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Threshold resummation of rapidity distributions at fixed partonic rapidity

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Abstract: We derive a general expression for the resummation of rapidity distributions for processes with a colorless final state, such as Drell-Yan or Higgs production, in the limit in which the center-of-mass energy goes on threshold, but with fixed rapidity of the Higgs or gauge boson in the partonic center-of-mass frame. The result is obtained by suitably generalizing the renormalization-group based approach to threshold resummation previously pursued by us. The ensuing expression is valid to all logarithmic orders but the resummation coefficients must be determined by comparing to fixed order results. We perform this comparison for the Drell-Yan process using the fixed-order next-to-next-to-leading (NNLO) result, thereby determining resummation coefficients up to next-to-next-to-leading logarithmic (NNLL) accuracy, for the quark-antiquark coefficient function in the quark nonsinglet channel. We provide a translation to direct QCD of a result for this resummation previously obtained using SCET methods, and we show that it agrees with our own.

Keywords: Resummation, Higher-Order Perturbative Calculations

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Contents

1 Introduction

The resummation of rapidity distributions for the Drell-Yan (DY) process, and more generally for the production of colorless final states, is as old as soft (Sudakov) resummation itself: indeed, at the next-to-leading logarithmic (NLL) level it was presented in ref. [1], one of the original papers that first performed Sudakov resummation for hard processes [1, 2]. In this paper, results are presented using the then-customary Feynman x_F variable to parametrize the rapidity of the final-state boson, suitable for the phenomenology of fixed-target DY. The results of ref. [1] were subsequently rewritten using variables more suited to collider kinematics in refs. [3, 4] (see also refs. [5, 6]), and more recently extended up to next-to-next-to-leading log (NNLL) matched to fixed next-to-next-to leading order (NNLO) [7] and even to next-to-leading power [8].

However, all these results are obtained in the limit in which the center-of-mass energy of the partonic process $\sqrt{s}$ goes on threshold, i.e. $s \to M^2$, where M is the mass of the final-state colorless object, which henceforth we will call gauge boson for short (as for DY). In this limit the rapidity goes to zero: the gauge boson is produced at rest. The meaning of this limit can be understood by introducing a pair of scaling variables x_1, x_2 such that $x_1 x_2 = \frac{M^2}{s} \equiv \tau$ and $\frac{x_1}{x_2} = \exp(2y)$, where y is the gauge boson rapidity, defined by $\tanh y = \frac{p_z}{E}$ with E and p_z respectively the energy of the gauge boson and its momentum along the partonic collision axis in the partonic center of mass frame. In the limit $s \to M^2$ of course $p_z \to 0$, thus $y \to 0$, and both scaling variables $x_i \to 1$. We will call this a double-soft limit: both scaling variables tend to their common kinematic threshold.

In this limit, the partonic differential rapidity distribution acquires at all perturbative orders logarithmically enhanced contributions, that can be put in to one-to-one correspondence to the contributions to the total partonic cross section [1]. Namely, they can be obtained from the resummation of the inclusive cross section, which resums contributions of the form $\ln(1-\tau)^2$, by the simple substitution $\ln(1-\tau)^2 \to \ln[(1-x_1)(1-x_2)]$. While of course the dependence on y is logarithmically subleading [9], because $\ln(1-x_i) = \ln(1-\tau) + O\left((1-\tau)^0\right)$, the resummation of ref. [1] predicts it fully, in the sense that if the limit $\tau \to 1$ is reached for fixed $\frac{x_1}{x_2}$, the dependence on this ratio (and thus on y) is predicted, to given logarithmic accuracy in τ, up to terms that are either constant or suppressed by powers of $1-\tau$.[1]

In this paper we consider the more general situation in which the partonic center of mass energy reaches its threshold value, but now for fixed nonzero p_z, or, equivalently, at fixed rapidity. In terms of the scaling variables x_1, x_2 this corresponds to the case in which $x_1 \to 1$ but x_2 is fixed. We will consequently call this a single-soft limit. Of course, results are symmetric upon the interchange of x_1 and x_2: we will henceforth take x_1 as the scaling variable that tends to the soft limit. In this limit resummation should include all contributions

[1]The fact that the rapidity dependence is subleading was proven with increasing degrees of detail in refs. [9, 10], and used in refs. [11–13] to perform soft resummation of the rapidity distribution by reducing it to that of the total cross section. The validity of the arguments of refs. [9, 10] was questioned in ref. [14] and established more rigorously in ref. [15], though the issue of a formal proof is somewhat academic, given that the explicit form of the rapidity dependence in the limit is known from refs. [1, 5], so whether it is subleading can be checked from the known result without need of a formal proof, see eq. (4.13) below.

that are logarithmically enhanced in $\ln(1-x_1)$, even if they are not logarithmically enhanced in the other scaling variable.

Sudakov resummation in this limit is akin to the resummation of transverse momentum distributions [16, 17], but with the role of the longitudinal and transverse momenta of the final-state gauge boson interchanged. We will consequently attack it here with methods similar to those used in ref. [17] to perform resummation of transverse-momentum distributions, in turn based on a multi-scale generalization [18] of the renormalization-group based approach to Sudakov resummation [19].

The Sudakov resummation for the inclusive Drell-Yan process of refs. [1, 2] was more recently re-derived using soft-collinear effective field theory (SCET) methods [12, 20, 21]. Specifically, in ref. [12] the rapidity distribution was also considered, but, as mentioned, its resummation was reduced to that of the inclusive cross section based on the observation that the rapidity dependence is subleading. The result of ref. [1] for resummation of the rapidity distribution in the double-soft limit was reobtained using SCET methods in ref. [14], where for the first time resummed results in the single-soft limit where also derived, up to NNLL. This SCET resummation of rapidity distributions in the single soft limit was recently re-derived in ref. [22], where it was also extended to N^{3}LL.

The comparison of direct QCD (dQCD) and SCET results is generally nontrivial: specifically, in the approach of ref. [14] resummed results are given in the space of physical momentum variables, and are expressed as the solution to renormalization group equations that can only be worked out in closed form order by order (in ref. [14] results up to order α_s^3 are explicitly given). A more direct comparison can be performed by using results of ref. [12], in which SCET resummation can be expressed in Mellin-transform space, thereby allowing for a direct comparison to the dQCD result. Even so, the comparison is not immediate: for DY, the dQCD and SCET results were shown to be equivalent in refs. [23, 24].

The purpose of this paper is to perform resummation of rapidity distributions in the single-soft limit. In particular, we will derive a general expression, valid at any logarithmic order, using the methods of ref. [19]. We will then obtain explicit expressions for DY up to NNLL accuracy, by comparing to the fixed NNLO result of ref. [25] for the quark-antiquark coefficient function in the quark-quark nonsinglet channel. We will finally provide a translation to dQCD of the results of ref. [14] and show their equivalence to our own. Note that our results correspond to what is usually referred to as N^{k}LL' accuracy in the SCET literature, which is half a perturbative more accurate than what is usually called NNLL in that context.

2 The kinematic structure of rapidity distributions

We will study the rapidity distributions for processes of the form

$$H_1 + H_2 \to Z + X, \tag{2.1}$$

where Z is a massive colorless particle, which we denote by Z because in the sequel we will specifically work out explicit expressions up to NNLL for Drell-Yan production, but which could equally be a Higgs boson (and in fact was denoted by H in ref. [17] where NNLL expressions for the Higgs p_T distribution were given). For definiteness, and without intended loss of generality, we will henceforth refer to this final-state particle as the Z.

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As explained in the introduction, we want to perform resummation of a rapidity distribution in the case in which the kinematics of the underlying partonic sub-process is characterized by the fact that the minimum energy of the partonic sub-process is higher than the Z mass, because the Z is assumed to be boosted, i.e. to have a non-vanishing longitudinal momentum in the center-of-mass frame of the partonic collision. In this situation, unlike in the case of transverse momentum distributions, discussed in ref. [17], the kinematic constraint on the partonic subprocess (namely the requirement that the Z is boosted) is not decoupled from the partonic kinematics itself, as they are both longitudinal. Because of this, it is crucial to discuss the kinematics of the process at the partonic level. Indeed, using standard QCD factorization for a hadronic process, the partonic center-of-mass frame is boosted with respect to the hadronic center of mass frame whenever the momentum fractions of the incoming partons are unequal. The nontrivial boost at the parton level, that changes the form of the soft limit in the partonic cross section, gets then mixed with this trivial boost at the level of the kinematics of the hadronic process.

Consequently, we will henceforth focus on purely partonic kinematics, and specifically study the process

$$f_1(p_1) + f_2(p_2) \to Z(p) + X(k)\,, \tag{2.2}$$

where f_i are incoming partons with momenta p_i, and p and k are respectively the momenta of the Z and of the system X that recoils against it, all given in the partonic center-of-mass frame. For explicit expressions of the factorized distribution at the hadronic level, as well as a discussion of the relation between hadronic and partonic kinematics (neither of which are relevant here) we refer e.g. to Section 4 of ref. [13].

2.1 Kinematics

In the partonic center of mass frame

$$p_1 = \frac{\sqrt{s}}{2}(1,0,0,1), \qquad p_2 = \frac{\sqrt{s}}{2}(1,0,0,-1), \tag{2.3}$$

$$p = \left(\sqrt{M^2+p_\mathrm{T}^2+p_z^2}, \vec{p}_\mathrm{T}, p_z\right) = \left(\sqrt{M^2+p_\mathrm{T}^2}\cosh y, \vec{p}_\mathrm{T}, \sqrt{M^2+p_\mathrm{T}^2}\sinh y\right), \tag{2.4}$$

where $\vec{p}_\mathrm{T}$ and y are respectively the transverse momentum and rapidity of the Z. The final-state kinematics is fully parametrized by the scale M^2, the rapidity y, and the scaling variable

$$\tau = \frac{M^2}{s}. \tag{2.5}$$

The rapidity range for fixed τ is restricted by the requirement that $|p_z|$ eq. (2.4) does not exceed the value allowed by the maximum available energy. Because

$$\sqrt{s} = \sqrt{M^2+p_\mathrm{T}^2+p_z^2} + \sqrt{m_X^2+p_\mathrm{T}^2+p_z^2} \geq \sqrt{M^2+p_z^2} + |p_z| \tag{2.6}$$

the minimum value of s corresponds to the minimum value of both the energy of the Z and of the system X with invariant mass m_X that recoils against it, reached when $p_\mathrm{T}=0$ and $m_X = 0$:

$$s \geq s^{\min}(p_z) = \left(\sqrt{M^2+p_z^2} + \sqrt{p_z^2}\right)^2. \tag{2.7}$$

This gives

$$|p_z| \leq |p_z|^{\max} = \frac{s - M^2}{2\sqrt{s}} = M\frac{1-\tau}{2\sqrt{\tau}}\,. \tag{2.8}$$

The maximum is attained when $p_{\mathrm{T}} = 0$, so using eq. (2.4) the condition is $|p_z|^{\max} = M|\sinh y^{\max}|$ leading to

$$\ln\sqrt{\tau} \leq y \leq \ln\frac{1}{\sqrt{\tau}}\,, \tag{2.9}$$

which can be equivalently obtained by rewriting eq. (2.6) in terms of rapidity

$$\sqrt{s} = \sqrt{M^2 + p_{\mathrm{T}}^2}\cosh y + \sqrt{m_X^2 + p_{\mathrm{T}}^2 + (M^2 + p_{\mathrm{T}}^2)\sinh^2 y} \geq M\cosh y + M|\sinh y|\,. \tag{2.10}$$

Resummation is performed in conjugate space, in which convolutions reduce to ordinary products. For the rapidity distribution, the appropriate integral transform is a double Mellin transform. We define a coefficient function in terms of the partonic differential cross section:

$$C(x_1, x_2, M^2) = \frac{1}{\sigma_0}\frac{1}{\tau}\frac{d^2\sigma}{dM^2 dy}, \tag{2.11}$$

where

$$x_1 = \sqrt{\tau}e^{y}\,, \qquad x_2 = \sqrt{\tau}e^{-y}\,, \tag{2.12}$$

so that

$$\tau = x_1 x_2\,, \qquad y = \frac{1}{2}\ln\frac{x_1}{x_2}, \tag{2.13}$$

and σ_0 is defined so that

$$C(x_1, x_2, M^2) = \delta(1 - x_1)\delta(1 - x_2) + \mathcal{O}(\alpha_s). \tag{2.14}$$

It is easy to check that the jacobian of the transformation from (x_1, x_2) to (y, τ) equals one, and that

$$0 \leq x_i \leq 1, \qquad i = 1, 2. \tag{2.15}$$

The coefficient function is a dimensionless function with a power expansion in α_s; as such, it also depends on the strong coupling α_s evaluated at a renormalization scale μ_R^2, and on the ratio of M^2 to renormalization and factorization scales. Furthermore, both the coefficient function and σ_0 depend on the choice of initial state partons and thus carry parton indices. In this section, for simplicity, we omit all these functional dependences and indices, and we only display the dependence on kinematic variables. The full set of arguments will be restored in the next section. The coefficient function in Mellin-Mellin space is given by

$$C(N_1, N_2, M^2) = \int_0^1 dx_1\, x_1^{N_1 - 1}\int_0^1 dx_2\, x_2^{N_2 - 1} C(x_1, x_2, M^2), \tag{2.16}$$

where by slight abuse of notation we denote both the original function and its transform with the same symbol C.

Using eq. (2.12), the integral transform eq. (2.16) can be equivalently viewed as a Fourier-Mellin transform [26] of the coefficient function with respect to y and τ. Indeed, defining

$$N_1 = N + i\frac{b}{2}; \qquad N_2 = N - i\frac{b}{2}, \tag{2.17}$$

so that

$$N(N_1, N_2) = \frac{1}{2}\left(N_1 + N_2\right), \qquad b(N_1, N_2) = \frac{N_1 - N_2}{i}, \tag{2.18}$$

and

$$\hat{C}(N, b, M^2) = C(N_1(N, b), N_2(N, b), M^2)\,, \tag{2.19}$$

$$\hat{C}(\tau, y, M^2) = C(x_1(\tau, y), x_2(\tau, y), M^2)\,. \tag{2.20}$$

Eq. (2.16) can be written

$$\hat{C}(N, b, M^2) = \int_0^1 d\tau\, \tau^{N-1} \int_{\ln\sqrt{\tau}}^{-\ln\sqrt{\tau}} dy\, e^{iby}\hat{C}(\tau, y, M^2). \tag{2.21}$$

The soft limits that we consider are defined as follows. The double-soft limit is the limit in which

$$s \to s^{\min}(0) = M^2, \tag{2.22}$$

with $s^{\min}(p_z)$ given in eq. (2.7). This means that

$$p_{\mathrm{T}} \to 0; \quad p_z \to 0. \tag{2.23}$$

In this limit

$$\tau \to 1; \quad y \to 0, \tag{2.24}$$

i.e.

$$x_1 \to 1, \quad x_2 \to 1. \tag{2.25}$$

In Mellin-Mellin space this means

$$\mathrm{Re}\, N_1 \to \infty; \quad \mathrm{Re}\, N_2 \to \infty. \tag{2.26}$$

In Fourier-Mellin space this corresponds to

$$\mathrm{Re}\, N \to \infty. \tag{2.27}$$

The single-soft limit is instead the limit in which

$$s \to s^{\min}(p_z) \tag{2.28}$$

for fixed p_z, or, equivalently for some fixed rapidity y. In terms of the variables x_i eq. (2.12) the condition of fixed y implies fixed ratio $\frac{x_1}{x_2} = e^{2y}$ and the condition of minimum s corresponds to maximum product $x_1 x_2$, which (assuming without loss of generality $x_1 \geq x_2$) for fixed e^{2y} means maximum $x_1 e^{-2y}$ i.e. $x_1 = 1$. So the single soft limit is the limit in which

$$x_1 \to 1; \quad \text{fixed } x_2. \tag{2.29}$$

In Mellin-Mellin space this means

$$N_1 \to \infty; \quad \text{fixed } N_2. \tag{2.30}$$

2.2 Phase space

We now discuss the phase space measure for the process eq. (2.2). We consider the Z rapidity distribution when the system X includes n final-state massless partons, with momenta $k_1, \ldots, k_n$:

$$p_1 + p_2 = p + k_1 + \ldots + k_n. \tag{2.31}$$

We are interested in the double-soft and single-soft limits.

2.2.1 Double-soft limit

In the double-soft limit the center-of mass energy is going on threshold eq. (2.22), hence the limit coincides with the threshold limit of the total cross section, discussed by us in ref. [19]. Namely, squaring both sides of eq. (2.31)

$$s(1-\tau) = \sum_{i,j=1}^{n} k_i \cdot k_j + 2\sum_{i=1}^{n} p \cdot k_i, \tag{2.32}$$

but $p \cdot k_i$ is positive semi-definite, and vanishes only when $k_i^0 = 0$, hence $k_i^0 \to 0$ for all i in the limit $\tau \to 1$, i.e. all emitted partons must be soft.

We write the phase space for the process as

$$d\phi_{n+1}(p_1 + p_2; p, k_1, \ldots, k_n) = \int \frac{dk^2}{2\pi}\, d\phi_2(p_1 + p_2; p, k)\, d\phi_n(k; k_1, \ldots, k_n), \tag{2.33}$$

with $n \geq 0$. Here $d\phi_2$ is the phase space for production from the incoming total momentum $p_1 + p_2$ of a massive final state with mass M and momentum p, and a system with momentum k recoiling against it; $d\phi_n$ is the phase space for the production, from incoming momentum k, of a final-state system containing n massless partons with momenta k_i.

We now work out the two-body phase space $d\phi_2$ in $d = 4 - 2\epsilon$ dimensions in terms of the rapidity of the Z. In the partonic center of mass frame we have

$$\begin{aligned} d\phi_2(p_1 + p_2; p, k) &= \frac{d^{d-1}p}{(2\pi)^{d-1} 2p^0} \frac{d^{d-1}k}{(2\pi)^{d-1} 2k^0} (2\pi)^d \delta^{(d)}(p_1 + p_2 - p - k) \\ &= \frac{d^{d-1}p}{4(2\pi)^{d-2} p^0 k^0} \delta(\sqrt{s} - p^0 - k^0). \end{aligned} \tag{2.34}$$

The integration measure can be rewritten as

$$d^{d-1}p = \frac{1}{2} dp_{\mathrm{T}}^2\, |\vec{p}_{\mathrm{T}}|^{d-4} dp_z\, d\Omega_{d-2} = \frac{\pi^{1-\epsilon}}{\Gamma(1-\epsilon)} dp_{\mathrm{T}}^2\, |\vec{p}_{\mathrm{T}}|^{-2\epsilon}\, dp_z, \tag{2.35}$$

where we have performed the azimuthal integration using the identity $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$.

Substituting in eq. (2.34) gives

$$d\phi_2(p_1 + p_2; p, k) = \frac{(4\pi)^\epsilon |\vec{p}_{\mathrm{T}}|^{-2\epsilon}}{16\pi\Gamma(1-\epsilon)} \frac{dp_{\mathrm{T}}^2\, dp_z}{p_0 k_0} \delta(\sqrt{s} - p_0 - k_0). \tag{2.36}$$

We first trade p_z for the rapidity y, using

$$p_z = \sqrt{p_{\mathrm{T}}^2 + M^2} \sinh y; \qquad dp_z = p_0 dy. \tag{2.37}$$

We obtain

$$d\phi_2(p_1+p_2;p,k) = \frac{(4\pi)^\epsilon |\vec{p}_\mathrm{T}|^{-2\epsilon}}{16\pi\Gamma(1-\epsilon)} \frac{dp_\mathrm{T}^2\, dy}{k_0} \delta(\sqrt{s}-p_0-k_0). \tag{2.38}$$

Next, we perform the p_T^2 integration using the delta function. We have

$$\delta(\sqrt{s}-p_0-k_0) = \frac{\delta(p_\mathrm{T}^2-\bar{p}_\mathrm{T}^2)}{J} \tag{2.39}$$

where

$$J = \left| \frac{d}{dp_\mathrm{T}^2}(\sqrt{s}-p_0-k_0) \right| = \left(1+\frac{p_0}{k_0}\right) \frac{dp_0}{dp_\mathrm{T}^2} = \frac{\sqrt{s}}{2p_0k_0} \frac{dp_0^2}{dp_\mathrm{T}^2} = \frac{\sqrt{s}}{2p_0k_0} \cosh^2 y, \tag{2.40}$$

and $\bar{p}_\mathrm{T}^2$ is the solution of

$$\sqrt{s} = p_0 + \sqrt{k^2+p_0^2-M^2} = \sqrt{M^2+\bar{p}_\mathrm{T}^2}\cosh y + \sqrt{k^2+(M^2+\bar{p}_\mathrm{T}^2)\cosh^2 y - M^2}. \tag{2.41}$$

We get

$$d\phi_2(p_1+p_2;p,k) = \frac{(4\pi)^\epsilon (\bar{p}_\mathrm{T}^2)^{-\epsilon}}{8\pi\Gamma(1-\epsilon)} \frac{p_0}{\sqrt{s}\cosh^2 y} dy. \tag{2.42}$$

Expressing p_0 as a function of s, M^2, k^2 through the first equality in eq. (2.41), we finally obtain

$$d\phi_2(q;p,k) = \frac{(4\pi)^\epsilon (\bar{p}_\mathrm{T}^2)^{-\epsilon}}{16\pi\Gamma(1-\epsilon)} \frac{s-k^2+M^2}{s} \frac{dy}{\cosh^2 y}. \tag{2.43}$$

The phase space eq. (2.33) becomes thus

$$\begin{aligned} &d\phi_{n+1}(p_1+p_2;p,k_1,\ldots,k_n) \\ &= \frac{dy}{\cosh^2 y} \frac{(4\pi)^\epsilon}{32\pi^2\Gamma(1-\epsilon)} \int_{k^2_\mathrm{min}}^{k^2_\mathrm{max}} dk^2\, (\bar{p}_\mathrm{T}^2)^{-\epsilon} \frac{s-k^2+M^2}{s} d\phi_n(k;k_1,\ldots,k_n). \end{aligned} \tag{2.44}$$

We can now work out the kinematic limits for the k^2 integration. Of course, $k^2_\mathrm{min}=0$: this happens whenever the final-state parton momenta k_i are all collinear to each other (or there is a single parton, of course). The upper bound is obtained by observing that

$$k^2 = s+M^2-2\sqrt{s}\sqrt{M^2+p_\mathrm{T}^2+p_z^2} = s+M^2-2\sqrt{s}\sqrt{M^2+p_\mathrm{T}^2}\cosh y, \tag{2.45}$$

so for given rapidity the maximum value of k^2 is obtained for $p_\mathrm{T}^2 \to 0$:

$$k^2_\mathrm{max} = \frac{M^2}{\tau}(1+\tau-2\sqrt{\tau}\cosh y) \tag{2.46}$$

$$= M^2 \frac{(1-x_1)(1-x_2)}{x_1x_2}. \tag{2.47}$$

Introducing a dimensionless variable v in order to interpolate between 0 and k^2_max we can set

$$k^2 = vk^2_\mathrm{max}; \qquad 0 \le v \le 1, \tag{2.48}$$

and rewrite the measure of integration over k^2 in the phase space eq. (2.44) as

$$dk^2 = \frac{M^2(1-x_1)(1-x_2)}{x_1 x_2} dv. \tag{2.49}$$

Furthermore, as in ref. [17] the phase space $d\phi_n(k; k_1, \ldots, k_n)$ can be rewritten following ref. [18] (in turn based on ref. [19]) as

$$d\phi_n(k; k_1, \ldots, k_n) = 2\pi \left[\frac{N(\epsilon)}{2\pi}\right]^{n-1} (k^2)^{n-2-(n-1)\epsilon} d\Omega^{(n-1)}(\epsilon), \tag{2.50}$$

where $N(\epsilon) = \frac{1}{2(4\pi)^{2-2\epsilon}}$ and

$$d\Omega^{(n-1)}(\epsilon) = d\Omega_1 \ldots d\Omega_{n-1} \int_0^1 dz_m z_m^{(n-3)-(n-2)\epsilon}(1-z_{n-1})^{1-2\epsilon} \cdots \int_0^1 dz_2 z_2^{-\epsilon}(1-z_2)^{1-2\epsilon}. \tag{2.51}$$

The definition of the variables z_i is irrelevant here and can be found in ref. [19] (where they are called z_i').

We can finally consider the double-soft limit. Equation (2.47) implies that $k^2 \to 0$ as both x_1 and x_2 approach 1. Furthermore, in eq. (2.50) the phase space $d\phi_n(k; k_1, \ldots, k_n)$ is rewritten in terms of a dimensionless integration measure, with all the dimensional dependence contained in powers of a soft scale

$$\Lambda_{\mathrm{ds}}^2 = k_{\mathrm{max}}^2 = M^2(1-x_1)(1-x_2)\left[1 + \mathcal{O}(1-x_1) + \mathcal{O}(1-x_2)\right]. \tag{2.52}$$

This is in fact the same as the phase space for deep-inelastic scattering with incoming momentum k whose soft limit was discussed in ref. [19].

2.2.2 Single-soft limit

In this limit, the rapidity of the Z is fixed, while $s \to s^{\mathrm{min}}(p_z)$, given by eq. (2.7). At leading order, the process eq. (2.31) must thus include at least one parton that recoils against the Z with momentum k_1 and the soft limit corresponds to $p_{\mathrm{T}} \to 0$ with fixed y (fixed p_z) in eq. (2.4) so

$$k_1 = k_1^{(0)} = (p_z, 0, 0, -p_z). \tag{2.53}$$

Note that while a fixed y value of course corresponds to a fixed p_z value, approaching the $s \to s^{\mathrm{min}}(p_z)$ limit at fixed y or fixed p_z correspond to different paths.

Using again eq. (2.32) it is clear that with more than one parton in the final state the soft limit is reached when $k_i \cdot k_j = 0$ for all i, j, in which case

$$\sum_{i=1}^{n} k_i = k_1^{(0)}, \tag{2.54}$$

i.e. all emitted partons must be collinear. They are not necessarily soft, though of course some of them will: the only constraint being that their longitudinal momentum fractions add in such a way that eq. (2.54) is satisfied.

The phase space can again be written in the form of eq. (2.33), except that now $n \geq 1$. In particular, eq. (2.44) still holds, and the upper bound of k^2 is still given by $k^2_{\max}$ eq. (2.46). However, the limit is now given by eq. (2.29), so $1-x_2$ is now generic. Consequently, eq. (2.50) now implies that the dimensional dependence is contained in powers of a soft scale

$$\Lambda^2_{\rm ss} = k^2_{\max} = M^2(1-x_1)\left[1+\mathcal{O}(1-x_1)\right]. \tag{2.55}$$

It is interesting to contrast this with the case in which the Z has fixed $p_{\rm T}$, discussed in ref. [17]. In that case, $p_{\rm T}$ is fixed and $p_z \to 0$ in the soft limit. Hence in that case eq. (2.53) is replaced by $k_1 = k_1^{(0)} = (p_{\rm T}, -p_{\rm T}, 0, 0)$ (choosing without loss of generality the x axis along $\vec{p}_{\rm T}$) and the condition $k_i \cdot k_j = 0$ when $i = 1$ can be satisfied in two different ways: either with $k^j_{\rm T} \to 0$ (soft partons), or with $k^j_{\rm T} \to k^1_{\rm T} > 0$ (collinear partons). The phase space in this case must be written by separating these two families of partons, that turn out to be characterized by two different soft scales, due to the fact that soft partons correspond to radiation from incoming partons (i.e. collinear to the incoming partons), while collinear partons correspond to radiation from the final-state parton that recoils against the Z [16] (i.e. collinear to the Z). In the case considered here, it is impossible to distinguish initial- and final-state radiation because $p_{\rm T} \to 0$ so partons collinear to the Z are also collinear to the initial state partons.

3 Resummation by renormalization group improvement

The main result of section 2 is that, due to the collinear nature of the soft limits at fixed rapidity, the rapidity distribution in the soft limits depends on a single soft scale. This is true both in the double-soft limit eqs. (2.25)–(2.26), with soft scale $\Lambda^2_{\rm ds}$ eq. (2.52), and in the single-soft limit eqs. (2.29)–(2.30), with soft scale $\Lambda^2_{\rm ss}$ eq. (2.55). Consequently, resummation is simpler than that of transverse momentum distributions that we derived in ref. [17], and can in fact be performed by a mild generalization of the method we developed in ref. [19] in the inclusive case. On the other hand, some technical complications are related to the need of taking a double Mellin transform, and, in the single-soft limit, of treating the two incoming partons asymmetrically.

Resummation is performed based on a suitably adapted version of the argument of refs. [17, 19]. Perturbative factorization implies that the Mellin-space hadronic cross section can be written as

$$\frac{d^2\sigma_H}{dM^2dY}(N_1,N_2,M^2) = \tau_H \sum_{ij} \sigma_0^{ij} C_{ij}\left(N_1,N_2,\frac{M^2}{\mu^2},\frac{M^2}{\mu_R^2},\alpha_s(\mu_R^2)\right) f_i(N_1,\mu^2) f_j(N_2,\mu^2) \tag{3.1}$$

where $\tau_H = M^2/S$, S is the hadronic center-of-mass energy squared, and $f_i(N,\mu^2)$ are (Mellin-transformed) parton distributions (PDFs). In view of the construction of a resummed result in the single-soft limit, in which the incoming partons are treated asymmetrically, it is useful to study the dependence of the coefficient function on the factorization scale of each of the two partons independently. Specifically, we write

$$\frac{d^2\sigma_H}{dM^2dY}(N_1,N_2,M^2) = \tau_H \sum_{ij} \sigma_0^{ij} C_{ij}\left(N_1,N_2,\frac{M^2}{\mu_1^2},\frac{M^2}{\mu_2^2},\frac{M^2}{\mu_R^2},\alpha_s(\mu_R^2)\right) f_i(N_1,\mu_1^2) f_j(N_2,\mu_2^2), \tag{3.2}$$

which is allowed because it is always possible to express PDFs at one scale in terms of those at a different scale, and re-define the coefficient functions C_{ij} accordingly. We are assuming here that $f_i(N,\mu^2)$ are evolution eigenstates, with eigenvalues $\gamma_i^{\rm AP}(N,\alpha_s(\mu^2))$, so that they do not mix upon evolution.

We perform resummation of the Mellin-transformed coefficient function for the ij partonic channel

$$C_{ij}\left(N_1,N_2,\frac{M^2}{\mu_1^2},\frac{M^2}{\mu_2^2},\frac{M^2}{\mu_R^2},\alpha_s(\mu_R^2)\right)=1+\mathcal{O}(\alpha_s). \tag{3.3}$$

Observing that the hadronic cross section cannot depend on either μ_1^2 or μ_2^2 we get a Callan-Symanzik-Altarelli-Parisi equation of the form

$$\begin{aligned}\mu_1^2\frac{d}{d\mu_1^2}C_{ij}&\left(N_1,N_2,\frac{M^2}{\mu_1^2},\frac{M^2}{\mu_2^2},\frac{M^2}{\mu_R^2},\alpha_s(\mu_R^2)\right)\\&=-\gamma_i^{\rm AP}(N_1,\alpha_s(\mu_1^2))C_{ij}\left(N_1,N_2,\frac{M^2}{\mu_1^2},\frac{M^2}{\mu_2^2},\frac{M^2}{\mu_R^2},\alpha_s(\mu_R^2)\right)\end{aligned} \tag{3.4}$$

and an analogous equation with $\mu_1^2\to\mu_2^2$ and $\gamma_i^{\rm AP}(N_1,\alpha_s(\mu_1^2))\to\gamma_j^{\rm AP}(N_2,\alpha_s(\mu_2^2))$.

In the remainder of this section we omit the parton indices i,j and denote with C a generic eigenstate coefficient function, which is the quantity that undergoes resummation through renormalization group improvement. We will then restore parton indices in the next section when considering the explicit construction of the resummed coefficient function in various limits.

Solving eq. (3.4), the coefficient function is found to have the form

$$\begin{aligned}C&\left(N_1,N_2,\frac{M^2}{\mu_1^2},\frac{M^2}{\mu_2^2},\frac{M^2}{\mu_R^2},\alpha_s(\mu_R^2)\right)\\&=C\left(N_1,N_2,1,\frac{M^2}{\mu_2^2},\frac{M^2}{\mu_R^2},\alpha_s(\mu_R^2)\right)\exp\int_{\mu_1^2}^{M^2}\frac{dk^2}{k^2}\gamma^{\rm AP}(N_1,\alpha_s(k^2)).\end{aligned} \tag{3.5}$$

Resummation is performed in terms of a physical anomalous dimension

$$\gamma\left(N_1,N_2,\frac{M^2}{\mu^2},\alpha_s(\mu^2)\right)=\frac{d}{d\ln M^2}\ln C\left(N_1,N_2,\frac{M^2}{\mu^2},\frac{M^2}{\mu^2},\frac{M^2}{\mu^2},\alpha_s(\mu^2)\right), \tag{3.6}$$

where in the $\overline{\rm MS}$ scheme there is a single common renormalization and factorization scale μ^2 identified with the scale arising when performing dimensional regularization in $4-2\epsilon$ space-time dimensions.

The argument then reproduces that of refs. [17, 19], the only difference being that now the coefficient function depends on two Mellin variables N_1, N_2: the coefficient function is written in terms of bare coefficient function and coupling $C^{(0)}$ and α_0

$$C\left(N_1,N_2,\frac{M^2}{\mu^2},\frac{M^2}{\mu^2},\frac{M^2}{\mu^2},\alpha_s(\mu^2)\right)=\lim_{\epsilon\to0}Z^C(N_1,N_2,\alpha_s(\mu^2),\epsilon)C^{(0)}(N_1,N_2,M^2,\alpha_0,\epsilon). \tag{3.7}$$

The physical anomalous dimension (in mass-independent factorization schemes) is determined by dimensional analysis from the bare coefficient function [19] and it is finite as $\epsilon \to 0$ so that

$$\gamma\left(N_1, N_2, \frac{M^2}{\mu^2}, \alpha_s(\mu^2)\right) = -\lim_{\epsilon\to 0} \epsilon \frac{d}{d\ln\alpha_0} \ln C^{(0)}(N_1, N_2, M^2, \alpha_0, \epsilon). \tag{3.8}$$

3.1 Double-soft resummation

In the double-soft limit, the kinematic analysis of the phase space presented in section 2, together with the argument of ref. [19], implies that the leading contribution to the hard coefficient function as $x_1 \to 1$ and $x_2 \to 1$ depends on x_1 and x_2 only through the fixed combination of the scale and the scaling variables

$$\Lambda^2_{\rm ds} = M^2(1-x_1)(1-x_2), \tag{3.9}$$

up to terms of relative order $(1-x_1)$ or $(1-x_2)$. It is then easy to prove (see appendix A) that the Mellin-space coefficient function only depends on N_1, N_2 through the dimensionful variable

$$\tilde{\Lambda}^2_{\rm ds} = \frac{M^2}{N_1 N_2}, \tag{3.10}$$

up to corrections of order $\frac{1}{N_1}$ or $\frac{1}{N_2}$. The bare coefficient function can be consequently expanded as

$$C^{(0)}(N_1 N_2, M^2, \alpha_0, \epsilon) = C^{(0,c)}(M^2, \alpha_0, \epsilon) C^{(0,l)}(\tilde{\Lambda}^2_{\rm ds}, \alpha_0, \epsilon) \tag{3.11}$$

$$C^{(0,c)}(M^2, \alpha_0, \epsilon) = \sum_n C^{(0,c)}_n(\epsilon) M^{-2n\epsilon} \alpha_0^n \tag{3.12}$$

$$C^{(0,l)}(\tilde{\Lambda}^2_{\rm ds}, \alpha_0, \epsilon) = \sum_n C^{(0,l)}_n(\epsilon) \tilde{\Lambda}_{\rm ds}^{-2n\epsilon} \alpha_0^n, \tag{3.13}$$

where $C^{(0,l)}$ collects contributions due to real emission, which have a nontrivial dependence on N_1, N_2, $C^{(0,c)}$ collects virtual contributions, that have Born kinematics, and it is assumed [27] that virtual and real soft-emission contributions fully factorize.

The factorization eq. (3.11) implies that in the double-soft limit the physical anomalous dimension is the sum of two contributions:

$$\gamma\left(N_1 N_2, \frac{M^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \bar{\gamma}^{(c)}\left(\frac{M^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) + \bar{\gamma}^{(l)}\left(\frac{\tilde{\Lambda}^2_{\rm ds}(N_1 N_2, M^2)}{\mu^2}, \alpha_s(\mu^2), \epsilon\right), \tag{3.14}$$

that are not separately finite and μ-independent. Note that the dependence of $\bar{\gamma}^{(l)}$ on N_1, N_2 is only through the scale $\tilde{\Lambda}^2_{\rm ds}$. Renormalization-group invariance, i.e. μ independence of the full anomalous dimension, then implies

$$-\frac{d}{d\ln\mu^2} \lim_{\epsilon\to 0} \bar{\gamma}^{(l)}\left(\frac{\tilde{\Lambda}^2_{\rm ds}}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \frac{d}{d\ln\mu^2} \lim_{\epsilon\to 0} \bar{\gamma}^{(c)}\left(\frac{M^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \bar{g}^{\rm ds}(\alpha_s(\mu^2)) \tag{3.15}$$

where $\bar{g}^{\rm ds}(\alpha_s)$ has an expansion in powers of α_s with finite coefficients. Solving eqs. (3.15)

$$\bar{\gamma}^{(c)}\left(\frac{M^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \int_{M^2}^{\mu^2} \frac{dk^2}{k^2} \bar{g}^{\rm ds}(\alpha_s(k^2)) + \bar{g}_0^{(c)}(\alpha_s(M^2), \epsilon) \tag{3.16}$$

$$\bar{\gamma}^{(l)}\left(\frac{\tilde{\Lambda}^2_{\rm ds}(N_1 N_2, M^2)}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = -\int_{\tilde{\Lambda}^2_{ds}}^{\mu^2} \frac{dk^2}{k^2} \bar{g}^{\rm ds}(\alpha_s(k^2)) + \bar{g}_0^{(l)}(\alpha_s(\tilde{\Lambda}^2_{\rm ds}), \epsilon) \tag{3.17}$$

and adding up the solutions we get

$$\gamma\left(N_1N_2,\frac{M^2}{\mu^2},\alpha_s(\mu^2)\right)=\lim_{\epsilon\to0}\left[\bar g_0^{(c)}(\alpha_s(M^2),\epsilon)+\bar g_0^{(l)}(\alpha_s(\tilde\Lambda^2_{\rm ds}),\epsilon)\right]+\int_{M^2}^{\tilde\Lambda^2_{ds}}\frac{d\mu^2}{\mu^2}\bar g^{\rm ds}(\alpha_s(\mu^2)) \quad (3.18)$$

$$=g_0^{\rm ds}(\alpha_s(M^2))+\int_{M^2}^{\tilde\Lambda^2_{ds}}\frac{d\mu^2}{\mu^2}g^{\rm ds}(\alpha_s(\mu^2)), \quad (3.19)$$

where we have made use of the fact that the singularity must cancel between $\bar g_0^{(l)}$ and $\bar g_0^{(c)}$, and

$$g_0^{\rm ds}(\alpha_s(M^2))=\bar g_0^{(c)}(\alpha_s(M^2),0)+\bar g_0^{(l)}(\alpha_s(M^2),0), \quad (3.20)$$

$$g^{\rm ds}(\alpha_s)=\bar g^{\rm ds}(\alpha_s)+\beta(\alpha_s)\frac{d\bar g_0^{(l)}}{d\alpha_s}. \quad (3.21)$$

In eq. (3.19) both $g_0^{\rm ds}(\alpha_s(M^2)$ and $g^{\rm ds}(\alpha_s(\mu^2))$ are power series in α_s with finite coefficients.

The physical anomalous dimension eq. (3.18) is now written as the sum of two contributions that are separately finite and μ independent:

$$\gamma\left(N_1N_2,\alpha_s(M^2)\right)=\gamma^{(c)}\left(\alpha_s(M^2)\right)+\gamma^{(l)}\left(N_1N_2,\alpha_s(M^2)\right) \quad (3.22)$$

$$\gamma^{(c)}\left(\alpha_s(M^2)\right)=g_0^{\rm ds}(\alpha_s(M^2)) \quad (3.23)$$

$$\gamma^{(l)}\left(N_1N_2,\alpha_s(M^2)\right)=\int_{M^2}^{M^2/(N_1N_2)}\frac{dk^2}{k^2}g^{\rm ds}(\alpha_s(k^2)). \quad (3.24)$$

We conclude that in the double-soft limit the finite renormalized coefficient function eq. (3.5) can be factorized as

$$C\left(N_1,N_2,\frac{M^2}{\mu^2},\frac{M^2}{\mu^2},\frac{M^2}{\mu^2},\alpha_s(\mu^2)\right)=C^{(c)}\left(\frac{M^2}{\mu^2},\alpha_s(\mu^2)\right)C^{(l)}\left(\frac{M^2/(N_1N_2)}{\mu^2},\alpha_s(\mu^2)\right)$$
$$+\mathcal{O}\left(\frac{1}{N_1},\frac{1}{N_2}\right) \quad (3.25)$$

where

$$M^2\frac{d}{dM^2}\ln C^{(c)}\left(\frac{M^2}{\mu^2},\alpha_s(\mu^2)\right)=\gamma^{(c)}\left(\alpha_s(M)\right) \quad (3.26)$$

$$M^2\frac{d}{dM^2}\ln C^{(l)}\left(\frac{M^2/(N_1N_2)}{\mu^2},\alpha_s(\mu^2)\right)=\gamma^{(l)}\left(N_1N_2,\alpha_s(M^2)\right). \quad (3.27)$$

Logarithmically enhanced terms are contained in $C^{(l)}(M^2/(N_1N_2\mu^2),\alpha_s(\mu^2))$, for which the expression eq. (3.24) of $\gamma^{(l)}$ provides the resummed prediction, while $C^{(c)}$ contains the constants. Note that the separation between $C^{(l)}$ and $C^{(c)}$ is ambiguous, reflecting the freedom in fixing the finite part after the cancellation of the divergence: a constant term can always be reabsorbed in a redefinition of $C^{(l)}$.

3.2 Single-soft resummation

In the single soft limit the argument proceeds in the same way, except that now, assuming for definiteness that the soft variable is x_1, the dimensional dependence is through the scale

$$\Lambda^2_{\rm ss}=M^2(1-x_1), \quad (3.28)$$

up to corrections suppressed by powers of $(1-x_1)$. The Mellin-space coefficient function then depends on N_1 through the dimensional variable

$$\tilde{\Lambda}^2_{\rm ss} = \frac{M^2}{N_1}. \tag{3.29}$$

The dependence on N_2 is parametric: the argument runs through unchanged, except that now N_1N_2 is everywhere replaced by N_1, and we end up with the physical anomalous dimension

$$\gamma(N_1, N_2, 1, \alpha_s(M^2), \epsilon) = g_0^{\rm ss}(\alpha_s(M^2), N_2) + \int_{M^2}^{\tilde{\Lambda}^2_{\rm ss}(M^2,N_1)} \frac{d\mu^2}{\mu^2} g^{\rm ss}(\alpha_s(\mu^2), N_2), \tag{3.30}$$

where $g_0^{\rm ss}(\alpha, N_2)$ and $g^{\rm ss}(\alpha, N_2)$ are power series in α with N_2-dependent coefficients.

The resummed coefficient function now factorizes as

$$\begin{aligned} C\left(N_1, N_2, \frac{M^2}{\mu^2}, \frac{M^2}{\mu^2}, \frac{M^2}{\mu^2}, \alpha_s(\mu^2)\right) &= C^{(c)}\left(N_2, \frac{M^2}{\mu^2}, \alpha_s(\mu^2)\right) C^{(l)}\left(\frac{M^2/N_1}{\mu^2}, N_2, \alpha_s(\mu^2)\right) \\ &\quad + \mathcal{O}\left(\frac{1}{N_1}\right) \end{aligned} \tag{3.31}$$

where

$$M^2 \frac{d}{dM^2} \ln C^{(c)}\left(\frac{M^2}{\mu^2}, N_2, \alpha_s(\mu^2)\right) = g_0^{\rm ss}(\alpha_s(M^2), N_2) \tag{3.32}$$

$$M^2 \frac{d}{dM^2} \ln C^{(l)}\left(\frac{M^2/N_1}{\mu^2}, N_2, \alpha_s(\mu^2)\right) = \int_{M^2}^{M^2/N_1} \frac{d\mu^2}{\mu^2} g^{\rm ss}(\alpha_s(\mu^2), N_2). \tag{3.33}$$

Terms that are logarithmically enhanced in N_1 are contained in $C^{(l)}\left(\frac{M^2/N_1}{\mu^2}, N_2, \alpha_s(\mu^2)\right)$ for which the expression eq. (3.33) of $\gamma^{(l)}$ provides the resummed prediction, while $C^{(c)}(\frac{M^2}{\mu^2}, N_2, \alpha_s(\mu^2))$ contain all terms that do not vanish as $N_1 \to \infty$ but remain finite in the limit, i.e. constant in N_1 in Mellin space, thus proportional to a $\delta(1-x_1)$ in physical space. All these terms depend parametrically on N_2. Again, the separation between $C^{(l)}$ and $C^{(c)}$ is ambiguous.

4 The resummed coefficient function

The renormalization group argument leads to a resummed expression for the logarithmic terms contained in the physical anomalous dimension, which in turn determines the ratio of the resummed coefficient function at two different scales. In order to obtain a resummed expression for the coefficient function itself it is necessary to separate off the dependence of the coefficient function on the factorization scale from that on the soft scale that is being resummed. This requires some care because the physical anomalous dimension eq. (3.6) is a function of the pair of Mellin variables N_1 and N_2, while the dependence on the factorization scale is given by an anomalous dimension that is a function of a single Mellin variable, and indeed, according to eq. (3.4), it provides the dependence on a single scale of the coefficient when the factorization scales of the incoming partons are taken to be independent. To this purpose, we first summarize how this is done in the inclusive case [19] in which only one scale and one Mellin variable is present, and then consider the double-soft limit (two Mellin variables and one scale) and the single-soft limit (two Mellin variables and two scales).

4.1 The inclusive coefficient function

In the inclusive case, the resummed physical anomalous dimension is given by eq. (3.19) but with $\Lambda_{\rm ds}^2$ replaced by the soft scale $\Lambda_{\rm s}^2 = \frac{M^2}{N^2}$. Logarithms of $\frac{1}{N}$ are then resummed in the anomalous dimension $\gamma^{(l)}$, given by eq. (3.24) but with $N_1 N_2$ replaced by N^2, still related to the coefficient function (now only dependent on N) through eq. (3.27). Integrating eq. (3.27) the resummed coefficient function is given by [19]

$$
\begin{aligned}
E^{\rm res}(N; M_0^2, M^2) &\equiv \ln C^{(l)}\left(\frac{M^2/N^2}{\mu^2}, \alpha_s(\mu^2)\right) - \ln C^{(l)}\left(\frac{M_0^2/N^2}{\mu^2}, \alpha_s(\mu^2)\right) \\
&= -\int_1^{N^2} \frac{dn}{n} \int_{M_0^2}^{M^2} \frac{dk^2}{k^2}\, g\left(\alpha_s(k^2/n)\right).
\end{aligned} \tag{4.1}
$$

Comparing to the solution to the renormalization group equation, which has of course the form eq. (3.5) but now with a single Mellin variable, it is clear that $\ln C$ is the sum of a term that depends on the factorization scale μ^2 and a term that only depends on α_s evaluated at the scale of the process. In order to rewrite the resummed result in this form we let [19]

$$
g(\alpha_s) = A(\alpha_s) - \frac{\partial D(\alpha_s(k^2))}{\partial \ln k^2}. \tag{4.2}
$$

where A and D are power series in α_s, so that

$$
E^{\rm res}(N;M_0^2,M^2) = \int_1^{N^2} \frac{dn}{n}\left[\left(-\int_{M_0^2}^{M^2} \frac{dk^2}{k^2} A(\alpha_s(k^2/n))\right) + D(\alpha_s(M^2/n)) - D(\alpha_s(M_0^2/n))\right]. \tag{4.3}
$$

We then conclude that the resummed coefficient function has the general structure

$$
\ln C^{(l)}\left(\frac{M^2/N^2}{\mu^2}, \alpha_s(\mu^2)\right) = \int_1^{N^2} \frac{dn}{n}\left[\left(-\int_{\bar\mu^2(\mu^2)}^{M^2} \frac{dk^2}{k^2} A(\alpha_s(k^2/n))\right) + D(\alpha_s(M^2/n))\right]. \tag{4.4}
$$

Both the lower integration limit $\bar\mu^2$ and the function $A(\alpha_s(k^2/n))$ in eq. (4.4) are fully determined by the renormalization group equation satisfied by the coefficient function and the known properties of the $\overline{\rm MS}$ anomalous dimensions in the soft limit.

Indeed, differentiating eq. (4.4) we get

$$
\mu^2 \frac{d}{d\mu^2} \ln C^{(l)}\left(\frac{M^2/N^2}{\mu^2}, \alpha_s(\mu^2)\right) = \int_1^{N^2} \frac{dn}{n} A(\alpha_s(\bar\mu^2/n)) \frac{d \ln \bar\mu^2}{d \ln \mu^2}, \tag{4.5}
$$

but on the other hand, eq. (3.4) implies that the right-hand side of eq. (4.5) must be matched to the standard anomalous dimension $\gamma^{\rm AP}(N, \alpha_s(\mu^2))$ in the soft limit. Because in this inclusive case there is a single scale, the relevant anomalous dimension is the sum of the two anomalous dimensions γ_i, γ_j that correspond to the incoming evolution eigenstates.

Noting that by construction only $\gamma^{(l)}$ and $C^{(l)}$ contain logarithmically enhanced contributions (constant contributions having been included in $\gamma^{(c)}$), the right-hand side of eq. (4.5) must be given by the logarithmically enhanced contributions to the anomalous dimension $\gamma^{\rm AP}$. Now, in the $\overline{\rm MS}$ scheme it can be proven [28] that to all perturbative orders

$$
\gamma_i^{\rm AP}(N, \alpha_s(\mu^2)) = -\ln N \sum_k \left(\frac{\alpha_s}{\pi}\right)^k A_k^{(i)} + O(N^0) + O\left(\frac{1}{N}\right), \tag{4.6}
$$

where $A_i^{(k)}$ are numerical coefficients and only the diagonal splitting functions P_{qq} and P_{gg} are logarithmically enhanced, so in the large-N limit the anomalous dimension eigenvectors are the quark singlet and gluon and $i = q, g$. The quantity on the right-hand side of eq. (4.6) is usually referred to as the cusp anomalous dimension.

Demanding that the right-hand side of eq. (4.5) coincides with (minus) the cusp anomalous dimension, with an extra factor of 2 to account for the presence of two partons of the same species in the initial state, we immediately find

$$\bar{\mu}^2 = n\mu^2 \tag{4.7}$$

$$A^{(i)}(\alpha_s) = \sum_k \left(\frac{\alpha_s}{\pi}\right)^k A_k^{(i)}, \tag{4.8}$$

We therefore conclude that there are two independent resummed coefficient functions, in the gluon-gluon and the quark-quark channel.

The full coefficient function in the soft limit is finally found by multiplying the resummed expression for $C^{(l)}\left(\frac{M^2/N^2}{\mu^2}, \alpha_s(\mu^2)\right)$ by $C^{(c)}\left(\frac{M^2}{\mu^2}, \alpha_s(\mu^2)\right)$, which in Mellin space is just a series in $\alpha_s(M^2)$ with constant coefficients:

$$\begin{aligned} C(N,\frac{M^2}{\mu^2},\alpha_s(\mu^2)) = &C^{(c)}\left(1,\alpha_s(M^2)\right) \\ &\times\exp\int_1^{N^2}\frac{dn}{n}\left[\left(-\int_{\mu^2}^{M^2/n}\frac{dk^2}{k^2}A(\alpha_s(k^2))\right)+D(\alpha_s(M^2/n))\right]\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right). \end{aligned} \tag{4.9}$$

Specifically choosing $\mu^2 = M^2$ and changing integration variables we get

$$\begin{aligned} C(N, 1, \alpha_s(M^2)) = &C^{(c)}\left(1, \alpha_s(M^2)\right) \\ &\times \exp\int_{M^2}^{M^2/N^2}\frac{dk^2}{k^2}\left[A(\alpha_s(k^2))\ln\frac{M^2/N^2}{k^2} + D(\alpha_s(k^2))\right]\left(1 + \mathcal{O}\left(\frac{1}{N}\right)\right). \end{aligned} \tag{4.10}$$

As well known (see e.g. [29]), N^{k}LL resummation is obtained by including both $C^{(c)}$ and $g(\alpha_s)$ up to order α_s^{k+1}, i.e. $A(\alpha_s)$ up to order α_s^{k+1} and $D(\alpha_s)$ up to order α_s^k. Assuming knowledge of the cusp anomalous dimension, all the remaining coefficients in D and $C^{(c)}$ are determined by comparing to a fixed N^{k}LO computation. Note that in the dQCD literature (see e.g. [30]) the function D is usually further divided into two separate functions B and D that originate from different kinds of soft radiation, though of course if they are determined by matching to fixed order no such separation is possible. Note also that in the SCET literature this is referred to as N^{k}LL', while N^{k}LL refers to including $C^{(c)}$ only up to order α_s^k (see e.g. [12]).

4.2 The coefficient function in the double-soft limit

Coming now to the rapidity distribution in the soft limit, we can use the expression eq. (3.24) of the resummed physical anomalous dimension: we get

$$\begin{aligned} E^{\mathrm{res,ds}}(N_1,N_2;M_0^2,M^2) \equiv &\ln C^{(l),\mathrm{ds}}\left(\frac{M^2/(N_1N_2)}{\mu^2},\alpha_s(\mu^2)\right) - \ln C^{(l),\mathrm{ds}}\left(\frac{M_0^2/(N_1N_2)}{\mu^2},\alpha_s(\mu^2)\right) \\ = &-\int_1^{N_1N_2}\frac{dn}{n}\int_{M_0^2}^{M^2}\frac{dk^2}{k^2}g^{\mathrm{ds}}(\alpha_s(k^2/n)). \end{aligned} \tag{4.11}$$

The right-hand side coincides with the inclusive result eq. (4.1) with the substitution $N^2 \to N_1 N_2$. Given the form eq. (3.5) of the solution to the renormalization group equation, by simply making a common scale choice $\mu_1^2 = \mu_2^2 = \mu^2$ we thus arrive through the same steps at the resummed coefficient function

$$\ln C^{(l),\mathrm{ds}}\left(\frac{M^2/(N_1N_2)}{\mu^2},\alpha_s(\mu^2)\right)=\int_1^{N_1N_2}\frac{dn}{n}\left[\left(-\int_{\bar\mu^2(\mu^2)}^{M^2}\frac{dk^2}{k^2}A(\alpha_s(k^2/n))\right)+D^{\mathrm{ds}}(\alpha_s(M^2/n))\right]. \tag{4.12}$$

Matching to the Callan-Symanzik-Altarelli-Parisi equation (3.4) is now performed based on the simple observation that, because of eqs. (2.17)

$$N_1N_2 = N^2 + \frac{b^2}{4} = N^2\left(1+O\left(\frac{b^2}{N^2}\right)\right). \tag{4.13}$$

This then immediately implies that consistency with the inclusive case (which indeed corresponds to $b=0$) implies that eqs. (4.7)–(4.8) still hold, and in particular the functions $C^{(c)}$, A and D are the same as in the inclusive case, so

$$\begin{aligned} C^{\mathrm{ds}}(N_1,N_2,\frac{M^2}{\mu^2},\frac{M^2}{\mu^2},\frac{M^2}{\mu^2},\alpha_s(\mu^2)) &= C^{(c)}\left(1,\alpha_s(M^2)\right) \\ &\times \exp\int_1^{N_1N_2}\frac{dn}{n}\left[\left(-\int_{\mu^2}^{M^2/n}\frac{dk^2}{k^2}A(\alpha_s(k^2))\right)+D(\alpha_s(M^2/n))\right]\left(1+\mathcal{O}\left(\frac{1}{N_1},\frac{1}{N_2}\right)\right). \end{aligned} \tag{4.14}$$

Hence double-soft resummation is simply obtained by performing the substitution $N^2 \to N_1N_2$ in the inclusive result:

$$C^{\mathrm{ds}}\left(N_1,N_2,\frac{M^2}{\mu^2},\frac{M^2}{\mu^2},\frac{M^2}{\mu^2},\alpha_s(\mu^2)\right)=C\left(N,\frac{M^2}{\mu^2},\alpha_s(\mu^2)\right)\left(1+\mathcal{O}\left(\frac{b^2}{N^2}\right)\right) \tag{4.15}$$

where $C(N,M^2/\mu^2,\alpha_s(\mu^2))$ is the coefficient function for the inclusive process, eq. (4.9), and $N^2=N_1N_2$ in the double-soft limit, eq. (4.13). In particular, as in the inclusive case, only the quark-quark and gluon-gluon coefficient survive the limit, and specifically for the Drell-Yan process only the quark-quark channel logarithmically enhanced. As mentioned in the introduction, the result eq. (4.14) was first obtained to NLL accuracy in ref. [1], where it was used to derive resummation in the double soft limit of the differential distribution with respect to the longitudinal momentum of the Z (sometimes called Feynman x). It was then rewritten in ref. [5] for the rapidity distribution, and extended to NNLO and next-to-leading power in refs. [7, 8]. In these references an alternate form of the resummed results eq. (4.14)–(4.16) is actually used; its equivalence to the resummed expression of this paper is proven in appendix B.

Finally, choosing again $\mu^2 = M^2$ we get

$$\begin{aligned} C^{\mathrm{ds}}(N_1,N_2,1,1,1,\alpha_s(M^2)) &= C^{(c)}\left(1,\alpha_s(M^2)\right) \\ &\times\exp\int_{M^2}^{\tilde\Lambda^2_{\mathrm{ds}}}\left[\frac{dk^2}{k^2}A(\alpha_s(k^2))\ln\frac{\tilde\Lambda^2_{\mathrm{ds}}}{k^2}+D(\alpha_s(k^2))\right]\left(1+\mathcal{O}\left(\frac{1}{N_1},\frac{1}{N_2}\right)\right), \end{aligned} \tag{4.16}$$

with $\tilde{\Lambda}^2_{\rm ds}$ the soft scale eq. (3.10). Of course, we may equivalently choose any soft scale that is proportional to $\tilde{\Lambda}^2_{\rm ds}$. This is clear from the structure of the renormalization group argument, since the latter only uses the fact that the kinematic dependence on N_1 and N_2 goes through the dimensionful variable $\tilde{\Lambda}^2_{\rm ds}$. It is also manifest from our final result eq. (4.16), as a change of soft scale by a constant factor can always reabsorbed by a change of the coefficients of the expansion of the function $D^{\rm ds}$ in powers of α_s. A common choice, suggested in ref. [1], as it simplifies the expression of $D^{\rm ds}$ by reabsorbing some subleading terms, is

$$\bar{\Lambda}^2_{\rm ds} = \frac{M^2}{\bar{N}_1 \bar{N}_2}; \quad \bar{N}_i \equiv e^{\gamma_E} N_i, \tag{4.17}$$

where γ_E is the Euler constant.

4.3 The coefficient function in the single-soft limit

The argument in the single-soft $N_1 \to \infty$ limit requires treating both the kinematic variables asymmetrically. The scale dependence of the coefficient function is now given by

$$\begin{aligned} E^{\rm res,\,ss}(N_1, N_2; M_0^2, M^2) &\equiv \ln C^{(l),\rm ss}\left(\frac{M^2/N_1}{\mu^2}, N_2, \alpha_s(\mu^2)\right) - \ln C^{(l),\rm ss}\left(\frac{M_0^2/N_1}{\mu^2}, N_2, \alpha_s(\mu^2)\right) \\ &= -\int_1^{N_1} \frac{dn}{n} \int_{M_0^2}^{M^2} \frac{d\mu^2}{\mu^2} g^{\rm ss}(\alpha_s(\mu^2/n), N_2), \end{aligned} \tag{4.18}$$

leading to

$$\begin{aligned} &\ln C^{(l),\rm ss}\left(\frac{M^2/N_1}{\mu^2}, N_2, \alpha_s(\mu^2)\right) \\ &= \int_1^{N_1} \frac{dn}{n} \left[\left(-\int_{\bar{\mu}^2(\mu^2)}^{M^2} \frac{dk^2}{k^2} A^{\rm ss}(\alpha_s(k^2/n, N_2))\right) + D^{\rm ss}(\alpha_s(M^2/n), N_2)\right]. \end{aligned} \tag{4.19}$$

We now note that, because the PDF $f_2(N_2, \mu_2^2)$ in eq. (3.2) only depends on N_2, changing its scale cannot introduce any large logs of the soft variable, which is M^2/N_1. It follows that

$$C^{\rm ss}\left(N_1, N_2, \frac{M^2}{\mu_1^2}, \frac{M^2}{\mu_2^2}, \frac{M^2}{\mu_R^2}, \alpha_s(\mu_R^2)\right) = C^{\rm ss}\left(N_1, N_2, \frac{M^2}{\mu_1^2}, \frac{M^2}{\mu_1^2}, \frac{M^2}{\mu_R^2}, \alpha_s(\mu_R^2)\right)(1 + \mathcal{O}(N_1^0)). \tag{4.20}$$

We thus consider the coefficient function with the choice $\mu_1^2 = \mu_R^2 = \mu^2$ but $\mu_2^2 = M^2$, which satisfies

$$\mu^2 \frac{d}{d\mu^2} \ln C^{\rm ss}_{ki}\left(N_1, N_2, \frac{M^2}{\mu^2}, 1, \frac{M^2}{\mu^2}, \alpha_s(\mu^2)\right) = \ln N_1 \sum_n \alpha_s^n(\mu^2) A_k^{(n)} + \mathcal{O}\left(N_1^0\right), \tag{4.21}$$

where we have restored the parton channel indices, and now k corresponds to the diagonal quark or gluon logarithmically enhanced channel, while i runs over all eigenstates of perturbative evolution. On the other hand, because of eq. (4.20),

$$\mu^2 \frac{d}{d\mu^2} \ln C^{(l),\rm ss}_{ki}\left(\frac{M^2/N_1}{\mu^2}, N_2, \alpha_s(\mu^2)\right) = \mu^2 \frac{d}{d\mu^2} \ln C^{\rm ss}_{ki}\left(N_1, N_2, \frac{M^2}{\mu^2}, 1, \frac{M^2}{\mu^2}, \alpha_s(\mu^2)\right) + \mathcal{O}(N_1^0), \tag{4.22}$$

where we have restored the parton indices ki also on the function $C_{ki}^{(l),\text{ss}}$ that contains the logarithmically enhanced terms.

Hence the μ_1^2 factorization scale dependence of the coefficient function in the single-soft limit must match the cusp anomalous dimension, and thus cannot depend on N_2. We consequently arrive at a resummed expression for the coefficient function with asymmetric scale choice in the single soft limit of the form

$$C_{ki}^{\text{ss}}\left(N_1, N_2, \frac{M^2}{\mu^2}, 1, \frac{M^2}{\mu^2}, \alpha_s(\mu^2)\right) = C_{ki}^{(c),\text{ss}}\left(1, \alpha_s(M^2), N_2\right) \\ \times \exp \int_1^{N_1} \frac{dn}{n}\left[\left(-\int_{\mu^2}^{M^2/n} \frac{dk^2}{k^2} A_k(\alpha_s(k^2))\right) + D_{ki}^{\text{ss}}(\alpha_s(M^2/n), N_2)\right]\left(1+\mathcal{O}\left(\frac{1}{N_1}\right)\right). \tag{4.23}$$

For the Drell-Yan process only quark channels are logarithmically enhanced, so $k = q$ or $k = \bar{q}$, while i runs over all eigenstates of perturbative evolution, with $C_{qi}^{\text{ss}} = C_{\bar{q}i}^{\text{ss}}$. We will denote henceforth the single-soft Drell-Yan coefficient functions as C_{qi}^{ss} for definiteness.

Finally restoring a symmetric scale choice $\mu^2 = M^2$ we get

$$C_{qi}^{\text{ss}}(N_1, N_2, 1, 1, 1, \alpha_s(M^2)) = C_{ki}^{(c),\text{ss}}\left(1, \alpha_s(M^2), N_2\right) \\ \times \exp \int_{M^2}^{\tilde{\Lambda}_{\text{ss}}^2} \frac{dk^2}{k^2}\left[A_k(\alpha_s(k^2)) \ln \frac{\tilde{\Lambda}_{\text{ss}}^2}{k^2} + D_{ki}^{\text{ss}}(\alpha_s(k^2), N_2)\right]\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right), \tag{4.24}$$

with the soft scale now given by $\tilde{\Lambda}_{\text{ss}}^2$ eq. (3.29). Again, we are free to redefine the soft scale by a constant factor, so it might be useful to take instead

$$\bar{\Lambda}_{\text{ss}}^2 = \frac{M^2}{\bar{N}_1}; \quad \bar{N}_1 \equiv e_E^{\gamma} N_1. \tag{4.25}$$

However, in the single-soft limit, the non-soft variable N_2 is just a fixed parameter. Hence, we may also use $\tilde{\Lambda}_{\text{ds}}^2$ eq. (3.10) or $\bar{\Lambda}_{\text{ds}}^2$ eq. (4.17) instead of $\tilde{\Lambda}_{\text{ss}}^2$ in the resummation formula eq. (4.24), with the only effect of changing the coefficients of the expansion of the function $D_{ki}^{\text{ss}}(\alpha_s(k^2), N_2)$ in powers of α_s.

5 Matching with fixed order calculations

The explicit values of the coefficients in the expansions of $C^{(c)}$, A, and D in powers of the strong coupling are obtained by expanding the resummed coefficient function at fixed order in α_s and requiring the resummed result to match the Mellin transform of the fixed-order computation. We perform the matching procedure explicitly up to NNLO, which allows us to obtain a resummed expression with NNLL accuracy, both in the double-soft and single-soft limit. In the single-soft case in this paper we will only consider the $C_{q\bar{q}}$ coefficient function, and only resum the quark nonsinglet channel.

To this purpose, we first compute explicitly the integrals in the resummed expressions eqs. (4.16), (4.24) and expand out the result up to order $\alpha_s^2(M^2)$. We then provide the fixed-order NNLO expressions, which were computed in refs. [25, 31] and are available as a public `FORTRAN` code [32], but must be transformed into the variables x_1, x_2 in terms of whose Mellin conjugates N_1, N_2 is expressed. We finally determine up to $\mathcal{O}(\alpha_s^2)$ the functions $C^{(c)}$ and D that, along with the cusp anomalous dimension, control the resummed result.

5.1 Explicit resummed results

We start by noting that the resummed expressions we are interested in, namely the double-soft result eq. (4.16) and the single-soft result eq. (4.24) have the same form if expressed in terms of a variable

$$\lambda = \beta_0 \alpha_s(M^2)\mathcal{L}, \tag{5.1}$$

where $\mathcal{L}$ is a large log, such as $\ln N_1 N_2$, and of the coefficients of the expansion of the function D, only with different values of the coefficients of this expansion.

Specifically, we can write the resummed coefficient function as

$$C_{qi}\left(N_1, N_2, \alpha_s(M^2)\right) = g_0(\alpha_s(M^2)) \exp\left(\frac{1}{\alpha_s(M^2)} g_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda) + \dots\right). \tag{5.2}$$

Note that this defines unambiguously the separation between logarithmic and soft contribution, i.e. it provides an unambiguous definition of $C^{(l)}$ and $C^{(c)}$, which are otherwise not uniquely defined, as one can always reshuffle a constant between them. As mentioned, we provide results for the Drell-Yan $C_{q\bar{q}}$ coefficient function, with only the quark nonsinglet channel included in the single-soft case. We will consequently henceforth omit the parton indices on the coefficient function. Also, throughout the section we choose the renormalization and factorization scales

$$\mu_1^2 = \mu_2^2 = \mu_R^2 = M^2, \tag{5.3}$$

and in order to lighten the notation we omit the corresponding arguments in the coefficient function.

Performing the integrals in eqs. (4.16), (4.24) we then get

$$g_1(\lambda) = \frac{A_1[\lambda + (1-\lambda)\ln(1-\lambda)]}{\pi\beta_0^2} \tag{5.4}$$

$$g_2(\lambda) = \frac{A_1\beta_1}{2\pi\beta_0^3}\left(2\lambda + \ln^2(1-\lambda) + 2\ln(1-\lambda)\right) - \frac{A_2}{\pi^2\beta_0^2}\left(\lambda + \ln(1-\lambda)\right) + \frac{D_1\ln(1-\lambda)}{\pi\beta_0} \tag{5.5}$$

$$\begin{aligned} g_3(\lambda) = {} & \frac{A_1\beta_1^2}{\pi\beta_0^4(1-\lambda)}\left(\frac{\lambda^2}{2} + \frac{1}{2}\ln^2(1-\lambda) + \lambda\ln(1-\lambda)\right) \\ & + \frac{A_1\beta_2}{\pi\beta_0^3(1-\lambda)}\left(\ln(1-\lambda) - \lambda\ln(1-\lambda) - \frac{\lambda^2}{2} + \lambda\right) \\ & - \frac{A_2\beta_1}{\pi^2\beta_0^3(1-\lambda)}\left(\frac{\lambda^2}{2} + \lambda + \ln(1-\lambda)\right) + \frac{A_3\lambda^2}{2\pi^3\beta_0^2(1-\lambda)} \\ & - \frac{D_2\lambda}{\pi^2\beta_0(1-\lambda)} + \frac{D_1\beta_1}{\pi\beta_0^2(1-\lambda)}\left(\lambda + \ln(1-\lambda)\right), \end{aligned} \tag{5.6}$$

where we have written the beta function as

$$\mu^2\frac{d\alpha_s(\mu^2)}{d\mu^2} = -\beta_0\alpha_s^2(\mu^2) - \beta_1\alpha_s^3(\mu^2) + O(\alpha_s^4) \tag{5.7}$$

$$\beta_0 = \frac{11C_A - 4T_F N_f}{12\pi} \tag{5.8}$$

$$\beta_1 = \frac{17C_A^2 - 10C_A T_F N_f - 6C_F T_F N_f}{24\pi^2}. \tag{5.9}$$

A_k are the coefficients in the expansion of the quark cusp anomalous dimension eq. (4.8) and D_i are the coefficients of the corresponding expansion

$$D(\alpha_s) = D_1 \frac{\alpha_s}{\pi} + D_2 \left(\frac{\alpha_s}{\pi}\right)^2 + \dots \tag{5.10}$$

of the D function. Note that D is a function of N_2 in the single-soft limit, with different values according to the specific soft scale choice, and just a constant in the double-soft limit. We also define the g_0 function as an expansion of the form

$$g_0(\alpha_s) = 1 + g_{01} \frac{\alpha_s}{\pi} + g_{02} \left(\frac{\alpha_s}{\pi}\right)^2 + g_{03} \left(\frac{\alpha_s}{\pi}\right)^3 + \dots . \tag{5.11}$$

We finally give the explicit expression of the coefficient function eq. (5.2), expanded out up to NNLO:

$$\begin{aligned} C\left(N_1, N_2, \alpha_s(M^2)\right) = 1 &+ \frac{\alpha_s}{\pi}\left(\frac{A_1}{2}\mathcal{L}^2 - D_1\mathcal{L} + g_{01}\right) \\ &+ \left(\frac{\alpha_s}{\pi}\right)^2 \left[\frac{A_1^2}{8}\mathcal{L}^4 + \mathcal{L}^3\left(\frac{1}{6}A_1\pi\beta_0 - \frac{1}{2}A_1 D_1\right)\right. \\ &\qquad + \mathcal{L}^2\left(\frac{1}{2}A_1 g_{01} + \frac{1}{2}A_2 - \frac{1}{2}\pi\beta_0 D_1 + \frac{1}{2}D_1^2\right) \\ &\qquad \left. + \mathcal{L}\left(-D_1 g_{01} - D_2\right) + g_{02}\right] + \mathcal{O}(\alpha_s^3). \end{aligned} \tag{5.12}$$

The coefficients A_i, D_i and g_{0i} corresponding to the various limits will be determined by matching eq. (5.12) to the fixed-order expression.

Note that all results given in this section are entirely generic, and hold for any resummed formula of the form of eqs. (4.16), (4.24) regardless of the particular limit (single soft or double soft), of the choice of soft scale, and of the specific coefficient function and parton channel that is being resummed. What will change in each case is the form (and functional dependence) of the coefficients D_i and g_{0i}.

5.2 The NNLO Drell-Yan differential distribution

The NLO Drell-Yan rapidity distribution was first computed in ref. [33] in terms of the variable τ eq. (2.5) and the scaling variable x_F, which is related to the difference of the variables x_1 and x_2 eq. (2.12). The NNLO result was first obtained in refs. [25, 31], where it is expressed in terms of τ and a variable u which in terms of τ and the rapidity y eq. (2.4) is given by

$$u = \frac{1}{1-\tau} \frac{1-\tau e^{2y}}{1+e^{2y}}. \tag{5.13}$$

The variable u is simply related to the scattering angle of the Z in the partonic center-of-mass frame in the single and double soft limits, as shown in appendix C.

In refs. [25, 31] the NLO cross section is also explicitly provided in terms of these variables, while the NNLO is available through the `Vrap` [32] code. In order to compare and match to the resummation, however, the fixed order result must be expressed in terms of the scaling variables x_i. At NLO this was first done in refs. [5, 6], and then more recently, correcting some errors, in ref. [15]. However, the NNLO expression in terms of x_i is not available.

The transformation between the variables x_1, x_2 and the variables τ and u is

$$x_1 = \sqrt{\tau}\sqrt{\frac{1-u(1-\tau)}{\tau+u(1-\tau)}}, \tag{5.14}$$

$$x_2 = \sqrt{\tau}\sqrt{\frac{\tau+u(1-\tau)}{1-u(1-\tau)}}, \tag{5.15}$$

with inverse

$$\tau = x_1 x_2, \tag{5.16}$$

$$u = \frac{x_2(1-x_1^2)}{(x_1+x_2)(1-x_1x_2)}. \tag{5.17}$$

The coefficient function $C(x_1, x_2, M^2)$ defined in eq. (2.11) enters the computation of hadronic cross sections through a convolution with a pair of parton distributions that respectively depend on x_1 and x_2. We can thus view it as a distribution acting on a test function $T(x_1, x_2)$. The results of refs. [25, 31] provide an expression for the coefficient function $\bar{C}(\tau, u, M^2)$ that satisfies

$$\int_0^1 d\tau \int_0^1 du\, \bar{C}(\tau, u, M^2) t(\tau, u) = \int_0^1 dx_1 \int_0^1 dx_2\, C(x_1, x_2, M^2) T(x_1, x_2) \tag{5.18}$$

where

$$t(\tau, u) = T(x_1(\tau, u), x_2(\tau, u)), \tag{5.19}$$

so that

$$C(x_1, x_2, M^2) = j(x_1, x_2)\bar{C}(\tau(x_1, x_2), u(x_1, x_2), M^2) \tag{5.20}$$

with

$$j(x_1, x_2) = \left|\frac{\partial(\tau, u)}{\partial(x_1, x_2)}\right|. \tag{5.21}$$

Equations (5.14), (5.17) show that the single-soft limit $x_1 \to 1$ corresponds to $u \to 0$, and $x_2 \to 1$ corresponds to $u \to 1$: indeed, as mentioned (see appendix C) u is related to the scattering angle and $u = 0$ and $u = 1$ are the collinear limits corresponding to the scattering angle of the final-state Z being along the directions of either of the incoming partons. The double-soft limit corresponds to $\tau \to 1$, and the transformation becomes singular in this limit, i.e. the jacobian $j(x_1, x_2)$ eq. (5.21) diverges, because when $\tau = 1$ a regular function of τ and u becomes u-independent.

The transformation of the coefficient function from (τ, u) to (x_1, x_2) space is accordingly nontrivial in the soft limit, because it requires mapping distributions in τ and u localized at $\tau = 1$ and at $u = 0$ or $u = 1$ in terms of distributions in x_1 and x_2 localized at $x_1 = 1$ and $x_2 = 1$ through a singular jacobian. A general expression of the transformations for all possible distributions of u only, τ only, or all products of distributions in u and τ is quite cumbersome. However, our task is facilitated by the fact that only specific combinations of distributions appear in the resummed result.

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Indeed, in the double-soft limit we know from the discussion in section 2.2.1 that the coefficient function only depends on $M^2(1-x_1)(1-x_2)$: hence, only logs of $(1-x_1)(1-x_2)$ can appear, multiplied by associate double distributions localized at $x_1 = 1$ and $x_2 = 1$. Hence, we only need to work out the (τ, u) expression of these distributions, which corresponds to specific combinations of double and single distributions in τ and u which we can then recognize in the NNLO coefficient function. In the single-soft limit τ cannot reach the value $\tau = 1$, hence only single distributions in u localized at either $u = 0$ or $u = 1$, and we only need to work out the expressions of these.

5.2.1 The NNLO coefficient function in (τ, u) space

We write the expansion of the coefficient function $\bar{C}(\tau, u, M^2)$ eqs. (5.18)–(5.20)

$$\bar{C}(\tau, u, \alpha_s) = \bar{C}^{(0)}(\tau, u) + \bar{C}^{(1)}(\tau, u)\frac{\alpha_s}{\pi} + \bar{C}^{(2)}(\tau, u)\left(\frac{\alpha_s}{\pi}\right)^2 + O(\alpha_s^3). \tag{5.22}$$

The LO term fixes the normalization of the coefficient function: as we shall show explicitly in section 5.2.2 below,

$$\bar{C}^{(0)}(\tau, u) = \delta(1-\tau)\frac{\delta(u) + \delta(1-u)}{2} \tag{5.23}$$

is consistent with the choice of normalization of eq. (2.11).

At higher orders, following ref. [25], we write the coefficient function as the sum of three contributions: $\bar{C}^{(i)}_{\text{Born}}$, which is purely virtual and has the same kinematics of the leading order; $\bar{C}^{(i)}_{\text{Boost}}$, which is proportional to $\delta(u) + \delta(1-u)$ and thus contains real radiation contributions that are collinear to either of the incoming partons; and $\bar{C}^{(i)}_{\text{Real}}$ that contains the rest of the coefficient function, corresponding to generic real emission contributions:

$$\bar{C}^{(n)}(\tau, u) = \delta(1-\tau)\frac{\delta(u) + \delta(1-u)}{2}\bar{C}^{(n)}_{\text{Born}} + \frac{\delta(u) + \delta(1-u)}{2}\bar{C}^{(n)}_{\text{Boost}}(\tau) + \bar{C}^{(n)}_{\text{Real}}(\tau, u). \tag{5.24}$$

At NLO we provide the full expression of the quark-quark coefficient function. At NNLO we provide the expression for the contributions that are distributional in at least one variable to $C_{q\bar{q}}$ in the quark-quark nonsinglet channel, i.e. such that the incoming quark or antiquark leg is the same that couples to the gauge boson. These have been extracted by us from the `FORTRAN` code [32], following the decomposition into individual contributions given in ref. [25], eqs. (5.1)–(5.3).

At NLO we have

$$\bar{C}^{(1)}_{\text{Born}} = C_F\left(\frac{\pi^2}{3} - 4\right) \tag{5.25}$$

$$\bar{C}^{(1)}_{\text{Boost}}(\tau) = \frac{C_F}{\tau}\left(4\left[\frac{\ln(1-\tau)}{1-\tau}\right]_+ - \tau(1+\tau^2)\frac{\ln\tau}{1-\tau} - 2(2+\tau+\tau^2)\ln(1-\tau) + \tau(1-\tau)\right) \tag{5.26}$$

$$\bar{C}^{(1)}_{\text{Real}}(\tau, u) = C_F\left(\left[\frac{1}{u}\right]_+ + \left[\frac{1}{1-u}\right]_+\right)\left(\left[\frac{1}{1-\tau}\right]_+ - \frac{1+\tau}{2}\right) - (1-\tau). \tag{5.27}$$

At NNLO we find

$$\bar{C}^{(2)}_{\text{Born}}=-\frac{2561}{144}+\frac{13}{9}\pi^2+\frac{1}{3}\zeta_3-\frac{19}{1620}\pi^4+\left(\frac{127}{72}-\frac{14}{81}\pi^2+\frac{2}{3}\zeta_3\right)N_f, \tag{5.28}$$

$$\begin{aligned}&\bar{C}^{(2)}_{\text{Boost}}(\tau)\\&=\frac{128}{9}\left[\frac{\ln^3(1-\tau)}{1-\tau}\right]_+ +\left(-\frac{44}{3}+\frac{8}{9}N_f\right)\left[\frac{\ln^2(1-\tau)}{1-\tau}\right]_+ +\left(\frac{4}{3}-\frac{40}{27}N_f-\frac{4}{27}\pi^2\right)\left[\frac{\ln(1-\tau)}{1-\tau}\right]_+\\&\quad+\left(-\frac{404}{27}+\frac{11\pi^2}{9}+N_f\left(\frac{56}{81}-\frac{2\pi^2}{27}\right)+\frac{190}{9}\zeta_3\right)\left[\frac{1}{1-\tau}\right]_+ +2\bar{\kappa}(\tau),\end{aligned} \tag{5.29}$$

$$\begin{aligned}\bar{C}^{(2)}_{\text{Real}}(\tau,u)=&\,\frac{16}{3}\left(\frac{1}{2}\left[\frac{1}{1-\tau}\right]_+\left[\frac{\ln^2 u}{u}\right]_+ +2\left[\frac{1}{u}\right]_+\left[\frac{\ln^2(1-\tau)}{1-\tau}\right]_+ +2\left[\frac{\ln u}{u}\right]_+\left[\frac{\ln(1-\tau)}{1-\tau}\right]_+\right.\\&\left.+2\frac{\ln u}{1-u}\left[\frac{\ln(1-\tau)}{1-\tau}\right]_+ +\left(\frac{\ln(1-u)\ln u}{1-u}+\frac{1}{2}\frac{\ln^2 u}{1-u}\right)\left[\frac{1}{1-\tau}\right]_+\right)\\&+\left(\frac{2}{9}N_f-\frac{11}{3}\right)\left(\left[\frac{1}{1-\tau}\right]_+\left[\frac{\ln u}{u}\right]_+ +2\left[\frac{1}{u}\right]_+\left[\frac{\ln(1-\tau)}{1-\tau}\right]_+ +\frac{\ln u}{1-u}\left[\frac{1}{1-\tau}\right]_+\right)\\&+\left(\frac{1}{3}-\frac{1}{27}\pi^2-\frac{10}{27}N_f\right)\left[\frac{1}{u}\right]_+\left[\frac{1}{1-\tau}\right]_+ -\frac{4}{3}(1+\tau)\left[\frac{\ln^2 u}{u}\right]_+\\&+\left(\left(\frac{11}{6}-\frac{1}{9}N_f\right)(1+\tau)-\frac{16}{3}(1+\tau)\ln(1-\tau)\right.\\&\left.-\frac{4}{9}(7+9\tau^2)\frac{\ln\tau}{1-\tau}+\frac{16}{9}(1+\tau^2)\frac{\ln\left(\frac{1+\tau}{2}\right)}{1-\tau}\right)\left[\frac{\ln u}{u}\right]_+\\&+\left(-\frac{16}{3}\ln^2(1-\tau)+\left(\frac{11}{3}-\frac{2}{9}N_f\right)(1+\tau)\ln(1-\tau)+\frac{1}{18}(37+61\tau^2)\frac{\ln^2\tau}{1-\tau}\right.\\&+\frac{4}{9}\left(8(1+\tau^2)\ln(1-\tau)-(3+5\tau^2)\ln\tau\right)\frac{\ln\frac{1+\tau}{2}}{1-\tau}\\&+\frac{1}{9}\left(26-8\tau+36\tau^2-2N_f(1+\tau^2)+4(1+\tau)(1-\tau)\ln 2-4(17+19\tau^2)\ln(1-\tau)\right)\frac{\ln(\tau)}{1-\tau}\\&\left.+\frac{4}{9}(1+\tau)(\mathrm{Li}_2(-\tau)-\mathrm{Li}_2(\tau))+\frac{1}{9}(14-17\tau)+\frac{7}{54}(1+\tau)\pi^2+\frac{2}{27}(1+4\tau)N_f\right)\left[\frac{1}{u}\right]_+\\&+(u\mapsto 1-u).\end{aligned} \tag{5.30}$$

The function $\bar{\kappa}(\tau)$ is given explicitly in appendix D.1.

5.2.2 The NNLO coefficient function in (x_1,x_2) space

We now turn to the expressions of the expansion coefficients of the coefficient function $C(x_1,x_2,M^2)$ eq. (2.11). In order to obtain it we have transformed all the combinations of distributions that appear in the NLO and NNLO coefficient functions from (τ,u) space to (x_1,x_2) space. We have performed the transformation by obtaining the relevant distributions from a generating function, following the approach used in ref. [19] for the inclusive coefficient

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function, and performing the transformation at the level of generating functions. The procedure is discussed in appendix E, and specifically all the distributional identities needed in order to perform the transformation are collected in appendix E.3.

At leading order we get

$$C^{(0)}(x_1,x_2) = \delta(1-x_1)\delta(1-x_2), \tag{5.31}$$

consistently with eq. (2.11), as already mentioned. Beyond leading order we separate again terms with Born kinematics, which, like the leading order, are proportional to a double delta distribution; terms that are proportional to a single Dirac delta in x_1 or x_2, and which therefore, based on the analysis of section 2.2.2, include contributions from radiation that is collinear to either of the incoming partons; and terms that correspond to generic real emission:

$$C^{(n)}(x_1,x_2)=\delta(1-x_1)\delta(1-x_2)C^{(n)}_{\text{Born}}+\delta(1-x_1)C^{(n)}_{\text{Boost}}(x_2)+\delta(1-x_2)C^{(n)}_{\text{Boost}}(x_1)+C^{(n)}_{\text{Real}}(x_1,x_2). \tag{5.32}$$

Note that, as it is clear from the relations between distributions from appendix E.3, the three contributions to eq. (5.32) are not respectively obtained by transforming the corresponding three contributions in eq. (5.24). This is due to the fact that it is generally possible to redefine a plus distribution by addition term proportional to a Dirac delta, corresponding to the fact that in Mellin space it is generally possible to redefine a log by addition of a constant. Hence the three terms can get reshuffled according to how one defines the plus distribution in each case.

At NLO we get

$$C^{(1)}_{\text{Born}} = \frac{C_F}{2}\left(\pi^2-8\right) \tag{5.33}$$

$$C^{(1)}_{\text{Boost}}(x_2) = C_F\left(\left[\frac{\ln(1-x_2)}{1-x_2}\right]_+ - \frac{1+x_2^2}{2}\frac{\ln\frac{1+x_2}{2}}{1-x_2} - \frac{1+x_2}{2}\ln(1-x_2) + \frac{1-x_2}{2}\right) \tag{5.34}$$

$$\begin{aligned} C^{(1)}_{\text{Real}} = C_F\Bigg(&\left[\frac{1}{1-x_1}\right]_+\left[\frac{1}{1-x_2}\right]_+ - \frac{1+x_1}{2}\left[\frac{1}{1-x_2}\right]_+ - \frac{1+x_2}{2}\left[\frac{1}{1-x_1}\right]_+ \\ &+ \frac{(x_1^2+x_2^2)((1+x_1^2)+(1+x_2^2)+2x_1x_2(3+x_1+x_2+x_1x_2))}{2(1+x_1)(1+x_2)(x_1+x_2)^2}\Bigg). \end{aligned} \tag{5.35}$$

At NNLO the individual terms are given by

$$C^{(2)}_{\text{Born}} = -\frac{2561}{144} + \frac{13}{9}\pi^2 + \frac{\zeta_3}{3} - \frac{19}{1620}\pi^4 + \left(\frac{127}{72} - \frac{14}{81}\pi^2 + \frac{2}{3}\zeta_3\right)N_f \tag{5.36}$$

$$\begin{aligned} C^{(2)}_{\text{Boost}}(x_2) &= \frac{64}{9}\left[\frac{\ln^3(1-x_2)}{1-x_2}\right]_+ + \left(\frac{4}{9}N_f - \frac{22}{3}\right)\left[\frac{\ln^2(1-x_2)}{1-x_2}\right]_+ \\ &+ \left(-\frac{2}{3} - \frac{2}{27}\pi^2 - \frac{20}{27}N_f\right)\left[\frac{\ln(1-x_2)}{1-x_2}\right]_+ \\ &+ \left(-\frac{202}{27} + \frac{11}{18}\pi^2 + \frac{95}{9}\zeta_3 + N_f\left(\frac{28}{81} - \frac{\pi^2}{27}\right)\right)\left[\frac{1}{1-x_2}\right]_+ + \frac{1}{2}\kappa(x_2), \end{aligned} \tag{5.37}$$

$$
\begin{aligned}
C^{(2)}_{\mathrm{Real}}(x_1,x_2) = &\Bigg\{ \frac{8}{3}\Bigg(\bigg[\frac{1}{1-x_2}\bigg]_+ - \frac{1+x_2}{2}\Bigg)\bigg[\frac{\ln^2(1-x_1)}{1-x_1}\bigg]_+ + (x_1 \leftrightarrow x_2)\Bigg\} \\
&+ \frac{16}{3}\bigg[\frac{\ln(1-x_1)}{1-x_1}\bigg]_+ \bigg[\frac{\ln(1-x_2)}{1-x_2}\bigg]_+ + \Bigg\{ \bigg[\frac{\ln(1-x_1)}{1-x_1}\bigg]_+ \Bigg(\bigg(-\frac{11}{3}+\frac{2}{9}N_f\bigg)\bigg[\frac{1}{1-x_2}\bigg]_+ \\
&- \frac{8}{9}(1+x_2^2)\frac{\ln\frac{1+x_2}{2}}{1-x_2} - \frac{4}{9}\left(1+3x_2^2\right)\frac{\ln x_2}{1-x_2} - \frac{8}{3}(1+x_2)\ln(1-x_2) \\
&+ \frac{11}{6}(1+x_2) - \frac{1}{9}(1+x_2)N_f\Bigg) + (x_1 \leftrightarrow x_2)\Bigg\} \\
&+ \bigg(\frac{1}{3} - \frac{10}{27}N_f - \frac{1}{27}\pi^2\bigg)\bigg[\frac{1}{1-x_1}\bigg]_+ \bigg[\frac{1}{1-x_2}\bigg]_+ \\
&\times \Bigg\{ \bigg[\frac{1}{1-x_1}\bigg]_+ \Bigg(-\frac{4}{9}(1+x_2)\left(\mathrm{Li}_2(x_2) - \mathrm{Li}_2(-x_2)\right) - \frac{4}{9}(1+x_2^2)\frac{\ln^2\frac{1+x_2}{2}}{1-x_2} \\
&+ \bigg(\frac{11}{6}(1+x_2^2) - \frac{1}{9}(1+x_2^2)N_f - \frac{8}{9}(1+x_2^2)\ln(1-x_2) + \frac{1}{9}(11+5x_2^2)\ln(x_2)\bigg)\frac{\ln\left(\frac{1+x_2}{2}\right)}{1-x_2} \\
&- \frac{1}{18}(5+13x_2^2)\frac{\ln^2(x_2)}{1-x_2} + \Bigg(-\frac{1}{9}(1+x_2^2)N_f + \frac{7}{9}(1-x_2^2)\ln(2) - \frac{16}{9}(1+x_2^2)\ln(1-x_2) \\
&\frac{19-16x_2+39x_2^2}{18} - \frac{1}{3}(1-x_2^2)\ln(1+x_2)\Bigg)\frac{\ln(x_2)}{1-x_2} - \frac{4}{3}(1+x_2)\ln^2(1-x_2) \\
&+ \bigg(-\frac{1}{9}N_f + \frac{11}{6}\bigg)(1+x_2)\ln(1-x_2) + \frac{2}{27}(1+4x_2)N_f + \frac{7}{54}(1+x_2)\pi^2 + \frac{1}{9}(14-17x_2)\Bigg) \\
&+ (x_1 \leftrightarrow x_2)\Bigg\}, \qquad (5.38)
\end{aligned}
$$

where again the explicit expression of $\kappa(x_2)$ is given in appendix D.1.

5.3 Resummation coefficients up to NNLO

The resummation coefficients can now be determined by performing a double Mellin transform of the (x_1, x_2)-space expression of the coefficient function $C(x_1, x_2, M^2)$ and comparing to the expanded resummed result eq. (5.12). The Mellin transform is

$$
\begin{aligned}
C(N_1, N_2, M^2) &= \int_0^1 dx_1 \int_0^1 dx_2 x_1^{N_1-1} x_2^{N_2-1} C(x_1, x_2, M^2) && (5.39) \\
&= 1 + \frac{\alpha_s(M^2)}{\pi} C^{(1)}(N_1, N_2) + \left(\frac{\alpha_s(M^2)}{\pi}\right)^2 C^{(1)}(N_1, N_2), && (5.40)
\end{aligned}
$$

where $C(x_1, x_2, M^2)$ is given in eqs. (5.32)–(5.37). Most of the Mellin transforms are all elementary and can be performed using well-known identities as collected for instance in appendix B of ref. [34]. In practice, we have used th `MT` code [35], and simplified the result using `HarmonicSums` [36].

We will choose as a soft scale both in the single-soft and the double-soft case $\bar{\Lambda}^2_{\mathrm{ds}}$ eq. (4.17). This means that the resummed result and its expansion will have an identical expression in

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the single-soft and double soft case, with the large log $\mathcal{L}$ everywhere given by

$$\mathcal{L} = \ln \bar{N}_1 \bar{N}_2. \tag{5.41}$$

We will denote with $D_i^{\rm ds}$, $g_{0i}^{\rm ds}$ and $D_i^{\rm ss}$, $g_{0i}^{\rm ss}$ the expansion coefficients eq. (5.10) and eq. (5.11) determined in the double-soft and single-soft limit respectively. The double-soft coefficients, as discussed in section 4.2, are expected to coincide with those found in the inclusive case, as we will explicitly check. Furthermore, with this choice of scale the single-soft result reduces to the double-soft one in the double-soft limit.

5.3.1 Double-soft limit

We expand the coefficients $C^{(n)}(N_1, N_2)$ of eq. (5.40) in powers of $\mathcal{L}$ according to

$$C^{(n)}(N_1, N_2) = \sum_{m=0}^{2n} C_m^{(n)} \mathcal{L}^m, \tag{5.42}$$

where all coefficients $C_m^{(n)}$ are evaluated in the limit $N_1 \to \infty$, $N_2 \to \infty$ and are consequently constant. Comparing the coefficients of the expansion eq. (5.42) of the fixed order expression to the expansion of the resummed result eq. (5.12 we get the matching relations

$$\text{NLO}: \quad C_2^{(1)} = \frac{A_1}{2}, \quad C_1^{(1)} = -D_1^{\rm ds}, \qquad C_0^{(1)} = g_{01}^{\rm ds}, \tag{5.43}$$

$$\text{NNLO}: \quad C_4^{(2)} = \frac{A_1^2}{8}, \quad C_3^{(2)} = \frac{4}{3}\frac{\pi\beta_0 A_1}{8}, \quad C_2^{(2)} = \frac{A_2 + g_{01}^{\rm ds} A_1}{2}, \quad C_1^{(2)} = -D_2^{\rm ds}, \quad C_0^{(2)} = g_{02}^{\rm ds}, \tag{5.44}$$

Comparing to the explicit expression of the fixed-order coefficient function we get

$$\text{NLO:} \qquad A_1 = 4/3, \qquad D_1^{\rm ds} = 0, \qquad g_{01}^{\rm ds} = \frac{16}{3}\left(\frac{\pi^2}{6} - 1\right); \tag{5.45}$$

$$\text{NNLO:} \qquad A_2 = \frac{67}{9} - \frac{\pi^2}{3} - \frac{10}{27}N_f, \tag{5.46}$$

$$D_2^{\rm ds} = -\frac{202}{27} + 7\zeta_3 + \frac{28}{81}N_f, \tag{5.47}$$

$$g_{02}^{\rm ds} = -\frac{2561}{144} + \frac{14}{9}\pi^2 + \frac{91}{9}\zeta_3 + \frac{23}{108}\pi^4 + N_f\left(\frac{2}{27}\zeta_3 - \frac{8}{27}\pi^2 + \frac{127}{72}\right). \tag{5.48}$$

The coefficients A_i of the expansion of the cusp anomalous dimension are well known (see e.g. ref. [30]), and in fact the NNLO cusp anomalous dimension A_3, which only enters at fixed N^{3}LO is needed for NNLL resummation.

As mentioned in the introduction and proven in section 4.2 the functions $D^{\rm ds}(\alpha_s)$ and $g_0^{\rm ds}(\alpha_s)$ coincide with their inclusive counterparts, and thus the coefficients in eqs. (5.45)–(5.48) must coincide with the NLO [1] and NNLO [37] inclusive Drell-Yan threshold resummation coefficients. We have explicitly checked that the resummation D function agrees with the inclusive case by using the expressions given in ref. [30]. Note that in this reference the perturbative expansion is performed in powers of $\frac{\alpha_s}{4\pi}$, and furthermore that the alternate form of the resummation given in appendix B eq. (B.2) is adopted, hence they must be compared using the relations given in the appendix, in particular eqs. (B.40)–(B.41). Finally, we have checked that the constants collected in $g_0^{\rm ds}(\alpha_s)$ also agree with the NNLO result, first obtained in ref. [38], by checking against the form given in ref. [15], eq. (A.5). This provides a nontrivial check of the consistency of the double-soft resummation.

5.3.2 Single-soft limit

We now expand the coefficients $C^{(n)}(N_1, N_2)$ of eq. (5.40) in powers of $\ln \bar{N}_1$ according to

$$C^{(n)}(N_1, N_2) = \sum_{m=0}^{2n} \bar{C}_m^{(n)}(N_2)(\ln \bar{N}_1)^m, \tag{5.49}$$

where the coefficient $\bar{C}_m^{(n)}$ are evaluated in the limit $N_1 \to \infty$, and are consequently a function of N_2 only. Furthermore, we substitute $\mathcal{L} = \ln \bar{N}_1 + \ln \bar{N}_2$ in the expanded resummed result eq. (5.12) and similarly expand in powers of $\ln \bar{N}_1$. Comparing the two expansions we get the matching relations at NLO

$$\bar{C}_2^{(1)} = \frac{A_1}{2}, \qquad \bar{C}_1^{(1)} = -B_1^{\rm ss} - A_1 \ln \bar{N}_2, \qquad \bar{C}_{10}^{(1)} = g_{01}^{\rm ss} - D_1^{\rm ss} \ln\left(\bar{N}_2\right) + \frac{1}{2} A_1 \ln^2(\bar{N}_2) \tag{5.50}$$

and at NNLO

$$\bar{C}_4^{(2)} = \frac{A_1^2}{8} \tag{5.51}$$

$$\bar{C}_3^{(2)} = \frac{\pi}{6}\beta_0 A_1 - \frac{1}{2} A_1 D_1^{\rm ss} + \frac{1}{2} A_1^2 \ln \bar{N}_2, \tag{5.52}$$

$$\bar{C}_2^{(2)} = \frac{1}{2}A_2 + \frac{1}{2}(D_1^{\rm ss})^2 + \frac{1}{2}A_1 g_{01}^{\rm ss} - \frac{\pi}{2}\beta_0 D_1^{\rm ss} + \left(\frac{1}{2}A_1 \pi \beta_0 - \frac{3}{2}A_1 D_1^{\rm ss}\right)\ln \bar{N}_2 + \frac{3}{4}A_1^2 \ln^2 \bar{N}_2, \tag{5.53}$$

$$\begin{aligned}\bar{C}_1^{(2)} = & -D_2^{\rm ss} - B_1^{\rm ss} g_{01}^{\rm ss} + \left(A_2 + (D_1^{\rm ss})^2 + A_1 g_{01}^{\rm ss} - \pi\beta_0 D_1^{\rm ss}\right)\ln \bar{N}_2 \\ & + \left(\frac{\pi}{2}\beta_0 A_1 - \frac{3}{2}A_1 D_1^{\rm ss}\right)\ln^2 \bar{N}_2 + \frac{2}{3}A_1 \ln^3 \bar{N}_2 \end{aligned} \tag{5.54}$$

$$\begin{aligned}\bar{C}_0^{(2)} = & \, g_{02}^{\rm ss} - (D_2^{\rm ss} + D_1^{\rm ss} g_{01}^{\rm ss})\ln \bar{N}_2 + \frac{1}{2}\left(A_2 + (D_1^{\rm ss})^2 + A_1 g_{01}^{\rm ss} - \pi\beta_0 D_1^{\rm ss}\right)\ln^2 \bar{N}_2 \\ & + \frac{A_1}{2}\left(\frac{\pi}{3}\beta_0 - D_1^{\rm ss}\right)\ln^3 \bar{N}_2 + \frac{1}{8}A_1^2 \ln^4 \bar{N}_2. \end{aligned} \tag{5.55}$$

The resummation coefficients are again obtained by substituting the explicit expressions of the fixed-order result coefficients $\bar{C}_{ij}$. Because we have adopted the same soft scale choice as in the double-soft limits, all coefficients must reduce to their double-soft counterparts in the $N_2 \to \infty$ limit. We get

$$\text{NLO}: \qquad g_{01}^{\rm ss}(N_2) = g_{01}^{\rm ds} + F(N_2), \qquad D_1^{\rm ss}(N_2) = \hat{\gamma}_{qq}^{(0)}(N_2), \tag{5.56}$$

$$\text{NNLO}: \qquad D_2^{\rm ss}(N_2) = D_2^{\rm ds} - \pi\beta_0 F(N_2) + \hat{\gamma}_{qq}^{(1)}(N_2), \tag{5.57}$$

where $g_{01}^{\rm ds}$ and $D_2^{\rm ds}$ were respectively given in eqs. (5.45) and (5.47).

The explicit expressions of $F(N_2)$ and $\hat{P}^{(i)}(N_2)$ are

$$\begin{aligned} F(N_2) =& C_{\rm F}\bigg[\ln\bar{N}_2\left(\frac{1}{2N_2(N_2+1)}-S_1(N_2)\right)+\left(\frac{(-1)^{N_2}(2N_2+1)}{2N_2(N_2+1)}-\ln 2\right)S_{-1}(N_2) \\ &+\frac{1}{2}\Big[S_1^2(N_2)-S_{-1}^2(N_2)+\ln^2\bar{N}_2-\ln^2 2\Big]+\frac{1}{2N_2(N_2+1)}\Big[1-S_1(N_2)\Big] \\ &+\frac{(-1)^{N_2}(2N_2+1)}{2N_2(N_2+1)}\ln 2\bigg] \end{aligned} \tag{5.58}$$

$$\hat{\gamma}^{(0)}_{qq}(N_2) = C_{\rm F}\left(\frac{1}{2N_2(N_2+1)}+\ln\bar{N}_2-S_1(N_2)\right) \tag{5.59}$$

$$\begin{aligned} \hat{\gamma}^{(1)}_{qq}(N_2) =& C_{\rm A}C_{\rm F}\bigg[\frac{N_2(N_2+1)\big[N_2(151N_2+85)+3\big]+18}{72N_2^3(N_2+1)^3}+\left(\frac{\pi^2}{12}-\frac{67}{36}\right)\Big(S_1(N_2)-\ln\bar{N}_2\Big) \\ &+\frac{11S_2(N_2)}{12}-\frac{S_3(N_2)}{2}+\frac{\pi^2}{72}\left(\frac{3}{N_2+1}-\frac{3}{N_2}-11\right)+\frac{\zeta_3}{2}\bigg]+C_{\rm F}^2\bigg\{S_1(N_2)\bigg[\frac{1}{2(N_2+1)^2}-\frac{1}{2N_2^2} \\ &+S_2(N_2)-\frac{\pi^2}{6}\bigg]-\left(\frac{1}{2N_2(N_2+1)}+\frac{3}{4}\right)S_2(N_2)+\frac{\pi^2}{4}\left(\frac{1}{3N_2(N_2+1)}+\frac{1}{2}\right)+\frac{3N_2^3+N_2^2-1}{4N_2^3(N_2+1)^3} \\ &+S_3(N_2)-\zeta_3\bigg\}+C_{\rm F}n_{\rm f}T_{\rm R}\left[\frac{3-N_2(11N_2+5)}{18N_2^2(N_2+1)^2}+\frac{5}{9}\big(S_1(N_2)-\ln\bar{N}_2\big)-\frac{S_2(N_2)}{3}-\frac{\pi^2}{18}\right]. \end{aligned} \tag{5.60}$$

The lengthy expression of the NNLO constant $g_{02}(N_2)$ is given in appendix D.2. We have explicitly checked that

$$\lim_{N\to\infty} F(N) = \lim_{N\to\infty} \hat{\gamma}^{(0)}_{qq}(N) = \lim_{N\to\infty} \hat{\gamma}^{(1)}_{qq}(N) = 0. \tag{5.61}$$

The functions $\hat{\gamma}^{(i)}_{qq}(N)$ are easily identified in terms of the coefficients in the expansion in powers of $\frac{\alpha_s}{\pi}$ of the leading and next-to-leading order nonsinglet quark-quark anomalous dimensions

$$\gamma_{qq}(N) = \frac{\alpha_s}{\pi}\gamma^{(0)}_{qq}(N) + \left(\frac{\alpha_s}{\pi}\right)^2\gamma^{(1)}_{qq}(N), \tag{5.62}$$

by separating off the contributions to the anomalous dimension that survive in the soft limit. Namely, by writing

$$\gamma^{(i)}_{qq}(N) = \hat{\gamma}^{(i)}_{qq}(N) + \gamma^{(i,0)}_{qq} - \ln N A^{(i)}_q, \tag{5.63}$$

where $\gamma^{(i,0)}_{qq}$ are constants, and we recalled the definition eq. (4.6) of the cusp anomalous dimension (with $\gamma_{qq} = \gamma^{\rm AP}_q$).

The physical meaning of this single-soft resummation contribution is understood recalling that, as discussed in section 2.2.2, the single-soft limit is purely collinear. These terms then resum collinear evolution in the non-soft variable from the hard scale up to the soft scale. The cusp contribution is soft-collinear, and thus not included in D because it is already included in the double logarithmic A term: indeed, it already contributes to the double soft limit. The constants $\gamma^{(i,0)}_{qq}$ are not included either, because we choose to define the constant according to eq. (5.2), i.e. not to exponentiate any constant. They could of course be equivalently included in D, reflecting the arbitrariness in defining the separation between constants and logs already discussed above. Finally, the remaining contribution to D at NNLO is proportional to the contribution $F(N_2)$ to the single soft constant $g^{\rm ss}_{01}(N_2)$ that vanishes in the double-soft limit: it is recognized to originate from evolving up to the soft scale the argument of α_s in this contribution.

The single-soft resummation of the nonsinglet coefficient function up to NNLL accuracy in dQCD found using the coefficients eqs. (5.56)–(5.60) and eq. (D.3) in the general expression eq. (5.2) with the form eqs. (5.4)–(5.6), (5.11) of the g_i functions is a new result of this paper. The result was also implicitly given in refs. [14, 22] using a SCET formalism. The translation of the SCET result into explicit dQCD expressions will be given in the next section.

6 Comparison to SCET

As mentioned in the introduction, resummation of the rapidity distribution for Drell-Yan and Higgs production in both the double-soft and single-soft limits was derived using SCET methods in ref. [14]. The SCET resummation in the single-soft limit was recently re-derived and extended to N^{3}LL in ref. [22]. Also as mentioned, the comparison of SCET and dQCD results is not immediate. In this section we will show that the SCET result agrees with the dQCD resummation eqs. (4.16), (4.24) in the double- and single-soft limits, and it predicts the respective B and D functions, specifically agreeing with the form of the double-soft D function (which starts at NNLL) of ref. [7].

Resummed results in SCET are obtained as the solution of suitable momentum-space renormalization group equations depending on a number of soft and hard scales. Comparison to dQCD thus requires solving these equations in closed form, transforming to Mellin space, and making a suitable choice of the various scales. We will follow through these steps, also exploiting the previous results on the SCET-dQCD comparison of inclusive resummed results performed by some of us in refs. [23, 24]. We first solve the relevant evolution equations; next we transform to Mellin space, and finally, following our previous comparison [23, 24] of the inclusive SCET result of ref. [12] to the dQCD language, we make the appropriate choice of soft, hard and factorization scales, substitute explicit expressions of the relevant SCET anomalous dimensions and prove the equivalence of the SECT result of ref. [14] to our result in both the double-soft and single-soft limit.

In this section, we mostly follow the notation originally used in the SCET literature, specifically refs. [12, 14], in order to ease comparison with these references, and we only depart from it when needed to make contact with our results presented in the previous sections.

6.1 SCET resummation from evolution

We start considering the single-soft limit, as the double-soft limit can be obtained from it as a particular case. The single soft limit is referred to as generalized threshold limit in ref. [14], and collinear approximation in ref. [22]. We specifically discuss results up to NNLL accuracy. Resummation in SCET is generally performed at the level of hadronic cross sections, because the partonic cross section is sometimes chosen to depend on a hadronic scale (see ref. [39]).

The expression for the resummed hadronic differential distribution is given in ref. [14] in the single-soft $z_1 \to 1$ limit in the form

$$\frac{d\sigma(z_1, z_2, M^2)}{d\tau_H dY} = \sum_{ij} H_{ij}(M^2, \mu^2) \int_0^{M^2(1-z_1)} dt\, f_i\left(z_1\left(1+\frac{t}{M^2}\right), \mu^2\right) \tilde{B}_j(t, z_2, \mu^2). \quad (6.1)$$

Here i, j are parton indices, and

$$\tau_H = \frac{M^2}{S} \tag{6.2}$$

$$z_1 = \sqrt{\tau_H}\, e^{Y} \tag{6.3}$$

$$z_2 = \sqrt{\tau_H}\, e^{-Y}. \tag{6.4}$$

In eq. (6.1), $f_i(z,\mu^2)$ is a PDF, $H_{ij}(M^2,\mu^2)$ is a hard function, and $\tilde{B}_j$ a modified beam function, all of which will be defined below. In the SCET formalism, resummation is performed by evolving the hard function H_{ij} from the factorization scale μ^2 to a hard scale $\mu_{\rm H}^2$, and the beam function $\tilde{B}_j$ from the factorization scale μ^2 to a soft scale $\mu_{\rm S}^2$.

The dependence of $H_{ij}(M^2,\mu^2)$ on μ^2 is governed by an evolution factor U_H: [40, 41]

$$H_{ij}(M^2,\mu^2) = U_H^j(M^2,\mu_{\rm H}^2,\mu^2)\, H_{ij}(M^2,\mu_{\rm H}^2), \tag{6.5}$$

where

$$U_H^j(M^2,\mu_{\rm H}^2,\mu^2) = \left(\frac{M^2}{\mu_{\rm H}^2}\right)^{2\eta_\Gamma^j(\mu_{\rm H}^2,\mu^2)} \exp\Big\{-4K_\Gamma^j(\mu_{\rm H}^2,\mu^2) + 2K_{\gamma_H^j}(\mu_{\rm H}^2,\mu^2)\Big\}, \tag{6.6}$$

and we have assumed that partons i and j evolve with the same anomalous dimension

$$\gamma_H^j(\alpha_s) = \sum_{n=0}^{\infty} \gamma_{H,n}^j \left(\frac{\alpha_s}{4\pi}\right)^{n+1}. \tag{6.7}$$

The coefficients of the expansion of γ_H^j up to order α_s^2 are given in eqs. (F.12)–(F.13).

In eq. (6.6), the evolution kernels K_Γ^j and η_Γ^j are defined as integrals of the cusp anomalous dimension $\Gamma_{\rm cusp}^j$ as

$$K_\Gamma^j(\mu_0^2,\mu^2) \equiv \int_{\alpha_s(\mu_0^2)}^{\alpha_s(\mu^2)} \frac{d\alpha}{2\beta(\alpha)}\, \Gamma_{\rm cusp}^j(\alpha) \int_{\alpha_s(\mu_0^2)}^{\alpha} \frac{d\alpha'}{2\beta(\alpha')} \tag{6.8}$$

$$\eta_\Gamma^j(\mu_0^2,\mu^2) \equiv \int_{\alpha_s(\mu_0^2)}^{\alpha_s(\mu^2)} \frac{d\alpha}{2\beta(\alpha)}\, \Gamma_{\rm cusp}^j(\alpha), \tag{6.9}$$

while, for a generic anomalous dimension γ,

$$K_\gamma(\mu_0^2,\mu^2) \equiv \int_{\alpha_s(\mu_0^2)}^{\alpha_s(\mu^2)} \frac{d\alpha}{2\beta(\alpha)}\, \gamma(\alpha). \tag{6.10}$$

In our case, $\gamma = \gamma_H^j$.

The scale dependence of the beam function $\tilde{B}_j(t,z_2,\mu^2)$ is given by [14, 40]

$$\tilde{B}_j(t,z_2,\mu^2) = \int_0^t dt'\, U_B^j(t-t',\mu_{\rm S}^2,\mu^2)\, \tilde{B}_j(t',z_2,\mu_{\rm S}^2), \tag{6.11}$$

where

$$U_B^j(t,\mu_{\rm S}^2,\mu^2) = \frac{\exp\Big\{K_B^j(\mu_{\rm S}^2,\mu^2) - \gamma_{\rm E}\,\eta_B^j(\mu_{\rm S}^2,\mu^2)\Big\}}{\Gamma(1+\eta_B^j(\mu_{\rm S}^2,\mu^2))} \left[\frac{\eta_B^j(\mu_{\rm S}^2,\mu^2)}{\mu_{\rm S}^2}\mathcal{L}^{\eta_B^j}\left(\frac{t}{\mu_{\rm S}^2}\right) + \delta(t)\right]. \tag{6.12}$$

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The evolution kernels now are

$$K_B^j(\mu_S^2,\mu^2) = 4K_\Gamma^j(\mu_S^2,\mu^2) + K_{\gamma_B^j}(\mu_S^2,\mu^2) \tag{6.13}$$

$$\eta_B^j(\mu_S^2,\mu^2) = -2\eta_\Gamma^j(\mu_S^2,\mu^2), \tag{6.14}$$

where K_Γ^j and η_Γ^j are given in eqs. (6.8) and (6.9), respectively. The relevant coefficients of the γ_B^j anomalous dimension are given in eqs. (F.14)–(F.15). Finally, $\hat{\mathcal{L}}^\eta(t,\mu^2)$ is a generating function for plus distributions, defined in eq. (F.10).

The resummed result is obtained by substituting in eq. (6.1) the expressions eqs. (6.5) and (6.11) of the hard and beam functions, respectively evolved to the hard scale μ_H^2 and the soft scale μ_S^2. We get

$$\begin{aligned}\frac{d\sigma(z_1,z_2,M^2)}{d\tau_H dY} &= \sum_{ij} H_{ij}(M^2,\mu_H^2)\,U_H^j(M^2,\mu_H^2,\mu^2)\,\frac{\exp\left\{K_B^j(\mu_S^2,\mu^2) - \gamma_E\,\eta_B^j(\mu_S^2,\mu^2)\right\}}{\Gamma(1+\eta_B^j(\mu_S^2,\mu^2))} \\ &\quad\times \int_0^{M^2(1-z_1)} dt\, f_i\left(z_1\left(1+\frac{t}{M^2}\right),\mu^2\right) \\ &\quad\times \int_0^t dt'\left[\delta(t-t') + \frac{\eta_B^j(\mu_S^2,\mu^2)}{\mu_S^2}\mathcal{L}^{\eta_B^j}\left(\frac{t-t'}{\mu_S^2}\right)\right]\tilde{B}_j(t',z_2,\mu_S^2).\end{aligned} \tag{6.15}$$

This resummed result can be simplified by changing the scale integration variable $w = t'/t$, and exploiting the scale transformation property of the distribution $\hat{\mathcal{L}}^\eta(t,\mu^2)$ given in eq. (F.6):

$$\begin{aligned}&\int_0^t dt'\left[\delta(t-t') + \frac{\eta_B^j}{\mu_S^2}\mathcal{L}^{\eta_B^j}\left(\frac{t-t'}{\mu_S^2}\right)\right]\tilde{B}_j(t',z_2,\mu_S^2) \\ &= \tilde{B}_j(t,z_2,\mu_S^2) + \eta_B^j\int_0^1 dw\,\frac{t}{\mu_S^2}\,\mathcal{L}^{\eta_B^j}\left(\frac{t}{\mu_S^2}(1-w)\right)\tilde{B}_j(tw,z_2,\mu_S^2) \\ &= \eta_B^j\int_0^1 dw\left(\frac{t}{\mu_S^2}\right)^{\eta_B^j}\frac{\tilde{B}_j(tw,z_2,\mu_S^2)-\tilde{B}_j(t,z_2,\mu_S^2)}{(1-w)^{1-\eta_B^j}} + \left(\frac{t}{\mu_S^2}\right)^{\eta_B^j}\tilde{B}_j(t,z_2,\mu_S^2) \\ &= \eta_B^j\left(\frac{t}{\mu_S^2}\right)^{\eta_B^j}\int_0^1 dw\,\frac{\tilde{B}_j(tw,z_2,\mu_S^2)}{(1-w)^{1-\eta_B^j}}.\end{aligned} \tag{6.16}$$

Substituting this simplified expression in eq. (6.16) in eq. (6.15), the resummed result now takes the form

$$\begin{aligned}\frac{d\sigma(z_1,z_2,M^2)}{d\tau_H dY} &= \sum_{ij} H_{ij}(M^2,\mu_H^2)\,U_H^j(M^2,\mu_H^2,\mu^2)\,\frac{\eta_B^j\,\exp\left\{K_B^j - \gamma_E\,\eta_B^j\right\}}{\Gamma(1+\eta_B^j)} \\ &\quad\times\int_0^{M^2(1-z_1)} dt\, f_i\left(z_1\left(1+\frac{t}{M^2}\right),\mu^2\right)\left(\frac{t}{\mu_S^2}\right)^{\eta_B^j}\int_0^1 dw\,\frac{\tilde{B}_j(tw,z_2,\mu_S^2)}{(1-w)^{1-\eta_B^j}},\end{aligned} \tag{6.17}$$

where we have omitted the arguments (μ_S^2,μ^2) of K_B^j and η_B^j.

The explicit expression of the beam function $\tilde{B}_j(t,x,\mu^2)$ is given by [14]

$$\tilde{B}_j(t,x,\mu^2) = \sum_k \int_x^1 \frac{dz}{z}\, f_k\left(\frac{x}{z},\mu_S^2\right)\tilde{\mathcal{I}}_{jk}(t,z,\mu^2), \tag{6.18}$$

where

$$\widetilde{\mathcal{I}}_{jk}(t,z,\mu^2) = \sum_{n=0}^{\infty} \left[\frac{\alpha_s(\mu^2)}{4\pi}\right]^n \widetilde{\mathcal{I}}_{jk}^{(n)}(t,z,\mu^2) \tag{6.19}$$

and NNLL accuracy (in the sense of SCET[2]) is achieved by including terms up to $n = 1$ in eq. (6.19). The relevant coefficients are

$$\widetilde{\mathcal{I}}_{jk}^{(0)}(t,z,\mu^2) = \delta(t)\delta_{jk}\delta(1-z)\,, \tag{6.20}$$

$$\widetilde{\mathcal{I}}_{jk}^{(1)}(t,z,\mu^2) = \hat{\mathcal{L}}_1(t,\mu^2)\,\Gamma_0^j\,\delta_{jk}\delta(1-z) + \hat{\mathcal{L}}_0(t,\mu^2)\left[4P_{jk}^{(0)}(z) - \frac{\gamma_{B,0}^j}{2}\delta_{jk}\delta(1-z)\right] + \delta(t)\,\tilde{I}_{jk}^{(1)}(z), \tag{6.21}$$

where

$$\Gamma_0^q = 4C_{\mathrm{F}}\,, \qquad \gamma_{B,0}^j = 6C_{\mathrm{F}} \tag{6.22}$$

$$\Gamma_0^g = 4C_{\mathrm{A}}\,, \qquad \gamma_{B,0}^j = 8\pi\beta_0, \tag{6.23}$$

$P_{jk}^{(0)}(z)$ are the leading order coefficients in the expansion in powers of $\frac{\alpha_s}{\pi}$ of the splitting functions,[3] and the explicit expressions of the coefficients $\tilde{I}_{jk}^{(1)}(z)$ are given in eq. (S51) of ref. [14].

The integral over the beam function in eq. (6.17) can now be performed explicitly, with the result

$$\int_0^1 dw\, \frac{\tilde{B}_j(tw, z_2, \mu_{\mathrm{S}}^2)}{(1-w)^{1-\eta_B^j}} = \sum_k \frac{1}{t}\int_{z_2}^1 \frac{dx_2}{x_2}\, f_k\!\left(\frac{z_2}{x_2},\mu_{\mathrm{S}}^2\right) \widetilde{\mathcal{F}}_{jk}(t,x_2,\mu_{\mathrm{S}}^2) + \mathcal{O}\!\left(\alpha_s^2(\mu_{\mathrm{S}}^2)\right), \tag{6.24}$$

where

$$\begin{aligned}\widetilde{\mathcal{F}}_{jk}(t,x_2,\mu_{\mathrm{S}}^2) = \delta_{jk}\delta(1-x_2) + \frac{\alpha_s(\mu_{\mathrm{S}}^2)}{4\pi}\Bigg\{&\Gamma_0^j\,\delta_{jk}\delta(1-x_2)\left(V_1 + V_0\ln\frac{t}{\mu_{\mathrm{S}}^2} + \frac{1}{2}\ln^2\frac{t}{\mu_{\mathrm{S}}^2}\right)\\ &+\left[4P_{jk}^{(0)}(x_2) - \frac{\gamma_{B,0}^j}{2}\delta_{jk}\delta(1-x_2)\right]\left(V_0 + \ln\frac{t}{\mu_{\mathrm{S}}^2}\right) + \tilde{I}_{jk}^{(1)}(x_2)\Bigg\}.\end{aligned} \tag{6.25}$$

In eq. (6.25),

$$V_0 = \int_0^1 dw\,\frac{\mathcal{L}_0(w)}{(1-w)^{1-\eta_B^j}} = -\psi_0(\eta_B^j) - \gamma_{\mathrm{E}}\,, \tag{6.26}$$

$$V_1 = \int_0^1 dw\,\frac{\mathcal{L}_1(w)}{(1-w)^{1-\eta_B^j}} = \frac{1}{2}\Big[-\psi_1(\eta_B^j) + \psi_0^2(\eta_B^j) + 2\gamma_{\mathrm{E}}\,\psi_0(\eta_B^j) + \gamma_{\mathrm{E}}{}^2 + \zeta_2\Big]. \tag{6.27}$$

Substituting the expression eq. (6.24) of the beam function into the resummed expression eq. (6.17), changing the integration variable from t to x_1 according to $t = M^2(1-x_1)$, so that $x_1 \in [0, z_1]$, and noting that

$$f_i\left(z_1\left(1+\frac{t}{M^2}\right),\mu^2\right) = \frac{1}{x_1} f_i\!\left(\frac{z_1}{x_1},\mu^2\right)(1+\mathcal{O}(1-x_1)), \tag{6.28}$$

[2]In ref. [14] also the term with $n = 2$ was given, needed in order to achieve NNLL' accuracy using the SCET nomenclature, which corresponds to the NNLL accuracy of our dQCD result. A comparison also including this term is left for future work.

[3]Note that the definition of $P_{jk}^{(0)}(z)$ in ref. [14] brings a factor of 4 with respect to the convention assumed in this paper. This is the reason why a factor a 4 appears in front of $P_{jk}^{(0)}(z)$ in eq. (6.21), which is not present in ref. [14].

we obtain the final resummed expression, up to NNLO accuracy in the single soft $x_1 \to 1$ limit

$$\begin{aligned}\frac{d\sigma(z_1, z_2, M^2)}{d\tau_H dY} &= \sum_{ijk} H_{ij}(M^2, \mu_{\rm H}^2)\, U_H^j(M^2, \mu_{\rm H}^2, \mu^2)\, \frac{\exp\left\{K_B^j - \gamma_{\rm E}\, \eta_B^j\right\}}{\Gamma(\eta_B^j)} \\ &\times \int_{z_1}^1 \frac{dx_1}{x_1}\, f_i\left(\frac{z_1}{x_1}, \mu^2\right) \int_{z_2}^1 \frac{dx_2}{x_2}\, f_k\left(\frac{z_2}{x_2}, \mu_{\rm S}^2\right) \widetilde{\mathcal{T}}_{jk}(x_1, x_2, M^2, \mu_{\rm S}^2)\,, \end{aligned} \tag{6.29}$$

where

$$\widetilde{\mathcal{T}}_{jk}(x_1, x_2, M^2, \mu_{\rm S}^2) \equiv \left(\frac{M^2}{\mu_{\rm S}^2}\right)^{\eta_B^j} (1-x_1)^{\eta_B^j - 1}\, \widetilde{\mathcal{F}}_{jk}(M^2(1-x_1), x_2, \mu_{\rm S}^2) + \mathcal{O}\left(\alpha_s^2(\mu_{\rm S}^2)\right). \tag{6.30}$$

In this resummed expression, the hard and soft contributions are evaluated at the scales $\mu_{\rm H}^2$ and $\mu_{\rm S}^2$, respectively.

We further observe that if the hard and soft scales are chosen to be independent of the hadronic scaling variables z_1, z_2, then eq. (6.29) expresses the hadronic cross section as a convolution of the PDFs f_i, f_k with a coefficient function

$$\begin{aligned} &C_{ik}^{\rm scet}\left(x_1, x_2, \frac{M^2}{\mu^2}, \frac{M^2}{\mu_{\rm S}^2}, \frac{M^2}{\mu_{\rm H}^2}, \alpha_s(\mu_{\rm H}^2)\right) \\ &= \frac{1}{M^2 \sigma_{ik}^0} \sum_j H_{ij}(M^2, \mu_{\rm H}^2) U_H^j(M^2, \mu_{\rm H}^2, \mu^2)\, \frac{\exp\left\{K_B^j - \gamma_{\rm E} \eta_B^j\right\}}{\Gamma(\eta_B^j)} \widetilde{\mathcal{T}}_{jk}(x_1, x_2, M^2, \mu_{\rm S}^2). \end{aligned} \tag{6.31}$$

Because of the definition eqs. (6.2), (6.3) of the hadronic scaling variables, it then follows that the partonic variables x_1, x_2 can be identified with the variables defined in eq. (2.12).

6.2 SCET resummation in Mellin space

The Mellin transform of the coefficient function is given by

$$\begin{aligned} &C_{ik}^{\rm scet}\left(N_1, N_2, \frac{M^2}{\mu^2}, \frac{M^2}{\mu_{\rm S}^2}, \frac{M^2}{\mu_{\rm H}^2}, \alpha_s(\mu_{\rm H}^2)\right) \\ &= \int_0^1 dx_1\, x_1^{N_1 - 1} \int_0^1 dx_2\, x_2^{N_2 - 1}\, C_{ik}^{\rm scet}\left(x_1, x_2, \frac{M^2}{\mu^2}, \frac{M^2}{\mu_{\rm S}^2}, \frac{M^2}{\mu_{\rm H}^2}, \alpha_s(\mu_{\rm H}^2)\right). \end{aligned} \tag{6.32}$$

We are specifically interested in evaluating the Mellin transform in the single-soft limit $N_1 \to \infty$, by only retaining terms that do not vanish in this limit.

The dependence of the coefficient function on x_1, x_2 is contained in the function $\widetilde{\mathcal{T}}_{jk}$, which can be expanded in powers of $\alpha_s(\mu_{\rm S}^2)$:

$$\widetilde{\mathcal{T}}_{jk} = \widetilde{\mathcal{T}}_{jk}^{(0)} + \frac{\alpha_s(\mu_{\rm S}^2)}{4\pi} \widetilde{\mathcal{T}}_{jk}^{(1)} + \mathcal{O}\left(\alpha_s^2(\mu_{\rm S}^2)\right). \tag{6.33}$$

Using eq. (6.25), the coefficients $\widetilde{\mathcal{T}}_{jk}^{(0,1)}$ read

$$\begin{aligned}
\widetilde{\mathcal{T}}_{jk}^{(0)}(x_1,x_2,M^2,\mu_{\rm S}^2) &= \left(\frac{M^2}{\mu_{\rm S}^2}\right)^{\eta_B^j}(1-x_1)^{\eta_B^j-1}\delta_{jk}\delta(1-x_2) \\
\widetilde{\mathcal{T}}_{jk}^{(1)}(x_1,x_2,M^2,\mu_{\rm S}^2) &= \left(\frac{M^2}{\mu_{\rm S}^2}\right)^{\eta_B^j}(1-x_1)^{\eta_B^j-1}\Bigg\{\Gamma_0^j\delta_{jk}\delta(1-x_2)\Bigg[\left(V_1+V_0\ln\frac{M^2}{\mu_{\rm S}^2}+\frac{1}{2}\ln^2\frac{M^2}{\mu_{\rm S}^2}\right) \\
&\quad+\left(V_0+\ln\frac{M^2}{\mu_{\rm S}^2}\right)\ln(1-x_1)+\frac{1}{2}\ln^2(1-x_1)\Bigg] \\
&\quad+\left[4P_{jk}^{(0)}(x_2)-\frac{\gamma_{B,0}^j}{2}\delta_{jk}\delta(1-x_2)\right]\left(V_0+\ln\frac{M^2}{\mu_{\rm S}^2}+\ln(1-x_1)\right)+\tilde{I}_{jk}^{(1)}(x_2)\Bigg\},
\end{aligned} \tag{6.34}$$

where η_B^j is defined in eq. (6.14), V_0 and V_1 are given in eqs. (6.26), (6.27), and Γ_0^j are given in eq. (6.23). We can now perform the Mellin transform in the soft limit using eq. (A.5) and its derivatives with respect to η. We find

$$\widetilde{\mathcal{T}}_{jk}^{(0)}(N_1,N_2,M^2,\mu_{\rm S}^2) = \delta_{jk}\left(\frac{M^2}{\mu_{\rm S}^2 N_1}\right)^{\eta_B^j} \tag{6.35}$$

$$\begin{aligned}
\widetilde{\mathcal{T}}_{jk}^{(1)}(N_1,N_2,M^2,\mu_{\rm S}^2) &= \left(\frac{M^2}{\mu_{\rm S}^2 N_1}\right)^{\eta_B^j}\Bigg\{\Gamma_0^j\delta_{jk}\left(\frac{1}{2}\ln^2\frac{M^2}{\mu_{\rm S}^2\bar{N}_1}+\frac{\pi^2}{12}\right) \\
&\quad+\left[4\gamma_{jk}^{(0)}(N_2)-\frac{\gamma_{B,0}^j}{2}\delta_{jk}\right]\ln\frac{M^2}{\mu_{\rm S}^2\bar{N}_1}+\tilde{I}_{jk}^{(1)}(N_2)\Bigg\},
\end{aligned} \tag{6.36}$$

where $\bar{N}_i = e^{\gamma_{\rm E}}N_i$, and $\gamma_{jk}^{(0)}(N_2)$ is the Mellin transform of $P_{jk}^{(0)}(z)$.

The Mellin-transformed resummed coefficient function in eq. (6.32) in the single-soft limit thus takes the form

$$\begin{aligned}
&C_{ik}^{\rm scet}\left(N_1,N_2,\frac{M^2}{\mu^2},\frac{M^2}{\mu_{\rm S}^2},\frac{M^2}{\mu_{\rm H}^2},\alpha_s(\mu_{\rm H}^2)\right) \\
&= \frac{1}{M^2\sigma_{ik}^0}\sum_j H_{ij}(M^2,\mu_{\rm H}^2)\,U_H^j(M^2,\mu_{\rm H}^2,\mu^2)\exp\left[4K_\Gamma^j(\mu_{\rm S}^2,\mu^2)+K_{\gamma_B^j}(\mu_{\rm S}^2,\mu^2)\right] \\
&\quad\times\left(\frac{M^2}{\mu_{\rm S}^2\bar{N}_1}\right)^{\eta_B^j}\tilde{s}^{jk}(N_1,N_2,\mu_{\rm S}^2)+\mathcal{O}\left(\frac{1}{N_1}\right),
\end{aligned} \tag{6.37}$$

where we have used the explicit expression eq. (6.13) for the evolution kernels K_B^j. The N_2 dependence is collected in the function

$$\begin{aligned}
\tilde{s}^{jk}(N_1,N_2,\mu_{\rm S}^2) &= \delta_{jk}+\frac{\alpha_s(\mu_{\rm S}^2)}{4\pi}\Bigg\{\Gamma_0^j\delta_{jk}\left(\frac{1}{2}\ln^2\frac{M^2}{\mu_{\rm S}^2\bar{N}_1\bar{N}_2}+\frac{\pi^2}{12}\right) \\
&\quad+\left[4\gamma_{jk}^{(0)}(N_2)-\frac{\gamma_{B,0}^j}{2}\delta_{jk}+4C_j\delta_{jk}\ln\bar{N}_2\right]\ln\frac{M^2}{\mu_{\rm S}^2\bar{N}_1\bar{N}_2}+4F_{jk}^{\rm scet}(N_2)\Bigg\}+\mathcal{O}\left(\alpha_s^2(\mu_{\rm S}^2)\right),
\end{aligned} \tag{6.38}$$

where

$$4F_{jk}^{\rm scet}(N_2) = \tilde{I}_{jk}^{(1)}(N_2)-2C_j\delta_{jk}\ln^2\bar{N}_2+\left[4\gamma_{jk}^{(0)}(N_2)-\frac{\gamma_{B,0}^j}{2}\delta_{jk}+4C_j\delta_{jk}\ln\bar{N}_2\right]\ln\bar{N}_2. \tag{6.39}$$

Note that from the definition of $\tilde{I}^{(1)}_{jk}(N_2)$, given in eq. (S51) of ref. [14], it follows that

$$\lim_{N_2\to\infty}\left[\tilde{I}^{(1)}_{jk}(N_2) - 2C_j\delta_{jk}\ln^2\bar{N}_2\right] = 0. \tag{6.40}$$

We get to the final Mellin-space resummed expression by trading the beam anomalous dimension for a contribution to the anomalous dimension that governs the evolution of the PDF, and also expressing the result in terms of PDFs that are all evaluated at the factorization scale μ^2. The first step can be accomplished by noting that (see appendix F.2)

$$\gamma^j_B(\alpha_s) = -2\gamma^j_H(\alpha_s) - 2\gamma^j_\phi(\alpha_s)\,, \tag{6.41}$$

where γ^j_ϕ is defined in eqs. (F.16)–(F.17). The evolution kernels K^j_Γ (Sudakov exponents) that originate from the hard and beam function can then be combined. Specifically, using the relation

$$K^j_\Gamma(\mu^2_{\rm S},\mu^2) - K^j_\Gamma(\mu^2_{\rm H},\mu^2) = -K^j_\Gamma(\mu^2_{\rm H},\mu^2_{\rm S}) + \eta^j_\Gamma(\mu^2_{\rm S},\mu^2)\ln\frac{\mu^2_{\rm H}}{\mu^2_{\rm S}}, \tag{6.42}$$

we find

$$\begin{aligned}&U^j_H(M^2,\mu^2_{\rm H},\mu^2)\,\exp\Big\{4K^j_\Gamma(\mu^2_{\rm S},\mu^2) + K_{\gamma^j_B}(\mu^2_{\rm S},\mu^2)\Big\}\left(\frac{M^2}{\mu^2_{\rm S}}\right)^{\eta^j_B}\\ &= \exp\Big\{2K_{\gamma^j_\phi}(\mu^2_{\rm S},\mu^2)\Big\}\,U_j(M^2,\mu^2_{\rm H},\mu^2_{\rm S},\mu^2),\end{aligned} \tag{6.43}$$

where

$$U_j(M^2,\mu^2_{\rm H},\mu^2_{\rm S},\mu^2) = \left(\frac{M^2}{\mu^2_{\rm H}}\right)^{-\eta^j_B}\exp\Big\{-4K^j_\Gamma(\mu^2_{\rm H},\mu^2_{\rm S}) + 2K_{\gamma^j_H}(\mu^2_{\rm H},\mu^2_{\rm S}) - 4K_{\gamma^j_\phi}(\mu^2_{\rm S},\mu^2)\Big\}, \tag{6.44}$$

and $K_{\gamma^j_\phi}$ is defined as in eq. (6.10) with $\gamma = \gamma^j_\phi$.

Substituting eq. (6.43) in eq. (6.37), we obtain

$$\begin{aligned}&C^{\rm scet}_{ik}\left(N_1,N_2,\frac{M^2}{\mu^2},\frac{M^2}{\mu^2_{\rm S}},\frac{M^2}{\mu^2_{\rm H}},\alpha_s(\mu^2_{\rm H})\right)\\ &= \frac{1}{M^2\sigma^0_{ik}}\sum_j H_{ij}(M^2,\mu^2_{\rm H})\exp\Big\{2K_{\gamma^j_\phi}(\mu^2_{\rm S},\mu^2)\Big\}U_j(M^2,\mu^2_{\rm H},\mu^2_{\rm S},\mu^2)\,\bar{N}_1^{-\eta^j_B}\,\tilde{s}^{jk}(N_1,N_2,\mu^2_{\rm S}).\end{aligned} \tag{6.45}$$

Finally, the PDF $f_k(N_2,\mu^2_{\rm S})$ can be evolved to the factorization scale μ^2 through standard perturbative evolution

$$f_k(N_2,\mu^2_{\rm S}) = \sum_l U^{\rm pdf}_{kl}(N_2,\mu^2,\mu^2_{\rm S})\,f_l(N_2,\mu^2), \tag{6.46}$$

thereby leading to an expression of the SCET coefficient function which is especially suited for the comparison with the dQCD result:

$$\begin{aligned}C^{\rm scet}_{ik}\left(N_1,N_2,\frac{M^2}{\mu^2},\frac{M^2}{\mu^2_{\rm S}},\frac{M^2}{\mu^2_{\rm H}},\alpha_s(\mu^2_{\rm H})\right) &= \frac{1}{M^2\sigma^0_{ik}}\sum_{jl} H_{ij}(M^2,\mu^2_{\rm H})U^{\rm pdf}_{lk}(N_2,\mu^2,\mu^2_{\rm S})\\ \times\exp\Big\{2K_{\gamma^j_\phi}(\mu^2_{\rm S},\mu^2)\Big\}U_j(M^2,\mu^2_{\rm H},\mu^2_{\rm S},\mu^2)\,\bar{N}_1^{-\eta^j_B}\,\tilde{s}^{jl}(N_1,N_2,\mu^2_{\rm S}).&\end{aligned} \tag{6.47}$$

6.3 Comparison to the dQCD result

We finally turn to the comparison of the SCET resummed result eq. (6.47) with the dQCD result first derived in this paper. As discussed in refs. [23, 24], agreement between SCET and dQCD resummation is obtained by setting the SCET soft scale $\mu_{\rm S}^2$ equal to the soft scale of QCD resummation. In order to facilitate the comparison, we choose the scale $\bar{\Lambda}_{\rm ds}^2$ eq. (4.17) that we had chosen as a common scale for both the single-soft and double soft result in section 5.3:

$$\mu_{\rm S}^2 = \bar{\Lambda}_{\rm ds}^2 \equiv \frac{M^2}{\bar{N}_1 \bar{N}_2}\,. \tag{6.48}$$

Furthermore, we fix the hard scale and the factorization scale as

$$\mu_{\rm H}^2 = \mu^2 = M^2. \tag{6.49}$$

With this choice of the hard scale, the evolution factor U_H eq. (6.6) equals 1, and eq. (6.47) becomes

$$\begin{aligned} C_{ik}^{\rm scet}\left(N_1,N_2,1,\frac{M^2}{\mu_{\rm S}^2},1,\alpha_s(M^2)\right) &= \frac{1}{M^2\sigma_{ik}^0}\sum_{jl} H_{ij}(M^2,M^2)U_{lk}^{\rm pdf}(N_2,M^2,\mu_{\rm S}^2) \\ &\times \exp\left\{2K_{\gamma_\phi^j}(\mu_{\rm S}^2,M^2)\right\}U_j(M^2,\mu_{\rm S})\bar{N}_1^{-\eta_B^j}\tilde{s}_{\rm DY}^{jl}(N_1,N_2,\mu_{\rm S}^2), \end{aligned} \tag{6.50}$$

where

$$U_j(M^2,\mu_{\rm S}^2) = \exp\left\{-4K_\Gamma^j(M^2,\mu_{\rm S}^2) + 2K_{\gamma_H^j}(M^2,\mu_{\rm S}^2) + 4K_{\gamma_\phi^j}(M^2,\mu_{\rm S}^2)\right\}. \tag{6.51}$$

We now note that $K_\Gamma^j(M^2,\mu_{\rm S}^2)$ eq. (6.8) is defined in terms of the cusp anomalous dimension $\Gamma_{\rm cusp}^j(\alpha)$, which coincides with the function $A^j(\alpha_s)$ which appears in the dQCD resummation formula eq. (4.24). Furthermore, the combination of anomalous dimensions that enters the kernels $K_{\gamma_H^j}$ and $K_{\gamma_\phi^j}$ can be arranged as (see appendix F.2)

$$\gamma_W^j(\alpha_s) = \gamma_H^j(\alpha_s) + 2\gamma_\phi^j(\alpha_s) = \sum_{n=0}^\infty \gamma_{W,n}^j\left(\frac{\alpha_s}{4\pi}\right)^{n+1}. \tag{6.52}$$

We note that the coefficient $\gamma_{W,0}^j$ vanishes, so that the expansion of γ_W^j starts at order α_s^2. The coefficient $\gamma_{W,1}^j$ is given in eq. (F.20). Thus

$$U_j(M^2,\mu_{\rm S}^2) = \exp\left\{\int_{M^2}^{\mu_{\rm S}^2}\frac{dk^2}{k^2}\left[A^j(\alpha_s(k^2))\,\ln\frac{M^2}{k^2} + \frac{\gamma_{W,1}^j}{16}\frac{\alpha_s^2(k^2)}{\pi^2}\right]\right\}. \tag{6.53}$$

The remaining evolution factors in eq. (6.50), using the definitions eq. (6.10) and eq. (6.14), are

$$\bar{N}_1^{-\eta_B^j} = \exp\left\{-\int_{M^2}^{\mu_{\rm S}^2}\frac{dk^2}{k^2}\,A^j(\alpha_s(k^2))\ln\bar{N}_1\right\} \tag{6.54}$$

$$\exp\left\{2K_{\gamma_\phi^j}(\mu_{\rm S}^2,\mu^2)\right\} = \exp\left\{-\int_{M^2}^{\mu_{\rm S}^2}\frac{dk^2}{k^2}\,\gamma_\phi^j(\alpha_s(k^2))\right\}. \tag{6.55}$$

Combining these three evolution factors eqs. (6.53), (6.54), (6.55) we obtain the full soft evolution function

$$
\begin{aligned}
&\exp\left\{2K_{\gamma_\phi^j}(\mu_{\rm S}^2,\mu^2)\right\} U_j(M^2,\mu_{\rm S}^2)\,\bar N_1^{-\eta_B^j}\\
&=\exp\left\{\int_{M^2}^{\mu_{\rm S}^2}\frac{dk^2}{k^2}\left[A^j(\alpha_s(k^2))\,\ln\frac{M^2/\bar N_1}{k^2}-\gamma_\phi^j(\alpha_s(k^2))+\frac{\gamma_{W,1}^j}{16}\frac{\alpha_s^2(k^2)}{\pi^2}\right]\right\}.
\end{aligned}
\tag{6.56}
$$

Furthermore, with the soft scale choice eq. (6.48) the matching function is

$$
\begin{aligned}
\tilde s_{\rm DY}^{jl}(N_1,N_2,\mu_{\rm S}^2)&=\delta_{jl}+\frac{\alpha_s(\mu_{\rm S}^2)}{4\pi}\left[\delta_{jl}\Gamma_0^j\frac{\pi^2}{12}+4F_{jl}^{\rm scet}(N_2)\right]\\
&=\exp\left\{\frac{\alpha_s(\mu_{\rm S}^2)}{\pi}\left[\delta_{jl}\frac{C_j\pi^2}{12}+F_{jl}^{\rm scet}(N_2)\right]\right\},
\end{aligned}
\tag{6.57}
$$

where we have used $\Gamma_0^j=4C_j$. Expanding $\alpha_s(\mu_{\rm S}^2)$ in powers of $\alpha_s(M^2)$ through

$$
\alpha_s(\mu_{\rm S}^2)=\alpha_s(M^2)-\beta_0\int_{M^2}^{\mu_{\rm S}^2}\frac{dk^2}{k^2}\alpha_s^2(k^2)
\tag{6.58}
$$

this can be rewritten as

$$
\begin{aligned}
\tilde s_{\rm DY}^{jl}(N_1,N_2,M^2/\bar N_1)=&\left[\delta_{jl}+\frac{\alpha_s(M^2)}{\pi}\left(\delta_{jl}\frac{C_j\pi^2}{12}+F_{jl}^{\rm scet}(N_2)\right)\right]\\
&\times\exp\left\{-\int_{M^2}^{\mu_{\rm S}^2}\frac{dk^2}{k^2}\left(\delta_{jl}\frac{C_j\beta_0\pi^3}{12}+\pi\beta_0F_{jl}^{\rm scet}(N_2)\right)\frac{\alpha_s^2(k^2)}{\pi^2}\right\}.
\end{aligned}
\tag{6.59}
$$

Collecting the above results, we finally obtain

$$
\begin{aligned}
C_{ik}^{\rm scet}\left(N_1,N_2,1,\frac{M^2}{\mu_{\rm S}^2},1,\alpha_s(M^2)\right)=&\frac{1}{M^2\sigma_{ik}^0}\sum_{jl}\hat H_{ij,l}(M^2)U_{lk}^{\rm pdf}(N_2,M^2,\mu_{\rm S}^2)\\
&\times\exp\left\{\int_{M^2}^{\mu_{\rm S}^2}\frac{dk^2}{k^2}\left\{\delta_{jl}A^j(\alpha_s(k^2))\ln\frac{M^2/\bar N_1}{k^2}-\delta_{jl}\gamma_\phi^j(\alpha_s(k^2))\right.\right.\\
&\left.\left.+\left[\delta_{jl}\left(\frac{\gamma_{W,1}^j}{16}-\frac{C_j\beta_0\pi^3}{12}\right)-\pi\beta_0F_{jl}^{\rm scet}(N_2)\right]\frac{\alpha_s^2(k^2)}{\pi^2}\right\}\right\}
\end{aligned}
\tag{6.60}
$$

where

$$
\hat H_{ij,l}(M^2)=H_{ij}(M^2,M^2)\left[\delta_{jl}+\frac{\alpha_s(M^2)}{\pi}\left(\delta_{jl}\frac{C_j\pi^2}{12}+F_{jl}^{\rm scet}(N_2)\right)\right].
\tag{6.61}
$$

Equation (6.60) can be compared with the dQCD result eq. (4.24) in any parton channel. In section 5.3 we have presented results for the resummation of the quark nonsinglet channel of the $C_{q\bar q}$ coefficient function for the Drell-Yan process, hence we now specialize to this case. To this purpose we make the following observations.

i) For the Drell-Yan process $C_{q\bar q}$ coefficient function eq. (6.61) becomes [12, 14]

$$
\hat H_{ij,l}(M^2)=\delta_{jl}(\delta_{iq}\delta_{j\bar q}+\delta_{i\bar q}\delta_{jq})\sigma_{ij}^0\left[1+\frac{\alpha_s(M^2)}{\pi}\left(H_1+\frac{C_{\rm F}\pi^2}{12}+F_{jl}^{\rm scet}(N_2)\right)\right],
\tag{6.62}
$$

where

$$H_1 = C_{\rm F}\left(\frac{7\zeta_2}{2} - 4\right). \tag{6.63}$$

ii) The exponential in the second and third lines of eq. (6.60) is diagonal in the indices j, l except for the term $F^{\rm scet}_{jl}(N_2)$. However, parton j is fixed to be a quark or an antiquark from point i), and in the non-singlet channel l is also a quark or antiquark, with

$$\tilde{I}^{(1)}_{q_j q_l}(N_2) = \tilde{I}^{(1)}_{\bar{q}_j \bar{q}_l}(N_2) = \delta_{jk}\tilde{I}^{(1)}_{qq}(N_2) \tag{6.64}$$

$$\gamma^{(0)}_{q_j q_l}(N_2) = \gamma^{(0)}_{\bar{q}_j \bar{q}_l}(N_2) = \delta_{jl}\gamma^{(0)}_{qq}(N_2). \tag{6.65}$$

iii) In the non-singlet channel, the evolution function $U^{\rm pdf}_{lk}$ in eq. (6.60) is diagonal in the indices l, k. Therefore, combining observations i) and ii), $j = k = l$ and $U^{\rm pdf}_{lk}$ becomes

$$U^{\rm pdf}_{lk}(N_2, M^2, \mu_{\rm S}^2) = \delta_{lk} \exp\left\{\int_{M^2}^{\mu_{\rm S}^2} \frac{dk^2}{k^2}\left(\frac{\alpha_s(k^2)}{\pi}\gamma^{(0)}_{qq}(N_2) + \frac{\alpha_s^2(k^2)}{\pi^2}\,\gamma^{(1)}_{qq}(N_2)\right)\right\}, \tag{6.66}$$

where $\gamma^{(0)}_{qq}(N_2)$ and $\gamma^{(1)}_{qq}(N_2)$ were introduced in eq. (5.62).

iv) The anomalous dimension $\gamma^q_\phi(\alpha_s)$ appearing in eq. (6.60) and whose coefficients are given in eqs. (F.16), (F.17) is defined by decomposing the quark-quark anomalous dimension as in eq. (5.63) and identifying

$$\gamma^q_\phi(\alpha_s) = \frac{\alpha_s}{\pi}\gamma^{(0,0)}_{qq} + \left(\frac{\alpha_s}{\pi}\right)^2 \gamma^{(1,0)}_{qq}. \tag{6.67}$$

Combining these observations, we can write eq. (6.60) as

$$\begin{aligned} C_{q\bar{q}}\left(N_1, N_2, 1, \frac{M^2}{\mu_{\rm S}^2}, 1, \alpha_s(M^2)\right) &= g_0^{\rm scet}(\alpha_s(M^2), N_2) \\ \times \exp\left\{\int_{M^2}^{M^2/(\bar{N}_1\bar{N}_2)} \frac{dk^2}{k^2}\left\{A^q(\alpha_s(k^2))\,\ln\frac{M^2/(\bar{N}_1\bar{N}_2)}{k^2} + \frac{\alpha_s(k^2)}{\pi}D_1^{\rm scet} + \frac{\alpha_s^2(k^2)}{\pi^2}D_2^{\rm scet}\right\}\right.&, \end{aligned} \tag{6.68}$$

where

$$g_0^{\rm scet}(\alpha_s(M^2), N_2) = 1 + \frac{\alpha_s(M^2)}{\pi}\left(H_1 + \frac{C_{\rm F}\pi^2}{12} + F^{\rm scet}_{qq}(N_2)\right), \tag{6.69}$$

$$D_1^{\rm scet} = \hat{\gamma}^{(0)}_{qq}(N_2), \tag{6.70}$$

$$D_2^{\rm scet} = \left(\frac{\gamma^q_{W,1}}{16} - \frac{C_{\rm F}\beta_0\pi^3}{12}\right) - \pi\beta_0 F^{\rm scet}_{qq}(N_2) + \hat{\gamma}^{(1)}_{qq}(N_2). \tag{6.71}$$

In eq. (6.69), H_1 is given in eq. (6.63), while $F^{\rm scet}_{qq}(N_2)$ is defined eq. (6.39) in terms of the function $\tilde{I}^{(1)}_{jk}(N_2)$ of ref. [14] and the beam and standard anomalous dimensions. Combining these definitions we get

$$F^{\rm scet}_{qq}(N_2) = F(N_2), \tag{6.72}$$

where $= F(N_2)$ was given in eq. (5.58), and using the explicit expression of H_1 we also verify that

$$H_1 + \frac{C_{\rm F}\pi^2}{12} = g_{01}^{\rm ds} \tag{6.73}$$

which then implies

$$g_0^{\rm scet}(\alpha_s(M^2), N_2) = 1 + \frac{\alpha_s(M^2)}{\pi} g_{01}^{\rm ds}. \tag{6.74}$$

Equations (6.73) and (6.74) verify that the constants agree between SCET and dQCD respectively in the double-soft and single-soft limit, up to $O(\alpha_s)$; the $O(\alpha_s^2)$ single-soft constant (see eq. (D.3) is not given in ref. [14].

In eq. (6.70) $\hat{\gamma}_{qq}^{(0)}(N_2)$ is the leading-order subtracted anomalous dimension eq. (5.59). Comparing to the dQCD expression eq. (5.56) verifies the agreement of the single-soft D function up to $O(\alpha_s)$. Finally, using the expression of $\gamma_{W,1}^q$ of eq. (F.20) it is easy to check that

$$\frac{\gamma_{W,1}^q}{16} - \frac{C_{\rm F}\beta_0\pi^3}{12} = D_2^{\rm ds}, \tag{6.75}$$

where $D_2^{\rm ds}$ was given in eq. (5.47). The contribution $\hat{\gamma}_{qq}^{(0)}(N_2)$ to eq. (6.71) is the next-leading-order subtracted anomalous dimension eq. (5.60), so it follows that the $O(\alpha_s^2)$ contribution to the SCET D-function also agrees with its dQCD counterpart of eq. (5.57). We conclude that we find full agreement between the SCET result eq. (6.68) and our result in the single soft limit eqs. (5.56)–(5.60).

7 Conclusion

In this paper we have derived a resummed result for rapidity distributions of a colorless final state object (such as a gauge or Higgs boson) in the limit in which the center-of-mass energy tends to the minimum value which is needed in order to impart to the final state gauge boson a fixed value of the rapidity in the center-of-mass frame of the partonic collision. This result is complementary to our previous [17] resummation of transverse momentum distributions, in which instead the gauge boson transverse momentum was kept fixed in the limit.

In comparison to that case, rapidity distributions are simpler in that the resummation is characterized by a single soft and a single hard scale while for transverse momentum distributions there are two. On the other hand, it is more subtle because the parton distributions do not depend on transverse momentum but they do depend on rapidity, hence the resummation dynamics gets tangled with parton evolution. In fact, resummation of rapidity distributions in this limit turns out to be collinear, and it is consequently characterized by the conventional anomalous dimensions, now however evolving parton distributions up to a suitable soft scale.

A byproduct of the work presented here is the derivation of a set of relations between partonic cross-sections expressed in the variables that are most naturally used for phase space parametrization, and thus for fixed order calculations, and those that are needed for QCD factorization and resummation, namely the scaling variables on which PDFs depend. Another significant byproduct is the relation of dQCD and SCET approaches to resummation, which we have pursued in the past [23, 24], and that was needed here in order to compare our results to previous approaches to the same problem using SCET techniques.

Our results are a first proof of concept, in that we have only presented full explicit expressions for one particular partonic subchannel and evolution eigenstate. The extension to the complete set of parton states using the methods presented here, which is required for full phenomenology, is in principle straghtforward, but in practice laborious, and will be left for future work. A natural spinoff of our results is also the application of these techniques to the resummation of semi-inclusive deep-inelastic scattering (SIDIS), a process that has attracted considerable attention in recent years [42–45] and whose resummation can be attacked with closely related methods [46–49]. This will also be the focus of forthcoming work.

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A Mellin transforms in the soft limit

The argument presented in section 3 implies that in the soft limit the coefficient function only depends on the scale M^2 and the scaling variable $\Lambda^2_{\rm ds} = M^2(1-x_1)(1-x_2)$ eq. (3.9), where in the single-soft limit $1-x_2$ is a fixed finite value, while in the double-soft limit both x_1 and x_2 are close to 1. This implies that in the soft limit the bare coefficient function has the form

$$C^{(0)}(x_1, x_2, M^2, \alpha_0, \epsilon) = \sum_{n=0}^{\infty} \alpha_0^n C_n^{(0)}(x_1, x_2, M^2, \epsilon) \tag{A.1}$$

where

$$C_n^{(0)}(x_1,x_2,M^2,\epsilon) = (M^2)^{-n\epsilon}\left[\delta(1-x_1)\delta(1-x_2) + \sum_{p=1}^{n} C_{np}^{(0)}(\epsilon)(1-x_1)^{-1+p\epsilon}(1-x_2)^{-1+p\epsilon}\right]. \tag{A.2}$$

The Mellin transform of this quantity is divergent due to soft and collinear singularities, and indeed the renormalization group argument of section 3 is presented for a dimensionally regularized coefficient function.

We now show that the Mellin transform of $C_n^{(0)}(x_1, x_2, M^2, \epsilon)$

$$C_n^{(0)}(N_1, N_2, M^2, \epsilon) = \int_0^1 dx_1\, x_1^{N_1-1} \int_0^1 dx_2\, x_2^{N_2-1} C_n^{(0)}(x_1, x_2, M^2, \epsilon) \tag{A.3}$$

is a function of the product $N_1 N_2$ up to terms that vanish as $N_1, N_2 \to \infty$. Indeed

$$\begin{aligned}
&\int_0^1 dx_1\, x_1^{N_1-1} \int_0^1 dx_2\, x_2^{N_2-1} (1-x_1)^{-1+p\epsilon}(1-x_2)^{-1+p\epsilon} \\
&\quad = \frac{\Gamma(N_1)\Gamma(p\epsilon)}{\Gamma(N_1+p\epsilon)} \frac{\Gamma(N_2)\Gamma(p\epsilon)}{\Gamma(N_2+p\epsilon)} \\
&\quad = \Gamma^2(p\epsilon)(N_1 N_2)^{-p\epsilon} + \mathcal{O}\left(\frac{1}{N_1}\right) + \mathcal{O}\left(\frac{1}{N_2}\right),
\end{aligned} \tag{A.4}$$

where we have used the well-known identity

$$\int_0^1 dx\, x^{N-1}(1-x)^{\eta-1} = \frac{\Gamma(N)\Gamma(\eta)}{\Gamma(N+\eta)} = \frac{\Gamma(\eta)}{N^\eta}\left[1 + \mathcal{O}(1/N)\right]. \tag{A.5}$$

Hence, in the limit $N_1 \to \infty, N_2 \to \infty$

$$C_n^{(0)}(N_1, N_2, M^2, \epsilon) = (M^2)^{-n\epsilon}\left[1 + \sum_{p=1}^{n} C_{np}^{(0)}(\epsilon)\Gamma^2(p\epsilon)(N_1 N_2)^{-p\epsilon}\right] + \mathcal{O}\left(\frac{1}{N_1}\right) + \mathcal{O}\left(\frac{1}{N_2}\right) \tag{A.6}$$

which is a function of $N_1 N_2$, thus establishing eq. (3.13).

B Alternate forms of the resummed coefficient function

The form of the resummed result in the inclusive case in the original paper ref. [1] (and also e.g. in ref. [30]) was given in a somewhat different form, whose relation to the form given in eq. (4.10) is discussed e.g. in ref. [19] eq. (3.19) (see also ref. [29] eqs. (110)–(111)), namely

$$\begin{aligned}
C(N,1,\alpha_s(M^2)) = {} & C^{(c)}\left(\alpha_s(M^2)\right) \\
& \times \exp\left[\int_0^1 dx \frac{x^{N-1}-1}{1-x}\left(\int_{M^2}^{(1-x)^2 M^2} \frac{dk^2}{k^2} A(\alpha_s(k^2))\right) + \bar{D}(\alpha_s((1-x)^2 M^2))\right],
\end{aligned} \tag{B.1}$$

where the function $\bar{D}$ is a series in α_s whose coefficients are determined order by order in terms of the coefficients of the expansions of A and D.

The double-soft resummed result, also given in ref. [1], was also written similarly up to NLL (where no D function appears for Drell-Yan) and then extended to higher logarithmic accuracy in ref. [7]:

$$\ln C^{(l)}\left(\frac{M^2}{\mu^2 N_1 N_2}, \alpha_s(\mu^2)\right) = I_1 + I_2 + J_1 + J_2 \tag{B.2}$$

$$I_1 = \int_0^1 dx_1 \int_0^1 dx_2 \frac{(x_1^{N_1-1}-1)(x_2^{N_2-1}-1)}{(1-x_1)(1-x_2)} A(\alpha_s(M^2(1-x_1)(1-x_2))) \tag{B.3}$$

$$I_2 = \int_0^1 dx \frac{(x^{N_1-1}-1)+(x^{N_2-1}-1)}{1-x} \int_{\mu^2}^{M^2(1-x)} \frac{dk^2}{k^2} A(\alpha_s(k^2)) \tag{B.4}$$

$$J_1 = \int_0^1 dx_1 \int_0^1 dx_2 \frac{(x_1^{N_1-1}-1)(x_2^{N_2-1}-1)}{(1-x_1)(1-x_2)} \frac{d}{d\ln M^2} D(\alpha_s(M^2(1-x_1)(1-x_2))) \tag{B.5}$$

$$J_2 = \int_0^1 dx \frac{(x^{N_1-1}-1)+(x^{N_2-1}-1)}{1-x} D(\alpha_s(M^2(1-x))). \tag{B.6}$$

Proving the equivalence of this to the double-soft resummation as given in eq. (4.16) is nontrivial because of the double-plus distribution appearing in I_1 eq. (B.3), which requires a double subtraction in order to be finite, as this equation shows. However, the interference of the subtraction in the Mellin transform with respect to x_1 with the Mellin with respect to x_2 and conversely generates extra contributions that are compensated by the terms in eq. (B.4), as we will now show.

Specifically, we prove that, up to terms that are either constant or power-suppressed in the large N_1, N_2 limit,

$$I_1 + I_2 + J_1 + J_2 = \int_1^{N_1 N_2} \frac{dn}{n} \left[\left(-\int_{\mu^2 n}^{M^2} \frac{dk^2}{k^2} A(\alpha_s(k^2/n))\right) + D^{\rm ds}(\alpha_s(M^2/n))\right]. \tag{B.7}$$

We also provide the relationship between the functions $D^{\rm ds}(\alpha_s)$ in this equation and $D(\alpha_s)$ in eq. (B.2).

The proof relies on the same identity used to prove the equivalence of the inclusive resummation formulae eq. (B.1) and eq. (4.10), namely

$$\int_0^1 dx \frac{x^{N-1}-1}{1-x} f(\ln(1-x)) = -\sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(1)}{k!} \frac{d^k}{dL^k} \int_0^{1-1/N} \frac{dx}{1-x} f(\ln(1-x)) + \mathcal{O}\left(\frac{1}{N}\right), \tag{B.8}$$

where $L = \ln \frac{1}{N}$, for any function $f(z)$ with a Taylor expansion in its argument. The leading log, next-to-leading log, . . . approximations are achieved by including terms up to $k = 0, 1, \ldots$ in the sum. Using eq. (B.8) we find, in the large N_1, N_2 limit,

$$I_1 = \sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(1)}{k!} \frac{d^k}{dL_1^k} \sum_{j=0}^{\infty} \frac{\Gamma^{(j)}(1)}{j!} \frac{d^j}{dL_2^j} G(L_1, L_2) \tag{B.9}$$

$$I_2 = -\sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(1)}{k!} \frac{d^k}{dL_1^k} H(L_1) - \sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(1)}{k!} \frac{d^k}{dL_2^k} H(L_2) \tag{B.10}$$

where

$$L_1 = \ln \frac{1}{N_1}; \qquad L_2 = \ln \frac{1}{N_2} \tag{B.11}$$

and

$$G(L_1,L_2)=\int_0^{1-\frac{1}{N_1}}\frac{dx_1}{1-x_1}\int_0^{1-\frac{1}{N_2}}\frac{dx_2}{1-x_2}A(\alpha_s(M^2(1-x_1)(1-x_2))) \tag{B.12}$$

$$H(L_1)=\int_0^{1-\frac{1}{N_1}}\frac{dx}{1-x}\int_{\mu^2}^{M^2(1-x)}\frac{dk^2}{k^2}A(\alpha_s(k^2)). \tag{B.13}$$

As a first step, we show that

$$G(L_1,L_2)=H(L_1)+H(L_2)-H(L), \tag{B.14}$$

where

$$L=L_1+L_2=\ln\frac{1}{N_1N_2}. \tag{B.15}$$

To this purpose, we trade the integration variable x_2 in $G(L_1,L_2)$ for

$$k^2=M^2(1-x_1)(1-x_2);\qquad \frac{M^2(1-x_1)}{N_2}\le k^2\le M^2(1-x_1). \tag{B.16}$$

We find

$$G(L_1,L_2)=\int_0^{1-\frac{1}{N_1}}\frac{dx_1}{1-x_1}\int_{\frac{M^2(1-x_1)}{N_2}}^{M^2(1-x_1)}\frac{dk^2}{k^2}A(\alpha_s(k^2)). \tag{B.17}$$

We may now split the k^2 integration range at $k^2=\mu^2$, in order to isolate a term which depends only on N_1 and coincides with $H(L_1)$:

$$\begin{aligned}G(L_1,L_2)&=\int_0^{1-\frac{1}{N_1}}\frac{dx_1}{1-x_1}\int_{\frac{M^2(1-x_1)}{N_2}}^{\mu^2}\frac{dk^2}{k^2}A(\alpha_s(k^2))+\int_0^{1-\frac{1}{N_1}}\frac{dx_1}{1-x_1}\int_{\mu^2}^{M^2(1-x_1)}\frac{dk^2}{k^2}A(\alpha_s(k^2))\\&=-\int_0^{1-\frac{1}{N_1}}\frac{dx_1}{1-x_1}\int_{\mu^2}^{\frac{M^2(1-x_1)}{N_2}}\frac{dk^2}{k^2}A(\alpha_s(k^2))+H(L_1).\end{aligned} \tag{B.18}$$

By rescaling $\frac{1-x_1}{N_2}=1-x$ in the first term, we can further isolate a N_2 dependent term, equal to $H(L_2)$, and a N_1N_2 dependent one:

$$\begin{aligned}G(L_1,L_2)&=-\int_{1-\frac{1}{N_2}}^{1-\frac{1}{N_1N_2}}\frac{dx}{1-x}\int_{\mu^2}^{M^2(1-x)}\frac{dk^2}{k^2}A(\alpha_s(k^2))+H(L_1)\\&=-\int_0^{1-\frac{1}{N_1N_2}}\frac{dx}{1-x}\int_{\mu^2}^{M^2(1-x)}\frac{dk^2}{k^2}A(\alpha_s(k^2))+\int_0^{1-\frac{1}{N_2}}\frac{dx}{1-x}\int_{\mu^2}^{M^2(1-x)}\frac{dk^2}{k^2}A(\alpha_s(k^2))\\&\quad+H(L_1)\\&=-H(L)+H(L_2)+H(L_1),\end{aligned} \tag{B.19}$$

as promised.

Next, we show that the contribution to I_1 from the term $H(L_1)+H(L_2)$ is exactly canceled by I_2. Indeed

$$\sum_{k=0}^{\infty}\frac{\Gamma^{(k)}(1)}{k!}\frac{d^k}{dL_1^k}\sum_{j=0}^{\infty}\frac{\Gamma^{(j)}(1)}{j!}\frac{d^j}{dL_2^j}[H(L_1)+H(L_2)]=\sum_{k=0}^{\infty}\frac{\Gamma^{(k)}(1)}{k!}\frac{d^kH(L_1)}{dL_1^k}+\sum_{j=0}^{\infty}\frac{\Gamma^{(j)}(1)}{j!}\frac{d^jH(L_2)}{dL_2^j} \tag{B.20}$$

which is precisely $-I_2$, as one can check by inspection of eq. (B.10). Therefore

$$I_1 + I_2 = -\sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(1)}{k!} \sum_{j=0}^{\infty} \frac{\Gamma^{(j)}(1)}{j!} \frac{d^k}{dL_1^k} \frac{d^j}{dL_2^j} H(L), \tag{B.21}$$

where

$$H(L) = \int_0^{1-\frac{1}{N_1 N_2}} \frac{dx}{1-x} \int_{\mu^2}^{M^2(1-x)} \frac{dk^2}{k^2} A(\alpha_s(k^2)) = \int_1^{N_1 N_2} \frac{dn}{n} \int_{\mu^2 n}^{M^2} \frac{dk^2}{k^2} A(\alpha_s(k^2/n)). \tag{B.22}$$

Now, because $L = L_1 + L_2$, eq. (B.21) can be rewritten

$$I_1 + I_2 = -\sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(1)}{k!} \sum_{j=0}^{\infty} \frac{\Gamma^{(j)}(1)}{j!} \frac{d^{k+j}}{dL^{k+j}} H(L) = -\sum_{m=0}^{\infty} \frac{\gamma_m}{m!} \frac{d^m H(L)}{dL^m} \tag{B.23}$$

where

$$\gamma_m = \sum_{k=0}^{m} \binom{m}{k} \Gamma^{(k)}(1) \Gamma^{(m-k)}(1) = \left. \frac{d^m \Gamma^2(z)}{dz^m} \right|_{z=1}. \tag{B.24}$$

In order to show that eq. (B.23) can be written in the form of eq. (B.7) we first separate off the term $m = 1$ in the sum, $-H(L)$, which is equal to the first term in eq. (B.7):

$$I_1 + I_2 = -\int_1^{N_1 N_2} \frac{dn}{n} \int_{\mu^2 n}^{M^2} \frac{dk^2}{k^2} A(\alpha_s(k^2/n)) + R\left(\frac{M^2}{\mu^2 N_1 N_2}, \alpha_s(\mu^2)\right) \tag{B.25}$$

with

$$R\left(\frac{M^2}{\mu^2 N_1 N_2}, \alpha_s(\mu^2)\right) = -\sum_{m=1}^{\infty} \frac{\gamma_m}{m!} \frac{d^m}{dL^m} \int_1^{N_1 N_2} \frac{dn}{n} \int_{\mu^2 n}^{M^2} \frac{dk^2}{k^2} A(\alpha_s(k^2/n)). \tag{B.26}$$

The function R is in fact μ^2-independent, up to constant terms in $N_1 N_2$. Indeed

$$\frac{d}{d\ln\mu^2} R\left(\frac{M^2}{\mu^2 N_1 N_2}, \alpha_s(\mu^2)\right) = \sum_{m=1}^{\infty} \frac{\gamma_m}{m!} \frac{d^m}{dL^m} \int_1^{N_1 N_2} \frac{dn}{n} A(\alpha_s(\mu^2)) = -2\gamma_{\mathrm{E}} A(\alpha_s(\mu^2)) \tag{B.27}$$

which is a constant in N_1, N_2, and can be neglected. We may therefore compute R for any choice of μ^2, for example $\mu^2 = M^2$:

$$R\left(\frac{1}{N_1 N_2}, \alpha_s(M^2)\right) = -\sum_{m=1}^{\infty} \frac{\gamma_m}{m!} \frac{d^m}{dL^m} \int_1^{N_1 N_2} \frac{dn}{n} \int_{M^2 n}^{M^2} \frac{dk^2}{k^2} A(\alpha_s(k^2/n)). \tag{B.28}$$

It is now useful to define new integration variables

$$t = \ln\frac{1}{n}; \qquad t' = \ln\frac{k^2}{M^2 n} \tag{B.29}$$

so that, for $\mu^2 = M^2$,

$$H(L) = \int_1^{N_1 N_2} \frac{dn}{n} \int_{M^2 n}^{M^2} \frac{dk^2}{k^2} A(\alpha_s(k^2/n)) = -\int_0^L dt \int_0^t dt' \, \mathcal{A}(t'), \tag{B.30}$$

where

$$\mathcal{A}(t') = A(\alpha_s(k^2/n)) = A(\alpha_s(M^2 e^{t'})). \tag{B.31}$$

We now show that for $m \geq 1$

$$\frac{d^m H(L)}{dL^m} = -\int_0^L dt\, \mathcal{A}^{(m-1)}(t) \tag{B.32}$$

up to terms that are either constant or vanishing in the large N_1, N_2 limit. This is obviously true for $m = 1$: differentiating eq. (B.30) with respect to L we get

$$\frac{dH(L)}{dL} = -\int_0^L dt'\, \mathcal{A}(t'). \tag{B.33}$$

Now assume that eq. (B.32) holds for a given value of m. Then

$$\frac{d^{m+1} H(L)}{dL^{m+1}} = -\mathcal{A}^{(m-1)}(L) = -\int_0^L dt\, \mathcal{A}^{(m)}(t) - \mathcal{A}^{(m-1)}(0) \tag{B.34}$$

and the last term can be dropped in the large N_1, N_2 limit. This completes the proof of eq. (B.32) for all $m \geq 1$.

We may now substitute eq. (B.32) in eq. (B.28), obtaining

$$\begin{aligned} R\left(\frac{1}{N_1 N_2}, \alpha_s(M^2)\right) &= -\sum_{m=1}^{\infty} \frac{\gamma_m}{m!} \frac{d^m H(L)}{dL^m} \\ &= \sum_{m=1}^{\infty} \frac{\gamma_m}{m!} \int_0^L dt\, \mathcal{A}^{(m-1)}(t) \\ &= -\int_1^{N_1 N_2} \frac{dn}{n} \sum_{m=1}^{\infty} \frac{\gamma_m}{m!} \left(\frac{d}{d\ln M^2}\right)^{(m-1)} A\left(\alpha_s(M^2/n)\right). \end{aligned} \tag{B.35}$$

So

$$I_1 + I_2 = \int_1^{N_1 N_2} \frac{dn}{n} \left[\left(-\int_{\mu^2 n}^{M^2} \frac{dk^2}{k^2} A(\alpha_s(k^2/n))\right) + D_A(\alpha_s(M^2/n))\right] \tag{B.36}$$

with

$$D_A(\alpha_s(M^2)) = -\sum_{m=1}^{\infty} \frac{\gamma_m}{m!} \left(\frac{d}{d\ln M^2}\right)^{(m-1)} A\left(\alpha_s(M^2)\right). \tag{B.37}$$

We now note that

$$J_1 + J_2 = \frac{d}{d\ln M^2}(I_1 + I_2)\Big|_{A\to D}, \tag{B.38}$$

as one can check by inspection of eqs. (B.3)–(B.6). Hence

$$J_1 + J_2 = \int_1^{N_1 N_2} \frac{dn}{n} \left[-D(\alpha_s(M^2/n)) + \frac{d}{d\ln M^2} D(\alpha_s(M^2/n))\right], \tag{B.39}$$

and therefore, combining eqs. (B.36) and (B.39).

$$I_1 + I_2 + J_1 + J_2 = \int_1^{N_1 N_2} \frac{dn}{n} \left[\left(-\int_{\mu^2 n}^{M^2} \frac{dk^2}{k^2} A(\alpha_s(k^2/n))\right) + D^{\rm ds}(\alpha_s(M^2/n))\right], \tag{B.40}$$

as we set out to prove. Also, as announced, we have found the relation between the function D appearing in the resummation eq. (B.2) and the function $D^{\rm ds}$ appearing in our resummed result eq. (4.16), namely

$$D^{\rm ds}(\alpha_s(M^2)) = D_A(\alpha_s(M^2)) - D(\alpha_s(M^2)) + \frac{d}{d\ln M^2} D(\alpha_s(M^2)). \tag{B.41}$$

C The variable u

Fixed-order differential cross sections are expressed in terms of the scaling variables $\tau = x_1 x_2$ and u, eq. (5.13). We show that u is simply related to the scattering angle θ of the Z with respect to the incident beam direction in the partonic center-of-mass frame. In this frame the Z momentum can be parametrized as

$$p = (E, 0, |\vec{p}| \sin\theta, |\vec{p}| \cos\theta); \qquad E = \sqrt{|\vec{p}|^2 + M^2}. \tag{C.1}$$

Hence

$$e^{2y} = \frac{E + |\vec{p}| \cos\theta}{E - |\vec{p}| \cos\theta}. \tag{C.2}$$

Using the definition eq. (5.13) of u we get

$$\begin{aligned} u &= \frac{1}{1-\tau} \frac{1 - \tau e^{2y}}{1 + e^{2y}} \\ &= \frac{1}{1-\tau} \frac{E(1-\tau) - |\vec{p}|(1+\tau)\cos\theta}{2E} \\ &= \frac{1}{2}\left[1 - \frac{|\vec{p}|}{E} \frac{1+\tau}{1-\tau} \cos\theta\right]. \end{aligned} \tag{C.3}$$

In the limit of vanishing invariant mass m_X^2 of emitted radiation, i.e. when emitted partons are collinear, the energy conservation relation

$$\sqrt{s} = |\vec{p}| + \sqrt{|\vec{p}|^2 + M^2} \tag{C.4}$$

can be solved for $|\vec{p}|$ to give

$$|\vec{p}| = \frac{s - M^2}{2M}; \qquad E = \frac{s + M^2}{2M}; \qquad \frac{|\vec{p}|}{E} = \frac{s - M^2}{s - M^2} = \frac{1-\tau}{1+\tau}. \tag{C.5}$$

Therefore

$$u = \frac{1 - \cos\theta}{2} \tag{C.6}$$

in this limit. This implieas that the collinear limits $\theta = 0$ and $\theta = \pi$ correspond respectively to $u = 0$ and $u = 1$.

D Single-soft constant coefficients

We give here the explicit expression of some lengthy coefficients that contribute to the soft limit as constants, i.e. such that they do not vanish but are also not logarithmically enhanced as $x_1 \to 1$ or $N_1 \to \infty$

D.1 Fixed-order direct space

We give the expression of $\bar{\kappa}(\tau)$ that enters the expression of the contribution $\bar{C}^{(2)}_{\text{Boost}}(\tau)$ eq. (5.29) to the NNLO coefficient function in (τ, u) space and $\kappa(x_2)$ that enters the expression of the contribution $C^{(2)}_{\text{Boost}}(x_2)$ eq. (5.37) to the NNLO coefficient function in (x_1, x_2) space.

We have

$$
\begin{aligned}
\bar{\kappa}(\tau) =\; & \frac{(16+30\tau^2)}{9}\frac{\mathrm{Li}_3\left(\frac{2\tau}{1+\tau}\right)-\zeta_3}{1-\tau} - \frac{2(1+\tau^2)}{9}\frac{\mathrm{Li}_3\left(\frac{1+\tau}{2}\right)-\zeta_3}{1-\tau} - \frac{4(1+\tau^2)}{3}\frac{\mathrm{Li}_3(\tau)-\zeta_3}{1-\tau} \\
& + \frac{10(1+\tau^2)}{9}\frac{\mathrm{Li}_3(-\tau)+\frac{3}{4}\zeta_3}{1-\tau} + \left(\frac{1+\tau^2}{9}N_f - \frac{8+27\tau-22\tau^2+22\tau^3-2\tau^4}{9\tau}\right)\frac{\mathrm{Li}_2\left(\frac{1+\tau}{2}\right)-\frac{\pi^2}{6}}{1-\tau} \\
& - \left(\frac{2-14\tau+11\tau^2-9\tau^3-2\tau^4}{9\tau} + \frac{(19-9\tau^2)}{9}\ln(1-\tau)\right)\frac{\mathrm{Li}_2(\tau)-\frac{\pi^2}{6}}{1-\tau} \\
& + \left(-\frac{16+14\tau-19\tau^2+\tau^3}{9\tau} + \frac{14}{9}(1+\tau)(1-\tau)\ln(1-\tau)\right)\frac{\mathrm{Li}_2(-\tau)+\frac{\pi^2}{12}}{1-\tau} + \frac{(1+9\tau^2)}{9}\frac{\mathrm{Li}_3\left(\frac{1-\tau}{1+\tau}\right)}{1-\tau} \\
& + \frac{5(1+\tau^2)}{18}\frac{\mathrm{Li}_3\left(1-\tau^2\right)}{1-\tau} + \frac{1+\tau}{3}\mathrm{Li}_3\left(\frac{2\tau}{1+\tau}\right) - \frac{8}{9}\frac{\mathrm{Li}_3\left(\frac{1-\tau}{2}\right)}{1-\tau} - \frac{25\left(1+\tau^2\right)}{9}\frac{\mathrm{Li}_3(1-\tau)}{1-\tau} \\
& + \left(\frac{(1+\tau)(1-\tau)(16+5\tau+4\tau^2)}{18\tau} + \frac{8(1+\tau^2)}{9}\ln\tau - (1+\tau^2)\ln\frac{1+\tau}{2}\right)\frac{\mathrm{Li}_2\left(\frac{1+\tau}{2}\right)}{1-\tau} \\
& + \left(\frac{2+3\tau+6\tau^2+2\tau^3}{9\tau} + \frac{2}{3}(1+\tau)\ln(1-\tau) + \frac{4}{9}(1-3\tau^2)\frac{\ln\tau}{1-\tau} + \frac{3+11\tau^2}{9}\frac{\ln\frac{1+\tau}{2}}{1-\tau}\right)\mathrm{Li}_2(\tau) \\
& - \left(\frac{-12-\frac{16}{\tau}-3\tau}{9} + \frac{2}{3}(1+\tau)\ln(1-\tau) + \frac{1-15\tau^2}{9}\frac{\ln\tau}{1-\tau} + \frac{3+11\tau^2}{9}\frac{\ln\frac{1+\tau}{2}}{1-\tau}\right)\mathrm{Li}_2(-\tau) \\
& - \frac{17+25\tau^2}{27}\frac{\ln^3\frac{1+\tau}{2}}{1-\tau} \\
& + \left(-\frac{1+\tau^2}{18}N_f + \frac{11}{12}(1+\tau^2) - \frac{1-7\tau^2}{18}\ln\left(\frac{1-\tau}{2}\right)\right)\frac{\ln^2\frac{1+\tau}{2}}{1-\tau} \\
& \Bigg[\left(-\frac{4(1+\tau^2)}{9}\ln 2 - \frac{40(1+\tau^2)}{9}\ln(1-\tau) + \frac{4}{9}\left(\tau^2-2\tau-2\right) + \frac{4}{3}\left(1+\tau^2\right)\ln(1+\tau)\right)\ln\tau \\
& + \frac{1}{9}\left(8\tau^2-31\tau+13\right) + \frac{1}{108}\left(11\tau^2+19\right)\pi^2 + \frac{1}{18}\left(49\tau^2-32\tau+49\right)\ln 2 \\
& + \left(-\frac{2}{9}(1-\tau) - \frac{1}{9}\left(\tau^2+1\right)\ln 2\right)N_f + \left(\frac{1}{9}\left(1+\tau^2\right)N_f - \frac{1}{18}\left(49\tau^2-32\tau+49\right)\right)\ln(1-\tau) \\
& + \frac{32(1+\tau^2)}{9}\ln^2(1-\tau) + \frac{17(1+\tau^2)}{18}\ln^2\tau\Bigg]\frac{\ln\left(\frac{1+\tau}{2}\right)}{1-\tau} \\
& - \frac{(107+211\tau^2)}{108}\frac{\ln^3(\tau)}{1-\tau} + \Bigg(\frac{7(1+\tau^2)}{36}N_f + \frac{1+\tau^2}{18}\ln 2 + \frac{16+23\tau^2}{3}\ln(1-\tau) \\
& - \frac{1}{72}\left(267\tau^2-76\tau+151\right) + \frac{1}{6}\left(43\tau^2+35\right)\ln(1-\tau) - \frac{2}{3}\left(1-\tau^2\right)\ln(1+\tau)\Bigg)\frac{\ln^2\tau}{1-\tau} \\
& + \Bigg(-\frac{1}{18}\left(69\tau^2-82\tau+43\right) - \frac{1}{36}\left(1-15\tau^2\right)\pi^2 - \frac{2}{9}\left(2\tau^2-5\tau+9\right)\ln 2 \\
& + \frac{1}{18}\left(7\tau^2-4\tau+7\right)N_f - \frac{83}{9}(1+\tau^2)\ln^2(1-\tau) \\
& - \left(\frac{4}{9}\left(\tau^2+1\right)N_f + \frac{1}{9}\left(-73\tau^2+16\tau-63\right) - \frac{8}{9}\left(1-\tau^2\right)\ln(1+\tau)\right)\ln(1-\tau)\Bigg)\frac{\ln\tau}{1-\tau} \\
& - \frac{32}{9}(1+\tau)\ln^3(1-\tau) + \left(\frac{11}{3} - \frac{2}{9}N_f\right)(1+\tau)\ln^2(1-\tau)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{4}{27}(4\tau+1)N_f-\frac{7\tau}{2}-\frac{7}{54}(1+\tau)\pi^2+\frac{28}{9}\right)\ln(1-\tau)+\frac{2}{27}(82\tau+19)\\
&+\left(\frac{1+\tau}{54}\pi^2+\frac{-19\tau-37}{162}\right)N_f-\frac{\left(8\tau^3+106\tau^2-7\tau+4\right)}{108\tau}\pi^2-\frac{101}{18}(\tau+1)\zeta_3
\end{aligned}
\tag{D.1}
$$

$$
\begin{aligned}
\kappa(x_2)=&\frac{4\left(8+15x_2^2\right)}{9}\frac{\mathrm{Li}_3\left(\frac{2x_2}{1+x_2}\right)-\zeta_3}{1-x_2}-\frac{4\left(1+x_2^2\right)}{9}\frac{\mathrm{Li}_3\left(\frac{1+x_2}{2}\right)-\zeta_3}{1-x_2}\\
&-\frac{8(1+x_2^2)}{3}\frac{\mathrm{Li}_3(x_2)-\zeta_3}{1-x_2}+\frac{20\left(1+x_2^2\right)}{9}\frac{\mathrm{Li}_3(-x_2)+\frac{3}{4}\zeta_3}{1-x_2}\\
&+\left(\frac{2(1+x_2^2)}{9}N_f-\frac{16+54x_2-44x_2^2+44x_2^3-4x_2^4}{9x_2}\right)\frac{\mathrm{Li}_2\left(\frac{1+x_2}{2}\right)-\frac{\pi^2}{6}}{1-x_2}\\
&+\left(\frac{2}{9}\left(9x_2^2-19\right)\ln(1-x_2)+\frac{2\left(2x_2^4+9x_2^3-11x_2^2+14x_2-2\right)}{9x_2}\right)\frac{\mathrm{Li}_2(x_2)-\frac{\pi^2}{6}}{1-x_2}\\
&+\left(\frac{28}{9}(1-x_2)(x_2+1)\ln(1-x_2)-\frac{2\left(x_2^3-19x_2^2+14x_2+16\right)}{9x_2}\right)\frac{\mathrm{Li}_2(-x_2)+\frac{\pi^2}{12}}{1-x_2}\\
&-\frac{16}{9}\frac{\mathrm{Li}_3\left(\frac{1-x_2}{2}\right)}{1-x_2}+\frac{2}{3}(1+x_2)\mathrm{Li}_3\left(\frac{2x_2}{1+x_2}\right)-\frac{50}{9}\left(1+x_2^2\right)\frac{\mathrm{Li}_3(1-x_2)}{1-x_2}\\
&+\frac{5}{9}\left(1+x_2^2\right)\frac{\mathrm{Li}_3\left(1-x_2^2\right)}{1-x_2}+\frac{2}{9}\frac{\left(1+9x_2^2\right)\mathrm{Li}_3\left(\frac{1-x_2}{1+x_2}\right)}{1-x_2}\\
&+\left(\frac{7}{3}+x_2+\frac{4x_2^2}{9}+\frac{16}{9x_2}-2\left(1+x_2^2\right)\frac{\ln\left(\frac{1+x_2}{2}\right)}{1-x_2}+\frac{16}{9}\left(1+x_2^2\right)\frac{\ln(x_2)}{1-x_2}\right)\mathrm{Li}_2\left(\frac{1+x_2}{2}\right)\\
&+\left(\frac{2\left(2x_2^3+6x_2^2+3x_2+2\right)}{9x_2}+\frac{28}{9}x_2^2\frac{\ln\left(\frac{1+x_2}{2}\right)}{1-x_2}+\frac{2}{3}(1+x_2)\ln(1-x_2)+\frac{2}{9}(7-15x_2^2)\frac{\ln(x_2)}{1-x_2}\right)\mathrm{Li}_2(x_2)\\
&+\left(\frac{2}{9}\left(3x_2+\frac{16}{x_2}+12\right)-\frac{28}{9}x_2^2\frac{\ln\left(\frac{1+x_2}{2}\right)}{(1-x_2)}-\frac{2}{3}(1+x_2)\ln(1-x_2)+\frac{4}{9}\left(9x_2^2-2\right)\frac{\ln(x_2)}{1-x_2}\right)\mathrm{Li}_2(-x_2)\\
&+\Bigg(-\frac{1}{9}N_f\left(x_2^2+1\right)+\frac{-2x_2^4+11x_2^3-15x_2^2+21x_2+18}{9x_2}-\frac{1}{3}(1+x_2)(1-x_2)\ln(x_2)\\
&+\frac{1}{3}(x_2+1)(1-x_2)\ln\left(\frac{1-x_2}{2}\right)+\frac{2}{9}\left(5x_2^2-2\right)\ln(1-x_2)+\frac{1}{9}\left(4-10x_2^2\right)\ln(2)\Bigg)\frac{\ln^2\left(\frac{1+x_2}{2}\right)}{1-x_2}\\
&+\Bigg(\frac{\pi^2}{27}\left(x_2^2+14\right)-\frac{-44x_2^4+62x_2^3+120x_2^2+44x_2-50}{27x_2}+\frac{64}{9}\left(x_2^2+1\right)\ln^2(1-x_2)+\frac{1}{9}\left(20x_2^2+14\right)\ln^2(x_2)\\
&+\frac{\left(147x_2^3-96x_2^2+147x_2+80\right)}{27x_2}\ln(2)+N_f\left(-\frac{4}{9}(1-x_2)-\frac{16}{27}\left(x_2^2+1\right)\ln(2)\right)\\
&+\left(\frac{2}{9}N_f\left(1+x_2^2\right)+\frac{4x_2^4-56x_2^3+8x_2^2-46x_2+24}{9x_2}\right)\ln(1-x_2)+\Bigg(-\frac{8}{9}\left(x_2^2+1\right)\ln(2)\\
&-\frac{80}{9}\left(x_2^2+1\right)\ln(1-x_2)-\frac{8x_2^4-7x_2^3-14x_2^2+17x_2-6\left(3x_2^2+5\right)x_2\ln(x_2+1)+20}{9x_2}\Bigg)\ln(x_2)\Bigg)\frac{\ln\left(\frac{1+x_2}{2}\right)}{1-x_2}\\
&-\frac{\left(247x_2^2+71\right)}{54}\frac{\ln^3(x_2)}{1-x_2}\\
&+\Bigg(-\frac{1}{18}(1-x_2)\left(7N_f\left(x_2^2+1\right)\right)+\frac{40x_2^4-241x_2^3+16x_2^2-165x_2+8}{36x_2}-2(1-x_2)(1+x_2)\ln(1+x_2)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{9}\left(x_2^2+1\right)\ln(2)+\frac{1}{3}\left(38x_2^2+40\right)\ln(1-x_2)\Bigg)\frac{\ln^2(x_2)}{1-x_2}\\
&+\Bigg(\frac{1}{9}N_f\left(7x_2^2-4x_2+7\right)-\frac{\pi^2}{9}\left(2-9x_2^2\right)-\frac{44x_2^4+97x_2^3-180x_2^2+7x_2+50}{27x_2}-\frac{1}{9}\left(169x_2^2+163\right)\ln^2(1-x_2)\\
&-\frac{4}{9}\left(2x_2^2-5x_2+9\right)\ln(2)+\ln(1-x_2)\Bigg(-\frac{8}{9}N_f\left(x_2^2+1\right)-\frac{16x_2^4-141x_2^3+8x_2^2-131x_2+8}{9x_2}\\
&+\frac{22}{9}(1-x_2)(1+x_2)\ln(1+x_2)\Bigg)\Bigg)\frac{\ln(x_2)}{1-x_2}-\frac{64}{9}(1+x_2)\ln^3(1-x_2)\\
&+\Bigg(\frac{-4x_2^3+41x_2^2+47x_2+4}{6x_2}-\frac{4}{9}N_f(x_2+1)\Bigg)\ln^2(1-x_2)\\
&+\Bigg(\frac{8}{27}N_f(4x_2+1)-\frac{5}{54}\pi^2(1+x_2)+\frac{-44x_2^3-123x_2^2+168x_2+50}{27x_2}\Bigg)\ln(1-x_2)\\
&+N_f\left(\frac{\pi^2}{27}(x_2+1)-\frac{19x_2+37}{81}\right)-\frac{101}{9}(x_2+1)\zeta_3+\frac{4}{27}(82x_2+19)-\frac{\left(8x_2^3+106x_2^2-7x_2+4\right)}{54x_2}\pi^2
\end{aligned}
\tag{D.2}
$$

D.2 Resummed Mellin space

We give the expression of $g_{02}^{\rm ss}(N_2)$, i.e. the NNLO constant eq. (5.11) in the resummed expression in the single-soft limit.

$$
\begin{aligned}
g_{02}^{\rm ss}(N_2)=&\frac{20}{9}\mathrm{Li}_4(1/2)+\frac{17}{54}\ln^4 2+\frac{341}{3240}\pi^4-\frac{17}{54}\pi^2\ln^2 2+\frac{65}{36}\zeta_3\ln 2\\
&+\frac{\left(66N_2^2+\left(66-34(-1)^{N_2}\right)N_2-17(-1)^{N_2}-7\right)}{54N_2(N_2+1)}\ln^3 2\\
&-\frac{\left(26N_2^4+\left(52+66(-1)^{N_2}\right)N_2^3+\left(42+95(-1)^{N_2}\right)N_2^2+\left(20+13(-1)^{N_2}\right)N_2-10(-1)^{N_2}+2\right)}{36N_2^2(N_2+1)^2}\ln^2 2\\
&+\frac{(-1)^{N_2}\left(26N_2^5+22N_2^4+10N_2^3+41N_2^2+31N_2+10\right)}{18N_2^3(N_2+1)^3}\ln 2+\frac{2}{9}\ln^4(\bar{N}_2)-\frac{1}{108}S(-1,N_2)^4\\
&+\frac{2}{9}S(1,N_2)^4+\left(\frac{(-1)^{N_2}(2N_2+1)}{54N_2(N_2+1)}-\frac{\ln 2}{27}\right)S(-1,N_2)^3+\frac{\left(11N_2^2+11N_2-8\right)}{18N_2(N_2+1)}S(1,N_2)^3\\
&+\left(\frac{\ln^2 2}{18}-\frac{5(-1)^{N_2}(2N_2+1)\ln 2}{9N_2(N_2+1)}+\frac{6N_2^4+12N_2^3-11N_2^2+31N_2+32}{36N_2^2(N_2+1)^2}+\frac{17\pi^2}{36}\right)S(1,N_2)^2\\
&-\frac{17}{9}S(-2,N_2)^2+\left(\frac{\ln 2}{9}-\frac{5(-1)^{N_2}(2N_2+1)}{9N_2(N_2+1)}\right)S(-1,N_2)S(1,N_2)^2-\frac{3}{4}S(2,N_2)^2-\frac{53}{18}S(-1,1,N_2)^2\\
&+\Bigg(\frac{-9N_2^2+\left(-9+74(-1)^{N_2}\right)N_2+37(-1)^{N_2}+9}{18N_2(N_2+1)}\ln 2\\
&+\frac{404N_2^4+808N_2^3+\left(493-2(-1)^{N_2}\right)N_2^2+\left(97-6(-1)^{N_2}\right)N_2-2\left(-5+(-1)^{N_2}\right)}{36N_2^2(N_2+1)^2}\Bigg)\frac{\pi^2}{6}\\
&+\frac{766N_2^2+\left(766+24(-1)^{N_2}\right)N_2+12(-1)^{N_2}+67}{72N_2(N_2+1)}\zeta_3\\
&+\left(\frac{31(-1)^{N_2}(2N_2+1)}{27N_2(N_2+1)}-\frac{68}{27}\ln 2\right)S(-3,N_2)\\
&+\left(-\frac{11}{9}\ln^2 2+\frac{\left(7N_2^2+7N_2-3\right)}{3N_2(N_2+1)}\ln 2+\frac{\pi^2}{54}-\frac{(-1)^{N_2}\left(66N_2^3+97N_2^2+19N_2-8\right)}{18N_2^2(N_2+1)^2}\right)S(-2,N_2)\\
&+\Bigg(\frac{17}{27}\ln^3 2+\frac{\left(33N_2^2+\left(33+2(-1)^{N_2}\right)N_2+(-1)^{N_2}+2\right)}{18N_2(N_2+1)}\ln^2 2
\end{aligned}
$$

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$$
\begin{aligned}
&-\frac{\left(13N_2^4+26N_2^3+21N_2^2+10N_2+1\right)}{9N_2^2(N_2+1)^2}\ln 2+\frac{(-1)^{N_2}\left(26N_2^5+22N_2^4+10N_2^3+41N_2^2+31N_2+10\right)}{18N_2^3(N_2+1)^3}\\
&+\left(\frac{37(-1)^{N_2}(2N_2+1)}{18N_2(N_2+1)}-\frac{37}{9}\ln 2\right)\frac{\pi^2}{6}-\frac{\zeta_3}{3}\bigg)S(-1,N_2)-\frac{101}{27}S(-3,N_2)S(-1,N_2)\\
&+\frac{\left(11N_2^2+11N_2-3\right)}{6N_2(N_2+1)}S(-2,N_2)S(-1,N_2)+\left(\frac{7}{27}\ln^3 2-\frac{\left(36(-1)^{N_2}N_2+18(-1)^{N_2}+1\right)}{18N_2(N_2+1)}\ln^2 2\right.\\
&-\frac{7(-1)^{N_2}\left(2N_2^2+2N_2+1\right)}{9N_2^2(N_2+1)^2}\ln 2+\frac{404N_2^5+793N_2^4+227N_2^3-66N_2^2-27N_2-30}{54N_2^3(N_2+1)^2}\\
&+\left(\frac{33N_2^2+\left(33+4(-1)^{N_2}\right)N_2+2(-1)^{N_2}-51}{18N_2(N_2+1)}-\ln 2\right)\frac{\pi^2}{6}-\frac{67}{36}\zeta_3\bigg)S(1,N_2)\\
&+\left(2\ln 2-\frac{16(-1)^{N_2}(2N_2+1)}{9N_2(N_2+1)}\right)S(-2,N_2)S(1,N_2)+S(-2,N_2)S(-1,N_2)S(1,N_2)\\
&+\left(-\frac{2\ln^2 2}{9}-\frac{\ln 2}{9N_2^2+9N_2}-\frac{7(-1)^{N_2}\left(2N_2^2+2N_2+1\right)}{9N_2^2(N_2+1)^2}\right)S(-1,N_2)S(1,N_2)\\
&+\left(\frac{-11N_2^2-11N_2+8}{18N_2^2+18N_2}-\frac{8}{9}S(1,N_2)\right)\ln^3\bar{N}_2\\
&+\left(-\frac{4}{9}\ln^2 2+\frac{4(-1)^{N_2}(2N_2+1)}{9N_2(N_2+1)}\ln 2-\frac{4}{9}S(-1,N_2)^2+\frac{4}{3}S(1,N_2)^2\right.\\
&+\frac{6N_2^4+12N_2^3-11N_2^2-17N_2+8}{36N_2^2(N_2+1)^2}+\frac{23}{54}\pi^2+\left(\frac{4(-1)^{N_2}(2N_2+1)}{9N_2(N_2+1)}-\frac{8}{9}\ln 2\right)S(-1,N_2)\\
&\left.+\frac{\left(11N_2^2+11N_2-8\right)}{6N_2(N_2+1)}S(1,N_2)\right)\ln^2\bar{N}_2\\
&+\left(-\frac{8}{9}S(1,N_2)^3+\frac{\left(-11N_2^2-11N_2+8\right)}{6N_2^2+6N_2}S(1,N_2)^2+\frac{8}{9}S(-1,N_2)^2S(1,N_2)\right.\\
&+\left(\frac{8}{9}\ln^2 2-\frac{8(-1)^{N_2}(2N_2+1)}{9N_2(N_2+1)}\ln 2-\frac{31}{27}\pi^2-\frac{6N_2^4+12N_2^3-11N_2^2+15N_2+24}{18N_2^2(N_2+1)^2}\right)S(1,N_2)\\
&+\left(\frac{16}{9}\ln 2-\frac{8(-1)^{N_2}(2N_2+1)}{9N_2(N_2+1)}\right)S(-1,N_2)S(1,N_2)+\frac{16}{9}S(2,N_2)S(1,N_2)\\
&+\frac{\left(33N_2^2+33N_2-8\right)}{18N_2(N_2+1)}S(-1,N_2)^2+\frac{33N_2^2+33N_2-8}{18N_2(N_2+1)}\ln^2 2\\
&+\frac{-404N_2^6-1212N_2^5-1050N_2^4-206N_2^3+21N_2^2+33N_2+30}{54N_2^3(N_2+1)^3}+\frac{\left(-21N_2^2-21N_2+31\right)}{9N_2^2+9N_2}\frac{\pi^2}{6}+\frac{65}{9}\zeta_3\\
&+\left(\frac{33N_2^2+33N_2-8}{9N_2(N_2+1)}\ln 2-\frac{(-1)^{N_2}\left(66N_2^3+99N_2^2+17N_2-8\right)}{18N_2^2(N_2+1)^2}\right)S(-1,N_2)\\
&\left.+\frac{\left(21N_2^2+21N_2-8\right)S(2,N_2)}{9N_2(N_2+1)}-\frac{2}{9}S(3,N_2)-\frac{(-1)^{N_2}\ln 2\left(66N_2^3+99N_2^2+17N_2-8\right)}{18N_2^2(N_2+1)^2}\right)\ln\bar{N}_2\\
&+\left(-\frac{\ln^2 2}{18}+\frac{(-1)^{N_2}(2N_2+1)}{18N_2(N_2+1)}\ln 2+\frac{1}{18}S(1,N_2)^2-\frac{37}{108}\pi^2\right.\\
&\left.-\frac{13N_2^4+26N_2^3+21N_2^2+10N_2+1}{18N_2^2(N_2+1)^2}\right)S(-1,N_2)^2-\frac{71}{18}S(-1,N_2)^2S(2,N_2)-\frac{5}{18}S(1,N_2)^2S(2,N_2)\\
&+\left(\frac{\ln^2 2}{9}-\frac{(-1)^{N_2}(2N_2+1)}{2N_2(N_2+1)}\ln 2-\frac{\pi^2}{27}-\frac{68N_2^4+136N_2^3+47N_2^2+3N_2+18}{36N_2^2(N_2+1)^2}\right)S(2,N_2)\\
&+\left(\frac{3(-1)^{N_2}(2N_2+1)}{2N_2(N_2+1)}-\frac{22}{9}\ln 2\right)S(-1,N_2)S(2,N_2)+\frac{\left(-33N_2^2-33N_2+5\right)}{18N_2^2+18N_2}S(1,N_2)S(2,N_2)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\left(10N_2^2+10N_2-19\right)}{18N_2(N_2+1)}S(3,N_2)+\frac{19}{9}S(1,N_2)S(3,N_2)-\frac{16}{9}S(4,N_2)-\frac{1}{6}S(-3,-1,N_2)\\
&+S(-2,-2,N_2)+\frac{\left(N_2^2+N_2-1\right)}{2N_2(N_2+1)}S(-2,-1,N_2)+S(1,N_2)S(-2,-1,N_2)\\
&+\left(\frac{8(-1)^{N_2}(2N_2+1)}{9N_2(N_2+1)}-\frac{8}{3}\ln 2\right)S(-2,1,N_2)-\frac{29}{9}S(-1,N_2)S(-2,1,N_2)\\
&+\left(\frac{20}{9}\ln^2 2+\frac{11}{9N_2^2+9N_2}\ln 2+\frac{(-1)^{N_2}\left(66N_2^3+95N_2^2+13N_2-10\right)}{18N_2^2(N_2+1)^2}-\frac{\pi^2}{27}\right)S(-1,1,N_2)\\
&+4S(-2,N_2)S(-1,1,N_2)+\frac{\left(-11N_2^2-11N_2+3\right)}{6N_2^2+6N_2}S(-1,N_2)S(-1,1,N_2)\\
&+\left(\frac{2(-1)^{N_2}(2N_2+1)}{N_2(N_2+1)}-\frac{22}{9}\ln 2\right)S(1,N_2)S(-1,1,N_2)-S(-1,N_2)S(1,N_2)S(-1,1,N_2)\\
&+\left(\frac{8}{3}\ln 2-\frac{2(-1)^{N_2}(2N_2+1)}{N_2(N_2+1)}\right)S(2,-1,N_2)+\frac{73}{18}S(-1,N_2)S(2,-1,N_2)\\
&+\frac{\left(-9N_2^2-9N_2+29\right)}{18N_2^2+18N_2}S(2,1,N_2)-\frac{29}{9}S(1,N_2)S(2,1,N_2)-\frac{19}{9}S(3,1,N_2)-S(-2,-1,1,N_2)\\
&+\frac{5}{9}S(-2,1,-1,N_2)+\frac{\left(33N_2^2+33N_2+13\right)}{18N_2^2+18N_2}S(-1,1,-1,N_2)-\frac{13}{9}S(1,N_2)S(-1,1,-1,N_2)\\
&+N_f\Bigg\{-\frac{2}{27}\ln^3 2+\frac{\left(10N_2^2+2\left(5+3(-1)^{N_2}\right)N_2+3(-1)^{N_2}\right)}{54N_2(N_2+1)}\ln^2 2\\
&+\frac{(-1)^{N_2}\left(-10N_2^3-6N_2^2+4N_2+3\right)}{27N_2^2(N_2+1)^2}\ln 2+\frac{\ln^3\bar{N}_2}{27}-\frac{1}{27}S(1,N_2)^3+\frac{5}{27}S(-1,N_2)^2\\
&+\frac{\left(-10N_2^2-10N_2+3\right)S(1,N_2)^2}{54N_2^2+54N_2}+\frac{1143N_2^6+3429N_2^5+3505N_2^4+1175N_2^3+4N_2^2+48N_2+36}{648N_2^3(N_2+1)^3}\\
&+\left(\frac{\ln 2}{9}+\frac{-106N_2^2-106N_2+3}{54N_2^2+54N_2}\right)\frac{\pi^2}{6}+\left(\frac{(-1)^{N_2}(2N_2+1)}{9N_2(N_2+1)}-\frac{2}{9}\ln 2\right)S(-2,N_2)\\
&+\left(-\frac{\ln^2 2}{9}+\frac{10}{27}\ln 2+\frac{(-1)^{N_2}\left(-10N_2^3-6N_2^2+4N_2+3\right)}{27N_2^2(N_2+1)^2}\right)S(-1,N_2)-\frac{1}{9}S(-2,N_2)S(-1,N_2)\\
&+\left(-\frac{28N_2^4+56N_2^3+4N_2^2-6N_2+9}{81N_2^2(N_2+1)^2}-\frac{\pi^2}{54}\right)S(1,N_2)\\
&+\left(\frac{-10N_2^2-10N_2+3}{54N_2^2+54N_2}-\frac{1}{9}S(1,N_2)\right)\ln^2(\bar{N}_2)+\frac{\left(10N_2^2+10N_2-3\right)S(2,N_2)}{54N_2(N_2+1)}+\frac{1}{9}S(1,N_2)S(2,N_2)\\
&+\Bigg(-\frac{\ln^2 2}{9}+\frac{(-1)^{N_2}(2N_2+1)}{9N_2(N_2+1)}\ln 2-\frac{1}{9}S(-1,N_2)^2+\frac{1}{9}S(1,N_2)^2+\frac{28N_2^4+56N_2^3+4N_2^2-6N_2+9}{81N_2^2(N_2+1)^2}\\
&+\frac{\pi^2}{27}+\left(\frac{(-1)^{N_2}(2N_2+1)}{9N_2(N_2+1)}-\frac{2}{9}\ln 2\right)S(-1,N_2)+\frac{\left(10N_2^2+10N_2-3\right)S(1,N_2)}{27N_2(N_2+1)}-\frac{2}{9}S(2,N_2)\Bigg)\ln\bar{N}_2\\
&-\frac{2}{27}S(3,N_2)-\frac{1}{9}S(-2,-1,N_2)+\frac{1}{9}S(-1,N_2)S(-1,1,N_2)+\frac{1}{9}S(2,1,N_2)\\
&-\frac{1}{9}S(-1,1,-1,N_2)-\frac{7}{54}\zeta_3-\frac{(-1)^{N_2}(2N_2+1)S(-1,1,N_2)}{9N_2(N_2+1)}\Bigg\}\\
&+\left(\frac{62\ln 2}{9}-\frac{2(-1)^{N_2}(2N_2+1)}{N_2(N_2+1)}\right)S(-1,1,1,N_2)+\frac{62}{9}S(-1,N_2)S(-1,1,1,N_2)+\frac{25}{18}S(-1,2,-1,N_2)\\
&+\frac{13}{3}S(2,1,1,N_2)+\frac{13}{9}S(-1,1,-1,1,N_2)-\frac{S(-1,N_2)^2S(1,N_2)}{18N_2(N_2+1)}\\
&-\frac{7683N_2^8+30732N_2^7+49082N_2^6+38508N_2^5+14631N_2^4+1928N_2^3-504N_2^2-372N_2-120}{432N_2^4(N_2+1)^4}.
\end{aligned}
\tag{D.3}
$$

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The notation $S[a_1,\ldots,a_n,N_2]$ denotes finite harmonic sums as defined in ref. [50], and viewed as analytic functions of the complex variable N_2. The numerical evaluation of the analytic continuation can be carried out using `ANCONT` [51].

E Transformation of distributions from (τ,u) to (x_1,x_2) space

As discussed in section 5.2, the fixed-order NLO and NNLO Drell-Yan cross section in refs. [25, 31] is expressed in terms of variables τ and u, which are related to the variables x_1, x_2 relevant for the resummation by eqs. (5.14)–(5.17). Hence the coefficient function must be rewritten accordingly, using eq. (5.20), which involves the jacobian of the transformation

$$j(x_1,x_2)=\left|\frac{\partial(\tau,u)}{\partial(x_1,x_2)}\right|=\frac{2x_1x_2(1+x_1x_2)}{(x_1+x_2)^2(1-x_1x_2)}, \tag{E.1}$$

which is singular when $\tau = x_1x_2 = 1$.

The coefficient function in (τ,u) space involves distributions in the soft limit $\tau\to 1$ and in the two collinear limits $u\to 0$ and $u\to 1$. In (x_1,x_2) space instead the soft limit corresponds to both $x_1\to 1$ and $x_2\to 1$, while the two collinear limits correspond to either $x_1\to 1$ or $x_2\to 1$. Transformations of distributions from (τ,u) space to (x_1,x_2) space is accordingly nontrivial.

We have to transform double distributions in τ and u, and single distributions in u localized at $u=1$ or $u=0$, while single distributions in τ are not relevant because $\tau=1$ corresponds to both $x_1=1$ and $x_2=1$ which can only be realized with $u=0$ or $u=1$. We discuss these two cases in turn.

E.1 Double distributions in τ and u

Our task is facilitated by the observation that the coefficient function in (x_1,x_2) in the double-soft limit only depends on the combination $M^2(1-x_1)(1-x_2)$, as shown in section 2.2.1. Recalling eq. (A.1)–(A.2), this is seen to imply that only distributions generated by differentiation of a generating function that depend on a power of $(1-x_1)(1-x_2)$ are relevant. The strategy that we adopt is thus to start with such a generating function, rewrite in (τ,u) space, and then work out the relations between distributions that follow from differentiation of the equation that expresses the equality of these two forms of the generating function. These will turn out to be indeed all and only the double distributions that appear in the (τ,u) space coefficient function,

We start by expressing the relevant variable in (τ,u) space:

$$(1-x_1)(1-x_2)=\frac{u(1-u)(1-\tau)^2}{G(\tau,u)}, \tag{E.2}$$

where

$$G(\tau,u)=\frac{\sqrt{\tau+u(1-u)(1-\tau)^2}\left(\sqrt{\tau+u(1-u)(1-\tau)^2}+\sqrt{\tau}\right)}{1+\tau}. \tag{E.3}$$

It follows that for a generic test function $T(x_1,x_2)$, regular in the range $0\le x_1\le 1, 0\le x_2\le 1$,

$$\begin{aligned}&\int_0^1 dx_1\int_0^1 dx_2\,T(x_1,x_2)(1-x_1)^{-1+\epsilon}(1-x_2)^{-1+\epsilon}\\ &=\int_0^1 d\tau\int_0^1 du\,t(\tau,u)\frac{J(\tau,u)G^{1-\epsilon}(\tau,u)}{1-\tau}(1-\tau)^{-1+2\epsilon}u^{-1+\epsilon}(1-u)^{-1+\epsilon}\end{aligned} \tag{E.4}$$

where $t(\tau, u) = T(x_1(\tau, u), x_2(\tau, u))$ and

$$J(\tau, u) = \left| \frac{\partial(x_1, x_2)}{\partial(\tau, u)} \right| = \frac{1 - \tau^2}{2[1 - u(1-\tau)][\tau + u(1-\tau)]} \tag{E.5}$$

is the inverse of the jacobian $j(x_1, x_2)$ eq. (E.1). Equation (E.4) provides the expression in (τ, u) space of the relevant (x_1, x_2) space generating function.

However, we wish to express the distributions in (τ, u) space that appear in the coefficient function in terms of distributions in (x_1, x_2) space, not the other way up. To this purpose, we note that, as already mentioned, when $\tau = 1$ it follows that $x_1 = x_2 = 1$, hence $t(1, u)$ is u-independent, which is why the jacobian $j(x_1, x_2)$ eq. (E.1) is singular as $\tau \to 1$. It follows that the function

$$H(\tau, u, \epsilon) \frac{J(\tau, u) G^{1-\epsilon}(\tau, u)}{1 - \tau} = \frac{(1 + x_1)(1 + x_2)}{2(1 + x_1 x_2)} \left[\frac{x_1 x_2 (1 + x_1)(1 + x_2)}{(x_1 + x_2)^2} \right]^{-\epsilon} \equiv \frac{1}{F(x_1, x_2, \epsilon)} \tag{E.6}$$

that appears on the right-hand side of eq. (E.4) has the same property, and is regular at the boundaries of the integration range. We can thus divide both sides of eq. (E.4) by $H(\tau, u, \epsilon) = \frac{1}{F(x_1, x_2, \epsilon)}$, as this just amounts to a redefinition of the test function. We thus get

$$\begin{aligned} &\int_0^1 d\tau \int_0^1 du\, t(\tau, u)(1-\tau)^{-1+2\epsilon} u^{-1+\epsilon}(1-u)^{-1+\epsilon} \\ &= \int_0^1 dx_1 \int_0^1 dx_2\, T(x_1, x_2)(1 - x_1)^{-1+\epsilon}(1 - x_2)^{-1+\epsilon} F(x_1, x_2, \epsilon). \end{aligned} \tag{E.7}$$

This is the desired relation between generating functions and it is our starting point.

We first express both sides of eq. (E.7) in terms of distributions, using the identity

$$(1 - z)^{-1+\epsilon} = \frac{\delta(1 - z)}{\epsilon} + \left[(1 - z)^{-1+\epsilon} \right]_+ . \tag{E.8}$$

We get

$$\begin{aligned} &\int_0^1 d\tau \int_0^1 du\, t(\tau, u) \left(\frac{\delta(1-\tau)}{2\epsilon} + \left[(1-\tau)^{-1+2\epsilon} \right]_+ \right) \\ &\times \left(\frac{\delta(u) + \delta(1-u)}{\epsilon} + \left[u^{-1+\epsilon} \right]_+ + \left[(1-u)^{-1+\epsilon} \right]_+ + (u(1-u))^{-1+\epsilon} - u^{-1+\epsilon} - (1-u)^{-1+\epsilon} \right) \\ &= \int_0^1 dx_1 \int_0^1 dx_2\, T(x_1, x_2) \left\{ \frac{\delta(1 - x_1)\delta(1 - x_2)}{\epsilon^2} \right. \\ &\quad + \frac{1}{\epsilon} \delta(1 - x_1) \left[(1 - x_2)^{-1+\epsilon} \right]_+ F(1, x_2, \epsilon) + \frac{1}{\epsilon} \delta(1 - x_2) \left[(1 - x_1)^{-1+\epsilon} \right]_+ F(x_1, 1, \epsilon) \\ &\quad \left. + \left[(1 - x_1)^{-1+\epsilon} \right]_+ \left[(1 - x_2)^{-1+\epsilon} \right]_+ F(x_1, x_2, \epsilon) \right\} \end{aligned} \tag{E.9}$$

Equation (E.9) can be simplified by observing that for any function or distribution $g(u)$, integrable in the range $0 \le u \le 1$, we have

$$\begin{aligned} &\int_0^1 d\tau \int_0^1 du\, \delta(1-\tau) t(\tau, u) g(u) = t(1, 0) \int_0^1 du\, g(u) \\ &= \int_0^1 dx_1 \int_0^1 dx_2\, T(x_1, x_2) \delta(1 - x_1)\delta(1 - x_2) \int_0^1 du\, g(u). \end{aligned} \tag{E.10}$$

In particular,

$$\int_0^1 d\tau \int_0^1 du\, \delta(1-\tau)t(\tau,u)\frac{\delta(u)+\delta(1-u)}{2} = \int_0^1 dx_1 \int_0^1 dx_2\, T(x_1,x_2)\delta(1-x_1)\delta(1-x_2)$$

$$\int_0^1 d\tau \int_0^1 du\, \delta(1-\tau)t(\tau,u)\left(\left[u^{-1+\epsilon}\right]_+ + \left[(1-u)^{-1+\epsilon}\right]_+\right) = 0$$

$$\begin{aligned}&\int_0^1 d\tau \int_0^1 du\, \delta(1-\tau)t(\tau,u)\left((u(1-u))^{-1+\epsilon} - u^{-1+\epsilon} - (1-u)^{-1+\epsilon}\right)\\ &\qquad = \frac{2}{\epsilon}\left[\frac{\Gamma^2(1+\epsilon)}{\Gamma(1+2\epsilon)} - 1\right]\int_0^1 dx_1 \int_0^1 dx_2\, T(x_1,x_2).\delta(1-x_1)\delta(1-x_2).\end{aligned} \tag{E.11}$$

Furthermore, the term

$$\int_0^1 d\tau \int_0^1 du\, [\delta(u)+\delta(1-u)]\, t(\tau,u)\left[(1-\tau)^{-1+\epsilon}\right]_+ \tag{E.12}$$

can easily be recast as an integral in x_1, x_2. Indeed,

$$d\tau du\, \delta(u) = dx_1 dx_2\, j(x_1,x_2)\frac{\delta(1-x_2)}{\left|\frac{\partial u(x_1,x_2)}{\partial x_2}\right|_{x_2=1}} = dx_1 dx_2 \delta(1-x_2) \tag{E.13}$$

$$d\tau du\, \delta(1-u) = dx_1 dx_2\, j(x_1,x_2)\frac{\delta(1-x_1)}{\left|\frac{\partial u(x_1,x_2)}{\partial x_1}\right|_{x_1=1}} = dx_1 dx_2 \delta(1-x_1). \tag{E.14}$$

Therefore

$$\begin{aligned}&\int_0^1 d\tau \int_0^1 du\, t(\tau,u)\, [\delta(u)+\delta(1-u)]\left[(1-\tau)^{-1+2\epsilon}\right]_+\\ &= \int_0^1 dx_1 \int_0^1 dx_2\, T(x_1,x_2)\left\{\delta(1-x_1)\left[(1-x_2)^{-1+2\epsilon}\right]_+ + \delta(1-x_2)\left[(1-x_1)^{-1+2\epsilon}\right]_+\right\}.\end{aligned} \tag{E.15}$$

Substituting eqs. (E.11), (E.15) in eq. (E.9) we find

$$\begin{aligned}&\int_0^1 d\tau \int_0^1 du\, t(\tau,u)\left[(1-\tau)^{-1+2\epsilon}\right]_+\\ &\left(\left[u^{-1+\epsilon}\right]_+ + \left[(1-u)^{-1+\epsilon}\right]_+ + (u(1-u))^{-1+\epsilon} - u^{-1+\epsilon} - (1-u)^{-1+\epsilon}\right)\\ &= \int_0^1 dx_1 \int_0^1 dx_2\, T(x_1,x_2)\left\{\frac{1}{\epsilon^2}\left[1 - \frac{\Gamma^2(1+\epsilon)}{\Gamma(1+2\epsilon)}\right]\delta(1-x_1)\delta(1-x_2)\right.\\ &\quad + \frac{1}{\epsilon}\delta(1-x_1)\left(\left[(1-x_2)^{-1+\epsilon}\right]_+ F(1,x_2,\epsilon) - \left[(1-x_2)^{-1+2\epsilon}\right]_+\right)\\ &\quad + \frac{1}{\epsilon}\delta(1-x_2)\left(\left[(1-x_1)^{-1+\epsilon}\right]_+ F(x_1,1,\epsilon) - \left[(1-x_1)^{-1+2\epsilon}\right]_+\right)\\ &\quad \left. + \left[(1-x_1)^{-1+\epsilon}\right]_+\left[(1-x_2)^{-1+\epsilon}\right]_+ F(x_1,x_2,\epsilon)\right\}\end{aligned} \tag{E.16}$$

This is the final form of the relation between generating functions that we use in order to generate the identities that we need, which follow by expanding both sides of eq. (E.16) in

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powers of ϵ around $\epsilon = 0$. The distributions that appear up to NNLO are found expanding up to order ϵ^2.

For $\epsilon = 0$ we get

$$\begin{aligned}
&\int_0^1 d\tau \int_0^1 du t(\tau,u) \left[\frac{1}{1-\tau}\right]_+ \left(\left[\frac{1}{u}\right]_+ + \left[\frac{1}{1-u}\right]_+\right) \\
&= \int_0^1 dx_1 \int_0^1 dx_2 T(x_1,x_2) \Bigg\{ \zeta_2 \delta(1-x_1)\delta(1-x_2) \\
&-\delta(1-x_1)\left(\left[\frac{\ln(1-x_2)}{1-x_2}\right]_+ - \frac{1}{1-x_2}\ln\frac{2x_2}{1+x_2}\right) - \delta(1-x_2)\left(\left[\frac{\ln(1-x_1)}{1-x_1}\right]_+ - \frac{1}{1-x_1}\ln\frac{2x_1}{1+x_1}\right) \\
&+\left[\frac{1}{1-x_1}\right]_+ \left[\frac{1}{1-x_2}\right]_+ + \frac{1}{(1+x_1)(1+x_2)} \Bigg\}.
\end{aligned} \tag{E.17}$$

This identity was already obtained in refs. [14] and [15] by different methods.

The order-ϵ term of the expansion gives

$$\begin{aligned}
&\int_0^1 d\tau \int_0^1 du\, t(\tau,u) \Bigg\{ 2\left[\frac{\ln(1-\tau)}{1-\tau}\right]_+ \left(\left[\frac{1}{u}\right]_+ + \left[\frac{1}{1-u}\right]_+\right) \\
&+ \left[\frac{1}{1-\tau}\right]_+ \left(\left[\frac{\ln u}{u}\right]_+ + \left[\frac{\ln(1-u)}{1-u}\right]_+ + \frac{\ln u}{1-u} + \frac{\ln(1-u)}{u}\right) \Bigg\} \\
&= \int_0^1 dx_1 \int_0^1 dx_2\, T(x_1,x_2) \Bigg\{ -2\zeta_3 \delta(1-x_1)\delta(1-x_2) \\
&+ \frac{1}{2}\delta(1-x_2)\left(\frac{1}{1-x_1}\ln^2\frac{2x_1}{1+x_1} + 2\frac{\ln(1-x_1)}{1-x_1}\ln\frac{2x_1}{1+x_1} - 3\left[\frac{\ln^2(1-x_1)}{1-x_1}\right]_+\right) \\
&+ \frac{1}{2}\delta(1-x_1)\left(\frac{1}{1-x_2}\ln^2\frac{2x_2}{1+x_2} + 2\frac{\ln(1-x_2)}{1-x_2}\ln\frac{2x_2}{1+x_2} - 3\left[\frac{\ln^2(1-x_2)}{1-x_2}\right]_+\right) \\
&+ \frac{2(1+x_1x_2)}{(1+x_1)(1+x_2)}\left(\left[\frac{\ln(1-x_1)}{1-x_1}\right]_+ \left[\frac{1}{1-x_2}\right]_+ + \left[\frac{1}{1-x_1}\right]_+ \left[\frac{\ln(1-x_2)}{1-x_2}\right]_+ \right. \\
&\left. + \left[\frac{1}{1-x_1}\right]_+ \left[\frac{1}{1-x_2}\right]_+ \ln\frac{x_1x_2(1+x_1)(1+x_2)}{(x_1+x_2)^2}\right) \Bigg\}.
\end{aligned} \tag{E.18}$$

The order-ϵ^2 leads to

$$\begin{aligned}
&\int_0^1 d\tau \int_0^1 du t(\tau,u) \Bigg\{ 2\left[\frac{\ln^2(1-\tau)}{1-\tau}\right]_+ \left(\left[\frac{1}{u}\right]_+ + \left[\frac{1}{1-u}\right]_+\right) \\
&+2\left[\frac{\ln(1-\tau)}{1-\tau}\right]_+ \left(\left[\frac{\ln u}{u}\right]_+ + \left[\frac{\ln(1-u)}{1-u}\right]_+ + \frac{\ln u}{1-u} + \frac{\ln(1-u)}{u}\right) \\
&+\frac{1}{2}\left[\frac{1}{1-\tau}\right]_+ \left(\left[\frac{\ln^2 u}{u}\right]_+ + \left[\frac{\ln^2(1-u)}{1-u}\right]_+ + \frac{\ln^2 u}{1-u} + \frac{\ln^2(1-u)}{u} + \frac{2\ln u \ln(1-u)}{u(1-u)}\right) \Bigg\} \\
&= \int_0^1 dx_1 \int_0^1 dx_2 T(x_1,x_2) \Bigg\{ \frac{9}{4}\zeta_4 \delta(1-x_1)\delta(1-x_2)
\end{aligned}$$

$$+\frac{\delta(1-x_2)}{6}\left(\frac{1}{1-x_1}\ln^3\frac{2x_1}{1+x_1}+3\frac{\ln(1-x_1)}{1-x_1}\ln^2\frac{2x_1}{1+x_1}+3\frac{\ln^2(1-x_1)}{1-x_1}\ln\frac{2x_1}{1+x_1}-7\left[\frac{\ln^3(1-x_1)}{1-x_1}\right]_+\right)$$
$$+\frac{\delta(1-x_1)}{6}\left(\frac{1}{1-x_2}\ln^3\frac{2x_2}{1+x_2}+3\frac{\ln(1-x_2)}{1-x_2}\ln^2\frac{2x_2}{1+x_2}+3\frac{\ln^2(1-x_2)}{1-x_2}\ln\frac{2x_2}{1+x_2}-7\left[\frac{\ln^3(1-x_2)}{1-x_2}\right]_+\right)$$
$$+\frac{1}{2}\frac{2(1+x_1x_2)}{(1+x_1)(1+x_2)}$$
$$\times\Bigg(\left[\frac{\ln^2(1-x_1)}{1-x_1}\right]_+\left[\frac{1}{1-x_2}\right]_++2\left[\frac{\ln(1-x_1)}{1-x_1}\right]_+\left[\frac{\ln(1-x_2)}{1-x_2}\right]_++\left[\frac{1}{1-x_1}\right]_+\left[\frac{\ln^2(1-x_2)}{1-x_2}\right]_+$$
$$+2\left(\left[\frac{\ln(1-x_1)}{1-x_1}\right]_+\left[\frac{1}{1-x_2}\right]_++\left[\frac{1}{1-x_1}\right]_+\left[\frac{\ln(1-x_2)}{1-x_2}\right]_+\right)\ln\frac{x_1x_2(1+x_1)(1+x_2)}{(x_1+x_2)^2}$$
$$+\left[\frac{1}{1-x_1}\right]_+\left[\frac{1}{1-x_2}\right]_+\ln^2\frac{x_1x_2(1+x_1)(1+x_2)}{(x_1+x_2)^2}\Bigg)\Bigg\}. \tag{E.19}$$

Note that some single distributions in τ multiplied by ordinary functions of τ and u do appear on the left-hand side of eqs. (E.17)–(E.19). However, we can conclude that no other single distributions in τ other than these can appear in the coefficient function, because, as already discussed, the limit $\tau\to 1$ corresponds to the double-soft limit, in which the coefficient function only depends on the variable $(1-x_1)(1-x_2)$ which appears in the generating function eq. (E.16). This expectation is borne out by comparing to the explicit form of the NNLO coefficient function eq. (5.24).

E.2 Single distributions in u

We now turn to single distributions in u. A relevant observation here is that the coefficient function is necessarily symmetric upon the interchange of u and $1-u$. Hence these distribution always appears in pairs. This said, all single distributions in u can be transformed into distributions in x_1 and x_2 by simply performing the change of variables with the jacobian eq. (E.1).

We illustrate the procedure with a simple example. Consider

$$\int_0^1 d\tau\int_0^1 du\left[\frac{1}{u}\right]_+ t(\tau,u)=\int_0^1 d\tau\int_0^1 du\,\frac{1}{u}[t(\tau,u)-t(\tau,0)]. \tag{E.20}$$

Changing integration variables to x_1, x_2 we get

$$\begin{aligned}&\int_0^1 d\tau\int_0^1 du\left[\frac{1}{u}\right]_+ t(\tau,u)\\&=\int_0^1 dx_1\int_0^1 dx_2\,\frac{2x_1x_2(1+x_1x_2)}{(x_1+x_2)^2(1-x_1x_2)}\frac{(x_1+x_2)(1-x_1x_2)}{x_2(1-x_1^2)}\left[T(x_1,x_2)-T(1,x_2)\right]\\&=\int_0^1 dx_2\int_0^1 dx_1\,\frac{h(x_1,x_2)}{1-x_1}\left[T(x_1,x_2)-T(1,x_2)\right],\end{aligned} \tag{E.21}$$

where

$$h(x_1,x_2)=\frac{2x_1(1+x_1x_2)}{(x_1+x_2)(1+x_1)} \tag{E.22}$$

is non-singular at $x_1=1$ for all values of x_2.

Equation (E.21) can be written as

$$\int_0^1 d\tau \int_0^1 du \left[\frac{1}{u}\right]_+ t(\tau,u) \tag{E.23}$$
$$= \int_0^1 dx_2 \int_0^1 dx_1 \frac{1}{1-x_1} \{[h(x_1,x_2)T(x_1,x_2) - h(1,x_2)T(1,x_2)] - T(1,x_2)[h(x_1,x_2) - h(1,x_2)]\}$$
$$= \int_0^1 dx_2 \int_0^1 dx_1 \left\{ h(x_1,x_2)\left[\frac{1}{1-x_1}\right]_+ - \delta(1-x_1)\int_0^1 dx \frac{h(x,x_2)-h(1,x_2)}{1-x} \right\} T(x_1,x_2).$$

The same argument is easily extended to distributions containing powers of $\ln u$, and to distributions in $1-u$.

E.3 Distributional identities up to NNLO

We now list all distributional identities that we have derived using the methods described above, and that we have uses in order to transform the coefficient function from (τ,u) space to (x_1,x_2) space. All identities written below should be understood with the left-hand side multiplied by a measure of integration $d\tau\, du$, and the right-hand side multiplied by an integration measure $dx_1\, dx_2$.

$$\delta(1-\tau)\frac{\delta(u)+\delta(1-u)}{2} = \delta(1-x_1)\delta(1-x_2) \tag{E.24}$$

$$\times \left[\frac{1}{1-\tau}\right]_+ (\delta(u)+\delta(1-u)) = \delta(1-x_1)\left[\frac{1}{1-x_2}\right]_+ + \delta(1-x_2)\left[\frac{1}{1-x_1}\right]_+ \tag{E.25}$$

$$\times \left[\frac{1}{1-\tau}\right]_+ \left(\left[\frac{1}{u}\right]_+ + \left[\frac{1}{1-u}\right]_+\right) = \left[\frac{1}{1-x_1}\right]_+ \left[\frac{1}{1-x_2}\right]_+ - \delta(1-x_1)\left(\left[\frac{\ln(1-x_2)}{1-x_2}\right]_+ + \frac{\ln\frac{2x_2}{1+x_2}}{1-x_2}\right)$$
$$-\delta(1-x_2)\left(\left[\frac{\ln(1-x_1)}{1-x_1}\right]_+ + \frac{\ln\frac{2x_1}{1+x_1}}{1-x_1}\right) + \frac{\pi^2}{6}\delta(1-x_1)\delta(1-x_2) + \frac{1}{(1+x_1)(1+x_2)} \tag{E.26}$$

$$\times 2\left[\frac{\ln(1-\tau)}{1-\tau}\right]_+ \left(\left[\frac{1}{u}\right]_+ + \left[\frac{1}{1-u}\right]_+\right) + \left[\frac{1}{1-\tau}\right]_+ \left(\left[\frac{\ln u}{u}\right]_+ + \left[\frac{\ln(1-u)}{1-u}\right]_+ + \frac{\ln u}{1-u} + \frac{\ln(1-u)}{u}\right)$$
$$= \left[\frac{1}{1-x_1}\right]_+ \left(\left[\frac{\ln(1-x_2)}{1-x_2}\right]_+ + \frac{\ln\frac{2x_2}{1+x_2}}{1-x_2}\right) + \left[\frac{1}{1-x_2}\right]_+ \left(\left[\frac{\ln(1-x_1)}{1-x_1}\right]_+ + \frac{\ln\frac{2x_1}{1+x_1}}{1-x_1}\right)$$
$$+\delta(1-x_1)\left(-\frac{3}{2}\left[\frac{\ln^2(1-x_2)}{1-x_2}\right]_+ + \frac{1}{2}\ln\frac{2x_2(1-x_2)^2}{1+x_2}\frac{\ln\frac{2x_2}{1+x_2}}{1-x_2}\right)$$
$$+\delta(1-x_2)\left(-\frac{3}{2}\left[\frac{\ln^2(1-x_1)}{1-x_1}\right]_+ + \frac{1}{2}\ln\frac{2x_1(1-x_1)^2}{1+x_1}\frac{\ln\frac{2x_1}{1+x_1}}{1-x_1}\right)$$
$$-2\zeta_3\delta(1-x_1)\delta(1-x_2) + \frac{1}{(1+x_1)(1+x_2)}\ln\frac{x_1x_2(1+x_1)(1+x_2)(1-x_1)(1-x_2)}{(x_1+x_2)^2}$$
$$+\frac{2}{(1-x_1)(1-x_2)}\ln\frac{(1+x_1)(1+x_2)}{2(x_1+x_2)} \tag{E.27}$$

$$2\left[\frac{\ln^2(1-\tau)}{1-\tau}\right]_+ \left(\left[\frac{1}{u}\right]_+ + \left[\frac{1}{1-u}\right]_+\right)$$
$$+2\left[\frac{\ln(1-\tau)}{1-\tau}\right]_+ \left(\left[\frac{\ln u}{u}\right]_+ + \left[\frac{\ln(1-u)}{1-u}\right]_+ + \frac{\ln u}{1-u} + \frac{\ln(1-u)}{u}\right)$$

$$+\frac{1}{2}\left[\frac{1}{1-\tau}\right]_+\left(\left[\frac{\ln^2 u}{u}\right]_+ +\left[\frac{\ln^2(1-u)}{1-u}\right]_+ +\frac{\ln^2 u}{1-u}+\frac{\ln^2(1-u)}{u}+2\frac{\ln u\ln(1-u)}{u(1-u)}\right)$$

$$=\left[\frac{\ln(1-x_1)}{1-x_1}\right]_+\left[\frac{\ln(1-x_2)}{1-x_2}\right]_+ +\frac{1}{2}\left[\frac{1}{1-x_1}\right]_+\left[\frac{\ln^2(1-x_2)}{1-x_2}\right]_+ +\frac{1}{2}\left[\frac{1}{1-x_2}\right]_+\left[\frac{\ln^2(1-x_1)}{1-x_1}\right]_+$$

$$+\left[\frac{\ln(1-x_1)}{1-x_1}\right]_+\frac{\ln\frac{2x_2}{1+x_2}}{1-x_2}+\left[\frac{\ln(1-x_2)}{1-x_2}\right]_+\frac{\ln\frac{2x_1}{1+x_1}}{1-x_1}$$

$$+\frac{1}{2}\left[\frac{1}{1-x_1}\right]_+\ln\frac{2x_2(1-x_2)^2}{1+x_2}\frac{\ln\frac{2x_2}{1+x_2}}{1-x_2}+\frac{1}{2}\left[\frac{1}{1-x_2}\right]_+\ln\frac{2x_1(1-x_1)^2}{1+x_1}\frac{\ln\frac{2x_1}{1+x_1}}{1-x_1}$$

$$+\delta(1-x_1)\left(-\frac{7}{6}\left[\frac{\ln^3(1-x_2)}{1-x_2}\right]_+ +\left(\frac{1}{2}\ln\frac{2x_2(1-x_2)}{1+x_2}\ln(1-x_2)+\frac{1}{6}\ln^2\frac{2x_2}{1+x_2}\right)\frac{\ln\frac{2x_2}{1+x_2}}{1-x_2}\right)$$

$$+\delta(1-x_2)\left(-\frac{7}{6}\left[\frac{\ln^3(1-x_1)}{1-x_1}\right]_+ +\left(\frac{1}{2}\ln\frac{2x_1(1-x_1)}{1+x_1}\ln(1-x_1)+\frac{1}{6}\ln^2\frac{2x_1}{1+x_1}\right)\frac{\ln\frac{2x_1}{1+x_1}}{1-x_1}\right)$$

$$+\frac{\pi^4}{40}\delta(1-x_1)\delta(1-x_2)+\frac{1}{2(1+x_1)(1+x_2)}\ln\frac{x_1x_2(1+x_1)(1+x_2)(1-x_1)(1-x_2)}{(x_1+x_2)^2}$$

$$+\frac{\ln\frac{2x_1}{1+x_1}}{1-x_1}\frac{\ln\frac{2x_2}{1+x_2}}{1-x_2}+2\frac{\ln(1-x_1)+\ln(1-x_2)}{(1-x_1)(1-x_2)}\ln\frac{(1+x_1)(1+x_2)}{2(x_1+x_2)}$$

$$+2\frac{1}{(1-x_1)(1-x_2)}\ln\frac{2x_1x_2}{x_1+x_2}\ln\frac{(1+x_1)(1+x_2)}{2(x_1+x_2)} \tag{E.28}$$

$$\delta(u)+\delta(1-u)=\delta(1-x_1)+\delta(1-x_2) \tag{E.29}$$

$$\times\left[\frac{1}{u}\right]_+ +\left[\frac{1}{1-u}\right]_+ =\left[\frac{1}{1-x_1}\right]_+ +\left[\frac{1}{1-x_2}\right]_+ +\delta(1-x_1)\ln\frac{2x_2}{(1+x_2)(1-x_2)}$$

$$+\delta(1-x_2)\ln\frac{2x_1}{(1+x_1)(1-x_1)}-\frac{x_1+x_2+2x_1x_2}{(1+x_1)(1+x_2)} \tag{E.30}$$

$$\left[\frac{\ln u}{u}\right]_+ +\left[\frac{\ln(1-u)}{1-u}\right]_+ =\left[\frac{\ln(1-x_1)}{1-x_1}\right]_+ +\left[\frac{\ln(1-x_2)}{1-x_2}\right]_+ +\left[\frac{1}{1-x_1}\right]_+\ln\frac{2x_2}{(1+x_2)(1-x_2)}$$

$$+\left[\frac{1}{1-x_2}\right]_+\ln\frac{2x_1}{(1+x_1)(1-x_1)}+\delta(1-x_1)\frac{1}{2}\ln^2\frac{2x_2}{(1+x_2)(1-x_2)}$$

$$+\delta(1-x_2)\frac{1}{2}\ln^2\frac{2x_1}{(1+x_1)(1-x_1)}+\frac{\ln\frac{1-x_1}{1-x_1x_2}}{1-x_2}+\frac{\ln\frac{1-x_2}{1-x_1x_2}}{1-x_1}$$

$$+\frac{2x_1(1+x_1x_2)}{(x_1+x_2)(1+x_1)}\frac{\ln\frac{(1+x_1)(1+x_2)}{2(1+x_1x_2)}}{1-x_1}+\frac{2x_2(1+x_1x_2)}{(x_1+x_2)(1+x_2)}\frac{\ln\frac{(1+x_1)(1+x_2)}{2(1+x_1x_2)}}{1-x_2}$$

$$+\frac{x_1-x_2-2x_1x_2}{(1+x_1)(x_1+x_2)}\ln\frac{2x_2(1-x_1)}{(1+x_2)(1-x_1x_2)}+\frac{x_2-x_1-2x_1x_2}{(1+x_2)(x_1+x_2)}\ln\frac{2x_1(1-x_2)}{(1+x_1)(1-x_1x_2)} \tag{E.31}$$

$$\left[\frac{\ln^2 u}{u}\right]_+ +\left[\frac{\ln^2(1-u)}{1-u}\right]_+ =\left[\frac{\ln^2(1-x_1)}{1-x_1}\right]_+ +\left[\frac{\ln^2(1-x_2)}{1-x_2}\right]_+$$

$$+2\left[\frac{\ln(1-x_1)}{1-x_1}\right]_+\ln\frac{2x_2}{(1+x_2)(1-x_2)}+2\left[\frac{\ln(1-x_2)}{1-x_2}\right]_+\ln\frac{2x_1}{(1+x_1)(1-x_1)}$$

$$+\left[\frac{1}{1-x_1}\right]_+\ln^2\frac{2x_2}{(1+x_2)(1-x_2)}+\left[\frac{1}{1-x_2}\right]_+\ln^2\frac{2x_1}{(1+x_1)(1-x_1)}$$

$$
\begin{aligned}
&+\delta(1-x_1)\frac{1}{3}\ln^3\frac{2x_2}{(1+x_2)(1-x_2)}+\delta(1-x_2)\frac{1}{3}\ln^3\frac{2x_1}{(1+x_1)(1-x_1)}\\
&+\frac{\ln^2(1-x_1x_2)-\ln^2(1-x_2)}{1-x_1}+\frac{\ln^2(1-x_1x_2)-\ln^2(1-x_1)}{1-x_2}\\
&+2\ln\frac{2x_2(1-x_1)}{1+x_2}\frac{\ln\frac{1-x_2}{1-x_1x_2}}{1-x_1}+2\ln\frac{2x_1(1-x_2)}{1+x_1}\frac{\ln\frac{1-x_1}{1-x_1x_2}}{1-x_2}\\
&+\frac{2x_1(1+x_1x_2)}{(1+x_1)(x_1+x_2)}\ln\frac{2x_2^2(1+x_1)(1-x_1)^2}{(1+x_2)(x_1+x_2)(1-x_1x_2)^2}\frac{\ln\frac{2(x_1+x_2)}{(1+x_1)(1+x_2)}}{1-x_1}\\
&+\frac{2x_2(1+x_1x_2)}{(1+x_2)(x_1+x_2)}\ln\frac{2x_1^2(1+x_2)(1-x_2)^2}{(1+x_1)(x_1+x_2)(1-x_1x_2)^2}\frac{\ln\frac{2(x_1+x_2)}{(1+x_2)(1+x_1)}}{1-x_2}\\
&+\frac{x_1-x_2-2x_1x_2}{(x_1+x_2)(1+x_1)}\ln^2\frac{2x_2(1-x_1)}{(1+x_2)(1-x_1x_2)}+\frac{x_2-x_1-2x_1x_2}{(x_1+x_2)(1+x_2)}\ln^2\frac{2x_1(1-x_2)}{(1+x_1)(1-x_1x_2)}
\end{aligned}
\tag{E.32}
$$

F Distributions and anomalous dimensions in SCET

We collect here some definitions and results used in the SCET resummation discussed in section 6. First, we derive some distributional identities; then we list several anomalous dimensions that enter the renormalization-group equations whose solution leads to resummation in SCET.

F.1 Distributional identities

In ref. [14], the distributions that contain soft logs are written in terms of a variable w, which vanishes in the soft limit:

$$
\mathcal{L}_n(w)=\left[\frac{\ln^n w}{w}\right]_+,\qquad n\geq 0\,. \tag{F.1}
$$

Their action over a space of regular test functions $f(w)$ is defined by

$$
\int_0^1 dw\,\mathcal{L}_n(w)f(w)=\int_0^1 dw\,\frac{\ln^n w}{w}\left[f(w)-f(0)\right]. \tag{F.2}
$$

The distributions eq. (F.1) can be obtained by differentiation of a generating function:

$$
\mathcal{L}_n(w)=\left[\frac{\mathrm{d}^n}{\mathrm{d}\eta^n}\mathcal{L}^\eta(w)\right]_{\eta=0}, \tag{F.3}
$$

where

$$
\mathcal{L}^\eta(w)=\left[\frac{1}{w^{1-\eta}}\right]_+ \tag{F.4}
$$

and

$$
\int_0^1 dw\,\mathcal{L}_n(w)f(w)=\int_0^1 dw\,w^{-1+\eta}\left[f(w)-f(0)\right]. \tag{F.5}
$$

This is useful because, for $\eta>-1$, the two integrals in eq. (F.5) are separately convergent.

Upon rescaling of its argument by a positive constant $\lambda>0$, the generating function eq. (F.4) transforms according to

$$
\lambda\mathcal{L}^\eta(\lambda x)=\lambda^\eta\mathcal{L}^\eta(x)+\frac{\lambda^\eta-1}{\eta}\delta(x)\,. \tag{F.6}
$$

Indeed, for any regular function $f(\lambda x)$,

$$\begin{aligned}\int_0^1 dx\,\lambda\mathcal{L}^\eta(\lambda x)f(\lambda x) &= \int_0^1 d(\lambda x)\,(\lambda x)^{-1+\eta}f(\lambda x) - f(0)\int_0^1 d(\lambda x)\,(\lambda x)^{-1+\eta} \\ &= \lambda^\eta\int_0^1 dx\,x^{-1+\eta}f(\lambda x) - \frac{1}{\eta}f(0)\,, \end{aligned} \tag{F.7}$$

from which we obtain

$$\begin{aligned}\int_0^1 dx\,\left[\lambda^\eta\mathcal{L}^\eta(x) + \frac{\lambda^\eta-1}{\eta}\delta(x)\right]f(\lambda x) &= \lambda^\eta\int_0^1 dx\,x^{-1+\eta}f(\lambda x) - f(0)\frac{\lambda^\eta}{\eta} + \frac{\lambda^\eta-1}{\eta}f(0) \\ &= \lambda^\eta\int_0^1 dx\,x^{-1+\eta}f(\lambda x) - \frac{1}{\eta}f(0)\,. \end{aligned} \tag{F.8}$$

As a consequence,

$$\lambda\mathcal{L}_n(\lambda x) = \sum_{k=0}^{n}\binom{n}{k}\ln^k\lambda\,\mathcal{L}_{n-k}(x) + \frac{\ln^{n+1}\lambda}{n+1}\delta(x)\,. \tag{F.9}$$

Accordingly, a scale-dependent distribution is also defined:

$$\hat{\mathcal{L}}^\eta\left(t,\mu^2\right) \equiv \frac{1}{\mu^2}\,\mathcal{L}^\eta\left(\frac{t}{\mu^2}\right). \tag{F.10}$$

F.2 Anomalous dimensions

We list the anomalous dimensions used in section 6 up to the order required to achieve NNLL resummation. These anomalous dimensions are expanded in powers of $\frac{\alpha_s}{4\pi}$:

$$\gamma(\alpha_s) = \gamma_0\left(\frac{\alpha_s}{4\pi}\right) + \gamma_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \ldots\,. \tag{F.11}$$

The first expansion coefficients of the quark anomalous dimension γ_H^q of the hard function, appearing in eq. (6.6), are given by [40]

$$\gamma^q_{H,0} = -6C_{\rm F} \tag{F.12}$$

$$\gamma^q_{H,1} = -C_{\rm F}\left[\left(\frac{82}{9}-52\zeta_3\right)C_{\rm A} + (3-4\pi^2+48\zeta_3)C_{\rm F} + \left(\frac{65}{9}+\pi^2\right)(4\pi\beta_0)\right], \tag{F.13}$$

with the beta-function coefficients defined as in eqs. (5.8)–(5.9).

The expansion coefficients of the anomalous dimension of the quark beam function γ_B^q (see eqs. (6.11)–(6.13)) are

$$\gamma^q_{B,0} = 6C_{\rm F} \tag{F.14}$$

$$\gamma^q_{B,1} = C_{\rm F}\left[\left(\frac{146}{9}-80\zeta_3\right)C_{\rm A} + (3-4\pi^2+48\zeta_3)C_{\rm F} + \left(\frac{121}{9}+\frac{2\pi^2}{3}\right)(4\pi\beta_0)\right] \tag{F.15}$$

while the corresponding coefficients of the anomalous dimension γ_ϕ^q are

$$\gamma^q_{\phi,0} = 3C_{\rm F}\,, \tag{F.16}$$

$$\gamma^q_{\phi,1} = C_{\rm F}^2\left(\frac{3}{2}-2\pi^2+24\zeta_3\right) + C_{\rm F}C_{\rm A}\left(\frac{17}{6}+\frac{22\pi^2}{9}-12\zeta_3\right) - C_{\rm F}T_{\rm R}n_{\rm f}\left(\frac{2}{3}+\frac{8\pi^2}{9}\right). \tag{F.17}$$

Using these results we find that the expansion coefficients of the anomalous dimension γ_W^q of eq. (6.52)

$$\gamma_W^q(\alpha_s) = \gamma_H^q(\alpha_s) + 2\gamma_\phi^q(\alpha_s) \tag{F.18}$$

are given by [12, 52]

$$\gamma_{W,0}^q = 0 \tag{F.19}$$

$$\gamma_{W,1}^q = C_A C_F\left(-\frac{808}{27} + \frac{11\pi^2}{9} + 28\zeta_3\right) + C_F T_R n_f\left(\frac{224}{27} - \frac{4\pi^2}{9}\right). \tag{F.20}$$

Data Availability Statement. This article has no associated data or the data will not be deposited.

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