|\nabla u\|_{L^\infty} t\|\nabla a\|_{L^2}^2.$$

so that (3.3.13) follows again by Grönwall's inequality and (3.3.8).

- **Proof of (3.3.14):** We multiply (3.3.16) by $\langle t\rangle^{1+2\delta_-}$ to obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(t^{1+2\delta_-}\int_{\mathbb{R}^2} \mu\Big(((\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\omega)^2 + (2\mathcal{R}_1\mathcal{R}_2\omega)^2\Big)dx\Big) + t^{1+2\delta_-}\|\nabla a\|_{L^2}^2 \\ &\qquad \lesssim \mu^* t^{2\delta_-}\|\omega\|_{L^2}^2 + \mu^* t^{1+2\delta_-}\|\nabla u\|_{L^\infty}\|\omega\|_{L^2}^2, \end{aligned}$$

where integration over $[0, t]$ yields

$$\frac{t^{1+2\delta_-}}{\mu^*}\|\omega\|_{L^2}^2 + \|t'^{\frac{1}{2}+\delta_-}\nabla a\|_{L^2_tL^2}^2 \lesssim \mu^*\|t'^{\delta_-}\omega\|_{L^2_tL^2}^2 + \mu^*\int_0^t t'^{1+2\delta_-}\|\omega\|_{L^2}^2\|\nabla u\|_{L^\infty}dt'.$$

Then (3.3.14) follows from Grönwall's inequality.

□

3.3.2. Step II. The time-independent Lipschitz estimate

In this subsection we establish the time-independent Lipschitz estimate for the fluid velocity. To do so, we follow the steps demonstrated in Figure 3.1. Throughout this subsection time evolution is neglected, so that all quantities only depend on the spacial variable $x \in \mathbb{R}^2$.

Proposition 3.3.5 (Time-independent Lipschitz estimate). *Let $a \in L^2 \cap W^{1,p}(\mathbb{R}^2)$, $p \in (2,\infty)$ and $\mu \in L^\infty(\mathbb{R}^2; [\mu_*, \mu^*])$, $0 < \mu_* \le \mu^*$. Assume further that $\partial_\tau \mu \in L^p(\mathbb{R}^2)$, where $\tau \in L^\infty \cap \dot{W}^{1,p}(\mathbb{R}^2; \mathbb{R}^2)$ is a non-degenerate vector field. Let $\phi \in H^2(\mathbb{R}^2)$ be the unique solution of (3.2.2) on $\mathbb{R}^2$. Then $\|\nabla^2\phi\|_{L^\infty}$ can be bounded in terms of $\omega := \Delta\phi$ and $\bar{\tau} := \frac{\tau}{|\tau|}$ as follows*

$$\|\nabla^2\phi\|_{L^\infty} \lesssim \|\omega\|_{L^p}^{1-\frac{2}{p}} \Big(\|\nabla a\|_{L^p} + \|(\nabla\bar{\tau}, \partial_{\bar{\tau}}\mu)\|_{L^p}\|\nabla^2\phi\|_{L^\infty} + \|\partial_{\bar{\tau}}\omega\|_{L^p}\Big)^{\frac{2}{p}}. \tag{3.3.21}$$

In the above, the term $\|(\nabla\bar{\tau}, \partial_{\bar{\tau}}\mu)\|_{L^p}\|\nabla^2\phi\|_{L^\infty}$ can be replaced by $\|(\nabla\bar{\tau}, \partial_{\bar{\tau}}\mu)\|_{L^{p_1}}\|\omega\|_{L^{p_2}}$ for $p_1, p_2 \in (p,\infty)$, $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$.

Proof. Our goal is to control $\|\nabla^2\phi\|_{L^\infty}$ by the right hand side of (3.3.21)

$$I := \|\omega\|_{L^p}^{1-\frac{2}{p}} \Big(\|\nabla a\|_{L^p} + \|(\nabla\bar{\tau}, \partial_{\bar{\tau}}\mu)\|_{L^p}\|\nabla^2\phi\|_{L^\infty} + \|\partial_{\bar{\tau}}\omega\|_{L^p}\Big)^{\frac{2}{p}}. \tag{3.3.22}$$

Preliminary Estimate in the tangential direction for $\|\partial_{\bar{\tau}}\nabla^2\phi\|_{L^p}$. We first apply (3.3.4) with $X = \bar{\tau}$ and $f = \omega$ to derive the following tangential regularity (noticing $\nabla^2\phi = \mathcal{R}^2\omega$)

$$\|\partial_{\bar{\tau}}\nabla^2\phi\|_{L^p} \lesssim \|\nabla\bar{\tau}\|_{L^p}\|\nabla^2\phi\|_{L^\infty} + \|\partial_{\bar{\tau}}\omega\|_{L^p}. \tag{3.3.23}$$

Step 1. Reduction to $\|\partial_n\nabla\phi\|_{L^\infty}$. Using formula (1a) from Lemma 3.2.1 we write

$$\|\nabla^2\phi\|_{L^\infty} \le \|\bar{\tau} \otimes \partial_{\bar{\tau}}\nabla\phi\|_{L^\infty} + \|n \otimes \partial_n\nabla\phi\|_{L^\infty} \le \|\partial_{\bar{\tau}}\nabla\phi\|_{L^\infty} + \|\partial_n\nabla\phi\|_{L^\infty}. \tag{3.3.24}$$

We can use the interpolation inequality (3.3.2) and the above estimate (3.3.23) to control the tangential derivative $\|\partial_{\bar{\tau}}\nabla\phi\|_{L^\infty}$ by I:

$$\|\partial_{\bar{\tau}}\nabla\phi\|_{L^\infty} \lesssim \|\partial_{\bar{\tau}}\nabla\phi\|_{L^p}^{1-\frac{2}{p}}\|\nabla(\partial_{\bar{\tau}}\nabla\phi)\|_{L^p}^{\frac{2}{p}} \lesssim \|\omega\|_{L^p}^{1-\frac{2}{p}}\Big(\|\nabla\bar{\tau}\|_{L^p}\|\nabla^2\phi\|_{L^\infty} + \|\partial_{\bar{\tau}}\omega\|_{L^p}\Big)^{\frac{2}{p}}, \tag{3.3.25}$$

where in the second inequality we used also the definition $\partial_{\bar{\tau}} = \bar{\tau} \cdot \nabla$ and L^p-boundedness of Riesz operator. It remains to control $\|\partial_n\nabla\phi\|_{L^\infty}$ by I.

Step 2. Reduction to $\|\alpha\|_{L^\infty}$. We consider the normal derivative of $\nabla\phi$. Recall the reformulation (3.2.13) in Lemma 3.2.1 such that

$$\partial_n\nabla\phi = \frac{\alpha}{\mu}n + 2(\bar{\tau} \cdot \partial_{\bar{\tau}}\nabla\phi)n + \partial_{\bar{\tau}}\nabla^\perp\phi. \tag{3.3.26}$$

The last two terms on the right hand side are related to tangential derivatives and can be bounded by I by Step 1. Since the first term satisfies $\|n\frac{\alpha}{\mu}\|_{L^\infty} \le \frac{1}{\mu_*}\|\alpha\|_{L^\infty}$, it remains to control $\|\alpha\|_{L^\infty}$ by I.

Step 3. Estimate for $\|\alpha\|_{W^{1,p}}$ and conclusion. Recall the definition (3.2.10) of α:

$$\begin{aligned} \alpha &= -(\bar{\tau}_2^2 - \bar{\tau}_1^2)\mu(\partial_{22} - \partial_{11})\phi - 2\bar{\tau}_1\bar{\tau}_2\mu(2\partial_{12}\phi) \\ &= -(\bar{\tau}_2^2 - \bar{\tau}_1^2)\mu(\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\omega - 2\bar{\tau}_1\bar{\tau}_2\mu(2\mathcal{R}_1\mathcal{R}_2\omega). \end{aligned} \tag{3.3.27}$$

We derive from the L^p-boundedness of the Riesz transform that

$$\|\alpha\|_{L^p} \leq C(p, \mu^*)\|\omega\|_{L^p}. \tag{3.3.28}$$

Applying $\partial_{\bar{\tau}}$ to (3.3.27) and recalling (3.3.23) we derive

$$\begin{aligned}\|\partial_{\bar{\tau}}\alpha\|_{L^p} &\lesssim_{\mu^*} (\|\nabla\bar{\tau}\|_{L^p} + \|\partial_{\bar{\tau}}\mu\|_{L^p})\|\nabla^2\phi\|_{L^\infty} + \|\partial_{\bar{\tau}}\nabla^2\phi\|_{L^p}\\ &\lesssim_{\mu^*} (\|\nabla\bar{\tau}\|_{L^p} + \|\partial_{\bar{\tau}}\mu\|_{L^p})\|\nabla^2\phi\|_{L^\infty} + \|\partial_{\bar{\tau}}\omega\|_{L^p}.\end{aligned}$$

Now we bound $\|\nabla\alpha\|_{L^p}$ by use of the relation between a and α in (3.2.14), the L^p-boundedness of the Riesz transform and (3.3.23), as follows

$$\begin{aligned}\|\nabla\alpha\|_{L^p} &\lesssim_{\mu^*} \|\nabla a\|_{L^p} + \|\partial_{\bar{\tau}}\alpha\|_{L^p} + \|\partial_{\bar{\tau}}(\mu\nabla^2\phi)\|_{L^p} + \|\nabla\bar{\tau}\|_{L^p}\|\nabla^2\phi\|_{L^\infty}\\ &\lesssim_{\mu^*} \|\nabla a\|_{L^p} + (\|\nabla\bar{\tau}\|_{L^p} + \|\partial_{\bar{\tau}}\mu\|_{L^p})\|\nabla^2\phi\|_{L^\infty} + \|\partial_{\bar{\tau}}\omega\|_{L^p}.\end{aligned} \tag{3.3.29}$$

Consequently, by use of the interpolation inequality

$$\|\alpha\|_{L^\infty} \lesssim \|\alpha\|_{L^p}^{1-\frac{2}{p}}\|\nabla\alpha\|_{L^p}^{\frac{2}{p}}$$

and the estimate (3.3.28) we achieve $\|\alpha\|_{L^\infty} \lesssim I$. Hence, $\|\partial_n\nabla\phi\|_{L^\infty}$ and $\|\nabla^2\phi\|_{L^\infty}$ are both controlled by I by Step 1 and Step 2. In particular, this proves the desired estimate (3.3.21). □

Corollary 3.3.6. *Under the hypotheses of Proposition 3.3.5, let $\epsilon_0 = \epsilon_0(\mu_*, \mu^*)$ be given in Lemma 3.1.1 - 2. Then for any $\epsilon \in (0, \epsilon_0]$, we have for $u := \nabla^\perp\phi$*

$$\|\nabla u\|_{L^\infty} \leq C(\mu_*, \mu^*, \epsilon)\|a\|_{L^{2+\epsilon}}^{\frac{\epsilon}{2+\epsilon}}\Big(\|\nabla a\|_{L^{2+\epsilon}} + \|(\nabla\bar{\tau}, \partial_{\bar{\tau}}\mu)\|_{L^{2+\epsilon}}\|(\nabla u, a)\|_{L^\infty}\Big)^{\frac{2}{2+\epsilon}}. \tag{3.3.30}$$

Proof. By definition of a and Lemma 3.1.1 - 2 we derive that

$$\|\omega\|_{L^{2+\epsilon}} = \|\mathcal{R}_\mu^{-1}\mathcal{R}_\mu\omega\|_{L^{2+\epsilon}} \lesssim_{\mu_*,\mu^*} \|a\|_{L^{2+\epsilon}}.$$

Now we rewrite

$$\partial_{\bar{\tau}}\omega = \mathcal{R}_\mu^{-1}\mathcal{R}_\mu\partial_{\bar{\tau}}\omega = \mathcal{R}_\mu^{-1}(\partial_{\bar{\tau}}a + [\mathcal{R}_\mu, \partial_{\bar{\tau}}]\omega).$$

By virtue of the commutator estimate (3.3.6) and again Lemma 3.1.1 - 2, we arrive at

$$\|\partial_{\bar{\tau}}\omega\|_{L^{2+\epsilon}} \lesssim_{\mu_*,\mu^*,\epsilon} \|\nabla a\|_{L^{2+\epsilon}} + (\|\nabla\bar{\tau}\|_{L^{2+\epsilon}} + \|\partial_{\bar{\tau}}\mu\|_{L^{2+\epsilon}})(\|\nabla u\|_{L^\infty} + \|a\|_{L^\infty}). \tag{3.3.31}$$

Choosing $p = 2+\epsilon$ in (3.3.21) and inserting the above estimates for $\|\omega\|_{L^{2+\epsilon}}$ and $\|\partial_{\bar{\tau}}\omega\|_{L^{2+\epsilon}}$ we arrive at (3.3.30). □

Remark 3.3.7 (Time-independent estimates of ∇u revisited). (i) In complex coordinates in $\mathbb{R}^2$:

$$z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2, \quad x_1 = \frac{1}{2}(z+\bar{z}), \quad x_2 = \frac{1}{2i}(z-\bar{z}),$$

we can express a, b in terms of μ, ω as follows (noticing $\partial_1 = (\partial_z + \partial_{\bar{z}})$, $\partial_2 = \frac{1}{i}(\partial_z - \partial_{\bar{z}})$, $\Delta = 4\partial_{z\bar{z}}$, $\frac{\partial_z}{\partial_{\bar{z}}} = \frac{4\partial_{zz}^2}{\Delta}$, $\frac{\partial_{\bar{z}}}{\partial_z} = \frac{4\partial_{\bar{z}\bar{z}}^2}{\Delta}$)

$$a = \mathcal{R}_\mu\omega = \Re[\frac{\partial_z}{\partial_{\bar{z}}}\mu\frac{\partial_{\bar{z}}}{\partial_z}\omega], \quad b = \mathcal{Q}_\mu\omega = \mathrm{Im}[\frac{\partial_z}{\partial_{\bar{z}}}\mu\frac{\partial_{\bar{z}}}{\partial_z}\omega].$$

Thus ω can be respresented in terms of a, b, μ as

$$\omega = \frac{\partial_z}{\partial_{\bar z}} \frac{1}{\mu} \frac{\partial_{\bar z}}{\partial_z}(a+ib).$$

If $\mu \in [\mu_*, \mu^*]$, then we can control a in terms of ω by use of the boundedness of the Riesz transform:

$$\|a\|_{L^p(\mathbb{R}^2)} \le C(p,\mu^*)\|\omega\|_{L^p(\mathbb{R}^2)}, \ \forall p \in (1,\infty). \tag{3.3.32}$$

We have already seen in Lemma 3.1.1 - 2 that the reverse estimate holds for $p = 2+\epsilon$, for $\epsilon \in [0, \epsilon_0]$ close to 2.

As is shown in [123, Corollary 1.9, Theorem 1.11] that $\nabla b, \nabla\omega \notin L^1_{\mathrm{loc}}$ for the stationary case with piecewise-constant viscosity coefficient, we don't have energy estimates for $\nabla b, \nabla \omega$ in the presence of jumping viscosity coefficient.

(ii) In [41, 42] J.-Y. Chemin established the celebrated (time-independent) Lipschitz estimate for the velocity field with a logarithm growth in the tangential regularity of ω with respect to the vector field τ:

$$\begin{aligned}\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\omega\|_{L^p(\mathbb{R}^2)} + \|\omega\|_{L^\infty(\mathbb{R}^2)} \times \\ \times \log\Big(e + \|\frac{1}{|\tau|}\|_{L^\infty(\mathbb{R}^2)} \frac{\|\omega\|_{L^\infty(\mathbb{R}^2)}\|\tau\|_{\mathcal{C}^\alpha(\mathbb{R}^2)} + \|\mathrm{div}\,(\tau\omega)\|_{\mathcal{C}^{\alpha-1}(\mathbb{R}^2)}}{\|\omega\|_{L^\infty(\mathbb{R}^2)}}\Big)\end{aligned}$$

for $p \in [1,\infty)$. This estimate comes essentially from the analysis of the elliptic equation $\Delta\phi = (-\partial^*_{\overline{\tau}}\partial_{\overline{\tau}} - \partial^*_n\partial_n)\phi = \omega$. When taking time into account, the logarithmic growth in the τ-norms, which grows exponentially in $\|\nabla u\|_{L^1_t L^\infty}$ as in (3.2.16), implies finally the linear growth in $\int_0^t \|\nabla u\|_{L^\infty}$ on the right hand side. An application of Grönwall's inequality yields the boundedness of $\|\nabla u\|_{L^\infty}$ on any bounded time interval. This is key in showing the regularity propagation for the vortex patch problem of incompressible Euler equations.

Our estimate (3.3.30) is essentially of interpolation type, and we do not have an a priori L^∞-estimate for ω. When taking into account of time, we can not avoid the exponential growth in $\|\nabla u\|_{L^1_t L^\infty}$ on the right hand side. That is why we need some smallness condition as (3.1.27).

3.3.3. STEP III. THE $L^1_t\mathrm{Lip}(\mathbb{R}^2)$-ESTIMATE

In this subsection we combine the results from the previous sections to deduce the $L^1 Lip$-estimate for the velocity vector field.

Proposition 3.3.8 ($L^1_t\mathrm{Lip}(\mathbb{R}^2)$-estimate). *Let (μ, u, τ) be a sufficiently smooth solution of $(\mu\mathrm{INS})$-(τ) on some time interval $[0,T^*)$, $T^*>0$. Then, under the assumptions of Theorem 3.1.3 there exists a constant $C>0$ depending only on μ_*, μ^*, ϵ such that for $t \in (0,T^*)$,*

$$\begin{aligned}\|\nabla u\|_{L^1_t L^\infty} + \|t'^{\frac12}\nabla u\|_{L^2_t L^\infty} \le & C\Big(\|u_0\|^{\frac{\epsilon}{2}}_{L^2}(\|u_0\|_{\dot H^{-1}} + \|\mu_0 - 1\|_{L^2}\|u_0\|_{L^2}) \\ & \times (\|\nabla u_0\|_{L^2} + \|(\nabla\bar\tau_0, \partial_{\bar\tau_0}\mu_0)\|^{\frac{2+\epsilon}{\epsilon}}_{L^{2+\epsilon}})\Big)^{\frac{2\epsilon}{(2+\epsilon)^2}} e^{C\|u_0\|^4_{L^2}}. \end{aligned} \tag{3.3.33}$$

Proof. Let $t \in (0,T^*)$ be arbitrary but fixed. The goal is to prove that the $L^1_t Lip$-norm of u can be controlled independently of t.

Step 1: Scaling consideration. For notational simplicity, we introduce

$$
\begin{aligned}
\sigma_{-1} &:= \sigma_{-1}(\mu_0, u_0) = \|u_0\|_{\dot{H}^{-1}(\mathbb{R}^2)} + \|\mu_0 - 1\|_{L^2(\mathbb{R}^2)}\|u_0\|_{L^2(\mathbb{R}^2)},\\
\sigma_0 &:= \sigma_0(u_0) = \|u_0\|_{L^2(\mathbb{R}^2)},\\
\sigma_1 &:= \sigma_1(\mu_0, u_0, \bar{\tau}_0) = \|u_0\|_{\dot{H}^1(\mathbb{R}^2)} + \|(\partial_{\bar{\tau}_0}\mu_0, \nabla\bar{\tau}_0)\|_{L^{2+\epsilon}(\mathbb{R}^2)}^{\frac{2+\epsilon}{\epsilon}},\\
\tilde{V}(t) &:= \tilde{V}(u(t)) = \exp(C(\|\nabla_x u(t', x)\|_{L^1_t L^\infty} + \|t'^{\frac{1}{2}}\nabla_x u(t', x)\|_{L^2_t L^\infty})),
\end{aligned}
\tag{3.3.34}
$$

where C is a big enough constant depending only on μ_*, μ^*, ϵ. We assume without loss of generality $\sigma_j > 0$, $j = -1, 0, 1$.

For $\lambda > 0$ we define the rescaled initial data

$$
\begin{aligned}
\mu_{0,\lambda}(x) &:= \mu_0(\lambda^{-1}x), \quad u_{0,\lambda}(x) := \lambda^{-1}u_0(\lambda^{-1}x),\\
\tau_{0,\lambda}(x) &:= \lambda^{-1}\tau_0(\lambda^{-1}x), \quad \bar{\tau}_{0,\lambda}(x) := \frac{\tau_{0,\lambda}}{|\tau_{0,\lambda}|}(x).
\end{aligned}
$$

It is straightforward to verify that (μ, u, π, τ) is a solution of (μINS)-(τ) with initial data (μ_0, u_0, τ_0) on some time interval $[0, T^*)$, if and only if the rescaled triplet

$$
(\mu_\lambda, u_\lambda, \pi_\lambda, \tau_\lambda)(t, x) := (\mu, \lambda^{-1}u, \lambda^{-2}\pi, \lambda^{-1}\tau)(\lambda^{-2}t, \lambda^{-1}x) \tag{3.3.35}
$$

solves (μINS)-(τ) with initial data $(\mu_{0,\lambda}, u_{0,\lambda}, \tau_{0,\lambda})$ on the time interval $[0, \lambda^2 T^*)$. Observe that after rescaling

$$
\begin{aligned}
\sigma_{-1,\lambda} &:= \sigma_{-1}(\mu_{0,\lambda}, u_{0,\lambda}) = \lambda\sigma_{-1},\\
\sigma_{0,\lambda} &:= \sigma_0(u_{0,\lambda}) = \sigma_0,\\
\sigma_{1,\lambda} &:= \sigma_1(\mu_{0,\lambda}, u_{0,\lambda}, \bar{\tau}_{0,\lambda}) = \lambda^{-1}\sigma_1,\\
\tilde{V}_\lambda(\lambda^2 t) &:= \tilde{V}(u_\lambda(\lambda^2 t)) = \tilde{V}(t), \quad t \in (0, T^*).
\end{aligned}
\tag{3.3.36}
$$

In the following we fix

$$
\lambda = \frac{\sigma_0}{\sigma_{-1}} = \frac{\|u_0(x)\|_{L^2(\mathbb{R}^2)}}{\|u_0(x)\|_{\dot{H}^{-1}(\mathbb{R}^2)} + \|\mu_0(x) - 1\|_{L^2(\mathbb{R}^2)}\|u_0(x)\|_{L^2(\mathbb{R}^2)}}, \tag{3.3.37}
$$

such that

$$
\sigma_{0,\lambda} + \sigma_{-1,\lambda} = \sigma_0 + \lambda\sigma_{-1} = 2\sigma_0, \quad \sigma_{1,\lambda} = \lambda^{-1}\sigma_1 = \sigma_0^{-1}(\sigma_1\sigma_{-1}). \tag{3.3.38}
$$

We consider the solution $(\mu_\lambda, u_\lambda, \tau_\lambda)$ of the system (μINS)-(τ) corresponding to the initial data $(\mu_{0,\lambda}, u_{0,\lambda}, \tau_{0,\lambda})$ on the time interval $[0, \lambda^2 T^*)$. We also define $\bar{\tau}_\lambda(t, x) = \frac{\tau_\lambda}{|\tau_\lambda|}(t, x)$.

Step 2: Preliminary estimates for a. We first summarize the energy estimates for a from Section 3.3.1 as follows (noticing $\|a\|_{L^2_t L^2} \lesssim \|\nabla u\|_{L^2_t L^2}$)

$$
\begin{aligned}
&\|(t'^{\delta}a, t'^{\frac{1}{2}+\delta}\nabla a)\|_{L^2_t L^2} \le C(\sigma_0 + \sigma_{-1})e^{C(\sigma_0^2 + \sigma_{-1}^2 + \sigma_0^4)}\tilde{V}(t),\\
&\|a\|_{L^2_t L^2} \le C\sigma_0, \quad \|(\nabla a, t'^{\frac{1}{2}}\Delta a)\|_{L^2_t L^2} \le C\sigma_1\tilde{V}(t),\ t \in (0, T^*),
\end{aligned}
\tag{3.3.39}
$$

where we used the initial condition $u_0 \in L^2 \cap \dot{H}^{-1} \subset \dot{H}^{-2\delta}$ for all $\delta \in (0, \frac{1}{2})$. In this chapter we choose δ such that

$$
\delta \in \Big(\frac{1}{2+\epsilon}, \min\Big\{\frac{4+\epsilon}{4(2+\epsilon)}, \frac{1}{\epsilon}\Big\}\Big) \subset \Big(\frac{1}{2+\epsilon}, \frac{1}{2}\Big). \tag{3.3.40}
$$

Thus the constant C in (3.3.39) depends only on μ_*, μ^*, ϵ. In the following we aim to achieve the $L^1_t W^{1,2+\epsilon}_x$-estimate for the rescaled a_λ by applying an interpolation idea.

Let $\omega_\lambda(t,x) = \nabla_x^\perp \cdot u_\lambda(t,x) = \lambda^{-2}\omega(\lambda^{-2}t, \lambda^{-1}x)$ be the rescaled vorticity and $a_\lambda(t,x) = (\mathcal{R}_{\mu_\lambda}\omega_\lambda)(t,x) = \lambda^{-2}a(\lambda^{-2}t, \lambda^{-1}x)$ be the rescaled version of a. Then by virtue of (3.3.36) and (3.3.38), the energy estimates in (3.3.39) are rescaled into

$$\begin{aligned} &\|(a_\lambda, t'^{\delta}a_\lambda, t'^{\frac12+\delta}\nabla a_\lambda)\|_{L^2_{\lambda^2 t}L^2} \le C\sigma_0 e^{C\sigma_0^4}\tilde{V}(t), \\ &\|(\nabla a_\lambda, t'^{\frac12}\Delta a_\lambda)\|_{L^2_{\lambda^2 t}L^2} \le C\sigma_0^{-1}(\sigma_{-1}\sigma_1)\tilde{V}(t), \end{aligned} \tag{3.3.41}$$

for $t \in (0, T^*)$, where we estimated $\sigma_0^2 \lesssim 1 + \sigma_0^4$ in the exponential. By the interpolation inequality (3.3.1) with $r = 2+\epsilon$:

$$\|g\|_{L^{2+\epsilon}} \lesssim \|g\|_{L^2}^{\frac{2}{2+\epsilon}} \|\nabla g\|_{L^2}^{\frac{\epsilon}{2+\epsilon}}, \tag{3.3.42}$$

we derive from (3.3.41) that

$$\begin{aligned} \|a_\lambda\|_{L^1_{\lambda^2 t}L^{2+\epsilon}} &\lesssim \Big\| \|a_\lambda\|_{L^2}^{\frac{2}{2+\epsilon}} \|t'^{\frac12+\delta}\nabla a_\lambda\|_{L^2}^{\frac{\epsilon}{2+\epsilon}} t'^{-(\frac12+\delta)\frac{\epsilon}{2+\epsilon}} \Big\|_{L^1(0,\lambda^2 t)} \\ &\lesssim \|t'^{\frac12+\delta}\nabla a_\lambda\|_{L^2_{\lambda^2 t}L^2}^{\frac{\epsilon}{2+\epsilon}} \Big\| \|a_\lambda\|_{L^2}^{\frac{2}{2+\epsilon}} t'^{-(\frac12+\delta)\frac{\epsilon}{2+\epsilon}} \Big\|_{L^{\frac{2(2+\epsilon)}{2(2+\epsilon)-\epsilon}}(0,\lambda^2 t)} \\ &\lesssim \sigma_0 e^{C\sigma_0^4}\tilde{V}(t), \quad t \in (0, T^*), \end{aligned} \tag{3.3.43}$$

where for the last inequality we used

- If $\lambda^2 t < 1$, then (by (3.3.40) such that $\frac12 - (\frac12+\delta)\frac{\epsilon}{2+\epsilon} > 0$, i.e. $\delta < \frac{1}{\epsilon}$)

$$\|a_\lambda\|_{L^1_{\lambda^2 t}L^{2+\epsilon}} \lesssim \sigma_0^{\frac{\epsilon}{2+\epsilon}} e^{C\sigma_0^4}\tilde{V}(t) \|a_\lambda\|_{L^2_{\lambda^2 t}L^2}^{\frac{2}{2+\epsilon}} \|t'^{-(\frac12+\delta)\frac{\epsilon}{2+\epsilon}}\|_{L^2(0,1)} \lesssim \sigma_0 e^{C\sigma_0^4}\tilde{V}(t).$$

- If $\lambda^2 t \ge 1$, then we decompose the interval $(0, \lambda^2 t)$ into $(0,1)$ and $(1, \lambda^2 t)$, such that (by (3.3.40): $\frac12 - (\frac12+\delta)\frac{\epsilon}{2+\epsilon} - \frac{2\delta}{2+\epsilon} < 0$, i.e. $\delta > \frac{1}{2+\epsilon}$)

$$\begin{aligned} \|a_\lambda\|_{L^1_{\lambda^2 t}L^{2+\epsilon}} \lesssim \sigma_0^{\frac{\epsilon}{2+\epsilon}} e^{C\sigma_0^4}\tilde{V}(t) \Big(&\|a_\lambda\|_{L^2_{\lambda^2 t}L^2}^{\frac{2}{2+\epsilon}} \|t'^{-(\frac12+\delta)\frac{\epsilon}{2+\epsilon}}\|_{L^2(0,1)} \\ &+ \|t'^{\delta}a_\lambda\|_{L^2_{\lambda^2 t}L^2}^{\frac{2}{2+\epsilon}} \|t'^{-(\frac12+\delta)\frac{\epsilon}{2+\epsilon}-\frac{2\delta}{2+\epsilon}}\|_{L^2(1,\infty)} \Big) \lesssim \sigma_0 e^{C\sigma_0^4}\tilde{V}(t). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} \|\nabla a_\lambda\|_{L^1_{\lambda^2 t}L^{2+\epsilon}} &\lesssim \Big\| \|\nabla a_\lambda\|_{L^2}^{\frac{2}{2+\epsilon}} \|t'^{\frac12}\Delta a_\lambda\|_{L^2}^{\frac{\epsilon}{2+\epsilon}} t'^{-\frac{\epsilon}{2(2+\epsilon)}} \Big\|_{L^1(0,\lambda^2 t)} \\ &\lesssim \|t'^{\frac12}\Delta a_\lambda\|_{L^2_{\lambda^2 t}L^2}^{\frac{\epsilon}{2+\epsilon}} \Big\| \|\nabla a_\lambda\|_{L^2}^{\frac{2}{2+\epsilon}} t'^{-\frac{\epsilon}{2(2+\epsilon)}} \Big\|_{L^{\frac{2(2+\epsilon)}{2(2+\epsilon)-\epsilon}}(0,\lambda^2 t)} \\ &\lesssim \sigma_0^{\theta_1}(\sigma_1\sigma_{-1})^{\theta_2} e^{C\sigma_0^4}\tilde{V}(t), \quad t \in (0, T^*), \end{aligned} \tag{3.3.44}$$

where $\theta_1 = \frac{2\frac{1-2\delta}{1+2\delta}-\epsilon}{2+\epsilon}$, $\theta_2 = \frac{2\frac{2\delta}{1+2\delta}+\epsilon}{2+\epsilon}$, and for the last inequality we estimated as follows:

- Firstly, for some $t_1 \in (0, \lambda^2 t]$, we can bound

$$\begin{aligned} \|\nabla a_\lambda\|_{L^1_{\lambda^2 t}L^{2+\epsilon}} \lesssim (\sigma_0^{-1}\sigma_1\sigma_{-1})^{\frac{\epsilon}{2+\epsilon}} \tilde{V}(t) \Big(&\|\nabla a_\lambda\|_{L^2_{\lambda^2 t}L^2}^{\frac{2}{2+\epsilon}} \|t'^{-\frac{\epsilon}{2(2+\epsilon)}}\|_{L^2(0,t_1)} \\ &+ \|t'^{\frac12+\delta}\nabla a_\lambda\|_{L^2_{\lambda^2 t}L^2}^{\frac{2}{2+\epsilon}} \|t'^{-\frac{\epsilon}{2(2+\epsilon)}-(\frac12+\delta)\frac{2}{2+\epsilon}}\|_{L^2(t_1,\lambda^2 t)} \Big) \\ \lesssim (\sigma_0^{-1}\sigma_1\sigma_{-1})^{\frac{\epsilon}{2+\epsilon}} e^{C\sigma_0^4}\tilde{V}(t) \Big(&(\sigma_0^{-1}\sigma_1\sigma_{-1})^{\frac{2}{2+\epsilon}} t_1^{\frac{1}{2+\epsilon}} + \sigma_0^{\frac{2}{2+\epsilon}} t_1^{-\frac{2\delta}{2+\epsilon}} \Big). \end{aligned}$$

- Secondly, if $\lambda^2 t \geq t_0 := (\frac{\sigma_0^2}{\sigma_1\sigma_{-1}})^{\frac{2}{1+2\delta}}$, then we take $t_1 = t_0$ above, while if $\lambda^2 t < t_0$ we can simply bound with the first term in the bracket with $t_1 = t_0$.

Now we can interpolate between (3.3.43) and (3.3.44) to achieve

$$\|a_\lambda\|_{L^1_{\lambda^2 t}L^\infty} \lesssim \|a_\lambda\|^{\frac{\epsilon}{2+\epsilon}}_{L^1_{\lambda^2 t}L^{2+\epsilon}} \|\nabla a_\lambda\|^{\frac{2}{2+\epsilon}}_{L^1_{\lambda^2 t}L^{2+\epsilon}} \lesssim \sigma_0^{\theta_3}(\sigma_1\sigma_{-1})^{\theta_4} e^{C\sigma_0^4}\tilde{V}(t), \quad t \in (0, T^*), \tag{3.3.45}$$

where $\theta_3 = \frac{\epsilon}{2+\epsilon} + \frac{2}{2+\epsilon}\theta_1 = \frac{4\frac{1-2\delta}{1+2\delta}+\epsilon^2}{(2+\epsilon)^2} > 0$, $\theta_4 = \frac{2}{2+\epsilon}\theta_2 = \frac{4\frac{2\delta}{1+2\delta}+2\epsilon}{(2+\epsilon)^2} > 0$.

Very similar calculations show that $\|t'^{\frac{1}{2}}a\|_{L^2_t L^{2+\epsilon}}$, $\|t'^{\frac{1}{2}}\nabla a\|_{L^2_t L^{2+\epsilon}}$ and $\|t'^{\frac{1}{2}}a\|_{L^2_t L^\infty}$ can also be bounded by the right hand sides of (3.3.43), (3.3.44) and (3.3.45), respectively. We omit the details here.

Step 3: $L^\infty L^{2+\epsilon}$-estimates for $(\nabla\bar{\tau}, \partial_{\bar{\tau}}\mu)$. We derive the evolution equation for $\bar{\tau} = \frac{\tau}{|\tau|}$ from the equation (τ) for τ as

$$\partial_t\bar{\tau} + u\cdot\nabla\bar{\tau} = \partial_{\bar{\tau}}u - \bar{\tau}(\bar{\tau}\otimes\bar{\tau} : \nabla u), \tag{3.3.46}$$

so that by an application of the gradient to this equation we find that

$$\|\nabla\bar{\tau}\|_{L^{2+\epsilon}} \lesssim \|\nabla\bar{\tau}_0\|_{L^{2+\epsilon}} + \int_0^t \|\nabla\bar{\tau}\|_{L^{2+\epsilon}}\|\nabla u\|_{L^\infty} + \|\partial_{\bar{\tau}}\nabla u\|_{L^{2+\epsilon}} dt'.$$

By virtue of (3.3.23) and (3.3.31) we have

$$\|\partial_{\bar{\tau}}\nabla u\|_{L^{2+\epsilon}} \lesssim \|(\nabla\bar{\tau}, \partial_{\bar{\tau}}\mu)\|_{L^{2+\epsilon}}(\|\nabla u\|_{L^\infty} + \|a\|_{L^\infty}) + \|\nabla a\|_{L^{2+\epsilon}}.$$

Next, we deduce the evolution equation for $\partial_{\bar{\tau}}\mu$ from the equations of $\partial_\tau\mu$ and $\frac{1}{|\tau|}$:

$$\partial_t\partial_{\bar{\tau}}\mu + u\cdot\nabla\partial_{\bar{\tau}}\mu = -\partial_{\bar{\tau}}\mu(\bar{\tau}\cdot\partial_{\bar{\tau}}u), \tag{3.3.47}$$

from which it follows that

$$\|\partial_{\bar{\tau}}\mu\|_{L^\infty_t L^{2+\epsilon}} \leq \|\partial_{\bar{\tau}_0}\mu_0\|_{L^{2+\epsilon}}V(t), \quad \text{with } \bar{\tau}_0 = \frac{\tau_0}{|\tau_0|}. \tag{3.3.48}$$

Inserting this bound into the above estimate for $\|\nabla\bar{\tau}\|_{L^{2+\epsilon}}$ yields

$$\|\nabla\bar{\tau}\|_{L^\infty_t L^{2+\epsilon}} \lesssim (\|(\nabla\bar{\tau}_0, \partial_{\bar{\tau}_0}\mu_0)\|_{L^{2+\epsilon}} + \|\nabla a\|_{L^1_t L^{2+\epsilon}}) \exp(C\|a\|_{L^1_t L^\infty})V(t). \tag{3.3.49}$$

By the definition (3.3.34), the choice of λ in (3.3.37) and the scaling relation (3.3.38) we obtain

$$\begin{aligned} &\|\nabla\bar{\tau}_\lambda\|_{L^\infty_{\lambda^2 t}L^{2+\epsilon}} + \|\partial_{\bar{\tau}_\lambda}\mu_\lambda\|_{L^\infty_{\lambda^2 t}L^{2+\epsilon}} \\ &\lesssim (\lambda^{-\frac{\epsilon}{2+\epsilon}}(\|\nabla\bar{\tau}_0\|_{L^{2+\epsilon}} + \|\partial_{\bar{\tau}_0}\mu_0\|_{L^{2+\epsilon}}) + \|\nabla a_\lambda\|_{L^1_{\lambda^2 t}L^{2+\epsilon}}) \exp(C\|a_\lambda\|_{L^1_{\lambda^2 t}L^\infty})\tilde{V}_\lambda(\lambda^2 t) \\ &= ((\sigma_0^{-1}\sigma_{-1}\sigma_1)^{\frac{\epsilon}{2+\epsilon}} + \|\nabla a_\lambda\|_{L^1_{\lambda^2 t}L^{2+\epsilon}}) \exp(C\|a_\lambda\|_{L^1_{\lambda^2 t}L^\infty})\tilde{V}(t), \quad t \in (0, T^*). \end{aligned} \tag{3.3.50}$$

Step 4: Lipschitz estimates for u. The time-independent Lipschitz estimate (3.3.30) for the rescaled solution $(\mu_\lambda, u_\lambda, \tau_\lambda)$ and Hölder's inequality with respect to the time variable yields

$$\|\nabla u_\lambda\|_{L^1_{\lambda^2 t}L^\infty} \lesssim \|a_\lambda\|^{\frac{\epsilon}{2+\epsilon}}_{L^1_{\lambda^2 t}L^{2+\epsilon}} \Big(\|\nabla a_\lambda\|_{L^1_{\lambda^2 t}L^{2+\epsilon}}$$

$$
\begin{aligned}
&\qquad + \|(\nabla\bar{\tau}_\lambda, \partial_{\bar{\tau}_\lambda}\mu_\lambda)\|_{L^\infty_{\lambda^2 t} L^{2+\epsilon}} \|(\nabla u_\lambda, a_\lambda)\|_{L^1_{\lambda^2 t} L^\infty}\Big)^{\frac{2}{2+\epsilon}}, \\
\|t'^{\frac12}\nabla u_\lambda\|_{L^2_{\lambda^2 t} L^\infty} &\lesssim \|t'^{\frac12} a_\lambda\|^{\frac{\epsilon}{2+\epsilon}}_{L^2_{\lambda^2 t} L^{2+\epsilon}} \Big(\|t'^{\frac12}\nabla a_\lambda\|_{L^2_{\lambda^2 t} L^{2+\epsilon}} \\
&\qquad + \|(\nabla\bar{\tau}_\lambda, \partial_{\bar{\tau}_\lambda}\mu_\lambda)\|_{L^\infty_{\lambda^2 t} L^{2+\epsilon}} \|t'^{\frac12}(\nabla u_\lambda, a_\lambda)\|_{L^2_{\lambda^2 t} L^\infty}\Big)^{\frac{2}{2+\epsilon}},
\end{aligned}
$$

for $t \in (0, T^*)$. By use of the estimates (3.3.43), (3.3.44), (3.3.45) (together with the version with respect to the time-weighted norm $L^2(tdt)$) and (3.3.50) above, we obtain

$$
\begin{aligned}
&\|\nabla u_\lambda\|_{L^1_{\lambda^2 t} L^\infty} + \|t'^{\frac12}\nabla u_\lambda\|_{L^2_{\lambda^2 t} L^\infty} \\
&\lesssim \sigma_0^{\frac{\epsilon}{2+\epsilon}} \Big((\sigma_0^{-1}\sigma_{-1}\sigma_1)^{\frac{\epsilon}{2+\epsilon}} + \sigma_0^{\theta_1}(\sigma_{-1}\sigma_1)^{\theta_2}\Big)^{\frac{2}{2+\epsilon}} e^{C\sigma_0^4}\tilde{V}(t)\exp\big(C\sigma_0^{\theta_3}(\sigma_{-1}\sigma_1)^{\theta_4} e^{C\sigma_0^4}\tilde{V}(t)\big) \\
&\lesssim \sigma_0^{\frac{\epsilon^2}{(2+\epsilon)^2}} (\sigma_{-1}\sigma_1)^{\frac{2\epsilon}{(2+\epsilon)^2}} e^{C\sigma_0^4}\tilde{V}(t)\exp\big(C\sigma_0^{\theta_3}(\sigma_{-1}\sigma_1)^{\theta_4} e^{C\sigma_0^4}\tilde{V}(t)\big), \quad t \in (0, T^*).
\end{aligned}
$$

We now perform the bootstrap argument. Let

$$A(t) := A(u(t)) = \|\nabla u\|_{L^1_t L^\infty} + \|t'^{\frac12}\nabla u\|_{L^2_t L^\infty}, \quad t \in (0, T^*)$$

denote a time-dependent nonnegative function, such that

$$\tilde{V}(t) = e^{CA(t)}, \quad A_\lambda(\lambda^2 t) := A(u_\lambda(\lambda^2 t)) = A(u(t)) = A(t), \quad t \in (0, T^*).$$

From the above it satisfies

$$A(t) \le C\sigma_0^{\frac{\epsilon^2}{(2+\epsilon)^2}} (\sigma_{-1}\sigma_1)^{\frac{2\epsilon}{(2+\epsilon)^2}} e^{C\sigma_0^4} \exp\big(CA(t) + C\sigma_0^{\theta_3}(\sigma_{-1}\sigma_1)^{\theta_4} e^{C\sigma_0^4} e^{CA(t)}\big).$$

Recall the definition of θ_3, θ_4 in (3.3.45) and the restriction of δ in (3.3.40), such that

$$
\begin{aligned}
\frac{\theta_3}{\theta_4} &= \frac{4\frac{1-2\delta}{1+2\delta} + \epsilon^2}{4\frac{2\delta}{1+2\delta} + 2\epsilon} = -\frac{4-\epsilon^2}{2(2+\epsilon)} + \frac{2+\epsilon}{\delta(4+2\epsilon)+\epsilon} \in \Big(\frac{\epsilon(2+3\epsilon)}{2(4+3\epsilon)}, \frac{\epsilon}{2}\Big) \\
&\quad \text{is close to } \frac{\epsilon}{2} \text{ if } \delta \to \Big(\frac{1}{2+\epsilon}\Big)_+, \\
\theta_4 &= \frac{4\frac{2\delta}{1+2\delta} + 2\epsilon}{(2+\epsilon)^2} = \frac{2}{2+\epsilon} - \Big(\frac{2}{2+\epsilon}\Big)^2 \frac{1}{1+2\delta} \in \Big(\frac{2}{4+\epsilon}, \frac{2(4+3\epsilon)}{(2+\epsilon)(8+3\epsilon)}\Big) \\
&\quad \text{is uniformly bounded in terms of } \mu_*, \mu^*.
\end{aligned}
$$

Under the smallness assumption

$$2C^2(\sigma_0^{\frac{\epsilon}{2}}\sigma_{-1}\sigma_1)^{\frac{2\epsilon}{(2+\epsilon)^2}} e^{C\sigma_0^4} + C\sqrt{e}\Big(\sigma_0^{\frac{\theta_3}{\theta_4}}\sigma_{-1}\sigma_1\Big)^{\theta_4} e^{C\sigma_0^4} \le \frac12, \tag{3.3.51}$$

by the standard bootstrap argument we have the uniform bound

$$A(t) \le 2C(\sigma_0^{\frac{\epsilon}{2}}\sigma_{-1}\sigma_1)^{\frac{2\epsilon}{(2+\epsilon)^2}} e^{C\sigma_0^4}, \quad \forall t \in (0, T^*).$$

Notice that if the smallness assumption (3.1.27): $\sigma_0^{\frac{\epsilon}{2}}\sigma_{-1}\sigma_1 e^{c\sigma_0^4} \le c_0$ is satisfied, then we can choose $\delta \in (\frac{1}{2+\epsilon}, \frac{4+\epsilon}{4(2+\epsilon)})$ (recalling (3.3.40)) close to $\frac{1}{2+\epsilon}$ such that $\frac{\theta_3}{\theta_4}$ is close to $\frac{\epsilon}{2}$, and hence (3.3.51) holds if c_0 is small enough. This completes the proof.

□

3.3.4. Proof of Theorem 3.1.3 and Corollary 3.1.5

In this subsection we prove Theorem 3.1.3 by use of the a priori estimates from the previous subsections.

Proof of Theorem 3.1.3. We start with the proof of existence. The idea is to smooth out the given initial data and then show the convergence of the approximation solution sequence by uniform bounds and compactness.

Step 1: Approximation solution sequence. Given the initial data as in the hypotheses of Theorem 3.1.3 we are going to smooth them out using the standard Friedrich's mollifier. Let $\eta \in C_c^\infty((0,\infty);[0,1])$ be a smooth cut-off function with $\int_{\mathbb{R}} \eta = 1$. Denote $\eta_j(x) = j^2\eta(j|x|)$, $j \in \mathbb{N}$. Define the regularized initial data by the convolution with η_j as

$$\mu_0^j = \mu_0 * \eta_j, \quad u_0^j = u_0 * \eta_j.$$

Then we have

$$\begin{aligned}&\mu_* \le \mu_0^j \le \mu^*,\ \|\mu_0^j - 1\|_{L^2} \le \|\mu_0 - 1\|_{L^2},\ \|u_0^j\|_H \le \|u_0\|_H,\ H = \dot{H}^1, L^2, \dot{H}^{-1},\\ &\|\partial_{\bar\tau_0}\mu_0^j\|_{L^{2+\epsilon}} \le \|(\partial_{\bar\tau_0}\mu_0) * \eta_j\|_{L^{2+\epsilon}} + \|[\partial_{\bar\tau_0}, \eta_j *]\mu_0\|_{L^{2+\epsilon}} \le \|\partial_{\bar\tau_0}\mu_0\|_{L^{2+\epsilon}} + C\mu^*\|\nabla\bar\tau_0\|_{L^{2+\epsilon}}.\end{aligned} \tag{3.3.52}$$

We regularize the transported velocity and the viscosity coefficient in the Cauchy problem of the coupled system (μINS)-(τ) as follows:

$$\begin{cases}\partial_t\mu + (u*\eta_j)\cdot\nabla\mu = 0, \quad \partial_t\tau + (u*\eta_j)\cdot\nabla\tau = \partial_\tau(u*\eta_j), \quad (t,x) \in (0,\infty)\times\mathbb{R}^2,\\ \partial_t u + (u*\eta_j)\cdot\nabla u - \operatorname{div}((\mu*\eta_j)Su) + \nabla\pi = 0, \quad \operatorname{div} u = 0,\\ (\mu, u, \tau)|_{t=0} = (\mu_0^j, u_0^j, \tau_0),\ \text{with } \bar\tau_0^j = \bar\tau_0.\end{cases} \tag{3.3.53}$$

By the classical existence theory (see e.g. [175]) there exists for big enough $j \in \mathbb{N}$ a smooth global-in-time solution $(\mu^j, u^j, \nabla\pi^j, \tau^j)$ of (3.3.53).

We remark that with the regularized "material derivative"

$$D_t^j = \partial_t + (u*\eta_j)\cdot\nabla,$$

the first two equations in (3.3.53) mean that $D_t^j\mu = 0$, and that ∂_τ commutes with D_t^j. Hence (3.3.53) implies the free transport of the tangential derivative $\partial_\tau\mu$

$$D_t^j(\partial_\tau\mu) = \partial_\tau(D_t^j\mu) = 0. \tag{3.3.54}$$

Consequently, similarly as in (3.3.46) and (3.3.47), $\bar\tau^j = \frac{\tau^j}{|\tau^j|}$ and $\partial_{\bar\tau^j}\mu^j$ satisfy the following equations:

$$\partial_t\bar\tau + (u*\eta_j)\cdot\nabla\bar\tau = \partial_{\bar\tau}(u*\eta_j) - \bar\tau(\bar\tau\otimes\bar\tau : \nabla u * \eta_j), \tag{3.3.55}$$

$$\partial_t\partial_{\bar\tau}\mu + (u*\eta_j)\cdot\nabla\partial_{\bar\tau}\mu = -\partial_{\bar\tau}\mu(\bar\tau\cdot\partial_{\bar\tau}(u*\eta_j)). \tag{3.3.56}$$

We notice that the τ-equation in (3.3.53) implies the boundedness and nondegenerity of the vector field τ^j

$$\|\tau^j\|_{L_t^\infty L^\infty} \le \|\tau_0\|_{L^\infty}V^j(t), \quad \|\frac{1}{|\tau^j|}\|_{L_t^\infty L^\infty} \le \|\frac{1}{|\tau_0|}\|_{L^\infty}V^j(t), \quad V^j(t) := \exp(C\|\nabla u^j\|_{L_t^1 L^\infty}),$$

as long as $V^j(t) < \infty$. We have this estimate for all time in (3.3.62) below, which implies the legitimacy of the definition of $\bar{\tau}^j$.

Step 2: Uniform bounds. We show that the a priori estimates in the previous Sections 3.3.1, 3.3.2 and 3.3.3 stay valid for solutions $(\mu^j, u^j, \nabla\pi^j, \tau^j)$ of (3.3.53). We denote $a^j := \mathcal{R}_{\mu^j * \eta_j}\omega^j$ with $\omega^j = \nabla^\perp \cdot u^j$. Recall the uniform bounds (3.3.52) for the initial data, and observe that $\mu_* \le \mu^j(t,x) \le \mu^*$.

Firstly, the energy estimates (3.3.8) and (3.3.9) for u^j follow exactly as before

$$\begin{aligned}&\|\langle t'\rangle^\delta u^j\|_{L^\infty_t L^2 \cap L^2_t \dot{H}^1}\\ &\le C(\|u_0\|_{L^2\cap \dot{H}^{-1}} + \|\mu_0 - 1\|_{L^2}\|u_0\|_{L^2}) e^{C(\|u_0\|^2_{L^2\cap\dot{H}^{-1}} + \|\mu_0-1\|^2_{L^2}\|u_0\|^2_{L^2} + \|u_0\|^4_{L^2})}, \qquad (3.3.57)\end{aligned}$$

where we choose $\delta \in (\frac{1}{2+\epsilon}, \min\{\frac{4+\epsilon}{4(2+\epsilon)}, \frac{1}{\epsilon}\})$ as in (3.3.40) and the constant $C = C(\mu_*, \mu^*, \epsilon)$. Next, an application of the curl operator to the regularized velocity equation $(3.3.53)_2$ yields the following analogue of the vorticity equation (3.3.10) for ω^j and a^j:

$$D^j_t \omega^j - \Delta a^j = -(\nabla^\perp u^j * \eta_j) : (\nabla u^j)^T, \quad a^j = \mathcal{R}_{\mu^j*\eta_j}\omega^j, \quad u^j = \nabla^\perp \Delta^{-1}\omega^j. \qquad (3.3.58)$$

From this we deduce the L^2-energy estimates (3.3.11), (3.3.12) and(3.3.14) as well as $H^1(\mathbb{R}^2)$-estimates (3.3.13) for a^j:

$$\|a^j\|_{L^\infty_t L^2\cap L^2_t\dot{H}^1} \le C\|\nabla u_0\|_{L^2} V^j(t), \quad \|t'^{\frac12} a^j\|_{L^\infty_t L^2\cap L^2_t\dot{H}^1} \le C\|u_0\|_{L^2} V^j(t), \qquad (3.3.59)$$

$$\begin{aligned}\|t'^{\frac12+\delta} a^j\|_{L^\infty_t L^2\cap L^2_t\dot{H}^1} &\le C(\|u_0\|_{L^2\cap\dot{H}^{-1}} + \|\mu_0 - 1\|_{L^2}\|u_0\|_{L^2})\\ &\quad \times e^{C(\|u_0\|^2_{L^2\cap\dot{H}^{-1}} + \|\mu_0-1\|^2_{L^2}\|u_0\|^2_{L^2} + \|u_0\|^4_{L^2})} V^j(t), \qquad (3.3.60)\end{aligned}$$

$$\|t'^{\frac12}\nabla a^j\|_{L^\infty_t L^2\cap L^2_t\dot{H}^1} \le C\|\nabla u_0\|_{L^2}\tilde{V}^j(t), \qquad (3.3.61)$$

with

$$V^j(t) = \exp(C\|\nabla u^j\|_{L^1_t L^\infty}), \quad \tilde{V}^j(t) = V^j(t)\exp(C\|t'^{\frac12}\nabla u^j\|_{L^2_t L^\infty}).$$

Indeed, as in the proof of (3.3.11), we take the L^2-inner product between (3.3.58) and $a^j = \mathcal{R}_{\mu^j*\eta_j}\omega^j$ to derive (3.3.16), with μ replaced by $\mu^j * \eta_j$, $u^j\cdot\nabla$ replaced by $(u^j*\eta_j)\cdot\nabla$ and the following additional terms on the right hand side:

$$\begin{aligned}&-\int_{\mathbb{R}^2} (\nabla^\perp u^j * \eta_j) : (\nabla u^j)^T \mathcal{R}_{\mu^j*\eta_j}\omega^j \, dx\\ &+\frac12\int_{\mathbb{R}^2} ([D^j_t, *\eta_j]\mu^j)\Big(((\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\omega^j)^2 + (2\mathcal{R}_1\mathcal{R}_2\omega^j)^2\Big)dx,\end{aligned}$$

which can be bounded by $\|\nabla u^j\|_{L^\infty}\|\omega^j\|^2_{L^2}$. The L^2-estimates (3.3.59) and (3.3.60) follow from (the modified version of) (3.3.16) immediately. Similarly, we take the L^2-inner product between (3.3.58) and $\mathcal{R}_{\mu^j*\eta_j}\Delta\mathcal{R}_{\mu^j*\eta_j}\omega^j$ to derive (3.3.19), with μ, u replaced by $\mu^j*\eta_j$, $u^j*\eta_j$ respectively, and with the following additional integral on the right hand side

$$\int_{\mathbb{R}^2} \mathcal{R}_{\mu^j*\eta_j}\Big((\nabla^\perp u^j * \eta_j) : (\nabla u^j)^T\Big)\Delta a^j \, dx,$$

which can be bounded by $\|\nabla u^j\|_{L^\infty}\|a^j\|_{L^2}\|\Delta a^j\|_{L^2}$. The H^1-estimate (3.3.61) follows.

Corollary 3.3.6 holds via the consideration of $\alpha^j = (\frac{\tau^j}{|\tau^j|} \otimes \frac{(\tau^j)^\perp}{|\tau^j|}) : ((\mu^j * \eta_j) S u^j)$. Then, along the same lines as in the proof for Proposition 3.3.8, we deduce under the smallness assumption (3.1.27) (with possibly a smaller c_0),

$$\begin{aligned}\|\nabla u^j\|_{L^1_t L^\infty} + \|t'^{\frac{1}{2}} \nabla u^j\|_{L^2_t L^\infty} \leq C(\mu_*, \mu^*, \epsilon)\Big(\|u_0\|_{L^2}^{\frac{\epsilon}{2}} (\|u_0\|_{\dot{H}^{-1}} + \|\mu_0 - 1\|_{L^2} \|u_0\|_{L^2})\\ \times (\|\nabla u_0\|_{L^2} + \|(\nabla \bar{\tau}_0, \partial_{\bar{\tau}_0} \mu_0)\|_{L^{2+\epsilon}}^{\frac{2+\epsilon}{\epsilon}})\Big)^{\frac{2\epsilon}{(2+\epsilon)^2}} e^{C\|u_0\|_{L^2}^4},\end{aligned} \tag{3.3.62}$$

where we have in between used the uniform bounds for $(\nabla \bar{\tau}^j, \partial_{\bar{\tau}^j} \mu^j)$ (recalling (3.3.48), (3.3.49) and (3.3.55), (3.3.56))

$$\|(\nabla \bar{\tau}^j, \partial_{\bar{\tau}^j} \mu^j)\|_{L^\infty_t L^{2+\epsilon}} \lesssim (\|(\nabla \bar{\tau}_0, \partial_{\bar{\tau}_0} \mu^j_0)\|_{L^{2+\epsilon}} + \|\nabla a^j\|_{L^1_t L^{2+\epsilon}}) \exp(C\|a^j\|_{L^1_t L^\infty}) V^j(t).$$

To conclude,

$$\begin{aligned}&\|(\langle t'\rangle^\delta u^j, \langle t'\rangle^{\frac{1}{2}+\delta} a^j, t'^{\frac{1}{2}} \nabla a^j)\|_{L^\infty_t L^2 \cap L^2_t \dot{H}^1} + \tilde{V}^j(t) + \|a^j\|_{L^1_t W^{1,2+\epsilon}} + \|(\nabla \bar{\tau}^j, \partial_{\bar{\tau}^j} \mu^j)\|_{L^\infty_t L^{2+\epsilon}}\\ &\leq C_0,\end{aligned} \tag{3.3.63}$$

for all $j \in \mathbb{N}$ and $t \in (0, \infty)$, where C_0 is some constant depending on the initial data. Applying (3.3.25), (3.3.29) with $p = 2 + \epsilon$ and using (3.3.31), (3.3.59) and (3.3.61) we deduce

$$\|(\alpha^j, \partial_{\tau^j} u^j)\|_{L^1_t W^{1,2+\epsilon}} \leq C_0 \tag{3.3.64}$$

uniformly in $t \in (0, \infty)$ and $j \in \mathbb{N}$.

Now we turn to the uniform estimates for the stress tensor

$$T_{\mu^j}(u^j, \pi^j) := (\mu^j * \eta_j) S u^j - \pi^j Id.$$

By Lemma 3.1.1 - 1 and the u-equation in (3.3.53) we have the following equality

$$\begin{aligned}\nabla^\perp a^j - \nabla \tilde{\pi}^j = \operatorname{div} T_{\mu^j}(u^j, \pi^j) = D^j_t u^j,\\ \text{with} \quad a^j = \mathcal{R}_{\mu^j * \eta_j} \omega^j, \quad \nabla \tilde{\pi}^j := \nabla(\pi^j - Q_{\mu^j * \eta_j} \omega^j).\end{aligned}$$

The curl-free part of the above equation (noticing $\operatorname{div} u^j = 0$)

$$-\nabla \tilde{\pi}^j = \nabla \Delta^{-1} \operatorname{div} D^j_t u^j = \nabla \Delta^{-1} \operatorname{div} ((u^j * \eta_j) \cdot \nabla u^j) = \nabla \Delta^{-1} \big((\nabla u^j * \eta_j) : (\nabla u^j)^T\big)$$

implies from (3.3.63) that for any $t \in (0, \infty)$,

$$\begin{aligned}\|\nabla \tilde{\pi}^j\|_{L^2_t L^2} &\lesssim \|(u^j * \eta_j) \cdot \nabla u^j\|_{L^2_t L^2}\\ &\lesssim \Big\| \|u^j\|_{L^2}^{\frac{\epsilon}{2+\epsilon}} \|\nabla u^j\|_{L^2}^{\frac{2}{2+\epsilon}} \|\nabla u^j\|_{L^{2+\epsilon}} \Big\|_{L^2(0,t)}\\ &\lesssim \|u^j\|_{L^\infty_t L^2}^{\frac{\epsilon}{2+\epsilon}} \|a^j\|_{L^2_t L^2}^{\frac{2}{2+\epsilon}} \|a^j\|_{L^\infty_t L^2}^{\frac{2}{2+\epsilon}} \|\nabla a^j\|_{L^2_t L^2}^{\frac{\epsilon}{2+\epsilon}} \leq C_0,\\ \|t'^{\frac{1}{2}} \nabla^2 \tilde{\pi}^j\|_{L^2_t L^2} &\lesssim \|t'^{\frac{1}{2}} (\nabla u^j * \eta_j) : (\nabla u^j)^T\|_{L^2_t L^2}\\ &\lesssim \|\nabla u^j\|_{L^\infty_t L^2} \|t'^{\frac{1}{2}} \nabla u^j\|_{L^2_t L^\infty} \leq C_0.\end{aligned} \tag{3.3.65}$$

Thus

$$\begin{aligned}&\|(\operatorname{div} T_{\mu^j}(u^j, \pi^j), t'^{\frac{1}{2}} \nabla \operatorname{div} T_{\mu^j}(u^j, \pi^j))\|_{L^2_t L^2} \leq \|(\nabla^\perp a, t'^{\frac{1}{2}} \Delta a^j)\|_{L^2_t L^2} + \|(\nabla \tilde{\pi}^j, t'^{\frac{1}{2}} \nabla^2 \tilde{\pi}^j)\|_{L^2_t L^2}\\ &\leq C_0.\end{aligned} \tag{3.3.66}$$

Step 3: Convergence. By virtue of the above uniform estimates and standard compactness arguments, there exists a subsequence of the approximation solution sequence, still denoted by $(\mu^j, u^j, \nabla\tilde{\pi}^j, \tau^j)$, converging to the limit $(\mu, u, \nabla\tilde{\pi}, \tau)$ which satisfies the properties stated in Theorem 3.1.3. Indeed,

$$\begin{array}{lll} \mu^j \overset{*}{\rightharpoonup} \mu & \text{in} & L^\infty([0,\infty)\times\mathbb{R}^2;[0,\infty)), \\ u^j \overset{*}{\rightharpoonup} u & \text{in} & L^\infty([0,\infty);L^2(\mathbb{R}^2)), \\ \nabla u^j \rightharpoonup \nabla u & \text{in} & L^2([0,\infty);L^2(\mathbb{R}^2)), \\ \tau^j \overset{*}{\rightharpoonup} \tau & \text{in} & L^\infty([0,\infty);L^\infty\cap\dot{W}^{1,2+\epsilon}(\mathbb{R}^2)), \\ \nabla\tilde{\pi}^j \rightharpoonup \nabla\tilde{\pi} & \text{in} & L^2((0,\infty);L^2(\mathbb{R}^2)). \end{array}$$

By the equation $(3.3.53)_1$ for μ^j and the above convergence properties of u^j (and thus of $u^j*\eta_j$), [175, Theorem 2.5] yields $\mu^j\to\mu$ in $\mathcal{C}([0,T];L^p_{\mathrm{loc}}))$ for any $p\in[1,\infty)$. Consequently, we have $\mu^j\to\mu$ almost everywhere on $[0,\infty)\times\mathbb{R}^2$ (up to subsequence), which implies that

$$(\mu^j*\eta_j)Su^j \rightharpoonup \mu Su \quad \text{in } L^2_tL^2_{\mathrm{loc}},\ \forall t>0.$$

Furthermore, by the u-equation in (3.3.53) and the uniform estimates in Step 2, $\partial_t u^j$ is uniformly bounded in $L^2_tL^2$, and hence u^j is relatively compact in $L^p_tL^4_{\mathrm{loc}}$ for all $p\in[1,\infty)$ and $t>0$. Together with the fact that u^j is uniformly bounded in $L^4_tL^4$ we conclude that

$$(u^j*\eta_j)\otimes u^j \rightharpoonup u\otimes u \quad \text{in } L^2_tL^2,\ \forall t>0.$$

Similarly $\mu^j(u^j*\eta_j) \overset{*}{\rightharpoonup} \mu u$, $\tau^j(u^j*\eta_j) \overset{*}{\rightharpoonup} \tau u$ in e.g. $L^\infty_tL^2$. It follows that $(\mu,u,\nabla\pi,\tau)$, with $\nabla\pi=\nabla\tilde{\pi}+\nabla b$ and $b=\mathcal{Q}_\mu(\nabla^\perp\cdot u)$, solves ($\mu$INS)-($\tau$) in the distribution sense. The properties (3.1.28), (3.1.29), (3.1.30) and (3.1.32) follow from the estimates in Step 2.

Step 4: Uniqueness. The uniqueness follows from the L^1_tLip-bound for the velocity field. More precisely, let $(\mu_i,u_i,\nabla\pi_i,\tau_i)$, $i=1,2$, be two solutions of (μINS)-(τ) satisfying (3.1.28). For the uniqueness of the viscosity function we make use of Lagrangian coordinates (see also [60, Section 4]). Let $\mathcal{X}_i:[0,\infty)\times\mathbb{R}^2\to\mathbb{R}^2$ denote the flow of u_i defined as $\mathcal{X}_i(t,\xi)=\xi+\int_0^t u_i(t',\mathcal{X}_i(t',\xi))dt'$, for $i=1,2$. Let $\tilde{\mu}_i(t,\xi)=\mu_i(t,\mathcal{X}_i(t,\xi))$. Then the transport equation $(\mu\mathrm{INS})_1$ implies that $\partial_t\tilde{\mu}_i=0$, and thus $\tilde{\mu}_i(t,\xi)=\mu_i(0,\xi)=\mu_0(\xi)$, $i=1,2$, for any $\xi\in\mathbb{R}^2$.

The uniqueness of the velocity follows from the energy estimate

$$\|\delta u\|^2_{L^\infty_tL^2}+\|\nabla\delta u\|^2_{L^2_tL^2}\lesssim\|(\delta u)(0)\|^2_{L^2}\exp\Big(\int_0^t\|\nabla u_1\|_{L^\infty}dt'\Big) \tag{3.3.67}$$

for the velocity difference $\delta u=u_2-u_1$. Indeed, (3.3.67) follows by testing the difference of the momentum equations $(\mu\mathrm{INS})_2$ for u_1,u_2 by δu and then applying Grönwall's inequality.

Finally we have $\nabla\pi_1=\nabla\pi_2$ from the momentum equations, and $\tau_1=\tau_2$ from the τ-equation. □

Now we prove Corollary 3.1.5.

Proof of Corollary 3.1.5- 1. The assumptions and hence the results of Theorem 3.1.3 hold. The regularity propagation of the viscosity coefficient $\nabla\mu\in L^\infty_tL^q$ follows immediately from the Lipschitz regularity of the velocity field $(3.1.28)_3$ and the evolution equation for $\nabla\mu$: $\partial_t\nabla\mu+u\cdot\nabla\mu=-\nabla u\cdot\nabla\mu$.

To prove (3.1.36), since $\nabla u = \mathcal{R}\mathcal{R}^\perp\omega$, it suffices to show $t^{\frac{1}{q}}\nabla\omega \in L^2((0,\infty)\times\mathbb{R}^2)$ with $q\in[2,\infty]$. Notice that $\mathcal{R}_\mu\nabla\omega = \nabla a - \mathcal{R}_{\nabla\mu}\omega$, so that the L^2-invertibility of $\mathcal{R}_\mu$ implies

$$\begin{aligned}\|t^{\frac{1}{q}}\nabla\omega\|_{L^2L^2} &\lesssim_{\mu_*} \|t^{\frac{1}{q}}\nabla a\|_{L^2L^2} + \|\nabla\mu\|_{L^\infty L^q}\|t^{\frac{1}{q}}\nabla u\|_{L^2L^{\frac{2q}{q-2}}}\\ &\lesssim_{\mu_*} \|\langle t\rangle^{\frac{1}{q}}\nabla a\|_{L^2L^2} + \|\nabla\mu\|_{L^\infty L^q}\|\nabla u\|_{L^2L^2}^{1-\frac{2}{q}}\|t^{\frac{1}{2}}\nabla u\|_{L^2L^\infty}^{\frac{2}{q}},\end{aligned}$$

where we used that $(\mathcal{R}_2\mathcal{R}_2-\mathcal{R}_1\mathcal{R}_1)\omega = -(\partial_2 u_1+\partial_1 u_2)$ and $2\mathcal{R}_1\mathcal{R}_2\omega = \partial_2 u_2 - \partial_1 u_1$ due to $\operatorname{div} u = 0$. By (3.3.11), (3.3.14) and (3.3.33), the right hand side is bounded, which concludes the proof of (3.1.36).

□

We follow the strategy performed for the density patch problem, cf. [166, Section 2] and [198, Theorem 1.3], to show the regularity propagation of the viscosity patch problem, provided with the Lipschitz continuity of the velocity field. The construction of nondegenerate tangent vector field is found in Appendix 3.D.

Proof of Corollary 3.1.5 - 2.

As the assumptions in Theorem 3.1.3 are fulfilled for the viscosity patch-type problem (3.1.37) stated in Corollary 3.1.5 - 2., there exists a unique global-in-time solution $(\mu, u, \nabla\pi)$ of (μINS), satisfying all the estimates in Theorem 3.1.3.

The Lipschitz regularity of the velocity field $(3.1.28)_3$ guarantees the existence of the flow $\mathcal{X}:[0,\infty)\times\mathbb{R}^2\to\mathbb{R}^2$, defined by the initial value problem $\mathcal{X}(t,\xi) = \xi + \int_0^t u(t',\mathcal{X}(t',\xi))dt'$, such that $\mathcal{X}(t,\cdot)\in \mathrm{Lip}(\mathbb{R}^2;\mathbb{R}^2)$ with $\|\nabla\mathcal{X}\|_{L^\infty_t L^\infty} \lesssim \exp(\|\nabla u\|_{L^1_t L^\infty}) < \infty$ for all $t\in[0,\infty)$. By classical transport theory we know that the fluid viscosity is given by $\mu(t,x) = \mu^+(t,x)1_{D_t}(x) + \mu^-(t,x)1_{D_t^c}(x)$ with the time-evolved domain $D_t = \mathcal{X}(t,D)$ and $\mu^\pm(t,x) = \mu_0^\pm(\mathcal{X}^{-1}(t,x))$, where $\mathcal{X}^{-1}(t,\cdot)$ denotes the inverse of the mapping $\mathcal{X}(t,\cdot)$ with respect to the spatial variable. From the fact that $\mu_0^+\in W^{1,2+\epsilon}(\overline{D})$ and $\mu_0^- - 1\in L^2\cap W^{1,2+\epsilon}(\overline{D}^c)$, we deduce $\mu^+(t,\cdot)\in W^{1,2+\epsilon}(\overline{D_t})$ and $\mu^-(t,\cdot)-1\in L^2\cap W^{1,2+\epsilon}(\overline{D_t}^c)$ for $t>0$.

Now we parametrize the boundary ∂D of the initial domain with a function $\gamma_0\in W^{2-\frac{1}{2+\epsilon},2+\epsilon}(\mathbb{S}^1)$ defined as

$$\gamma_0:\mathbb{S}^1\to\partial D,\quad \text{such that}\quad \partial_s\gamma_0(s) = \tau_0(\gamma_0(s)).$$

Then the boundary of D_t can be parametrized by $\mathcal{X}(t,\gamma_0):\mathbb{S}^1\to\partial D_t$. Differentiating with respect to s yields

$$\partial_s(\mathcal{X}(t,\gamma_0(s))) = \tau_0(\gamma_0(s))\cdot\nabla\mathcal{X}(t,\gamma_0(s)) = \tau(t,\mathcal{X}(t,\gamma_0(s))). \tag{3.3.68}$$

Due to the uniform bound of $\tau\in L^\infty_t(L^\infty\cap\dot W^{1,2+\epsilon})$, the trace theorem implies the right hand side of (3.3.68) lies in $W^{1-\frac{1}{2+\epsilon},2+\epsilon}(\mathbb{S}^1)$. This shows that the parametrization $\mathcal{X}(t,\gamma_0)$ is contained in $W^{2-\frac{1}{2+\epsilon},2+\epsilon}(\mathbb{S}^1)$. By another application of the trace theorem we conclude that $\partial D_t\in W^{2,2+\epsilon}(\mathbb{R}^2)$. The regular patch structure is preserved.

Finally, due to the continuity of u and $T(u,\pi)n$ provided by (3.1.28) and (3.1.32), respectively, on the interface $\Gamma_t = \partial D_t$, the solution $(\mu,u,\nabla\pi)$ also solves (3.1.10) with $\Omega_t^+ = D_t$, $\Omega_t^- = \overline{D_t}^c$. □

3.3.5. Proof of Theorems 3.1.7 and 3.1.9

Recall the main estimates obtained in Steps I, II, III, which were used in the conclusion of L^1_tLip-estimate for the system (μINS)-(τ) in Subsection 3.3.3:

- Step I. Estimates for $\|a\|_{L^1_t W^{1,2+\epsilon}}$, $\|t'^{\frac12} a\|_{L^2_t W^{1,2+\epsilon}}$, $\|a\|_{L^1_t L^\infty}$ in terms of

$$\tilde V(t) = V(t) \exp(C\|t'^{\frac12}\nabla u\|_{L^2_t L^\infty}), \text{ with } V(t) = \exp(C\|\nabla u\|_{L^1_t L^\infty}). \tag{3.3.69}$$

- Step II. Estimates which follow from Corollary 3.3.6

$$\begin{aligned} \|\nabla u\|_{L^1_t L^\infty} &\le \|a\|^{\frac{\epsilon}{2+\epsilon}}_{L^1_t L^{2+\epsilon}} \Big(\|\nabla a\|_{L^1_t L^{2+\epsilon}} \\ &\quad + \|(\nabla \bar\tau, \partial_{\bar\tau}\mu)\|_{L^\infty_t L^{2+\epsilon}} \|(\nabla u, a)\|_{L^1_t L^\infty} \Big)^{\frac{2}{2+\epsilon}}, \\ \|t'^{\frac12}\nabla u\|_{L^2_t L^\infty} &\le \|t'^{\frac12} a\|^{\frac{\epsilon}{2+\epsilon}}_{L^2_t L^{2+\epsilon}} \Big(\|t'^{\frac12}\nabla a\|_{L^2_t L^{2+\epsilon}} \\ &\quad + \|(\nabla \bar\tau, \partial_{\bar\tau}\mu)\|_{L^\infty_t L^{2+\epsilon}} \|t'^{\frac12}(\nabla u, a)\|_{L^2_t L^\infty} \Big)^{\frac{2}{2+\epsilon}}, \end{aligned} \tag{3.3.70}$$

  where by use of the transport equations $(\mu\text{INS})_1$, (τ) for μ, τ respectively,

$$\|(\nabla \bar\tau, \partial_{\bar\tau}\mu)\|_{L^\infty_t L^{2+\epsilon}} \lesssim (\|(\nabla \bar\tau_0, \partial_{\bar\tau_0}\mu_0)\|_{L^{2+\epsilon}} + \|\nabla a\|_{L^1_t L^{2+\epsilon}}) \exp(C\|a\|_{L^1_t L^\infty}) V(t). \tag{3.3.71}$$

- Step III. Inequality for $A(t) = \|\nabla u\|_{L^1_t L^\infty} + \|t'^{\frac12}\nabla u\|_{L^2_t L^\infty}$ of type $A(t) \le C\sigma \exp(CA(t) + \tilde\sigma \exp(CA(t)))$, with $\sigma, \tilde\sigma$ depends only on the initial data.

As the estimates from Step II hold universally, in order to prove Theorems 3.1.7 and 3.1.9, it suffices to derive the $W^{1,2+\epsilon}$-estimates for a in Step I, such that the bootstrap argument in Step III works.

We first outline the proof of Theorem 3.1.7. Different from system (μINS) where we derived directly the H^1-energy estimates for a in Step I, for the Boussinesq equations (B) we first derive the H^1-energy estimate for $a_\vartheta = a - \mathcal{R}_{-1}\vartheta$, which takes into account the buoyancy force ϑe_2.

Proof of Theorem 3.1.7 . We aim to establish a priori estimates for

$$\|\vartheta\|_{L^\infty_t L^1 \cap L^r} + \|u\|_{L^\infty_t L^2 \cap L^2_t \dot H^1} + \|a\|_{L^\infty_t L^2 \cap L^2_t \dot H^1} + \|t'^{\frac12}\nabla a\|_{L^\infty_t L^2 \cap L^2_t \dot H^1}.$$

Firstly, the transport equation with divergence-free velocity vector for the temperature $(\text{B})_1$ yields

$$\|\vartheta\|_{L^\infty_t L^{r_1}} = \|\vartheta_0\|_{L^{r_1}}, \quad \forall r_1 \in [1, r] \supset [1, 2+\epsilon], \tag{3.3.72}$$

where the last inclusion relation follows from the assumption $\epsilon \in (0, \min\{\epsilon_0, r-2\}]$.

Compared to the system (μINS), there is an additional term ϑe_2 on the right hand side of the velocity equation $(\text{B})_2$. Consequently, the vorticity equation (3.1.21) is replaced by

$$\partial_t \omega + u \cdot \nabla \omega - \Delta a = \partial_1 \vartheta, \text{ with } \omega = \nabla^\perp \cdot u, \quad a = \mathcal{R}_\mu \omega, \tag{3.3.73}$$

which is derived by applying the curl operator to the velocity equation $(\text{B})_2$. We follow the proofs of Proposition 3.3.3 and Proposition 3.3.4 to derive the energy estimates for u and a.

Taking the L^2-inner product between the velocity equation $(B)_2$ and u we derive by Cauchy-Schwarz inequality, Young's inequality and (3.3.72)

$$\|u\|^2_{L^\infty_t L^2} + \|\nabla u\|^2_{L^2_t L^2} \lesssim_{\mu_*} \|(u_0, t\,\vartheta_0)\|^2_{L^2}. \tag{3.3.74}$$

Next, using the same arguments as in the proof for (3.3.11) we deduce from the vorticity equation (3.3.73) the following estimate

$$\|a\|^2_{L^\infty_t L^2} + \|\nabla a\|^2_{L^2_t L^2} \lesssim_{\mu_*,\mu^*} \|(\omega_0, t^{\frac{1}{2}}\vartheta_0)\|^2_{L^2} V(t), \quad V(t) = \exp(C\|\nabla u\|_{L^1_t L^\infty}). \tag{3.3.75}$$

H^1-estimate for Γ. To obtain higher order energy estimates for a, motivated by e.g. [124], we define the quantity

$$\Gamma = \omega - \mathcal{R}_\mu^{-1}\mathcal{R}_{-1}\vartheta, \quad \text{ with } \mathcal{R}_{-1} = \partial_1(-\Delta)^{-1}.$$

From the energy estimate (3.3.75) above and the relation $\nabla\mathcal{R}_\mu\Gamma = \nabla a - \nabla\mathcal{R}_{-1}\vartheta = \nabla a_\vartheta$ we deduce from (3.3.72)

$$\|\nabla\mathcal{R}_\mu\Gamma\|^2_{L^2_t L^2} \lesssim \|\nabla a\|^2_{L^2_t L^2} + \|\vartheta\|^2_{L^2_t L^2} \lesssim_{\mu_*,\mu^*} \|(\omega_0, t^{\frac{1}{2}}\vartheta_0)\|^2_{L^2} V(t). \tag{3.3.76}$$

Now we derive the $\dot{H}^1$-energy estimate for $\mathcal{R}_\mu\Gamma = a_\vartheta$, similarly as in the proof for (3.3.13). Applying the operator $\mathcal{R}_\mu^{-1}\mathcal{R}_{-1}$ to the temperature equation $(B)_1$, subtracting the resulting equation from the vorticity equation (3.3.73) and using $\Delta a + \partial_1\vartheta = \Delta(\mathcal{R}_\mu\Gamma)$, we obtain

$$\partial_t\Gamma + u\cdot\nabla\Gamma - \Delta\mathcal{R}_\mu\Gamma = [\mathcal{R}_\mu^{-1}\mathcal{R}_{-1}, \frac{D}{Dt}]\vartheta. \tag{3.3.77}$$

We take the L^2-inner product between (3.3.77) and $\mathcal{R}_\mu\dot{\Gamma}$ and perform similar calculations as for (3.3.13) to derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla\mathcal{R}_\mu\Gamma|^2dx + \int_{\mathbb{R}^2}\mu\Big(((\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\dot{\Gamma})^2 + (2\mathcal{R}_1\mathcal{R}_2\dot{\Gamma})^2\Big)dx \\
&= \int_{\mathbb{R}^2}([\mathcal{R}_\mu^{-1}\mathcal{R}_{-1}, \frac{D}{Dt}]\vartheta)\cdot(\mathcal{R}_\mu\dot{\Gamma})dx - \int_{\mathbb{R}^2}\nabla\mathcal{R}_\mu\Gamma\cdot\nabla u\cdot\nabla\mathcal{R}_\mu\Gamma dx \\
&\quad + \int_{\mathbb{R}^2}(\Delta\mathcal{R}_\mu\Gamma)\cdot([\mathcal{R}_\mu, \frac{D}{Dt}]\Gamma)dx.
\end{aligned} \tag{3.3.78}$$

By the commutator estimate (3.3.3) we have

$$\|[\mathcal{R}_\mu, \frac{D}{Dt}]f\|_{L^2} \lesssim_{\mu^*,p_1,p_2} \|\nabla u\|_{L^{p_2}}\|f\|_{L^{p_1}}, \quad \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}, \quad p_1\in[2,\infty),\, p_2\in(2,\infty]. \tag{3.3.79}$$

Hence

$$\begin{aligned}
\|[\mathcal{R}_\mu, \frac{D}{Dt}]\Gamma\|_{L^2} &\le \|[\mathcal{R}_\mu, \frac{D}{Dt}]\omega\|_{L^2} + \|[\mathcal{R}_\mu, \frac{D}{Dt}]\mathcal{R}_\mu^{-1}\mathcal{R}_{-1}\vartheta\|_{L^2} \\
&\lesssim \|\nabla u\|_{L^\infty}\|\omega\|_{L^2} + \|\nabla u\|_{L^{\frac{2(2+\epsilon)}{\epsilon}}}\|\mathcal{R}_\mu^{-1}\mathcal{R}_{-1}\vartheta\|_{L^{2+\epsilon}},
\end{aligned}$$

where by Sobolev embedding we can bound

$$\|\mathcal{R}_\mu^{-1}\mathcal{R}_{-1}\vartheta\|_{L^{2+\epsilon}} \lesssim \|\mathcal{R}_{-1}\vartheta\|_{L^{2+\epsilon}} \lesssim \|\nabla\mathcal{R}_{-1}\vartheta\|_{L^{\frac{2(2+\epsilon)}{4+\epsilon}}} \lesssim \|\vartheta\|_{L^{\frac{2(2+\epsilon)}{4+\epsilon}}}.$$

Similarly the commutator

$$
\begin{aligned}
[\mathcal{R}_\mu^{-1}\mathcal{R}_{-1}, \frac{D}{Dt}]\vartheta &= \mathcal{R}_\mu^{-1}[\mathcal{R}_{-1}, \frac{D}{Dt}]\vartheta + [\mathcal{R}_\mu^{-1}, \frac{D}{Dt}]\mathcal{R}_{-1}\vartheta \\
&= \mathcal{R}_\mu^{-1}(\mathcal{R}_{-1}\mathrm{div}\,(u\vartheta) - u\cdot\nabla\mathcal{R}_{-1}\vartheta) - \mathcal{R}_\mu^{-1}[\mathcal{R}_\mu, \frac{D}{Dt}]\mathcal{R}_\mu^{-1}\mathcal{R}_{-1}\vartheta
\end{aligned}
$$

can be bounded by

$$
\|[\mathcal{R}_\mu^{-1}\mathcal{R}_{-1}, \frac{D}{Dt}]\vartheta\|_{L^2} \lesssim \|u\|_{L^\infty}\|\vartheta\|_{L^2} + \|\nabla u\|_{L^{\frac{2(2+\epsilon)}{\epsilon}}}\|\vartheta\|_{L^{\frac{2(2+\epsilon)}{4+\epsilon}}},
$$

where we used the $L^{2+\epsilon}$-boundedness of $\mathcal{R}_\mu^{-1}$. To conclude, we obtain together with Young's inequality and the identity $\Delta\mathcal{R}_\mu\Gamma = \dot\Gamma + [\mathcal{R}_\mu^{-1}\mathcal{R}_{-1}, \frac{D}{Dt}]\vartheta$

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla\mathcal{R}_\mu\Gamma\|_{L^2}^2 + \|(\dot\Gamma, \Delta\mathcal{R}_\mu\Gamma)\|_{L^2}^2 \\
&\lesssim_{\mu_*,\mu^*} \|[\mathcal{R}_\mu^{-1}\mathcal{R}_{-1}, \frac{D}{Dt}]\vartheta\|_{L^2}^2 + \|\nabla u\|_{L^\infty}\|\nabla\mathcal{R}_\mu\Gamma\|_{L^2}^2 + \|[\mathcal{R}_\mu, \frac{D}{Dt}]\Gamma\|_{L^2}^2 \\
&\lesssim_{\mu_*,\mu^*} \|\nabla u\|_{L^\infty}\|\nabla\mathcal{R}_\mu\Gamma\|_{L^2}^2 + \|\nabla u\|_{L^\infty}^2\|\omega\|_{L^2}^2 + \|u\|_{L^\infty}^2\|\vartheta\|_{L^2}^2 + \|\nabla u\|^2_{L^{\frac{2(2+\epsilon)}{\epsilon}}}\|\vartheta\|^2_{L^{\frac{2(2+\epsilon)}{4+\epsilon}}}.
\end{aligned}
$$

We multiply the above inequality by t and make use of Grönwall's inequality and interpolation inequality to obtain

$$
\begin{aligned}
&\|t^{\frac12}\nabla\mathcal{R}_\mu\Gamma(t)\|_{L^2}^2 + \|t'^{\frac12}(\dot\Gamma, \Delta\mathcal{R}_\mu\Gamma)\|_{L^2_tL^2}^2 \\
&\lesssim_{\mu_*,\mu^*} \Big(\|\nabla\mathcal{R}_\mu\Gamma\|_{L^2_tL^2}^2 + \|\omega\|_{L^\infty_tL^2}^2 + \int_0^t \Big[t'\|u\|_{L^\infty}^2\|\vartheta\|_{L^2}^2 + t'\|\nabla u\|^2_{L^{\frac{2(2+\epsilon)}{\epsilon}}}\|\vartheta\|^2_{L^{\frac{2(2+\epsilon)}{4+\epsilon}}}\Big]dt'\Big)\tilde V(t) \\
&\lesssim_{\mu_*,\mu^*} (\|\nabla\mathcal{R}_\mu\Gamma\|_{L^2_tL^2}^2 + \|a\|_{L^\infty_tL^2}^2)\tilde V(t) + \|u\|_{L^\infty_tL^2}\|\nabla u\|_{L^1_tL^\infty}\|t'^{\frac12}\vartheta\|_{L^\infty_tL^2}^2\tilde V(t) \\
&\quad + \|t'^{\frac12}\nabla u\|_{L^2_tL^\infty}^{\frac{4}{2+\epsilon}}\|\nabla u\|_{L^2_tL^2}^{\frac{2\epsilon}{2+\epsilon}}\|t'^{\frac12}\vartheta\|_{L^\infty_tL^1}^{\frac{4}{2+\epsilon}}\|t'\vartheta\|_{L^\infty_tL^2}^{\frac{2\epsilon}{2+\epsilon}}\tilde V(t),
\end{aligned}
$$

where $\tilde V(t)$ was defined in (3.3.69). Inserting the estimates (3.3.72), (3.3.74), (3.3.75) and (3.3.76), we conclude the time weighted $\dot H^1$-estimate for $\mathcal{R}_\mu\Gamma$

$$
\begin{aligned}
&\|t'^{\frac12}\nabla\mathcal{R}_\mu\Gamma\|_{L^\infty_tL^2}^2 + \|t'^{\frac12}(\dot\Gamma, \Delta\mathcal{R}_\mu\Gamma)\|_{L^2_tL^2}^2 \\
&\lesssim \Big(\|(\omega_0, t^{\frac12}\vartheta_0)\|_{L^2}^2 + \|(u_0, t\vartheta_0)\|_{L^2}\|t^{\frac12}\vartheta_0\|_{L^2}^2 + \|(u_0, t\vartheta_0)\|_{L^2}^{\frac{2\epsilon}{2+\epsilon}}\|t^{\frac12}\vartheta_0\|_{L^1}^{\frac{4}{2+\epsilon}}\|t\vartheta_0\|_{L^2}^{\frac{2\epsilon}{2+\epsilon}}\Big)\tilde V(t).
\end{aligned}
\tag{3.3.80}
$$

$W^{1,2+\epsilon}(\mathbb{R}^2)$-estimate for a. We set

$$
\begin{aligned}
&\sigma_0 = \|u_0\|_{L^2}, \quad \sigma_\vartheta = \sigma_\vartheta(t) = \|t^{\frac12}\vartheta_0\|_{L^1} + \|t\vartheta_0\|_{L^2} + \|t^{\frac32-\frac{1}{2+\epsilon}}\vartheta_0\|_{L^{2+\epsilon}}, \\
&\tilde\sigma_0 = \tilde\sigma_0(t) = \sigma_0 + \sigma_\vartheta, \\
&t^{\frac12}\tilde\sigma_1 = t^{\frac12}\|\omega_0\|_{L^2} + t^{\frac{\epsilon}{2(2+\epsilon)}}\|(\nabla\bar\tau_0, \partial_{\bar\tau_0}\mu_0)\|_{L^{2+\epsilon}} + \sigma_\vartheta(1 + \sigma_0^{\frac12} + \sigma_\vartheta^{\frac12}).
\end{aligned}
$$

Notice that the Boussinesq equations (B) are invariant under the following scaling:

$$
(\vartheta_\lambda, u_\lambda, \pi_\lambda)(t,x) = (\lambda^{-3}\vartheta, \lambda^{-1}u, \lambda^{-2}\pi)(\lambda^{-2}t, \lambda^{-1}x), \quad \lambda > 0,
$$

and hence $\sigma_0, \sigma_\vartheta, t^{\frac12}\tilde\sigma_1, V(t), \tilde V(t)$ are also scaling invariant. Let us recall the estimates (3.3.74), (3.3.75) and (3.3.80) we established above (noticing $\sigma^{\frac{\epsilon}{2+\epsilon}} \lesssim 1 + \sigma^{\frac12}$):

$$
\|a\|_{L^2_tL^2} \le C\tilde\sigma_0, \quad \|(\nabla a, \nabla\mathcal{R}_\mu\Gamma)\|_{L^2_tL^2} \le C\tilde\sigma_1 V(t), \quad \|t'^{\frac12}\Delta\mathcal{R}_\mu\Gamma\|_{L^2_tL^2} \le C\tilde\sigma_1\tilde V(t).
$$

Using interpolation and Hölder's inequality we estimate

$$\begin{aligned}
\|a\|_{L^1_t L^{2+\epsilon}} &\lesssim t^{\frac12}\|a\|_{L^2_t L^{2+\epsilon}} \lesssim t^{\frac12}\|a\|^{\frac{2}{2+\epsilon}}_{L^2_t L^2}\|\nabla a\|^{\frac{\epsilon}{2+\epsilon}}_{L^2_t L^2} \lesssim t^{\frac{1}{2+\epsilon}}\tilde\sigma_0^{\frac{2}{2+\epsilon}}(t^{\frac12}\tilde\sigma_1)^{\frac{\epsilon}{2+\epsilon}}V(t),\\
\|\nabla a\|_{L^1_t L^{2+\epsilon}} &\le \|\nabla\mathcal{R}_\mu\Gamma\|_{L^1_t L^{2+\epsilon}} + \|\nabla\mathcal{R}_{-1}\vartheta\|_{L^1_t L^{2+\epsilon}}\\
&\lesssim \|\nabla\mathcal{R}_\mu\Gamma\|^{\frac{2}{2+\epsilon}}_{L^2_t L^2}\|t'^{\frac12}\Delta\mathcal{R}_\mu\Gamma\|^{\frac{\epsilon}{2+\epsilon}}_{L^2_t L^2} t^{\frac12-\frac{\epsilon}{2(2+\epsilon)}} + t\|\vartheta_0\|_{L^{2+\epsilon}}\\
&\lesssim t^{-\frac{\epsilon}{2(2+\epsilon)}}(t^{\frac12}\tilde\sigma_1+\sigma_\vartheta)\tilde V(t) \lesssim t^{-\frac{\epsilon}{2(2+\epsilon)}}(t^{\frac12}\tilde\sigma_1)\tilde V(t),\\
\|a\|_{L^1_t L^\infty} &\lesssim \|a\|^{\frac{\epsilon}{2+\epsilon}}_{L^1_t L^{2+\epsilon}}\|\nabla a\|^{\frac{2}{2+\epsilon}}_{L^1_t L^{2+\epsilon}} \lesssim \tilde\sigma_0^{\frac{2\epsilon}{(2+\epsilon)^2}}(t^{\frac12}\tilde\sigma_1)^{\frac{\epsilon^2}{(2+\epsilon)^2}+\frac{2}{2+\epsilon}}\tilde V(t),
\end{aligned}$$

and similarly for the quantities $\|t'^{\frac12}a\|_{L^2_t L^{2+\epsilon}}$, $\|t'^{\frac12}\nabla a\|_{L^2_t L^{2+\epsilon}}$ and $\|t'^{\frac12}a\|_{L^2_t L^\infty}$.

Conclusion. Recall (3.3.71):

$$\|(\nabla\bar\tau,\partial_{\bar\tau}\mu)\|_{L^\infty_t L^{2+\epsilon}} \le t^{-\frac{\epsilon}{2(2+\epsilon)}}(t^{\frac12}\tilde\sigma_1)\tilde V(t)\exp(C\|a\|_{L^1_t L^\infty}),$$

and (3.3.70):

$$\begin{aligned}
&\|\nabla u\|_{L^1_t L^\infty} + \|t'^{\frac12}\nabla u\|_{L^2_t L^\infty}\\
&\lesssim \Big(t^{\frac{1}{2+\epsilon}}\tilde\sigma_0^{\frac{2}{2+\epsilon}}(t^{\frac12}\tilde\sigma_1)^{\frac{\epsilon}{2+\epsilon}}\Big)^{\frac{\epsilon}{2+\epsilon}}\Big(t^{-\frac{\epsilon}{2(2+\epsilon)}}(t^{\frac12}\tilde\sigma_1)\Big)^{\frac{2}{2+\epsilon}}\tilde V(t)\exp\big(C\tilde\sigma_0^{\frac{2\epsilon}{(2+\epsilon)^2}}(t^{\frac12}\tilde\sigma_1)^{\frac{\epsilon^2}{(2+\epsilon)^2}+\frac{2}{2+\epsilon}}\tilde V(t)\big).
\end{aligned}$$

With $A(t) = \|\nabla u\|_{L^1_t L^\infty} + \|t'^{\frac12}\nabla u\|_{L^2_t L^\infty}$, the above shows that

$$A(t) \le C\tilde\sigma_0^{\frac{2\epsilon}{(2+\epsilon)^2}}(t^{\frac12}\tilde\sigma_1)^{\frac{\epsilon^2}{(2+\epsilon)^2}+\frac{2}{2+\epsilon}}\exp\big(CA(t) + C\tilde\sigma_0^{\frac{2\epsilon}{(2+\epsilon)^2}}(t^{\frac12}\tilde\sigma_1)^{\frac{\epsilon^2}{(2+\epsilon)^2}+\frac{2}{2+\epsilon}}e^{CA(t)}\big).$$

For $T>0$ satisfying (3.1.41) the following smallness condition is satisfied

$$(2C^2 + C\sqrt{e})\tilde\sigma_0(T)^{\frac{2\epsilon}{(2+\epsilon)^2}}(T^{\frac12}\tilde\sigma_1(T))^{\frac{\epsilon^2}{(2+\epsilon)^2}+\frac{2}{2+\epsilon}} \le \frac12, \tag{3.3.81}$$

and we obtain via a bootstrap argument the uniform bound

$$A(T) \le 2C\tilde\sigma_0(T)^{\frac{2\epsilon}{(2+\epsilon)^2}}(T^{\frac12}\tilde\sigma_1(T))^{\frac{\epsilon^2}{(2+\epsilon)^2}+\frac{2}{2+\epsilon}} \le \frac{1}{2C}.$$

Finally, following the proof of Theorem 3.1.3 in Subsection 3.3.4 we complete the proof of Theorem 3.1.7. □

Proof of Theorem 3.1.9. Firstly, since the density function $\rho(t,x)$ and the viscosity coefficient $\mu(t,x) = \mu_{\mathrm{den}}(\rho(t,x))$ both satisfy the free transport equation, the initial lower and upper bounds are preserved by the Navier-Stokes flow a priori

$$0 < \rho_* \le \rho(t,x) \le \rho^*, \quad 0 < \mu_* \le \mu(t,x) \le \mu^*.$$

In the following, the constant C depends only on the positive constants $\rho_*, \rho^*, \mu_*, \mu^*, \epsilon$ and $\|\mu'_{\mathrm{den}}\|_{L^\infty([\rho_*,\rho^*])}$, and may vary from line to line.

With appropriately adapted modifications, we set as in Subsection 3.3.3

$$\sigma_0 = \|u_0\|_{L^2} + \|\rho_0 - 1\|_{L^2}\|\nabla u_0\|_{L^2},$$

$$\sigma_1 = \|\nabla u_0\|_{L^2} + \|(\nabla \bar{\tau}_0, \partial_{\bar{\tau}_0}\mu_0)\|_{L^{2+\epsilon}}^{\frac{2+\epsilon}{\epsilon}},$$
$$\sigma_{-1} = \|u_0\|_{\dot{H}^{-1}} + \|\rho_0 - 1\|_{L^2}\|u_0\|_{L^2},$$
$$V(t) = \exp(C\|\nabla u\|_{L^1_t L^\infty}), \quad \tilde{V}(t) = \exp\big(C(\|\nabla u\|_{L^1_t L^\infty} + \|t'^{\frac{1}{2}}\nabla u\|_{L^2_t L^\infty})\big).$$

The energy estimates for u in Proposition 3.3.3 are still valid for equations (INS) with some modifications:

$$\|\sqrt{\rho}u\|_{L^\infty_t L^2} + \|\nabla u\|_{L^2_t L^2} \leq C(\mu_*)\|u_0\|_{L^2}, \tag{3.3.82}$$
$$\|\langle t\rangle^\delta u\|_{L^2} + \|\langle t'\rangle^\delta \nabla u\|_{L^2_t L^2} \leq C(\mu_*, \mu^*)(\sigma_0 + \sigma_{-1})e^{C(\sigma_0^2+\sigma_{-1}^2)\exp(C\|u_0\|_{L^2}^2)V(t)}V(t), \tag{3.3.83}$$

where we take $\delta \in (\frac{1}{2+\epsilon}, \min\{\frac{4+\epsilon}{4(2+\epsilon)}, \frac{1}{\epsilon}\}) \subset (\frac{1}{2+\epsilon}, \frac{1}{2})$ as in (3.3.40). Indeed, (3.3.82) is the classical energy estimate which follows from taking the L^2-inner product between u-equation and u itself, see e.g. [175]. The estimate (3.3.83) was also established in e.g. [11, 242], and we sketch its proof at the end of Appendix 3.C, incorporating the necessary modifications to the proof of Proposition 3.3.3 and emphasizing the explicit dependence on the initial norms.

Higher-order energy estimates. We claim the following estimates (in analogy to Proposition 3.3.4)

$$\|\nabla u\|_{L^\infty_t L^2} + \|\dot{u}\|_{L^2_t L^2} \leq C(\mu_*, \mu^*, \rho_*, \rho^*)\|\nabla u_0\|_{L^2} e^{C\|u_0\|_{L^2}^2} V(t), \tag{3.3.84}$$
$$\|t'^{\frac{1}{2}}\nabla u\|_{L^\infty_t L^2} + \|t'^{\frac{1}{2}}\dot{u}\|_{L^2_t L^2} \leq C(\mu_*, \mu^*, \rho_*, \rho^*)\|u_0\|_{L^2} e^{C\|u_0\|_{L^2}^2} V(t), \tag{3.3.85}$$
$$\|t'^{\frac{1}{2}}\sqrt{\rho}\dot{u}\|_{L^\infty_t L^2} + \|t'^{\frac{1}{2}}\dot{\omega}\|_{L^2_t L^2} \leq C(\mu_*, \mu^*, \rho_*, \rho^*)\|\nabla u_0\|_{L^2} e^{C\|u_0\|_{L^2}^2} \tilde{V}(t), \tag{3.3.86}$$
$$\|t'^{\frac{1}{2}+\delta} a\|_{L^\infty_t L^2} + \|t'^{\frac{1}{2}+\delta}\nabla a\|_{L^2_t L^2} \leq C(\mu_*, \mu^*, \rho_*, \rho^*)(\sigma_0 + \sigma_{-1})e^{C(\sigma_0^2+\sigma_{-1}^2)\exp(C\|u_0\|_{L^2}^2)V(t)}V(t). \tag{3.3.87}$$

We only explain the main ideas of their proofs.

(3.3.84) is established in e.g. [11]: taking the L^2 inner product of $(\mathrm{INS})_2$ with $\dot{u}$, performing integration by parts, using the duality between

$$\pi = -(-\Delta)^{-1}\mathrm{div}\,\mathrm{div}\,(\mu S u) + (-\Delta)^{-1}\mathrm{div}\,(\rho\dot{u}) \in L^2 + \mathrm{BMO}$$
$$\text{and} \quad \mathrm{div}\,\dot{u} = \partial_i u_j \partial_j u_i \in L^2 \cap \text{Hardy space } \mathcal{H}^1,$$

and finally applying Young's inequality and then Grönwall's inequality yield (3.3.84). The time-weighted version (3.3.85) of (3.3.84) follows similarly. The decay estimate (3.3.87) follows from (3.3.83).

We now show the time-weighted L^2-estimate for $\dot{u}$ in (3.3.86). With the decomposition (3.1.11), the momentum equation $(\mathrm{INS})_2$ reads

$$\rho\dot{u} - \nabla^\perp a + \nabla\tilde{\pi} = 0, \quad \tilde{\pi} = \pi - b. \tag{3.3.88}$$

We apply $\frac{D}{Dt}$ onto both sides, take the L^2-innder product with $\dot{u}$ and use the transport equation $\frac{D}{Dt}\rho = 0$ to derive

$$\int_{\mathbb{R}^2} \rho\frac{D}{Dt}\dot{u}\cdot\dot{u}dx - \int_{\mathbb{R}^2}\frac{D}{Dt}\nabla^\perp a\cdot\dot{u}dx + \int_{\mathbb{R}^2}\frac{D}{Dt}\nabla\tilde{\pi}\cdot\dot{u}dx = 0.$$

In the following we reformulate each integral one by one.

- By $(\mathrm{INS})_1$ the first integral is equal to $\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\rho|\dot{u}|^2dx$.

- The second integral can be rewritten as

$$-\int_{\mathbb{R}^2}\frac{D}{Dt}\nabla^\perp a\cdot\dot u dx = -\int_{\mathbb{R}^2}[\frac{D}{Dt},\nabla^\perp]a\cdot\dot u dx - \int_{\mathbb{R}^2}\nabla^\perp\Big(\mathcal{R}_\mu\dot\omega + [\frac{D}{Dt},\mathcal{R}_\mu]\omega\Big)\cdot\dot u dx$$
$$= \int_{\mathbb{R}^2}\mu\Big(((\mathcal{R}_2\mathcal{R}_2-\mathcal{R}_1\mathcal{R}_1)\dot\omega)^2 + (2\mathcal{R}_1\mathcal{R}_2\dot\omega)^2\Big)dx + \int_{\mathbb{R}^2}[\frac{D}{Dt},\mathcal{R}_\mu]\omega\,\dot\omega dx$$
$$+\int_{\mathbb{R}^2}\dot u\cdot\nabla^\perp u\cdot\nabla a dx + \int_{\mathbb{R}^2}\Big(\mathcal{R}_\mu\dot\omega + [\frac{D}{Dt},\mathcal{R}_\mu]\omega\Big)[\nabla^\perp\cdot,\frac{D}{Dt}]u\,dx$$

- Using integration by parts and the fact that $\operatorname{div} u = 0$, $\operatorname{div}\dot u = \nabla u : (\nabla u)^T$ we obtain

$$\int_{\mathbb{R}^2}\frac{D}{Dt}\nabla\tilde\pi\cdot\dot u dx = \int_{\mathbb{R}^2}\nabla\frac{D}{Dt}\tilde\pi\cdot\dot u + [\frac{D}{Dt},\nabla]\tilde\pi\cdot\dot u\,dx$$
$$= -\frac{d}{dt}\int_{\mathbb{R}^2}\tilde\pi\nabla u:(\nabla u)^T dx + \int_{\mathbb{R}^2}\tilde\pi\frac{D}{Dt}(\nabla u:(\nabla u)^T)dx + \int_{\mathbb{R}^2}\tilde\pi\nabla u:(\nabla\dot u)^T dx$$
$$= -\frac{d}{dt}\int_{\mathbb{R}^2}\tilde\pi\nabla u:(\nabla u)^T dx + 3\int_{\mathbb{R}^2}\tilde\pi\nabla u:(\nabla\dot u)^T dx,$$

where we used in the third line that

$$\frac{D}{Dt}(\nabla u:(\nabla u)^T) = 2\nabla u:(\nabla\dot u)^T - 2(\partial_i u\cdot\nabla u)\cdot\nabla u_i = 2\nabla u:(\nabla\dot u)^T,$$

which follows from explicit computation

$$(\partial_i u\cdot\nabla u)\cdot\nabla u_i = (\partial_1 u_1)^3 + (\partial_2 u_2)^3 + 3\partial_1 u_2\partial_2 u_1\operatorname{div} u$$
$$= ((\partial_1 u_1)^2 + (\partial_2 u_2)^2 + 2\partial_1 u_2\partial_2 u_1)\operatorname{div} u = 0.$$

Summing up, we showed that

$$\frac{d}{dt}\Big(\frac12\int_{\mathbb{R}^2}\rho|\dot u|^2dx - \int_{\mathbb{R}^2}\tilde\pi\nabla u:(\nabla u)^T dx\Big) + \int_{\mathbb{R}^2}\mu\Big(((\mathcal{R}_2\mathcal{R}_2-\mathcal{R}_1\mathcal{R}_1)\dot\omega)^2 + (2\mathcal{R}_1\mathcal{R}_2\dot\omega)^2\Big)dx$$
$$= -\int_{\mathbb{R}^2}\Big([\frac{D}{Dt},\mathcal{R}_\mu]\omega\,(\dot\omega + \nabla^\perp u:(\nabla u)^T) + \dot u\cdot\nabla^\perp u\cdot\nabla a + \mathcal{R}_\mu\dot\omega(\nabla^\perp u:(\nabla u)^T)\Big)dx$$
$$-3\int_{\mathbb{R}^2}\tilde\pi\nabla u:(\nabla\dot u)^T dx. \tag{3.3.89}$$

Applying the commutator estimate (3.3.3) we see that the first integral on the right hand side is bounded up to a constant by

$$\|\nabla u\|_{L^\infty}(\|\nabla u\|_{L^2}(\|\dot\omega\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla u\|_{L^2}) + \|\dot u\|_{L^2}\|\nabla a\|_{L^2})$$
$$\lesssim \|\nabla u\|_{L^\infty}(\|\nabla u\|_{L^2}\|\dot\omega\|_{L^2} + \|\dot u\|_{L^2}\|\rho\dot u\|_{L^2}) + \|\nabla u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2,$$

where the above inequality holds due to $\nabla^\perp a = \mathbb{P}(\rho\dot u)$ with the Helmholtz projection $\mathbb{P}$ by (3.3.88). The formula $\nabla\tilde\pi = -\nabla\Delta^{-1}\operatorname{div}(\rho\dot u)$, the fact that $\nabla u : (\nabla\dot u)^T = \operatorname{div}(\dot u\cdot\nabla u)$ and integration by parts yield

$$|\int_{\mathbb{R}^2}\tilde\pi\nabla u:(\nabla\dot u)^T dx| = |-\int\nabla\tilde\pi\cdot(\dot u\cdot\nabla u)dx| \lesssim \|\rho\dot u\|_{L^2}\|\dot u\|_{L^2}\|\nabla u\|_{L^\infty}.$$

We insert these estimates into (3.3.89), multiply the resulting inequality by t and integrate in time to derive

$$\|t^{\frac12}\sqrt{\rho}\dot u\|_{L^2}^2 + \|t'^{\frac12}\dot\omega\|_{L^2_tL^2}^2$$

$$
\begin{aligned}
&\lesssim \int_0^t \|\sqrt{\rho}\dot u\|_{L^2}^2 dt' + \int_0^t \Big|\int_{\mathbb{R}^2} \tilde\pi \nabla u : (\nabla u)^T dx\Big| dt' + t\Big|\int_{\mathbb{R}^2} \tilde\pi \nabla u : (\nabla u)^T dx\Big| \\
&\quad + \int_0^t \|t'^{\frac12}\nabla u\|_{L^\infty}(\|\nabla u\|_{L^2}\|t'^{\frac12}\dot\omega\|_{L^2} + \|t'^{\frac12}\rho\dot u\|_{L^2}\|\dot u\|_{L^2})dt' + \|t'^{\frac12}\nabla u\|_{L_t^2L^\infty}^2\|\nabla u\|_{L_t^\infty L^2}^2 \\
&\lesssim_{\rho_*,\rho^*} \|\sqrt{\rho}\dot u\|_{L_t^2L^2}^2 + \|\rho\dot u\|_{L_t^2L^2}\|\nabla u\|_{L_t^\infty L^2}\|\nabla u\|_{L_t^2L^2} + \|t^{\frac12}\rho\dot u\|_{L^2}\|t'^{\frac12}\nabla u\|_{L_t^\infty L^2}\|\nabla u\|_{L_t^\infty L^2} \\
&\quad + \|t'^{\frac12}\nabla u\|_{L_t^2L^\infty}\|\nabla u\|_{L_t^\infty L^2}\|t'^{\frac12}\dot\omega\|_{L_t^2L^2} + \int_0^t \|t'^{\frac12}\nabla u\|_{L^\infty}^2\|t'^{\frac12}\sqrt{\rho}\dot u\|_{L^2}^2 dt' \\
&\quad + \|t'^{\frac12}\nabla u\|_{L_t^2L^\infty}^2\|\nabla u\|_{L_t^\infty L^2}^2.
\end{aligned}
$$

In the second inequality, again, we used the duality between BMO and the Hardy space $\mathcal{H}^1$, the continuous embedding $\dot H^1(\mathbb{R}^2) \subset \mathrm{BMO}(\mathbb{R}^2)$ together with $\nabla\tilde\pi = (\mathrm{Id} - \mathbb{P})(\rho\dot u)$, and the inequality $\|\nabla u : (\nabla u)^T\|_{\mathcal{H}^1} \lesssim \|\nabla u\|_{L^2}^2$ from [47], to derive

$$
\Big|\int_{\mathbb{R}^2} \tilde\pi\nabla u : (\nabla u)^T dx\Big| \lesssim \|\tilde\pi\|_{\mathrm{BMO}}\|\nabla u\|_{L^2}^2 \lesssim \|\rho\dot u\|_{L^2}\|\nabla u\|_{L^2}^2.
$$

We find by Young's and Grönwall's inequality

$$
\begin{aligned}
&\|t'^{\frac12}\sqrt{\rho}\dot u\|_{L_t^\infty L^2}^2 + \|t'^{\frac12}\dot\omega\|_{L^2}^2 \\
&\lesssim_{\rho_*,\rho^*} \tilde V(t)\Big(\|\sqrt{\rho}\dot u\|_{L_t^2L^2}^2 + \|\nabla u\|_{L_t^\infty L^2}^2(\|\nabla u\|_{L_t^2L^2}^2 + \|t'^{\frac12}\nabla u\|_{L_t^\infty L^2}^2 + \|t'^{\frac12}\nabla u\|_{L_t^2L^\infty}^2)\Big).
\end{aligned}
$$

Inserting the estimates (3.3.82), (3.3.84) and (3.3.85) results in (3.3.86).

$W^{1,2+\epsilon}(\mathbb{R}^2)$-estimate for a. First, notice that it follows from the Helmholtz-decomposition $\nabla\dot u = \mathcal{R}\mathcal{R}^\perp\dot\omega + \mathcal{R}\mathcal{R}(\nabla u : (\nabla u)^T)$ that

$$
\|t'^{\frac12}\nabla\dot u\|_{L_t^2L^2} \lesssim \|t'^{\frac12}\dot\omega\|_{L_t^2L^2} + \|t'^{\frac12}\nabla u\|_{L_t^2L^\infty}\|\nabla u\|_{L_t^\infty L^2} \lesssim \|\nabla u_0\|_{L^2}e^{C\|u_0\|_{L^2}^2}\tilde V(t). \tag{3.3.90}
$$

We derive from (3.3.84), (3.3.85), (3.3.86), (3.3.90) and (3.3.87) the following estimates for a:

$$
\begin{aligned}
&\|(t'^{\delta}a, t'^{\frac12+\delta}\nabla a)\|_{L_t^2L^2} \le C(\sigma_0 + \sigma_{-1})e^{C(\sigma_0^2+\sigma_{-1}^2)\exp(C\|u_0\|_{L^2}^2)V(t)}V(t), \\
&\|a\|_{L_t^2L^2} \le C\sigma_0, \quad \|(\dot u, t'^{\frac12}\nabla\dot u)\|_{L_t^2L^2} \le C\sigma_1 e^{C\|u_0\|_{L^2}^2}\tilde V(t).
\end{aligned}
$$

where a and $\dot u$ are related by $\nabla^\perp a = \mathbb{P}(\rho\dot u)$. These estimates are the analogue of (3.3.39) in Subsection 3.3.3, up to exponential factors and the replacement of ∇a by $\dot u$. Thus we can proceed exactly as in Subsection 3.3.3. Scaling with $\lambda = \frac{\sigma_0}{\sigma_{-1}}$ yields the following estimates for a_λ:

$$
\begin{aligned}
\|a_\lambda\|_{L^1_{\lambda^2 t}L^{2+\epsilon}} + \|t'^{\frac12}a_\lambda\|_{L^2_{\lambda^2 t}L^{2+\epsilon}} &\lesssim \sigma_0 e^{C\sigma_0^2\exp(C\|u_0\|_{L^2}^2)V(t)}\tilde V(t), \\
\|\nabla a_\lambda\|_{L^1_{\lambda^2 t}L^{2+\epsilon}} + \|t'^{\frac12}\nabla a_\lambda\|_{L^2_{\lambda^2 t}L^{2+\epsilon}} &\lesssim \sigma_0^{\theta_1}(\sigma_{-1}\sigma_1)^{\theta_2}e^{C\sigma_0^2\exp(C\|u_0\|_{L^2}^2)V(t)}\tilde V(t), \\
\|a_\lambda\|_{L^1_{\lambda^2 t}L^\infty} + \|t'^{\frac12}a_\lambda\|_{L^2_{\lambda^2 t}L^\infty} &\lesssim \sigma_0^{\theta_3}(\sigma_{-1}\sigma_1)^{\theta_4}e^{C\sigma_0^2\exp(C\|u_0\|_{L^2}^2)V(t)}\tilde V(t),
\end{aligned}
$$

with the same exponents $\theta_1, \theta_2, \theta_3, \theta_4$ as in Subsection 3.3.3.

Conclusion. With $A(t) = \|\nabla u\|_{L_t^1L^\infty} + \|t'^{\frac12}\nabla u\|_{L_t^2L^\infty}$, we have derived

$$
A(t) \le C\sigma_0^{\frac{\epsilon^2}{(2+\epsilon)^2}}(\sigma_{-1}\sigma_1)^{\frac{2\epsilon}{(2+\epsilon)^2}}e^{C\sigma_0^2\exp(C\|u_0\|_{L^2}^2+CA(t))}
$$

$$\times \exp\bigl(CA(t) + C\sigma_0^{\theta_3}(\sigma_{-1}\sigma_1)^{\theta_4} e^{C\sigma_0^2 \exp(C\|u_0\|_{L^2}^2 + CA(t))} e^{CA(t)}\bigr).$$

If the initial data satisfies

$$2C^2\bigl(\sigma_0^{\frac{\epsilon}{2}}\sigma_{-1}\sigma_1\bigr)^{\frac{2\epsilon}{(2+\epsilon)^2}} e^{C\sqrt{e}\sigma_0^2 \exp(C\|u_0\|_{L^2}^2)} + C\sqrt{e}\bigl(\sigma_0^{\frac{\theta_3}{\theta_4}}\sigma_{-1}\sigma_1\bigr)^{\theta_4} e^{C\sqrt{e}\sigma_0^2 \exp(C\|u_0\|_{L^2}^2)} \leq \frac{1}{2}, \quad (3.3.91)$$

then with a bootstrap argument we arrive at the uniform bound

$$A(t) \leq 2C\bigl(\sigma_0^{\frac{\epsilon}{2}}\sigma_{-1}\sigma_1\bigr)^{\frac{2\epsilon}{(2+\epsilon)^2}} e^{C\sqrt{e}\sigma_0^2 \exp(C\|u_0\|_{L^2}^2)}.$$

Notice that as before, the smallness condition (3.1.42) with sufficiently small c_3 implies the condition (3.3.91) above. Following the proof of Theorem 3.1.3 in Subsection 3.3.4 completes the first part of the proof of Theorem 3.1.9. The statement about the density-patch is proved exactly as for Corollary 3.1.5 - 2. We omit the details here.

□

3.A. Appendix: Proof of Lemma 3.1.1 - 2

We sketch the proof of the invertibility in $L^{2+\epsilon}(\mathbb{R}^2)$ of the operator

$$\mathcal{R}_\mu = (\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\mu(\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1) + (2\mathcal{R}_1\mathcal{R}_2)\mu(2\mathcal{R}_1\mathcal{R}_2),$$

given the positive lower and upper boundedness of the coefficient: $\mu \in [\mu_*, \mu^*]$, in three steps. The ideas can be generalized to a wider class of elliptic operators.

Step 1: L^2-invertibility. This is another proof of (3.1.24), by use of the ellipticity of the operator $\mathcal{L}_\mu$.

Firstly, we define the homogeneous space $\dot{H}^2(\mathbb{R}^2)$ in such a way that it is complete, for example by factoring out polynomials of order 1. Then $\dot{H}^2(\mathbb{R}^2)$ is a Hilbert space, on which we define the bilinear, symmetric form

$$\begin{aligned}
&\mathfrak{a} : \dot{H}^2(\mathbb{R}^2) \times \dot{H}^2(\mathbb{R}^2) \to \mathbb{R}, \\
&(v, w) \mapsto \int_{\mathbb{R}^2} \mu\Bigl((\partial_{22} - \partial_{11})v(\partial_{22} - \partial_{11})w + 4\partial_{12}v\partial_{12}w\Bigr)dx.
\end{aligned}$$

The bilinear form $\mathfrak{a}$ is bounded and coercive with lower and upper bounds as follows

$$\mathfrak{a}(v, v) \geq \frac{\mu_*}{2}\|\nabla^2 v\|_{L^2}^2, \quad |\mathfrak{a}(v, w)| \leq 2\mu^*\|\nabla^2 v\|_{L^2}\|\nabla^2 w\|_{L^2}, \quad \forall v, w \in \dot{H}^2(\mathbb{R}^2).$$

By the Lax-Milgram lemma there exists for all $g \in \dot{H}^{-2}(\mathbb{R}^2)$, the dual space of $\dot{H}^2(\mathbb{R}^2)$, a unique element $v \in \dot{H}^2(\mathbb{R}^2)$ such that

$$\mathfrak{a}(v, w) = \langle w, g \rangle_{\dot{H}^2 \times \dot{H}^{-2}}, \quad \forall w \in \dot{H}^2(\mathbb{R}^2). \quad (3.A.1)$$

That is, for any $g \in \dot{H}^{-2}(\mathbb{R}^2)$, there exists a unique weak solution $v \in \dot{H}^2(\mathbb{R}^2)$ of the elliptic equation

$$\mathcal{L}_\mu v = g, \quad \text{with } \mathcal{L}_\mu = (\partial_{22} - \partial_{11})\mu(\partial_{22} - \partial_{11}) + (2\partial_{12})\mu(2\partial_{12}).$$

Now we define the bounded operator $\operatorname{div}_2 : L^2(\mathbb{R}^2;\mathbb{R}^3) \to \dot{H}^{-2}(\mathbb{R}^2;\mathbb{R})$ as follows. For $G = (G_1, G_2, G_3)^T \in L^2(\mathbb{R}^2;\mathbb{R}^3)$, we define $\operatorname{div}_2 G \in \dot{H}^{-2}(\mathbb{R}^2)$ by

$$\langle w, \operatorname{div}_2 G\rangle_{\dot{H}^2\times\dot{H}^{-2}} = \int_{\mathbb{R}^2} \Big(G_1\partial_{11}w + G_2\partial_{22}w + G_3\partial_{12}w\Big)dx, \quad \forall w \in \dot{H}^2(\mathbb{R}^2).$$

Then the operator

$$\mathfrak{A} : L^2(\mathbb{R}^2;\mathbb{R}^3) \to L^2(\mathbb{R}^2;\mathbb{R}^3), \quad G \mapsto \nabla^2\mathcal{L}_\mu^{-1}\operatorname{div}_2 G,$$

is bounded on $L^2(\mathbb{R}^2;\mathbb{R}^3)$, where we identify $\nabla^2 \cong (\partial_{11}, \partial_{22}, \partial_{12})^T$. Indeed, for $G \in L^2(\mathbb{R}^2;\mathbb{R}^3)$, let $v_G \in \dot{H}^2(\mathbb{R}^2)$ be the Lax-Milgram solution of $\mathcal{L}_\mu v_G = \operatorname{div}_2 G$ in the sense of (3.A.1). Choosing $w = v_G$ in (3.A.1) and using the coercivity of the sesquilinear form $\mathfrak{a}$ yields the boundedness of $\mathfrak{A}$ on $L^2(\mathbb{R}^2;\mathbb{R}^3)$ as follows

$$\begin{aligned}\frac{\mu_*}{2}\|\nabla^2 v_G\|_{L^2}^2 &\le \operatorname{Re}\mathfrak{a}(v_G, v_G) = \operatorname{Re}\langle v_G, \operatorname{div}_2 G\rangle_{\dot{H}^2\times\dot{H}^{-2}} \\ &\le \|v_G\|_{\dot{H}^2}\|\operatorname{div}_2 G\|_{\dot{H}^{-2}} \lesssim \|\nabla^2 v_G\|_{L^2}\|G\|_{L^2}.\end{aligned}$$

Step 2: $L^{2+\epsilon}$-invertibility. In order to prove that the operator $\mathfrak{A}$ is bounded on $L^{2+\epsilon}(\mathbb{R}^2;\mathbb{R}^3)$, $\epsilon \in (0, \epsilon_0]$ for some $\epsilon_0 > 0$ we are going to make use of Z. Shen's theorem [220, Theorem 3.1], which is a version of the Calderón-Zygmund Lemma. More precisely, if there exists some constant $C > 0$ such that the following holds for all $x_0 \in \mathbb{R}^2$, $r > 0$ and $G \in L^\infty(\mathbb{R}^2;\mathbb{R}^3)$ with compact support outside $B_{3r}(x_0)$

$$\Big(\frac{1}{r^2}\int_{B_r(x_0)} |\mathfrak{A}G|^q dx\Big)^{\frac{1}{q}} \le C\Big(\frac{1}{4r^2}\int_{B_{3r}(x_0)} |\mathfrak{A}G|^2 dx\Big)^{\frac{1}{2}}, \tag{3.A.2}$$

then $\mathfrak{A}$ is bounded on $L^p(\mathbb{R}^2;\mathbb{R}^3)$ for any $p \in (2, q)$.

We sketch the proof of (3.A.2). For this let $x_0 \in \mathbb{R}^2$, $r > 0$ and $G \in L^\infty(\mathbb{R}^2;\mathbb{R}^3)$ have compact support with $G \equiv 0$ in $B_{3r}(x_0)$. Then $v_G = \mathcal{L}_\mu^{-1}\operatorname{div}_2 G$ is the solution to

$$\mathfrak{a}(v_G, w) = \langle w, \operatorname{div}_2 G\rangle_{\dot{H}^2\times\dot{H}^{-2}} = 0 \quad \forall w \in C_c^\infty(B_{2r}(x_0)),$$

and hence, $\mathcal{L}_\mu v_G = 0$ in $B_{2r}(x_0)$ in the sense of distributions. Thus A. Barton's higher order version of Meyer's reverse Hölder estimate [21, Theorem 24] yields the existence of some $q \in (2, \infty)$ such that (3.A.2) holds.

Consequently, $\mathfrak{A} = \nabla^2\mathcal{L}_\mu^{-1}\operatorname{div}_2$ is bounded on $L^{2+\epsilon}(\mathbb{R}^2;\mathbb{R}^3)$, $\forall\epsilon \in (0, \epsilon_0]$ for some $\epsilon_0 > 0$. In particular, $\mathcal{R}_\mu^{-1} = \Delta\mathcal{L}_\mu^{-1}\Delta$ is bounded on $L^{2+\epsilon}(\mathbb{R}^2)$, which concludes the proof.

3.B. Appendix: Proof of Lemma 3.3.2: Commutator estimates

Proof of Lemma 3.3.2. The proof of the first estimate (3.3.3) can be found in A. P. Calderón's article [33, Theorem 1].

We sketch the proof of the second statement in Lemma 3.3.2. Recall Bony's decomposition for a product into low-high frequency, high-low frequency and remainder parts below:

$$FG = T_F G + T_G F + R(F, G),$$

and we refer to [18] for the precise definitions of the paraproduct $T_F G$ and the remainder term $R(F,G)$. We apply Bony's decomposition to the product $\partial_X \mathcal{R}^2 g = X_k(\mathcal{R}^2\partial_k g)$ and $\mathrm{div}\,(Xg) = \partial_k(X_k g)$, for $X = (X_1, X_2)^T$, to achieve

$$\begin{aligned}\partial_X \mathcal{R}^2 g = [T_{X_k}, \mathcal{R}^2\partial_k]g + T_{\mathcal{R}^2\partial_k g}X_k + R(X_k, \mathcal{R}^2\partial_k g)\\ + \mathcal{R}^2\mathrm{div}\,(Xg) - \mathcal{R}^2\partial_k R(X_k, g) - \mathcal{R}^2\partial_k T_g X_k,\end{aligned}$$

where we used the Einstein's summation convention to omit $\sum_k$ above. Observe that for $q > 2$ (see for example [18] or the proofs of [198, Lemma 5.1] and [55, Lemma 2.10])

$$\|(T_{\partial_k h}X_k, \partial_k T_h X_k, R(X_k, \partial_k h), \partial_k R(X_k, h), [T_{X_k}, \mathcal{R}^2\partial_k]h)\|_{L^q} \lesssim \|\nabla X\|_{L^q}\|h\|_{L^\infty}. \quad (3.B.1)$$

This (with $h = \mathcal{R}^2 g$ or g), together with

$$\|\mathcal{R}^2\mathrm{div}\,(Xg)\|_{L^p} \lesssim \|\partial_X g\|_{L^p} + \|\nabla X\|_{L^p}\|g\|_{L^\infty},$$

and the fact that $\|g\|_{L^\infty} = \|(\mathcal{R}_1\mathcal{R}_1 + \mathcal{R}_2\mathcal{R}_2)g\|_{L^\infty} \le 2\|\mathcal{R}^2 g\|_{L^\infty}$ yields (3.3.4), (3.3.5).

Next, we show (3.3.6). Denoting $\mathcal{P}_1 = \mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1$, $\mathcal{P}_2 = 2\mathcal{R}_1\mathcal{R}_2$, such that $\mathcal{R}_\mu = \mathcal{P}_1\mu\mathcal{P}_1 + \mathcal{P}_2\mu\mathcal{P}_2$, and using the identity $\partial_X h = \mathrm{div}\,(Xh) - h\mathrm{div}\,X$, we calculate

$$\begin{aligned}&[\mathcal{R}_\mu, \partial_X]g\\ &= \mathcal{P}_1\mu[\mathcal{P}_1, \partial_X]g + \mathcal{P}_1[\mu, \partial_X]\mathcal{P}_1 g + [\mathcal{P}_1, \partial_X]\mu\mathcal{P}_1 g + \mathcal{P}_2\mu[\mathcal{P}_2, \partial_X]g + \mathcal{P}_2[\mu, \partial_X]\mathcal{P}_2 g + [\mathcal{P}_2, \partial_X]\mu\mathcal{P}_2 g\\ &= -\mathcal{P}_1\mu\big(\partial_X\mathcal{P}_1 g - \mathcal{P}_1\mathrm{div}\,(Xg) + \mathcal{P}_1(g\mathrm{div}\,X)\big) - \big(\partial_X\mathcal{P}_1\mu\mathcal{P}_1 g - \mathcal{P}_1\mathrm{div}\,(X\mu\mathcal{P}_1 g) + \mathcal{P}_1(\mu\mathcal{P}_1 g\,\mathrm{div}\,X)\big)\\ &\quad - \mathcal{P}_2\mu\big(\partial_X\mathcal{P}_2 g - \mathcal{P}_2\mathrm{div}\,(Xg) + \mathcal{P}_2(g\mathrm{div}\,X)\big) - \big(\partial_X\mathcal{P}_2\mu\mathcal{P}_2 g - \mathcal{P}_2\mathrm{div}\,(X\mu\mathcal{P}_2 g) + \mathcal{P}_2(\mu\mathcal{P}_2 g\,\mathrm{div}\,X)\big)\\ &\quad - \mathcal{P}_1(\partial_X\mu\mathcal{P}_1 g) - \mathcal{P}_2(\partial_X\mu\mathcal{P}_2 g)\\ &= -\Big(\mathcal{P}_1\mu(\partial_X\mathcal{P}_1 g - \mathcal{P}_1\mathrm{div}\,(Xg)) + \mathcal{P}_2\mu(\partial_X\mathcal{P}_2 g - \mathcal{P}_2\mathrm{div}\,(Xg))\Big)\\ &\quad - \Big(\mathcal{R}_\mu(g\mathrm{div}\,X) + \mathcal{P}_1(\mu\mathcal{P}_1 g\,\mathrm{div}\,X) + \mathcal{P}_2(\mu\mathcal{P}_2 g\,\mathrm{div}\,X)\Big)\\ &\quad - \Big(\big(\partial_X\mathcal{P}_1\mu\mathcal{P}_1 g - \mathcal{P}_1\mathrm{div}\,(X\mu\mathcal{P}_1 g)\big) + \big(\partial_X\mathcal{P}_2\mu\mathcal{P}_2 g - \mathcal{P}_2\mathrm{div}\,(X\mu\mathcal{P}_2 g)\big)\Big)\\ &\quad - \Big(\mathcal{P}_1(\partial_X\mu\mathcal{P}_1 g) + \mathcal{P}_2(\partial_X\mu\mathcal{P}_2 g)\Big).\end{aligned}$$

We apply (3.3.5) and the L^p-boundedness of Riesz operators to bound the first and second brackets on the right hand side in $L^p(\mathbb{R}^2)$ by $\|\nabla X\|_{L^p}\|\mathcal{R}^2 g\|_{L^\infty}$, respectively. The fourth bracket is bounded in $L^p(\mathbb{R}^2)$ by $\|\partial_X\mu\|_{L^q}\|\mathcal{R}^2 g\|_{L^{\frac{qp}{q-p}}}$. Similarly as above, we use Bony's decomposition to rewrite the third bracket on the right hand side above as

$$\begin{aligned}&[T_{X_k}, \partial_k\mathcal{P}_1]\mu\mathcal{P}_1 g + T_{\partial_k\mathcal{P}_1\mu\mathcal{P}_1 g}X_k + R(X_k, \partial_k\mathcal{P}_1\mu\mathcal{P}_1 g)\\ &- \mathcal{P}_1\partial_k\big(T_{\mu\mathcal{P}_1 g}X_k + R(X_k, \mu\mathcal{P}_1 g)\big)\\ &+ [T_{X_k}, \partial_k\mathcal{P}_2]\mu\mathcal{P}_2 g + T_{\partial_k\mathcal{P}_2\mu\mathcal{P}_2 g}X_k + R(X_k, \partial_k\mathcal{P}_2\mu\mathcal{P}_2 g)\\ &- \mathcal{P}_2\partial_k\big(T_{\mu\mathcal{P}_2 g}X_k + R(X_k, \mu\mathcal{P}_2 g)\big),\end{aligned}$$

where by (3.B.1) all terms can be bounded in $L^p(\mathbb{R}^2)$ by $\|\nabla X\|_{L^p}\|\mathcal{R}^2 g\|_{L^\infty}$, except for

$$\begin{aligned}&T_{\partial_k\mathcal{P}_1\mu\mathcal{P}_1 g}X_k + R(X_k, \partial_k\mathcal{P}_1\mu\mathcal{P}_1 g) + T_{\partial_k\mathcal{P}_2\mu\mathcal{P}_2 g}X_k + R(X_k, \partial_k\mathcal{P}_2\mu\mathcal{P}_2 g)\\ &= T_{\partial_k\mathcal{R}_\mu g}X_k + R(X_k, \partial_k\mathcal{R}_\mu g).\end{aligned}$$

Again by (3.B.1), these last terms satisfy

$$\|T_{\partial_k\mathcal{R}_\mu g}X_k\|_{L^q} + \|R(X_k, \partial_k\mathcal{R}_\mu g)\|_{L^q} \lesssim \|\nabla X\|_{L^q}\|\mathcal{R}_\mu g\|_{L^\infty}.$$

This finishes the proof of (3.3.6). □

3.C. Appendix: Proof of Proposition 3.3.3: Energy estimates for the velocity field

In this section we prove Proposition 3.3.3, and at the end we mention the changes in the proof of (3.3.83) for the density-dependent Navier-Stokes equations (INS). We recall the definition of the Fourier transform of a Schwartz function $f(x) \in \mathcal{S}(\mathbb{R}^2)$ as

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\,dx, \quad \xi \in \mathbb{R}^2,$$

and we define the Fourier transform of a tempered distribution $g \in \mathcal{S}'(\mathbb{R}^2)$ by duality: $\langle \hat{g}, f\rangle_{\mathcal{S}',\mathcal{S}} = \langle g, \hat{f}\rangle_{\mathcal{S}',\mathcal{S}}$.

Proof of Proposition 3.3.3.

- **Proof of (3.3.8):** Multiplying the momentum equation $(\mu\text{INS})_2$ by u, integrating over $\mathbb{R}^2$ and using integration by parts result in

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2}^2 + 2\mu_*\|\nabla u(t)\|_{L^2}^2 \leq 0. \tag{3.C.1}$$

  The estimate (3.3.8) then follows from integrating in time over $[0, t]$.
- **Proof of (3.3.9):** We claim the following decay estimate

$$\|u(t)\|_{L^2} \leq C_\delta \bar{\sigma}_0 e^{C(\bar{\sigma}_0^2 + \|u_0\|_{L^2}^4)} \langle t\rangle^{-\delta_-}, \tag{3.C.2}$$

  where $\delta_- \in (0, \delta)$, $\bar{\sigma}_0 = \|u_0\|_{L^2\cap \dot{H}^{-2\delta}} + \|\mu_0 - 1\|_{L^2}\|u_0\|_{L^2}$, and C_δ is a constant depending only on δ_-, δ, μ_*.

  Now multiplying both sides of (3.C.1) by $\langle t\rangle^{2\delta'} = (e+t)^{2\delta'}$, $\delta' > 0$ and integrating in time we obtain

$$\|\langle t\rangle^{\delta'} u\|_{L^2}^2 + 2\mu_*\|\langle t'\rangle^{\delta'}\nabla u\|_{L^2_t L^2}^2 \lesssim \|u_0\|_{L^2}^2 + \int_0^t \langle t'\rangle^{2\delta'-1}\|u(t')\|_{L^2}^2 dt'.$$

  Thus (3.3.9) follows from the claim (3.C.2) by choosing $\delta' \in (0, \delta_-)$.

  Proof of the claim (3.C.2): We now turn to showing (3.C.2). The idea is to use a time-dependent cut-off in frequency space similar to M. E. Schonbek's Fourier splitting method [215, 216] (see also [242, pp. 310-311] or [11]). Let $g(t)$ be a positive function to be determined later, and let $S(t)$ denote the low-frequency set with respect to $g(t)$:

$$S(t) = \Big\{\xi \in \mathbb{R}^2 : |\xi| \leq \sqrt{\frac{1}{2\mu_*} g(t)}\Big\}.$$

  Then we deduce from (3.C.1) that (noticing $\widehat{\partial_{x_j} f}(\xi) = i\xi_j \hat{f}(\xi)$)

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 + g^2(t)\|u(t)\|_{L^2}^2 \leq g^2(t)\int_{S(t)} |\hat{u}(t,\xi)|^2 d\xi. \tag{3.C.3}$$

  Now we rewrite the velocity equation as $(\mu\text{INS})_2$: $(\partial_t - \Delta)u = -u\cdot\nabla u + \operatorname{div}((\mu-1)Su) - \nabla\pi$ and apply Duhamel's formula to obtain

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-t')\Delta}\mathbb{P}\Big(\operatorname{div}((\mu-1)Su) - u\cdot\nabla u\Big)(t')dt', \tag{3.C.4}$$

where $\mathbb{P} = \mathrm{Id} + \nabla(-\Delta)^{\perp}\mathrm{div}$ denotes the Leray-Helmholtz projector. Then (3.C.4) implies for any fixed time $t > 0$,

$$|\hat{u}(t,\xi)| \lesssim e^{-t|\xi|^2}|\hat{u}_0(\xi)| + \int_0^t e^{-(t-t')|\xi|^2}|\xi||\mathcal{F}((\mu-1)Su) - \mathcal{F}(u\otimes u)|(t')dt',$$

and thus (noticing $\int_{S(t)} |\xi|^2\, d\xi \lesssim \frac{1}{(\mu_*)^2} g^4(t)$)

$$\begin{aligned} &g^2(t)\int_{S(t)} |\hat{u}(t,\xi)|^2 d\xi \\ &\lesssim_{\mu_*} g^2(t)\int_{S(t)} e^{-2t|\xi|^2}|\hat{u}_0(\xi)|^2 d\xi + g^6(t)\Big(\int_0^t \|\mathcal{F}((\mu-1)Su - u\otimes u)(t')\|_{L^\infty_\xi} dt'\Big)^2. \end{aligned}$$

The first integral on the right hand side satisfies

$$\begin{aligned} g^2(t)\int_{S(t)} e^{-2t|\xi|^2}|\hat{u}_0(\xi)|^2 d\xi &\le g^2(t)\int_{\mathbb{R}^2} \langle t\rangle^{-2\delta}\Big(e^{-2t|\xi|^2}(\langle t\rangle|\xi|^2)^{2\delta}\Big)\big(|\xi|^{-4\delta}|\hat{u}_0(\xi)|^2\big)\, d\xi \\ &\lesssim 1_{\{t\le 1\}} g^2(t)\|u_0\|^2_{L^2} + 1_{\{t\ge 1\}} g^2(t) t^{-2\delta}\|u_0\|^2_{\dot{H}^{-2\delta}}, \end{aligned}$$

and the second one can be bounded as

$$\begin{aligned} &g^6(t)\Big(\int_0^t \|\mathcal{F}((\mu-1)Su - u\otimes u)(t')\|_{L^\infty_\xi} dt'\Big)^2 \\ &\lesssim g^6(t)\Big(\int_0^t \|((\mu-1)Su - u\otimes u)(t')\|_{L^1_x} dt'\Big)^2 \\ &\lesssim g^6(t) t\|\mu - 1\|^2_{L^\infty_t L^2}\|\nabla u\|^2_{L^2_t L^2} + g^6(t)\Big(\int_0^t \|u(t')\|^2_{L^2} dt'\Big)^2 \\ &\lesssim g^6(t) t\|\mu_0 - 1\|^2_{L^2}\|u_0\|^2_{L^2} + g^6(t)\|u\|^4_{L^2_t L^2}. \end{aligned}$$

Inserting these estimates into (3.C.3) we obtain

$$\begin{aligned} \exp\Big(\int_0^t g^2(t')dt'\Big)\|u(t)\|^2_{L^2} \lesssim &\|u_0\|^2_{L^2} + \|u_0\|^2_{L^2}\int_0^1 \exp\Big(\int_0^{t'} g^2(t'')dt''\Big) g^2(t')dt' \\ &+ \|u_0\|^2_{\dot{H}^{-2\delta}}\int_1^t \exp\Big(\int_0^{t'} g^2(t'')dt''\Big) g^2(t') t'^{-2\delta} dt' \\ &+ \|\mu_0 - 1\|^2_{L^2}\|u_0\|^2_{L^2}\int_0^t \exp\Big(\int_0^{t'} g^2(t'')dt''\Big) g^6(t') t' dt' \\ &+ \int_0^t \exp\Big(\int_0^{t'} g^2(t'')dt''\Big) g^6(t')\|u\|^4_{L^2_t L^2} dt'. \end{aligned} \tag{3.C.5}$$

We first choose $g^2(t) = \frac{3}{\langle t\rangle \log\langle t\rangle}$ with $\langle t\rangle = e + t$ such that $\int_0^t g^2 = 3\log\log\langle t\rangle$ and $e^{\int_0^t g^2} = (\log\langle t\rangle)^3$. Then (3.C.5), together with the a priori energy estimate: $\|u\|^2_{L^2_t L^2} \le \|u\|^2_{L^\infty_t L^2}\langle t\rangle \le \|u_0\|^2_{L^2}\langle t\rangle$, implies (recalling $\bar{\sigma}_0 = \|u_0\|_{L^2\cap \dot{H}^{-2\delta}} + \|\mu_0 - 1\|_{L^2}\|u_0\|_{L^2}$)

$$\begin{aligned} (\log\langle t\rangle)^3\|u(t)\|^2_{L^2} &\lesssim \bar{\sigma}_0^2 + \int_0^t (\log\langle t'\rangle)^3\langle t'\rangle^{-3}(\log\langle t'\rangle)^{-3}\|u\|^4_{L^2_{t'} L^2} dt' \\ &\lesssim \bar{\sigma}_0^2 + \|u_0\|^4_{L^2}\log\langle t\rangle, \end{aligned}$$

and hence $\|u(t)\|^2_{L^2} \lesssim (\bar{\sigma}_0^2 + \|u_0\|^4_{L^2})(\log\langle t\rangle)^{-2}$. This yields, by use of the inequality $\int_0^t (\log\langle t'\rangle)^{-2} dt' \le C\langle t\rangle(\log\langle t\rangle)^{-2}$ from [242, Lemma 4.1 (iii)], that

$$\|u\|^2_{L^2_t L^2} \lesssim (\bar{\sigma}_0^2 + \|u_0\|^4_{L^2})\langle t\rangle(\log\langle t\rangle)^{-2}.$$

Then we choose $g^2(t) = 2\delta_- \langle t\rangle^{-1}$ in (3.C.5), to derive, by use of $\int_0^t g^2 = 2\delta_-(\log\langle t\rangle - 1)$ and $e^{\int_0^t g^2} = e^{-2\delta_-}\langle t\rangle^{2\delta_-}$, that

$$\langle t\rangle^{2\delta_-}\|u(t)\|_{L^2}^2 \lesssim \bar\sigma_0^2 + \int_0^t \langle t'\rangle^{2\delta_- - 3}\|u\|_{L^2_{t'}L^2}^4 dt' \tag{3.C.6}$$

$$\lesssim \bar\sigma_0^2 + (\bar\sigma_0^2 + \|u_0\|_{L^2}^4)\int_0^t \langle t'\rangle^{2\delta_- - 2}(\log\langle t'\rangle)^{-2}\|u\|_{L^2_{t'}L^2}^2 dt'. \tag{3.C.7}$$

We now define

$$y(t) = \int_{t-1}^t \|u(t')\|_{L^2}^2\langle t'\rangle^{2\delta_-}dt', \quad t\geq 1, \quad \text{and} \quad Y(t) = \max_{1\leq t'\leq t} y(t').$$

Notice that by the above definition $\|u\|_{L^2_tL^2}^2 \leq CY(t)\int_0^t\langle t'\rangle^{-2\delta_-}dt' = CY(t)\frac{\langle t\rangle^{1-2\delta_-}}{1-2\delta_-}$. Using this inequality after integrating (3.C.7) over $[t-1,t]$, we obtain

$$y(t) \lesssim \bar\sigma_0^2 + (\bar\sigma_0^2 + \|u_0\|_{L^2}^4)\int_{t-1}^t\int_0^{t'}\langle t''\rangle^{2\delta_- - 2}(\log\langle t''\rangle)^{-2}\|u\|_{L^2_{t''}L^2}^2 dt''dt'$$
$$\lesssim \bar\sigma_0^2 + (\bar\sigma_0^2 + \|u_0\|_{L^2}^4)\int_0^t \langle t'\rangle^{-1}(\log\langle t'\rangle)^{-2}Y(t')dt',$$

and therefore by Grönwall's inequality and the inequality $\int_0^t\langle t'\rangle^{-1}(\log\langle t'\rangle)^{-2}dt' \leq 1$ from [242, Lemma 4.1 (i)] it follows that $Y(t) \lesssim \bar\sigma_0^2 e^{C(\bar\sigma_0^2 + \|u_0\|_{L^2}^4)}$. Finally

$$\|u\|_{L^2_tL^2}^2 \leq CY(t)\frac{\langle t\rangle^{1-2\delta_-}}{1-2\delta_-} \lesssim \bar\sigma_0^2 e^{C(\bar\sigma_0^2+\|u_0\|_{L^2}^4)}\langle t\rangle^{1-2\delta_-}.$$

Applying this inequality to (3.C.6) we finally arrive at

$$\langle t\rangle^{2\delta_-}\|u(t)\|_{L^2}^2 \lesssim \bar\sigma_0^2 + \bar\sigma_0^4 e^{C(\bar\sigma_0^2+\|u_0\|_{L^2}^4)} \lesssim \bar\sigma_0^2 e^{C(\bar\sigma_0^2+\|u_0\|_{L^2}^4)}.$$

This completes the proof of (3.C.2).

□

In order to show (3.3.83) for the system (INS), we replace the formula (3.C.4) by

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-t')\Delta}\mathbb{P}\Big(\mathrm{div}\,((\mu-1)Su) + (1-\rho)\dot u - u\cdot\nabla u\Big)(t')dt'.$$

The additional term can be estimated as

$$\Big(\int_0^t \|\mathcal{F}((1-\rho)\dot u)\|_{L^\infty}dt'\Big)^2 \lesssim \|1-\rho_0\|_{L^2}^2\log\langle t\rangle\|\langle t'\rangle^{\frac12}\dot u\|_{L^2_tL^2}^2$$
$$\lesssim \|1-\rho_0\|_{L^2}^2\|u_0\|_{H^1}^2\log\langle t\rangle V(t)e^{C\|u_0\|_{L^2}^2},$$

where the second inequality follows from (3.3.84), (3.3.85). We then proceed similarly as above. Indeed, (3.C.5) becomes

$$e^{\int_0^t g^2}\|u(t)\|_{L^2}^2$$
$$\lesssim \bar\sigma_0^2\Big(1 + \int_0^1 e^{\int_0^{t'}g^2dt''}g^2(t')dt' + \int_1^t e^{\int_0^{t'}g^2dt''}g^2(t')t'^{-2\delta}dt' + \int_0^t e^{\int_0^{t'}g^2dt''}g^6(t')t'dt'\Big)$$
$$+ \|1-\rho_0\|_{L^2}^2\|u_0\|_{H^1}^2 e^{C\|u_0\|_{L^2}^2}V(t)\int_0^t e^{\int_0^{t'}g^2dt''}g^4(t')\log\langle t'\rangle dt' + \int_0^t e^{\int_0^{t'}g^2dt''}g^6(t')\|u\|_{L^2_tL^2}^4dt'. \tag{3.C.8}$$

We choose $g^2(t) = \frac{3}{\langle t\rangle \log\langle t\rangle}$ to derive

$$\|u(t)\|_{L^2}^2 \lesssim \Big(\Sigma_t^2 + \|u_0\|_{L^2}^4\Big)(\log\langle t\rangle)^{-2}, \text{ with } \Sigma_t^2 := \bar\sigma_0^2 + \|1-\rho_0\|_{L^2}^2\|u_0\|_{H^1}^2 e^{C\|u_0\|_{L^2}^2} V(t),$$

which implies

$$\|u\|_{L_t^2L^2}^2 \lesssim \Big(\Sigma_t^2 + \|u_0\|_{L^2}^4\Big)\langle t\rangle(\log\langle t\rangle)^{-2}.$$

Then we choose $g^2(t) = 2\delta_-\langle t\rangle^{-1}$ to deduce

$$\langle t\rangle^{2\delta_-}\|u(t)\|_{L^2}^2 \lesssim \Sigma_t^2 + \Big(\Sigma_t^2 + \|u_0\|_{L^2}^4\Big)\int_0^t \langle t'\rangle^{2\delta_- -2}(\log\langle t'\rangle)^{-2}\|u\|_{L_{t'}^2L^2}^2 dt',$$

from which we deduce as above by help of $y(t), Y(t)$ that

$$\|u\|_{L_t^2L^2}^2 \lesssim \Sigma_t^2 e^{C\big(\Sigma_t^2 + \|u_0\|_{L^2}^4\big)}\langle t\rangle^{1-2\delta_-}, \text{ and hence } \|\langle t\rangle^{\delta_-}u(t)\|_{L^2}^2 \lesssim \Sigma_t^2 e^{C\big(\Sigma_t^2+\|u_0\|_{L^2}^4\big)}.$$

This implies, with $\sigma_0 = \|u_0\|_{L^2} + \|1-\rho_0\|_{L^2}\|\nabla u_0\|_{L^2}$ and $\sigma_{-1} = \|u_0\|_{\dot H^{-1}} + \|1-\rho_0\|_{L^2}\|u_0\|_{L^2}$, such that $\Sigma_t^2 \le (\sigma_0^2 + \sigma_{-1}^2)e^{C\|u_0\|_{L^2}^2}V(t)$ and $\|u_0\|_{L^2}^4 \le \sigma_0^2 e^{C\|u_0\|_{L^2}^2}$,

$$\|\langle t\rangle^{\delta_-}u(t)\|_{L^2}^2 \lesssim (\sigma_0^2+\sigma_{-1}^2)V(t)e^{C(\sigma_0^2+\sigma_{-1}^2)\exp(C\|u_0\|_{L^2}^2)V(t)}.$$

3.D. APPENDIX: CONSTRUCTION OF A NONDEGENERATE TANGENTIAL VECTOR FIELD

Given a bounded simply connected domain D with $W^{2,2+\epsilon}$-boundary, there are many different ways to construct a nondegenerate vector field $\tau_0 \in L^\infty \cap \dot W^{1,2+\epsilon}(\mathbb{R}^2;\mathbb{R}^2)$ which is tangent to the boundary ∂D. One way can be described as follows.

We begin with the simplest case in which $D = B = B_1(0)$ is the unit disk in $\mathbb{R}^2$ with the origin as the center. Intuitively, we aim to construct a nondegenerate regular vector field $\tau_B \in L^\infty \cap \dot W^{1,2+\epsilon}(\mathbb{R}^2;\mathbb{R}^2)$, such that the renormalized unit vector field

$$\bar\tau_B(x) = \frac{\tau_B}{|\tau_B|}(x) = \begin{cases} \begin{pmatrix} -\frac{x_2}{|x|} \\ \frac{x_1}{|x|}\end{pmatrix} =: e_\theta, \text{ near the boundary,} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} =: e_1, \text{ away from the boundary,}\end{cases}$$

is tangent to the boundary $\partial D = \partial B = \{x \in \mathbb{R}^2 \,|\, |x| = 1\}$. To this end, we connect the tangential vector e_θ at $|x| = \frac34, \frac54$ to the unit vector e_1 at $|x| = \frac14, \frac74$ respectively as follows

$$\tau_B(r\cos\theta, r\sin\theta) = \begin{cases} \begin{pmatrix} \sin\big(3\pi(r-\frac34) - 2\theta(r-\frac14)\big) \\ \cos\big(3\pi(r-\frac34) - 2\theta(r-\frac14)\big)\end{pmatrix} =: \tau_B^-(r\cos\theta, r\sin\theta), & r \in [\frac14, \frac34], \\ \begin{pmatrix} -\sin\big(3\pi(r-\frac54) - 2\theta(r-\frac74)\big) \\ \cos\big(3\pi(r-\frac54) - 2\theta(r-\frac74)\big)\end{pmatrix} =: \tau_B^+(r\cos\theta, r\sin\theta), & r \in [\frac54, \frac74], \\ e_\theta, & r \in [\frac34, \frac54], \\ e_1, & r \in [0, \frac14] \cup [\frac74, \infty). \end{cases} \tag{3.D.1}$$

Then $|\tau_B| = 1$, $\tau_B \in \mathrm{Lip}(\mathbb{R}^2;\mathbb{R}^2)$, and τ_B is the constant vector field e_1 outside the ball $B_{\frac{7}{4}}(0)$, such that τ_B is the searched nondegenerate regular vector field.

Now, for a general bounded, simply connected $W^{2,2+\epsilon}$-domain $D \subset \mathbb{R}^2$, we construct the nondegenerate tangential vector field $\tau_D \in L^\infty \cap \dot{W}^{1,2+\epsilon}(\mathbb{R}^2;\mathbb{R}^2)$ as follows: let $\gamma = \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} : [0,2\pi) \to \partial D$ be an injective $W^{2-\frac{1}{2+\epsilon},2+\epsilon}$-parametrization of the boundary ∂D such that $|\gamma'(s)| \neq 0$ for all $s \in [0,2\pi)$. We define a continuous function $\theta = \theta(x) \in W^{1-\frac{1}{2+\epsilon},2+\epsilon}(\partial D,[0,\infty))$ as follows: for any $x \in \partial D$ there exists a unique $s \in [0,2\pi)$ such that $x = \gamma(s)$, and we define $\theta = \theta(x) \in [0,\infty)$ such that

$$\frac{\gamma'(s)}{|\gamma'(s)|} = e_\theta = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}.$$

Next, let $d = d(x) = \mathrm{dist}(x,\partial D)$ be the distance of a point $x \in \mathbb{R}^2$ to the boundary ∂D. For $\delta > 0$, let $\Pi : \{x \in \mathbb{R}^2 : d(x) < \delta\} \to \partial D$ denote the projection onto the boundary. It is well-defined for sufficiently small δ, and we fix it in the following. Similarly as before we define the nondegenerate regular vector field

$$\tau_D(x) = \begin{cases} \tau_B^-\big((1-\tfrac{1}{\delta}d(x))\cos\theta(\Pi x),(1-\tfrac{1}{\delta}d(x))\sin\theta(\Pi x)\big), & x \in D,\ \tfrac{1}{\delta}d(x) \in [\tfrac14,\tfrac34],\\ \tau_B^+\big((1+\tfrac{1}{\delta}d(x))\cos\theta(\Pi x),(1+\tfrac{1}{\delta}d(x))\sin\theta(\Pi x)\big), & x \in \mathbb{R}^2\setminus D,\ 1+\tfrac{1}{\delta}d(x) \in [\tfrac54,\tfrac74],\\ e_{\theta(\Pi x)}, & \tfrac{1}{\delta}d(x) \in [0,\tfrac14],\\ e_1, & \tfrac{1}{\delta}d(x) \in [\tfrac34,\infty). \end{cases} \tag{3.D.2}$$

We may generalize the above construction to the viscosity layer problem (3.1.38). For the special case (3.1.39), there are different choices of initial nondegenerate regular vector fields, e.g.

- For each $j = 1,...,N$ let $\delta^{(j)} < \frac13 \min(r^{(j+1)} - r^{(j)}, r^{(j)} - r^{(j-1)})$ with $r^{(0)} := 0$, let $\chi^{(j)} : \mathbb{R}^2 \to [0,1]$ be a smooth cut-off function such that

$$\chi^{(j)}(x) = \begin{cases} 1, \text{ if } \mathrm{dist}(x,\partial D^{(j)}) < \delta^{(j)},\\ 0, \text{ if } \mathrm{dist}(x,\partial D^{(j-1)}) < \delta^{(j-1)} \text{ or } \mathrm{dist}(x,\partial D^{(j+1)}) < \delta^{(j+1)}, \end{cases}$$

  with $\sum_j \chi^{(j)} = 1$, and let $\tau^{(j)}(x) = \tau_B^{(j)}(\frac{x}{r^{(j)}})$, where $\tau_B^{(j)}(y)$ is defined as in (3.D.1) with $r = |y|$ replaced by $1 - \frac{1-|y|}{\delta^{(j)}/r^{(j)}}$. Then $\tau_0(x) = \frac{1}{N}\sum_{j=1}^N \chi^{(j)}\tau^{(j)}$ is one choice, such that $\partial_{\tau_0}\mu_0 = 0$, $\tau_0 = e_1$ in $(\cup_{j=1}^N \mathrm{Supp}(\chi^{(j)}))^C$ (away from the boundaries $\cup_{j=1}^N \partial D^{(j)}$), and $\|\nabla\bar\tau_0\|_{L^{2+\epsilon}}^{\frac{2+\epsilon}{\epsilon}} \sim \big(\sum_{j=1,\cdots,N} \frac{r^{(j)}}{(\delta^{(j)})^{1+\epsilon}}\big)^{\frac{1}{\epsilon}}$.
  This construction can be easily generalized to other more general cases where the profiles of different boundaries vary largely, such that the distances between every two layers play an important role in the construction and consequently in the estimates.
- Alternatively, we can simply connect $e_1|_{r\in[0,\frac18 r^{(1)}]}, e_\theta|_{r\in[r^{(1)},r^{(N)}]}, e_1|_{r\in[\frac{15}{8}r^{(N)},\infty)}$ smoothly, similarly as in (3.D.1), such that $\|\nabla\bar\tau_0\|_{L^{2+\epsilon}}^{\frac{2+\epsilon}{\epsilon}} \sim \frac{1}{r^{(1)}}$. This reduces the smallness condition (3.1.27) to (3.1.40).

CHAPTER 4

The compressible Navier-Stokes equations with variable viscosity

The results presented in this chapter are based on joint research with Xian Liao and Sagbo Marcel Zodji.

4.1. Introduction

This chapter addresses the global-in-time existence and uniqueness of discontinuous solutions to the equations governing the motion of compressible fluids with density-dependent viscosity coefficients. Such variable-viscosity models arise in many physical contexts, including multiphase flows and particle suspension [85, 145]. We focus on the general setting where the viscosity coefficients may be *discontinuous* and exhibit *large variations*. More precisely, we consider the two-dimensional compressible Navier-Stokes equations

$$\begin{cases}\partial_t\rho + \operatorname{div}(\rho u) = 0,\\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) = \operatorname{div}(\mu(\rho)Su) + \nabla(\lambda(\rho)\operatorname{div} u) - \nabla P(\rho).\end{cases} \tag{CNS}$$

In the above, $t > 0$ denotes the time variable and $x = (x_1, x_2)^T \in \mathbb{R}^2$ is the space variable. The system is posed on the whole plane $\mathbb{R}^2$, and the unknowns are the density function $\rho = \rho(t, x) \in (0, \infty)$ and the velocity vector field $u = u(t, x) = (u_1(t, x), u_2(t, x))^T \in \mathbb{R}^2$. The constitutive laws – namely the pressure $P = P(\rho)$, the shear viscosity coefficient $\mu = \mu(\rho)$, and the bulk viscosity coefficient $\lambda = \lambda(\rho)$ – are prescribed as smooth ($W^{2,\infty}$-regularity is enough here) functions of the density. Finally, Su represents twice the symmetric part of the velocity gradient

$$Su = \nabla u + (\nabla u)^T,$$

and we denote the Cauchy stress tensor appearing on the right hand side of $(\mathrm{CNS})_2$ by

$$T = T(u,\rho) = \mu(\rho)Su + (\lambda(\rho)\operatorname{div} u - P(\rho))\mathrm{Id}\,, \tag{T}$$

where Id is the identity matrix in $\mathbb{R}^{2\times 2}$. The system (CNS) is supplemented with initial data

$$\rho_{|t=0} = \rho_0 \in L^\infty(\mathbb{R}^2; (0,\infty)), \quad u_{|t=0} = u_0 \in H^1(\mathbb{R}^2; \mathbb{R}^2), \tag{4.1.1}$$

and we assume the existence of a positive equilibrium density $\tilde\rho > 0$ at spatial infinity such that

$$\rho_0 - \tilde\rho \in L^2(\mathbb{R}^2; \mathbb{R}). \tag{4.1.2}$$

This chapter is structured as follows. In Subsection 4.1.1 we review the relevant literature, focusing on the typical approaches for handling discontinuous densities when the viscosity coefficients are constant, almost constant or specifically chosen functions. The main result in this chapter is Theorem 4.1.1 (cf. Theorem 4), which is presented in Subsection 4.1.2. In Subsection 4.1.3 we introduce our new approach to treat general variable viscosity coefficients that may exhibit large discontinuities. The proof strategy of Theorem 4.1.1 is explained in more detail in Section 4.2. Section 4.3 is devoted to the proofs; we first establish the a priori estimates in Subsections 4.3.1–4.3.4, and then prove Theorem 4.1.1 and Corollary 4.1.3 in Subsection 4.3.5. Appendices 4.A and 4.B present the proofs of our key Lemmas 4.1.7 and 4.1.9, respectively. In Appendix 4.C we carry out auxiliary calculations for the energy estimates.

4.1.1. Related results

The mathematical analysis of system (CNS) dates back to the work of J. Nash [193], who established the existence and uniqueness of local-in-time classical solutions with Hölder regularity. This was followed by several local-in-time well-posedness results in various functional settings, including Sobolev regularity frameworks for initial–boundary value problems (see e.g. [146, 224, 230] for a few representative examples). The first global-in-time existence results for solutions to (CNS) were obtained by A. Matsumura and T. Nishida around 1980. The authors assume that the initial data are a small perturbation of an equilibrium state in H^3 when the viscosity coefficients are constant [187], and in H^4 for the case with density-dependent viscosity coefficients [186]. Since then, these smallness and regularity assumptions have been relaxed to critical Besov spaces $(\rho_0 - \tilde{\rho}, u_0) \in \dot{B}^{d/p}_{p,1}(\mathbb{R}^d) \times \dot{B}^{d/p-1}_{p,1}(\mathbb{R}^d)$, for both the constant-viscosity case and the density-dependent case [39, 44, 45, 52, 118]. In this regularity class, solutions are sufficiently regular to guarantee uniqueness. Moreover, they fulfill the standard energy balance

$$\int_{\mathbb{R}^d} \Big(\rho\frac{|u|^2}{2} + H(\rho)\Big)(t,x)\,dx + \int_0^t \int_{\mathbb{R}^d} \Big(\frac{\mu(\rho)}{2}|Su|^2 + \lambda(\rho)(\operatorname{div} u)^2\Big)(t',x)\mathrm{d}t'\,dx \\ \leq \int_{\mathbb{R}^d} \Big(\rho_0\frac{|u_0|^2}{2} + H(\rho_0)\Big)(x)\,dx, \tag{4.1.3}$$

where $H(\rho)$ is the potential energy defined as the solution to the initial value problem for the ODE $\rho H'(\rho) - H(\rho) = P(\rho) - P(\tilde{\rho})$ and $H(\tilde{\rho}) = 0$. Given the existence theory of weak solutions à la Leray in the incompressible setting (see [163] for results in the constant-viscosity case and [175] for the density-dependent case), it is natural to investigate whether (CNS) admits weak solutions satisfying the energy balance (4.1.3). However, this problem is still open for large initial data. The main obstacle lies in proving strong compactness for the density approximation sequence, which is required to pass to the limit in the nonlinear terms in the existence proof. This does not follow directly from the DiPerna–Lions theory (see [76]), since for the compressible Navier–Stokes equations one typically lacks the required bounds on $\operatorname{div} u$.

Let us expand this discussion further. In the *constant-viscosity* case, one can (formally) apply the operator $-(-\Delta)^{-1}\operatorname{div}$ to the momentum equation $(\mathrm{CNS})_2$ to obtain

$$-(-\Delta)^{-1}\operatorname{div}\big(\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u)\big) = (2\mu+\lambda)\operatorname{div} u - \big(P(\rho) - P(\tilde{\rho})\big). \tag{4.1.4}$$

The expression on the right hand side is a simple algebraic relation linking the velocity divergence and the pressure fluctuation, known as "effective viscous flux", denoted by F. It was introduced by D. Hoff and J. Smoller [130] in the context of one-dimensional parabolic systems (see also [218]), and turns out to play a fundamental role in the analysis of the viscous compressible model: Firstly, it was used to study the propagation of density oscillations and discontinuities for the one-dimensional model [126, 218]. Afterwards, for the multidimensional model with constant viscosity [58, 74, 127], one observes that in an *intermediate* regularity class (less regular than the critical Besov regularity), the inertial force on the left hand side of $(\mathrm{CNS})_2$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) \ \text{ belongs to } \ L^1_{\mathrm{loc}}([0,\infty), L^p(\mathbb{R}^d;\mathbb{R}^d)),$$

with some $p \in (d,\infty)$. Thus, (4.1.4) implies that the effective flux F on the right hand side is Hölder continuous in space of order $\dot{\mathcal{C}}^{1-\frac{d}{p}}(\mathbb{R}^2)$, even though the pressure (or density) belongs merely to $L^\infty(\mathbb{R}^d)$ and may be discontinuous.

This Hölder continuity of F yields a quantitative control on density oscillations and hence the strong compactness of the density approximation sequence. This compactness mechanism was introduced by P.-L. Lions in [176] to prove the existence of finite energy weak solutions to (CNS) with constant viscosity, which was later extended by E. Feireisl, A. Novotný, and H. Petzeltová [93]; see also [83, 91, 205]. The regularity of F also allows to propagate discontinuity interfaces of the density function, thereby addressing the density-patch problem for compressible fluids (see [55, 125, 129, 169, 202]).

By contrast, for the compressible viscous fluid model (CNS) with *general* density-dependent viscosity coefficients, due to the lack of control on F, and hence on the density oscillation by the above argument, the global-in-time existence of finite-energy weak solutions for large initial data remains widely open. Nevertheless, such results are available in certain *specific* regimes; we now present some examples.

Some compressible Navier-Stokes equations with *specific* density-dependent viscosity coefficients

- *Case with constant shear viscosity $\mu > 0$ and variable bulk viscosity $\lambda = \lambda(\rho)$.* In this case the mathematical structure of the model does not change significantly from the constant-viscosity case. In particular, the effective flux

  $$F = \big(2\mu + \lambda(\rho)\big)\operatorname{div} u - \big(P(\rho) - P(\tilde\rho)\big)$$

  retains the same regularity as in the constant-viscosity case. For instance, global-in-time solutions with large initial data have been constructed in [139, 201, 235].
- *Case with almost-constant shear viscosity $|\mu(\rho) - \mu(\tilde\rho)| \ll 1$.* In this perturbative setting one rewrites the viscous part of the momentum equation $(\mathrm{CNS})_2$ as

  $$\operatorname{div}(\mu(\rho)Su) + \nabla\big(\lambda(\rho)\operatorname{div} u\big) = \mu(\tilde\rho)\Delta u + \nabla\big((\mu(\tilde\rho) + \lambda(\rho))\operatorname{div} u\big) + \operatorname{div}\big((\mu(\rho) - \mu(\tilde\rho))Su\big),$$

  where the last term on the right hand side is treated as a perturbation. This decomposition was recently used by the third author [252], where the density function is bounded and discontinuous with Hölder continuity on both sides of a flow-driven $\mathcal{C}^{1+\alpha}$ free interface.
- *Case with specific viscosity coefficients.* D. Bresch and his collaborators [28–30] (see also [188]) assumed the following algebraic relation between the shear and bulk viscosities

  $$\lambda(\rho) = 2\big(\rho\mu'(\rho) - \mu(\rho)\big),$$

  to establish an additional entropy balance

  $$\frac{d}{dt}\int_{\mathbb{R}^d}\left(\frac{\rho}{2}\left|u + 2\frac{\nabla\mu(\rho)}{\rho}\right|^2 + H(\rho)\right)dx + \int_{\mathbb{R}^d}\left(\frac{P'(\rho)\mu'(\rho)}{\rho}|\nabla\rho|^2 + \frac{1}{4}\mu(\rho)\left|\nabla u - \nabla u^T\right|^2\right)dx = 0,$$

which provides Sobolev regularity and hence strong compactness for the density function. However, to handle the vacuum states and to pass to the limit in the viscous term, additional conditions were imposed on the shear viscosity, which unfortunately exclude interesting viscosity coefficients of the form $\mu(\rho) = \rho^a$ with $a \in [0, 1 - 1/d)$, as well as those appearing in the context of suspension models (see [104]).

We remark that when the shear viscosity $\mu(\rho)$ is not constant, the structure of the model changes significantly. Even in the two aforementioned specific cases [28–30, 188, 252], the analysis is delicate. As shown in [252], the effective flux

$$F = \big(2\mu(\rho) + \lambda(\rho)\big)\,\mathrm{div}\, u - \big(P(\rho) - P(\tilde{\rho})\big)$$

is *discontinuous* and therefore ceases to be a suitable candidate for transferring the $L^p(\mathbb{R}^d)$-bound of the inertial forces to the compactness of the density function.

A similar problem arises for the *anisotropic* Navier-Stokes model, where, because of nonlocality, the pointwise relation between the pressure fluctuation and the velocity divergence via the effective flux is lost. This obstacle is the reason why the existence of weak and strong solutions is currently limited to the case where the anisotropic viscosity coefficients are close to one another (see [23, 27, 31, 240]).

Such smallness assumptions on the viscosity fluctuations also appear in the *incompressible* setting [75, 101, 198]. However, in the recent work [168], the first two authors succeeded in removing this smallness condition for the two-dimensional incompressible model and established global-in-time well-posedness in a framework that allows for large viscosity variations. The key new idea there is to introduce two "effective replacements" for the fluid vorticity

$$\omega = \nabla^{\perp} \cdot u = \partial_1 u_2 - \partial_2 u_1.$$

We now explain this idea for the incompressible model in more detail.

The two-dimensional *incompressible* Navier-Stokes equations with general variable viscosity

Recall that in the incompressible setting ($\mathrm{div}\, u = 0$), if the viscosity coefficient is constant, the viscous term in the momentum equation becomes

$$\mathrm{div}\,(\mu S u) = \mu \Delta u = \mu \nabla^{\perp} \omega.$$

Similarly as in the above discussion for the effective flux F, the $L^p(\mathbb{R}^2)$-bound for the inertial force implies now the Hölder continuity of ω and hence of the velocity gradient. In the past decade, a large number of works addressed the well-posedness for the two-dimensional incompressible inhomogeneous Navier-Stokes equations with constant viscosity coefficient, see for instance [51, 59, 116, 199, 249]. The propagation of the density-interface regularity in the density-patch problem is also established, see [71, 100, 166, 167].

However, if the viscosity coefficient is variable, especially when it admits discontinuities, the vorticity ω becomes discontinuous. This requires revisiting the structure of the model in order to identify its *effective replacements*. The two replacements introduced in [168] arise naturally from the mathematical and physical perspectives:

- "Nonlocal vorticity" a.

  Mathematically, it is natural to apply the Helmholtz decomposition to the vector field $\operatorname{div}(\mu(\rho)Su) \in \mathbb{R}^2$

  $$\operatorname{div}(\mu(\rho)Su) = \nabla^\perp a + \nabla\tilde{b}.$$

  The $L^p(\mathbb{R}^2)$-bound of the inertial force implies then the Hölder continuity of a.

  Structurally, a can be expressed in terms of ω via *nonlocal* operators. A crucial equivalence between a and the velocity gradient ∇u in $L^{2+\epsilon}(\mathbb{R}^2)$ for some $\epsilon > 0$ is established in Chapter 3.

- "Shear stress" α.

  Physically, it is natural to decompose the stress vector $\mu(\rho)Su\,n \in \mathbb{R}^2$ into its tangential and normal components, corresponding to the shear and normal stresses:

  $$\mu(\rho)Su\,n = \alpha\,\bar{\tau} + \tilde{\beta}\,n.$$

  One can choose any coordinate system $(\bar{\tau}, n)$ in $\mathbb{R}^2$, and in particular for density-patch-type problems, the tangential and normal vector fields of the freely transported density interface provide natural choices.

  As shown in Chapter 3, the shear stress α differs from a up to tangential vectors or tangential derivatives, and therefore inherits the Hölder regularity of a. This in turn implies the desired boundedness of the velocity gradient ∇u, thanks to the key observation that α and ∇u are related *pointwisely*.

The two-dimensional *compressible* Navier-Stokes equations (CNS) with *general* variable viscosities

In this chapter we consider the two-dimensional compressible model (CNS) with general variable viscosity coefficients, aiming to establish global-in-time well-posedness in the presence of large viscosity variations and density discontinuities. As in Chapter 3, the strategy is to introduce *effective replacements* for the classical vorticity-effective flux pair (ω, F) in the compressible setting.

We introduce our nonlocal and localized vorticity–effective flux pairs as follows:

- "Nonlocal vorticity–effective flux pair" (a, b).

  We apply the Helmholtz decomposition to the divergence of the Cauchy stress tensor $\operatorname{div} T \in \mathbb{R}^2$ on the right hand side of $(\mathrm{CNS})_2$ as

  $$\operatorname{div} T(u,\rho) = \operatorname{div}(\mu(\rho)Su) + \nabla(\lambda(\rho)\operatorname{div} u) - \nabla(P(\rho) - P(\tilde{\rho})) = \nabla^\perp a + \nabla b.$$

  The pair (a, b) is expected to be Hölder continuous.

  Structurally, the pair $(a, b + P(\rho) - P(\tilde{\rho}))$ is related to the velocity gradient ∇u via *nonlocal* operators. This is expected to yield an $L^{2+\epsilon}(\mathbb{R}^2)$-bound for ∇u. The presence of the pressure fluctuation makes the analysis of the compressible model more involved than the incompressible model.

- "Shear–normal stress pair" (α, β).

  The stress vector $Tn \in \mathbb{R}^2$ is decomposed along tangential and normal directions

  $$T(u,\rho)n = \alpha\bar{\tau} + \beta n,$$

in a flow-driven coordinate system $(\bar{\tau}, n)$. The shear-normal stress pair (α, β) is expected to differ from (a, b) up to tangential regularity terms and therefore to be Hölder continuous. Structurally, ∇u can be represented *pointwisely* in terms of $(\alpha, \beta + P(\rho) - P(\tilde{\rho}))$, up to tangential derivatives. This observation plays a crucial role in establishing the Lipschitz boundedness of the velocity field.

We emphasize that, in order to handle the pressure fluctuations, we introduce further in this chapter

- a *nonlocal* representation of $\operatorname{div} u$ in terms of $b + P(\rho) - P(\tilde{\rho})$, up to terms involving a,
- a *pointwise* representation of $\operatorname{div} u$ in terms of $\beta + P(\rho) - P(\tilde{\rho})$, up to tangential regularity terms,

such that the damping effect in equations for the density function can be quantified in the $L^{2+\epsilon}(\mathbb{R}^2)$-setting. The details of the decompositions as well as the explicit nonlocal and pointwise relations are found in Section 4.1.3 below. We believe that this new development of *Helmholtz* and *shear-normal* decompositions is subtle and robust enough to be extended to the study of other fluid models with variable physical coefficients.

Thanks to these newly identified structures, we establish the global-in-time well-posedness of (CNS) with large viscosity fluctuations. This result improves upon [252], which is limited to small viscosity fluctuations. It also goes beyond the work [28], as it handles densities that are discontinuous across interfaces and allows for more general viscosity laws. Moreover, we succeed in propagating the tangential regularity of the density, thereby solving the density-patch-type problem for compressible fluids without vaccum.

It should be noted that, for the constant-viscosity model, one first establishes $L^p(\mathbb{R}^2)$-estimates for the inertial force before propagating density regularity or deriving the Lipschitz bound for the velocity. Unfortunately, in the presence of large variation in the viscosity coeffcients, we are not able to proceed in the same way. We are indeed led to combine

the $L^p(\mathbb{R}^2)$-bound for the inertia, the lower and upper bounds for the density function, the tangential regularity of the density, and the Lipschitz bound for the velocity field

simutaneously in a bootstrap argument. A well-known fact is that tangential regularity estimates typically depend *exponentially* on the time integral of the Lipschitz norm of the velocity field: $\exp(\|\nabla u\|_{L^1((0,t),L^\infty(\mathbb{R}^2;\mathbb{R}^4))})$. Without uniform-in-time control of this norm, one may encounter exponential-in-time growth, which might preclude a global existence result. Previous works [125, 252] relied on the exponential-in-time decay of the density jump to counterbalance this growth. In the present work, we establish a uniform-in-time $L^1((0,\infty), L^\infty(\mathbb{R}^2;\mathbb{R}^4))$-estimate for the velocity gradient ∇u, under low-frequency and smallness assumptions on the initial energy. In addition to the standard decay arguments known for the compressible model, a new insight of our approach is the derivation of a damping effect for the divergence of the velocity and for the tangential regularity of the density.

4.1.2. Main results

Recall the Cauchy problem (CNS)-(4.1.1).

We first quantify the boundedness assumption for the initial density as well as the assumptions for the constitutive laws in the model. Let ρ_*, ρ^* be two fixed positive constants satisfying

$\rho_* \le \tilde{\rho} \le \rho^*$. Assume that the initial density is bounded from above and below as

$$0 < \rho_* \le \rho_0(x) \le \rho^*, \quad \text{a.e.} \quad x \in \mathbb{R}^2. \tag{4.1.5}$$

Recall the viscosity coefficients $\mu(\rho), \lambda(\rho)$ and the pressure law $P(\rho)$ in (CNS). We can correspondingly define the constants

$$\mu_* = \inf_{s\in(\frac14\rho_*,4\rho^*)} \mu(s), \quad \mu^* = \sup_{s\in(\frac14\rho_*,4\rho^*)} \mu(s),$$
$$\pi_* = \inf_{s\in(\frac14\rho_*,4\rho^*)} sP'(s), \quad \tilde{\pi}_* = \inf_{s\in(\frac14\rho_*,4\rho^*)} P'(s).$$

Setting $\nu(\rho) = 2\mu(\rho) + \lambda(\rho)$, we define the constants

$$\nu_* = 2\mu_* + \inf_{s\in(\frac14\rho_*,4\rho^*)} \lambda(s), \quad \nu^* = 2\mu^* + \sup_{s\in(\frac14\rho_*,4\rho^*)} \lambda(s).$$

We assume that our fluids are strictly viscous in the sense that

$$0 < \mu_* \le \mu^* < \infty, \quad 0 < \nu_* \le \nu^* < \infty, \tag{4.1.6}$$

and that the pressure law is strictly increasing in the sense that

$$0 < \pi_*, \tilde{\pi}_* < \infty. \tag{4.1.7}$$

We aim to show $\rho(t,x) \in [\frac14\rho_*, 4\rho^*]$ globally in time, such that $\mu(\rho) \in [\mu_*, \mu^*]$ and $\nu(\rho) \in [\nu_*, \nu^*]$ hold for all times.

As discussed in Subsection 4.1.1 above, a key to represent the velocity gradient *pointwisely* is some flow-driven coordinate system $(\bar{\tau}, n)$ and the "tangential" regularity along the "tangential" vector field τ (with $\bar{\tau} = \frac{\tau}{|\tau|}$ as its normalization)[1]. Here the flow-driven vector field $\tau = \tau(t,x) \in \mathbb{R}^2$ should satisfy the following evolution equation[2]

$$\partial_t \tau + (u \cdot \nabla)\tau = (\tau \cdot \nabla)u. \tag{τ}$$

We assume initial tangential regularity, i.e. the regularity of the nondegenerate tangential vector field itself as well as of the tangential derivative of the density function

$$|\tau_0|, |\tau_0|^{-1} \in L^\infty(\mathbb{R}^2;\mathbb{R}), \quad \nabla\tau_0 \in L^{2+\epsilon}(\mathbb{R}^2;\mathbb{R}^{2\times2}), \quad \text{and} \quad \partial_{\tau_0}\rho_0 \in L^{2+\epsilon}(\mathbb{R}^2;\mathbb{R}), \tag{4.1.8}$$

for some $\epsilon > 0$. Here and in the following, $\partial_X = X \cdot \nabla$ denotes the directional derivative along some vector field $X \in \mathbb{R}^2$. We are thus led to consider the coupled system (CNS)-(τ), and we aim to show the boundedness of the velocity gradient and the propagation of the tangential regularity (4.1.8) simutaneously. Our main result reads as follows.

Theorem 4.1.1 (Global-in-time well-posedness of (CNS)-(τ)). *Let $\rho_* \le \tilde{\rho} \le \rho^*$ be three fixed positive constants, and let the model assumptions (4.1.6)-(4.1.7) hold. Then there exists a positive constant $\epsilon_0 \in (0,2]$, depending only on $\mu_*, \mu^*, \nu_*, \nu^*, \pi_*$, such that the following holds.*

[1]We refer to the relevant direction as the "tangential" direction; in fact, it can be any arbitrarily chosen regular direction. In the case of a free sharp density interface, it can be taken as the tangential vector to the interface, see Corollary 4.1.3–2 below. If the density is smooth, one may simply take the initial vector field τ_0 to be $(0,1)^T$ or $(1,0)^T$, see Corollary 4.1.3–3 below.

[2]It means that the material derivative $D_t = \partial_t + u \cdot \nabla$ and the tangential derivative $\partial_\tau = \tau \cdot \nabla$ commutes: $[D_t, \partial_\tau] = 0$.

For any $\epsilon \in (0, \epsilon_0]$ *and* $\delta \in (\frac{1}{2} - \frac{1}{4}\frac{\epsilon}{2+\epsilon}, \frac{1}{2})$, *there exists* $c > 0$ *depending on* $\rho_*, \rho^*, \mu_*, \mu^*, \nu_*, \nu^*, \pi_*$, $\tilde{\pi}_*$, $\|(\mu, \lambda, P)\|_{W^{2,\infty}(\frac{1}{4}\rho_*, 4\rho^*)}$, ϵ, δ, *such that if the initial data* (ρ_0, u_0, τ_0) *satisfy* (4.1.1)-(4.1.2)-(4.1.5)-(4.1.8),

$$(\rho_0 - \tilde{\rho}, \rho_0 u_0) \in \dot{H}^{-2\delta}(\mathbb{R}^2; \mathbb{R}^{1+2}) \tag{4.1.9}$$

and

$$\big(\|(\rho_0 - \tilde{\rho}, \rho_0 u_0)\|_{L^2 \cap \dot{H}^{-2\delta}(\mathbb{R}^2)} + \|\nabla u_0\|_{L^2(\mathbb{R}^2)}\big)\big(1 + \|(\nabla \bar{\tau}_0, \partial_{\bar{\tau}_0}\rho_0)\|_{L^{2+\epsilon}(\mathbb{R}^2)}\big) \leq c, \tag{4.1.10}$$

where $\bar{\tau}_0 = \frac{\tau_0}{|\tau_0|}$, *then the system* (CNS)-(τ) *has a unique global-in-time solution* (ρ, u, τ) *satisfying*

$$\begin{aligned}
&\rho \in [\frac{1}{4}\rho_*, 4\rho^*], \quad \rho - \tilde{\rho} \in \mathcal{C}_b([0,\infty); L^2(\mathbb{R}^2)),\\
&u \in \mathcal{C}_b([0,\infty); L^2(\mathbb{R}^2;\mathbb{R}^2)) \cap L^2((0,\infty); \dot{H}^1(\mathbb{R}^2;\mathbb{R}^2)),\\
&\nabla u \in L^\infty\big((0,\infty); L^2(\mathbb{R}^2;\mathbb{R}^{2\times2})\big) \cap L^1\big((0,\infty);\ (L^{2+\epsilon} \cap L^\infty)(\mathbb{R}^2;\mathbb{R}^{2\times2})\big),\\
&t^{\frac{3}{4}}\nabla u \in L^\infty\big((0,\infty); L^\infty(\mathbb{R}^2;\mathbb{R}^{2\times2})\big), \quad t^{\frac{1}{2}}\nabla u \in L^2\big((0,\infty); L^\infty(\mathbb{R}^2;\mathbb{R}^{2\times2})\big),\\
&\tau \in \mathcal{C}_b([0,\infty);\ (L^\infty \cap \dot{W}^{1,2+\epsilon})(\mathbb{R}^2;\mathbb{R}^2)), \quad |\tau|^{-1} \in L^\infty((0,\infty)\times\mathbb{R}^2),\\
&\partial_\tau \rho \in (L^1 \cap L^\infty)((0,\infty); L^{2+\epsilon}(\mathbb{R}^2)), \quad \partial_\tau \nabla u,\ \nabla\partial_\tau u \in L^1\big((0,\infty); L^{2+\epsilon}(\mathbb{R}^2;\mathbb{R}^{2\times2})\big).
\end{aligned} \tag{4.1.11}$$

To the best of our knowledge, this is the first global-in-time well-posedness result for the higher-dimensional compressible Navier–Stokes equations with such general density-dependent viscosity coefficients. Theorem 4.1.1 is proved in Section 4.3, with the proof strategy outlined in Section 4.2. The fundamental structural analysis of the model can be found in Subsection 4.1.3, and the detailed proofs of some auxiliary results are postponed to the appendix.

Remark 4.1.2. We comment here on the assumptions and results of Theorem 4.1.1.

- *Low-frequency assumption* (4.1.9). The low-frequency assumption (4.1.9) can be replaced simply by the stronger assumption $(\rho_0 - \tilde{\rho}, \rho_0 u_0) \in L^1(\mathbb{R}^2;\mathbb{R}^{1+2})$, by virtue of the embedding $(L^1 \cap L^2)(\mathbb{R}^2) \subset \dot{H}^{-2\delta}(\mathbb{R}^2)$, $\forall \delta \in (0, \frac{1}{2})$. It is the source of the algebraic time decay of the velocity field. Especially, the technical lower bound $\frac{1}{2} - \frac{1}{4}\frac{\epsilon}{2+\epsilon}$ for δ in terms of ϵ ensures the $L^1((0,\infty); L^{2+\epsilon}(\mathbb{R}^2))$-boundedness of some quantities such as $a, \operatorname{div} u, \nabla u$, which follow essentially from interpolation of energy norms (see Section 4.3).
- *Smallness condition* (4.1.10) *while general variable viscosity coefficients.* The smallness condition (4.1.10) specifies the dependence of the small $L^2(\mathbb{R}^2)$-based norms for the initial density-velocity pair $(\rho_0 - \tilde{\rho}, u_0)$ on the $L^{2+\epsilon}(\mathbb{R}^2)$-based tangential regularity norms for the initial density-tangent vector pair $(\bar{\tau}_0, \partial_{\bar{\tau}_0}\rho_0)$, up to universal constants in the fluid model. Technically, it is introduced to compensate the exponential-in-time growth $\exp(\|\nabla u\|_{L^1((0,\infty);L^\infty(\mathbb{R}^2;\mathbb{R}^4))})$ when propagating tangential regularity of the density function. The latter ensures finally the compactness of the approximating density sequence.

  We emphasize that (4.1.10) does not require any smallness of the $L^\infty(\mathbb{R}^2)$-norm of the density fluctuation $\rho_0 - \tilde{\rho}$, and thus, large variations in the variable viscosity coefficients are allowed. To the best of our knowledge, this is the first mathematical result devoted to such general viscous compressible fluid models.
- *Further uniform-in-time bounds.* Thanks to the $L^1((0,\infty); L^\infty(\mathbb{R}^2))$-bound of the velocity gradient and the estimates in Section 4.3, we have further global-in-time estimates such as

$$a,\ \nabla a,\ \nabla b,\ \nabla\alpha,\ \nabla\beta,\ \operatorname{div} u,\ \nabla u,\ \dot{u} \in L^1((0,\infty); L^{2+\epsilon}(\mathbb{R}^2)), \quad \nabla\dot{u} \in L^1((1,\infty); L^{2+\epsilon}(\mathbb{R}^2)).$$

As a consequence of Theorem 4.1.1 we have the following results for the system (CNS), which is proved in Subsection 4.3.5.3.

Corollary 4.1.3. *Let $\rho_*, \tilde\rho, \rho^*$ and ϵ, δ, c be as in Theorem 4.1.1, where the model assumptions* (4.1.6)-(4.1.7) *hold.*

1. *(Global-in-time well-posedness of* (CNS)*). If the initial data (ρ_0, u_0) satisfy* (4.1.1)-(4.1.2)-(4.1.5)-(4.1.9)*, as well as* (4.1.8)-(4.1.10) *for some vector field $\tau_0 \in \mathbb{R}^2$, then the system* (CNS) *admits a unique global-in-time solution (ρ, u) such that*

$$\begin{aligned}&\rho \in [\frac{1}{4}\rho_*, 4\rho^*], \quad \rho - \tilde\rho \in \mathcal{C}_b([0,\infty); L^2(\mathbb{R}^2)),\\ &u \in \mathcal{C}_b([0,\infty); L^2(\mathbb{R}^2;\mathbb{R}^2)) \cap L^2((0,\infty); \dot H^1(\mathbb{R}^2;\mathbb{R}^2)),\\ &\nabla u \in L^\infty((0,\infty); L^2(\mathbb{R}^2;\mathbb{R}^{2\times 2})) \cap L^1((0,\infty);\ (L^{2+\epsilon}\cap L^\infty)(\mathbb{R}^2;\mathbb{R}^{2\times 2})),\\ &t^{\frac34}\nabla u \in L^\infty((0,\infty); L^\infty(\mathbb{R}^2;\mathbb{R}^{2\times 2})), \quad t^{\frac12}\nabla u \in L^2((0,\infty); L^\infty(\mathbb{R}^2;\mathbb{R}^{2\times 2})).\end{aligned} \tag{4.1.12}$$

2. *(Density-patch-type problem for* (CNS)*). Let $\Omega_0 \subset \mathbb{R}^2$ be a simply connected bounded domain with $W^{2,2+\epsilon}$-boundary, and let the initial density be of the patch-type*

$$\rho_0(x) = \rho_0^+(x)1_{\Omega_0}(x) + \rho_0^-(x)1_{\Omega_0^c}(x) \in [\rho_*, \rho^*],$$

with $\rho_0^+ \in W^{1,2+\epsilon}(\Omega_0)$, $\rho_0^- - \tilde\rho \in (L^2 \cap W^{1,2+\epsilon})(\overline{\Omega_0}^c)$. Then there exists a nondegenerate vector field $\tau_0 \in \mathbb{R}^2$ tangential to the boundary $\partial\Omega_0$, such that (4.1.8) *holds.*

If the initial data (ρ_0, u_0) together with τ_0 satisfy further (4.1.1)-(4.1.9)-(4.1.10)*, then the compressible Navier-Stokes equations* (CNS) *have a unique solution (ρ, u) satisfying for all times $t > 0$,*

$$\rho(t,x) = \rho^+(t,x)1_{\Omega_t}(x) + \rho^-(t,x)1_{\Omega_t^c}(x),$$

where $\Omega_t \subset \mathbb{R}^2$ is a simply connected bounded domain with a $W^{2,2+\epsilon}$-boundary, and functions $\rho^+(t,\cdot) \in W^{1,2+\epsilon}(\Omega_t)$, $\rho^-(t,\cdot) - \tilde\rho \in L^2 \cap W^{1,2+\epsilon}(\overline{\Omega_t}^c)$.

3. *(*(CNS) *with regular density). Let (ρ_0, u_0) satisfy* (4.1.1)-(4.1.2)-(4.1.5)-(4.1.9)*. If furthermore the initial density satisfies $\rho_0 \in \dot W^{1,2+\epsilon}(\mathbb{R}^2)$ and the smallness condition*

$$\big(\|(\rho_0 - \tilde\rho, \rho_0 u_0)\|_{L^2\cap \dot H^{-2\delta}(\mathbb{R}^2)} + \|\nabla u_0\|_{L^2(\mathbb{R}^2)}\big)(1 + \|\rho_0\|_{\dot W^{1,2+\epsilon}(\mathbb{R}^2)}) \le c, \tag{4.1.13}$$

then the compressible Navier-Stokes equations (CNS) *have a unique global-in-time solution (ρ, u), which satisfies $\rho \in (L^1 \cap L^\infty)((0,\infty); \dot W^{1,2+\epsilon}(\mathbb{R}^2))$.*

Remark 4.1.4. We comment on the results in Corollary 4.1.3.

1. *(Uniqueness and tangential regularity propagation).* If the initial tangential regularity assumptions (4.1.8)-(4.1.10) are satisfied with respect to two different vector fields $\tau_0^{(1)}, \tau_0^{(2)}$, then the corresponding solutions $(\rho^{(1)}, u^{(1)}, \tau^{(1)})$ and $(\rho^{(2)}, u^{(2)}, \tau^{(2)})$ given by Theorem 4.1.1 coincide in the sense that $(\rho^{(1)}, u^{(1)}) = (\rho^{(2)}, u^{(2)})$.

   Notice also that given any solution (ρ, u) of (CNS) satisfying (4.1.12), the tangential regularity along any flow-driven nondegenerate regular vector field (4.1.8) is preserved (see Propositions 4.3.3 and 4.3.7 below).

2. *($W^{2,2+\epsilon}$-regularity propagation for density-patch problem).* In Corollary 4.1.3–2, we prove the propagation of $W^{2,2+\epsilon}(\mathbb{R}^2)$-regularity for density patches by the two-dimensional compressible Navier–Stokes equations (CNS) in the absence of vacuum and under the smallness assumption (4.1.10).

3. *(Explicit smallness condition for smooth initial data).* The smallness condition (4.1.13) in Corollary 4.1.3–3 provides an explicit dependence of $L^2(\mathbb{R}^2)$-based initial density-velocity norms on the $\dot{W}^{1,2+\epsilon}(\mathbb{R}^2)$-norm of ρ_0, which, to the best of our knowledge, is the first explicit relation between these two norms in the literature. We remark that the norm $\|\rho_0\|_{\dot{W}^{1,2+\epsilon}(\mathbb{R}^2)}$ can be replaced by $\|\partial_1\rho_0\|_{L^{2+\epsilon}(\mathbb{R}^2)}$ or by $\|\partial_2\rho_0\|_{L^{2+\epsilon}(\mathbb{R}^2)}$ or by $\|(\nabla\bar{\tau}_0, \partial_{\bar{\tau}_0}\rho_0)\|_{L^{2+\epsilon}(\mathbb{R}^2)}$ for any nondegenerate regular vector field τ_0.

Notation

Recall the notation conventions introduced at the beginning of Section 1.2 in Chapter 1. Additionally,

- Classical vorticity-effective flux pair (ω, F):

$$\omega = \operatorname{curl} u = \nabla^\perp \cdot u = \partial_1 u_2 - \partial_2 u_1, \tag{ω}$$

$$F = \nu(\rho)\operatorname{div} u - (P(\rho) - \tilde{P}). \tag{F}$$

  Here and in the following, given the equilibrium state of the density $\tilde{\rho}$, the equilibrium states of the pressure and viscosity coefficients are defined as $\tilde{\mu} = \mu(\tilde{\rho})$, $\tilde{\nu} = \nu(\tilde{\rho})$ and $\tilde{P} = P(\tilde{\rho})$.
- (Classical) energy:

$$\mathcal{E}(t) = \int_{\mathbb{R}^2} \Big(\rho\frac{|u|^2}{2} + H(\rho)\Big)(t,x)\,dx \tag{$\mathcal{E}$}$$

  where

$$H(\rho) = \rho\int_{\tilde{\rho}}^{\rho} \frac{P(s) - \tilde{P}}{s^2}\,\mathrm{ds}.$$

- Directional and material derivatives: For a function f we denote
  - $\partial_X f = X\cdot\nabla f$ as the directional derivative along the vector field $X \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$.
  - $\dot{f} = D_t f$ with $D_t = \partial_t + u\cdot\nabla$, as the material derivative associated to the flow with velocity field $u \in \mathbb{R}^2$.
- (Time-weighted) higher-order energies: With or without some appropriately defined time weight $\sigma = \sigma(t)$, by (time-weighted) $\dot{H}^1$-energy for u we mean

$$\|\sigma(t')\nabla u\|^2_{L^\infty_t L^2} + \|\sigma(t')\sqrt{\rho}\dot{u}\|^2_{L^2_t L^2},$$

  and by (time-weighted) L^2-energy for $\dot{u}$ we mean

$$\|\sigma(t')\sqrt{\rho}\dot{u}\|^2_{L^\infty_t L^2} + \|\sigma(t')\nabla\dot{u}\|^2_{L^2_t L^2},$$

  while by (time-weighted) $\dot{H}^1$-energy for $\dot{u}$ we mean

$$\|\sigma(t')\nabla\dot{u}\|^2_{L^\infty_t L^2} + \|\sigma(t')\sqrt{\rho}\ddot{u}\|^2_{L^2_t L^2}.$$

  Typical choices for the time weight $\sigma(t)$ could be

$$t, \quad \min\{1,t\}, \quad \langle t\rangle := e + t.$$

- Lipschitz bound of the velocity field u: $\|\nabla u\|_{L^1_t L^\infty}$.

- Tangential and normal vectors: Given the nondegenerate vector field $\tau \in \mathbb{R}^2$,

  its normalized version $\overline{\tau} = \dfrac{\tau}{|\tau|}$ is referred to as "tangential" vector field;

  and its orthogonal vector $n = -\overline{\tau}^{\perp} = \begin{pmatrix} \overline{\tau}_2 \\ -\overline{\tau}_1 \end{pmatrix}$ is referred to as "normal" vector field.

- Tangential regularity: By ρ-tangential regularity we mean

$$(\nabla\overline{\tau},\ \partial_{\overline{\tau}}\rho) \in L^\infty_t L^{2+\epsilon}, \tag{ρTR}$$

  and by u-tangential regularity we mean

$$(\partial_{\overline{\tau}}\rho, \partial_{\overline{\tau}}\nabla u,\ \nabla\partial_{\overline{\tau}}u) \in L^1_t L^{2+\epsilon}. \tag{uTR}$$

  We mention $\partial_{\overline{\tau}}\rho$ twice here, reflecting the fact that the density plays a dual role: as a "coefficient" through $\mu(\rho)$ and $\nu(\rho)$ (cf. (ρTR)), and as a "potential" through $P(\rho)$ (cf. (uTR)).

4.1.3. Helmholtz and shear-normal decompositions of the Cauchy stress

Recall the role played by the classical vorticity–effective flux pair in establishing the Lipschitz bound of the velocity u, which we mentioned in Section 4.1.1. In the *constant-viscosity* case when $\mu(\rho) = \tilde{\mu} > 0$ and $\nu(\rho) = \tilde{\nu} > 0$, we have the following Helmholtz decomposition for the divergence of the Cauchy stress tensor in terms of the vorticity (ω) and the effective flux (F):

$$\operatorname{div} T(\rho, u) = \operatorname{div}(\tilde{\mu} S u) + \nabla(\tilde{\lambda}\operatorname{div} u - P(\rho)) = \nabla^{\perp}(\tilde{\mu}\omega) + \nabla F. \tag{4.1.14}$$

When applying the curl operator $\nabla^{\perp}\cdot$ respectively the divergence operator div to the momentum equation $(\mathrm{CNS})_2$, one derives the two Poisson equations

$$\tilde{\mu}\Delta\omega = \nabla^{\perp}\cdot(\rho\dot{u}) \quad \text{respectively} \quad \Delta F = \operatorname{div}(\rho\dot{u}). \tag{4.1.15}$$

In the framework of D. Hoff [127], one has $\rho\dot{u} \in L^p(\mathbb{R}^2)$ for any $p \in [2, \infty)$ by use of some energy functionals (see (4.2.4) below), which implies, thanks to (4.1.15), that both ω and F are Hölder continuous for positive times, even if the initial data are discontinuous. From this, D. Hoff [125] deduces the Lipschitz bound for the velocity field by decomposing the velocity gradient as

$$\nabla u = \mathcal{R}\mathcal{R}^{\perp}\omega + \mathcal{R}\mathcal{R}\operatorname{div} u = \mathcal{R}\mathcal{R}^{\perp}\omega + \frac{1}{\tilde{\nu}}\mathcal{R}\mathcal{R}F + \frac{1}{\tilde{\nu}}\mathcal{R}\mathcal{R}(P(\rho) - \tilde{P}). \tag{4.1.16}$$

The first two terms in this decomposition are Hölder continuous. The third term is less regular. To prove its boundedness, additional regularity of the density must be assumed, such as tangential regularity [169] or piecewise Hölder continuity on both sides of a regular curve in the plane [125].

In this section we apply the Helmholtz and shear-normal decompositions to the Cauchy stress tensor (T), and introduce the *effective* nonlocal and localized replacements (a, b), (α, β) of the classical vorticity–effective flux pair (ω, F). This is the key to understand the structure of the Cauchy stress tensor in the presence of large variations in the viscosity coefficients. The time-dependence is largely ignored in the analysis here.

4.1.3.1. LEMMA 4.1.5: NONLOCAL VORTICITY–EFFECTIVE FLUX PAIR (a, b)

In the *variable-viscosity* case we introduce the following nonlocal vorticity–effective pair (a, b).

Lemma 4.1.5 (Helmholtz decomposition of $\operatorname{div} T$ in terms of the nonlocal vorticity–effective flux pair (a, b)). *Let $u = u(x) : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector-valued function, and let $\phi = \phi(x), \psi = \psi(x) : \mathbb{R}^2 \to \mathbb{R}$ be two scalar functions such that $u = \nabla^\perp \phi + \nabla \psi$. Let $\rho = \rho(x) : \mathbb{R}^2 \to \mathbb{R}$ be a scalar function, and let $\mu, \lambda, P \in W^{2,\infty}(\mathbb{R}; \mathbb{R})$ be regular scalar functions. Let $T \in \mathbb{R}^{2\times 2}$ be the matrix defined in* (T).

(1) *Then we have the following Helmholtz decomposition*

$$\operatorname{div} T(\rho, u) = \nabla^\perp a + \nabla b, \tag{4.1.17}$$

where the scalar functions a and b are related to ϕ, ψ and $P(\rho)$ as follows

$$\Delta a = \mathcal{L}_\mu \phi - \mathcal{J}_\mu \psi, \tag{4.1.18}$$

$$\Delta(b + P(\rho)) = \mathcal{L}_{\mu,\lambda} \psi + \mathcal{J}_\mu \phi, \tag{4.1.19}$$

with the operators $\mathcal{L}_\mu, \mathcal{J}_\mu, \mathcal{L}_{\mu,\lambda}$ defined by

$$\mathcal{L}_\mu = (\partial_{x_2x_2} - \partial_{x_1x_1})\mu(\rho)(\partial_{x_2x_2} - \partial_{x_1x_1}) + (2\partial_{x_1x_2})\mu(\rho)(2\partial_{x_1x_2}), \tag{4.1.20}$$

$$\mathcal{J}_\mu = (\partial_{x_2x_2} - \partial_{x_1x_1})\mu(\rho)(2\partial_{x_1x_2}) - (2\partial_{x_1x_2})\mu(\rho)(\partial_{x_2x_2} - \partial_{x_1x_1}), \tag{4.1.21}$$

$$\mathcal{L}_{\mu,\lambda} = \mathcal{L}_\mu + \Delta(\mu(\rho) + \lambda(\rho))\Delta. \tag{4.1.22}$$

Here $\mu(\rho)$ is understood as a multiplication operator by the function $(\mu(\rho))(x)$. Similarly for $\lambda(\rho)$. In the $L^2(\mathbb{R}^2)$-based functional setting, with positive and bounded coefficients $\mu(\rho)$ and $\nu(\rho)$, the operators $\mathcal{L}_\mu$ and $\mathcal{L}_{\mu,\lambda}$ are fourth-order elliptic operators and the operator $\mathcal{J}_\mu$ is antisymmetric.

(2) *Furthermore, if $T \in L^2(\mathbb{R}^2; \mathbb{R}^{2\times 2})$, then $a, b \in L^2(\mathbb{R}^2; \mathbb{R})$ can be determined by T as follows:*

$$a = -(-\Delta)^{-1}\nabla^\perp \cdot \operatorname{div} T = -(-\Delta)^{-1}(\nabla^\perp \otimes \nabla) : T = (\mathcal{R}^\perp \otimes \mathcal{R}) : T, \tag{4.1.23}$$

$$b = -(-\Delta)^{-1}\nabla \cdot \operatorname{div} T = -(-\Delta)^{-1}(\nabla \otimes \nabla) : T = (\mathcal{R} \otimes \mathcal{R}) : T, \tag{4.1.24}$$

where $\mathcal{R} = \frac{\frac{1}{i}\nabla}{\sqrt{-\Delta}}$ and $\mathcal{R}^\perp = \frac{\frac{1}{i}\nabla^\perp}{\sqrt{-\Delta}}$ are the Riesz operators.

(3) *If $\rho \in L^\infty(\mathbb{R}^2; \mathbb{R})$, $\rho - \tilde{\rho} \in L^2(\mathbb{R}^2; \mathbb{R})$ for some $\tilde{\rho} \in \mathbb{R}$ and $\nabla u \in L^2(\mathbb{R}^2; \mathbb{R}^{2\times 2})$, then a, b can be represented in terms of $\mu(\rho), \lambda(\rho), \omega, \operatorname{div} u, P(\rho) - \tilde{P}$ and Riesz operators in the $L^2(\mathbb{R}^2)$-functional setting as*

$$a = \mathcal{R}_\mu \omega - \mathcal{Q}_\mu \operatorname{div} u, \tag{4.1.25}$$

$$b = \mathcal{R}_{\mu,\lambda} \operatorname{div} u + \mathcal{Q}_\mu \omega - (P(\rho) - \tilde{P}), \tag{4.1.26}$$

with the operators $\mathcal{R}_\mu, \mathcal{Q}_\mu, \mathcal{R}_{\mu,\lambda}$ defined by

$$\mathcal{R}_\mu = (\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\mu(\rho)(\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1) + (2\mathcal{R}_1\mathcal{R}_2)\mu(\rho)(2\mathcal{R}_1\mathcal{R}_2), \tag{4.1.27}$$

$$\mathcal{Q}_\mu = (\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)\mu(\rho)(2\mathcal{R}_1\mathcal{R}_2) - (2\mathcal{R}_1\mathcal{R}_2)\mu(\rho)(\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1), \tag{4.1.28}$$

$$\mathcal{R}_{\mu,\lambda} = \mathcal{R}_\mu + \mu(\rho) + \lambda(\rho). \tag{4.1.29}$$

Here the scalar fluid vorticity and divergence are $\omega = \nabla^\perp \cdot u = \Delta\phi$ and $\operatorname{div} u = \nabla \cdot u = \Delta\psi$.

Proof. We calculate

$$\operatorname{div}(\mu(\rho)Su) = \begin{pmatrix} \partial_{x_1}(-2\mu(\rho)\partial_{x_1x_2}\phi) + \partial_{x_2}(\mu(\rho)(\partial_{x_1x_1}\phi - \partial_{x_2x_2}\phi)) \\ \partial_{x_1}(\mu(\rho)(\partial_{x_1x_1}\phi - \partial_{x_2x_2}\phi)) + \partial_{x_2}(2\mu(\rho)\partial_{x_1x_2}\phi) \end{pmatrix} + \begin{pmatrix} \partial_{x_1}(2\mu(\rho)\partial_{x_1x_1}\psi) + \partial_{x_2}(2\mu\partial_{x_1x_2}\psi) \\ \partial_{x_1}(2\mu(\rho)\partial_{x_1x_2}\psi) + \partial_{x_2}(2\mu(\rho)\partial_{x_2x_2}\psi) \end{pmatrix},$$

so that a direct computation yields

$$\Delta a = \nabla^\perp \cdot \operatorname{div} T(\rho,u) = \mathcal{L}_\mu\phi - \mathcal{J}_\mu\psi,$$
$$\Delta b = \operatorname{div}\operatorname{div} T(\rho,u) = \mathcal{J}_\mu\phi + \mathcal{L}_{\mu,\lambda}\psi - \Delta P(\rho),$$

from which we obtain (4.1.18)-(4.1.19). The ellipticity of the operator $\mathcal{L}_\mu$ with positive and bounded coefficient $\mu(\rho)$ is proved in [168, Lemma 1.1], and the ellipticity of $\mathcal{L}_{\mu,\lambda}$ with positive and bounded coefficient $\nu(\rho) = 2\mu(\rho) + \lambda(\rho)$ follows similarly. The representations (4.1.23)-(4.1.24)-(4.1.25)-(4.1.26) follow immediately from (4.1.18)-(4.1.19). □

Remark 4.1.6. 1. *(Constant-viscosity case).* It is easy to observe that if $\mu(\rho) = \tilde{\mu}$ is constant, then by the above definitions

$$\mathcal{R}_\mu = \tilde{\mu}, \quad \mathcal{Q}_\mu = 0, \quad \mathcal{R}_{\mu,\lambda} = 2\tilde{\mu} + \lambda(\rho),$$

are all local operators. Recalling the definitions of the vorticity–effective flux pair and the classical Helmholtz decomposition (4.1.14), we have

- $a = \tilde{\mu}\omega$ is the vorticity up to a multiplicative constant,
- $b = (2\tilde{\mu} + \lambda(\rho))\operatorname{div} u - (P(\rho) - \tilde{P}) = F$ is the effective flux of the compressible Navier-Stokes equations.

2. *(Incompressible or irrotational case).* We have indeed shown the following (formal) identities

- Case $\psi = 0$:

$$\operatorname{div}(\mu(\rho)S\nabla^\perp\phi) = \nabla^\perp\big(-(-\Delta)^{-1}\mathcal{L}_\mu\phi\big) + \nabla\big(-(-\Delta)^{-1}\mathcal{J}_\mu\phi\big), \qquad (4.1.30)$$

or equivalently,

$$\operatorname{div}(\mu(\rho)(\mathcal{R}\mathcal{R}^\perp + \mathcal{R}^\perp\mathcal{R})f) = \nabla^\perp(\mathcal{R}_\mu f) + \nabla(\mathcal{Q}_\mu f).$$

Under the incompressiblity condition $\operatorname{div} u = 0$, that is $u = \nabla^\perp\phi + \nabla\psi$ with $\psi = 0$, we have

$$a = \mathcal{R}_\mu\omega.$$

We introduced this "global good unknown" a in Chapter 3.

- Case $\phi = 0$:

$$\operatorname{div}(\mu(\rho)S\nabla\psi) = \nabla^\perp\big((-\Delta)^{-1}\mathcal{J}_\mu\psi\big) + \nabla\big(-(-\Delta)^{-1}\mathcal{L}_\mu\psi + \mu(\rho)\Delta\psi\big), \qquad (4.1.31)$$

or equivalently,

$$\operatorname{div}(2\mu(\rho)\mathcal{R}\mathcal{R}g) = \nabla^\perp\big(-\mathcal{Q}_\mu g\big) + \nabla\big((\mathcal{R}_\mu + \mu(\rho))g\big).$$

3. *(Reformulation of $\|\sqrt{\frac{\mu(\rho)}{2}}Sv\|_{L^2}^2$ in terms of $\mathcal{R}_\mu, \mathcal{Q}_\mu$ and $\operatorname{curl} v, \operatorname{div} v$).* For a general vector field $v \in H^1(\mathbb{R}^2;\mathbb{R}^2)$ we can write by use of $v = \nabla^\perp(-(-\Delta)^{-1}\operatorname{curl} v) + \nabla(-(-\Delta)^{-1}\operatorname{div} v)$, (4.1.30) and (4.1.31) that

$$\begin{aligned}
&\int_{\mathbb{R}^2} \frac{\mu(\rho)}{2}|Sv|^2\,dx = \langle \frac{1}{2}\mu(\rho)Sv, 2\nabla v\rangle = -\langle \operatorname{div}(\mu(\rho)Sv), v\rangle_{\dot H^{-1}\times \dot H^1}\\
&= \Big\langle \nabla^\perp\Big(\mathcal{R}_\mu \operatorname{curl} v - \mathcal{Q}_\mu \operatorname{div} v\Big) + \nabla\Big(\mathcal{Q}_\mu \operatorname{curl} v + \mathcal{R}_\mu \operatorname{div} v + \mu(\rho)\operatorname{div} v\Big),\\
&\qquad\qquad \nabla^\perp((-\Delta)^{-1}\operatorname{curl} v) + \nabla((-\Delta)^{-1}\operatorname{div} v)\Big\rangle_{\dot H^{-1}\times \dot H^1}\\
&= \Big\langle \mathcal{R}_\mu \operatorname{curl} v - \mathcal{Q}_\mu \operatorname{div} v, \operatorname{curl} v\Big\rangle + \Big\langle \mathcal{Q}_\mu \operatorname{curl} v + \mathcal{R}_\mu \operatorname{div} v + \mu(\rho)\operatorname{div} v, \operatorname{div} v\Big\rangle \qquad (4.1.32)\\
&= \langle \mathcal{R}_\mu \operatorname{curl} v, \operatorname{curl} v\rangle + \langle(\mathcal{R}_\mu + \mu(\rho))\operatorname{div} v, \operatorname{div} v\rangle + 2\langle \mathcal{Q}_\mu \operatorname{curl} v, \operatorname{div} v\rangle.
\end{aligned}$$

4. *(Relation to $\rho\dot u$.* By use of the density equation $(\mathrm{CNS})_1$, we can formulate the momentum equation $(\mathrm{CNS})_2$ as

$$\rho\dot u = \operatorname{div} T(\rho,u) = \nabla^\perp a + \nabla b. \qquad (4.1.33)$$

We derive similar Poisson equations (4.1.15) for the functions (a,b) when (ρ,u) is a solution of (CNS):

$$\begin{aligned}
\Delta a &= \nabla^\perp\cdot(\operatorname{div} T) = \nabla^\perp\cdot(\rho\dot u), &(4.1.34)\\
\Delta b &= \nabla\cdot(\operatorname{div} T) = \operatorname{div}(\rho\dot u). &(4.1.35)
\end{aligned}$$

4.1.3.2. Lemma 4.1.7: Estimates for nonlocal operators

It is straightforward to derive the following $L^p(\mathbb{R}^2)$-estimate from the decomposition (4.1.16) of ∇u in the *constant-viscosity* case:

$$\|\nabla u\|_{L^p(\mathbb{R}^2)} \le C(p,\tilde\nu)\|(\omega, F + P(\rho) - \tilde P)\|_{L^p(\mathbb{R}^2)}, \quad \forall p\in(1,\infty).$$

The situation becomes more delicate in the presence of variable viscosity coefficients. Recall from [168, Lemma 1.2] that for any two positive constants $d_* \le d^*$ there exists a positive number $\bar\epsilon_0 > 0$ depending only on d_*, d^* such that for any positive bounded function $d = d(x) : \mathbb{R}^2 \to [d_*, d^*]$, the operator $\mathcal{R}_d$ is an isomorphism in $L^{2+\epsilon}(\mathbb{R}^2)$ for any $\epsilon\in[0,\bar\epsilon_0]$, with operator bounds

$$\|(\mathcal{R}_d, \mathcal{R}_d^{-1})\|_{L^{2+\epsilon}(\mathbb{R}^2)\to L^{2+\epsilon}(\mathbb{R}^2)} \le C(d_*, d^*), \quad \forall\epsilon\in[0,\bar\epsilon_0].$$

Above, the operator $\mathcal{R}_d = (\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)d(\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1) + (2\mathcal{R}_1\mathcal{R}_2)d(2\mathcal{R}_1\mathcal{R}_2)$ is defined as in (4.1.27). We can apply this estimate directly with $d = \mu(\rho)$ to (4.1.25):

$$\mathcal{R}_\mu\omega = a + \mathcal{Q}_\mu \operatorname{div} u. \qquad (4.1.36)$$

The resulting bound, together with the boundedness of the Riesz operator on $L^p(\mathbb{R}^2)$ for $p\in(1,\infty)$, implies

$$\|\omega\|_{L^{2+\epsilon}(\mathbb{R}^2)} \le C(\mu_*, \mu^*)\|(a, \operatorname{div} u)\|_{L^{2+\epsilon}(\mathbb{R}^2)}, \quad \forall\epsilon\in[0,\bar\epsilon_0],$$

and hence the same bound holds for $\|\nabla u\|_{L^{2+\epsilon}(\mathbb{R}^2)}$, up to a constant (see (4.1.37) below).

Furthermore, we establish an $L^{2+\epsilon}(\mathbb{R}^2)$-estimate for ∇u in terms of $(a, b + P(\rho) - \tilde P)$ in (4.1.38) below. We also prove some commutator estimates (see Part (II) below), which yield an $L^{2+\epsilon}(\mathbb{R}^2)$-estimate for $\partial_{\bar{\tau}}\nabla u$ in (4.1.39). The invertibility of certain nonlocal operators implies in particular the nonlocal representation of $\operatorname{div} u$ in terms of $(a, b+P(\rho)-\tilde{P})$ in (4.1.42), which finally leads to the damping effect in the density function via the operator $\mathcal{A}$ (see (4.1.43), (4.2.13) below).

Lemma 4.1.7 (Estimates for the nonlocal operators). *Assume the hypotheses of Lemma 4.1.5. Let $\mu_*, \mu^*, \nu_*, \nu^*, \pi_*$ be five positive constants with $\mu_* \le \mu^*$ and $2(\mu^*-\mu_*) \le \nu^* - \nu_*$, and let the functions in the Cauchy stress tensor* (T) *satisfy $\mu(\rho) \in L^\infty(\mathbb{R}^2; [\mu_*, \mu^*])$, $\lambda(\rho) \in L^\infty(\mathbb{R}^2; [\nu_* - 2\mu_*, \nu^* - 2\mu^*])$ and $\rho P'(\rho) \ge \pi_*$.*

(I) *There exists $\epsilon_0 > 0$ depending only on $\mu_*, \mu^*, \nu_*, \nu^*, \pi_*$, such that*

(1) *$\mathcal{R}_\mu$ defined in* (4.1.27) *and its inverse $\mathcal{R}_\mu^{-1}$ are bounded linear operators in $L^{2+\epsilon}(\mathbb{R}^2)$, $\forall \epsilon \in [0, \epsilon_0]$.*

In particular, the equality (4.1.36) *implies*

$$\|\nabla u\|_{L^{2+\epsilon}(\mathbb{R}^2)} \le C(\mu_*, \mu^*)\|(a, \operatorname{div} u)\|_{L^{2+\epsilon}(\mathbb{R}^2)}, \quad \forall \epsilon \in [0, \epsilon_0]. \tag{4.1.37}$$

(2) *Define $\mathcal{M} = \begin{pmatrix} \mathcal{R}_\mu & -\mathcal{Q}_\mu \\ \mathcal{Q}_\mu & \mathcal{R}_{\mu,\lambda} \end{pmatrix}$ with $\mathcal{Q}_\mu$, $\mathcal{R}_{\mu,\lambda}$ given in* (4.1.28), (4.1.29). *Then $\mathcal{M}$ and its inverse $\mathcal{M}^{-1}$ are bounded linear operators in $L^{2+\epsilon}(\mathbb{R}^2; \mathbb{R}^2)$, $\forall \epsilon \in [0, \epsilon_0]$.*

In particular, we have $\mathcal{M}\begin{pmatrix} \omega \\ \operatorname{div} u \end{pmatrix} = \begin{pmatrix} a \\ b + P(\rho) - \tilde{P} \end{pmatrix}$ and

$$\|\nabla u\|_{L^{2+\epsilon}(\mathbb{R}^2)} \le C(\mu_*, \mu^*, \nu_*, \nu^*)\|(a, b+P(\rho)-\tilde{P})\|_{L^{2+\epsilon}(\mathbb{R}^2)}, \ \forall \epsilon \in [0, \epsilon_0]. \tag{4.1.38}$$

Furthermore, for any vector field $\bar{\tau} \in \dot{W}^{1,2+\epsilon}(\mathbb{R}^2; \mathbb{R}^2)$ with $|\bar{\tau}| = 1$,

$$\begin{aligned} \|(\partial_{\bar{\tau}}\nabla u, \nabla\partial_{\bar{\tau}} u)\|_{L^{2+\epsilon}(\mathbb{R}^2)} &\le C(\|(\nabla a, \nabla b, \partial_{\bar{\tau}}\rho)\|_{L^{2+\epsilon}(\mathbb{R}^2)} \\ &\quad + \|(\nabla\bar{\tau}, \partial_{\bar{\tau}}\rho)\|_{L^{2+\epsilon}(\mathbb{R}^2)}\|(\nabla u, a, b+P(\rho)-\tilde{P})\|_{L^\infty(\mathbb{R}^2)}), \end{aligned} \tag{4.1.39}$$

where the constant C depends on $\mu_*, \mu^*, \nu_*, \nu^*, \|(\mu'(\rho), \lambda'(\rho))\|_{L^\infty(\mathbb{R}^2)}$.

(3) *Define $\mathcal{N} = \mathcal{R}_{\mu,\lambda} + \mathcal{Q}_\mu \mathcal{R}_\mu^{-1}\mathcal{Q}_\mu$. Then $\mathcal{N}$ and its inverse $\mathcal{N}^{-1}$ are bounded and invertible in $L^{2+\epsilon}(\mathbb{R}^2)$, $\forall \epsilon \in [0, \epsilon_0]$. In fact, $\mathcal{N}^{-1}$ is the projection of $\mathcal{M}^{-1}$ on the second component in the sense that*

$$\mathcal{N}^{-1} = (0 \quad 1) \cdot \mathcal{M}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{4.1.40}$$

Conversely, $\mathcal{M}^{-1}$ reads in terms of $\mathcal{N}^{-1}$ as

$$\mathcal{M}^{-1} = \begin{pmatrix} \mathcal{R}_\mu^{-1} + \mathcal{R}_\mu^{-1}\mathcal{Q}_\mu^*\mathcal{N}^{-1}\mathcal{Q}_\mu\mathcal{R}_\mu^{-1} & \mathcal{R}_\mu^{-1}\mathcal{Q}_\mu\mathcal{N}^{-1} \\ \mathcal{N}^{-1}\mathcal{Q}_\mu^*\mathcal{R}_\mu^{-1} & \mathcal{N}^{-1} \end{pmatrix} \text{ with } \mathcal{Q}_\mu^* = -\mathcal{Q}_\mu. \tag{4.1.41}$$

We can represent $\operatorname{div} u$ *as*

$$\operatorname{div} u = \mathcal{N}^{-1}(P(\rho) - \tilde{P}) + \mathcal{N}^{-1}(b - \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}a). \tag{4.1.42}$$

Furthermore, there exists $w > 0$ depending only on $\mu_*, \mu^*, \nu_*, \nu^*$, π_* such that the operator

$$\mathcal{A} = -\sqrt{\rho P'(\rho)}\mathcal{N}^{-1}\sqrt{\rho P'(\rho)} \tag{4.1.43}$$

is bounded and dissipative in $L^{2+\epsilon}(\mathbb{R}^2)$ in the sense that

$$\langle \mathcal{A}f, |f|^\epsilon f\rangle_{L^2(\mathbb{R}^2)} \le -w\|f\|_{L^{2+\epsilon}(\mathbb{R}^2)}^{2+\epsilon}, \quad \forall f \in L^{2+\epsilon}(\mathbb{R}^2), \quad \forall \epsilon \in [0, \epsilon_0]. \tag{4.1.44}$$

(II) *The Riesz operator $\mathcal{R}$ and the operators $\mathcal{R}_\mu, \mathcal{Q}_\mu$, and $\mathcal{M}$ are all bounded in $L^p(\mathbb{R}^2)$, $p \in (1,\infty)$. If $p, q \in (1,\infty)$ and $r \in [1,\infty]$ satisfy the relation $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$, then the following commutator estimates hold.*

(i) *For $f \in L^p(\mathbb{R}^2)$ and $X \in C_c^1(\mathbb{R}^2;\mathbb{R}^2)$ we have*

$$\|[\mathcal{R}^2, \partial_X]f\|_{L^q(\mathbb{R}^2)} \le C(q,p,r)\|\nabla X\|_{L^r(\mathbb{R}^2)}\|f\|_{L^p(\mathbb{R}^2)}. \tag{4.1.45}$$

(ii) *If additionally $\partial_X \rho \in L^q(\mathbb{R}^2)$ and $q > 2$, then for $f, g \in C_c^1(\mathbb{R}^2)$ and $v, X \in C_c^1(\mathbb{R}^2;\mathbb{R}^2)$,*

$$\|\partial_X \nabla v\|_{L^q(\mathbb{R}^2)} \le C(q)\big(\|\partial_X(\operatorname{curl} v, \operatorname{div} v)\|_{L^q(\mathbb{R}^2)} + \|\nabla X\|_{L^q}\|\nabla v\|_{L^\infty(\mathbb{R}^2)}\big) \tag{4.1.46}$$

$$\begin{aligned}&\|[\mathcal{R}_\mu, \partial_X]f - [\mathcal{Q}_\mu, \partial_X]g\|_{L^q(\mathbb{R}^2)} \qquad (4.1.47)\\ &\le C\|(\nabla X, \partial_X\rho)\|_{L^q(\mathbb{R}^2)}\|(f, g, \mathcal{P}_1 f - \mathcal{P}_2 g, \mathcal{P}_2 f + \mathcal{P}_1 g, \mathcal{R}_\mu f - \mathcal{Q}_\mu g)\|_{L^\infty(\mathbb{R}^2)},\end{aligned}$$

$$\begin{aligned}&\|[\mathcal{Q}_\mu, \partial_X]f + [\mathcal{R}_\mu, \partial_X]g\|_{L^q(\mathbb{R}^2)} \qquad (4.1.48)\\ &\le C\|(\nabla X, \partial_X\rho)\|_{L^q(\mathbb{R}^2)}\|(f, g, \mathcal{P}_1 f - \mathcal{P}_2 g, \mathcal{P}_2 f + \mathcal{P}_1 g, \mathcal{Q}_\mu f + \mathcal{R}_\mu g)\|_{L^\infty(\mathbb{R}^2)},\end{aligned}$$

$$\begin{aligned}&\left\|[\mathcal{M}, \partial_X]\begin{pmatrix} f \\ g\end{pmatrix}\right\|_{L^q(\mathbb{R}^2)} \qquad (4.1.49)\\ &\le C\|(\nabla X, \partial_X\rho)\|_{L^q(\mathbb{R}^2)}\left\|\left(f, g, \mathcal{P}_1 f - \mathcal{P}_2 g, \mathcal{P}_2 f + \mathcal{P}_1 g, \mathcal{M}\begin{pmatrix} f \\ g\end{pmatrix}\right)\right\|_{L^\infty(\mathbb{R}^2)},\end{aligned}$$

where $\mathcal{P}_1 = \mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1$, $\mathcal{P}_2 = 2\mathcal{R}_1\mathcal{R}_2$. The constant C in (4.1.47), (4.1.48) *and* (4.1.49) *depends only on $q, \mu_*, \mu^*, \nu_*, \nu^*, \|(\mu'(\rho), \lambda'(\rho))\|_{L^\infty(\mathbb{R}^2)}$.*

The proof of Lemma 4.1.7 can be found in Appendix 4.A. Let us give some remarks.

Remark 4.1.8. 1. *(Formal calculation involving $\mathcal{M}^{-1}$ and $\mathcal{N}^{-1}$).* We derive (formally) the representation (4.1.41) of $\mathcal{M}^{-1}$. Indeed, a straightforward computation by use of the definitions of $\mathcal{M}$ and $\mathcal{N}$ leads to

$$\begin{pmatrix} f \\ g\end{pmatrix} = \mathcal{M}^{-1}\begin{pmatrix} F \\ G\end{pmatrix} = \begin{pmatrix} \mathcal{R}_\mu^{-1}F + \mathcal{R}_\mu^{-1}\mathcal{Q}_\mu g \\ \mathcal{N}^{-1}(G - \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}F)\end{pmatrix},$$

and in particular

- if $F = 0$, then
$$\begin{pmatrix} f \\ g\end{pmatrix} = \mathcal{M}^{-1}\begin{pmatrix} 0 \\ G\end{pmatrix} = \begin{pmatrix} \mathcal{R}_\mu^{-1}\mathcal{Q}_\mu g \\ \mathcal{N}^{-1}G\end{pmatrix},$$
which verifies the definition of $\mathcal{N}^{-1}$ in (4.1.40);
- if $\begin{pmatrix} F \\ G\end{pmatrix} = \begin{pmatrix} 0 \\ P(\rho) - \tilde{P}\end{pmatrix} + \begin{pmatrix} a \\ b\end{pmatrix}$, then
$$\begin{pmatrix} f \\ g\end{pmatrix} = \begin{pmatrix} \mathcal{R}_\mu^{-1}a + \mathcal{R}_\mu^{-1}\mathcal{Q}_\mu g \\ \mathcal{N}^{-1}(P(\rho) - \tilde{P}) + \mathcal{N}^{-1}(b - \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}a)\end{pmatrix},$$
which verifies the representation (4.1.42) for $\operatorname{div} u$ as well as the representation $\omega = \mathcal{R}_\mu^{-1}a + \mathcal{R}_\mu^{-1}\mathcal{Q}_\mu \operatorname{div} u$ from (4.1.36).

2. *(Heuristic explanation of Part (I) in the $L^2(\mathbb{R}^2)$-setting, i.e. $\epsilon = 0$).*

- (1): For $h \in L^2(\mathbb{R}^2)$ it is straightforward to compute the $L^2(\mathbb{R}^2)$ inner product

$$\begin{aligned}\langle \mathcal{R}_\mu h, h\rangle = \Big\langle \mu(\rho)(\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)h, (\mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1)h\Big\rangle \\ + \Big\langle \mu(\rho)(2\mathcal{R}_1\mathcal{R}_2)h, (2\mathcal{R}_1\mathcal{R}_2)h\Big\rangle \in [\mu_*, \mu^*]\|h\|^2_{L^2(\mathbb{R}^2)},\end{aligned}$$

  which implies the boundedness and invertibility of $\mathcal{R}_\mu$ in $L^2(\mathbb{R}^2)$.

- (2): For $\begin{pmatrix} f \\ g\end{pmatrix} \in L^2(\mathbb{R}^2;\mathbb{R}^2)$ we find by use of (4.1.32) that

$$\begin{aligned}\Big\langle \mathcal{M}\begin{pmatrix} f \\ g\end{pmatrix}, \begin{pmatrix} f \\ g\end{pmatrix}\Big\rangle &= \int_{\mathbb{R}^2} \frac{\mu(\rho)}{2}|Sv|^2 + \lambda(\rho)g^2\,dx \\ &\in [\min\{\mu_*, \nu_*\}, \max\{\mu^*, \nu^*\}]\Big\|\begin{pmatrix} f \\ g\end{pmatrix}\Big\|^2_{L^2(\mathbb{R}^2)},\end{aligned} \tag{4.1.50}$$

  where we have defined $v = \nabla^\perp(-(-\Delta)^{-1}f) + \nabla(-(-\Delta)^{-1}g) \in \dot{H}^1(\mathbb{R}^2;\mathbb{R}^2)$ such that $\operatorname{curl} v = f$, $\operatorname{div} v = g$, and hence $\frac{1}{2}\|Sv\|^2_{L^2} = \|f\|^2_{L^2} + 2\|g\|^2_{L^2}$. This leads to the boundedness and invertibility of $\mathcal{M}$ in $L^2(\mathbb{R}^2;\mathbb{R}^2)$.

  Correspondingly, for $g \in L^2(\mathbb{R}^2;\mathbb{R})$, let $f = \mathcal{R}_\mu^{-1}\mathcal{Q}_\mu g$, then $\mathcal{M}\begin{pmatrix} f \\ g\end{pmatrix} = \begin{pmatrix} 0 \\ \mathcal{N}g\end{pmatrix}$ and hence

$$\langle \mathcal{N}g, g\rangle = \Big\langle \mathcal{M}\begin{pmatrix} f \\ g\end{pmatrix}, \begin{pmatrix} f \\ g\end{pmatrix}\Big\rangle = \int_{\mathbb{R}^2} \frac{\mu(\rho)}{2}|Sv|^2 + \lambda(\rho)g^2\ dx \geq \nu_*\|g\|^2_{L^2(\mathbb{R}^2)}. \tag{4.1.51}$$

- (3): Given the boundedness of $\mathcal{M}^{-1}$ and hence $\mathcal{N}^{-1}$ in $L^2(\mathbb{R}^2)$, it is also straightforward to verify (4.1.44) when $\epsilon = 0$, since (4.1.52) implies

$$\begin{aligned}\langle \mathcal{A}f, f\rangle &= -\langle \mathcal{N}^{-1}\sqrt{\rho P'(\rho)}f, \sqrt{\rho P'(\rho)}f\rangle \\ &\leq -\nu_*\|\mathcal{N}^{-1}\sqrt{\rho P'(\rho)}f\|^2_{L^2} \\ &\leq -c_*\|f\|^2_{L^2},\end{aligned} \tag{4.1.52}$$

  for some positive constant $c_* > 0$ depending only on $\mu_*, \mu^*, \nu_*, \nu^*, \pi_*$.

3. *(Remark on the application of Part (II)).* We will later use (4.1.48) and (4.1.49) with $(f, g) = (\omega, \operatorname{div} u)$, such that on the right hand side we have

$$\begin{aligned}&\mathcal{P}_1\omega - \mathcal{P}_2\operatorname{div} u = -\partial_2 u_1 - \partial_1 u_2, \quad \mathcal{P}_2\omega + \mathcal{P}_1\operatorname{div} u = \partial_2 u_2 - \partial_1 u_1, \\ &\mathcal{R}_\mu\omega - \mathcal{Q}_\mu\operatorname{div} u = a, \quad \mathcal{Q}_\mu\omega + \mathcal{R}_\mu\operatorname{div} u = b + P(\rho) - \tilde{P} - (\mu(\rho) + \lambda(\rho))\operatorname{div} u.\end{aligned} \tag{4.1.53}$$

4.1.3.3. Lemma 4.1.9: Shear–normal stress pair (α, β)

We shall make use of $(a, b, \partial_{\bar{\tau}}\rho)$ and the tangential regularity $(\nabla\bar{\tau}, \partial_{\bar{\tau}}\rho)$ to show the Lipschitz bound of the velocity field. We mention $\partial_{\bar{\tau}}\rho$ twice here, since the density function plays the role of "potential" via the pressure term $P(\rho)$ as well as of "coefficient" via the viscosity coefficients $\mu(\rho), \nu(\rho)$.

To this end, we rewrite ∇u in $(\bar{\tau}, n)$-coordinates *pointwisely* in the following lemma. It is in the spirit of [168, Lemma 1.7], and is proved in Appendix 4.B by long but straightforward computations.

Lemma 4.1.9 (Shear–normal decomposition of Tn and ∇u in terms of shear–normal stress pair (α,β)). *Assume the hypotheses of Lemma 4.1.5. Let $\tau = \begin{pmatrix}\tau_1\\ \tau_2\end{pmatrix}(x)$ be a vector field satisfying*

$$\tau \in L^\infty(\mathbb{R}^2;\mathbb{R}^2), \quad \nabla\tau \in L^q(\mathbb{R}^2;\mathbb{R}^{2\times 2}), \text{ for some } q\in(1,\infty), \quad \frac{1}{|\tau|}\in L^\infty(\mathbb{R}^2). \tag{4.1.54}$$

Denote the unit tangential and normal vectors $\bar\tau = \begin{pmatrix}\frac{\tau_1}{|\tau|}\\ \frac{\tau_2}{|\tau|}\end{pmatrix}$, $n = -\bar\tau^\perp = \begin{pmatrix}\bar\tau_2\\ -\bar\tau_1\end{pmatrix}$.

We have the following representations in the $(\bar\tau, n)$-coordinates:

(1) *We have the shear–normal decomposition of the stress vector Tn*

$$Tn = \alpha\bar\tau + \beta n, \tag{4.1.55}$$

where the shear–normal stress pair (α,β) is given by

$$\alpha = (\bar\tau\otimes n) : T, \quad \beta = (n\otimes n) : T.$$

By use of the expression (T)*, a straightforward computation yields the relationship between the vorticity–effective flux pair (ω, F) and its shear–normal reformulation version (α,β):*

$$\alpha = \mu(\rho)(\omega + 2n\cdot\partial_{\bar\tau}u), \tag{4.1.56}$$
$$\beta = F - 2\mu(\rho)\bar\tau\cdot\partial_{\bar\tau}u. \tag{4.1.57}$$

(2) *We represent the normal derivative $\partial_n u$ in terms of $\alpha, \beta + P(\rho) - \tilde P$ and the tangential derivative $\partial_{\bar\tau}u$ as*

$$\partial_n u = \bar\tau\frac{\alpha}{\mu(\rho)} + n\frac{\beta + P(\rho)-\tilde P}{\nu(\rho)} + 2\Big(\frac{\mu(\rho)}{\nu(\rho)}n\otimes\bar\tau - \bar\tau\otimes n\Big)\partial_{\bar\tau}u + \partial_{\bar\tau}u^\perp, \tag{4.1.58}$$

which leads to the following shear–normal decomposition of the velocity gradient

$$\begin{aligned}\nabla u = \Big(\bar\tau\frac{\alpha}{\mu(\rho)} + n\frac{\beta+P(\rho)-\tilde P}{\nu(\rho)}\Big)\otimes n + (\partial_{\bar\tau}u)^\perp\otimes n + \partial_{\bar\tau}u\otimes\bar\tau \\ + 2\Big[\Big(\frac{\mu(\rho)}{\nu(\rho)}n\otimes\bar\tau - \bar\tau\otimes n\Big)\cdot\partial_{\bar\tau}u\Big]\otimes n.\end{aligned} \tag{4.1.59}$$

In particular, we have the pointwise representation for $\operatorname{div} u$

$$\operatorname{div} u = \frac{\beta+P(\rho)-\tilde P}{\nu(\rho)} + \frac{2\mu(\rho)\bar\tau\cdot\partial_{\bar\tau}u}{\nu(\rho)}. \tag{4.1.60}$$

(3) *We represent (α,β) in terms of (a,b) up to tangential regularity terms as*

$$\begin{aligned}\nabla\alpha &= \nabla a + \mathcal{R}\mathcal{R}\cdot(\bar\tau\partial_{\bar\tau}\alpha)\\ &+ \mathcal{R}\mathcal{R}\cdot\Big(\bar\tau\big((\bar\tau_2^2-\bar\tau_1^2)\partial_{\bar\tau}\gamma_1 + 2\bar\tau_1\bar\tau_2\partial_{\bar\tau}\gamma_2\big) - 2n\big(2\bar\tau_1\bar\tau_2\partial_{\bar\tau}\gamma_1 - (\bar\tau_2^2-\bar\tau_1^2)\partial_{\bar\tau}\gamma_2\big)\Big)\\ &+ \mathcal{R}\mathcal{R}_1\big(-\partial_2(2\bar\tau_1\bar\tau_2)\,\gamma_1 + \partial_2(\bar\tau_2^2-\bar\tau_1^2)\,\gamma_2\big) - \mathcal{R}\mathcal{R}_2\big(-\partial_1(2\bar\tau_1\bar\tau_2)\,\gamma_1 + \partial_1(\bar\tau_2^2-\bar\tau_1^2)\,\gamma_2\big)\\ &+ \mathcal{R}\mathcal{R}\cdot\big(n\partial_n(\bar\tau_2^2-\bar\tau_1^2)\,\gamma_1 + n\partial_n(2\bar\tau_1\bar\tau_2)\,\gamma_2\big),\end{aligned} \tag{4.1.61}$$

$$
\begin{aligned}
\nabla\beta = \nabla b &+ \mathcal{R}\mathcal{R}\cdot\left(\bar\tau\partial_{\bar\tau}(-2\bar\tau_1\bar\tau_2\gamma_1 + (\bar\tau_2^2-\bar\tau_1^2)\gamma_2)\right)\\
&- \mathcal{R}\mathcal{R}\cdot\Big(\bar\tau(2\bar\tau_2\bar\tau_2\partial_{\bar\tau}\gamma_1 - (\bar\tau_2^2-\bar\tau_1^2)\partial_{\bar\tau}\gamma_2) + 2n\big((\bar\tau_2^2-\bar\tau_1^2)\partial_{\bar\tau}\gamma_1 + 2\bar\tau_1\bar\tau_2\partial_{\bar\tau}\gamma_2\big)\Big)\\
&- \mathcal{R}\mathcal{R}_1\Big(\partial_2(\bar\tau_2^2-\bar\tau_1^2)\,\gamma_1 + \partial_2(2\bar\tau_1\bar\tau_2)\,\gamma_2\Big) + \mathcal{R}\mathcal{R}_2\Big(\partial_1(\bar\tau_2^2-\bar\tau_1^2)\,\gamma_1 + \partial_1(2\bar\tau_1\bar\tau_2)\,\gamma_2\Big)\\
&+ \mathcal{R}\mathcal{R}\cdot\left(-n\partial_n(2\bar\tau_1\bar\tau_2)\,\gamma_1 + n\partial_n(\bar\tau_2^2-\bar\tau_1^2)\,\gamma_2\right),
\end{aligned}
\tag{4.1.62}
$$

and

$$
\alpha = a + (\bar\tau\otimes n - \mathcal{R}^\perp\otimes\mathcal{R}) : (\mu(\rho)Su), \tag{4.1.63}
$$

$$
\beta = b - \frac{1}{2}(\bar\tau\otimes\bar\tau - n\otimes n + \mathcal{R}\otimes\mathcal{R} - \mathcal{R}^\perp\otimes\mathcal{R}^\perp) : (\mu(\rho)Su). \tag{4.1.64}
$$

In the above, (γ_1,γ_2) *is given by*

$$
\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} = \begin{pmatrix}\mu(\rho)(\partial_{x_2}u_1 + \partial_{x_1}u_2)\\ \mu(\rho)(\partial_{x_1}u_1 - \partial_{x_2}u_2)\end{pmatrix}, \tag{4.1.65}
$$

such that $T(\rho,u) = \begin{pmatrix}\gamma_2 & \gamma_1\\ \gamma_1 & -\gamma_2\end{pmatrix} + ((\mu(\rho)+\lambda(\rho))\operatorname{div} u - (P(\rho)-\tilde P))\mathrm{Id}\,.$
Therefore, up to tangential regularity terms, $\|\nabla\alpha\|_{L^q(\mathbb{R}^2)}$ *and* $\|\nabla a\|_{L^q(\mathbb{R}^2)}$ *resp.* $\|\nabla\beta\|_{L^q(\mathbb{R}^2)}$ *and* $\|\nabla b\|_{L^q(\mathbb{R}^2)}$ *are equivalent in the following sense*

$$
\|\nabla\alpha - \nabla a\|_{L^q(\mathbb{R}^2)} \le C_*\big(\|(\nabla\bar\tau,\partial_{\bar\tau}\rho)\|_{L^q(\mathbb{R}^2)}\|\nabla u\|_{L^\infty(\mathbb{R}^2)} + \|\partial_{\bar\tau}\nabla u\|_{L^q(\mathbb{R}^2)}\big), \tag{4.1.66}
$$

$$
\|\nabla\beta - \nabla b\|_{L^q(\mathbb{R}^2)} \le C_*\big(\|(\nabla\bar\tau,\partial_{\bar\tau}\rho)\|_{L^q(\mathbb{R}^2)}\|\nabla u\|_{L^\infty(\mathbb{R}^2)} + \|\partial_{\bar\tau}\nabla u\|_{L^q(\mathbb{R}^2)}\big), \tag{4.1.67}
$$

where the constant C_* *depends only on* $q,\mu_*,\mu^*,\nu_*,\nu^*,\|(\mu'(\rho),\lambda'(\rho))\|_{L^\infty(\mathbb{R}^2)}$.

Remark 4.1.10 (Proof ideas for the preliminary Lipschitz bound via (α,β)). Our starting point for the Lipschitz bound for the velocity field is the *pointwise* decomposition (4.1.59), which directly implies

$$
\|\nabla u\|_{L^1_tL^\infty} \le C(\mu_*,\mu^*,\nu_*,\nu^*)\big\|(\alpha,\beta+P(\rho)-\tilde P,\partial_{\bar\tau}u)\big\|_{L^1_tL^\infty}. \tag{4.1.68}
$$

The shear–normal stress pair (α,β) is related to (a,b) by (4.1.63)-(4.1.64), which implies also the time-dependent version of (4.1.66)-(4.1.67) as

$$
\|\nabla\alpha - \nabla a\|_{L^1_tL^{2+\epsilon}} \le C_*\big(\|(\nabla\bar\tau,\partial_{\bar\tau}\rho)\|_{L^\infty_tL^{2+\epsilon}}\|\nabla u\|_{L^1_tL^\infty} + \|\partial_{\bar\tau}\nabla u\|_{L^1_tL^{2+\epsilon}}\big), \tag{4.1.69}
$$

$$
\|\nabla\beta - \nabla b\|_{L^1_tL^{2+\epsilon}} \le C_*\big(\|(\nabla\bar\tau,\partial_{\bar\tau}\rho)\|_{L^\infty_tL^{2+\epsilon}}\|\nabla u\|_{L^1_tL^\infty} + \|\partial_{\bar\tau}\nabla u\|_{L^1_tL^{2+\epsilon}}\big). \tag{4.1.70}
$$

This, together with (4.1.39), yields

$$
\begin{aligned}
&\|(\nabla\alpha,\nabla\beta,\partial_{\bar\tau}\nabla u,\nabla\partial_{\bar\tau}u)\|_{L^1_tL^{2+\epsilon}}\\
&\le C_*\big(\|(\nabla a,\nabla b,\partial_{\bar\tau}\rho)\|_{L^1_tL^{2+\epsilon}} + \|(\nabla\bar\tau,\partial_{\bar\tau}\rho)\|_{L^\infty_tL^{2+\epsilon}}\|(\nabla u,a,b+P(\rho)-\tilde P)\|_{L^1_tL^\infty}\big).
\end{aligned}
\tag{4.1.71}
$$

The bound for $\|(\alpha,\partial_{\bar\tau}u)\|_{L^1_tL^\infty}$ on the right hand side of (4.1.68) follows directly by interpolating between (4.1.71) and $\|\nabla u\|_{L^1_tL^{2+\epsilon}}$. The estimate for $\|\beta+P(\rho)-\tilde P\|_{L^\infty(\mathbb{R}^2)}$ is more delicate, and we have to consider low and high frequencies separately and make use of the damping effect of the density function. Details are found in the proof of Proposition 4.3.8 below.

Remark 4.1.11 (Representations of $\operatorname{div} u$). We summarize several representations for $\operatorname{div} u$, which are used to establish various estimates in this chapter.

1. *(Equivalence of* $\operatorname{div} u$ *and* $b+P(\rho)-\tilde{P}$ *in* $L^1_t L^{2+\epsilon}$ *up to energy perturbation).* We can rewrite the expression of $b+P(\rho)-\tilde{P}$ given in (4.1.26) as

$$b+P(\rho)-\tilde{P}=\mathcal{R}_{\mu,\lambda}\operatorname{div} u+\mathcal{Q}_\mu\omega=\mathcal{R}_\mu\operatorname{div} u+(\mu(\rho)+\lambda(\rho))\operatorname{div} u+\mathcal{Q}_\mu\omega \tag{4.1.72}$$

$$=\nu(\rho)\operatorname{div} u+\Big(\mathcal{R}_{\mu-\tilde{\mu}}\operatorname{div} u-(\mu(\rho)-\tilde{\mu})\operatorname{div} u+\mathcal{Q}_{\mu-\tilde{\mu}}\omega\Big), \tag{4.1.73}$$

where we used the facts that $\mathcal{R}_{\tilde{\mu}}=\tilde{\mu}$ and $\mathcal{Q}_\mu=\mathcal{Q}_{\mu-\tilde{\mu}}$, and thus,

$$\operatorname{div} u=\frac{b+P(\rho)-\tilde{P}}{\nu(\rho)}-\frac{(\mathcal{R}_{\mu-\tilde{\mu}}-(\mu(\rho)-\tilde{\mu}))\operatorname{div} u+\mathcal{Q}_{\mu-\tilde{\mu}}\omega}{\nu(\rho)}. \tag{4.1.74}$$

Thus $\|b+P(\rho)-\tilde{P}\|_{L^1_t L^{2+\epsilon}}$ and $\|\operatorname{div} u\|_{L^1_t L^{2+\epsilon}}$ are equivalent, up to a perturbation, which can be controlled by the initial energy and the Lipschitz bound of the velocity field

$$\|\mu(\rho)-\tilde{\mu}\|_{L^\infty_t L^{2+\epsilon}}\|\nabla u\|_{L^1_t L^\infty}\le C_*\|\rho-\tilde{\rho}\|^{\frac{2}{2+\epsilon}}_{L^\infty_t L^2}\|\nabla u\|_{L^1_t L^\infty}\le C_* E_0^{\frac{1}{2+\epsilon}}\|\nabla u\|_{L^1_t L^\infty},$$

where the constant C_* depends only on $\rho_*,\rho^*,\mu_*,\mu^*$, $\|(\mu'(\rho),P'(\rho),P(\rho))\|_{L^\infty(\mathbb{R}^2)}$.

2. *((4.1.42): Nonlocal representation of* $\operatorname{div} u$*, leading to the decay estimate for density tangential regularity).* In general, whenever regularity is relevant, the above "perturbative" viewpoint is not useful anymore, as we do not assume any smallness of $\|\mu(\rho_0)-\tilde{\mu}\|_{L^\infty(\mathbb{R}^2)}$ in this chapter. For instance, when applying the tangential derivative $\partial_{\bar{\tau}}$ to (4.1.74), the term $\|\frac{1}{\nu(\rho)}\mathcal{R}_{\mu-\tilde{\mu}}\partial_{\bar{\tau}}(\operatorname{div} u)\|_{L^{2+\epsilon}(\mathbb{R}^2)}$ on the right hand side of (4.1.74) is not a perturbation of the "leading" term $\|\partial_{\bar{\tau}}(\operatorname{div} u)\|_{L^{2+\epsilon}(\mathbb{R}^2)}$, when $\|\mu(\rho)-\tilde{\mu}\|_{L^\infty(\mathbb{R}^2)}$ is not small.

   Instead, we make use of the nonlocal representation (4.1.42) to explore the damping effect for the density function in order to establish the tangential regularity (see Step III below). The representation (4.1.42) and the operator (4.1.43) are key to eliminating the smallness assumption on $\|\mu(\rho)-\tilde{\mu}\|_{L^\infty(\mathbb{R}^2)}$ and propagating regularity.

3. *((4.1.60): Pointwise representation of* $\operatorname{div} u$*, yielding uniform bounds).* The local counterpart of (4.1.42) is given by (4.1.60). It is particularly useful in the $L^\infty_{t,x}$-setting, see Propositions 4.3.8, 4.3.9 below.

4. *(($\dot{\mathrm{P}}$): Representation of* $\operatorname{div} u$ *in terms of* $\ddot{u}$*, leading to decay estimates for* $\operatorname{div} u$*).* In order to establish suitable decay estimates for $\operatorname{div} u$, we make use of the representation of $\operatorname{div} u$ by $\ddot{u}$ in ($\dot{\mathrm{P}}$) below, such that

$$\|\rho P'(\rho)\operatorname{div} u+(-\Delta)^{-1}\operatorname{div}(\rho\ddot{u})\|_{L^q(\mathbb{R}^2)}\le C_*\|(u\otimes\dot{u},\nabla u\otimes\nabla u,\nabla\dot{u})\|_{L^q(\mathbb{R}^2)},\quad q\in(1,\infty).$$

   This inequality implies that the bound for $\|\operatorname{div} u\|_{L^1_t L^{2+\epsilon}}$ can be derived from the $\dot{H}^1$-energy estimate for $\dot{u}$, see Proposition 4.3.6 below.

5. *(($\dot{\mathrm{P}}'$): Evolutionary equation for* $\operatorname{div} u$ *with damping effect, leading to further decay estimates for* $\operatorname{div} u$*).* We require a uniform-in-time bound for $\|t'^{\frac{3}{4}}\operatorname{div} u\|_{L^\infty_t L^{2+\epsilon}}$, which cannot be obtained solely from the time-weighted L^2_t-estimate for $\ddot{u}$ combined with ($\dot{\mathrm{P}}$). However, the "artificial" damping equation ($\dot{\mathrm{P}}'$) below for $\operatorname{div} u$ enables us to derive the desired time-weighted L^∞_t-estimate for $\operatorname{div} u$, see Proposition 4.3.6 below.

4.2. Outline of the proof strategy

Recall the definitions and notations from Section 4.1.

The main challenge in this chapter is to deal with variable viscosity coefficients $(\mu(\rho))(t,x)$ and $(\nu(\rho))(t,x)$, which may be discontinuous and exhibit large variations. The classical vorticity–effective flux pair (ω, F), which plays an important role in the analysis of (CNS) with constant viscosity coefficients in the literature, is not convenient here. In Subsection 4.1.3 we introduced

(a,b) and (α,β) as its nonlocal and localized replacements,

to establish the $L^{2+\epsilon}(\mathbb{R}^2)$-estimate (Step II below) and the $L^\infty(\mathbb{R}^2)$-estimate (Step IV below) of the velocity gradient.

In the constant-viscosity case, the damping effect for the density function, which leads to the necessary time-decay estimates, can be realized thanks to the pointwise linear conncection between $\operatorname{div} u$ and the monotonic function $P(\rho)$ via the effective flux (F). In the variable-viscosity case here, this straightforward representation of $\operatorname{div} u$ is not efficient anymore, and we make use of

(4.1.42) and (4.1.60), as the nonlocal and pointwise representations for $\operatorname{div} u$,

to establish the $L^{2+\epsilon}(\mathbb{R}^2)$-estimate for the tangential regularity (ρTR), (uTR) (Step III below) and the uniform $L^\infty(\mathbb{R}^2)$-bounds for ρ (Step IV below).

We shall establish the desired Lipschitz bound for the velocity field u in the following four steps.

$L^2(\mathbb{R}^2)$ setting: Step I Energy estimates

$\downarrow$

$L^{2+\epsilon}(\mathbb{R}^2)$ setting: Step II $\|\nabla u\|_{L^1_t L^{2+\epsilon}}$ by use of (a,b) & Step III (ρTR), (uTR) by use of $\mathcal{A}$

$\downarrow$

$L^\infty(\mathbb{R}^2)$ setting: Step IV $\|\nabla u\|_{L^1_t L^\infty}$ by use of (α,β).

More precisely, we can summarize each step as follows.

Step I Establish time-decay estimates by use of the low-frequency assumption $(\rho_0 - \tilde{\rho}, \rho_0 u_0) \in (\dot{H}^{-2\delta} \cap L^2)(\mathbb{R}^2)$, as well as higher order energy estimates for u and $\dot{u}$ in terms of the Lipschitz norm of u.

Since the momentum equation can be formulated as (4.1.33):

$$\rho\dot{u} = \operatorname{div} T(\rho, u) = \nabla^\perp a + \nabla b, \tag{CNS$_{ab}$}$$

the $L^p(\mathbb{R}^2)$-estimates for $(\nabla a, \nabla b)$, $p \in [2,\infty)$, follow from the energy estimates for $\dot{u}$ by the Helmholtz decomposition and interpolation.

Step II Establish the estimate for $\|\nabla u\|_{L^1_t L^{2+\epsilon}}$ in terms of $\|a\|_{L^1_t L^{2+\epsilon}}$ and $\|\operatorname{div} u\|_{L^1_t L^{2+\epsilon}}$.

The time-decay estimate for $\|a\|_{L^{2+\epsilon}(\mathbb{R}^2)}$ follows from a further study of the incompressible counterpart of the energy estimates. In order to achieve sufficient time decay of $\|\operatorname{div} u\|_{L^{2+\epsilon}(\mathbb{R}^2)}$, we have to further explore the "low-high" frequency connection provided by the system (CNS) itself (see (P), ($\dot{\text{P}}$), ($\dot{\text{P}}'$) below).

Step III Establish (ρTR), (uTR) and, in particular, the improved time-integrability $\partial_{\bar{\tau}}\rho \in L^1_t L^{2+\epsilon}$, by use of the damping effect via the *nonlocal* representation (4.1.42):

$$\operatorname{div} u = \mathcal{N}^{-1}(P(\rho) - \tilde{P}) + \mathcal{N}^{-1}(b - \mathcal{Q}_\mu \mathcal{R}_\mu^{-1} a). \tag{4.2.1}$$

Here, $\mathcal{N} = \mathcal{R}_{\mu,\lambda} + \mathcal{Q}_\mu \mathcal{R}_\mu^{-1} \mathcal{Q}_\mu$ was introduced in Lemma 4.1.7, and the associated dissipative operator (4.1.43):

$$\mathcal{A} = -\sqrt{\rho P'(\rho)} \mathcal{N}^{-1} \sqrt{\rho P'(\rho)} \tag{$\mathcal{A}$}$$

makes the damping effect transparent (see (4.2.13) below).

Step IV Close the Lipschitz estimate for u and conclude the global-in-time bounds with a bootstrap argument under a suitable small energy assumption. Here we employ the shear–normal stress pair (α, β) defined by

$$T(\rho, u)n = \alpha\bar{\tau} + \beta n,$$

and the fact that ∇u can be *pointwisely* represented by (α, β). Recall that (α, β) is at the same time nonlocally related to (a, b) up to tangential regularity as

$$\nabla(\alpha, \beta) = \nabla(a, b) + \mathcal{R}\mathcal{R} \cdot (\text{Tangential Regularity Terms (TRT)}),$$

where $\|(TRT)\|_{L^1_t L^{2+\epsilon}}$ can be bounded by (ρTR), (uTR) up to the Lipschitz norm of u (see Remark 4.1.10).

In the following we outline Step I - Step IV and the proof sketch of Theorem 4.1.1 in more detail. The rigorous proofs can be found in Section 4.3. As we shall use a bootstrap argument to establish the global-in-time estimates in Step IV, we assume a priori in Step I - Step III that

$$\rho \in [\frac{1}{4}\rho_*, 4\rho^*], \quad \nabla u \in L^1((0,\infty); L^\infty(\mathbb{R}^2)). \tag{4.2.2}$$

Hence, the viscosity coeffcients and pressure satisfy a priori

$$\mu(\rho) \in [\mu_*, \mu^*], \quad \nu(\rho) \in [\nu_*, \nu^*], \quad \rho P'(\rho) \geq \pi_*, \quad P'(\rho) \geq \tilde{\pi}_*,$$

with the positive constants $\mu_*, \mu^*, \nu_*, \nu^*, \pi_*, \tilde{\pi}_*$ defined in Subsection 4.1.2. Correspondingly, the positive constants ϵ_0, C, w given in Part (I) of Lemma 4.1.7 are all fixed positive constants. Next, for any fixed $\epsilon \in (0, \epsilon_0]$, we take $q = 2 + \epsilon$, such that the constants in Part (II) of Lemma 4.1.7 and in Lemma 4.1.9 are all fixed positive constants.

The proof strategy of the Lipschitz bound and the a priori estimates are illustrated in Figure 4.1.

4.2.1. Step I Energy estimates

It is well-known that smooth and fast decaying solutions to the Navier–Stokes equations (CNS) satisfy the energy inequality (4.1.3):

$$\mathcal{E}(t) \leq \mathcal{E}(0) = \int_{\mathbb{R}^2} \Big(\rho_0 \frac{|u_0|^2}{2} + H(\rho_0)\Big)(x)dx, \quad \forall t > 0, \tag{4.2.3}$$

where the energy functional is given by $(\mathcal{E})$.

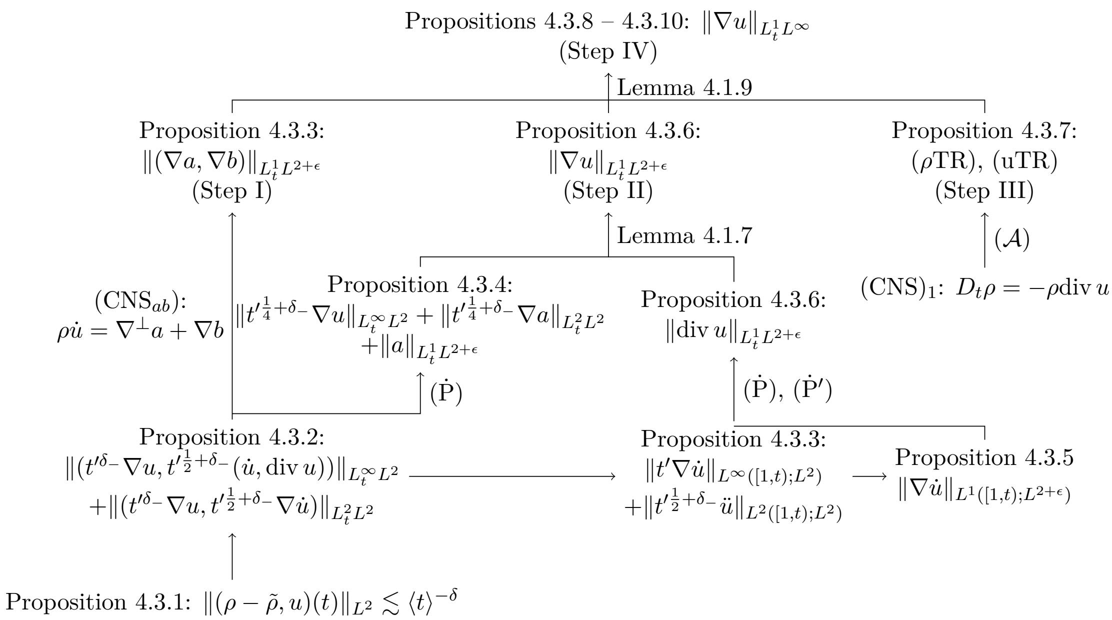

Figure 4.1.: Outline of the Lipschitz bound proof

4.2.1.1. Proposition 4.3.1: Time-decay energy estimates

A large number of works have been dedicated to investigating the optimal time decay rates of global smooth solutions to (CNS). For classical solutions whose initial data are small in $L^1 \cap H^3(\mathbb{R}^3)$, A. Matsumura and T. Nishida [186] first obtained the optimal decay rates of the $L^2(\mathbb{R}^3)$-norm in three space dimensions

$$\|(\rho-\tilde{\rho}, u)(t)\|_{L^2(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{4}}.$$

The general $L^p(\mathbb{R}^d)$-decay rates in the constant viscosity case were established later by G. Ponce [206]

$$\|\nabla^k(\rho-\tilde{\rho}, u)(t)\|_{L^p(\mathbb{R}^d)} \leq C(1+t)^{-\frac{d}{2}(1-\frac{1}{p})-\frac{k}{2}},$$

for $p \in [2,\infty], k = 0,1,2$ and dimensions $d = 2,3$. Further investigations of the decay rates have been made in e.g. [81, 82, 114, 131, 152, 164, 178] for constant viscosity, and in [66, 67] for density-dependent viscosity. The optimal decay rates in the critical L^p-framework were established in [63, 245, 246]. All the aforementioned works deal with regular solutions where the density is at least continuous. The decay rates for low-regularity weak solutions of (CNS) which allow for discontinuous densities were established by X. Hu and G. Wu [135] in the *constant viscosity* case and in three spatial dimensions.

We aim to show a similar time-decay as in [135] for the case with variable viscosity coefficients. More precisely, under the low frequency assumption on the initial data

$$(\rho_0-\tilde{\rho}, \rho_0 u_0) \in \dot{H}^{-2\delta}(\mathbb{R}^2;\mathbb{R}^{1+2}) \quad \text{for some } \delta \in (0, \frac{1}{2}),$$

we are able to prove the following decay rate for (CNS)

$$\|(\rho-\tilde{\rho}, u)(t)\|_{L^2(\mathbb{R}^2)} \leq C(1+t)^{-\delta_-},$$

where δ_- denotes any positive number strictly smaller than δ and C depends only on the initial norm $\|(\rho_0-\tilde{\rho}, \rho_0 u_0)\|_{L^2\cap\dot{H}^{-2\delta}(\mathbb{R}^2;\mathbb{R}^{1+2})}$ and $\rho_*, \rho^*, \mu_*, \mu^*, \nu_*, \nu^*, \delta_-, \delta$ (see Proposition 4.3.1 below).

4.2.1.2. Proposition 4.3.2: Higher-order energy estimates

In his pioneer work [127] for the *constant-viscosity* case, D. Hoff introduced the energy functionals

$$\begin{aligned} A_1(t) &= \sup_{[0,t]} \sigma \|\nabla u\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \sigma \|\sqrt{\rho}\dot{u}\|^2_{L^2(\mathbb{R}^2)} \mathrm{d}t', \\ A_2(t) &= \sup_{[0,t]} \sigma^2 \|\sqrt{\rho}\dot{u}\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \sigma^2 \|\nabla \dot{u}\|^2_{L^2(\mathbb{R}^2)} \mathrm{d}t', \end{aligned} \tag{4.2.4}$$

with the material derivative of the velocity $\dot{u} = (\partial_t + u \cdot \nabla)u$ and the time weight $\sigma(t) = \min\{1, t\}$. Notice that these functionals appear naturally by taking the L^2-inner product between the momentum equation $(\mathrm{CNS})_2$ and $\dot{u}$ resp. $\ddot{u} = (\partial_t + u \cdot \nabla)\dot{u}$. He establishes bounds for the energy functionals (4.2.4) provided that the L^2-norm of the initial velocity is small and the density is bounded away from zero and from above, along with some technical assumptions. With the boundedness of these energy functionals one in particular has $\rho\dot{u} \in L^p(\mathbb{R}^2)$ for any $p \in [2, \infty)$.

Here in the *variable-viscosity* case, provided with initial data $u_0 \in H^1(\mathbb{R}^2)$, we can derive similar bounds on the following time-weighted higher order energy functionals (see Proposition 4.3.2 below)

$$\|(t'^{\delta_-} \nabla u, t'^{\frac{1}{2}+\delta_-} \sqrt{\rho}\dot{u}, t'^{\frac{1}{2}+\delta_-} \sqrt{\rho P'(\rho)} \mathrm{div}\, u)\|^2_{L^\infty_t L^2} + \|(t'^{\delta_-} \sqrt{\rho}\dot{u}, t'^{\frac{1}{2}+\delta_-} \nabla \dot{u})\|^2_{L^2_t L^2}. \tag{4.2.5}$$

We point out here that the bound depends on ρ_*, ρ^* and *the Lipschitz norm* of the velocity, which requires the bootstrap assumptions (4.2.2). Thus the energy estimates are not yet closed. The closing of the energy and Lipschitz estimates is performed later with a classical (but rather delicate) bootstrap argument in Step IV.

We can derive a $L^1_t L^{2+\epsilon}$-bound for $\dot{u}$ directly from these higher order energy estimates and Gagliardo-Nirenberg's inequality

$$\|\dot{u}\|_{L^1_t L^{2+\epsilon}} \le C \|\langle t' \rangle^{\delta_-} \dot{u}\|^{\frac{2}{2+\epsilon}}_{L^2_t L^2} \|t'^{\frac{1}{2}+\delta_-} \nabla \dot{u}\|^{\frac{\epsilon}{2+\epsilon}}_{L^2_t L^2}, \tag{4.2.6}$$

where we used (4.3.1) below. However, it is completely unclear whether this could lead to $\nabla u \in L^1_t L^\infty$ or even $\nabla u \in L^1_t L^p$ for some $p > 2$, due to the presence of variable viscosity coefficients. We establish an $L^p(\mathbb{R}^2)$-estimate for ∇u with $p = 2 + \epsilon$ in Step II below.

4.2.1.3. Proposition 4.3.3: Estimate for $\|(\nabla a, \nabla b)\|_{L^1_t L^{2+\epsilon}}$

With similar ideas as discussed in Subsection 4.2.1.2, we can derive higher order energy estimate for $\dot{u}$. We apply the material derivative D_t to the momentum equation $(\mathrm{CNS})_2$, and then take the L^2-inner product with $\ddot{u}$. This yields after long computations (see Proposition 4.3.3) a bound for[3]

$$\|t'^{\frac{1}{2}+\delta_-} \nabla \dot{u}\|^2_{L^\infty((1,t); L^2(\mathbb{R}^2))} + \left\|t'^{\frac{1}{2}+\delta_-} \sqrt{\rho}\ddot{u}\right\|^2_{L^2((1,t); L^2(\mathbb{R}^2))}, \tag{4.2.7}$$

and eventually by interpolation a bound for

$$\|(\nabla a, \nabla b)\|_{L^1((0,\infty); L^{2+\epsilon}(\mathbb{R}^2))}.$$

[3]We integrate over a time interval away from the initial time, as we do not assume higher regularity of the initial velocity.

4.2.2. STEP II ESTIMATE FOR $\|\nabla u\|_{L^1_t L^{2+\epsilon}}$

In this step we further investigate the time decay of $\|\nabla u\|_{L^{2+\epsilon}(\mathbb{R}^2)}$. Our idea is to explore the "low-high" frequency connection provided by the system (CNS) itself. Such an idea has played an important role in e.g. [61, 120, 135] to show time decay estimates. To this end we employ the following identities.

- The "low-frequency" term $P(\rho)-\tilde{P}$ can be represented by the "high-frequency" terms $\dot{u}, \nabla u$, when one applies $-(-\Delta)^{-1}\mathrm{div}$ to the momentum equation $(\mathrm{CNS})_2$:

$$P(\rho)-\tilde{P}=(-\Delta)^{-1}\mathrm{div}\,(\rho\dot{u})+(\mathcal{R}\otimes\mathcal{R}):(\mu(\rho)Su)+\lambda(\rho)\mathrm{div}\,u. \tag{P}$$

  Then one can benefit from the decay properties of $\dot{u}, \nabla u$ to obtain better time decay for $P(\rho)-\tilde{P}$.
- We apply D_t to (P) and make use of $(\mathrm{CNS})_1$: $\dot{\rho}=-\rho\mathrm{div}\,u$, to achieve

$$-\rho P'(\rho)\mathrm{div}\,u=\dot{P(\rho)}=(-\Delta)^{-1}\mathrm{div}\,(\rho\ddot{u})+g_1, \tag{$\dot{\mathrm{P}}$}$$

  where

$$\begin{aligned}&g_1=u\cdot\nabla(-\Delta)^{-1}\mathrm{div}\,(\rho\dot{u})-(-\Delta)^{-1}\mathrm{div}\,\partial_j(u_j\rho\dot{u})\\&+(\mathcal{R}\otimes\mathcal{R}):\left(-\mu'(\rho)\rho\mathrm{div}\,uSu+\mu(\rho)S\dot{u}+\mu(\rho)[u\cdot\nabla,S]u\right)+[u\cdot\nabla,(\mathcal{R}\otimes\mathcal{R})]:(\mu(\rho)Su)\\&-\lambda'(\rho)\rho(\mathrm{div}\,u)^2+\lambda(\rho)\mathrm{div}\,\dot{u}+\lambda(\rho)[u\cdot\nabla,\mathrm{div}\,]u.\end{aligned}$$

  The decay estimates for the "high-frequency" terms $\ddot{u}, \nabla\dot{u}$ may lead to further decay property of the "low-frequency" term $\mathrm{div}\,u$.
- If we rewrite $\lambda(\rho)\mathrm{div}\,u=\nu(\rho)\mathrm{div}\,u-(\mathcal{R}\otimes\mathcal{R}):(2\mu(\rho)\mathrm{div}\,u\mathrm{Id}\,)$, then the application of D_t to (P) and the fact that $-\dot{P(\rho)}=\rho P'(\rho)\mathrm{div}\,u$ yield

$$\Big(D_t+\frac{\rho P'(\rho)}{\nu(\rho)}\Big)(\nu(\rho)\mathrm{div}\,u)=\tilde{g}_1, \tag{$\dot{\mathrm{P}}'$}$$

  where

$$\begin{aligned}\tilde{g}_1=&-(-\Delta)^{-1}\mathrm{div}\,(\rho\ddot{u})-u\cdot\nabla(-\Delta)^{-1}\mathrm{div}\,(\rho\dot{u})+(-\Delta)^{-1}\mathrm{div}\,\partial_j(u_j\rho\dot{u})\\&-D_t\big((\mathcal{R}\otimes\mathcal{R}):(\mu(\rho)(Su-2(\mathrm{div}\,u)\mathrm{Id})\big).\end{aligned}$$

  The evolutionary equation ($\dot{\mathrm{P}}'$) is equivalent to ($\dot{\mathrm{P}}$), and represents a damping effect for $\mathrm{div}\,u$.

4.2.2.1. PROPOSITION 4.3.4: IMPROVED ESTIMATE FOR $\|a\|_{L^1_t L^{2+\epsilon}}$

Recall from Subsection 4.2.1.2 that we derive the following $\dot{H}^1$-energy estimate for u by taking the L^2-inner product between the momentum equation $(\mathrm{CNS})_2$ and $\dot{u}$ (see (4.3.11) below)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\Big(\frac{\mu(\rho)}{2}|Su|^2+\lambda(\rho)|\mathrm{div}\,u|^2\Big)\,dx+\int_{\mathbb{R}^2}\rho|\dot{u}|^2\,dx\\&\le C\|(\nabla u,P(\rho)-\tilde{P})\|_{L^2(\mathbb{R}^2)}\|\nabla u\|_{L^2(\mathbb{R}^2)}\|\nabla u\|_{L^\infty(\mathbb{R}^2)}\\&\quad+\frac{d}{dt}\int_{\mathbb{R}^2}(P(\rho)-\tilde{P})\mathrm{div}\,u\,dx+\int_{\mathbb{R}^2}\rho P'(\rho)|\mathrm{div}\,u|^2\,dx.\end{aligned}$$

The last integral on the right hand side results in a "loss" of $t^{-\frac{1}{2}}$-decay when performing time-weighted energy estimates directly.Hence we can only control norms such as $\|t'^{\delta-}\nabla u\|_{L^\infty_t L^2}$ in (4.2.5) while not $\|t'^{\frac{1}{2}+\delta-}\nabla u\|_{L^\infty_t L^2}$.

Our idea to improve this decay estimate is to use ($\dot{\mathrm{P}}$) in order to rewrite the last integral as

$$\int_{\mathbb{R}^2}(-\Delta)^{-1}\mathrm{div}\,(\rho\dot{u})\mathrm{div}\,\dot{u}dx = \langle \mathcal{R}\cdot(\rho\dot{u}), \mathcal{R}\cdot\dot{u}\rangle, \text{ up to unproblematic terms.}$$

Its difference with the integral $\int_{\mathbb{R}^2}\rho|\dot{u}|^2dx = \langle \rho\dot{u}, \dot{u}\rangle$ on the left hand side yields a control on

$$\|\nabla a\|^2_{L^2(\mathbb{R}^2)} = \|\mathcal{R}^\perp\cdot(\rho\dot{u})\|^2_{L^2(\mathbb{R}^2)}, \text{ up to unproblematic terms.}$$

Therefore, we derive the "incompressible" counterpart of (4.2.5) with *improved time decay*

$$\|t'^{\frac{1}{4}+\delta-}\nabla u\|^2_{L^\infty_t L^2} + \|t'^{\frac{1}{4}+\delta-}\nabla a\|^2_{L^2_t L^2}. \tag{4.2.8}$$

This suffices to control $\|a\|_{L^1_t L^{2+\epsilon}}$ in terms of the Lipschitz norm of the velocity (see Proposition 4.3.4).

4.2.2.2. Proposition 4.3.5: Estimate for $\|\nabla\dot{u}\|_{L^1((1,t);L^{2+\epsilon})}$

Since we do not have any control of $\|\nabla^2\dot{u}\|_{L^2(\mathbb{R}^2)}$ due to possible discontinuities of the viscosity coefficients, we cannot derive the $\|\nabla\dot{u}\|_{L^{2+\epsilon}(\mathbb{R}^2)}$-bound simply by interpolation. We have to go via the "coordinates" (a,b) as follows. Recall the quantities (4.2.7).

- By virtue of the invertibility of the operator $\mathcal{M}$ in $L^{2+\epsilon}(\mathbb{R}^2)$ by Lemma 4.1.7, $\|\nabla\dot{u}\|_{L^{2+\epsilon}(\mathbb{R}^2)}$ is controlled by $\|(\dot{a}, \dot{b}+\dot{P(\rho)})\|_{L^{2+\epsilon}(\mathbb{R}^2)}$.
- By virtue of the decomposition of $\rho\dot{u}$ in (CNS_{ab}), $\|(\nabla\dot{a},\nabla\dot{b})\|_{L^2(\mathbb{R}^2)}$ can be bounded by $\|\ddot{u}\|_{L^2(\mathbb{R}^2)}$ up to lower-order error terms.
- The bound for $\|\dot{a}\|_{L^{2+\epsilon}(\mathbb{R}^2)}$ follows direclty from interpolation.

  We have to further decompose $\dot{b}+\dot{P(\rho)}$ to control its $L^{2+\epsilon}(\mathbb{R}^2)$-norm:
  - the "low-frequency" part is bounded by $\|\dot{b}+\dot{P(\rho)}\|_{L^2(\mathbb{R}^2)}$, and hence, thanks to the boundedness of $\mathcal{M}$ in $L^2(\mathbb{R}^2)$, by $\|\nabla\dot{u}\|_{L^2(\mathbb{R}^2)}$ up to lower-order errors;
  - the "high-frequency" part is bounded by $\|\nabla\dot{b}\|_{L^2(\mathbb{R}^2)}$ and $\|\dot{P(\rho)}\|_{L^\infty(\mathbb{R}^2)}$, where the latter quantity is bounded by $C_*\|\nabla u\|_{L^\infty(\mathbb{R}^2)}$.

4.2.2.3. Proposition 4.3.6: Estimate for $\|\nabla u\|_{L^1_t L^{2+\epsilon}}$

Thanks to (4.1.38), the bound for $\|\nabla u\|_{L^1_t L^{2+\epsilon}}$ follows from the control on $\|a\|_{L^1_t L^{2+\epsilon}}$ and $\|\mathrm{div}\,u\|_{L^1_t L^{2+\epsilon}}$. In order to control $\|\mathrm{div}\,u\|_{L^1_t L^{2+\epsilon}}$, one cannot simply make use of the interpolation inequality $\|\mathrm{div}\,u\|_{L^{2+\epsilon}(\mathbb{R}^2)} \le \|\mathrm{div}\,u\|^{\frac{2}{2+\epsilon}}_{L^2(\mathbb{R}^2)}\|\mathrm{div}\,u\|^{\frac{\epsilon}{2+\epsilon}}_{L^\infty(\mathbb{R}^2)}$, since the energy bound in (4.2.5) does not give enough time-decay for $\|\mathrm{div}\,u\|_{L^2(\mathbb{R}^2)}$. Our idea is to make use of the representation ($\dot{\mathrm{P}}$) and the energy control of $\|(\dot{u},\ddot{u})\|_{L^2(\mathbb{R}^2)}$ and $\|\nabla\dot{u}\|_{L^{2+\epsilon}(\mathbb{R}^2)}$ obtained in Proposition 4.3.5.

We will also need a uniform-in-time bound for $\|t^{\frac{3}{4}}\mathrm{div}\,u\|_{L^\infty_t L^{2+\epsilon}}$. This requires more work, since we have time-weighted $L^2_t L^2$-control on $\ddot{u}$ while no $L^\infty_t L^2$-control. The latter would follow from higher order energy estimates, which we would like to avoid here, due to lenghty computations. Instead, we explore here the evolution equation ($\dot{\mathrm{P}}'$) for $\mathrm{div}\,u$. Then the obtained bounds for $u, \nabla u, \dot{u}, \nabla\dot{u}, \ddot{u}$ and the damping effect yield the desired uniform-in-time bound.

4.2.3. Step III Proposition 4.3.7: Estimates for tangential regularity

In the constant viscosity case, the effective flux (F) satisfies $\nabla F \in L^p(\mathbb{R}^2)$, $p \in [2,\infty)$, thanks to the elliptic equation $\Delta F = \operatorname{div}(\rho\dot{u})$ in (4.1.15). This, together with interpolation estimates, can be used to control the L^∞-norm of F in the density bounds. More precisely, recalling the mass conservation $(\text{CNS})_1$: $D_t \log\frac{\rho}{\tilde\rho} = -\operatorname{div} u$, we write

$$\operatorname{div} u = \frac{P(\rho)-\tilde P}{\nu} + \frac{F}{\nu}, \tag{4.2.9}$$

to obtain

$$D_t \log\frac{\rho}{\tilde\rho} + h(\rho)\log\frac{\rho}{\tilde\rho} = -\frac{F}{\nu}, \quad \text{with} \quad h(\rho) := \frac{P(\rho)-\tilde P}{\nu\log\frac{\rho}{\tilde\rho}} \geq h_* > 0 \tag{4.2.10}$$

for some constant h_* depending only on the upper and lower bounds of the density and viscosity coefficients, thanks to the assumption $P'(\rho) > 0$. This equation, together with the integrability properties of F, help to establish the upper and lower bounds as well as the time decay estimate of the density function. This is referred to as damping effect.

Unfortunately, for density-dependent viscosity coefficients one can not expect regularity of F due to possible jumps in the viscosity coefficients. Instead, we make use of the newly introduced nonlocal representation (4.1.42) of $\operatorname{div} u$ to rewrite $(\text{CNS})_1$ as

$$D_t\rho + \rho\mathcal{N}^{-1}(P(\rho)-\tilde P) = -\rho\mathcal{N}^{-1}(b - \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}a). \tag{4.2.11}$$

The damping effect is also present here, since the nonlocal operator $\mathcal{N}^{-1}$ is positive by Lemma 4.1.7. We investigate this damping effect in the equation for the tangential derivative of the density function. We apply the tangential derivative $\partial_{\bar\tau}$ to (4.2.11) to obtain

$$D_t\partial_{\bar\tau}\rho + \rho\mathcal{N}^{-1}\partial_{\bar\tau}(P(\rho)) = \Theta_0, \tag{4.2.12}$$

where

$$\begin{aligned}\Theta_0 = {} & -[\partial_{\bar\tau}, D_t]\rho - \partial_{\bar\tau}\rho\operatorname{div} u - \rho\mathcal{N}^{-1}(\partial_{\bar\tau} b - \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}\partial_{\bar\tau} a) \\ & + \rho\mathcal{N}^{-1}[\partial_{\bar\tau}, \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}]a - \rho[\partial_{\bar\tau}, \mathcal{N}^{-1}](P(\rho) - \tilde P + b - \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}a).\end{aligned}$$

We multiply the equation by $\sqrt{\frac{P'(\rho)}{\rho}}$ to derive a convenient equation for $\theta = \sqrt{\frac{P'(\rho)}{\rho}}\partial_{\bar\tau}\rho$,

$$D_t\theta = \mathcal{A}\theta + \Theta, \quad \text{with } \mathcal{A} = -\sqrt{\rho P'(\rho)}\mathcal{N}^{-1}\sqrt{\rho P'(\rho)} \text{ defined in (4.1.43)}, \tag{4.2.13}$$

where

$$\Theta = \left[D_t, \sqrt{\frac{P'(\rho)}{\rho}}\right]\partial_{\bar\tau}\rho + \sqrt{\frac{P'(\rho)}{\rho}}\Theta_0.$$

In Proposition 4.3.7 we employ this evolution equation and the dissipative property (4.1.44) of $\mathcal{A}$ to establish an $(L_t^\infty \cap L_t^1)L^{2+\epsilon}$-estimate for θ and hence for $\partial_{\bar\tau}\rho$. The same bound for (ρTR), (uTR) follows accordingly.

4.2.4. Step IV The Lipschitz bound

4.2.4.1. Proposition 4.3.8: Preliminary Lipschitz bound

By Remark 4.1.10, it suffices to control $\|(\alpha, \beta + P(\rho) - \tilde{P}, \partial_{\bar{\tau}} u)\|_{L^1_t L^\infty}$ to obtain the Lipschitz bound. Firstly, the bound for $\|(\alpha, \partial_{\bar{\tau}} u)\|_{L^1_t L^\infty}$ follows from interpolation bewteen $\|\nabla u\|_{L^1_t L^{2+\epsilon}}$ and the tangential regularity terms

$$\|(\nabla a, \nabla b, \partial_{\bar{\tau}}\rho, \partial_{\bar{\tau}}\nabla u)\|_{L^1_t L^{2+\epsilon}} + \|(\nabla\bar{\tau}, \partial_{\bar{\tau}}\rho)\|_{L^\infty_t L^{2+\epsilon}}\|(\nabla u, a, b + P(\rho) - \tilde{P})\|_{L^1_t L^\infty}.$$

The bound for $\|\beta + P(\rho) - \tilde{P}\|_{L^1_t L^\infty}$ requires more work. To realize a similar interpolation idea, we perform a high-low frequency decomposition and explore again the damping effect so as to control the high frequency part of the pressure fluctuation. Instead of the nonlocal viewpoint (the usage of $\mathcal{A}$), we insert the local representation (4.1.60): $\operatorname{div} u = \frac{\beta + P(\rho) - \tilde{P}}{\nu(\rho)} + \frac{2\mu(\rho)\bar{\tau}\cdot\partial_{\bar{\tau}} u}{\nu(\rho)}$ into the "renormalized" mass conservation law $D_t(P(\rho) - \tilde{P}) = -\rho P'(\rho)\operatorname{div} u$, and apply the high-frequency cutoff operator $\mathrm{Id} - \varphi_N(D)$, with the frequency threshold 2^N defined later, to obtain

$$\begin{aligned} D_t(P(\rho) - \tilde{P})^h + d(\rho)(P(\rho) - \tilde{P})^h = g_2, \\ \text{where} \quad f^h := (\mathrm{Id} - \varphi_N(D))f, \quad d(\rho) = \frac{\rho P'(\rho)}{\nu(\rho)}, \end{aligned} \tag{4.2.14}$$

with

$$\begin{aligned} g_2 = -d(\rho)\beta^h - \big(d(\rho)2\mu(\rho)\bar{\tau}\cdot\partial_{\bar{\tau}} u\big)^h - [u\cdot\nabla, \varphi_N(D)](P(\rho) - \tilde{P}) \\ -[d(\rho), \varphi_N(D)](P(\rho) - \tilde{P} + \beta). \end{aligned} \tag{4.2.15}$$

Here we used that the identity operator Id plays no role in the commutators $[D_t, \mathrm{Id} - \varphi_N(D)] = [u\cdot\nabla, -\varphi_N(D)]$. One then uses similar tangential regularity terms as above to control g_2 in $L^1_t L^\infty(\mathbb{R}^2)$ up to a parameter 2^N, which is afterwards determined to balance the control terms. Hence, the bound for $\|\nabla u\|_{L^1_t L^\infty}$ follows.

4.2.4.2. Proposition 4.3.9: Preliminary density bounds

We derive the lower and upper bounds for the density function from the mass equation, with $\operatorname{div} u$ replaced by its pointwise representation (4.1.60):

$$(D_t + h(\rho))\log\frac{\rho}{\tilde{\rho}} = -\frac{1}{\nu(\rho)}\Big(\beta + 2\mu(\rho)\bar{\tau}\cdot\partial_{\bar{\tau}} u\Big), \quad h(\rho) = \frac{P(\rho) - \tilde{P}}{\nu(\rho)\log\frac{\rho}{\tilde{\rho}}}, \tag{4.2.16}$$

where the function $h(\rho)$ is the same as in (4.2.10). The right hand side can be bouned similarly as above by interpolation. This eventually yields the density bounds.

4.2.4.3. Proposition 4.3.10: Bootstrap argument and Lipschitz bound

We put all the obtained estimates together, and use a bootstrap argument to close the estimates by virtue of the small energy assumption.

4.2.5. Proof sketch of Theorem 4.1.1

We follow the standard procedure to show the global-in-time wellposedness: We first construct a sequence of global-in-time approximate solutions $(\rho^\varepsilon, u^\varepsilon, \tau^\varepsilon)_{\varepsilon>0}$ with regularized initial data, and then show its convergence to the unique solution (ρ, u, τ) of the system (CNS)-(τ), with initial data (ρ_0, u_0, τ_0) satisfying the assumptions in Theorem 4.1.1.

4.2.5.1. Theorem 4.3.11: Local-in-time well-posedness and continuation criteria

We first establish the local-in-time well-posedness and continuation criteria of the system (CNS)-(τ) in Theorem 4.3.11, without any low-frequency or smallness assumptions on the initial data.

The proof strategy is as in [251], which is devoted to a compressible Navier–Stokes-type model governing the motion of two compressible fluids[4]. The usage of the new decompositions introduced in Section 4.1.3 helps to remove the smallness restriction in the viscosity fluctuations in [251].

4.2.5.2. Conclusion of Theorem 4.1.1: Compactness of approximate sequence

With regularized initial data, the local-in-time solution $(\rho^\varepsilon, u^\varepsilon, \tau^\varepsilon)$ established in Theorem 4.3.11 is regular enough, such that all the estimates obtained in Step I - Step IV hold. Therefore by the continuation criteria in Theorem 4.3.11, this solution can be continued globally-in-time.

To show the compactness of the approximate solution sequence, we adapt the proof strategy of [235] (see also [139]) where the shear viscosity μ is constant, to our case when μ is variable, making use of the pointwise representation (4.1.59) of $\operatorname{div} u$.

4.3. Proofs

The goal of this section is to prove Theorem 4.1.1 and Corollary 4.1.3. To this end, we first establish a priori estimates in a series of propositions: Subsections 4.3.1, 4.3.2, 4.3.3 and 4.3.4 are devoted to Steps I, II, III, IV, illustrated in Section 4.2, respectively. We then complete the proofs of Theorem 4.1.1 and Corollary 4.1.3 in Subsection 4.3.5.

Recall the assumptions in Theorem 4.1.1. Under the bootstrap assumption $\rho \in [\frac{1}{4}\rho_*, 4\rho^*]$ (see (4.2.2)), the parameter ϵ_0 depending on $\mu_*, \mu^*, \nu_*, \nu^*, \pi_*$ is given by Lemma 4.1.7. Without loss of generality we restrict to the case $\epsilon_0 \in (0, 2]$. Fix any number $\epsilon \in (0, \epsilon_0]$, and then fix $\delta \in (\frac{1}{2} - \frac{1}{4}\frac{\epsilon}{2+\epsilon}, \frac{1}{2}) \subset (\frac{1}{2+\epsilon}, \frac{1}{\epsilon})$. Let $\delta_- \in (\frac{1}{2} - \frac{1}{4}\frac{\epsilon}{2+\epsilon}, \delta)$ be some fixed number close to δ, then

$$\big\|\langle t\rangle^{-\delta-\frac{2}{2+\epsilon}} t^{-(\frac{1}{2}+\delta_-)\frac{\epsilon}{2+\epsilon}}\big\|_{L^2(0,\infty)} + \big\|\langle t\rangle^{-\delta-\frac{2}{2+\epsilon}} t^{-(\frac{1}{4}+\delta_-)\frac{\epsilon}{2+\epsilon}}\big\|_{L^2(0,\infty)} \leq C(\epsilon, \delta). \tag{4.3.1}$$

For notational simplicity, we introduce

[4] The model has the structure of a Navier–Stokes system, supplemented by a transport equation for the volume fraction of one phase, with constitutive laws depending on this quantity. If this dependence is dropped, the equations reduce to the classical Navier–Stokes system.

- Initial norms:
  - $E_0 = \|(\rho_0 - \tilde{\rho}, \rho_0 u_0)\|_{L^2(\mathbb{R}^2)}$;
  - $E_\delta = \|(\rho_0 - \tilde{\rho}, \rho_0 u_0)\|_{(L^2\cap \dot{H}^{-2\delta})(\mathbb{R}^2)}$;
  - $E_1 = E_\delta + \|\nabla u_0\|_{L^2(\mathbb{R}^2)}$;

  Recall the smallness assumption (4.1.10). Without loss of generality we assume in this section
$$E_0 \leq 1; \tag{4.3.2}$$
- Positive constants:
  - C_*, which may vary from line to line, denotes some positive constant depending on
$$\rho_*, \rho^*, \mu_*, \mu^*, \nu_*, \nu^*, \pi_*, \tilde{\pi}_*, \|(\mu,\lambda,P)\|_{W^{2,\infty}([\frac{1}{4}\rho_*, 4\rho^*])}, \epsilon, \delta;$$
  - $C_0 = C_* e^{C_* E_\delta^2} E_1$;
  - $\widetilde{C}_0 = C_* e^{C_* E_\delta^2}(1+E_1)^{\frac{\epsilon}{2+\epsilon}} E_1^{\frac{2}{2+\epsilon}}$
- Time-space norms:
  - $|||f|||_\epsilon \equiv |||f|||_\epsilon(t) = \|f\|_{L^1_t L^{2+\epsilon}} + \|t'^{\frac{1}{2}} f\|_{L^2_t L^{2+\epsilon}} + \|t'^{\frac{3}{4}} f\|_{L^\infty_t L^{2+\epsilon}}$;
  - $|||f|||_\infty \equiv |||f|||_\infty(t) = \|f\|_{L^1_t L^\infty} + \|t'^{\frac{1}{2}} f\|_{L^2_t L^\infty} + \|t'^{\frac{3}{4}} f\|_{L^\infty_t L^\infty}$;
- Generalized Lipschitz bounds:
  - $W(t) = \Big|\Big|\Big|\Big(\nabla u, a, b + P(\rho) - \tilde{P}\Big)\Big|\Big|\Big|_\infty(t)$;
  - $V(t) = \exp(C_* |||\nabla u|||_\infty(t))$;

  Thus $\exp(C_*\|\nabla u\|_{L^1_t L^\infty}) + |||\nabla u|||_\infty(t) \leq V(t) \leq \exp(C_* W(t))$.

4.3.1. Step I Energy estimates

4.3.1.1. Classical energy and decay of $\|(\rho - \tilde{\rho}, u)(t)\|_{L^2}$

We have the following energy and decay estimates, which are proven in Appendix 4.C.1.

Proposition 4.3.1. *Let (ρ, u) be a sufficiently smooth and fast decaying solution of* (CNS) *with initial data satisfying $(\rho_0 - \tilde{\rho}, u_0) \in L^2(\mathbb{R}^2; \mathbb{R}^{1+2})$, $0 < \rho_* \leq \rho_0 \leq \rho^*$ and (4.3.2).*

1. *Then taking the $L^2(\mathbb{R}^2)$-inner product between the momentum equation* $(\text{CNS})_2$ *and u yields*
$$\frac{d}{dt}\int_{\mathbb{R}^2}\Big(\frac{1}{2}\rho|u|^2 + H(\rho)\Big)\,dx + \int_{\mathbb{R}^2}\Big(\frac{\mu(\rho)}{2}|Su|^2 + \lambda(\rho)|\operatorname{div} u|^2\Big)\,dx = 0. \tag{4.3.3}$$
   Integrating in time implies that the energy defined in $(\mathcal{E})$ is dissipated as follows
$$\mathcal{E}(t) + \int_0^t\int_{\mathbb{R}^2}\Big(\frac{\mu(\rho)}{2}|Su|^2 + \lambda(\rho)|\operatorname{div} u|^2\Big)(t',x)dt'\,dx = \mathcal{E}(0). \tag{4.3.4}$$
2. *If additionally $(\rho_0 - \tilde{\rho}, \rho_0 u_0) \in \dot{H}^{-2\delta}(\mathbb{R}^2;\mathbb{R}^3)$ for some $\delta \in (0, \frac{1}{2})$ and*
$$\rho(t,x) \in [\frac{1}{4}\rho_*, 4\rho^*], \quad \forall t \geq 0,\ x \in \mathbb{R}^2, \tag{4.3.5}$$
   then for any fixed $\delta_- \in (0,\delta)$ and for all $t > 0$ we have
$$\|(\rho - \tilde{\rho}, u)(t)\|_{L^2} + \|\nabla u\|_{L^2_t L^2} \leq C_* E_0, \tag{4.3.6}$$
$$\|(\rho - \tilde{\rho}, u)(t)\|_{L^2} \lesssim_{\delta_-} C_* e^{C_* E_\delta^2} E_\delta \langle t\rangle^{-\delta_-}. \tag{4.3.7}$$

4.3.1.2. $\dot H^1$-energy estimate for u and L^2-energy estimate for $\dot u$

As a consequence of the decay estimate (4.3.7) we establish the following higher order energy estimates.

Proposition 4.3.2. *Let the assumptions of Proposition 4.3.1 hold and additionally $u_0 \in H^1(\mathbb{R}^2)$. Then for $t > 0$,*

$$\|\langle t'\rangle^{\delta_-}\nabla u\|_{L^2_tL^2} \lesssim_{\delta_-} C_* e^{C_*E_\delta^2}E_\delta, \tag{4.3.8}$$

$$\|(\nabla u, t'^{\frac12}\sqrt{\rho}\dot u, t'^{\frac12}\sqrt{\rho P'(\rho)}\operatorname{div} u)\|_{L^\infty_tL^2} + \|(\sqrt{\rho}\dot u, t'^{\frac12}\nabla\dot u)\|_{L^2_tL^2} \le C_*\big(E_0 + \|\nabla u_0\|_{L^2}\big)V(t), \tag{4.3.9}$$

$$\|(\langle t'\rangle^{\delta_-}\nabla u, t'^{\frac12+\delta_-}\sqrt{\rho}\dot u, t'^{\frac12+\delta_-}\sqrt{\rho P'(\rho)}\operatorname{div} u)\|_{L^\infty_tL^2} + \|(\langle t'\rangle^{\delta_-}\sqrt{\rho}\dot u, t'^{\frac12+\delta_-}\nabla\dot u)\|_{L^2_tL^2} \lesssim_{\delta_-} C_0V(t). \tag{4.3.10}$$

Proof. **Proof of (4.3.8).** Multiplying the energy balance (4.3.3) by $\langle t\rangle^{2\delta_-}$, integrating in time, and using (4.3.7) yields

$$\begin{aligned}&\langle t\rangle^{2\delta_-}\int_{\mathbb{R}^2}\Big(\frac12\rho|u|^2 + H(\rho)\Big)\,dx + \int_0^t\langle t'\rangle^{2\delta_-}\int_{\mathbb{R}^2}\Big(\frac{\mu(\rho)}{2}|Su|^2 + \lambda(\rho)|\operatorname{div} u|^2\Big)\,dxdt'\\ &= \mathcal{E}(0) + \int_0^t\langle t'\rangle^{2\delta_- -1}\int_{\mathbb{R}^2}\Big(\frac12\rho|u|^2 + H(\rho)\Big)\,dxdt' \lesssim_{\delta_-} C_*e^{C_*E_\delta^2}E_\delta^2,\end{aligned}$$

which directly implies (4.3.8).

Proof of (4.3.9). Below in Appendix 4.C.2 we establish the following higher order energy inequalities:

$$\begin{aligned}&\frac12\frac{d}{dt}\int_{\mathbb{R}^2}\Big(\frac{\mu(\rho)}{2}|Su|^2 + \lambda(\rho)|\operatorname{div} u|^2\Big)\,dx + \int_{\mathbb{R}^2}\rho|\dot u|^2\,dx\\ &\le C_*\|(\nabla u, P(\rho)-\tilde P)\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{L^\infty} + \frac{d}{dt}\int_{\mathbb{R}^2}(P(\rho)-\tilde P)\operatorname{div} u\,dx + \int_{\mathbb{R}^2}\rho P'(\rho)|\operatorname{div} u|^2\,dx,\end{aligned} \tag{4.3.11}$$

$$\begin{aligned}&\frac12\frac{d}{dt}\int_{\mathbb{R}^2}\rho|\dot u|^2 + \rho P'(\rho)|\operatorname{div} u|^2\,dx + \int_{\mathbb{R}^2}\frac{\mu(\rho)}{2}|S\dot u|^2 + \lambda(\rho)|\operatorname{div}\dot u|^2\,dx\\ &\le C_*(\|(\nabla u, P(\rho)-\tilde P)\|_{L^2}\|\nabla u\|_{L^\infty}\|\nabla\dot u\|_{L^2} + \|\operatorname{div} u\|_{L^2}\|\nabla u\|_{L^2}\|\nabla u\|_{L^\infty}).\end{aligned} \tag{4.3.12}$$

We multiply (4.3.12) by t, add the resulting inequality to (4.3.11), apply Young's inequality and integrate in time to obtain

$$\begin{aligned}&\frac{\nu_*}{4}\|\nabla u(t)\|_{L^2}^2 + \|t^{\frac12}(\sqrt{\rho}\dot u, \sqrt{\rho P'(\rho)}\operatorname{div} u)(t)\|_{L^2}^2 + \frac12\|\sqrt{\rho}\dot u\|_{L^2_tL^2}^2 + \frac{\nu_*}{4}\|t'^{\frac12}\nabla\dot u\|_{L^2_tL^2}^2\\ &\le \int_0^t\frac{d}{dt'}\int_{\mathbb{R}^2}(P(\rho)-\tilde P)\operatorname{div} u\,dxdt' + C_*\int_0^t\|(\nabla u, P(\rho)-\tilde P)\|_{L^2}^2(\|\nabla u\|_{L^\infty} + \|t'^{\frac12}\nabla u\|_{L^\infty}^2)dt'\\ &\quad + C_*(\|\operatorname{div} u\|_{L^2_tL^2}^2 + \|t'^{\frac12}\operatorname{div} u\|_{L^\infty_tL^2}\|\nabla u\|_{L^2_tL^2}\|t'^{\frac12}\nabla u\|_{L^2_tL^\infty} + \frac{\nu_*}{4}\|\nabla u_0\|_{L^2}^2,\end{aligned}$$

where we apply integration by parts to the first integral on the right hand side to bound it by

$$\begin{aligned}&\Big|\int_0^t\frac{d}{dt'}\int_{\mathbb{R}^2}(P(\rho)-\tilde P)\operatorname{div} u\,dxdt'\Big|\\ &\le \|P(\rho)-\tilde P\|_{L^\infty_tL^2}\|\operatorname{div} u(t)\|_{L^2} + \|P(\rho_0)-\tilde P\|_{L^2}\|\operatorname{div} u_0\|_{L^2}.\end{aligned}$$

We first apply Gronwall's inequality and then Young's inequality to derive

$$\frac{\nu_*}{8}\|\nabla u\|_{L^\infty_tL^2}^2 + \frac12\|t'^{\frac12}(\sqrt{\rho}\dot u, \sqrt{\rho P'(\rho)}\operatorname{div} u)\|_{L^\infty_tL^2}^2 + \frac12\|\sqrt{\rho}\dot u\|_{L^2_tL^2}^2 + \frac{\nu_*}{4}\|t'^{\frac12}\nabla\dot u\|_{L^2_tL^2}^2$$

$$\leq C_*\Big(\|P(\rho)-\tilde{P}\|^2_{L^\infty_t L^2} + \|P(\rho_0)-\tilde{P}\|_{L^2}\|\mathrm{div}\, u_0\|_{L^2} + \|\nabla u_0\|^2_{L^2}$$
$$+ \|\mathrm{div}\, u\|^2_{L^2_t L^2} + \|\nabla u\|^2_{L^2_t L^2}\Big)V(t).$$

Inserting the energy inequality (4.3.3) implies (4.3.9).

Proof of (4.3.10)**.** We multiply (4.3.11) by $\langle t\rangle^{2\delta_-}$ and (4.3.12) by $t^{1+2\delta_-}$, and add the resulting inequalities to obtain

$$\frac{\nu_*}{2}\|\langle t\rangle^{\delta_-}\nabla u(t)\|^2_{L^2} + \frac{1}{2}\|t^{\frac12+\delta_-}(\sqrt{\rho}\dot{u}, \sqrt{\rho P'(\rho)}\mathrm{div}\, u)(t)\|^2_{L^2}$$
$$+ (\frac12 - \delta_-)\|\langle t\rangle^{\delta_-}\sqrt{\rho}\dot{u}\|^2_{L^2_t L^2} + \frac{\nu_*}{4}\|t^{\frac12+\delta_-}\nabla\dot{u}\|^2_{L^2_t L^2}$$
$$\leq \int_0^t \langle t'\rangle^{2\delta_-}\frac{d}{dt'}\int_{\mathbb{R}^2}(P(\rho)-\tilde{P})\mathrm{div}\, u\, dxdt'$$
$$+ C_*\int_0^t \|\langle t'\rangle^{\delta_-}(\nabla u, P(\rho)-\tilde{P})\|^2_{L^2}(\|t'^{\frac12}\nabla u\|^2_{L^\infty} + \|\nabla u\|_{L^\infty})dt'$$
$$+ C_*(\|t'^{\frac12+\delta_-}\mathrm{div}\, u\|_{L^\infty_t L^2}\|t'^{\delta_-}\nabla u\|_{L^2_z L^2}\|t'^{\frac12}\nabla u\|_{L^2_t L^\infty} + \|\langle t'\rangle^{\delta_-}\nabla u\|^2_{L^2_t L^2}) + \frac{\nu_*}{2}\|\nabla u_0\|^2_{L^2}.$$

Using integration by parts we control $\int_0^t \langle t'\rangle^{2\delta_-}\frac{d}{dt}\int_{\mathbb{R}^2}(P(\rho)-\tilde{P})\mathrm{div}\, u\, dxdt'$ by

$$\|\langle t'\rangle^{\delta_-}(P(\rho)-\tilde{P})\|_{L^\infty_t L^2}\|\langle t\rangle^{\delta_-}\mathrm{div}\, u(t)\|_{L^2} + \|P(\rho_0)-\tilde{P}\|_{L^2}\|\mathrm{div}\, u_0\|_{L^2}$$
$$+ 2\delta_-\|\langle t'\rangle^{-1-(\delta-\delta_-)}\|_{L^2(0,t)}\|\langle t'\rangle^{\delta}(P(\rho)-\tilde{P})\|_{L^\infty_t L^2}\|\langle t'\rangle^{\delta_-}\mathrm{div}\, u\|_{L^2_t L^2},$$

where $\|\langle t'\rangle^{-1-(\delta-\delta_-)}\|_{L^2(0,t)}$ is uniformly bounded in t since $\delta_- < \delta$. We first apply Young's inequality to the $\|\langle t'\rangle^{\delta_-}\nabla u\|_{L^\infty_t L^2}$- and $\|t^{\frac12+\delta_-}\mathrm{div}\, u(t)\|_{L^2}$-terms and then use Gronwall's inequality to derive

$$\frac{\nu_*}{4}\|\langle t'\rangle^{\delta_-}\nabla u\|^2_{L^\infty_t L^2} + \frac14\|t'^{\frac12+\delta_-}(\sqrt{\rho}\dot{u}, \sqrt{\rho P'(\rho)}\mathrm{div}\, u)(t)\|^2_{L^\infty_t L^2}$$
$$+ (\frac12 - \delta_-)\|\langle t\rangle^{\delta_-}\sqrt{\rho}\dot{u}\|^2_{L^2_t L^2} + \frac{\nu_*}{4}\|t^{\frac12+\delta_-}\nabla\dot{u}\|^2_{L^2_t L^2}$$
$$\leq C_*\Big(\|\langle t'\rangle^{\delta_-}(P(\rho)-\tilde{P})\|^2_{L^\infty_t L^2} + \|P(\rho_0)-\tilde{P}\|_{L^2}\|\mathrm{div}\, u_0\|_{L^2}$$
$$+ \|\langle t'\rangle^{\delta_-}\nabla u\|^2_{L^2_t L^2} + \|\nabla u_0\|^2_{L^2}\Big)V(t).$$

We derive (4.3.10) from (4.3.7) and (4.3.8). □

4.3.1.3. $\dot{H}^1$-energy estimate for $\dot{u}$ and estimate for $|||(\nabla a, \nabla b)|||_\epsilon$

Proposition 4.3.3 (Estimates for $\|\dot{u}\|_{\dot{H}^1}$ and $|||(\nabla a, \nabla b)|||_\epsilon$). *Let $\epsilon \in (0,2]$ and $\delta \in (\frac{1}{2+\epsilon}, \frac12)$, $\delta_- \in (\frac{1}{2+\epsilon}, \delta)$. Under the assumptions of Proposition 4.3.2, we have for $t > 0$,*

$$\frac{\nu_*}{4}\|t'^{\frac12+\delta_-}\sigma^{\frac12-\delta_-}\nabla\dot{u}\|_{L^\infty_t L^2} + \|t'^{\frac12+\delta_-}\sigma^{\frac12-\delta_-}\sqrt{\rho}\ddot{u}\|_{L^2_t L^2} \leq C_0 V(t), \tag{4.3.13}$$

and consequently

$$|||(\nabla a, \nabla b)|||_\epsilon \leq C_0 V(t). \tag{4.3.14}$$

Here, σ is the time-weight $\sigma(t) = \min\{1, t\}$.

Proof. **Proof of** (4.3.13)**.** Recall the definition of the tensor (T): $T = \mu(\rho)Su + (\lambda(\rho)\mathrm{div}\, u - (P(\rho) - \tilde{P}))\mathrm{Id}$. Below in Appendix 4.C.2 we show that

$$\begin{aligned} &\|\sqrt{\rho}\ddot{u}\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\Big(\frac{\mu(\rho)}{2}|S\dot{u}|^2 + \lambda(\rho)|\mathrm{div}\,\dot{u}|^2\Big)\,dx \\ &\leq \frac{d}{dt}I + C_*(\|\nabla\dot{u}\|_{L^2} + \|\nabla u\|_{L^2}\|\nabla u\|_{L^\infty} + \|\nabla u\|_{L^\infty}\|\nabla\dot{u}\|_{L^2} + \|\nabla u\|_{L^2}\|\nabla u\|_{L^\infty}^2)\|\nabla\dot{u}\|_{L^2}, \end{aligned} \tag{4.3.15}$$

where I can be bounded by

$$|I| \leq C_*\|(\nabla u, P(\rho) - \tilde{P})\|_{L^2}\|\nabla u\|_{L^\infty}\|\nabla\dot{u}\|_{L^2} + C_*\|\mathrm{div}\, u\|_{L^2}\|\mathrm{div}\,\dot{u}\|_{L^2}, \tag{4.3.16}$$

We multiply (4.3.15) by $t^{1+2\delta_-}\sigma^{1-2\delta_-}$, and integrate in time to derive

$$\begin{aligned} &\|t'^{\frac{1}{2}+\delta_-}\sigma^{\frac{1}{2}-\delta_-}\sqrt{\rho}\ddot{u}\|_{L^2_tL^2}^2 + \frac{1}{2}t^{1+2\delta_-}\sigma^{1-2\delta_-}\int_{\mathbb{R}^2}\Big(\frac{\mu(\rho)}{2}|S\dot{u}|^2 + \lambda(\rho)|\mathrm{div}\,\dot{u}|^2\Big)(t)\,dx \\ &\leq C_*\|t'^{\frac{1}{2}}\nabla\dot{u}\|_{L^2_tL^2}^2 + \int_0^t t'^{1+2\delta_-}\sigma^{1-2\delta_-}\frac{d}{dt'}I dt' + C_*\int_0^t\|\nabla u\|_{L^\infty}\|t'^{\frac{1}{2}+\delta_-}\sigma^{\frac{1}{2}-\delta_-}\nabla\dot{u}\|_{L^2}^2dt' \\ &\quad + C_*(\|t'^{\frac{1}{2}+\delta_-}\nabla\dot{u}\|_{L^2_tL^2} + \|t'^{\delta_-}\nabla u\|_{L^\infty_tL^2}\|t'^{\frac{1}{2}}\nabla u\|_{L^2_tL^\infty} \\ &\qquad + \|t'^{\frac{1}{4}}\nabla u\|_{L^\infty_tL^2}\|t'^{\frac{3}{4}}\nabla u\|_{L^\infty_tL^\infty}\|t'^{\frac{1}{2}}\nabla u\|_{L^2_tL^\infty})\|t'^{\frac{1}{2}}\langle t'\rangle^{\delta_-}\nabla\dot{u}\|_{L^2_tL^2}. \end{aligned}$$

Notice that integration by parts yields

$$\begin{aligned} &\int_0^t t'^{1+2\delta_-}\sigma^{1-2\delta_-}\frac{d}{dt'}I dt' \\ &\leq C_*\|\langle t'\rangle^{\delta_-}(\nabla u, P(\rho) - \tilde{P})\|_{L^\infty_tL^2}(\|t'^{\frac{3}{4}}\nabla u\|_{L^\infty_tL^\infty}\|t^{\frac{1}{2}+\delta_-}\sigma^{\frac{1}{2}-\delta_-}\nabla\dot{u}(t)\|_{L^2} \\ &\qquad + \|t'^{\frac{1}{2}}\nabla u\|_{L^2_tL^\infty}\|t'^{\frac{1}{2}}\nabla\dot{u}\|_{L^2_tL^2}) \\ &\quad + C_*\|t'^{\frac{1}{2}+\delta_-}\mathrm{div}\, u\|_{L^\infty_tL^2}\|t^{\frac{1}{2}+\delta_-}\sigma^{\frac{1}{2}-\delta_-}\mathrm{div}\,\dot{u}(t)\|_{L^2} + C_*\|t'^{\delta_-}\nabla u\|_{L^2_tL^2}\|t'^{\frac{1}{2}}\nabla\dot{u}\|_{L^2_tL^2}. \end{aligned}$$

Thus, we apply Young's inequality related to the term $\|t^{\frac{1}{2}+\delta_-}\sigma^{\frac{1}{2}-\delta_-}\nabla\dot{u}(t)\|_{L^2}$ and then use Gronwall's inequality to derive (choosing $\delta_- \in (\frac{1}{4}, \frac{1}{2})$)

$$\begin{aligned} &\|t'^{\frac{1}{2}+\delta_-}\sigma^{\frac{1}{2}-\delta_-}\sqrt{\rho}\ddot{u}\|_{L^2_tL^2}^2 + \frac{\nu_*}{4}\|t^{\frac{1}{2}+\delta_-}\sigma^{\frac{1}{2}-\delta_-}\nabla\dot{u}(t)\|_{L^2}^2 \\ &\leq C_*\Big(\|\langle t'\rangle^{\delta_-}(\nabla u, P(\rho) - \tilde{P})\|_{L^\infty_tL^2}^2 + \|t'^{\frac{1}{2}+\delta_-}\mathrm{div}\, u\|_{L^\infty_tL^2}^2 + \|(t'^{\delta_-}\nabla u, t'^{\frac{1}{2}}\langle t'\rangle^{\delta_-}\nabla\dot{u})\|_{L^2_tL^2}^2\Big)V(t). \end{aligned}$$

Then we apply the energy estimates (4.3.9), (4.3.10) to obtain the desired bound (4.3.13).

Proof of (4.3.14)**.** First notice that the Helmholtz decomposition (CNS$_{ab}$): $\rho\dot{u} = \nabla^\perp a + \nabla b$ implies

$$\|(\nabla a, \nabla b)\|_{L^p} \leq C_p\|\rho\dot{u}\|_{L^p}, \quad \forall p \in (1, \infty). \tag{4.3.17}$$

Together with the interpolation intequality $\|f\|_{L^{2+\epsilon}} \leq C_\epsilon\|f\|_{L^2}^{\frac{2}{2+\epsilon}}\|\nabla f\|_{L^2}^{\frac{\epsilon}{2+\epsilon}}$, we have

$$\|(\nabla a, \nabla b)\|_{L^{2+\epsilon}} \leq C_*\|\dot{u}\|_{L^2}^{\frac{2}{2+\epsilon}}\|\nabla\dot{u}\|_{L^2}^{\frac{\epsilon}{2+\epsilon}},$$

and therefore

$$\|(\nabla a, \nabla b)\|_{L^1_tL^{2+\epsilon}} + \|t'^{\frac{1}{2}}(\nabla a, \nabla b)\|_{L^2_tL^{2+\epsilon}} \leq C_*\|\langle t'\rangle^{\delta_-}\dot{u}\|_{L^2_tL^2}^{\frac{2}{2+\epsilon}}\|t'^{\frac{1}{2}+\delta_-}\nabla\dot{u}\|_{L^2_tL^2}^{\frac{\epsilon}{2+\epsilon}},$$

$$\|t'^{\frac{3}{4}}(\nabla a, \nabla b)\|_{L^\infty_t L^{2+\epsilon}} \leq C_* \|t'^{\frac{1}{2}} \langle t' \rangle^{\delta_-} \dot{u}\|^{\frac{2}{2+\epsilon}}_{L^\infty_t L^2} \|t'^{\frac{1}{2}+\delta_-} \sigma^{\frac{1}{2}-\delta_-} \nabla \dot{u}\|^{\frac{\epsilon}{2+\epsilon}}_{L^\infty_t L^2},$$

where $\delta_- \in (\frac{1}{2+\epsilon}, \delta)$ and we used the facts (4.3.1), $(\frac{1}{2} - (\frac{1}{2} + \delta_-)\frac{\epsilon}{2+\epsilon})\frac{2+\epsilon}{2} \in [0, \delta_-]$ and $(\frac{3}{4} - \frac{\epsilon}{2+\epsilon})\frac{2+\epsilon}{2}, (\frac{3}{4} - \frac{\epsilon}{2+\epsilon}(\frac{1}{2} + \delta_-))\frac{2+\epsilon}{2} \in [\frac{1}{2}, \frac{1}{2} + \delta_-]$. Hence (4.3.14) follows from (4.3.10) and (4.3.13).

□

4.3.2. Step II Estimate for $|||\nabla u|||_\epsilon$

The goal of this subsection is to establish the a priori estimate for $|||\nabla u|||_\epsilon$. In view of the $L^{2+\epsilon}(\mathbb{R}^2)$-estimate (4.1.37) in Lemma 4.1.7 we have

$$|||\nabla u|||_\epsilon \leq C_* |||(a, \operatorname{div} u)|||_\epsilon.$$

In Subsections 4.3.2.1 and 4.3.2.3 below we establish a priori estimates for $|||a|||_\epsilon$ and $|||\operatorname{div} u|||_\epsilon$ respectively.

4.3.2.1. Improved $\dot{H}^1$-energy estimate for u and estimate for $|||a|||_\epsilon$

Proposition 4.3.4 (Improved $\dot{H}^1$ estimate). *Let $\epsilon \in (0, 2]$, and $\delta \in (\frac{1}{2} - \frac{1}{4}\frac{\epsilon}{2+\epsilon}, \frac{1}{2})$, $\delta_- \in (\frac{1}{2} - \frac{1}{4}\frac{\epsilon}{2+\epsilon}, \delta)$. Under the assumptions of Proposition 4.3.2, there holds for $t > 0$*

$$\|t'^{\frac{1}{4}+\delta_-} \nabla u\|_{L^\infty_t L^2} + \|t'^{\frac{1}{4}+\delta_-} \nabla a\|_{L^2_t L^2} \leq C_0 V(t), \tag{4.3.18}$$

and consequently,

$$|||a|||_\epsilon \leq C_0 V(t). \tag{4.3.19}$$

Proof. We write

$$(-\Delta)^{-1}\operatorname{div}(\rho \ddot{u}) = \partial_t (-\Delta)^{-1}\operatorname{div}(\rho \dot{u}) + (-\Delta)^{-1}\operatorname{div} \partial_j (u_j \rho \dot{u}),$$

so that by taking the L^2-inner product between ($\dot{\text{P}}$) and $-\operatorname{div} u$, we derive

$$\int_{\mathbb{R}^2} \rho P'(\rho) |\operatorname{div} u|^2 dx + \frac{d}{dt}\int_{\mathbb{R}^2} (-\Delta)^{-1}\operatorname{div}(\rho \dot{u}) \operatorname{div} u \, dx - \int_{\mathbb{R}^2} (-\Delta)^{-1}\operatorname{div}(\rho \dot{u}) \operatorname{div} \dot{u} \, dx$$
$$\leq C\|\rho \dot{u}\|_{L^2} \|u\|_{L^2} \|\nabla u\|_{L^\infty} + C_*(\|\nabla u\|^2_{L^2} \|\nabla u\|_{L^\infty} + \|\nabla \dot{u}\|_{L^2} \|\nabla u\|_{L^2}),$$

where we used

$$\int_{\mathbb{R}^2} \partial_t (-\Delta)^{-1}\operatorname{div}(\rho \dot{u}) \operatorname{div} u \, dx = \frac{d}{dt}\int_{\mathbb{R}^2} (-\Delta)^{-1}\operatorname{div}(\rho \dot{u}) \operatorname{div} u \, dx - \int_{\mathbb{R}^2} (-\Delta)^{-1}\operatorname{div}(\rho \dot{u}) \operatorname{div} \dot{u} \, dx$$
$$+ \int_{\mathbb{R}^2} (-\Delta)^{-1}\operatorname{div}(\rho \dot{u}) \operatorname{div}(u \cdot \nabla u) \, dx.$$

Adding the above inequality to (4.3.11) while noticing that (noting $v = \mathcal{R}(\mathcal{R} \cdot v) + \mathcal{R}^\perp(\mathcal{R}^\perp \cdot v)$ for $v \in L^2(\mathbb{R}^2; \mathbb{R}^2)$ and $\rho \dot{u} = \nabla^\perp a + \nabla b$)

$$\int_{\mathbb{R}^2} \rho |\dot{u}|^2 \, dx - \int_{\mathbb{R}^2} (-\Delta)^{-1}\operatorname{div}(\rho \dot{u}) \operatorname{div} \dot{u} \, dx = \langle \rho \dot{u}, \dot{u} \rangle - \langle \mathcal{R} \cdot (\rho \dot{u}), \mathcal{R} \cdot \dot{u} \rangle = \langle \mathcal{R}^\perp \cdot (\rho \dot{u}), \mathcal{R}^\perp \cdot \dot{u} \rangle$$
$$= \frac{1}{\tilde{\rho}} \|\mathcal{R}^\perp \cdot (\rho \dot{u})\|^2_{L^2(\mathbb{R}^2)} - \frac{1}{\tilde{\rho}} \langle \mathcal{R}^\perp \cdot (\rho \dot{u}), \mathcal{R}^\perp \cdot ((\rho - \tilde{\rho}) \dot{u}) \rangle \geq \frac{1}{\tilde{\rho}} \|\nabla a\|^2_{L^2(\mathbb{R}^2)} - C_* \|\rho - \tilde{\rho}\|_{L^2} \|\dot{u}\|^2_{L^4},$$

we obtain (by Gagliardo-Nirenberg's inequality $\|\dot u\|^2_{L^4(\mathbb{R}^2)} \le C\|\dot u\|_{L^2}\|\nabla\dot u\|_{L^2(\mathbb{R}^2)}$)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[\int_{\mathbb{R}^2}\frac{\mu(\rho)}{2}|Su|^2+\lambda(\rho)|\mathrm{div}\,u|^2\,dx\Big]+\frac{1}{\tilde\rho}\|\nabla a\|^2_{L^2(\mathbb{R}^2)}\\&\le C_*\|\rho-\tilde\rho\|_{L^2}\|\dot u\|_{L^2}\|\nabla\dot u\|_{L^2}+\frac{d}{dt}\Big[\int_{\mathbb{R}^2}(P(\rho)-\tilde P)\mathrm{div}\,u\,dx-\int_{\mathbb{R}^2}(-\Delta)^{-1}\mathrm{div}\,(\rho\dot u)\mathrm{div}\,u\,dx\Big]\\&\quad+C\|\rho\dot u\|_{L^2}\|u\|_{L^2}\|\nabla u\|_{L^\infty}+C_*(\|(\nabla u,P(\rho)-\tilde P)\|_{L^2}(\|\nabla u\|_{L^2}\|\nabla u\|_{L^\infty}+\|\nabla\dot u\|_{L^2}).\end{aligned}$$

Multiplying this inequality by $t^{\frac12+2\delta_-}$ and integrating in time implies

$$\begin{aligned}&\frac{\nu_*}{2}\|t^{\frac14+\delta_-}\nabla u(t)\|^2_{L^2}+\frac{1}{\tilde\rho}\|t'^{\frac14+\delta_-}\nabla a\|^2_{L^2_tL^2}\\&\le C_*\|t'^{\delta_--\frac14}\nabla u\|^2_{L^2_tL^2}+C_*\|\rho-\tilde\rho\|_{L^\infty_tL^2}\|t'^{\delta_-}\dot u\|_{L^2_tL^2}\|t'^{\frac12+\delta_-}\nabla\dot u\|_{L^2_tL^2}\\&\quad+\int_0^t t'^{\frac12+2\delta_-}\frac{d}{dt}\Big[\int_{\mathbb{R}^2}(P(\rho)-\tilde P)\mathrm{div}\,u\,dx-\int_{\mathbb{R}^2}(-\Delta)^{-1}\mathrm{div}\,(\rho\dot u)\mathrm{div}\,u\,dx\Big]dt'\\&\quad+C\|t'^{\delta_-}\rho\dot u\|_{L^2_tL^2}\|t'^{\delta_-}u\|_{L^\infty_tL^2}\|t'^{\frac12}\nabla u\|_{L^2_tL^\infty}\\&\quad+C_*(\|t'^{\delta_-}(\nabla u,\rho-\tilde\rho)\|_{L^\infty_tL^2}(\|t'^{\delta_-}\nabla u\|_{L^2_tL^2}\|t'^{\frac12}\nabla u\|_{L^2_tL^\infty}+\|t'^{\frac12+\delta_-}\nabla\dot u\|_{L^2_tL^2}).\end{aligned}$$

Using integration by parts with respect to the time variable, the first term in the second line above can be bounded by

$$\begin{aligned}&C\|t'^{\delta_-}(P(\rho)-\tilde P,u)\|_{L^\infty_tL^2}\|t'^{\frac12+\delta_-}(\mathrm{div}\,u,\rho\dot u)\|_{L^\infty_tL^2}\\&+C_{\delta,\delta_-}\|t'^{\delta}(P(\rho)-\tilde P,u)\|_{L^\infty_tL^2}\|\langle t'\rangle^{\delta_-}(\mathrm{div}\,u,\rho\dot u)\|_{L^2_tL^2},\end{aligned}$$

where we used that $\langle t'\rangle^{-\delta_-}t'^{2\delta_--\frac12-\delta}\in L^2(0,t)$ uniformly in t as long as $\delta_-<\delta$. Plugging this inequality back into the H^1-energy estimate for u and inserting the estimates from Proposition 4.3.2, we achieve the claimed inequality.

Proof of (4.3.19)**.** Interpolation and the fact that $\|a\|_{L^p}\le C_p\|\nabla u\|_{L^p}, p\in(1,\infty)$, imply

$$\|a\|_{L^{2+\epsilon}}\le C_*\|\nabla u\|^{\frac{2}{2+\epsilon}}_{L^2}\|\nabla a\|^{\frac{\epsilon}{2+\epsilon}}_{L^2}.$$

Thus,

$$\begin{aligned}&\|a\|_{L^1_tL^{2+\epsilon}}+\|t'^{\frac12}a\|_{L^2_tL^{2+\epsilon}}\le C_*\|\langle t'\rangle^{\delta_-}\nabla u\|^{\frac{2}{2+\epsilon}}_{L^2_tL^2}\|t'^{\frac14+\delta_-}\nabla a\|^{\frac{\epsilon}{2+\epsilon}}_{L^2_tL^2},\\&\|t'^{\frac34}a\|_{L^\infty_tL^{2+\epsilon}}\le C_*\|\langle t'\rangle^{\frac14+\delta_-}\nabla u\|^{\frac{2}{2+\epsilon}}_{L^\infty_tL^2}\|t'^{\frac12+\delta_-}\dot u\|^{\frac{\epsilon}{2+\epsilon}}_{L^\infty_tL^2},\end{aligned}$$

where we used the facts (4.3.1), $(\frac12-(\frac14+\delta_-)\frac{\epsilon}{2+\epsilon})\frac{2+\epsilon}{2}\in[0,\delta_-]$, $(\frac34-(\frac12+\delta_-)\frac{\epsilon}{2+\epsilon})\frac{2+\epsilon}{2}\in[0,\frac14+\delta_-]$, since $\delta_-+\frac14\frac{\epsilon}{2+\epsilon}>\frac12$. We also used the estimate (4.3.17) of ∇a in terms of $\dot u$ in the second line above. Now (4.3.19) follows from inserting (4.3.8), (4.3.10), and (4.3.18) .

□

4.3.2.2. TIME-DECAY $L^{2+\epsilon}(\mathbb{R}^2)$-ESTIMATE FOR $\nabla\dot u$

Proposition 4.3.5. *Let $\epsilon\in(0,\epsilon_0]$ with $\epsilon_0\in(0,2]$ given in Lemma 4.1.7. Under the assumptions of Proposition 4.3.2 with $\delta\in(\frac{1}{2+\epsilon},\frac12)$, we have for $\gamma>0$ and $t\ge1$,*

$$\|\nabla\dot u\|_{L^1(1,t;L^{2+\epsilon})}+\|t'^{\frac12}\nabla\dot u\|_{L^2(1,t;L^{2+\epsilon})}+\Big(\int_1^t e^{-\gamma(t-t')}(t'^{\frac34}\|\nabla\dot u\|_{L^{2+\epsilon}})^2\,dt'\Big)^{\frac12}\lesssim_\gamma\widetilde C_0V(t). \tag{4.3.20}$$

Proof. Recall the operator $\mathcal{M} = \begin{pmatrix} \mathcal{R}_\mu & -\mathcal{Q}_\mu \\ \mathcal{Q}_\mu & \mathcal{R}_{\mu,\lambda} \end{pmatrix}$ with $\mathcal{M}\begin{pmatrix} \omega \\ \operatorname{div} u \end{pmatrix} = \begin{pmatrix} a \\ b + P(\rho) - \tilde{P} \end{pmatrix}$ in Lemma 4.1.7. Then

$$\begin{aligned}
\mathcal{M}\begin{pmatrix} \nabla^\perp \cdot \dot{u} \\ \operatorname{div} \dot{u} \end{pmatrix} &= \begin{pmatrix} \mathcal{R}_\mu(\nabla^\perp \cdot \dot{u}) - \mathcal{Q}_\mu \operatorname{div} \dot{u} \\ \mathcal{Q}_\mu(\nabla^\perp \cdot \dot{u}) + \mathcal{R}_{\mu,\lambda} \operatorname{div} \dot{u} \end{pmatrix} \\
&= \begin{pmatrix} \dot{a} \\ \dot{b} + \dot{P(\rho)} \end{pmatrix} + \begin{pmatrix} [\mathcal{R}_\mu, D_t]\omega - [\mathcal{Q}_\mu, D_t]\operatorname{div} u + \mathcal{R}_\mu(\nabla^\perp u_j \cdot \partial_j u) - \mathcal{Q}_\mu(\nabla u_j \cdot \partial_j u) \\ [\mathcal{R}_{\mu,\lambda}, D_t]\operatorname{div} u + [\mathcal{Q}_\mu, D_t]\omega + \mathcal{R}_{\mu,\lambda}(\nabla u_j \cdot \partial_j u) + \mathcal{Q}_\mu(\nabla^\perp u_j \cdot \partial_j u) \end{pmatrix}.
\end{aligned}$$

Thus, the commutator estimate (4.1.45) with $X = u$, $r = \infty$ and $q = p = 2$ resp. $2 + \epsilon$ and $D_t \mu(\rho) = -\rho\mu'(\rho)\operatorname{div} u$

- together with the boundedness of $\mathcal{M}$ in $L^2(\mathbb{R}^2)$, imply

$$\|(\dot{a}, \dot{b} + \dot{P(\rho)})\|_{L^2} \leq C_*(\|\nabla \dot{u}\|_{L^2} + \|\nabla u\|_{L^2}\|\nabla u\|_{L^\infty}). \tag{4.3.21}$$

- together with the $L^{2+\epsilon}$ invertibility of $\mathcal{M}$, yield

$$\|\nabla \dot{u}\|_{L^{2+\epsilon}} \leq C_*(\|(\dot{a}, \dot{b} + \dot{P(\rho)})\|_{L^{2+\epsilon}} + \|\nabla u\|_{L^{2+\epsilon}}\|\nabla u\|_{L^\infty}). \tag{4.3.22}$$

Using interpolation we deduce

$$\begin{aligned}
&\|\dot{a}\|_{L^{2+\epsilon}} \leq C_* \|\dot{a}\|_{L^2}^{\frac{2}{2+\epsilon}} \|\nabla \dot{a}\|_{L^2}^{\frac{\epsilon}{2+\epsilon}}, \\
&\|\dot{b} + \dot{P(\rho)}\|_{L^{2+\epsilon}} \leq C_* \|\dot{b} + \dot{P(\rho)}\|_{L^2}^{\frac{2}{2+\epsilon}} (\|\nabla \dot{b}\|_{L^2} + \|\operatorname{div} u\|_{L^\infty})^{\frac{\epsilon}{2+\epsilon}}.
\end{aligned}$$

This, together with (4.3.21) and (4.3.22), implies

$$\|\nabla \dot{u}\|_{L^{2+\epsilon}} \leq C_*(\|\nabla \dot{u}\|_{L^2} + \|\nabla u\|_{L^2}\|\nabla u\|_{L^\infty})^{\frac{2}{2+\epsilon}} (\|(\nabla \dot{a}, \nabla \dot{b})\|_{L^2} + (1 + \|\nabla u\|_{L^\infty})\|\nabla u\|_{L^\infty})^{\frac{\epsilon}{2+\epsilon}}.$$

We apply D_t to the momentum equation $(\text{CNS})_2$ to achieve

$$\rho \ddot{u} - (\rho \operatorname{div} u)\dot{u} = \nabla^\perp \dot{a} + \nabla \dot{b} - \nabla^\perp u \cdot \nabla a - \nabla u \cdot \nabla b.$$

Since $\|(\nabla a, \nabla b)\|_{L^2} \leq C_* \|\dot{u}\|_{L^2}$, we derive

$$\|(\nabla \dot{a}, \nabla \dot{b})\|_{L^2} \leq C_*(\|\rho \ddot{u}\|_{L^2} + \|\nabla u\|_{L^\infty}\|\rho \dot{u}\|_{L^2}). \tag{4.3.23}$$

Hence inserting the estimates from Propositions 4.3.2 and 4.3.3, we obtain for $t \geq 1$,

$$\begin{aligned}
&\|\nabla \dot{u}\|_{L^1(1,t;L^{2+\epsilon})} + \|t'^{\frac{1}{2}} \nabla \dot{u}\|_{L^2(1,t;L^{2+\epsilon})} \\
&\leq C_* \Big(\|t'^{\frac{1}{2}+\delta-} \nabla \dot{u}\|_{L^2_t L^2} + \|t'^{\frac{1}{4}+\delta-} \nabla u\|_{L^\infty_t L^2} \|\!|\nabla u|\!\|_\infty \Big)^{\frac{2}{2+\epsilon}} \\
&\quad \times \Big(\|t'^{\frac{1}{2}+\delta-} \sigma^{\frac{1}{2}-\delta-} \rho \ddot{u}\|_{L^2_t L^2} + (1 + \|t'^{\frac{3}{4}} \nabla u\|_{L^\infty_t L^\infty} + \|t'^{\frac{1}{2}} \rho \dot{u}\|_{L^\infty_t L^2}) \|\!|\nabla u|\!\|_\infty \Big)^{\frac{\epsilon}{2+\epsilon}} \\
&\leq C_0^{\frac{2}{2+\epsilon}} (C_0 + 1)^{\frac{\epsilon}{2+\epsilon}} V(t).
\end{aligned}$$

and, using $(\int_1^t e^{-\gamma(t-t')} (f_1 + f_2)^2 dt')^{\frac{1}{2}} \leq \gamma^{-\frac{1}{2+\epsilon}} \|f_1\|_{L^{\frac{2(2+\epsilon)}{\epsilon}}(1,t)} + \gamma^{-\frac{1}{2}} \|f_2\|_{L^\infty(1,t)}$ for functions $(f_1, f_2) = (f_1, f_2)(t')$,

$$\Big(\int_1^t e^{-\gamma(t-t')} (t'^{\frac{3}{4}} \|\nabla \dot{u}\|_{L^{2+\epsilon}})^2 \, dt' \Big)^{\frac{1}{2}}$$

$$
\begin{aligned}
&\lesssim_\gamma \left(\|t'^{\frac{1}{2}+\delta_-}\sigma^{\frac{1}{2}-\delta_-}\nabla\dot u\|_{L^\infty_t L^2} + \|t'^{\delta_-}\nabla u\|_{L^\infty_t L^2}\|t'^{\frac{3}{4}}\nabla u\|_{L^\infty_t L^\infty}\right)^{\frac{2}{2+\epsilon}}\\
&\quad \times \left(\|t'^{\frac{1}{2}+\delta_-}\sigma^{\frac{1}{2}-\delta_-}\ddot u\|_{L^2_t L^2} + (1+\|(\nabla u, t'^{\frac{1}{2}}\dot u)\|_{L^\infty_t L^2})\|t'^{\frac{3}{4}}\nabla u\|_{L^\infty_t L^\infty}\right)^{\frac{\epsilon}{2+\epsilon}}\\
&\lesssim_\gamma C_0^{\frac{2}{2+\epsilon}}(C_0+1)^{\frac{\epsilon}{2+\epsilon}}V(t).
\end{aligned}
$$

□

Notice that we have used (4.3.21) to bound $\|\dot b + \dot{P(\rho)}\|_{L^2}$, which “a priori” has faster time decay than $\|\dot{P(\rho)}\|_{L^2} \sim \|\mathrm{div}\, u\|_{L^2}$. That is the reason why we show the decay estimate for $\nabla\dot u$ first and then for $\mathrm{div}\, u$ later, see below.

4.3.2.3. Estimates for $|\!|\!|\mathrm{div}\, u|\!|\!|_\epsilon$ and hence for $|\!|\!|\nabla u|\!|\!|_\epsilon$

Proposition 4.3.6. *Let $\epsilon \in (0, \epsilon_0]$ with $\epsilon_0 \in (0, 2]$ given in Lemma 4.1.7. Under the assumptions of Proposition 4.3.2 with $\delta \in (\frac{1}{2} - \frac{1}{4}\frac{\epsilon}{2+\epsilon}, \frac{1}{2})$, we have for $t > 0$*

$$|\!|\!|\mathrm{div}\, u|\!|\!|_\epsilon \le \widetilde C_0 V(t), \tag{4.3.24}$$

and consequently,

$$|\!|\!|\nabla u|\!|\!|_\epsilon \le \widetilde C_0 V(t). \tag{4.3.25}$$

Proof. We first notice that on the finite time interval $[0, 1]$ we can use the interpolation inequality to estimate

$$|\!|\!|\mathrm{div}\, u|\!|\!|_\epsilon(t=1) \le \|\nabla u\|_{(L^2\cap L^\infty)(0,1;L^2)}^{\frac{2}{2+\epsilon}} |\!|\!|\nabla u|\!|\!|_\infty^{\frac{\epsilon}{2+\epsilon}}(t=1) \le C_0^{\frac{2}{2+\epsilon}}V(t).$$

In the following we restrict ourselves to large times $t \ge 1$. We are going to show the time integrability of $\|\mathrm{div}\, u\|_{L^{2+\epsilon}}$ by use of Proposition 4.3.5 first, and then establish its time-weighted uniform bound by use of ($\dot{\mathrm{P}}'$).

Estimates for $\|\mathrm{div}\, u\|_{L^1([1,t);L^{2+\epsilon})}$ and $\|t'^{\frac{1}{2}}\mathrm{div}\, u\|_{L^2([1,t);L^{2+\epsilon})}$. Recall ($\dot{\mathrm{P}}$). We apply Gagliardo-Nirenberg's inequality to $(-\Delta)^{-1}\mathrm{div}\,(\rho\ddot u) = -\rho P'(\rho)\mathrm{div}\, u - g_1$ to obtain

$$\|\mathrm{div}\, u\|_{L^{2+\epsilon}} \le C_*\big(\|\mathrm{div}\, u\|_{L^2} + \|g_1\|_{L^2}\big)^{\frac{2}{2+\epsilon}}\|\ddot u\|_{L^2}^{\frac{\epsilon}{2+\epsilon}} + \|g_1\|_{L^{2+\epsilon}}.$$

Since $\|g_1\|_{L^r} \le C(\|u\|_{L^\infty}\|\dot u\|_{L^r} + \|\nabla u\|_{L^\infty}\|\nabla u\|_{L^r} + \|\nabla\dot u\|_{L^r})$ for all $r \in (1, \infty)$, it follows that

$$
\begin{aligned}
\|\mathrm{div}\, u\|_{L^{2+\epsilon}} \le C_*\Big(&\big(\|\mathrm{div}\, u\|_{L^2} + \|u\|_{L^\infty}\|\dot u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla u\|_{L^2} + \|\nabla\dot u\|_{L^2}\big)^{\frac{2}{2+\epsilon}}\|\ddot u\|_{L^2}^{\frac{\epsilon}{2+\epsilon}}\\
&+ \|u\|_{L^\infty}\|\dot u\|_{L^{2+\epsilon}} + \|\nabla u\|_{L^\infty}\|\nabla u\|_{L^{2+\epsilon}} + \|\nabla\dot u\|_{L^{2+\epsilon}}\Big).
\end{aligned}
$$

Hence, by (4.3.1),

$$
\begin{aligned}
&\|\mathrm{div}\, u\|_{L^1([1,t);L^{2+\epsilon})} + \|t'^{\frac{1}{2}}\mathrm{div}\, u\|_{L^2([1,t);L^{2+\epsilon})}\\
&\le C_*\Big(\big(\|t'^{\delta_-}\mathrm{div}\, u\|_{L^2_t L^2} + \|u\|_{L^\infty([1,t);L^\infty)}\|t'^{\delta_-}\dot u\|_{L^2_t L^2}\\
&\quad + \|t'^{\frac{3}{4}}\nabla u\|_{L^\infty_t L^\infty}\|\nabla u\|_{L^2_t L^2} + \|t'^{\frac{1}{2}+\delta_-}\nabla\dot u\|_{L^2_t L^2}\big)^{\frac{2}{2+\epsilon}}\|t'^{\frac{1}{2}+\delta_-}\sigma^{\frac{1}{2}-\delta_-}\ddot u\|_{L^2_t L^2}^{\frac{\epsilon}{2+\epsilon}}\\
&\quad + \|u\|_{L^\infty([1,t);L^\infty)}\big(\|\dot u\|_{L^1([1,t);L^{2+\epsilon})} + \|t'^{\frac{1}{2}}\dot u\|_{L^2([1,t);L^{2+\epsilon})}\big) + \|t'^{\frac{3}{4}}\nabla u\|_{L^\infty_t L^\infty}\|\nabla u\|_{L^2([1,t);L^{2+\epsilon})}
\end{aligned}
$$

$$+\|\nabla\dot{u}\|_{L^1([1,t);L^{2+\epsilon})}+\|t'^{\frac{1}{2}}\nabla\dot{u}\|_{L^2([1,t);L^{2+\epsilon})}\Big),$$

and we can further use the interpolation inequalities

$$\begin{aligned}\|u\|_{L^\infty([1,t);L^\infty)}&\lesssim\|u\|^{\frac{1}{2}}_{L^\infty([1,t);L^2)}\|t'^{\frac{3}{4}}\nabla u\|^{\frac{1}{2}}_{L^\infty_t L^\infty},\\ \|\nabla u\|_{L^2([1,t);L^{2+\epsilon})}&\lesssim\|\nabla u\|^{\frac{2}{2+\epsilon}}_{L^2_t L^2}\|t'^{\frac{1}{2}}\nabla u\|^{\frac{\epsilon}{2+\epsilon}}_{L^2_t L^\infty},\\ \|\dot{u}\|_{L^1([1,t);L^{2+\epsilon})}+\|t'^{\frac{1}{2}}\dot{u}\|_{L^2([1,t);L^{2+\epsilon})}&\lesssim\|t'^{\delta_-}\dot{u}\|^{\frac{2}{2+\epsilon}}_{L^2([1,t);L^2)}\|t'^{\frac{1}{2}+\delta_-}\nabla\dot{u}\|^{\frac{\epsilon}{2+\epsilon}}_{L^2([1,t);L^2)},\end{aligned}\tag{4.3.26}$$

and (4.3.20) to obtain

$$\|\operatorname{div}u\|_{L^1(1,t;L^{2+\epsilon})}+\|t'^{\frac{1}{2}}\operatorname{div}u\|_{L^2(1,t;L^{2+\epsilon})}\le\widetilde{C}_0V(t).$$

Estimate for $\|t'^{\frac{3}{4}}\operatorname{div}u\|_{L^\infty([1,t);L^{2+\epsilon})}$. Recall ($\dot{\mathrm{P}}'$), where $\tilde{g}_1$ can be bounded by (similarly as the derivation for $\|\operatorname{div}u\|_{L^{2+\epsilon}}$-bound above)

$$\begin{aligned}\|\tilde{g}_1\|_{L^{2+\epsilon}}\le C_*\Big(&(\|(\operatorname{div}u,\nabla\dot{u})\|_{L^2}+\|u\|_{L^\infty}\|\dot{u}\|_{L^2}+\|\nabla u\|_{L^\infty}\|\nabla u\|_{L^2})^{\frac{2}{2+\epsilon}}\|\ddot{u}\|^{\frac{\epsilon}{2+\epsilon}}_{L^2}\\&+\|u\|_{L^\infty}\|\dot{u}\|_{L^{2+\epsilon}}+\|\nabla u\|_{L^\infty}\|\nabla u\|_{L^{2+\epsilon}}+\|\nabla\dot{u}\|_{L^{2+\epsilon}}\Big).\end{aligned}$$

We multiply both sides of ($\dot{\mathrm{P}}'$) by $(2+\epsilon)\nu(\rho)\operatorname{div}u|\nu(\rho)\operatorname{div}u|^\epsilon$ to achieve

$$D_t(|\nu(\rho)\operatorname{div}u|^{2+\epsilon})+(2+\epsilon)\frac{\rho P'(\rho)}{\nu(\rho)}|\nu(\rho)\operatorname{div}u|^{2+\epsilon}=(2+\epsilon)\nu(\rho)\operatorname{div}u|\nu(\rho)\operatorname{div}u|^\epsilon\tilde{g}_1.$$

We integrate the above equation with respect to x and integrate by parts, to obtain by Hölder's inequality that

$$\frac{d}{dt}\|\nu(\rho)\operatorname{div}u\|^{2+\epsilon}_{L^{2+\epsilon}}+c_*\|\nu(\rho)\operatorname{div}u\|^{2+\epsilon}_{L^{2+\epsilon}}\le(2+\epsilon)\|(\tilde{g}_1,\nu(\rho)(\operatorname{div}u)^2)\|_{L^{2+\epsilon}}\|\nu(\rho)\operatorname{div}u\|^{1+\epsilon}_{L^{2+\epsilon}},$$

and further multiplication by $t^{\frac{3}{2}}\|\nu(\rho)\operatorname{div}u\|^{-\epsilon}_{L^{2+\epsilon}}$ and an application of Young's inequality yield

$$\begin{aligned}&\frac{d}{dt}(t^{\frac{3}{4}}\|\nu(\rho)\operatorname{div}u\|_{L^{2+\epsilon}})^2+c_*(t^{\frac{3}{4}}\|\nu(\rho)\operatorname{div}u\|_{L^{2+\epsilon}})^2\\&\le(t^{\frac{3}{4}}\|(\tilde{g}_1,(\operatorname{div}u)^2)\|_{L^{2+\epsilon}})^2+\frac{3}{2}t^{\frac{1}{2}}\|\nu(\rho)\operatorname{div}u\|^2_{L^{2+\epsilon}}.\end{aligned}$$

We integrate over $(1,t)$ to derive

$$\begin{aligned}(t^{\frac{3}{4}}\|\operatorname{div}u(t)\|_{L^{2+\epsilon}})^2\le{}&C_*e^{-c_*(t-1)}\|\operatorname{div}u|_{t=1}\|^2_{L^{2+\epsilon}}\\&+C_*\int_1^t e^{-c_*(t-t')}\Big[(t'^{\frac{3}{4}}\|(\tilde{g}_1,(\operatorname{div}u)^2)\|_{L^{2+\epsilon}})^2+\frac{3}{2}t'^{\frac{1}{2}}\|\operatorname{div}u\|^2_{L^{2+\epsilon}}\Big]dt'.\end{aligned}$$

where by Proposition 4.3.5 with $\gamma=c_*$, (4.3.26) and the fact that $E_0\le C_0$,

$$\begin{aligned}&\int_1^t e^{-c_*(t-t')}(t'^{\frac{3}{4}}\|(\tilde{g}_1,(\operatorname{div}u)^2)\|_{L^{2+\epsilon}})^2dt'\\&\le C_*\Big((\|t'^{\frac{1}{2}+\delta_-}(\operatorname{div}u,\sigma^{\frac{1}{2}-\delta_-}\nabla\dot{u})\|_{L^\infty_t L^2}+\|u\|_{L^\infty([1,t);L^\infty)}\|t'^{\frac{1}{2}+\delta_-}\dot{u}\|_{L^\infty_t L^2}\\&\quad+\|t'^{\frac{3}{4}}\nabla u\|_{L^\infty_t L^\infty}\|\nabla u\|_{L^\infty_t L^2})^{\frac{2}{2+\epsilon}}\|t'^{\frac{1}{2}+\delta_-}\sigma^{\frac{1}{2}-\delta_-}\ddot{u}\|^{\frac{\epsilon}{2+\epsilon}}_{L^2_t L^2}+\|u\|_{L^\infty(1,t;L^\infty)}\|t'^{\frac{3}{4}}\dot{u}\|_{L^{\frac{2(2+\epsilon)}{\epsilon}}(1,t;L^{2+\epsilon})}\end{aligned}$$

$$+ \|t'^{\frac{3}{4}}\nabla u\|_{L^\infty_t L^\infty} \|\nabla u\|^{\frac{2}{2+\epsilon}}_{L^\infty_t L^2} \|t'^{\frac{1}{2}}\nabla u\|^{\frac{\epsilon}{2+\epsilon}}_{L^2_t L^\infty} + \Big(\int_1^t e^{-c_*(t-t')}(t'^{\frac{3}{4}}\|\nabla \dot{u}\|_{L^{2+\epsilon}})^2 dt'\Big)^{\frac{1}{2}}\Big)^2$$
$$\le C_*\Big(C_0^{\frac{1}{2+\epsilon}}(C_0^{\frac{1}{2}}+1)\Big)^2 V(t).$$

Combining this with

$$\int_1^t e^{-c_*(t-t')} t'^{\frac{1}{2}} \|\operatorname{div} u\|^2_{L^{2+\epsilon}} dt' \le \|t'^{\frac{1}{2}} \operatorname{div} u\|^2_{L^2_t L^{2+\epsilon}} \le \widetilde{C}_0^2 V(t),$$

yields for $t \ge 1$,

$$t^{\frac{3}{4}} \|\operatorname{div} u(t)\|_{L^{2+\epsilon}} \le C_*\Big(\|\operatorname{div} u|_{t=1}\|_{L^{2+\epsilon}} + \widetilde{C}_0 V(t)\Big).$$

We simply estimate

$$\|\operatorname{div} u|_{t=1}\|_{L^{2+\epsilon}} \le \|t'^{\frac{3}{4}} \operatorname{div} u\|_{L^\infty(0,2;L^{2+\epsilon})} \le \|\nabla u\|^{\frac{2}{2+\epsilon}}_{L^\infty_t L^2} \|t'^{\frac{3}{4}}\nabla u\|^{\frac{\epsilon}{2+\epsilon}}_{L^\infty_t L^\infty} \le C_* C_0^{\frac{1}{2+\epsilon}} V(t),$$

so that

$$\|t'^{\frac{3}{4}} \operatorname{div} u\|_{L^\infty([1,t);L^{2+\epsilon})} \le \widetilde{C}_0 V(t).$$

Summing up, we have shown (4.3.24).

Combining now (4.1.37), (4.3.19), and (4.3.24) we get (4.3.25). □

4.3.3. Step III Preliminary estimates for tangential regularity

We establish the preliminary estimates for the tangential regularity in (ρTR)-(uTR) in this subsection.

Proposition 4.3.7. *Let $\epsilon \in (0, \epsilon_0]$ with $\epsilon_0 \in (0, 2]$ given in Lemma 4.1.7. Let (ρ, u, τ) be a sufficiently smooth and fast decaying solution of* (CNS)-(τ). *Then*

$$\|\tau\|_{L^\infty} \le \|\tau_0\|_{L^\infty} \exp(C\|\nabla u\|_{L^1_t L^\infty}), \quad \||\tau|^{-1}\|_{L^\infty} \le \||\tau_0|^{-1}\|_{L^\infty} \exp(C\|\nabla u\|_{L^1_t L^\infty}), \tag{4.3.27}$$

and

$$\|(\nabla\bar{\tau}, \partial_{\bar{\tau}}\rho)(t)\|_{L^{2+\epsilon}} + \|\!|(\partial_{\bar{\tau}}\rho, \partial_{\bar{\tau}}\nabla u, \nabla\partial_{\bar{\tau}} u)\|\!|_\epsilon$$
$$\le C_*\Big(\|(\nabla\bar{\tau}_0, \partial_{\bar{\tau}_0}\rho_0)\|_{L^{2+\epsilon}} + \|\!|(\nabla a, \nabla b)\|\!|_\epsilon\Big) \exp(C_* W(t)). \tag{4.3.28}$$

Proof. The (classical) estimates (4.3.27) follow from the equation (τ) itself, see e.g. [53, (4.3)].

From (τ) we derive the equation for $\bar{\tau}$ to be

$$D_t\bar{\tau} = \partial_{\bar{\tau}} u - \bar{\tau}(\bar{\tau}\cdot\partial_{\bar{\tau}} u). \tag{4.3.29}$$

We apply ∇ and take the L^2-inner product with $|\nabla\bar{\tau}|^\epsilon \nabla\bar{\tau}$ to derive

$$\|\nabla\bar{\tau}(t)\|_{L^{2+\epsilon}} \le \|\nabla\bar{\tau}_0\|_{L^{2+\epsilon}} + C\int_0^t \Big(\|\nabla\bar{\tau}\|_{L^{2+\epsilon}}\|\nabla u\|_{L^\infty} + \|\partial_{\bar{\tau}}\nabla u\|_{L^{2+\epsilon}}\Big) dt', \tag{4.3.30}$$

where (4.1.39) and (4.1.71) imply the estimate for $\|\partial_{\bar{\tau}}\nabla u\|_{L^{2+\epsilon}}$.

Estimate for $\|\partial_{\bar\tau}\rho\|_\epsilon$. Recall the equation (4.2.13) for the "renormalized" tangential derivative of the density $\theta = \sqrt{\frac{P'(\rho)}{\rho}}\partial_{\bar\tau}\rho$:

$$D_t\theta = \mathcal{A}\theta + \Theta, \tag{4.3.31}$$

where $\mathcal{A}$ is given in Lemma 4.1.7 and

$$\Theta = \left[D_t, \sqrt{\frac{P'(\rho)}{\rho}}\right]\partial_{\bar\tau}\rho + \sqrt{\frac{P'(\rho)}{\rho}}\Big(-[\partial_{\bar\tau}, D_t]\rho - \partial_{\bar\tau}\rho\operatorname{div} u - \rho\mathcal{N}^{-1}(\partial_{\bar\tau}b - \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}\partial_{\bar\tau}a)$$
$$+ \rho\mathcal{N}^{-1}[\partial_{\bar\tau}, \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}]a - \rho[\partial_{\bar\tau}, \mathcal{N}^{-1}](P(\rho) - \tilde P + b - \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}a)\Big).$$

We take the L^2-inner product between (4.3.31) and $(2+\epsilon)|\theta|^\epsilon\theta$ and apply the decay estimate (4.1.44) to derive

$$\frac{d}{dt}\|\theta\|_{L^{2+\epsilon}} \le -w\|\theta\|_{L^{2+\epsilon}} + \|\Theta\|_{L^{2+\epsilon}} + \frac{1}{2+\epsilon}\|\operatorname{div} u\|_{L^\infty}\|\theta\|_{L^{2+\epsilon}}, \tag{4.3.32}$$

from which we multiply by e^{tw} and integrate in time twice, using Young's integral inequality, to derive

$$\|\theta\|_{L^1_tL^{2+\epsilon}} \le C_*\Big(\|\partial_{\bar\tau_0}\rho_0\|_{L^{2+\epsilon}} + \|\Theta\|_{L^1_tL^{2+\epsilon}} + \int_0^t \|\operatorname{div} u\|_{L^\infty}\|\theta\|_{L^{2+\epsilon}}dt'\Big). \tag{4.3.33}$$

We now establish a bound for $\|\Theta\|_{L^{2+\epsilon}}$:

- By the equation $(\text{CNS})_1$, $\left\|\left[D_t, \sqrt{\frac{P'(\rho)}{\rho}}\right]\partial_{\bar\tau}\rho\right\|_{L^{2+\epsilon}} \le C_*\|\operatorname{div} u\|_{L^\infty}\|\partial_{\bar\tau}\rho\|_{L^{2+\epsilon}}$;
- By the equation (4.3.29), $\|[\partial_{\bar\tau}, D_t]\rho\|_{L^{2+\epsilon}} = \|(D_t\bar\tau - \partial_{\bar\tau}u)\cdot\nabla\rho\|_{L^{2+\epsilon}} = \|(\bar\tau\cdot\partial_{\bar\tau}u)\partial_{\bar\tau}\rho\|_{L^{2+\epsilon}} \le \|\nabla u\|_{L^\infty}\|\partial_{\bar\tau}\rho\|_{L^{2+\epsilon}}$;
- By Hölder's inequality, $\|\partial_{\bar\tau}\rho\operatorname{div} u\|_{L^{2+\epsilon}} \le \|\operatorname{div} u\|_{L^\infty}\|\partial_{\bar\tau}\rho\|_{L^{2+\epsilon}}$;
- By Lemma 4.1.7, $\|\rho\mathcal{N}^{-1}(\partial_{\bar\tau}b - \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}\partial_{\bar\tau}a)\|_{L^{2+\epsilon}} \le C_*\|(\partial_{\bar\tau}a, \partial_{\bar\tau}b)\|_{L^{2+\epsilon}} \le C_*\|(\nabla a, \nabla b)\|_{L^{2+\epsilon}}$;
- Recalling $\mathcal{N}^{-1}(P(\rho) - \tilde P + b - \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}a) = \operatorname{div} u$ and $\mathcal{N} = \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}\mathcal{Q}_\mu + \mathcal{R}_{\mu,\lambda}$, we rewrite the last two commutators in Θ as

$$\begin{aligned}
&\rho\mathcal{N}^{-1}[\partial_{\bar\tau}, \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}]a - \rho[\partial_{\bar\tau}, \mathcal{N}^{-1}](P(\rho) - \tilde P + b - \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}a)\\
&= \rho\mathcal{N}^{-1}\Big([\partial_{\bar\tau}, \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}]a + [\partial_{\bar\tau}, \mathcal{N}]\mathcal{N}^{-1}(P(\rho) - \tilde P + b - \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}a)\Big)\\
&= \rho\mathcal{N}^{-1}\Big([\partial_{\bar\tau}, \mathcal{Q}_\mu]\mathcal{R}_\mu^{-1}a + \mathcal{Q}_\mu[\partial_{\bar\tau}, \mathcal{R}_\mu^{-1}]a + [\partial_{\bar\tau}, \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}\mathcal{Q}_\mu]\operatorname{div} u + [\partial_{\bar\tau}, \mathcal{R}_{\mu,\lambda}]\operatorname{div} u\Big)\\
&= \rho\mathcal{N}^{-1}\Big([\partial_{\bar\tau}, \mathcal{Q}_\mu](\mathcal{R}_\mu^{-1}a + \mathcal{R}_\mu^{-1}\mathcal{Q}_\mu\operatorname{div} u) - \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}[\partial_{\bar\tau}, \mathcal{R}_\mu](\mathcal{R}_\mu^{-1}a + \mathcal{R}_\mu^{-1}\mathcal{Q}_\mu\operatorname{div} u)\\
&\qquad + \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}[\partial_{\bar\tau}, \mathcal{Q}_\mu]\operatorname{div} u + [\partial_{\bar\tau}, \mathcal{R}_{\mu,\lambda}]\operatorname{div} u\Big)\\
&= \rho\mathcal{N}^{-1}\Big(([\partial_{\bar\tau}, \mathcal{Q}_\mu]\omega + [\partial_{\bar\tau}, \mathcal{R}_{\mu,\lambda}]\operatorname{div} u) - \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}([\partial_{\bar\tau}, \mathcal{R}_\mu]\omega - [\partial_{\bar\tau}, \mathcal{Q}_\mu]\operatorname{div} u)\Big).
\end{aligned}$$

  where we used $\mathcal{R}_\mu^{-1}a + \mathcal{R}_\mu^{-1}\mathcal{Q}_\mu\operatorname{div} u = \omega$. Since $b + P(\rho) - \tilde P = \mathcal{Q}_\mu\omega + \mathcal{R}_{\mu,\lambda}\operatorname{div} u$ and $a = \mathcal{R}_\mu\omega - \mathcal{Q}_\mu\operatorname{div} u$, Lemma 4.1.7 implies

$$\begin{aligned}
&\|\rho\mathcal{N}^{-1}[\partial_{\bar\tau}, \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}]a - \rho[\partial_{\bar\tau}, \mathcal{N}^{-1}](P(\rho) - \tilde P + b - \mathcal{Q}_\mu\mathcal{R}_\mu^{-1}a)\|_{L^{2+\epsilon}}\\
&\le C_*\|(\nabla\bar\tau, \partial_{\bar\tau}\mu(\rho))\|_{L^{2+\epsilon}}\|(\nabla u, a, b + P(\rho) - \tilde P)\|_{L^\infty}.
\end{aligned}$$

Summing up, we integrate in time to obtain

$$\|\Theta\|_{L^1_t L^{2+\epsilon}} \le C_*\Big(\|(\nabla a, \nabla b)\|_{L^1_t L^{2+\epsilon}} + \int_0^t \|(\nabla \bar\tau, \partial_{\bar\tau}\rho)\|_{L^{2+\epsilon}} \|(\nabla u, a, b + P(\rho) - \tilde P)\|_{L^\infty} dt'\Big),$$

and hence (4.3.33) implies

$$\begin{aligned} \|\partial_{\bar\tau}\rho\|_{L^1_t L^{2+\epsilon}} \le C_*\Big(&\|\partial_{\bar\tau_0}\rho_0\|_{L^{2+\epsilon}} + \|(\nabla a, \nabla b)\|_{L^1_t L^{2+\epsilon}} \\ &+ \int_0^t \|(\nabla \bar\tau, \partial_{\bar\tau}\rho)\|_{L^{2+\epsilon}} \|(\nabla u, a, b + P(\rho) - \tilde P)\|_{L^\infty} dt'\Big). \end{aligned} \tag{4.3.34}$$

Similarly, one can derive from (4.3.32) the following estimates

$$\begin{aligned} \|t'^{\frac12}\partial_{\bar\tau}\rho\|_{L^2_t L^{2+\epsilon}} &\le C_*\Big(\|\partial_{\bar\tau}\rho\|^{\frac12}_{L^1_t L^{2+\epsilon}} \|\partial_{\bar\tau}\rho\|^{\frac12}_{L^\infty_t L^{2+\epsilon}} + \|t'^{\frac12}(\nabla a, \nabla b)\|_{L^2_t L^{2+\epsilon}} \\ &\qquad + \|(\nabla\bar\tau, \partial_{\bar\tau}\rho)\|_{L^\infty_t L^{2+\epsilon}} \|t'^{\frac12}(\nabla u, a, b + P(\rho) - \tilde P)\|_{L^2_t L^\infty}\Big), \\ \|t'^{\frac34}\partial_{\bar\tau}\rho\|_{L^\infty_t L^{2+\epsilon}} &\le C_*\Big(\|t'^{\frac12}\partial_{\bar\tau}\rho\|^{\frac12}_{L^2_t L^{2+\epsilon}} \|\partial_{\bar\tau}\rho\|^{\frac12}_{L^\infty_t L^{2+\epsilon}} + \|t'^{\frac34}(\nabla a, \nabla b)\|_{L^\infty_t L^{2+\epsilon}} \\ &\qquad + \|(\nabla\bar\tau, \partial_{\bar\tau}\rho)\|_{L^\infty_t L^{2+\epsilon}} \|t'^{\frac34}(\nabla u, a, b + P(\rho) - \tilde P)\|_{L^\infty_t L^\infty}\Big). \end{aligned}$$

Adding the above inequalities to (4.3.34) and applying Young's inequality yields

$$|||\partial_{\bar\tau}\rho|||_\epsilon \le C_*\Big(\|\partial_{\bar\tau_0}\rho_0\|_{L^{2+\epsilon}} + \|\partial_{\bar\tau}\rho\|_{L^\infty_t L^{2+\epsilon}} + |||(\nabla a, \nabla b)|||_\epsilon + \|(\nabla\bar\tau, \partial_{\bar\tau}\rho)\|_{L^\infty_t L^{2+\epsilon}} W(t)\Big). \tag{4.3.35}$$

Conclusion. We insert (4.3.34) into (4.1.71) to get the same bound (4.3.34) for $\|\partial_{\bar\tau}\nabla u\|_{L^1_t L^{2+\epsilon}}$, and hence (4.3.30) and (4.3.32) imply

$$\begin{aligned} \|(\nabla\bar\tau, \partial_{\bar\tau}\rho)(t)\|_{L^{2+\epsilon}} \le C_*\Big(&\|(\nabla\bar\tau_0, \partial_{\bar\tau_0}\rho_0)\|_{L^{2+\epsilon}} + \|(\nabla a, \nabla b)\|_{L^1_t L^{2+\epsilon}} \\ &+ \int_0^t \|(\nabla \bar\tau, \partial_{\bar\tau}\rho)\|_{L^{2+\epsilon}} \|(\nabla u, a, b + P(\rho) - \tilde P)\|_{L^\infty} dt'\Big). \end{aligned}$$

Gronwall's inequality and (4.3.35), (4.1.71) finally lead to (4.3.28).

□

4.3.4. Step IV Lipschitz bound and conclusion of a priori estimates

The goal of this subsection is to prove the a priori Lipschitz bound for the velocity field and to close all of the preceding estimates.

4.3.4.1. Preliminary Lipschitz bound

Proposition 4.3.8 (Preliminary Lipschitz estimate). *Under the assumptions of Proposition 4.3.7 we have*

$$\begin{aligned} W(t) \le C_*(1 + \|P(\rho_0) - \tilde P\|_{L^\infty}) + C_*(|||\nabla u|||_\epsilon + \|P(\rho) - \tilde P\|_{L^\infty_t L^{2+\epsilon}} |||\nabla u|||_\infty)^{\frac{\epsilon}{2+\epsilon}} \\ \times (|||(\nabla a, \nabla b)|||_\epsilon + \|(\nabla\bar\tau, \partial_{\bar\tau}\rho)\|_{L^\infty_t L^{2+\epsilon}} W(t))^{\frac{2}{2+\epsilon}}. \end{aligned} \tag{4.3.36}$$

Proof. In view of the velocity gradient decomposition (4.1.59) we have

$$|||\nabla u|||_\infty \le C_*\Big(|||(\partial_{\bar\tau}u,\alpha)|||_\infty + \Big|\Big|\Big|\beta + P(\rho) - \tilde P\Big|\Big|\Big|_\infty\Big). \tag{4.3.37}$$

By the interpolation inequality $\|f\|_{L^\infty} \le C_\epsilon \|f\|_{L^{2+\epsilon}}^{\frac{\epsilon}{2+\epsilon}} \|\nabla f\|_{L^{2+\epsilon}}^{\frac{2}{2+\epsilon}}$ and (4.1.66), the first term is bounded by

$$|||(\partial_{\bar\tau}u, a, \alpha)|||_\infty \le C_* |||\nabla u|||_\epsilon^{\frac{\epsilon}{2+\epsilon}} \Big(|||(\nabla a, \partial_{\bar\tau}\nabla u)|||_\epsilon + \|(\nabla\bar\tau, \partial_{\bar\tau}\rho)\|_{L^\infty_t L^{2+\epsilon}} |||\nabla u|||_\infty\Big)^{\frac{2}{2+\epsilon}}. \tag{4.3.38}$$

Fix $t_0 > 0$. In the following we consider time points $t \in (0, t_0)$ and establish estimates independent of t_0.

Let $\varphi \in C^\infty(\mathbb{R}^2; [0,1])$ be a smooth function such that $\varphi(\xi) = \begin{cases} 1, & |\xi| < 1, \\ 0, & |\xi| \ge 2, \end{cases}$ and set $\varphi_N = \varphi(2^{-N}\cdot)$ for some $N = N(t_0) \in \mathbb{R}$ to be determined later. We define the low and high frequencies of a function f, respectively, as

$$f^l := \varphi_N(D) f = \mathcal{F}^{-1}(\varphi_N \hat f) = \check\varphi_N * f,$$
$$f^h := (\mathrm{Id} - \varphi_N(D)) f = \mathcal{F}^{-1}((1-\varphi_N)\hat f).$$

A direct computation shows that

$$\begin{aligned}\|\beta + P(\rho) - \tilde P\|_{L^\infty} &\le \|(\beta + P(\rho) - \tilde P)^l\|_{L^\infty} + \|\beta^h\|_{L^\infty} + \|(P(\rho)-\tilde P)^h\|_{L^\infty} \\ &\le C_*(2^{N\frac{2}{2+\epsilon}}\|\beta + P(\rho) - \tilde P\|_{L^{2+\epsilon}} + 2^{-N\frac{\epsilon}{2+\epsilon}}\|\nabla\beta\|_{L^{2+\epsilon}}) + \|(P(\rho)-\tilde P)^h\|_{L^\infty},\end{aligned} \tag{4.3.39}$$

where we can bound

- $\beta + P(\rho) - \tilde P = \nu(\rho)\mathrm{div}\, u - 2\mu(\rho)\bar\tau\cdot\partial_{\bar\tau}u$ by

$$\|\beta + P(\rho) - \tilde P\|_{L^{2+\epsilon}} \le C_*\|\nabla u\|_{L^{2+\epsilon}}; \tag{4.3.40}$$

- $\nabla\beta$ by (recalling (4.1.67))

$$\|\nabla\beta\|_{L^{2+\epsilon}} \le \|\nabla b\|_{L^{2+\epsilon}} + C_*(\|(\nabla\bar\tau, \partial_{\bar\tau}\rho)\|_{L^{2+\epsilon}}\|\nabla u\|_{L^\infty} + \|\partial_{\bar\tau}\nabla u\|_{L^{2+\epsilon}}).$$

It remains to establish a bound for $\|(P(\rho)-\tilde P)^h\|_{L^\infty}$. Our idea is to explore the damping effect of the density function in the system (4.2.14). We estimate g_2 defined in (4.2.15) one by one (see e.g. [18, Chapter 2]):

- $\|d(\rho)\beta^h\|_{L^\infty} \le C_* 2^{-N\frac{\epsilon}{2+\epsilon}}\|\nabla\beta\|_{L^{2+\epsilon}}$;
- $\|d(\rho)2\mu(\rho)\bar\tau\cdot\partial_{\bar\tau}u\|_{L^\infty} \le C_*\|\partial_{\bar\tau}u\|_{L^\infty}$;
- $\|[u\cdot\nabla, \varphi_N(D)](P(\rho)-\tilde P)\|_{L^\infty} \le C_* 2^{N\frac{2}{2+\epsilon}}\|\nabla u\|_{L^\infty}\|P(\rho)-\tilde P\|_{L^{2+\epsilon}}$;
- $\|\varphi_N(D)(d(\rho)(P(\rho)-\tilde P+\beta)) - d(\rho)\varphi_N(D)(P(\rho)-\tilde P+\beta)\|_{L^\infty} \le C_* 2^{N\frac{2}{2+\epsilon}}\|\nabla u\|_{L^{2+\epsilon}}$ by (4.3.40).

Integrating (4.2.14) in time and taking the L^∞_x-norm we derive

$$\begin{aligned}&\|(P(\rho)-\tilde P)^h(t)\|_{L^\infty} \le e^{-c_* t}\|P(\rho_0) - \tilde P\|_{L^\infty} \\ &+ C_* e^{-c_*\cdot} *_t \big(2^{-N\frac{\epsilon}{2+\epsilon}}\|\nabla\beta\|_{L^{2+\epsilon}} + \|\nabla u\|_{L^{2+\epsilon}}^{\frac{\epsilon}{2+\epsilon}}(\|\partial_{\bar\tau}\nabla u\|_{L^{2+\epsilon}} + \|\nabla\bar\tau\|_{L^{2+\epsilon}}\|\nabla u\|_{L^\infty})^{\frac{2}{2+\epsilon}} \\ &\qquad + 2^{N\frac{2}{2+\epsilon}}\|\nabla u\|_{L^\infty}\|P(\rho)-\tilde P\|_{L^{2+\epsilon}} + 2^{N\frac{2}{2+\epsilon}}\|\nabla u\|_{L^{2+\epsilon}}).\end{aligned}$$

Inserting this into (4.3.39) and integrating the resulting inequality over $(0, t_0)$ we get

$$\begin{aligned}
&\|\beta + P(\rho) - \tilde{P}\|_{L^1_{t_0} L^\infty} \\
&\le C_*\Big(2^{N\frac{2}{2+\epsilon}}(\|\nabla u\|_{L^1_{t_0}L^{2+\epsilon}} + \|P(\rho)-\tilde{P}\|_{L^\infty_{t_0}L^{2+\epsilon}}\|\nabla u\|_{L^1_{t_0}L^\infty}) + 2^{-N\frac{\epsilon}{2+\epsilon}}\|\nabla\beta\|_{L^1_{t_0}L^{2+\epsilon}} \\
&\quad + \|P(\rho_0)-\tilde{P}\|_{L^\infty} + \|\nabla u\|^{\frac{\epsilon}{2+\epsilon}}_{L^1_{t_0}L^{2+\epsilon}}\big(\|\partial_{\bar{\tau}}\nabla u\|_{L^1_{t_0}L^{2+\epsilon}} + \|\nabla\bar{\tau}\|_{L^\infty_{t_0}L^{2+\epsilon}}\|\nabla u\|_{L^1_{t_0}L^\infty}\big)^{\frac{2}{2+\epsilon}}\Big).
\end{aligned}$$

We chose N as

$$2^N = \frac{\|\nabla\beta\|_{L^1_{t_0}L^{2+\epsilon}}}{\|\nabla u\|_{L^1_{t_0}L^{2+\epsilon}} + \|\nabla u\|_{L^1_{t_0}L^\infty}\|P(\rho)-\tilde{P}\|_{L^\infty_{t_0}L^{2+\epsilon}}},$$

so that (recalling (4.1.67))

$$\begin{aligned}
&\|\beta + P(\rho) - \tilde{P}\|_{L^1_{t_0}L^\infty} \\
&\le C_*\|P(\rho_0)-\tilde{P}\|_{L^\infty} + C_*\big(\|\nabla u\|_{L^1_{t_0}L^{2+\epsilon}} + \|P(\rho)-\tilde{P}\|_{L^\infty_{t_0}L^{2+\epsilon}}\|\nabla u\|_{L^1_{t_0}L^\infty}\big)^{\frac{\epsilon}{2+\epsilon}} \\
&\qquad \times \big(\|(\nabla b, \partial_{\bar{\tau}}\nabla u)\|_{L^1_{t_0}L^{2+\epsilon}} + \|(\nabla\bar{\tau}, \partial_{\bar{\tau}}\rho)\|_{L^\infty_{t_0}L^{2+\epsilon}}\|\nabla u\|_{L^1_{t_0}L^\infty}\big)^{\frac{2}{2+\epsilon}}.
\end{aligned}$$

To obtain the time-weighted $L^2_t(t'^{\frac{1}{2}}dt'; L^\infty)$- and $L^\infty_t(t'^{\frac{3}{4}}dt'; L^\infty)$-estimates we multiply (4.2.14) by $(P(\rho)-\tilde{P})^h$ and apply Young's inequality to obtain

$$(D_t + d(\rho))|(P(\rho)-\tilde{P})^h|^2 \le \frac{1}{d(\rho)}(g_2)^2,$$

which we multiply by t resp. $t^{\frac{3}{2}}$, so that with similar arguments as above and suitable choices for $N = N(t_0)$ we eventually get

$$\begin{aligned}
&\|t^{\frac{1}{2}}(\beta + P(\rho) - \tilde{P})\|_{L^2_{t_0}L^\infty} \\
&\le C_*\|(P(\rho)-\tilde{P})^h\|^{\frac{1}{2}}_{L^1_{t_0}L^\infty} + C_*\big(\|t^{\frac{1}{2}}\nabla u\|_{L^2_{t_0}L^{2+\epsilon}} + \|t^{\frac{1}{2}}\nabla u\|_{L^2_{t_0}L^\infty}\|P(\rho)-\tilde{P}\|_{L^\infty_{t_0}L^{2+\epsilon}}\big)^{\frac{\epsilon}{2+\epsilon}} \\
&\quad \times \big(\|t^{\frac{1}{2}}(\nabla b, \partial_{\bar{\tau}}\nabla u)\|_{L^2_{t_0}L^{2+\epsilon}} + \|(\nabla\bar{\tau}, \partial_{\bar{\tau}}\rho)\|_{L^\infty_{t_0}L^{2+\epsilon}}\|t^{\frac{1}{2}}\nabla u\|_{L^2_{t_0}L^\infty}\big)^{\frac{2}{2+\epsilon}}, \\
&\|t^{\frac{3}{4}}(\beta + P(\rho) - \tilde{P})\|_{L^\infty_{t_0}L^\infty} \\
&\le C_*\|t^{\frac{1}{2}}(P(\rho)-\tilde{P})^h\|^{\frac{1}{2}}_{L^2_{t_0}L^\infty} + C_*\big(\|t^{\frac{3}{4}}\nabla u\|_{L^\infty_{t_0}L^{2+\epsilon}} + \|t^{\frac{3}{4}}\nabla u\|_{L^\infty_{t_0}L^\infty}\|P(\rho)-\tilde{P}\|_{L^\infty_{t_0}L^{2+\epsilon}}\big)^{\frac{\epsilon}{2+\epsilon}} \\
&\quad \times \big(\|t^{\frac{3}{4}}(\nabla b, \partial_{\bar{\tau}}\nabla u)\|_{L^\infty_{t_0}L^{2+\epsilon}} + \|(\nabla\bar{\tau}, \partial_{\bar{\tau}}\rho)\|_{L^\infty_{t_0}L^{2+\epsilon}}\|t^{\frac{3}{4}}\nabla u\|_{L^\infty_{t_0}L^\infty}\big)^{\frac{2}{2+\epsilon}}.
\end{aligned}$$

Adding the estimates together, applying Young's inequality, recalling (4.1.71) and noticing that t_0 was arbitrary, we achieve the estimate (4.3.36) for $\left\|\left|(\beta + P(\rho) - \tilde{P}, b + P(\rho) - \tilde{P})\right|\right\|_\infty$ for any $t > 0$. Here we also used that exactly the same calculations yield the same bound for $\left\|\left|b + P(\rho) - \tilde{P}\right|\right\|_\infty$ (one only has to replace β by b in (4.3.39)).

Therefore, by virtue of (4.3.37) and (4.3.38), the esimate (4.3.36) holds. □

4.3.4.2. Preliminary density bounds

Proposition 4.3.9 (Preliminary density bounds). *Let the assumptions of Proposition 4.3.7 hold with the initial density satisfying $\rho_0 \in [\rho_*, \rho^*]$. Define $\rho_{\sup}(t) := \sup_{\mathbb{R}^2}\rho(t,\cdot)$ and*

$\rho_{\inf}(t) := \inf_{\mathbb{R}^2} \rho(t,\cdot)$. *Then*

$$\rho_{\sup}(t) \le \tilde\rho\Big(\frac{\rho^*}{\tilde\rho}\Big)^{e^{-c_* t}} \exp\Big(C_* E_0^{\frac{\epsilon}{2+2\epsilon}} (\|(\nabla b, \partial_{\bar\tau}\nabla u)\|_{L^1_t L^{2+\epsilon}} + \|(\nabla\bar\tau, \partial_{\bar\tau}\rho)\|_{L^\infty_t L^{2+\epsilon}} \|\nabla u\|_{L^1_t L^\infty})^{\frac{2+\epsilon}{2+2\epsilon}}\Big),$$
$$\rho_{\inf}(t) \ge \tilde\rho\Big(\frac{\rho_*}{\tilde\rho}\Big)^{e^{-C_* t}} \exp\Big(-C_* E_0^{\frac{\epsilon}{2+2\epsilon}} (\|(\nabla b, \partial_{\bar\tau}\nabla u)\|_{L^1_t L^{2+\epsilon}} + \|(\nabla\bar\tau, \partial_{\bar\tau}\rho)\|_{L^\infty_t L^{2+\epsilon}} \|\nabla u\|_{L^1_t L^\infty})^{\frac{2+\epsilon}{2+2\epsilon}}\Big). \tag{4.3.41}$$

Proof. Recall (4.2.16). Denoting by $\mathcal{X}(t,x)$ the flow map $\partial_t \mathcal{X}(t,x) = u(t,\mathcal{X}(t,x))$, it follows that

$$\begin{aligned}&\frac{d}{dt}\Big(e^{\int_0^t h(\rho(s,\mathcal{X}(s,x)))ds} \log\frac{\rho}{\tilde\rho}(t,\mathcal{X}(t,x))\Big)\\ &= e^{\int_0^t h(\rho(s,\mathcal{X}(s,x)))ds}\Big(-\frac{1}{\nu(\rho)}\Big(\beta + 2\mu(\rho)\bar\tau\cdot\partial_{\bar\tau}u\Big)\Big)(t,\mathcal{X}(t,x)),\end{aligned}$$

and hence

$$\begin{aligned}&\frac{\rho(t,\mathcal{X}(t,x))}{\tilde\rho}\Big(\frac{\rho_0(x)}{\tilde\rho}\Big)^{-e^{-\int_0^t h(\rho(s,\mathcal{X}(s,x)))ds}}\\ &= \exp\Big(\int_0^t e^{-\int_s^t h(\rho(s',\mathcal{X}(s',x)))ds'}\Big(-\frac{\beta + 2\mu(\rho)\bar\tau\cdot\partial_{\bar\tau}u}{\nu(\rho)}\Big)(s,\mathcal{X}(s,x))ds\Big).\end{aligned}$$

This implies

$$\begin{aligned}\rho_{\sup}(t) &\le \tilde\rho\Big(\frac{\rho^*}{\tilde\rho}\Big)^{e^{-c_* t}} \exp\Big(C_*\int_0^t e^{-c_*(t-s)}\|(\beta,\partial_{\bar\tau}u)\|_{L^\infty} ds\Big),\\ \rho_{\inf}(t) &\ge \tilde\rho\Big(\frac{\rho_*}{\tilde\rho}\Big)^{e^{-C_* t}} \exp\Big(-C_*\int_0^t e^{-c_*(t-s)}\|(\beta,\partial_{\bar\tau}u)\|_{L^\infty} ds\Big).\end{aligned}$$

Using the interpolation inequality $\|f\|_{L^\infty} \le C_\epsilon \|f\|_{L^2}^{\frac{\epsilon}{2+2\epsilon}} \|\nabla f\|_{L^{2+\epsilon}}^{\frac{2+\epsilon}{2+2\epsilon}}$, the definition of $\beta = (n\otimes n) : T$ and (4.1.67), the integral in the exponential can be bounded by

$$\begin{aligned}&C_*\int_0^t e^{-c_*(t-s)}\|(\beta,\partial_{\bar\tau}u)\|_{L^2}^{\frac{\epsilon}{2+2\epsilon}} \|(\nabla\beta,\nabla\partial_{\bar\tau}u)\|_{L^{2+\epsilon}}^{\frac{2+\epsilon}{2+2\epsilon}} ds\\ &\le C_*(\|\nabla u\|_{L^2_t L^2} + \|P(\rho)-\tilde P\|_{L^\infty_t L^2})^{\frac{\epsilon}{2+2\epsilon}} (\|(\nabla\beta,\partial_{\bar\tau}\nabla u)\|_{L^1_t L^{2+\epsilon}} + \|\nabla\bar\tau\|_{L^\infty_t L^{2+\epsilon}}\|\nabla u\|_{L^1_t L^\infty})^{\frac{2+\epsilon}{2+2\epsilon}}\\ &\le C_* E_0^{\frac{\epsilon}{2+2\epsilon}} (\|(\nabla b,\partial_{\bar\tau}\nabla u)\|_{L^1_t L^{2+\epsilon}} + \|(\nabla\bar\tau,\partial_{\bar\tau}\rho)\|_{L^\infty_t L^{2+\epsilon}}\|\nabla u\|_{L^1_t L^\infty})^{\frac{2+\epsilon}{2+2\epsilon}},\end{aligned}$$

which implies (4.3.41). □

4.3.4.3. Bootstrap argument and conclusion of a priori estimates

Proposition 4.3.10. *Under the assumptions of Propositions 4.3.6 and 4.3.7, we have for* $t > 0$,

$$|\!|\!|\nabla u|\!|\!|_\infty(t) \le C, \quad \frac{1}{4}\rho_* \le \rho(t) \le 4\rho^*,$$

provided with the smallness condition (4.1.10) *for c sufficiently small, and hence a priori estimates in* (4.1.11) *hold.*

Proof. Recall

- (4.3.14): $\|\|(\nabla a, \nabla b)\|\|_\epsilon \le C_0 V(t) \le C_0 e^{C_* W(t)}$.
- (4.3.25): $\|\|\nabla u\|\|_\epsilon \le \widetilde{C}_0 e^{C_* W(t)}$.
- (4.3.28): $\|(\nabla\bar\tau, \partial_{\bar\tau}\rho)(t)\|_{L^{2+\epsilon}} + \|\|(\partial_{\bar\tau}\rho, \partial_{\bar\tau}\nabla u)\|\|_\epsilon \le C_*\big(\|(\nabla\bar\tau_0, \partial_{\bar\tau_0}\rho_0)\|_{L^{2+\epsilon}} + C_0\big)e^{C_* W(t)}$.
- (4.3.36): $W(t) \le C_* + \widetilde{C}_0^{\frac{\epsilon}{2+\epsilon}}(C_0 + \|(\nabla\bar\tau_0, \partial_{\bar\tau_0}\rho_0)\|_{L^{2+\epsilon}})^{\frac{2}{2+\epsilon}}\ e^{C_* W(t)}$ (by use of $\|P(\rho) - \tilde P\|_{L^{2+\epsilon}} \le C_* E_0^{\frac{2}{2+\epsilon}}$)
- (4.3.41): $\frac{\rho_*}{\rho_{\inf}(t)} + \frac{\rho_{\sup}(t)}{\rho^*} \le \exp\Big(C_* E_0^{\frac{\epsilon}{2+2\epsilon}}(C_0 + \|(\nabla\bar\tau_0, \partial_{\bar\tau_0}\rho_0)\|_{L^{2+\epsilon}})^{\frac{2+\epsilon}{2+2\epsilon}} e^{C_* W(t)}\Big)$.

The claim now follows from a standard bootstrap argument. Indeed, if we assume that

$$\frac{\rho_*}{\rho_{\inf}(t)} + \frac{\rho_{\sup}(t)}{\rho^*} \le 4,$$

then it follows that

$$C_* \le C$$

for some constant $C > 0$ depending only on $\rho_*, \rho^*, \|(\mu, \lambda, P)\|_{W^{2,\infty}(\frac{1}{4}\rho_*, 4\rho^*)}, \mu_*, \nu_*, \tilde\pi_*$. If we further assume that

$$W(t) \le 4C$$

then

- (4.3.36): $W(t) \le C + e^{CE_\delta^2 + 4C^2}(1 + E_1)^{\frac{2}{2+\epsilon} + (\frac{\epsilon}{2+\epsilon})^2}\big(E_1^{\frac{\epsilon}{2+\epsilon}}(1 + \|(\nabla\bar\tau_0, \partial_{\bar\tau_0}\rho_0)\|_{L^{2+\epsilon}(\mathbb{R}^2)})\big)^{\frac{2}{2+\epsilon}}$,
- (4.3.41): $\frac{\rho_*}{\rho_{\inf}(t)} + \frac{\rho_{\sup}(t)}{\rho^*} \le \exp\Big(Ce^{CE_\delta^2 + 4C^2}(1+E_1)^{\frac{2+\epsilon}{2+2\epsilon}}\big(E_0^{\frac{\epsilon}{2+\epsilon}}(1+\|(\nabla\bar\tau_0, \partial_{\bar\tau_0}\rho_0)\|_{L^{2+\epsilon}(\mathbb{R}^2)})\big)^{\frac{2+\epsilon}{2+2\epsilon}}\Big)$.

If

$$e^{CE_\delta^2}(1 + E_1)^{\frac{3}{2}} E_1^{\frac{\epsilon}{2+\epsilon}}(1 + \|(\nabla\bar\tau_0, \partial_{\bar\tau_0}\rho_0)\|_{L^{2+\epsilon}(\mathbb{R}^2)}) \le c$$

is sufficiently small, then we can conclude $\frac{\rho_*}{\rho_{\inf}(t)} + \frac{\rho_{\sup}(t)}{\rho^*} \le 2$ and $W(t) \le 2C$, and hence the above bootstrap argument assumptions are satisfied. Consequently all the a priori estimates in (4.1.11) follow. □

4.3.5. Proofs of Theorem 4.1.1 and Corollary 4.1.3

This section is devoted to the proofs of Theorem 4.1.1 and Corollary 4.1.3.

We start by smoothing out the initial density and velocity with a family of positive mollifiers $(\eta^\varepsilon)_{\varepsilon>0}$:

$$\rho_0^\varepsilon = \eta^\varepsilon * \rho_0, \quad u_0^\varepsilon = \eta^\varepsilon * u_0. \tag{4.3.42}$$

The existence of a local-in-time smooth solution to the system (CNS) with the initial data (4.3.42) is well known (see, e.g., [50] for Besov regularity), with $\rho - \tilde\rho \in \mathcal{C}([0, T_\varepsilon), B^s_{2,1}(\mathbb{R}^2))$, $u \in \mathcal{C}([0, T_\varepsilon), \dot B^{s-1}_{2,1}(\mathbb{R}^2))$, and $\nabla^2 u, \partial_t u \in L^1((0, T_\varepsilon), \dot B^{s-1}_{2,1}(\mathbb{R}^2))$. To ensure that all terms arising in the computations for the a priori estimates make sense, we must assume $s \geqslant 3$. Therefore, additional estimates must be performed to control the norms of the solution in order to establish the continuation criteria. However, we find it more convenient to construct approximative solutions in a regularity class closely matching ours, ensuring that the continuation criteria are a direct consequence of the a priori estimates. This is stated in Theorem 4.3.11 below.

4.3.5.1. Theorem 4.3.11: Local-in-time solutions and continuation criteria

Theorem 4.3.11 (Local-in-time well-posedness of (CNS) − (τ) **with large initial data).** *Consider the Cauchy problem* (CNS) – (τ) *with initial data* (ρ_0, u_0, τ_0) *satisfying the assumptions* (4.1.1)-(4.1.2)-(4.1.5)-(4.1.8)*, where the model satisfies* (4.1.6)-(4.1.7) *and* $\epsilon \in (0, \epsilon_0]$ *is as in the hypotheses of Theorem 4.1.1. If furthermore the following initial compatibility condition is satisfied:*

$$\operatorname{div}\left(\mu(\rho_0)Su_0 + \left(\lambda(\rho_0)\operatorname{div} u_0 - P(\rho_0)\right)Id\right) \in L^2(\mathbb{R}^2;\mathbb{R}^2), \tag{4.3.43}$$

then there exists a time $T > 0$ *and a unique solution* (ρ, u, τ) *of the Cauchy problem* (CNS) – (τ) *on* $[0,T]$ *with initial data* (ρ_0, u_0, τ_0)*, satisfying the following properties:*

- *Energy bounds:*

$$\begin{aligned}
&\rho - \tilde{\rho} \in \mathcal{C}([0,T], L^2(\mathbb{R}^2)) \cap L^\infty((0,T)\times\mathbb{R}^2; [\tfrac{1}{4}\rho_*, 4\rho^*]),\\
&u \in \mathcal{C}([0,T], H^1(\mathbb{R}^2)), \quad \dot{u} \in \mathcal{C}([0,T], L^2(\mathbb{R}^2)),\\
&\sqrt{\sigma}\nabla\dot{u},\ \sigma\ddot{u} \in L^\infty((0,T), L^2(\mathbb{R}^2)), \quad \nabla\dot{u},\ \sqrt{\sigma}\ddot{u},\ \sigma\nabla\ddot{u} \in L^2((0,T)\times\mathbb{R}^2),
\end{aligned}$$

  with $\sigma(t) = \min\{1,t\}$*;*

- *Uniform bounds for viscosity coefficients:*

$$\mu_* \le \mu(\rho(t,x)) \le \mu^*,\ \nu_* - 2\mu_* \le \lambda(\rho(t,x)) \le \nu^* - 2\mu^*, \quad \textit{for a.e. } (t,x) \in (0,T)\times\mathbb{R}^2. \tag{4.3.44}$$

- *Lipschitz bound and tangential regularity:*

$$\begin{aligned}
&\tau \in \mathcal{C}([0,T], L^\infty \cap \dot{W}^{1,2+\epsilon}), \quad \partial_\tau\rho \in \mathcal{C}([0,T], L^{2+\epsilon}(\mathbb{R}^2)),\\
&\nabla u \in L^2((0,T), L^\infty(\mathbb{R}^2)), \quad \partial_\tau\nabla u \in L^2((0,T), L^{2+\epsilon}(\mathbb{R}^2)).
\end{aligned}$$

Furthermore, the following blow up criteria hold: If (ρ, u, τ) *is the solution defined up to a maximal time* T^**, with* $T^* < \infty$*, then either of the following two cases holds:*

- *Viscosity bounds:*

$$\begin{aligned}
&\inf_{(t,x)\in(0,T^*)\times\mathbb{R}^2}\mu(\rho) < \mu_*, \quad \textit{or} \quad \sup_{(t,x)\in(0,T^*)\times\mathbb{R}^2}\mu(\rho) > \mu^*,\\
\textit{or} \quad &\inf_{(t,x)\in(0,T^*)\times\mathbb{R}^2}\lambda(\rho) < \nu_* - 2\mu_*, \quad \textit{or} \quad \sup_{(t,x)\in(0,T^*)\times\mathbb{R}^2}\lambda(\rho) > \nu^* - 2\mu^*.
\end{aligned} \tag{4.3.45}$$

- *Blow-up of solution norms:*

$$\begin{aligned}
&\limsup_{t\to T^*}\left[\left\|\left(\rho(t),\, \frac{1}{\rho(t)},\, |\tau(t)|,\, \frac{1}{|\tau(t)|}\right)\right\|_{L^\infty} + \left\|\left(\nabla\tau(t),\, \partial_{\tau(t)}\rho(t)\right)\right\|_{L^{2+\epsilon}} + \|(\nabla u(t),\, \dot{u}(t))\|_{L^2}\right]\\
&= \infty.
\end{aligned} \tag{4.3.46}$$

Remark 4.3.12. The low-frequency assumption (4.1.9) and the smallness condition (4.1.10) are not required for the local-in-time well-posedness result in Theorem 4.3.11. The compatibility condition (4.3.43) can also be removed by an approximation argument.

The viscosity bounds (4.3.44) ensure the validity of the $L^{2+\epsilon}(\mathbb{R}^2)$-estimates in (I)-(1) and (I)-(2) in Lemma 4.1.7. If the bounds do not hold (as in (4.3.45)), it is not clear that the tangential regularity can be propagated. Conversely, if both the blowup criteria (4.3.45) and (4.3.46) are not true, then we can extend the solution further.

Proof. We follow the main steps of the proof in [251], emphasizing the new approach based on the decompositions introduced in Section 4.1.3 in Step 2, which allows us to remove the smallness restriction in [251].

Step 1. Change of variables in Lagrangian coordinates. Let $\mathcal{X} = \mathcal{X}(t,y)$ (a priori) denote the associated flow map of the compressible fluid with the velocity field $u(t,x)$

$$\frac{d}{dt}\mathcal{X}(t,y) = u(t,\mathcal{X}(t,y)), \quad \mathcal{X}(0,y) = y.$$

In the Lagrangian coordinates (t,y), the density function and the velocity field are denoted by

$$\zeta(t,y) = \rho(t,\mathcal{X}(t,y)), \quad v(t,y) = u(t,\mathcal{X}(t,y)),$$

in such a way that the flow map $\mathcal{X}$ henceforth denoted by $\mathcal{X}_v$ takes the form:

$$\mathcal{X}_v(t,y) = y + \int_0^t v(t',y)dt'.$$

We can reformulate the system (CNS)-(τ) as follows.

1. Let $D_y\mathcal{X}_v = (\partial_{y_i}(\mathcal{X}_v)_j)_{i,j=1,2}$, and $J_v = \det(D\mathcal{X}_v)$. Then the density equation $(\text{CNS})_1$ becomes
$$\partial_t(\zeta J_v) = 0, \tag{4.3.47}$$
which implies immediately the solution
$$\zeta(t,y) = \zeta_v(t,y) = \rho_0(y)J_v^{-1}(t,y).$$
2. By virtue of the chain rule $D_y(f(t,\mathcal{X}_v(t,y))) = D_y\mathcal{X}_v(t,y)(D_xf)(t,\mathcal{X}_v(t,y))$, and its transpose counterpart $\nabla_y(f(t,\mathcal{X}(t,y))) = \big((\nabla_xf)(t,\mathcal{X}_v(t,y))\big)(\nabla_y\mathcal{X}_v(t,y))$, we introduce in the Lagrangian coordinates,
$$S_v(w) = \nabla w\mathcal{B}_v + \mathcal{B}_v^T Dw, \quad \mathrm{div}_v(w) = \mathcal{B}_v : \nabla w,$$
where $\mathcal{B}_v = (\nabla\mathcal{X}_v)^{-1}$ is the inverse matrix of $\nabla\mathcal{X}_v$. Then the velocity equation $(\text{CNS})_2$ becomes
$$\rho_0\partial_t v = \mathrm{div}\,\Big[\,\mathrm{Adj}(\nabla\mathcal{X}_v)\Big(\mu(\zeta_v)S_v(v) + \big(\lambda(\zeta_v)\,\mathrm{div}_v(v) - P(\zeta_v) + \tilde{P}\big)\mathrm{Id}\,\Big)\Big], \tag{4.3.48}$$
where $\mathrm{Adj}(\nabla\mathcal{X}_v)$ is the adjugate matrix of $\nabla\mathcal{X}_v$.
3. By virtue of the equation (τ) of the tangential vector τ, in the Lagrangian coordinates,
$$\tau(t,\mathcal{X}_v(t,y)) = \tau_0(t,y)\cdot\nabla\mathcal{X}_v(t,y) = (\partial_{\tau_0}\mathcal{X}_v)(t,y). \tag{4.3.49}$$

We now linearize the second equation (4.3.48) to obtain

$$\rho_0\partial_t v - \mathrm{div}\,(\mu_0 Sv) - \nabla(\lambda_0\,\mathrm{div}\,v) = \mathrm{div}\,(f(v)), \tag{4.3.50}$$

where the viscosity coefficients $\mu_0 = \mu(\rho_0), \lambda_0 = \lambda(\rho_0)$, and the source term $f(v)$ reads as

$$f(v) = (\mathrm{Adj}(\nabla\mathcal{X}_v) - \mathrm{Id}\,)\Big(\mu(\zeta_v)S_v(v) + \big(\lambda(\zeta_v)\,\mathrm{div}_v(v) - P(\zeta_v) + \tilde{P}\big)\mathrm{Id}\Big) \tag{4.3.51}$$

$$+ (\mu(\zeta_v) - \mu_0)S_v(v) + \big((\lambda(\zeta_v) - \lambda_0)\mathrm{div}_v v - (P(\zeta_v) - \tilde{P})\big)\mathrm{Id} \tag{4.3.52}$$

$$+ \mu_0(S_v(v) - Sv) + \lambda_0(\mathrm{div}_v v - \mathrm{div}\,v)\mathrm{Id}\,.$$

Notice that $f(v)$ depends only on the initial density ρ_0 and v.

Next we establish the a priori Lipschitz bound for v.

Step 2. A priori estimates via (a, b) **and** (α, β)**.** We follow the main line of the procedure illustrated in Figure 4.1. To this end, one has to introduce the corresponding nonlocal and localized vorticity-effective flux pairs of the equation (4.3.50), still denoted by (a, b) and (α, β) by a slightly abuse of notation in the following. On contrast to the interpolation-type estimate (4.3.36), we here largely make use of the following "linear" L^∞-estimate in terms of tangential regularity terms

$$\|\mathcal{R}^2 h\|_{L^\infty(\mathbb{R}^2)} \lesssim_\epsilon \|h\|_{L^2\cap L^\infty} + \|\partial_{\bar\tau_0} h\|_{L^{2+\epsilon}} + \|\nabla\bar\tau_0\|_{L^{2+\epsilon}}\|h\|_{L^\infty}. \tag{4.3.53}$$

where $\bar\tau_0$ is the normalized initial tangential vector.

1. Energy estimates. We test the equation (4.3.50) by $v, \partial_t v, \partial_{tt} v$ to obtain

$$\sup_{[0,t]}\|(v, \nabla v, \partial_t v)\|_{L^2}^2 + \int_0^t \|(\nabla v,\ \partial_t v,\ \nabla\partial_t v)\|_{L^2}^2$$
$$\le C\Big(\|(v, \nabla v, \partial_t v)|_{t=0}\|_{L^2}^2 + \int_0^t \|(f(v), \partial_t f(v))\|_{L^2}^2 dt'\Big), \tag{4.3.54}$$

where C depends on $\rho_*, \rho^*, \mu_*, \mu^*, \nu_*, \nu^*$. Similarly, we differentiate equation (4.3.50) with respect to time once and twice. By multiplying the resulting equations by $\sigma\partial_{tt}v$ and $\sigma^2\partial_{ttt}v$ respectively, with $\sigma = \min\{1, t\}$, we obtain the following higher-order energy estimates

$$\begin{aligned}
&\sup_{[0,t]}\|(\sqrt{\sigma}\nabla\partial_t v, \sigma\partial_{tt} v)\|_{L^2}^2 + \int_0^t \|(\sqrt{\sigma}\partial_{tt} v, \sigma\nabla\partial_{tt} v)\|_{L^2}^2 \\
&\le C\int_0^t \|(\nabla\partial_t v,\ \partial_t f(v),\ \sigma\partial_{tt} f(v))\|_{L^2}^2 \\
&\le C\Big(\|(v, \nabla v, \partial_t v)|_{t=0}\|_{L^2}^2 + \int_0^t \|(f(v), \partial_t f(v), \sigma\partial_{tt} f(v))\|_{L^2}^2 dt'\Big).
\end{aligned} \tag{4.3.55}$$

2. Nonlocal vorticity-effective flux pair (a, b). Let

$$a = \mathcal{R}_{\mu_0}(\nabla^\perp\cdot v) - \mathcal{Q}_{\mu_0}\operatorname{div} v, \quad b = \mathcal{R}_{\mu_0,\lambda_0}\operatorname{div} v + \mathcal{Q}_{\mu_0}(\nabla^\perp\cdot v), \tag{4.3.56}$$

such that $\operatorname{div}(\mu_0 Sv) + \nabla(\lambda_0 \operatorname{div} v) = \nabla^\perp a + \nabla b$. Then (4.3.50) reads as

$$\nabla^\perp a + \nabla b = \operatorname{div}(g(v)),$$

where

$$g(v) = -f(v) - (-\Delta)^{-1}\nabla(\rho_0\partial_t v) = -f(v) - (-\Delta)^{-1}\nabla\big[\nabla^\perp a + \nabla b + \operatorname{div}(f(v))\big], \tag{4.3.57}$$

and thus we have the following two Poisson equations for a and b

$$\Delta a = \nabla^\perp\cdot\operatorname{div}(g(v)), \quad \Delta b = \operatorname{div}\operatorname{div}(g(v)). \tag{4.3.58}$$

From these we have the following estimates

- $L^\infty(\mathbb{R}^2)$-estimate. We apply (4.3.53) to derive for $a = (\mathcal{R}^\perp\otimes\mathcal{R}) : g(v)$ and $b = (\mathcal{R}\otimes\mathcal{R}) : g(v)$ that

$$\|(a, b, \mathcal{R}^2(g(v)))\|_{L^\infty} \le C(\epsilon)\big(\|g(v)\|_{L^2\cap L^\infty} + \|\partial_{\bar\tau_0} g(v)\|_{L^{2+\epsilon}} + \|\nabla\bar\tau_0\|_{L^{2+\epsilon}}\|g(v)\|_{L^\infty}\big). \tag{4.3.59}$$

- $L^{2+\epsilon}(\mathbb{R}^2)$-estimates. We apply the commutator estimate of type (4.1.46) with $X = \bar\tau_0$ to achieve

$$\|(\partial_{\bar\tau_0} a, \partial_{\bar\tau_0} b)\|_{L^{2+\epsilon}} \lesssim_\epsilon \|\partial_{\bar\tau_0} g(v)\|_{L^{2+\epsilon}} + \|\nabla\bar\tau_0\|_{L^{2+\epsilon}}\|\mathcal{R}^2 g(v)\|_{L^\infty}.$$

We further apply the commutator estimate (4.1.49) with $X = \bar\tau_0$ and the fact that $\mathcal{M}\begin{pmatrix}\nabla^\perp\cdot v\\ \operatorname{div} v\end{pmatrix} = \begin{pmatrix}a\\ b\end{pmatrix}$ to obtain

$$\|(\nabla\partial_{\bar\tau_0} v, \partial_{\bar\tau_0}\nabla v)\|_{L^{2+\epsilon}} \le C\Big(\|(\partial_{\bar\tau_0} a, \partial_{\bar\tau_0} b)\|_{L^{2+\epsilon}} + R_0\|(\nabla v, a, b)\|_{L^\infty}\Big),$$

where

$$R_0 = \|(\partial_{\bar\tau_0}\mu_0,\, \partial_{\bar\tau_0}\lambda_0,\, \nabla\bar\tau_0)\|_{L^{2+\epsilon}}. \tag{4.3.60}$$

Thus by virtue of the above three estimates,

$$\begin{aligned}\|(\nabla\partial_{\bar\tau_0} v, \partial_{\bar\tau_0}\nabla v)\|_{L^{2+\epsilon}} &\le C(\epsilon, \mu_*, \mu^*, \nu_*, \nu^*)\Big(\|\partial_{\bar\tau_0} g(v)\|_{L^{2+\epsilon}}\\ &+ R_0\big(\|\nabla v\|_{L^\infty} + \|g(v)\|_{L^2\cap L^\infty} + \|\nabla\bar\tau_0\|_{L^{2+\epsilon}}\|g(v)\|_{L^\infty} + \|\partial_{\bar\tau_0} g(v)\|_{L^{2+\epsilon}}\big)\Big).\end{aligned} \tag{4.3.61}$$

3. Shear-normal stress pair (α, β). Let $n_0 = \bar\tau_0^\perp$, and let α and β denote the shear-normal stress pair:

$$\alpha = \mu(Svn_0)\cdot\bar\tau_0, \quad \beta = \mu(Svn_0)\cdot n_0 + \lambda \operatorname{div} v.$$

Then following the computations detailed in Appendix 4.B, we derive

- An expression similar as (4.1.58) for $\partial_{n_0} v$ (with $\nu = 2\mu + \lambda$)

$$\partial_{n_0} v = \bar\tau_0\frac{\alpha}{\mu} + n_0\frac{\beta}{\nu} + 2\Big(\frac{\mu}{\nu} n_0\otimes\bar\tau_0 - \bar\tau_0\otimes n_0\Big)\partial_{\bar\tau_0} v + \partial_{\bar\tau_0} v^\perp.$$

- The relations (4.1.61)-(4.1.62) for $\nabla\alpha - \nabla a$ resp. $\nabla\beta - \nabla b$, with $\bar\tau$, n replaced by $\bar\tau_0, n_0$ and

$$\begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} = \begin{pmatrix}\mu(\partial_{x_2} v_1 + \partial_{x_1} v_2)\\ \mu(\partial_{x_1} v_1 - \partial_{x_2} v_2)\end{pmatrix}.$$

From these relations we have the following $L^\infty(\mathbb{R}^2)$-estimate (by virtue of the interpolation inequality $\|\cdot\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\cdot\|_{L^2(\mathbb{R}^2)}^{\frac{\epsilon}{2+2\epsilon}}\|\nabla\cdot\|_{L^{2+\epsilon}(\mathbb{R}^2)}^{\frac{2+\epsilon}{2+2\epsilon}}$)

$$\begin{aligned}\|\nabla v\|_{L^\infty} &\le \|(\partial_{\bar\tau_0} v, \partial_{n_0} v)\|_{L^\infty} \lesssim \|(\alpha, \beta, \partial_{\bar\tau_0} v)\|_{L^\infty}\\ &\lesssim \|(a, b)\|_{L^\infty} + \|(\alpha - a, \beta - b, \partial_{\bar\tau_0} v)\|_{L^\infty}\\ &\lesssim \|(a, b)\|_{L^\infty} + \|(\alpha - a, \beta - b, \partial_{\bar\tau_0} v)\|_{L^2}^{\frac{\epsilon}{2+2\epsilon}}\|(\nabla\bar\tau_0\otimes\nabla v, \nabla\partial_{\bar\tau_0} v)\|_{L^{2+\epsilon}}^{\frac{2+\epsilon}{2+2\epsilon}}\\ &\lesssim \|(a, b)\|_{L^\infty} + \|\nabla v\|_{L^2}^{\frac{\epsilon}{2+2\epsilon}}\big(\|\nabla\bar\tau_0\|_{L^{2+\epsilon}}\|\nabla v\|_{L^\infty} + \|\nabla\partial_{\bar\tau_0} v\|_{L^{2+\epsilon}}\big)^{\frac{2+\epsilon}{2+2\epsilon}}.\end{aligned} \tag{4.3.62}$$

4. Conclusion of a priori estimates. We therefore conclude from the estimates (4.3.59), (4.3.61), (4.3.62) and Young's inequality that

$$\|\nabla v\|_{L^\infty} \lesssim (1 + R_0^{\frac{2+\epsilon}{\epsilon}})\|\nabla v\|_{L^2} + (1 + R_0^2)\|g(v)\|_{L^2\cap L^\infty} + (1 + R_0)\|\partial_{\bar\tau_0} g(v)\|_{L^{2+\epsilon}}.$$

By the definition of $g(v)$ in (4.3.57), we have the following estimates:

- $\|g(v)\|_{L^2} \lesssim \|(f(v), a, b)\|_{L^2} \lesssim_{\mu^*, \nu^*} \|(f(v), \nabla v)\|_{L^2}$;

- $\|g(v)\|_{L^\infty} \le \|f(v)\|_{L^\infty} + \|(-\Delta)^{-1}\nabla(\rho_0\partial_t v)\|_{L^\infty}$
  $\lesssim_\epsilon \|f(v)\|_{L^\infty} + \|(-\Delta)^{-1}\nabla(\rho_0\partial_t v)\|_{L^2}^{\frac{\epsilon}{2+2\epsilon}} \|\nabla(-\Delta)^{-1}\nabla(\rho_0\partial_t v)\|_{L^{2+\epsilon}}^{\frac{2+\epsilon}{2+2\epsilon}}$ by interpolation, and thus
  $\|g(v)\|_{L^\infty} \lesssim_\epsilon \|f(v)\|_{L^\infty} + \|(-\Delta)^{-1}\nabla(\nabla^\perp a + \nabla b + \mathrm{div}\,(f(v)))\|_{L^2}^{\frac{\epsilon}{2+2\epsilon}} \|\rho_0\partial_t v\|_{L^{2+\epsilon}}^{\frac{2+\epsilon}{2+2\epsilon}}$, which, by use of Young's inequality, leads to
$$\|g(v)\|_{L^\infty} \lesssim_{\epsilon,\mu^*,\nu^*} \|f(v)\|_{L^\infty} + \|(f(v),\nabla v)\|_{L^2} + \|\rho_0\partial_t v\|_{L^{2+\epsilon}}.$$
- $\|\partial_{\bar\tau_0} g(v)\|_{L^{2+\epsilon}} \lesssim_\epsilon \|(\partial_{\bar\tau_0} f(v),\ \rho_0\partial_t v)\|_{L^{2+\epsilon}}$.

We conclude from $\epsilon \le \epsilon_0 \le 2$ that
$$\begin{aligned}\|\nabla v\|_{L^\infty} \le C(\epsilon,\mu^*,\nu^*)(1+R_0^{1+\frac{2}{\epsilon}})\Big[&\|(f(v),\nabla v)\|_{L^2} + \|f(v)\|_{L^\infty}\\ &+\|(\partial_{\bar\tau_0} f(v),\ \rho_0\partial_t v)\|_{L^{2+\epsilon}}\Big],\end{aligned} \tag{4.3.63}$$
and hence (4.3.61) yields
$$\begin{aligned}&\|(\nabla\partial_{\bar\tau_0} v, \partial_{\bar\tau_0}\nabla v)\|_{L^{2+\epsilon}}\\ &\le C(\epsilon,\mu^*,\nu^*)(1+R_0^{2+\frac{2}{\epsilon}})\Big[\|(f(v),\nabla v)\|_{L^2} + \|f(v)\|_{L^\infty} + \|(\partial_{\bar\tau_0} f(v),\ \rho_0\partial_t v)\|_{L^{2+\epsilon}}\Big].\end{aligned} \tag{4.3.64}$$

Recall the energy estimates (4.3.54)-(4.3.55) such that
$$\begin{aligned}\sigma^{\frac{\epsilon}{2+\epsilon}}\|\partial_t v\|_{L^{2+\epsilon}}^2 &\lesssim_\epsilon \|\partial_t v\|_{L^2}^{\frac{4}{2+\epsilon}} \|\sqrt{\sigma}\nabla\partial_t v\|_{L^2}^{\frac{2\epsilon}{2+\epsilon}}\\ &\lesssim \|(v,\nabla v,\partial_t v)|_{t=0}\|_{L^2}^2 + \int_0^t \|(f(v),\partial_t f(v),\sigma\partial_{tt} f(v))\|_{L^2}^2 dt'.\end{aligned}$$

Then (4.3.54)-(4.3.55) and (4.3.63)-(4.3.64) yield the bounds
$$\begin{aligned}&\sup_{[0,t]}\Big[\|v,\nabla v,\partial_t v,\sqrt{\sigma}\nabla\partial_t v,\sigma\partial_{tt} v\|_{L^2}^2 + \sigma^{\frac{\epsilon}{2+\epsilon}}(\|\nabla v\|_{L^\infty}^2 + \|(\nabla\partial_{\bar\tau_0} v,\partial_{\bar\tau_0}\nabla v)\|_{L^{2+\epsilon}}^2)\Big]\\ &+\int_0^t \|(\nabla\partial_t v,\sqrt{\sigma}\partial_{tt} v,\sigma\nabla\partial_{tt} v)\|_{L^2}^2 dt'\\ &\le C(1+R_0^{2+\frac{4}{\epsilon}})\Big[\|(v,\nabla v,\partial_t v)|_{t=0}\|_{L^2}^2 + \sup_{[0,t]}\sigma^{\frac{\epsilon}{2+\epsilon}}(\|f(v)\|_{L^2\cap L^\infty}^2 + \|\partial_{\bar\tau_0} f(v)\|_{L^{2+\epsilon}}^2)\\ &\quad+\int_0^t \|(f(v),\partial_t f(v),\sigma\partial_{tt} f(v))\|_{L^2}^2 dt'\Big].\end{aligned} \tag{4.3.65}$$

Step 3. Conclusion. Estimate (4.3.65) specifies the regularity class
$$E_T = \Big\{v \in \mathcal{C}([0,T],H^1(\mathbb{R}^2)) \cap \mathcal{C}^1([0,T],L^2(\mathbb{R}^2)) \colon \|v\|_{E_T} < \infty\Big\},$$
with
$$\begin{aligned}\|v\|_{E_T}^2 = &\sup_{[0,T]}\Big[\|v,\nabla v,\partial_t v,\sqrt{\sigma}\nabla\partial_t v,\sigma\partial_{tt} v\|_{L^2}^2 + \sigma^{\frac{\epsilon}{2+\epsilon}}(\|\nabla v\|_{L^\infty}^2 + \|(\nabla\partial_{\bar\tau_0} v,\partial_{\bar\tau_0}\nabla v)\|_{L^{2+\epsilon}}^2)\Big]\\ &+\int_0^T \|(\nabla\partial_t v,\sqrt{\sigma}\partial_{tt} v,\sigma\nabla\partial_{tt} v)\|_{L^2}^2 dt,\end{aligned}$$

in which the fixed-point argument shall work. For any $v \in E_T$, we have $\nabla v \in L^{2+}((0,T), L^\infty(\mathbb{R}^2))$ and hence (see the expression of $f(v)$ in (4.3.51))

$$\sup_{[0,T]} \sigma^{\frac{\epsilon}{2+\epsilon}}(\|f(v)\|^2_{L^2\cap L^\infty} + \|\partial_{\bar{\tau}_0} f(v)\|^2_{L^{2+\epsilon}}) + \int_0^T \|(f(v), \partial_t f(v), \sigma\partial_{tt} f(v))\|^2_{L^2} dt$$
$$\leq C_*(\|\rho_0 - \tilde{\rho}\|_{L^2\cap L^\infty} + \|\partial_{\bar{\tau}_0}\rho_0\|_{L^{2+\epsilon}}) + C(T, \|v\|_{E_T})\|v\|^2_{E_T}.$$

Here, $C(T, \|v\|_{E_T})$ is a positive constant depending on ρ_*, ρ^*, μ_*, μ^*, ν_*, ν^*, R_0 and $\|v\|_{E_T}$, verifying
$C(T, \|v\|_{E_T}) \to 0$ as $T \to 0$. Moreover for all $v, w \in E_T$, we have

$$\int_0^T \|f(v) - f(w)\|^2_{L^2} \leqslant C(T, \|(v,w)\|_{E_T}) \int_0^T \|\nabla v - \nabla w\|^2_{L^2}.$$

With these estimates in hand, one may follow the argument in the proof of [251, Theorem 3.1], first proving the existence of solutions in the regularity class by a homotopy method and then implementing the fixed-point argument. The remainder of the proof is unchanged, apart from minor adaptive modifications. By going back to Eulerian coordinates, the results of the existence, uniqueness, energy bounds and uniform bounds on a small time interval $[0,T]$ for the system (CNS) in Theorem 4.3.11 follow. The tangential vector given by (4.3.49) satisfies (τ). The tangential regularity results in Eulerian coordinates then follow from

$$\tau_0 \in L^{2+\epsilon}, \quad \partial_{\tau_0}(\rho(t, \mathcal{X}(t,y))) \in \mathcal{C}([0,T]; L^{2+\epsilon}),$$
$$(\nabla_y \partial_{\tau_0}, \partial_{\tau_0}\nabla_y)(u(t, \mathcal{X}(t,y))) \in L^2((0,T); L^{2+\epsilon}),$$

and the identities

$$\tau(t, \mathcal{X}(t,y)) = \partial_{\tau_0}\mathcal{X}(t,y),$$
$$(\partial_\tau h)(t, \mathcal{X}(t,y)) = \tau(t, \mathcal{X}(t,y)) \cdot (\nabla_x h)(t, \mathcal{X}(t,y))) = \partial_{\tau_0}(h(t, \mathcal{X}(t,y))).$$

Finally, if the norms in (4.3.46) are finite, then we can repeat the above argument to extend the solution beyond T_*, which is contradiction to the maximality of the finite exitence time T_*. □

4.3.5.2. Conclusion of Theorem 4.1.1: Compactness of approximate sequence

By Theorem 4.3.11, there exists a unique solution $(\rho^\varepsilon, u^\varepsilon, \tau^\varepsilon)$ to the Cauchy problem (CNS)-(τ) associated with the initial data (4.3.42), defined on the maximal time interval $[0, T_\varepsilon)$. On this interval, the regularity stated in Theorem 4.3.11 as well as the uniform bounds (4.3.44) are sufficient to justify the computations carried out in Sections 4.3.1-4.3.4, under the additional assumptions (4.1.9)-(4.1.10). This leads to the uniform bounds (4.3.44) and the existence of a constant M, independent of ε, such that for all $t \in [0, T_\varepsilon)$, we have

$$\Big\|\Big(\rho^\varepsilon(t), \frac{1}{\rho^\varepsilon(t)}, |\tau^\varepsilon(t)|, \frac{1}{|\tau^\varepsilon(t)|}\Big)\Big\|_{L^\infty} + \|(\nabla\tau^\varepsilon(t), (\partial_{\tau^\varepsilon}\rho^\varepsilon)(t))\|_{L^{2+\epsilon}}$$
$$+ \|(\rho^\varepsilon - \tilde{\rho}, u^\varepsilon, \nabla u^\varepsilon(t), \sqrt{\sigma}(t)\dot{u}^\varepsilon(t))\|_{L^2} + \|\partial_t u^\varepsilon\|_{L^2((0,t);L^2)} + \|\!|\nabla u^\varepsilon|\!\|_\infty(t) \leq M. \quad (4.3.66)$$

In particular, the blowup criteria in Theorem 4.3.11 are not satisfied, implying that $T_\varepsilon = \infty$. In other words, the solution $(\rho^\varepsilon, u^\varepsilon, \tau^\varepsilon)$ is globally-in-time defined and satisfies (4.3.44)-(4.3.66) uniformly in time and ε. It suffices to prove that a subsequence of $(\rho^\varepsilon, u^\varepsilon, \tau^\varepsilon)$ convergence to a unique solution of (CNS)-(τ).

We first recall that $(\rho^\varepsilon, u^\varepsilon, \tau^\varepsilon)$ satisfies (CNS)-(τ)

$$\begin{cases} \partial_t\rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \nabla P(\rho^\varepsilon) = \operatorname{div}\big(\mu(\rho^\varepsilon) S u^\varepsilon\big) + \nabla\big(\lambda(\rho^\varepsilon)\operatorname{div} u^\varepsilon\big), \\ \partial_t\tau^\varepsilon + u^\varepsilon\cdot\nabla\tau^\varepsilon = (\tau^\varepsilon\cdot\nabla)u^\varepsilon. \end{cases} \tag{4.3.67}$$

As a consequence of the uniform bounds (4.3.66), there exists a triplet $(\rho, u, \tau) \in L^\infty_{t,x} \times L^\infty_t H^1 \times (L^\infty_{t,x} \cap L^\infty_t \dot W^{1,2+\epsilon})$ such that, up to a subsequence,

$$\begin{aligned} &\rho^\varepsilon - \tilde\rho \overset{*}{\rightharpoonup} \rho - \tilde\rho \ \text{ in } \ L^\infty_t(L^2\cap L^\infty), \quad u^\varepsilon \overset{*}{\rightharpoonup} u \ \text{ in } \ L^\infty_t H^1, \\ &\partial_t u^\varepsilon \rightharpoonup \partial_t u \ \text{ in } \ L^2_t L^2, \quad \tau^\varepsilon \overset{*}{\rightharpoonup} \tau \quad \text{in } \ L^\infty_t(L^\infty \cap \dot W^{1,2+\epsilon}), \end{aligned}$$

which, together with Aubin–Lions lemma and interpolation inequality, yields the following strong convergences:

$$u^\varepsilon \longrightarrow u \ \text{ in } \ L^p_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^2), \quad \tau^\varepsilon \longrightarrow \tau \ \text{ in } \ L^p_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^2), \quad \forall p<\infty.$$

Thus we can take the limit in (4.3.67), to obtain the following system for the triplet (ρ, u, τ)

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla\overline{P(\rho)} = \operatorname{div}\big(\overline{\mu(\rho)Su}\big) + \nabla\big(\overline{\lambda(\rho)\operatorname{div} u}\big), \\ \partial_t\tau + u\cdot\nabla\tau = (\tau\cdot\nabla)u, \end{cases} \tag{4.3.68}$$

where $\overline{P(\rho)} - \tilde P$, $\overline{\mu(\rho)Su}$, and $\overline{\lambda(\rho)\operatorname{div} u}$ are the weak-* limits of $P(\rho^\varepsilon) - \tilde P$, $\mu(\rho^\varepsilon)Su^\varepsilon$, and $\lambda(\rho^\varepsilon)\operatorname{div} u^\varepsilon$, respectively in $L^\infty_t L^2$. We claim that

$$\overline{P(\rho)} = P(\rho), \ \overline{\mu(\rho)Su} = \mu(\rho)Su, \ \text{ and } \ \overline{\lambda(\rho)\operatorname{div} u} = \lambda(\rho)\operatorname{div} u. \tag{4.3.69}$$

To show this claim, we adapt the strategy of [235] (see also [139]) where the shear viscosity μ is constant, to our case when μ is variable, making use of the pointwise representation (4.1.59) of $\operatorname{div} u$. Firstly, according to the DiPerna-Lions theory, the density indeed belongs to $\mathcal{C}_b([0,\infty); L^p)$, for all $p\in[2,\infty)$ and satisfies the renormalized continuity equation:

$$\partial_t b(\rho) + \operatorname{div}(b(\rho)u) + \big(\rho b'(\rho) - b(\rho)\big)\operatorname{div} u = 0, \quad \forall b\in W^{1,\infty}_{\mathrm{loc}}([0,\infty)), \tag{4.3.70}$$

and in particular,

$$\partial_t\rho^2 + \operatorname{div}(\rho^2 u) + \rho^2\operatorname{div} u = 0.$$

Now let $\beta^\varepsilon = (n^\varepsilon\otimes n^\varepsilon) : (\mu(\rho^\varepsilon)Su^\varepsilon + (\lambda(\rho^\varepsilon)\operatorname{div} u^\varepsilon - P(\rho^\varepsilon))\mathrm{Id}\,)$ be associated to the system (4.3.67), such that the pointwise representation (4.1.59) reads

$$\operatorname{div} u^\varepsilon = \frac{1}{\nu(\rho^\varepsilon)}\Big(\beta^\varepsilon + P(\rho^\varepsilon) - \tilde P + 2\,\mu(\rho^\varepsilon)\,\overline{\tau}^\varepsilon\cdot\partial_{\overline{\tau}^\varepsilon}u^\varepsilon\Big). \tag{4.3.71}$$

Recall the uniform estimates $|||\nabla b^\varepsilon|||_\epsilon \le M$ (thanks to (4.3.14)) and the relation (4.1.62), such that $|||\nabla\beta^\varepsilon|||_\epsilon \le M$, as well as the uniform bounds $\|\beta^\varepsilon\|_{L^\infty L^2} \le M$. Thus

$$\beta^\varepsilon \to \beta = (n\otimes n) : (\overline{\mu(\rho)Su} + (\overline{\lambda(\rho)\operatorname{div} u} - \overline{P(\rho)})\mathrm{Id}\,) \text{ in } L^1_{\mathrm{loc}}(L^2\cap L^\infty),$$

and the limit of (4.3.71) becomes (noticing the strong compactness of $\{\tau^\varepsilon\}_\varepsilon$ and $\{\partial_{\overline{\tau}^\varepsilon}u^\varepsilon\}_\varepsilon$)

$$\operatorname{div} u = \overline{\Big(\frac{1}{\nu(\rho)}\Big)}(\beta - \tilde P) + 2\overline{\Big(\frac{\mu(\rho)}{\nu(\rho)}\Big)}\overline{\tau}\cdot\partial_{\overline{\tau}}u + \overline{\Big(\frac{P(\rho)}{\nu(\rho)}\Big)}.$$

Hence the renormalized continuity equation reads

$$\partial_t \rho^2 + \operatorname{div}(\rho^2 u) = -\rho^2 \overline{\Big(\frac{1}{\nu(\rho)}\Big)}(\beta - \tilde{P}) - 2\rho^2 \overline{\Big(\frac{\mu(\rho)}{\nu(\rho)}\Big)\overline{\tau}} \cdot \partial_{\overline{\tau}} u - \rho^2 \overline{\Big(\frac{P(\rho)}{\nu(\rho)}\Big)}. \tag{4.3.72}$$

Notice that the approximate sequence also satisfies the renormalized equation (4.3.70), with $\operatorname{div} u^\varepsilon$ represented by (4.3.71), and we take the limit to obtain

$$\partial_t \overline{\rho^2} + \operatorname{div}\,(\overline{\rho^2} u) = -\overline{\Big(\frac{\rho^2}{\nu(\rho)}\Big)}(\beta - \tilde{P}) - 2\overline{\Big(\frac{\rho^2\mu(\rho)}{\nu(\rho)}\Big)\overline{\tau}} \cdot \partial_{\overline{\tau}} u - \overline{\Big(\frac{\rho^2(P(\rho))}{\nu(\rho)}\Big)}, \tag{4.3.73}$$

Taking the difference between (4.3.72) and (4.3.73) yields

$$\partial_t (\overline{\rho^2} - \rho^2) + \operatorname{div}\,[u(\overline{\rho^2} - \rho^2)] \tag{4.3.74}$$

$$= -\Big[\overline{\Big(\frac{\rho^2}{\nu(\rho)}\Big)} - \rho^2\overline{\Big(\frac{1}{\nu(\rho)}\Big)}\Big](\beta - \tilde{P}) - 2\Big[\overline{\Big(\frac{\rho^2\mu(\rho)}{\nu(\rho)}\Big)} - \rho^2\overline{\Big(\frac{\mu(\rho)}{\nu(\rho)}\Big)}\Big]\overline{\tau} \cdot \partial_{\overline{\tau}} u$$

$$- \overline{\Big(\frac{\rho^2 P(\rho)}{\nu(\rho)}\Big)} + \rho^2\overline{\Big(\frac{P(\rho)}{\nu(\rho)}\Big)}. \tag{4.3.75}$$

Note that by the convexity of the square function, the difference $\overline{\rho^2} - \rho^2$ is nonnegative. Furthermore, for any $g = g(\rho) \in W^{2,\infty}$ on the range of ρ, there exists a constant $C > 0$ such that[5]

$$\big|\overline{g(\rho)} - g(\rho)\big| \leq C(\overline{\rho^2} - \rho^2). \tag{4.3.76}$$

Therefore the quantities on the right hand side of (4.3.74) are bounded as

$$\Big|\overline{\Big(\frac{\rho^2}{\nu(\rho)}\Big)} - \rho^2\overline{\Big(\frac{1}{\nu(\rho)}\Big)}\Big| = \Big|\Big[\overline{\Big(\frac{\rho^2}{\nu(\rho)}\Big)} - \frac{\rho^2}{\nu(\rho)}\Big] + \rho^2\Big[\frac{1}{\nu(\rho)} - \overline{\Big(\frac{1}{\nu(\rho)}\Big)}\Big]\Big| \leq C(\overline{\rho^2} - \rho^2);$$

and similarly for

$$\rho^2\overline{\Big(\frac{P(\rho)}{\nu(\rho)}\Big)} - \overline{\Big(\frac{\rho^2 P(\rho)}{\nu(\rho)}\Big)}, \quad \text{and} \quad \overline{\Big(\frac{\rho^2\mu(\rho)}{\nu(\rho)}\Big)} - \rho^2\overline{\Big(\frac{\mu(\rho)}{\nu(\rho)}\Big)}.$$

As a result, we have

$$\partial_t (\overline{\rho^2} - \rho^2) + \operatorname{div}\,[u(\overline{\rho^2} - \rho^2)] \leq C\big(1 + \|(\beta,\, \partial_{\overline{\tau}} u)\|_{L^\infty}\big)(\overline{\rho^2} - \rho^2).$$

Since β and $\partial_{\overline{\tau}} u$ belong to $L^1_{\mathrm{loc}}([0,\infty), L^\infty(\mathbb{R}^2))$, and $\overline{\rho^2}|_{t=0} = \rho_0^2$, it follows that

$$\overline{\rho^2} = \rho^2 \quad \text{a.e. on } (0,\infty) \times \mathbb{R}^2.$$

Consequently, $\rho^\varepsilon \to \rho$ strongly in $L^1_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^2)$, and hence in $L^p_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^2)$ for every $p < \infty$. This strong convergence implies the claim (4.3.69), and hence (ρ, u, τ) satisfies (CNS)-(τ) and all the a priori bounds. This completes the existence and regularity part of the proof of Theorem 4.1.1. Given that the velocity u is Lipschitz, the uniqueness part follows from a change of variables to Lagrangian coordinates, followed by a standard stability estimate. We refer to Proposition 3.1 of [252].

[5]Indeed, as $g(\rho^\varepsilon) - g(\rho) = g'(\rho)(\rho^\varepsilon - \rho) + (\rho^\varepsilon - \rho)^2 \int_0^1 (1-s) g''\big(\rho + s(\rho^\varepsilon - \rho)\big) ds$, there exists $C > 0$ such that

$$-C(\rho^\varepsilon - \rho)^2 \leq g(\rho^\varepsilon) - g(\rho) - g'(\rho)(\rho^\varepsilon - \rho) \leq C(\rho^\varepsilon - \rho)^2.$$

Passing to the limit in ε yields (4.3.76), since $(\rho^\varepsilon - \rho)^2$ converges weakly to $\overline{\rho^2} - \rho^2$.

4.3.5.3. Proof of Corollary 4.1.3

The global-in-time well-posedness of the system (CNS) in Corollary 4.1.3-1 follows immediately from Theorem 4.1.1. In particular, the uniqueness result holds whenever the velocity field is Lipschitz in the sense of (4.1.12). We now show the remaining results in Corollary 4.1.3 by appropriately choosing the initial "tangential" vector field τ_0.

Proof of Corollary 4.1.3-2. We first show the existence of some nondegenerate tangential vector field τ_0 of $\partial\Omega_0$, such that (4.1.8) holds. Indeed, we mimic the construction in [168, Appendix D] to contruct two such vector fields $\tau_0^{(1)}, \tau_0^{(2)} \in L^\infty \cap \dot{W}^{1,2+\epsilon}(\mathbb{R}^2;\mathbb{R}^2)$ satisfying $|\tau_0^{(1)}|^{-1}, |\tau_0^{(2)}|^{-1} \in L^\infty(\mathbb{R}^2)$ and

$$\tau_0^{(1)}(x) \text{ and } \tau_0^{(2)}(x) \text{ are } \begin{cases} \text{both tangential to the boundary } \partial\Omega_0, & \text{if } x \in \partial\Omega_0, \\ \text{linearly independent}, & \text{if } x \in \mathbb{R}^2 \setminus \partial\Omega_0. \end{cases} \tag{4.3.77}$$

To this end, let $\gamma : [0, 2\pi) \to \partial\Omega_0$ be an injective $W^{2-\frac{1}{2+\epsilon},2+\epsilon}$-parametrization of the boundary $\partial\Omega_0$ such that $|\gamma'(s)| \neq 0$ for all $s \in [0, 2\pi)$. We define a continuous function $\theta \in W^{1-\frac{1}{2+\epsilon},2+\epsilon}(\partial\Omega_0, [0, 2\pi))$ as follows: for any $x \in \partial\Omega_0$ there exists a unique $s \in [0, 2\pi)$ such that $x = \gamma(s)$, and we define $\theta = \theta(x) \in [0, 2\pi)$ such that $\frac{\gamma'(s)}{|\gamma'(s)|} = e_\theta$. Next, let $d(x) = \mathrm{dist}(x, \partial\Omega_0)$ be the distance of a point $x \in \mathbb{R}^2$ to the boundary. For $\eta > 0$, let $\Pi : \{x \in \mathbb{R}^2 : d(x) < \eta\} \to \partial\Omega_0$ denote the projection onto the boundary. It is well-defined for sufficiently small η. We define

$$\tau_0^{(1)}(x) = \begin{cases} \begin{pmatrix} -\sin\big(3\pi\frac{d(x)}{\eta} + 2\theta(\Pi x)(\frac{1}{2} - \frac{d(x)}{\eta})\big) \\ \cos\big(3\pi\frac{d(x)}{\eta} + 2\theta(\Pi x)(\frac{1}{2} - \frac{d(x)}{\eta})\big) \end{pmatrix}, & \frac{d(x)}{\eta} \in [0, \frac{1}{2}), \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} =: e_1, & \frac{d(x)}{\eta} \in [\frac{1}{2}, \infty), \end{cases}$$

$$\tau_0^{(2)}(x) = \begin{cases} \begin{pmatrix} -\sin\big(4\pi\frac{d(x)}{\eta} + 2\theta(\Pi x)(\frac{1}{2} - \frac{d(x)}{\eta})\big) \\ \cos\big(4\pi\frac{d(x)}{\eta} + 2\theta(\Pi x)(\frac{1}{2} - \frac{d(x)}{\eta})\big) \end{pmatrix}, & \frac{d(x)}{\eta} \in [0, \frac{1}{2}), \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} =: e_2, & \frac{d(x)}{\eta} \in [\frac{1}{2}, \infty), \end{cases}$$

such that for $j = 1, 2$,

$$\tau_0^{(j)}(x) = \begin{cases} e_{\theta(x)}, & \text{for } x \in \partial\Omega_0, \\ e_j, & \text{away from the boundary } \partial\Omega_0. \end{cases}$$

It is straightforward to verify the regularity properties and (4.3.77).

Now we show the preservation of the patch-structure and the regularity propagation. If the initial data (ρ_0, u_0) satisfy (4.1.1)-(4.1.8)-(4.1.9)-(4.1.10) with respect to some vector field τ_0, then Corollary 4.1.3–1 ensures a unique solution (ρ, u) of (CNS), satisfying (4.1.12). Hence there exist two flow-driven vector fields $\tau^{(1)}, \tau^{(2)}$ satisfying the equation (τ) with initial data $\tau_0^{(1)}$, $\tau_0^{(2)}$, and furthermore, by Propositions 4.3.3 and 4.3.7, the tangential regularity is preserved:

$$\begin{aligned} &\tau^{(j)} \in L^\infty([0,\infty); L^\infty \cap \dot{W}^{1,2+\epsilon}(\mathbb{R}^2;\mathbb{R}^2)), \ |\tau^{(j)}|^{-1} \in L^\infty([0,\infty) \times \mathbb{R}^2), \\ &\partial_{\tau^{(j)}}\rho \in L^\infty([0,\infty); L^{2+\epsilon}(\mathbb{R}^2)), \quad j = 1, 2. \end{aligned}$$

If we define $\Omega_t = \mathcal{X}(t, \Omega_0)$ for $t \geq 0$, where $\mathcal{X}(t,x) = \mathcal{X}_t(x) \in \mathbb{R}^2$ denotes the flow map associated to the solution u, then Ω_t is a simply connected, bounded $W^{2,2+\epsilon}$-domain for each $t \geq 0$, and (4.3.77) holds with $(\tau_0^{(1)}, \tau_0^{(2)}, \Omega_0)$ replaced by $(\tau^{(1)}(t), \tau^{(2)}(t), \Omega_t)$. Indeed, to show the linear independence of $\tau^{(1)}(t,x)$ and $\tau^{(2)}(t,x)$ for $x \in \mathbb{R}^2 \setminus \partial\Omega_t$, note that the solution τ of (τ) with initial data τ_0 is given by $\tau(t,x) = (\partial_{\tau_0}\mathcal{X}_t)(\mathcal{X}_t^{-1}(x))$ (recalling (4.3.49)). Hence, for any $c \in \mathbb{R}$ we have

$$\begin{aligned}(\tau^{(1)} - c\tau^{(2)})(t,x) &= (\partial_{\tau_0^{(1)} - c\tau_0^{(2)}}\mathcal{X}_t)(\mathcal{X}_t^{-1}(x)) \\ &= \left((\nabla\mathcal{X}_t)(\mathcal{X}_t^{-1}(x))\right)^T(\tau_0^{(1)} - c\tau_0^{(2)})(\mathcal{X}_t^{-1}(x)) \neq 0,\end{aligned}$$

due to the facts that $\det\nabla\mathcal{X}_t \neq 0$, $\mathcal{X}_t^{-1}(x) \in \mathbb{R}^2 \setminus \partial\Omega_0$ for $x \in \mathbb{R}^2 \setminus \partial\Omega_t$, and the linear independence of the initial vectors $\tau_0^{(1)}, \tau_0^{(2)}$ on $\mathbb{R}^2 \setminus \partial\Omega_0$.

The representation $\nabla = \overline{\tau^{(1)}}(t)\partial_{\overline{\tau^{(1)}}(t)} + \overline{\tau^{(2)}}(t)\partial_{\overline{\tau^{(2)}}(t)}$ thus holds on $\mathbb{R}^2 \setminus \partial\Omega_t$, where we have set $\overline{\tau^{(j)}} = \frac{\tau^{(j)}}{|\tau^{(j)}|}$, $j = 1, 2$, implying the piecewise regularity of the density $\rho(t) \in W^{1,2+\epsilon}(\Omega_t)$, $\rho(t) - \tilde{\rho} \in W^{1,2+\epsilon}(\overline{\Omega_t}^c)$.

□

Proof of Corollary 4.1.3-3. We follow the argument as in the proof of Corollary 4.1.3-2. Firstly, the unique solution (ρ, u) of (CNS) follows from applying Corollary 4.1.3-1 with the initial data (ρ_0, u_0) together with the initial "tangential" vector field $\tau_0 = (1,0)^T$. Secondly, the initial tangential regularity along the vector field $(0,1)^T$ is also preserved by virtue of Propositions 4.3.3 and 4.3.7. Thanks to the linear independence of the two vector fields $(1,0)^T$ and $(0,1)^T$, the global regularity $\nabla\rho \in (L^\infty \cap L^1)((0,\infty); L^{2+\epsilon})$ follows. □

4.A. Appendix: Proof of Lemma 4.1.7

(1) is proved in [168, Lemma 1.2].

Proof of (2). We follow the same ideas as in [168, Lemma 1.2].

Step 1: L^2-invertibility of $\mathcal{M}$. We first define the homogeneouse space $\dot{H}^2(\mathbb{R}^2)$ in such a way that it is complete, for example by factoring out polynomials of order 1. Then $\dot{H}^2(\mathbb{R}^2)$ is a Hilbert space.

Recall the operators from Lemma 4.1.5. We define the bilinear form

$$\mathfrak{a} : \dot{H}^2(\mathbb{R}^2;\mathbb{R}^2) \times \dot{H}^2(\mathbb{R}^2;\mathbb{R}^2) \to \mathbb{R}, \quad \mathfrak{a}\left(\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \begin{pmatrix}\Phi\\ \Psi\end{pmatrix}\right) = \left\langle \begin{pmatrix}\Phi\\ \Psi\end{pmatrix}, \mathfrak{L}\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle_{\dot{H}^2\times\dot{H}^{-2}},$$

with

$$\mathfrak{L} = \begin{pmatrix}\mathcal{L}_\mu & -\mathcal{J}_\mu\\ \mathcal{J}_\mu & \mathcal{L}_{\mu,\lambda}\end{pmatrix}.$$

Then

$$\begin{aligned}\left|\mathfrak{a}\left(\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \begin{pmatrix}\Phi\\ \Psi\end{pmatrix}\right)\right| &\leq C\max\{\mu^*, \nu^*\}\left\|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\|_{\dot{H}^2}\left\|\begin{pmatrix}\Phi\\ \Psi\end{pmatrix}\right\|_{\dot{H}^2},\\ \mathfrak{a}\left(\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \begin{pmatrix}\phi\\ \psi\end{pmatrix}\right) &\geq \min\{\mu_*, \nu_*\}\left\|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\|_{\dot{H}^2}^2,\end{aligned} \tag{4.A.1}$$

for $\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \begin{pmatrix}\Phi\\ \Psi\end{pmatrix} \in \dot{H}^2(\mathbb{R}^2;\mathbb{R}^2)$. Indeed, when we define $\begin{pmatrix}f\\ g\end{pmatrix} = \begin{pmatrix}\Delta\phi\\ \Delta\psi\end{pmatrix}$ and $\begin{pmatrix}F\\ G\end{pmatrix} = \begin{pmatrix}\Delta\Phi\\ \Delta\Psi\end{pmatrix}$, then by the definition of $\mathcal{M}$ and integration by parts

$$\mathfrak{a}(\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \begin{pmatrix}\Phi\\ \Psi\end{pmatrix}) = \Big\langle \mathcal{M}\begin{pmatrix}f\\ g\end{pmatrix}, \begin{pmatrix}F\\ G\end{pmatrix}\Big\rangle.$$

The first inequality follows from the boundedness of μ, ν and and the second inequality comes from (4.1.50).

By the bounds (4.A.1) and the Lax-Milgram lemma, there exists for all $g \in \dot{H}^{-2}(\mathbb{R}^2;\mathbb{R}^2)$ a unique element $\begin{pmatrix}\phi\\ \psi\end{pmatrix} \in \dot{H}^2(\mathbb{R}^2;\mathbb{R}^2)$ such that

$$\mathfrak{a}(\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \begin{pmatrix}\Phi\\ \Psi\end{pmatrix}) = \langle \begin{pmatrix}\Phi\\ \Psi\end{pmatrix}, g\rangle_{\dot{H}^2\times\dot{H}^{-2}}, \quad \forall \begin{pmatrix}\Phi\\ \Psi\end{pmatrix} \in \dot{H}^2(\mathbb{R}^2;\mathbb{R}^2),$$

i.e. $\begin{pmatrix}\phi\\ \psi\end{pmatrix}$ solves the elliptic equation $\mathfrak{L}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = g$. We claim that the operator $\mathfrak{A}$ defined by

$$\mathfrak{A}h := \Delta\mathfrak{L}^{-1}\Delta h$$

is bounded on $L^2(\mathbb{R}^2;\mathbb{R}^2)$. Indeed, for $h \in L^2(\mathbb{R}^2;\mathbb{R}^2)$ let $\begin{pmatrix}\phi\\ \psi\end{pmatrix}_h \in \dot{H}^2(\mathbb{R}^2;\mathbb{R}^2)$ be the unique solution of $\mathfrak{L}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \Delta h$ in the sense of Lax-Milgram. Then, by the coercivity of $\mathfrak{a}$ we have

$$\begin{aligned}\min\{\mu_*, \nu_*\}\Big\|\begin{pmatrix}\phi\\ \psi\end{pmatrix}_h\Big\|^2_{\dot{H}^2} &\le \mathfrak{a}(\begin{pmatrix}\phi\\ \psi\end{pmatrix}_h, \begin{pmatrix}\phi\\ \psi\end{pmatrix}_h) = \Big\langle \begin{pmatrix}\phi\\ \psi\end{pmatrix}_h, \Delta h\Big\rangle_{\dot{H}^2\times\dot{H}^{-2}}\\ &\le \Big\|\begin{pmatrix}\phi\\ \psi\end{pmatrix}_h\Big\|_{\dot{H}^2}\|\Delta h\|_{\dot{H}^{-2}} \le \Big\|\begin{pmatrix}\phi\\ \psi\end{pmatrix}_h\Big\|_{\dot{H}^2}\|h\|_{L^2},\end{aligned}$$

and therefore, by the relation $\mathfrak{A}h = \Delta\begin{pmatrix}\phi\\ \psi\end{pmatrix}_h$,

$$\|\mathfrak{A}h\|_{L^2} \le \|\begin{pmatrix}\phi\\ \psi\end{pmatrix}_h\|_{\dot{H}^2} \le \max\{\mu_*^{-1}, \nu_*^{-1}\}\|h\|_{L^2},$$

which proves the $L^2(\mathbb{R}^2;\mathbb{R}^2)$-boundedness of $\mathfrak{A}$.

Step 2: $L^{2+\epsilon}$-invertibility of $\mathcal{M}$. In order to prove that the operator $\mathfrak{A}$ is bounded on $L^{2+\epsilon}(\mathbb{R}^2;\mathbb{R}^2)$ for some $\epsilon > 0$ we are going to make use of Z. Shen's theorem [220, Theorem 3.1], which is a version of the Calderón-Zygmund Lemma. More precisely, if there exists some constant $C > 0$ such that the following holds for all $x_0 \in \mathbb{R}^2$, $r > 0$ and $h \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ with compact support outside $B_{3r}(x_0)$

$$\Big(\frac{1}{r^2}\int_{B_r(x_0)} |\mathfrak{A}h|^q dx\Big)^{\frac{1}{q}} \le C\Big(\frac{1}{4r^2}\int_{B_{3r}(x_0)} |\mathfrak{A}h|^2 dx\Big)^{\frac{1}{2}}, \tag{4.A.2}$$

then $\mathfrak{A}$ is bounded on $L^p(\mathbb{R}^2;\mathbb{R}^2)$ for any $p \in (2, q)$.

To show (4.A.2), let $x_0 \in \mathbb{R}^2$, $r > 0$ and $h \in L^\infty(\mathbb{R}^2;\mathbb{R}^2)$ have compact support with $h \equiv 0$ in $B_{3r}(x_0)$. Then $\begin{pmatrix}\phi\\ \psi\end{pmatrix}_h = \mathfrak{L}^{-1}\Delta h$ is the solution to

$$\mathfrak{a}(\begin{pmatrix}\phi\\ \psi\end{pmatrix}_h, \begin{pmatrix}\Phi\\ \Psi\end{pmatrix}) = \Big\langle \begin{pmatrix}\Phi\\ \Psi\end{pmatrix}, \Delta h\Big\rangle_{\dot H^2\times \dot H^{-2}} = 0, \quad \forall \begin{pmatrix}\Phi\\ \Psi\end{pmatrix} \in C_c^\infty(B_{2r}(x_0);\mathbb{R}^2),$$

and hence, $\mathfrak{L}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_h = 0$ in $B_{2r}(x_0)$ in the sense of distributions. Thus, A. Barton's higher order version of Meyer's reverse Hölder estimate [21, Theorem 24] yields the existence of some $q \in (2,\infty)$ such that (4.A.2) holds.

Consequently, $\mathfrak{A}$ is bounded on $L^{2+\epsilon}(\mathbb{R}^2;\mathbb{R}^2)$ for any $\epsilon \in [0,\epsilon_0]$, for some $\epsilon_0 > 0$ depending only on the upper and lower bounds of $\mathfrak{a}$, i.e. only on $\mu_*, \mu^*, \nu_*, \nu^*$. In particular, since $\mathfrak{A}\mathcal{M} = (\Delta\mathfrak{L}^{-1}\Delta)(\Delta^{-1}\mathfrak{L}\Delta^{-1}) = \mathrm{Id} = (\Delta^{-1}\mathfrak{L}\Delta^{-1})(\Delta\mathfrak{L}^{-1}\Delta) = \mathcal{M}\mathfrak{A}$, it follows that $\mathcal{M}^{-1} = \mathfrak{A}$ is bounded on $L^{2+\epsilon}(\mathbb{R}^2;\mathbb{R}^2)$.

Step 3: Proofs of (4.1.38) **and** (4.1.39)**.** The estimate (4.1.38) follows directly from the relation $\begin{pmatrix}\omega\\ \operatorname{div} u\end{pmatrix} = \mathcal{M}^{-1}\begin{pmatrix}a\\ b + P(\rho) - \tilde P\end{pmatrix}$, Step 2 and the representation $\nabla u = \mathcal{R}\mathcal{R}^\perp\omega + \mathcal{R}\mathcal{R}\operatorname{div} u$.

Secondly, for a normed vector field $\bar\tau \in \dot W^{1,2+\epsilon}(\mathbb{R}^2;\mathbb{R}^2)$ we have

$$\partial_{\bar\tau}\begin{pmatrix}\omega\\ \operatorname{div} u\end{pmatrix} = \mathcal{M}^{-1}\mathcal{M}\partial_{\bar\tau}\begin{pmatrix}\omega\\ \operatorname{div} u\end{pmatrix} = \mathcal{M}^{-1}\Big(\begin{pmatrix}\partial_{\bar\tau} a\\ \partial_{\bar\tau}(b + P(\rho) - \tilde P)\end{pmatrix} + [\mathcal{M}, \partial_{\bar\tau}]\begin{pmatrix}\omega\\ \operatorname{div} u\end{pmatrix}\Big),$$

where we apply the commutator estimate (4.1.49) with $(g_1, g_2) = (\omega, \operatorname{div} u)$, using (4.1.53). Then (4.1.39) follows from (4.1.46). □

Proof of (3). The boundedness and invertibility of $\mathcal{N}$ follows from (1) and (2).

We now show (4.1.44). We first recall some classical results on dissipative operators and contraction semi-groups. Using the same terminology as in [182], a *semi inner-product* on a vector space X is a map $[\cdot,\cdot] : X \times X \to \mathbb{C}$ satisfying for all $x, y, z \in X$ and $\lambda \in \mathbb{C}$:

$$[x + y, z] = [x, z] + [y, z], \quad [\lambda x, y] = \lambda[x, y],$$
$$[x, x] > 0 \text{ if } x \neq 0, \quad |[x, y]|^2 \leq [x, x][y, y].$$

We have the following facts (see [182])

1. A bounded operator $\mathcal{T}$ on a semi inner-product space $(X, [\cdot,\cdot])$ generates a semi-group of contraction operators if and only if it is dissipative, i.e. $\mathrm{Re}[\mathcal{T}x, x] \leq 0$ for any $x \in X$.
2. For $p \in (1,\infty)$, the map

$$[f, g]_p := \int_{\mathbb{R}^2} f \frac{|g|^{p-2} g}{\|g\|_{L^p}^{p-2}}\, dx, \quad f, g \in L^p(\mathbb{R}^2) \tag{4.A.3}$$

   is a semi inner-product on $L^p(\mathbb{R}^2)$.

The goal is then to prove the existence of some $w > 0$ such that

$$\mathrm{Re}[\mathcal{A}f, f]_{2+\epsilon} \leq -w\|f\|_{L^{2+\epsilon}}^2, \quad \forall f \in L^{2+\epsilon}(\mathbb{R}^2), \tag{4.A.4}$$

where $[\cdot,\cdot]_{2+\epsilon}$ denotes the semi inner-product defined in (4.A.3) with $p = 2+\epsilon$. According to Fact 1 above, this is equivalent to the following contraction property of the semi-group:

$$\|e^{t\mathcal{A}}\|_{L^{2+\epsilon}\to L^{2+\epsilon}} \le e^{-wt}, \quad \forall t\ge 0. \tag{4.A.5}$$

To show this, we proceed in three steps:

- *(Decay in L^2).* By virtue of the dissipation of the operator $\mathcal{A}$ in $L^2(\mathbb{R}^2)$ given in (4.1.52), there exists some $c>0$ depending only on $\mu_*, \mu^*, \nu_*, \nu^*$, such that

$$-\langle \mathcal{A}f, f\rangle = \langle \mathcal{N}^{-1}(\sqrt{\rho P'(\rho)}f), \sqrt{\rho P'(\rho)}f\rangle \ge c\|\sqrt{\rho P'(\rho)}f\|_{L^2}^2 \ge c_*\|f\|_{L^2}^2$$

  with $c_* := c\inf_{s\in(\frac14\rho_*,4\rho^*)} sP'(s)$, so that the operator $\mathcal{T} := \mathcal{A}+c_*$ is dissipative in $L^2(\mathbb{R}^2)$. Fact 1 above implies that $\|e^{t\mathcal{T}}\|_{L^2\to L^2}\le 1$, or equivalently

$$\|e^{t\mathcal{A}}\|_{L^2\to L^2} \le e^{-c_*t}, \quad \forall t\ge 0. \tag{4.A.6}$$

- *(Decay in $L^{2+\epsilon_0}$).* Since $\mathcal{A}$ is bounded in $L^{2+\epsilon_0}$, we can simply estimate

$$\|e^{t\mathcal{A}}\|_{L^{2+\epsilon_0}\to L^{2+\epsilon_0}} \le e^{t\|\mathcal{A}\|_{L^{2+\epsilon_0}\to L^{2+\epsilon_0}}}, \quad \forall t\ge 0. \tag{4.A.7}$$

- *(Interpolation).* Let $\epsilon\in(0,\epsilon_0)$ and let $\theta\in(0,1)$ be such that $\frac{1}{2+\epsilon} = \frac{1-\theta}{2} + \frac{\theta}{2+\epsilon_0}$. By the Riesz-Thorin interpolation theorem it follows that

$$\begin{aligned}\|e^{t\mathcal{A}}\|_{L^{2+\epsilon}\to L^{2+\epsilon}} &\le \Big(\|e^{t\mathcal{A}}\|_{L^2\to L^2}\Big)^{1-\theta}\Big(\|e^{t\mathcal{A}}\|_{L^{2+\epsilon_0}\to L^{2+\epsilon_0}}\Big)^{\theta}\\ &\le \exp\Big(t[-(1-\theta)c_* + \theta\|\mathcal{A}\|_{L^{2+\epsilon_0}\to L^{2+\epsilon_0}}]\Big),\end{aligned}$$

  where we used (4.A.6) and (4.A.7) in the second inequality. If we set

$$w := -[-(1-\theta)c_* + \theta\|\mathcal{A}\|_{L^{2+\epsilon_0}\to L^{2+\epsilon_0}}] = c_* - \theta(c_* + \|\mathcal{A}\|_{L^{2+\epsilon_0}\to L^{2+\epsilon_0}})$$

  and choose $\theta\in(0,\frac{c_*}{c_*+\|\mathcal{A}\|_{L^{2+\epsilon_0}\to L^{2+\epsilon_0}}})$ so that $w>0$, the desired decay estimate (4.A.5) follows.

Then (4.1.44) follows after possibly decreasing ϵ_0 depending on the size of c_* (which in turn depends on $\inf_{s\in(\frac14\rho_*,4\rho^*)} sP'(s)$). □

(i) is proved in [33].

Proof of (ii). Recall Bony's decomposition for any product into low-high frequency, high-low frequency and remainder parts below:

$$FG = T_FG + T_GF + R(F,G),$$

and we refer to [18] for the precise definitions of the paraproduct T_FG and the remainder term $R(F,G)$. Observe that for $q>2$ (see for example [18] or the proofs of [198, Lemma 5.1])

$$\|(T_{\partial_kh}X_k, \partial_kT_hX_k, R(X_k,\partial_kh), \partial_kR(X_k,h), [T_{X_k},\mathcal{R}\mathcal{R}\partial_k]h)\|_{L^q} \le C_q\|\nabla X\|_{L^q}\|h\|_{L^\infty}. \tag{4.A.8}$$

Furthermore, for any indices $(i,j,l,m)\in\{1,2\}^4$ and functions h_1,h_2 we have

$$\|[\partial_X,\mathcal{R}_i\mathcal{R}_j]h_1 + [\partial_X,\mathcal{R}_l\mathcal{R}_m]h_2\|_{L^q} \le C_q\|\nabla X\|_{L^q}\|(h_1,h_2,\mathcal{R}_i\mathcal{R}_jh_1+\mathcal{R}_l\mathcal{R}_mh_2)\|_{L^\infty}. \tag{4.A.9}$$

Indeed, this follows by combining the identities

$$\begin{aligned}
[\partial_X, \mathcal{R}\mathcal{R}]h &= [T_{X_k}, \mathcal{R}\mathcal{R}\partial_k]h + T_{\mathcal{R}\mathcal{R}\partial_k h}X_k + R(X_k, \mathcal{R}\mathcal{R}\partial_k h) - \mathcal{R}\mathcal{R}\partial_k R(X_k, h) \\
&\quad - \mathcal{R}\mathcal{R}\partial_k T_h X_k + \mathcal{R}\mathcal{R}(\operatorname{div} Xh) \\
T_{\mathcal{R}_i\mathcal{R}_j\partial_k h_1}X_k &+ R(X_k, \mathcal{R}_i\mathcal{R}_j\partial_k h_1) + T_{\mathcal{R}_l\mathcal{R}_m\partial_k h_2}X_k + R(X_k, \mathcal{R}_l\mathcal{R}_m\partial_k h_2) \\
&= T_{\partial_k(\mathcal{R}_i\mathcal{R}_j h_1 + \mathcal{R}_l\mathcal{R}_m h_2)}X_k + R(X_k, \partial_k(\mathcal{R}_i\mathcal{R}_j h_1 + \mathcal{R}_l\mathcal{R}_m h_2))
\end{aligned}$$

with the estimate (4.A.8).

Proof of (4.1.46)**.** We write $\nabla v = \mathcal{R}\mathcal{R}^\perp \operatorname{curl} v + \mathcal{R}\mathcal{R}\operatorname{div} v$, so that

$$\partial_X \nabla v = \mathcal{R}\mathcal{R}^\perp \partial_X \operatorname{curl} v + \mathcal{R}\mathcal{R}\partial_X \operatorname{div} v + [\partial_X, \mathcal{R}\mathcal{R}^\perp] \operatorname{curl} v + [\partial_X, \mathcal{R}\mathcal{R}]\operatorname{div} v.$$

The first two terms on the right hand side are bounded in $L^q(\mathbb{R}^2)$ by $\|\partial_X(\operatorname{curl} v, \operatorname{div} v)\|_{L^q}$, and to the last two terms we apply (4.A.9) with $(h_1, h_2) = (\pm \operatorname{curl} v, \operatorname{div} v)$ and suitable indices (i, j, l, m).

Proof of (4.1.47)**.** Denoting $\mathcal{P}_1 = \mathcal{R}_2\mathcal{R}_2 - \mathcal{R}_1\mathcal{R}_1$, $\mathcal{P}_2 = 2\mathcal{R}_1\mathcal{R}_2$, such that $\mathcal{R}_\mu = \mathcal{P}_1\mu(\rho)\mathcal{P}_1 + \mathcal{P}_2\mu(\rho)\mathcal{P}_2$, $\mathcal{Q}_\mu = \mathcal{P}_1\mu(\rho)\mathcal{P}_2 - \mathcal{P}_2\mu(\rho)\mathcal{P}_1$ we have

$$\begin{aligned}
[\partial_X, \mathcal{R}_\mu] &= [\partial_X, \mathcal{P}_1](\mu(\rho)\mathcal{P}_1) + \mathcal{P}_1(\partial_X\mu(\rho))\mathcal{P}_1 + \mathcal{P}_1\mu(\rho)[\partial_X, \mathcal{P}_1] \\
&\quad + [\partial_X, \mathcal{P}_2](\mu(\rho)\mathcal{P}_2) + \mathcal{P}_2(\partial_X\mu(\rho))\mathcal{P}_2 + \mathcal{P}_2\mu(\rho)[\partial_X, \mathcal{P}_2], \\
[\partial_X, \mathcal{Q}_\mu] &= [\partial_X, \mathcal{P}_1](\mu(\rho)\mathcal{P}_2) + \mathcal{P}_1(\partial_X\mu(\rho))\mathcal{P}_2 + \mathcal{P}_1\mu(\rho)[\partial_X, \mathcal{P}_2] \\
&\quad - [\partial_X, \mathcal{P}_2](\mu(\rho)\mathcal{P}_1) - \mathcal{P}_2(\partial_X\mu(\rho))\mathcal{P}_1 - \mathcal{P}_2\mu(\rho)[\partial_X, \mathcal{P}_1],
\end{aligned}$$

so that

$$\begin{aligned}
&[\partial_X, \mathcal{R}_\mu]f - [\partial_X, \mathcal{Q}_\mu]g \\
&= [\partial_X, \mathcal{P}_1](\mu(\rho)(\mathcal{P}_1 f - \mathcal{P}_2 g)) + [\partial_X, \mathcal{P}_2](\mu(\rho)(\mathcal{P}_2 f + \mathcal{P}_1 g)) + \mathcal{P}_1(\mu(\rho)([\partial_X, \mathcal{P}_1]f - [\partial_X, \mathcal{P}_2]g)) \\
&+ \mathcal{P}_2(\mu(\rho)([\partial_X, \mathcal{P}_2]f + [\partial_X, \mathcal{P}_1]g)) + \mathcal{P}_1(\partial_X\mu(\rho)(\mathcal{P}_1 f - \mathcal{P}_2 g)) + \mathcal{P}_2(\partial_X\mu(\rho)(\mathcal{P}_2 f + \mathcal{P}_1 g)).
\end{aligned}$$

To the right hand side we apply (4.A.9) three times: to the first two terms with $(h_1, h_2) = (\mu(\rho)(\mathcal{P}_1 f - \mathcal{P}_2 g), \mu(\rho)(\mathcal{P}_2 f + \mathcal{P}_1 g))$, observing that $\mathcal{P}_1\mu(\rho)(\mathcal{P}_1 f - \mathcal{P}_2 g) + \mathcal{P}_2\mu(\rho)(\mathcal{P}_2 f + \mathcal{P}_1 g) = \mathcal{R}_\mu f - \mathcal{Q}_\mu g$; to the third term with $(h_1, h_2) = (f, -g)$; and to the fourth term with $(h_1, h_2) = (f, g)$. The remaining two terms are bounded in $L^q(\mathbb{R}^2)$ by $\|\partial_X\mu(\rho)\|_{L^q}\|(\mathcal{P}_1 f - \mathcal{P}_2 g, \mathcal{P}_2 f + \mathcal{P}_1 g)\|_{L^\infty}$. This proves (4.1.47).

The estimate (4.1.48) is proved similarly as (4.1.47).

The estimate (4.1.49) is achieved by combining (4.1.47) and (4.1.48) and noticing that

$$[\partial_X, \mathcal{M}]\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} [\partial_X, \mathcal{R}_\mu]f - [\partial_X, \mathcal{Q}_\mu]g \\ [\partial_X, \mathcal{Q}_\mu]f + [\partial_X, \mathcal{R}_{\mu,\lambda}]g \end{pmatrix}.$$

□

4.B. Appendix: Proof of Lemma 4.1.9

Define the adjoint operators of the directional derivatives $\partial_{\overline{\tau}}, \partial_n$ below

$$\partial_{\overline{\tau}}^* = -\operatorname{div} \overline{\tau}, \quad \partial_n^* = -\operatorname{div} n, \tag{4.B.1}$$

where the operator $\operatorname{div} v$ is understood as $\operatorname{div} v(f) = \operatorname{div}(vf) = \sum_{j=1}^{2} \partial_j(v_j f)$, for $v = \bar\tau$, n. Using the identities

$$\nabla = \bar\tau\partial_{\bar\tau} + n\partial_n = -\partial_{\bar\tau}^*(\bar\tau\cdot) - \partial_n^*(n\cdot),$$

we first calculate

$$\nabla\otimes\nabla = -\partial_{\bar\tau}^*(\bar\tau\otimes\bar\tau)\partial_{\bar\tau} - \partial_{\bar\tau}^*(\bar\tau\otimes n)\partial_n - \partial_n^*(n\otimes\bar\tau)\partial_{\bar\tau} - \partial_n^*(n\otimes n)\partial_n, \tag{4.B.2}$$

$$\nabla^\perp\otimes\nabla = \partial_{\bar\tau}^*(n\otimes\bar\tau)\partial_{\bar\tau} + \partial_{\bar\tau}^*(n\otimes n)\partial_n - \partial_n^*(\bar\tau\otimes\bar\tau)\partial_{\bar\tau} - \partial_n^*(\bar\tau\otimes n)\partial_n, \tag{4.B.3}$$

$$\Delta = \nabla\cdot\nabla = -\partial_{\bar\tau}^*\partial_{\bar\tau} - \partial_n^*\partial_n. \tag{4.B.4}$$

Proof of Lemma 4.1.9.

(I) By use of the definitions of α, β in (4.1.55) and the representation (4.1.65) of $T(\rho, u)$ in terms of (γ_1, γ_2), we can write

$$\alpha = (\bar\tau_2^2 - \bar\tau_1^2)\mu(\rho)(\partial_2 u_1 + \partial_1 u_2) + (2\bar\tau_1\bar\tau_2)\mu(\rho)(\partial_1 u_1 - \partial_2 u_2), \tag{4.B.5}$$

$$\begin{aligned}\beta = {}&(\bar\tau_2^2 - \bar\tau_1^2)\mu(\rho)(\partial_1 u_1 - \partial_2 u_2) - 2\bar\tau_1\bar\tau_2\mu(\rho)(\partial_2 u_1 + \partial_1 u_2)\\ &+ (\mu(\rho) + \lambda(\rho))\operatorname{div} u - (P(\rho) - \tilde P).\end{aligned} \tag{4.B.6}$$

Then a direct computation using (4.B.5), (4.B.6) shows that

$$\begin{aligned}&\omega = \frac{\alpha}{\mu(\rho)} - 2n\cdot\partial_{\bar\tau}u, \quad \operatorname{div} u = \frac{\beta + (P(\rho) - \tilde P)}{\nu(\rho)} + \frac{\mu(\rho)}{\nu(\rho)}2\bar\tau\cdot\partial_{\bar\tau}u,\\ &\beta - F = -2\mu(\rho)\bar\tau\cdot\partial_{\bar\tau}u,\end{aligned} \tag{4.B.7}$$

which implies (4.1.56) and (4.1.57).

(II) We calculate

$$-\partial_{\bar\tau}u^\perp + \partial_n u = n(\operatorname{div} u) + \bar\tau\omega,$$

where we insert (4.B.7) to obtain (4.1.58).

The representation of the gradient in (4.1.59) follows from the identity $\nabla u = \partial_{\bar\tau}u\otimes\bar\tau + \partial_n u\otimes n$ and (4.1.58).

(III) Applying Δ to (4.1.23) and using (4.B.3) we obtain

$$\begin{aligned}\Delta a &= (\nabla^\perp\otimes\nabla) : T = (\nabla^\perp\otimes\nabla) : \begin{pmatrix}\gamma_2 & \gamma_1\\ \gamma_1 & -\gamma_2\end{pmatrix}\\ &= \partial_{\bar\tau}^*((\bar\tau_2^2 - \bar\tau_1^2)\partial_{\bar\tau}\gamma_1) + \partial_{\bar\tau}^*(2\bar\tau_1\bar\tau_2\partial_{\bar\tau}\gamma_2) - \partial_{\bar\tau}^*(2\bar\tau_1\bar\tau_2\partial_n\gamma_1) + \partial_{\bar\tau}^*((\bar\tau_2^2 - \bar\tau_1^2)\partial_n\gamma_2)\\ &\quad - \partial_n^*(2\bar\tau_1\bar\tau_2\partial_{\bar\tau}\gamma_1) + \partial_n^*((\bar\tau_2^2 - \bar\tau_1^2)\partial_{\bar\tau}\gamma_2) - \partial_n^*\big((\bar\tau_2^2 - \bar\tau_1^2)\partial_n\gamma_1 + 2\bar\tau_1\bar\tau_2\partial_n\gamma_2\big),\end{aligned}$$

where

- by the formula

$$\partial_{\bar\tau}^*(h\partial_n f) = \partial_n^*(h\partial_{\bar\tau}f) - \partial_1(\partial_2 hf) + \partial_2(\partial_1 hf),$$

for functions h, f, we have

$$\begin{aligned}&- \partial_{\bar\tau}^*(2\bar\tau_1\bar\tau_2\partial_n\gamma_1) + \partial_{\bar\tau}^*((\bar\tau_2^2 - \bar\tau_1^2)\partial_n\gamma_2) = -\partial_n^*(2\bar\tau_1\bar\tau_2\partial_{\bar\tau}\gamma_1) + \partial_n^*((\bar\tau_2^2 - \bar\tau_1^2)\partial_{\bar\tau}\gamma_2)\\ &\quad - \partial_1\big(-\partial_2(2\bar\tau_1\bar\tau_2)\gamma_1 + \partial_2(\bar\tau_2^2 - \bar\tau_1^2)\gamma_2\big) + \partial_2\big(-\partial_1(2\bar\tau_1\bar\tau_2)\gamma_1 + \partial_1(\bar\tau_2^2 - \bar\tau_1^2)\gamma_2\big);\end{aligned}$$

- by (4.B.4), the last term can be expressed in terms of α from (4.1.55) as

$$
\begin{aligned}
&-\partial_n^*((\bar\tau_2^2-\bar\tau_1^2)\partial_n\gamma_1+2\bar\tau_1\bar\tau_2\partial_n\gamma_2)\\
&=-\partial_n^*\partial_n((\bar\tau_2^2-\bar\tau_1^2)\gamma_1+2\bar\tau_1\bar\tau_2\gamma_2)+\partial_n^*(\partial_n(\bar\tau_2^2-\bar\tau_1^2)\gamma_1+\partial_n(2\bar\tau_1\bar\tau_2)\gamma_2)\\
&=-\partial_n^*\partial_n\alpha+\partial_n^*(\partial_n(\bar\tau_2^2-\bar\tau_1^2)\gamma_1+\partial_n(2\bar\tau_1\bar\tau_2)\gamma_2)\\
&=\Delta\alpha+\partial_{\bar\tau}^*\partial_{\bar\tau}\alpha+\partial_n^*(\partial_n(\bar\tau_2^2-\bar\tau_1^2)\gamma_1+\partial_n(2\bar\tau_1\bar\tau_2)\gamma_2).
\end{aligned}
$$

Inserting these identities back into the expression for Δa results in (4.1.61) after applying $\nabla\Delta^{-1}$ to both sides.

Similarly, applying Δ to (4.1.24) and using (4.B.2) we compute

$$
\begin{aligned}
\Delta b&=(\nabla\otimes\nabla):T=(\nabla\otimes\nabla):\Big[\begin{pmatrix}\gamma_2&\gamma_1\\ \gamma_1&-\gamma_2\end{pmatrix}+((\mu(\rho)+\lambda(\rho))\operatorname{div}u-(P(\rho)-\tilde P))\mathrm{Id}\Big]\\
&=-\partial_{\bar\tau}^*(2\bar\tau_2\bar\tau_2\partial_{\bar\tau}\gamma_1)+\partial_{\bar\tau}^*((\bar\tau_2^2-\bar\tau_1^2)\partial_{\bar\tau}\gamma_2)-\partial_{\bar\tau}^*((\bar\tau_2^2-\bar\tau_1^2)\partial_n\gamma_1)-\partial_{\bar\tau}^*(2\bar\tau_1\bar\tau_2\partial_n\gamma_2)\\
&\quad-\partial_n^*((\bar\tau_2^2-\bar\tau_1^2)\partial_{\bar\tau}\gamma_1)-\partial_n^*(2\bar\tau_1\bar\tau_2\partial_{\bar\tau}\gamma_2)-\partial_n^*(-2\bar\tau_1\bar\tau_2\partial_n\gamma_1+(\bar\tau_2^2-\bar\tau_1^2)\partial_n\gamma_2)\\
&\quad+\Delta((\mu(\rho)+\lambda(\rho))\operatorname{div}u-(P(\rho)-\tilde P)),
\end{aligned}
$$

where

- as in the previous calculations

$$
\begin{aligned}
&-\partial_{\bar\tau}^*((\bar\tau_2^2-\bar\tau_1^2)\partial_n\gamma_1)-\partial_{\bar\tau}^*(2\bar\tau_1\bar\tau_2\partial_n\gamma_2)=-\partial_n^*((\bar\tau_2^2-\bar\tau_1^2)\partial_{\bar\tau}\gamma_1)-\partial_n^*(2\bar\tau_1\bar\tau_2\partial_{\bar\tau}\gamma_2)\\
&\quad+\partial_1\Big(\partial_2(\bar\tau_2^2-\bar\tau_1^2)\gamma_1+\partial_2(2\bar\tau_1\bar\tau_2)\gamma_2\Big)-\partial_2\Big(\partial_1(\bar\tau_2^2-\bar\tau_1^2)\gamma_1+\partial_1(2\bar\tau_1\bar\tau_2)\gamma_2\Big);
\end{aligned}
$$

- by (4.B.4), the last two terms on the right hand side can be expressed in terms of β from (4.1.55) as

$$
\begin{aligned}
&-\partial_n^*(-2\bar\tau_1\bar\tau_2\partial_n\gamma_1+(\bar\tau_2^2-\bar\tau_1^2)\partial_n\gamma_2)+\Delta((\mu(\rho)+\lambda(\rho))\operatorname{div}u-(P(\rho)-\tilde P))\\
&=-\partial_n^*\partial_n(-2\bar\tau_1\bar\tau_2\gamma_1+(\bar\tau_2^2-\bar\tau_1^2)\gamma_2)+\partial_n^*(-\partial_n(2\bar\tau_1\bar\tau_2)\gamma_1+\partial_n(\bar\tau_2^2-\bar\tau_1^2)\gamma_2)\\
&\quad+\Delta((\mu(\rho)+\lambda(\rho))\operatorname{div}u-(P(\rho)-\tilde P))\\
&=\Delta\beta+\partial_{\bar\tau}^*\partial_{\bar\tau}(-2\bar\tau_1\bar\tau_2\gamma_1+(\bar\tau_2^2-\bar\tau_1^2)\gamma_2)+\partial_n^*(-\partial_n(2\bar\tau_1\bar\tau_2)\gamma_1+\partial_n(\bar\tau_2^2-\bar\tau_1^2)\gamma_2).
\end{aligned}
$$

so that we obtain (4.1.62) by applying $\nabla\Delta^{-1}$ to both sides.

The identity (4.1.64) is obtained directly from the definition of b in (4.1.26) and (4.B.6) as follows

$$
\begin{aligned}
\beta&=b+(\bar\tau_2^2-\bar\tau_1^2)\mu(\rho)(\partial_1u_1-\partial_2u_2)-2\bar\tau_1\bar\tau_2\mu(\rho)(\partial_2u_1+\partial_1u_2)-\mathcal R_\mu\operatorname{div}u-\mathcal Q_\mu\omega\\
&=b-\Big[((\bar\tau_2^2-\bar\tau_1^2)+(\mathcal R_2\mathcal R_2-\mathcal R_1\mathcal R_1))\mu(\rho)(\partial_2u_2-\partial_1u_1)\\
&\qquad+(2\bar\tau_1\bar\tau_2+2\mathcal R_1\mathcal R_2)\mu(\rho)(\partial_2u_1+\partial_1u_2)\Big]\\
&=b-\frac12(\bar\tau\otimes\bar\tau-n\otimes n+\mathcal R\otimes\mathcal R-\mathcal R^\perp\otimes\mathcal R^\perp):(\mu(\rho)Su).
\end{aligned}
$$

□

4.C. APPENDIX: AUXILIARY CALCULATIONS FOR THE ENERGY ESTIMATES

4.C.1. PROOF OF PROPOSITION 4.3.1

Proof of Proposition 4.3.1. **Proof of** (4.3.7)**.** We follow Sections 2 and 3 of [135]. We set

$$L = \begin{pmatrix} 0 & \mathrm{div} \\ P'(\tilde{\rho})\nabla & -\frac{\tilde{\mu}}{\tilde{\rho}}\Delta - \frac{\tilde{\mu}+\tilde{\lambda}}{\tilde{\rho}}\nabla \mathrm{div} \end{pmatrix}, \quad U = \begin{pmatrix} n \\ m \end{pmatrix} := \begin{pmatrix} \rho - \tilde{\rho} \\ \rho u \end{pmatrix},$$

and rewrite (CNS) as

$$\frac{d}{dt}U + LU = \begin{pmatrix} 0 \\ N \end{pmatrix}, \quad U|_{t=0} = U_0 := \begin{pmatrix} n_0 \\ m_0 \end{pmatrix}, \tag{4.C.1}$$

with the nonlinear part

$$N = -\nabla(P(\rho) - \tilde{P} - P'(\tilde{\rho})n) - \mathrm{div}\,(m \otimes u) + \mathrm{div}\,((\mu(\rho) - \tilde{\mu})Su) - \tilde{\mu}\Delta(\frac{nu}{\tilde{\rho}})$$
$$+\nabla((\lambda(\rho) - \tilde{\lambda})\mathrm{div}\, u) - (\tilde{\mu} + \tilde{\lambda})\nabla \mathrm{div}\,(\frac{nu}{\tilde{\rho}}).$$

We compute the eigenvalues of the symbol of the operator $-L$ below

$$-\widehat{L} = -\begin{pmatrix} 0 & i\xi_1 & i\xi_2 \\ iP'(\tilde{\rho})\xi_1 & \frac{\tilde{\mu}}{\tilde{\rho}}|\xi|^2 + \frac{\tilde{\mu}+\tilde{\lambda}}{\tilde{\rho}}(\xi_1)^2 & \frac{\tilde{\mu}+\tilde{\lambda}}{\tilde{\rho}}\xi_1\xi_2 \\ iP'(\tilde{\rho})\xi_2 & \frac{\tilde{\mu}+\tilde{\lambda}}{\tilde{\rho}}\xi_1\xi_2 & \frac{\tilde{\mu}}{\tilde{\rho}}|\xi|^2 + \frac{\tilde{\mu}+\tilde{\lambda}}{\tilde{\rho}}(\xi_2)^2 \end{pmatrix}$$

to be (recalling $\tilde{\nu} = 2\tilde{\mu} + \tilde{\lambda}$)

$$z_0(\xi) = -\frac{\tilde{\mu}|\xi|^2}{\tilde{\rho}}, \quad z_\pm(\xi) = -\frac{1}{2\tilde{\rho}}\Big(\tilde{\nu}|\xi|^2 \pm |\xi|\sqrt{\tilde{\nu}^2|\xi|^2 - 4P'(\tilde{\rho})\tilde{\rho}^2}\Big).$$

Thus

$$\Re[z_\pm](\xi) = -\frac{1}{2\tilde{\rho}}\tilde{\nu}|\xi|^2, \text{ if } |\xi| \le \frac{2\tilde{\rho}\sqrt{P'(\tilde{\rho})}}{\tilde{\nu}},$$
$$\text{and } z_+(\xi) \sim -\frac{\tilde{\nu}}{\tilde{\rho}}|\xi|^2, \quad z_-(\xi) \sim -\tilde{\rho}P'(\tilde{\rho}), \text{ if } |\xi| \gg \frac{2\tilde{\rho}\sqrt{P'(\tilde{\rho})}}{\tilde{\nu}}.$$

Therefore there exists a constant c such that

$$|e^{-t\hat{L}(\xi)}f(\xi)| = \begin{cases} \mathcal{O}(1)e^{-c|\xi|^2 t}|f(\xi)|, & |\xi| \le 1, \\ \mathcal{O}(1)e^{-ct}|f(\xi)|, & |\xi \ge 1. \end{cases} \tag{4.C.2}$$

Step 1: Decay of the low frequencies $\|(n^l, m^l)(t)\|_{L^2_x}$**.** Let $g(t) > 0$ be the frequency threshold, which is determined later. As (4.3.7) follows immediately from (4.3.4) for small time region $t \in [0, 1]$, in the following we restrict ourselves to the large time case $t \ge 1$. For computational simplicity, we first assume for some absolute constant C that

$$g(t) \le C, \quad |g'(t)| \le Cg(t)^2 \le C^2 g(t), \quad \forall t \ge 1, \text{ and } \lim_{t\to\infty} g(t) = 0. \tag{4.C.3}$$

This assumption will be seen to be satisfied with the specific choices of $g(t)$ later (see (4.C.15)-(4.C.16) below).

We define the time-dependent low and high frequency part of a function f as

$$f^l(t,\cdot) = \mathcal{F}(\chi(\cdot,t)\hat{f}(t,\cdot)), \quad f^h(t,\cdot) = \mathcal{F}((1-\chi(\cdot,t))\hat{f}(t,\cdot))$$

where $\chi = \chi(t,\xi) \in [0,1]$ is a smooth approximation of $1_{\{|\xi|\leq g(t)\}}$ (for more details see below). Then

$$\|f^h\|_{L^2} \leq C\frac{1}{g(t)}\|\nabla f\|_{L^2}, \quad \|f^l\|_{L^2} \leq Cg(t)\|f\|_{L^1} \tag{4.C.4}$$

for a time-independent constant C.

By Duhamel's formula, the solution U to (4.C.1) is given by

$$U(t) = e^{-tL}U_0 + \int_0^t e^{-(t-t')L}\begin{pmatrix} 0 \\ N(t') \end{pmatrix} dt',$$

so that the Fourier transform $\hat{U}$ of the solution is

$$\hat{U}(t,\xi) = e^{-t\hat{L}(\xi)}\hat{U}_0(\xi) + \int_0^t e^{-(t-t')\hat{L}(\xi)}\begin{pmatrix} 0 \\ \hat{N}(t',\xi) \end{pmatrix} dt'.$$

We square, then integrate over $\{|\xi| \leq g(t)\}$, and apply (4.C.2) to obtain for $t \geq T \geq 1$,

$$\begin{aligned}\int_{\{|\xi|\leq g(t)\}} |\hat{U}(t,\xi)|^2 d\xi \leq C_* &\int_{\{|\xi|\leq g(t)\}} e^{-c|\xi|^2 t}|\hat{U}_0(\xi)|^2 d\xi \\ &+ C_* \int_{\{|\xi|\leq g(t)\}} \Big(\int_0^t e^{-c|\xi|^2(t-t')}|\hat{N}(t',\xi)|dt'\Big)^2 d\xi.\end{aligned}$$

The first term on the right hand side can be controlled by the initial data

$$\int_{\{|\xi|\leq g(t)\}} e^{-c|\xi|^2 t}|\hat{U}_0(\xi)|^2 d\xi = \int_{\{|\xi|\leq g(t)\}} e^{-c|\xi|^2 t}|\xi|^{4\delta}|\xi|^{-4\delta}|\hat{U}_0(\xi)|^2 d\xi \leq (g(t))^{4\delta}\|U_0\|^2_{\dot{H}^{-2\delta}},$$

and the second term is bounded by

$$\begin{aligned}&\int_{\{|\xi|\leq g(t)\}} |\xi|^2 d\xi \Big(\int_0^t \Big\|\mathcal{F}\Big(P(\rho)-\tilde{P}-P'(\tilde{\rho})n, m\otimes u, (\mu(\rho)-\tilde{\mu})Su, (\lambda(\rho)-\tilde{\lambda})\operatorname{div} u\Big)\Big\|_{L^\infty_\xi} dt'\Big)^2 \\ &\quad + \int_{\{|\xi|\leq g(t)\}} |\xi|^4 d\xi \Big(\int_0^t \frac{2\tilde{\mu}+\tilde{\lambda}}{\tilde{\rho}}\|\mathcal{F}(nu)\|_{L^\infty_\xi} dt'\Big)^2 \\ &\lesssim g(t)^4 \Big(\int_0^t \Big\|\Big(P(\rho)-\tilde{P}-P'(\tilde{\rho})n, m\otimes u, (\mu(\rho)-\tilde{\mu})Su, (\lambda(\rho)-\tilde{\lambda})\operatorname{div} u\Big)\Big\|_{L^1} dt'\Big)^2 \\ &\quad + g(t)^6\Big(\int_0^t \|nu\|_{L^1} dt'\Big)^2 \\ &\lesssim_* g(t)^4\|(n,u,\nabla u)\|^4_{L^2(0,t;L^2)}.\end{aligned}$$

where we used that $P(\rho)-\tilde{P}-P'(\tilde{\rho})n \sim_* n^2$, $\mu(\rho)-\tilde{\mu} \sim_* n$, $\lambda(\rho)-\tilde{\lambda} \sim_* n$ and $|g(t)| \leq C$.

In total we obtain for $t \geq 1$,

$$\|U^l(t)\|^2_{L^2} \lesssim_* g(t)^{4\delta}\|U_0\|^2_{\dot{H}^{-2\delta}} + g(t)^4 E_0^4 + g(t)^4\|(n,u)\|^4_{L^2(0,t;L^2)}. \tag{4.C.5}$$

Step 2: Perturbed energy functional. Let $\Lambda = \sqrt{-\Delta}$ be defined as Fourier multiplier. We claim that

$$-\frac{d}{dt}\int_{\mathbb{R}^2}(\Lambda^{-2}\operatorname{div} m^h)n^h dx + \frac{P'(\rho_*)}{4}\|n\|^2_{L^2} \leq C_*(1+E_0^2)\|(n^l, m^h, \nabla u)\|^2_{L^2}. \tag{4.C.6}$$

To show this, we apply $\Lambda^{-2}\mathrm{div}\,\mathcal{F}^{-1}((1-\chi)\mathcal{F})$ to the momentum equation $(\mathrm{CNS})_2$ and use that $\Lambda^{-2}\Delta = -1$ to obtain

$$(P(\rho)-\tilde{P})^h = (\partial_t\Lambda^{-2}\mathrm{div}\, m)^h + \Lambda^{-2}\mathrm{div}\,\mathrm{div}\,(u\otimes m)^h - \Lambda^{-2}\mathrm{div}\,\mathrm{div}\,(\mu(\rho)Su)^h + (\lambda(\rho)\mathrm{div}\, u)^h.$$

Taking the L^2-inner product with n^h yields

$$\begin{aligned}\int_{\mathbb{R}^2}(P(\rho)-\tilde{P})^h n^h dx &= \int_{\mathbb{R}^2}(\partial_t\Lambda^{-2}\mathrm{div}\, m)^h n^h dx \\ &+ \int_{\mathbb{R}^2}\Big(\Lambda^{-2}\mathrm{div}\,\mathrm{div}\,(u\otimes m)^h - \Lambda^{-2}\mathrm{div}\,\mathrm{div}\,(\mu(\rho)Su)^h + (\lambda(\rho)\mathrm{div}\, u)^h\Big)n^h dx,\end{aligned}$$

where

- exactly as in the proof of [135, Lemma 3.2] we deduce

$$\int_{\mathbb{R}^2}(P(\rho)-\tilde{P})^h n^h dx \geq \frac{P'(\rho_*)}{2}\|n\|_{L^2}^2 - C_*\|n^l\|_{L^2}^2;$$

- using $\|u\otimes m\|_{L^2} \leq C_*\|u\|_{L^4}^2 \leq C_*\|u\|_{L^2}\|\nabla u\|_{L^2}$ and $\|u\|_{L^2}^2 \leq C_* E_0^2$, the second integral on the right hand side is bounded from above by

$$\frac{P'(\rho_*)}{8}\|n\|_{L^2}^2 + C_*(1+E_0^2)\|\nabla u\|_{L^2}^2;$$

- by Plancherel's formula and the Fourier transform of the density equation $(\mathrm{CNS})_1$: $\partial_t\hat{n} = -\widehat{\mathrm{div}\, m} = -i\xi\cdot\hat{m}$, we get

$$\begin{aligned}\int_{\mathbb{R}^2}(\partial_t\Lambda^{-2}\mathrm{div}\, m)^h n^h dx = &\frac{d}{dt}\int_{\mathbb{R}^2}(\Lambda^{-2}\mathrm{div}\, m^h)n^h d\xi + \int_{\mathbb{R}^2}|\xi|^{-2}i\xi\cdot\hat{m}2(1-\chi)\partial_t\chi\hat{n}d\xi \\ &- \int_{\mathbb{R}^2}|\xi|^{-2}\xi\cdot\hat{m}(1-\chi)^2\xi\cdot\hat{m}d\xi. \qquad (4.C.7)\end{aligned}$$

We choose the function χ as a smooth approximation of $1_{\{|\xi|\leq g(t)\}}$, for example as

$$\chi(t,\xi) = \psi(|\xi|-g(t)), \quad \text{where} \quad \psi\in C^\infty(\mathbb{R};[0,1]),\ \psi = \begin{cases}1, & \text{on } (-\infty,0],\\ 0, & \text{on } [1,\infty),\end{cases}$$

with ψ being independent of time and the parameters. Then the function $1-\chi(t,\cdot)$ is supported on $\{\xi||\xi|\geq g(t)\}$ and $\partial_t\chi(t,\xi) = -\psi'(|\xi|-g(t))g'(t)$. It follows from (4.C.3) that

$$\|(1-\chi(t,\xi))|\xi|^{-1}\partial_t(1-\chi(t,\xi))\|_{L^\infty_\xi} \leq \|\psi'\|_{L^\infty(\mathbb{R})}|\frac{1}{g(t)}g'(t)| \leq C.$$

Hence, the second integral on the right hand side of (4.C.7) is bounded by $C\|m^h\|_{L^2}\|n\|_{L^2}$, which implies

$$\int_{\mathbb{R}^2}(\partial_t\Lambda^{-2}\mathrm{div}\, m)^h n^h dx \leq \frac{d}{dt}\int_{\mathbb{R}^2}(\Lambda^{-2}\mathrm{div}\, m^h)n^h dx + C_*\|m^h\|_{L^2}^2 + \frac{P'(\rho_*)}{8}\|n\|_{L^2}^2.$$

Combining the above inequalities yields (4.C.6).

We define the energy functional and perturbed energy functional

$$\mathcal{E}(t) = \int_{\mathbb{R}^2}\frac{1}{2}\rho|u|^2 + H(\rho)dx, \quad \mathcal{E}_M(t) = M\mathcal{E}(t) - g(t)^2\int_{\mathbb{R}^2}(\Lambda^{-2}\mathrm{div}\, m^h)n^h dx,$$

where $M \geq 1$ is sufficiently large, to be determined later. Notice that $\mathcal{E}(t) \sim_* \|(n,u)\|_{L^2}^2$. By the energy balance (4.3.3) and (4.C.6) we have for $M' \geq 1$ (to be determined later)

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}_M(t) + \frac{g(t)^2}{M'}\mathcal{E}_M(t) \\
&= M\frac{d}{dt}\mathcal{E}(t) - g(t)^2\frac{d}{dt}\int_{\mathbb{R}^2}(\Lambda^{-2}\mathrm{div}\, m^h)n^h\,dx - (2g(t)g'(t) + \frac{g(t)^4}{M'})\int_{\mathbb{R}^2}(\Lambda^{-2}\mathrm{div}\, m^h)n^h\,dx \\
&\quad + g(t)^2\frac{M}{M'}\mathcal{E}(t) \\
&\leq (-Mc_* + g(t)^2C_*(1+E_0^2))\|\nabla u\|_{L^2}^2 + g(t)^2\Big[-\frac{P'(\rho_*)}{4}\|n\|_{L^2}^2 + C_*(1+E_0^2)\|(n^l,m^h)\|_{L^2}^2\Big] \\
&\quad + C(|g'(t)| + \frac{g(t)^3}{M'})\|m^h\|_{L^2}\|n\|_{L^2} + \frac{M}{M'}C_*g(t)^2\|(n,u)\|_{L^2}^2. \qquad (4.C.8)
\end{aligned}$$

Step 3. Conclusion. Assume (4.3.2): $E_0 \leq 1$.

We first show that $\|(m^l, \frac{1}{g(t)}\nabla u)\|_{L^2}$ can control $\|(u^l, m)\|_{L^2}$ for $t \geq T_*$, where $T_* \geq 1$ is large enough: We write $u = \frac{m}{\tilde\rho} - \frac{nu}{\tilde\rho}$ and apply (4.C.4) and (4.3.3) to obtain

$$\|u^l(t)\|_{L^2} \leq \frac{1}{\tilde\rho}\|m^l(t)\|_{L^2} + \frac{1}{\tilde\rho}\|(nu)^l(t)\|_{L^2} \leq \frac{1}{\tilde\rho}\|m^l(t)\|_{L^2} + C_*g(t)E_0\|(u^l,u^h)(t)\|_{L^2}. \quad (4.C.9)$$

We take $T_* \geq 1$ big enough (depending only on C_*) such that

$$C_*g(t) \leq \frac{1}{2}, \quad \forall t \geq T_*, \qquad (4.C.10)$$

which is possible by (4.C.3): $\lim_{t\to\infty} g(t) = 0$ (and more precisely by the choices (4.C.15)-(4.C.16) below). Then

$$\|u^l(t)\|_{L^2} \leq \frac{2}{\tilde\rho}\|m^l(t)\|_{L^2} + \|u^h\|_{L^2} \leq \frac{2}{\tilde\rho}\|m^l(t)\|_{L^2} + C\frac{1}{g(t)}\|\nabla u\|_{L^2}, \quad t \geq T_* \geq 1,$$

and hence

$$\|m\|_{L^2} \leq \rho^*\|u\|_{L^2} \leq 2\frac{\rho^*}{\tilde\rho}\|m^l\|_{L^2} + C\frac{1}{g(t)}\|\nabla u\|_{L^2}, \quad t \geq T_* \geq 1.$$

We can then control $\|(m^h,u)\|_{L^2}$ in (4.C.8) by $\|(m^l, \frac{1}{g(t)}\nabla u)\|_{L^2}$. Applying Young's inequality to the term involving $\|m^h\|_{L^2}\|n\|_{L^2}$ in (4.C.8) (recalling $|g'(t)| \leq Cg(t)^2$) yields (with a possibly bigger C_*)

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}_M(t) + \frac{g(t)^2}{M'}\mathcal{E}_M(t) \\
&\leq \Big(-Mc_* + (g(t)^2 + 1 + \frac{M}{M'})C_*(1+E_0^2)\Big)\|\nabla u\|_{L^2}^2 + g(t)^2(-\frac{P'(\rho_*)}{8} + C_*\frac{M}{M'})\|n\|_{L^2}^2 \\
&\quad + g(t)^2C_*(1+E_0^2+\frac{M}{M'})\|(n^l,m^l)\|_{L^2}^2, \quad \forall t \geq T_* \geq 1. \qquad (4.C.11)
\end{aligned}$$

Recall $\mathcal{E}_M(t) = M\mathcal{E}(t) - g(t)^2\int_{\mathbb{R}^2}(\Lambda^{-2}\mathrm{div}\, m^h)n^h dx$, with $\mathcal{E}(t) \sim_* \|(n,u)\|_{L^2}^2$ and (recalling $g(t) \leq C$)

$$\Big|g(t)^2\int_{\mathbb{R}^2}(\Lambda^{-2}\mathrm{div}\, m^h)n^h dx\Big| \leq g(t)\|(m^h,n^h)\|_{L^2}^2 \leq C\|(n,u)\|_{L^2}^2.$$

We choose $M, \frac{M'}{M} \geq 1$ big enough (depending only on C_*) such that

$$\mathcal{E}_M(t) \sim_* \|(n,u)\|_{L^2}^2, \quad t \geq 1, \tag{4.C.12}$$

$$\frac{d}{dt}\mathcal{E}_M(t) + g(t)^2\mathcal{E}_M(t) \lesssim_* g(t)^2 \|(n^l, m^l)\|_{L^2}^2, \quad t \geq T_* \geq 1. \tag{4.C.13}$$

Recalling the low frequency estimate (4.C.5), we derive from (4.C.13) that

$$\frac{d}{dt}\mathcal{E}_M + g(t)^2\mathcal{E}_M \lesssim_* g(t)^{2+4\delta}\|U_0\|_{\dot{H}^{-2\delta}}^2 + g(t)^6 E_0^4 + g(t)^6 \|(n,u)\|_{L^2(0,t;L^2)}^4, \quad t \geq T_* \geq 1. \tag{4.C.14}$$

We now can follow the standard argument as in [242, pp. 310-311] or [11, Proposition 2.2] to achieve (4.3.7) for $t \geq T_* \geq 1$. More precisely, we first take (with $\langle t \rangle = e + t$)

$$g(t)^2 = \frac{3}{\langle t \rangle \ln\langle t \rangle}, \text{ such that } \int_0^t g^2 = 3 \ln\ln\langle t \rangle, \quad e^{\int_0^t g^2} = (\ln\langle t \rangle)^3, \tag{4.C.15}$$

and derive from (4.C.14) first that (noticing $T_* \lesssim C_*$, $\|(n,u)\|_{L^2(0,t;L^2)}^2 \lesssim_* tE_0^2$)

$$\mathcal{E}_M(t) \leq C_* E_\delta^2 (\ln\langle t \rangle)^{-2}, \quad t \geq T_*.$$

We now insert this decay estimate: $\|(n,u)\|_{L^2} \lesssim E_\delta (\ln\langle t \rangle)^{-1}$ back into (4.C.14) to achieve

$$\frac{d}{dt}\mathcal{E}_M + g(t)^2\mathcal{E}_M \leq C_*\Big(g(t)^{2+4\delta}E_\delta^2 + g(t)^6 E_0^4 + g(t)^6 E_\delta^2 \langle t \rangle (\ln\langle t \rangle)^{-2} \|(n,u)\|_{L^2(0,t;L^2)}^2\Big),$$

and by taking

$$g(t)^2 = \frac{2\delta_-}{\langle t \rangle} \text{ such that } \int_0^t g^2 = 2\delta_-(\log\langle t \rangle - 1), \quad e^{\int_0^t g^2} = e^{-2\delta_-}\langle t \rangle^{2\delta_-}, \tag{4.C.16}$$

and following Wiegner's arguments in [242, pp. 310-311] while carefully tracking the constants, we arrive at (4.3.7) for $t \geq T_*$ big enough. The case $t \leq T_*$ follows from (4.3.6). □

4.C.2. Calculations for higher order energies

Recall the definition of the stress tensor (T)

$$T = \mu(\rho)Su + (\lambda(\rho)\operatorname{div} u - (P(\rho) - \tilde{P}))\mathrm{Id}.$$

We apply material derivative to obtain

$$\begin{aligned} K &:= \dot{T} - \mu(\rho)S\dot{u} - \lambda(\rho)\operatorname{div}\dot{u}\mathrm{Id} + \dot{P(\rho)}\mathrm{Id} \\ &= \dot{\mu}(\rho)Su + \dot{\lambda}(\rho)\operatorname{div} u\mathrm{Id} + \mu(\rho)[D_t, S]u + \lambda(\rho)[D_t, \operatorname{div}]u\mathrm{Id}, \end{aligned} \tag{4.C.17}$$

where all the terms on the right hand side are of the form $\nabla u \otimes \nabla u$.

We will also use frequently the following identity

$$\int_{\mathbb{R}^2} f D_t g\, dx = \frac{d}{dt}\int_{\mathbb{R}^2} fg\, dx - \int_{\mathbb{R}^2} (\operatorname{div} u) fg\, dx - \int_{\mathbb{R}^2} D_t f g\, dx, \tag{4.C.18}$$

which implies in particular

$$\int_{\mathbb{R}^2} \varphi(\rho) D_t f dx = \frac{d}{dt}\int_{\mathbb{R}^2} \varphi(\rho) f dx - \int_{\mathbb{R}^2} ((-\varphi'(\rho)\rho + \varphi(\rho))\operatorname{div} u) f dx. \tag{4.C.19}$$

Proof of (4.3.11). We multiply the momentum equation $(\mathrm{CNS})_2$ by $\dot u$ to derive

$$\int_{\mathbb{R}^2}\rho|\dot u|^2dx-\int_{\mathbb{R}^2}\operatorname{div}T\cdot\dot u dx=0,$$

where we use integration by parts to the second term on the left hand side to derive

$$\begin{aligned}&-\int_{\mathbb{R}^2}\operatorname{div}T\cdot\dot u dx=\int_{\mathbb{R}^2}T:\nabla\dot u dx=\int_{\mathbb{R}^2}T:D_t\nabla u dx+\int_{\mathbb{R}^2}T:[\nabla,D_t]u dx\\&=\frac12\int_{\mathbb{R}^2}\Big(\frac{\mu(\rho)}{2}D_t|Su|^2+\lambda(\rho)D_t|\operatorname{div}u|^2\Big)dx-\int_{\mathbb{R}^2}(P(\rho)-\tilde P)D_t\operatorname{div}u dx\\&\quad+\int_{\mathbb{R}^2}\Big(\frac{\mu(\rho)}{2}Su:[S,D_t]u+\lambda(\rho)\operatorname{div}u[\operatorname{div},D_t]u-(P(\rho)-\tilde P)[\operatorname{div},D_t]u\Big)dx.\end{aligned}$$

We apply the formula (4.C.19) with $\varphi\in\{\mu,\lambda,P-\tilde P\}$ and $f\in\{|Su|^2,|\operatorname{div}u|^2,\operatorname{div}u\}$ and observe $[\nabla,D_t]=(\nabla u)^T\nabla$ to obtain (4.3.11). □

Proof of (4.3.12). We apply the material derivative D_t to the momentum equation $(\mathrm{CNS})_2$ and then take the $L^2(\mathbb{R}^2)$-inner product with $\dot u$ to derive

$$\int_{\mathbb{R}^2}D_t(\rho\dot u)\cdot\dot u dx-\int_{\mathbb{R}^2}D_t\operatorname{div}T\cdot\dot u dx=0, \tag{4.C.20}$$

where the first term on the left is equal to

$$\int_{\mathbb{R}^2}\dot\rho|\dot u|^2dx+\int_{\mathbb{R}^2}\frac{\rho}{2}D_t|\dot u|^2dx=-\int_{\mathbb{R}^2}\rho\operatorname{div}u|\dot u|^2dx+\frac12\frac{d}{dt}\int_{\mathbb{R}^2}\rho|\dot u|^2dx,$$

while we apply integration by parts multiple times to the second term and use $\operatorname{div}T=\rho\dot u$ and (4.C.17) to derive

$$\begin{aligned}&-\int_{\mathbb{R}^2}D_t\operatorname{div}T\cdot\dot u dx=-\int_{\mathbb{R}^2}\operatorname{div}\dot T\cdot\dot u dx-\int_{\mathbb{R}^2}[D_t,\operatorname{div}]T\cdot\dot u dx\\&=\int_{\mathbb{R}^2}\dot T:\nabla\dot u dx-\int_{\mathbb{R}^2}T:(\dot u\otimes\nabla\operatorname{div}u+\nabla\dot u\nabla u)dx\\&=\int_{\mathbb{R}^2}\frac{\mu(\rho)}{2}|S\dot u|^2+\lambda(\rho)|\operatorname{div}\dot u|^2+\dot{P(\rho)}\operatorname{div}\dot u\,dx+\int_{\mathbb{R}^2}K:\nabla\dot u dx\\&\quad+\int_{\mathbb{R}^2}\operatorname{div}u(\rho|\dot u|^2+T:\nabla\dot u)-T:(\nabla\dot u\nabla u)dx.\end{aligned}$$

We rewrite

$$\int_{\mathbb{R}^2}\dot{P(\rho)}\operatorname{div}\dot u\,dx=\int_{\mathbb{R}^2}\rho P'(\rho)\frac12D_t|\operatorname{div}u|^2dx-\int_{\mathbb{R}^2}\dot{P(\rho)}[\operatorname{div},D_t]u dx$$

and apply (4.C.19) to the first term on the right hand side. We insert all the resulting terms into (4.C.20), observing the cancellation of the integrals containing $\rho\operatorname{div}u|\dot u|^2$. This yields (4.3.12). □

Proof of (4.3.15). We apply the material derivative D_t to the momentum equation to derive

$$\rho\ddot u-\operatorname{div}\dot T=-\dot\rho\dot u+[D_t,\operatorname{div}]T,$$

where we take the $L^2(\mathbb{R}^2)$-inner product with $\rho\ddot u$ to obtain

$$\|\sqrt{\rho}\ddot u\|_{L^2}^2-\int_{\mathbb{R}^2}\operatorname{div}\dot T\cdot\ddot u\,dx=-\int_{\mathbb{R}^2}\dot\rho\dot u\cdot\ddot u\,dx+\int_{\mathbb{R}^2}[D_t,\operatorname{div}]T\cdot\ddot u\,dx. \tag{4.C.21}$$

We further compute as follows:

- For the second integral on the left hand side we use integration by parts and (4.C.17) to get

$$\begin{aligned}
&-\int_{\mathbb{R}^2} \operatorname{div} \dot T \cdot \ddot u dx = \int_{\mathbb{R}^2} \dot T : D_t \nabla \dot u dx + \int_{\mathbb{R}^2} \dot T : [\nabla, D_t]\dot u dx \\
&= \frac{1}{2}\int_{\mathbb{R}^2} \Big(\frac{\mu(\rho)}{2} D_t |S\dot u|^2 + \lambda(\rho) D_t |\operatorname{div} \dot u|^2 \Big) dx \\
&\quad + \int_{\mathbb{R}^2} (K - \dot{P(\rho)} \mathrm{Id}\,) : D_t \nabla \dot u dx + \int_{\mathbb{R}^2} \dot T : [\nabla, D_t]\dot u dx.
\end{aligned}$$

- For the second integral on the right hand side we use integration by parts and the identity $-\operatorname{div} T \operatorname{div} u = -\rho \dot u \operatorname{div} u = \dot\rho \dot u$ to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} [D_t, \operatorname{div}\,] T \cdot \ddot u \, dx = \int_{\mathbb{R}^2} T : (\ddot u \otimes \nabla \operatorname{div} u + \nabla \ddot u \nabla u) dx \\
&= -\int_{\mathbb{R}^2} \operatorname{div} T \cdot \ddot u \operatorname{div} u \, dx - \int_{\mathbb{R}^2} T : \nabla \ddot u \operatorname{div} u \, dx + \int_{\mathbb{R}^2} T : (\nabla \ddot u \nabla u) dx \\
&= \int_{\mathbb{R}^2} \dot\rho \dot u \cdot \ddot u \, dx - \int_{\mathbb{R}^2} T : \big(D_t \nabla \dot u \operatorname{div} u + [\nabla, D_t]\dot u \operatorname{div} u - D_t \nabla \dot u \nabla u - [\nabla, D_t]\dot u \nabla u) dx.
\end{aligned}$$

We insert these two expressions back into (4.C.21) and apply (4.C.18), (4.C.19) to obtain

$$\begin{aligned}
&\|\sqrt{\rho}\ddot u\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} \Big(\frac{\mu(\rho)}{2} |S\dot u|^2 + \lambda(\rho) |\operatorname{div} \dot u|^2 \Big) dx \\
&= \frac{1}{2}\int_{\mathbb{R}^2} (\operatorname{div} u) \Big((-\rho\mu'(\rho) + \mu(\rho)) \frac{1}{2} |S\dot u|^2 + (-\rho\lambda(\rho) + \lambda(\rho)) |\operatorname{div} \dot u|^2 \Big) dx \\
&\quad + \frac{d}{dt} I + \int_{\mathbb{R}^2} (\operatorname{div} u)(K - \dot{P(\rho)} \mathrm{Id}) : \nabla \dot u \, dx + \int_{\mathbb{R}^2} D_t (K - \dot{P(\rho)} \mathrm{Id}) : \nabla \dot u \, dx \\
&\quad - \int_{\mathbb{R}^2} \dot T : [\nabla, D_t]\dot u \, dx - \int_{\mathbb{R}^2} T : ([\nabla, D_t]\dot u \operatorname{div} u - [\nabla, D_t]\dot u \nabla u) dx \\
&\quad + \int_{\mathbb{R}^2} (\operatorname{div} u) T : (\nabla \dot u \operatorname{div} u - \nabla \dot u \nabla u) \, dx \\
&\quad + \int_{\mathbb{R}^2} (\dot T \operatorname{div} u + T \operatorname{div} \dot u + T[D_t, \operatorname{div}\,]u) : \nabla \dot u \, dx - \int_{\mathbb{R}^2} (\dot T \nabla u + T \nabla \dot u + T[D_t, \nabla]u) : \nabla \dot u \, dx,
\end{aligned}$$

where I defined below satisfies the bound (4.3.16)

$$I = -\int_{\mathbb{R}^2} (K - \dot{P(\rho)} \mathrm{Id}) : \nabla \dot u + T : (\nabla \dot u \operatorname{div} u - \nabla \dot u \nabla u) \, dx. \tag{4.C.22}$$

Except $\frac{d}{dt} I$, all other terms on the right hand side are of the form

$$(\nabla \dot u, \nabla u \otimes \nabla u, \nabla u \otimes \nabla \dot u, \nabla u \otimes \nabla u \otimes \nabla u) \otimes \nabla \dot u,$$

up to bounded coefficients depending on ρ. Thus (4.3.15) follows. □

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Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Arbeit selbstständig und nur unter Zuhilfenahme der ausgewiesenen Hilfsmittel angefertigt habe. Sämtliche Stellen der Arbeit, die im Wortlaut oder dem Sinn nach anderen gedruckten oder im Internet veröffentlichten Werken entnommen sind, habe ich durch genaue Quellenangaben kenntlich gemacht.

Karlsruhe, den 26.08.2025