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Calculus of Variations

A non-local approach to the generalized Stokes operator with bounded measurable coefficients

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Abstract

We establish functional analytic properties of the Stokes operator with bounded measurable coefficients on $\mathrm{L}^p_\sigma(\mathbb{R}^d)$, $d \geq 2$, for $|1/p - 1/2| < 1/d$. These include optimal resolvent bounds, the property of maximal L^q-regularity, the boundedness of its H^∞-calculus, and characterizations of fractional power domains. We further give regularity estimates on the gradient of the solution to the Stokes resolvent problem with bounded measurable coefficients. As a key to these results we establish a non-local Caccioppoli inequality to solutions of the Stokes resolvent problem.

Mathematics Subject Classification 47F10 · 47A60 · 35Q35

1 Introduction

This article is concerned with the investigation of the Stokes resolvent problem

$$\begin{cases} \lambda u - \operatorname{div}(\mu \nabla u) + \nabla \phi = f & \text{in } \mathbb{R}^d, \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}^d \end{cases} \tag{1.1}$$

for λ in some complex sector $\mathrm{S}_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta\}$ for some suitable $\theta \in (\pi/2, \pi)$ depending on d and the coefficients μ. The coefficients $\mu^{ij}_{\alpha\beta}$ are assumed to be essentially bounded and complex valued; ellipticity is enforced by a Gårding type inequality. The equation (1.1) is the resolvent equation for the Stokes operator with bounded measurable coefficients, which is formally given by

$$Au = -\operatorname{div}(\mu \nabla u) + \nabla \phi, \quad \operatorname{div}(u) = 0 \quad \text{in} \quad \mathbb{R}^d.$$

On the space $\mathrm{L}^2_\sigma(\mathbb{R}^d)$ - the space of solenoidal L^2-integrable vector fields - the operator A can be realized via a densely defined, closed, and sectorial sesquilinear form and thus is sectorial of some angle $\omega \in [0, \pi/2)$. In particular, there exists $C > 0$ such that for all $\theta \in (0, \pi - \omega)$

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and all $\lambda \in \mathrm{S}_\omega$ we have with $p = 2$

$$\|\lambda u\|_{\mathrm{L}^p_\sigma} = \|\lambda(\lambda + A)^{-1} f\|_{\mathrm{L}^p_\sigma} \leq C\|f\|_{\mathrm{L}^p_\sigma} \qquad (f \in \mathrm{L}^p_\sigma(\mathbb{R}^d)). \tag{1.2}$$

A natural question is, whether the estimate (1.2) has analogues in L^p_σ-spaces for numbers $p \neq 2$. In the elliptic situation, i.e., if the operator $Lu = -\operatorname{div}(\mu \nabla u)$ is considered, this question is well understood. Indeed, it is known [1, 2, 5, 10, 33] that there exists $\varepsilon > 0$ such that for all $p \in (1, \infty)$ that satisfy

$$\Big|\frac{1}{p} - \frac{1}{2}\Big| < \frac{1}{d} + \varepsilon \tag{1.3}$$

resolvent bounds of the form (1.2) are valid for the operator L. Moreover, the condition (1.3) is sharp among the class of all elliptic systems with bounded measurable coefficients. Indeed, Davies [6] constructed in $d \geq 3$ dimensions and for each $p > 2d/(d-2)$ coefficients μ such that the semigroup operators e^{-tL}, $t > 0$, do not even map $\mathrm{L}^p(\mathbb{R}^d; \mathbb{C}^d)$ into itself. Thus, in particular (1.2) fails for such p. To the best knowledge of the author, if the Stokes operator A is concerned, the resolvent bounds (1.2) for some $p \neq 2$ are completely unknown.

It is well-known, that the resolvent bounds (1.2) are the starting point for the investigation of further functional analytic properties of A. An immediate question is, whether A has the property of maximal L^q-regularity and whether its H^∞-calculus is bounded. These properties are of great interest for the study of fractional powers of A and eventually for well-posedness results of nonlinear problems. In particular, rough coefficients are of great interest in the investigation of non-Newtonian fluids in the regime of rough data.

If the coefficients μ are smooth enough, then all these properties mentioned above were established by Prüss and Simonett [27] and Prüss [26] for all $1 < p < \infty$. Moreover, Solonnikov [31] investigated operators that were not in divergence form. However, there is a big methodological difference between the smooth situation and the situation of mere essentially bounded coefficients as techniques like freezing the coefficients and arguing for variable coefficients via perturbation become unavailable. In the elliptic and rough situation, variational techniques replace the method of freezing the coefficients. For instance, using Davies' method one establishes so-called off-diagonal estimates for the semigroup $(\mathrm{e}^{-tL})_{t \geq 0}$ which eventually imply the desired resolvent bounds in L^p for p satisfying (1.3), see [1, 2, 5]. Another method is by using Caccioppoli's inequality for the resolvent equations $\lambda u + Lu = f$ combined with Sobolev's embedding theorem to deduce certain weak reverse Hölder estimates that - by virtue of an L^p-extrapolation theorem of Shen [28] - also imply the resolvent bounds in L^p for p satisfying (1.3), see [29, 33].

The use of these variational methods is again problematic if the Stokes operator A is considered. The reason is, that the derivation of off-diagonal estimates as well as of the Caccioppoli inequality rely on testing the resolvent equation by some appropriate function multiplied by a cut-off function. Due to the multiplication by the cut-off function, the test function is not divergence-free anymore, so that the pressure appears in the inequalities and has to be treated. Recently, Chang and Kang [3] proved that it is impossible to establish a parabolic Caccioppoli inequality for the Stokes system on the half-space that has the same form as its elliptic counterpart. Up to now, there has not been a satisfactory way of how to handle this pressure term and the purpose of this work is to propose an argument for its treatment.

As the pressure embodies the non-local part in the resolvent problem (1.1) some non-local terms will enter the inequalities. Kuusi, Mingione, and Sire investigated non-local elliptic integrodifferential operators of fractional type in [24] and established a non-local Caccioppoli inequality in this situation. Inspired by their paper, the author extended in [35] their non-local

Caccioppoli inequality to the resolvent equation of such operators and further extended the proof of Shen's L^p-extrapolation theorem to these non-local estimates. In this paper, we will proceed similarly and establish a non-local Caccioppoli inequality for the Stokes resolvent problem (1.1). As this non-local Caccioppoli inequality is only valid for solenoidal right-hand sides f, we further need to adapt the extrapolation argument of Shen as this requires general right-hand sides in $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^d)$.

Let us introduce some notation to state the main results of this article:

Assumption 1.1 The coefficients $\mu = (\mu_{\alpha\beta}^{ij})_{\alpha,\beta,i,j=1}^d$ with $\mu_{\alpha\beta}^{ij} \in \mathrm{L}^\infty(\mathbb{R}^d; \mathbb{C})$ for all $1 \leq \alpha, \beta, i, j \leq d$ satisfy for some $\mu_\bullet, \mu^\bullet > 0$ the inequalities

$$\Re \sum_{\alpha,\beta,i,j=1}^d \int_{\mathbb{R}^d} \mu_{\alpha\beta}^{ij} \partial_\beta u_j \overline{\partial_\alpha u_i} \, \mathrm{d}x \geq \mu_\bullet \|\nabla u\|_{\mathrm{L}^2}^2 \qquad (u \in \mathrm{H}^1(\mathbb{R}^d; \mathbb{C}^d)) \tag{1.4}$$

and

$$\| |\mu| \|_{\mathrm{L}^\infty} \leq \mu^\bullet, \tag{1.5}$$

where $|\mu|$ denotes the Euclidean norm of μ.

The generalized Stokes operator A is realized on $\mathrm{L}^2_\sigma(\mathbb{R}^d)$ as follows. Define the form domain $\mathrm{H}^1_\sigma(\mathbb{R}^d) := \{f \in \mathrm{H}^1(\mathbb{R}^d; \mathbb{C}^d) : \operatorname{div}(f) = 0\}$ and the sesquilinear form

$$\mathfrak{a} : \mathrm{H}^1_\sigma(\mathbb{R}^d) \times \mathrm{H}^1_\sigma(\mathbb{R}^d) \to \mathbb{C}, \quad (u, v) \mapsto \sum_{\alpha,\beta,i,j=1}^d \int_{\mathbb{R}^d} \mu_{\alpha\beta}^{ij} \partial_\beta u_j \overline{\partial_\alpha v_i} \, \mathrm{d}x. \tag{1.6}$$

The domain of A on $\mathrm{L}^2_\sigma(\mathbb{R}^d)$ is then given by

$$\mathcal{D}(A) := \Big\{ u \in \mathrm{H}^1_\sigma(\mathbb{R}^d) : \exists f \in \mathrm{L}^2_\sigma(\mathbb{R}^d), \forall v \in \mathrm{H}^1_\sigma(\mathbb{R}^d) \text{ we have } \mathfrak{a}(u, v) = \int_{\mathbb{R}^d} f \cdot \overline{v} \, \mathrm{d}x \Big\}.$$

Now, for $u \in \mathcal{D}(A)$ define $Au := f$.

This definition delivers a closed and densely defined operator on $\mathrm{L}^2_\sigma(\mathbb{R}^d)$. Moreover, the Assumption 1.1 together with the inequality of Cauchy–Schwarz implies that

$$|\Im(\mathfrak{a}(u, u))| \leq \frac{\mu^\bullet}{\mu_\bullet} \Re(\mathfrak{a}(u, u)) \qquad (u \in \mathrm{H}^1(\mathbb{R}^d; \mathbb{C}^d)).$$

This means that, with

$$\omega_0 := \arctan\Big(\frac{\mu^\bullet}{\mu_\bullet}\Big) \in (0, \tfrac{\pi}{2}), \tag{1.7}$$

the numerical range of $\mathfrak{a}$ is contained in $\overline{\mathrm{S}_{\omega_0}}$, i.e., that

$$\mathfrak{a}(u, u) \in \overline{\mathrm{S}_{\omega_0}} \qquad (u \in \mathrm{H}^1(\mathbb{R}^d; \mathbb{C}^d)). \tag{1.8}$$

An application of the Lax–Milgram lemma then implies that the resolvent equation

$$\lambda \int_{\mathbb{R}^d} u \cdot \overline{v} \, \mathrm{d}x + \mathfrak{a}(u, v) = G(v) \qquad (v \in \mathrm{H}^1_\sigma(\mathbb{R}^d)) \tag{1.9}$$

has a unique solution u for every $\lambda \in \mathrm{S}_\theta$ with $0 < \theta < \pi - \omega_0$ and every anti-linear functional $G \in (\mathrm{H}^1_\sigma(\mathbb{R}^d))^*$. In particular, the spectrum of A is contained in $\overline{\mathrm{S}_{\omega_0}}$. By a standard argument, see, e.g., [30, Lem. II.2.2.1], one can extend (1.9) to test functions $v \in \mathrm{C}_c^\infty(\mathbb{R}^d; \mathbb{C}^d)$ by introducing a pressure function ϕ. More precisely, there exists $\phi \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$, that is unique up to the addition of constants, such that

$$\lambda \int_{\mathbb{R}^d} u \cdot \overline{v} \, \mathrm{d}x + \mathfrak{a}(u, v) - \int_{\mathbb{R}^d} \phi \, \overline{\mathrm{div}(v)} \, \mathrm{d}x = G(v) \qquad (v \in \mathrm{C}_c^\infty(\mathbb{R}^d; \mathbb{C}^d)).$$

This pressure function ϕ will be called the *pressure associated to* u, in the following. Notice that we will, by abuse of notation, also insert non-solenoidal H^1-vector fields into $\mathfrak{a}$.

In the literature of non-Newtonian fluids, see, e.g., [8, 18–20], operators arising from the sesquilinear form

$$(u, v) \mapsto \sum_{\alpha,\beta,i,j=1}^d \int_{\mathbb{R}^d} \vartheta^{\alpha i}_{\beta j} D(u)_{\beta j} \overline{D(v)_{\alpha i}} \, \mathrm{d}x$$

are often considered. Here, $\vartheta^{\alpha i}_{\beta j} \in \mathrm{L}^\infty(\mathbb{R}^d; \mathbb{C})$ denote coefficients that satisfy

$$\sum_{\alpha,\beta,i,j=1}^d \Re(\vartheta^{\alpha i}_{\beta j}(x) \eta_{\beta j} \overline{\eta_{\alpha i}}) \geq c|\eta|^2 \tag{1.10}$$

for some constant $c > 0$, for almost every $x \in \mathbb{R}^d$, and all symmetric $\mathbb{C}^{d \times d}$-matrices η. Moreover, $D(u) := \frac{1}{2}(Du + [Du]^\top)$ denotes the symmetric Jacobian of u. Such sesquilinear forms can be realized within the framework of this article by noting that

$$\begin{aligned} &\sum_{\alpha,\beta,i,j=1}^d \int_{\mathbb{R}^d} \vartheta^{\alpha i}_{\beta j} D(u)_{\beta j} \overline{D(v)_{\alpha i}} \, \mathrm{d}x \\ &\quad = \sum_{\alpha,\beta,i,j=1}^d \int_{\mathbb{R}^d} \frac{1}{4}(\vartheta^{\alpha i}_{\beta j} + \vartheta^{\alpha i}_{j \beta} + \vartheta^{i \alpha}_{\beta j} + \vartheta^{i \alpha}_{j \beta}) \partial_\beta u_j \overline{\partial_\alpha v_i} \, \mathrm{d}x. \end{aligned}$$

In particular, $\vartheta^{\alpha i}_{\beta j} \in \mathrm{L}^\infty(\mathbb{R}^d; \mathbb{C})$ and (1.10) combined with Korn's inequality imply that μ defined by

$$\mu^{ij}_{\alpha\beta} := \frac{1}{4}(\vartheta^{\alpha i}_{\beta j} + \vartheta^{\alpha i}_{j \beta} + \vartheta^{i \alpha}_{\beta j} + \vartheta^{i \alpha}_{j \beta})$$

satisfies Assumption 1.1 for some constants $\mu_\bullet, \mu^\bullet > 0$.

The non-local Caccioppoli inequality we prove in this article reads as follows.

Theorem 1.2 *Let μ satisfy Assumption 1.1 for some constants $\mu_\bullet, \mu^\bullet > 0$ and let $\omega_0 \in (0, \frac{\pi}{2})$ be given by (1.7). Then for all $\theta \in (0, \pi - \omega_0)$ and all $0 < \nu < d + 2$ there exists $C > 0$ such that for all $\lambda \in \mathrm{S}_\theta$, $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$, $F \in \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^{d \times d})$ the solution $u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ to*

$$\lambda \int_{\mathbb{R}^d} u \cdot \overline{v} \, \mathrm{d}x + \mathfrak{a}(u, v) = \int_{\mathbb{R}^d} f \cdot \overline{v} \, \mathrm{d}x - \sum_{\alpha,\beta=1}^d \int_{\mathbb{R}^d} F_{\alpha\beta} \, \overline{\partial_\alpha v_\beta} \, \mathrm{d}x \qquad (v \in \mathrm{H}^1_\sigma(\mathbb{R}^d))$$

satisfies for all balls $B = B(x_0, r)$ and all sequences $(c_k)_{k \in \mathbb{N}_0}$ with $c_k \in \mathbb{C}^d$

$$\begin{aligned} &|\lambda| \sum_{k=0}^\infty 2^{-\nu k} \int_{B(x_0, 2^k r)} |u|^2 \, \mathrm{d}x + \sum_{k=0}^\infty 2^{-\nu k} \int_{B(x_0, 2^k r)} |\nabla u|^2 \, \mathrm{d}x \\ &\quad \leq \frac{C}{r^2} \sum_{k=0}^\infty 2^{-(\nu+2)k} \int_{B(x_0, 2^{k+1} r) \setminus B(x_0, 2^k r)} |u + c_k|^2 \, \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+ C|\lambda| \sum_{k=0}^{\infty} |c_k| 2^{-\nu k} \int_{B(x_0, 2^{k+1} r)} |u| \, \mathrm{d}x \\
&+ \frac{C}{|\lambda|} \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0, 2^{k+1} r)} |f|^2 \, \mathrm{d}x + C \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0, 2^{k+1} r)} |F|^2 \, \mathrm{d}x \\
&+ C \sum_{k=0}^{\infty} 2^{-\nu k} |B(x_0, 2^{k+1} r)| |c_k|^2 .
\end{aligned}
$$

The constant C depends on $\mu_\bullet$, $\mu^\bullet$, d, θ, and ν.

As described above, the non-local Caccioppoli inequality allows us to establish resolvent bounds in $\mathrm{L}^p_\sigma(\mathbb{R}^d)$. More precisely, we prove the following result.

Theorem 1.3 *Let μ satisfy Assumption 1.1 for some constants $\mu_\bullet, \mu^\bullet > 0$ and let $\omega_0 \in (0, \frac{\pi}{2})$ be given by (1.7). Then for all p satisfying*

$$
\Big| \frac{1}{p} - \frac{1}{2} \Big| < \frac{1}{d} \tag{1.11}
$$

and all $\theta \in (0, \pi - \omega_0)$ there exists $C > 0$ such that for all $\lambda \in \mathrm{S}_\theta$ we have

$$
\|\lambda (\lambda + A)^{-1} f\|_{\mathrm{L}^p_\sigma} \leq C \|f\|_{\mathrm{L}^p_\sigma} \qquad (f \in \mathrm{L}^2_\sigma(\mathbb{R}^d) \cap \mathrm{L}^p_\sigma(\mathbb{R}^d)).
$$

The constant $C > 0$ depends only on d, $\mu_\bullet$, $\mu^\bullet$, p, and θ.

Additionally to the L^p-resolvent estimates in Theorem 1.3 we establish further regularity estimates for solutions to the Stokes resolvent problem.

Theorem 1.4 *Let μ satisfy Assumption 1.1 for some constants $\mu_\bullet, \mu^\bullet > 0$ and let $\omega_0 \in (0, \frac{\pi}{2})$ be given by (1.7). Then for all p satisfying*

$$
\frac{2d}{d+2} < p \leq 2 \tag{1.12}
$$

and all $\theta \in (0, \pi - \omega_0)$ there exists $C > 0$ such that for all $\lambda \in \mathrm{S}_\theta$ we have

$$
|\lambda|^{1/2} \|\nabla (\lambda + A)^{-1} f\|_{\mathrm{L}^p} \leq C \|f\|_{\mathrm{L}^p_\sigma} \qquad (f \in \mathrm{L}^2_\sigma(\mathbb{R}^d) \cap \mathrm{L}^p_\sigma(\mathbb{R}^d)).
$$

The constant $C > 0$ depends only on d, $\mu_\bullet$, $\mu^\bullet$, p, and θ.

Theorem 1.3 allows us to realize the operator A as a sectorial operator on the L^p_σ-spaces for p satisfying (1.11). It is well-known that this is equivalent to the fact that $-A$ generates a bounded analytic semigroup $(\mathrm{e}^{-tA})_{t \geq 0}$ on $\mathrm{L}^p_\sigma(\mathbb{R}^d)$. Additionally, Theorem 1.4 tells us that this semigroup satisfies gradient estimates of the form

$$
t^{1/2} \|\nabla \mathrm{e}^{-tA} f\|_{\mathrm{L}^p} \leq C \|f\|_{\mathrm{L}^p_\sigma} \qquad (t > 0, \, f \in \mathrm{L}^p_\sigma(\mathbb{R}^d)).
$$

We mention here the results of Kaplický, Málek, and Stará [19] and of Kaplický and Wolf [20] who prove Meyers-type higher integrability results to obtain even integrability properties for the gradient of the instationary solution for p being slightly larger than 2.

We further prove that the L^p_σ-realizations of A have the property of maximal L^q-regularity as the following theorem states.

Theorem 1.5 *Let μ satisfy Assumption 1.1 for some constants $\mu_\bullet, \mu^\bullet > 0$. Then for all p satisfying (1.11) the L^p_σ-realization of A has maximal L^q-regularity for any $1 < q < \infty$. More precisely, for any $f \in \mathrm{L}^q(0, \infty; \mathrm{L}^p_\sigma(\mathbb{R}^d))$ the unique mild solution u to the Cauchy problem*

$$\begin{cases} u'(t) + Au(t) = f(t), & t > 0, \\ \qquad u(0) = 0 \end{cases}$$

satisfies $u(t) \in \mathcal{D}(A)$ for almost every $t > 0$, $u', Au \in \mathrm{L}^q(0, \infty; \mathrm{L}^p_\sigma(\mathbb{R}^d))$, and there exists a constant $C > 0$ depending only on d, $\mu_\bullet$, $\mu^\bullet$, p, and q such that

$$\|u'\|_{\mathrm{L}^q(0,\infty;\mathrm{L}^p_\sigma)} + \|Au\|_{\mathrm{L}^q(0,\infty;\mathrm{L}^p_\sigma)} \leq C\|f\|_{\mathrm{L}^q(0,\infty;\mathrm{L}^p_\sigma)}.$$

Remark 1.6 We stress that an embedding of the form $\mathcal{D}(A) \subset \mathrm{W}^{2,p}(\mathbb{R}^d; \mathbb{C}^d)$ is in general wrong for L^∞-coefficients μ. Thus, Theorem 1.5 does *not* give information about second derivatives of the solution u.

Finally, we show that the H^∞-calculus of the L^p_σ-realization of A is bounded and characterize certain fractional power domains. We refer to Section 6 for an introduction of the H^∞-calculus.

Theorem 1.7 *Let μ satisfy Assumption 1.1 for some constants $\mu_\bullet, \mu^\bullet > 0$ and let $\omega_0 \in (0, \frac{\pi}{2})$ be given by (1.7). Then for all p satisfying (1.11) the L^p_σ-realization of A has a bounded $\mathrm{H}^\infty(\mathrm{S}_\theta)$-calculus for $\theta \in (\omega_0, \pi)$. Moreover, for all $s \in (0, \frac{1}{2})$ such that*

$$0 < s < \frac{1}{2} - \frac{d}{2}\Big|\frac{1}{2} - \frac{1}{p}\Big|$$

the fractional power domains of A^s are given by

$$\mathcal{D}(A^s) = \mathrm{H}^{2s,p}_\sigma(\mathbb{R}^d),$$

with equivalent norms. Here $\mathrm{H}^{2s,p}_\sigma(\mathbb{R}^d)$ denotes the space of all solenoidal vector fields in the Bessel potential space $\mathrm{H}^{2s,p}(\mathbb{R}^d; \mathbb{C}^d)$.

We close this introduction by stating some standard notation. Throughout, the space dimension d is assumed to satisfy $d \geq 2$. The natural numbers $\mathbb{N}$ are given by $\{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a ball $B = B(x_0, r)$ and some number $\alpha > 0$ we denote by αB the dilated ball $B(x_0, \alpha r)$. The symbol $|\cdot|$ is reserved for the d-dimensional Lebesgue measure or for the Euclidean norm of a finite dimensional vector or matrix. Constants $C > 0$ will be generic and might change its values from line to line. In some occasions, we add subscripts, e.g., C_d, $C_{\mu^\bullet}$ to indicate the dependence of C on certain quantities. The mean value of a locally integrable function f on a bounded measurable set $\mathcal{A}$ with $|\mathcal{A}| > 0$ is denoted by

$$f_{\mathcal{A}} := \fint_{\mathcal{A}} f \, \mathrm{d}x := \frac{1}{|\mathcal{A}|} \int_{\mathcal{A}} f \, \mathrm{d}x.$$

For the rest of this work, we agree on summing over repeated indices.

2 Proof of the non-local Caccioppoli inequality

Let $B \subset \mathbb{R}^d$ denote a ball centered in $x_0 \in \mathbb{R}^d$ with radius $r > 0$. If $u \in \mathrm{H}^1(2B)$ is harmonic, then the classical Caccioppoli inequality for u reads as

$$\int_B |\nabla u|^2 \, \mathrm{d}x \leq \frac{C}{r^2} \int_{2B} |u|^2 \, \mathrm{d}x, \tag{2.1}$$

where $C > 0$ denotes a dimensional constant. Its proof is very simple as it follows after three lines of calculation after testing the equation $-\Delta u = 0$ in $2B$ by the test function $\eta^2 u$, where $\eta \in \mathrm{C}_c^\infty(2B)$ satisfies $0 \leq \eta \leq 1$, $\eta \equiv 1$ in B, and $\|\nabla \eta\|_{\mathrm{L}^\infty} \leq 2/r$. It is well-known that this inequality can be generalized to solutions to elliptic systems in divergence form with bounded measurable coefficients. One can even go further and consider weak solutions $u \in \mathrm{H}^1(2B)$ to the resolvent equation $\lambda u - \operatorname{div}(\mu \nabla u) = f$ for $\lambda \in \mathrm{S}_\theta$. Testing with the same test function as above then delivers the inequality

$$|\lambda| \int_B |u|^2 \, \mathrm{d}x + \int_B |\nabla u|^2 \, \mathrm{d}x \leq \frac{C}{r^2} \int_{2B} |u|^2 \, \mathrm{d}x + \frac{C}{|\lambda|} \int_{2B} |f|^2 \, \mathrm{d}x. \tag{2.2}$$

Here, the constant $C > 0$ depends on d, θ, and the ellipticity and boundedness constants of the coefficients. Clearly, θ has to be chosen appropriately depending on the ellipticity and boundedness constants of the coefficients.

When the Stokes problem with, say, continuous coefficients is concerned, the classical Caccioppoli inequality (2.1) was proven by Giaquinta and Modica in [14, Thm. 1.1]. Its proof bases again on testing the equation by $\eta^2 u$ as above. However, as the test function is not divergence free, the pressure will appear and needs to be handled in the estimates. With a similar argument, Choe and Kozono [4] established the Caccioppoli inequality for the Stokes resolvent problem with coefficient matrix $\mu = \mathrm{Id}$ and resolvent parameter $\lambda \in \mathrm{i}\mathbb{R}$. In this case, the Caccioppoli inequality reads

$$|\lambda| \int_B |u|^2 \, \mathrm{d}x + \int_B |\nabla u|^2 \, \mathrm{d}x \leq \frac{C(1 + |\lambda| r^2)}{r^2} \int_{2B} |u|^2 \, \mathrm{d}x + \frac{C}{|\lambda|} \int_{2B} |f|^2 \, \mathrm{d}x. \tag{2.3}$$

If one compares the elliptic estimate (2.2) with the estimate for the Stokes resolvent (2.3), one readily sees the difference in the prefactor in front of the L^2-integral of u on the right-hand side. The additional term $|\lambda| r^2$ results from the treatment of the pressure term. In some situations, e.g., in the derivation of weak reverse Hölder inequalities that are *uniform* with respect to λ, it is important that the constant in front of the L^2-integral of u on the right-hand side is *independent* of λ and it would be desirable if the constant would simply be C/r^2 as in the elliptic case (2.2). As the pressure reflects to a great extend the *non-local* behavior of the solution, it is, however, not very surprising that something odd happens if the non-local term is "pressed" into a local estimate. The goal of this section is to take the opposite viewpoint, namely, to prove a non-local counterpart of the Caccioppoli inequality and to recover the prefactor C/r^2 in front of the L^2-integrals of u on the right-hand side. The precise result we prove is formulated in Theorem 1.2.

To prepare the arguments we first introduce some technical tools. First of all, recall that the Helmholtz projection $\mathbb{P}$ is given on the whole space by

$$\mathbb{P} f = \mathcal{F}^{-1} \Big[\mathrm{Id} - \frac{\xi \otimes \xi}{|\xi|^2} \Big] \mathcal{F} f \qquad \Leftrightarrow \qquad (\mathrm{Id} - \mathbb{P}) f = \mathcal{F}^{-1} \frac{\xi \otimes \xi}{|\xi|^2} \mathcal{F} f.$$

Here, $\mathcal{F}$ denotes the Fourier transform, f denotes an element in $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^d)$, and $\xi \otimes \xi := \xi \xi^\top$. Notice that $\mathbb{P}$ is the orthogonal projection onto $\mathrm{L}^2_\sigma(\mathbb{R}^d)$. In particular, for

$f \in \mathrm{H}^1(\mathbb{R}^d; \mathbb{C}^d)$ one has

$$\operatorname{div}((\mathrm{Id} - \mathbb{P}) f) = \operatorname{div}(f).$$

Since $\xi \mapsto \frac{\xi \otimes \xi}{|\xi|^2}$ is a Mikhlin symbol, by [32, Prop. VI.4.2] there exists a kernel function $k : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d$ such that

$$[(\mathrm{Id} - \mathbb{P}) f](x) = \int_{\mathbb{R}^d} k(x-y) f(y) \, \mathrm{d}y \qquad (f \in \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^d),\ x \notin \operatorname{supp}(f)) \tag{2.4}$$

and such that there exists $C_d > 0$, depending only on d, such that

$$|\partial_\alpha k(x)| \leq \frac{C_d}{|x|^{d+|\alpha|}} \qquad (x \in \mathbb{R}^d \setminus \{0\} \text{ and } \alpha \in \mathbb{N}_0^d \text{ with } |\alpha| \leq 1). \tag{2.5}$$

Observe that $\mathbb{P}$ and $\mathrm{Id} - \mathbb{P}$ commute with partial derivatives whenever the function f is regular enough. In particular, $(\mathrm{Id} - \mathbb{P})\partial_\alpha$ is a convolution operator associated to the kernel $\partial_\alpha k$.

To proceed, let $\mathrm{L}^2_0(\mathcal{A}) := \{f \in \mathrm{L}^2(\mathcal{A}) : f_{\mathcal{A}} = 0\}$. Let $C_1 := B(0,1) \setminus \overline{B(0,1/2)}$. The Bogovskiĭ operator $\mathcal{B}_1 : \mathrm{L}^2_0(C_1) \to \mathrm{H}^1_0(C_1; \mathbb{C}^d)$ denotes the solution operator to the divergence equation for functions $f \in \mathrm{L}^2_0(C_1)$

$$\begin{cases} \operatorname{div}(u) = f & \text{in } C_1, \\ u = 0 & \text{on } \partial C_1. \end{cases}$$

Thus, we have $\operatorname{div}(\mathcal{B}_1 f) = f$. Furthermore, $\mathcal{B}_1$ is a bounded operator from $\mathrm{L}^2_0(C_1)$ onto $\mathrm{H}^1_0(C_1; \mathbb{C}^d)$, i.e., there exists a constant $C_{Bog} > 0$ such that

$$\|\mathcal{B}_1 f\|_{\mathrm{H}^1(C_1)} \leq C_{Bog} \|f\|_{\mathrm{L}^2(C_1)} \qquad (f \in \mathrm{L}^2_0(C_1)).$$

See, e.g., Galdi [13, Sect. III.3] for a construction of such an operator. Now, if C_α denotes the annulus $B(0,\alpha) \setminus \overline{B(0,\alpha/2)}$ for some $\alpha > 0$ and if $f \in \mathrm{L}^2_0(C_\alpha)$, the rescaled function $f_\alpha(x) := \alpha f(\alpha x)$ lies in $\mathrm{L}^2_0(C_1)$. Define the rescaled Bogovskiĭ operator on C_α as

$$[\mathcal{B}_\alpha f](x) := [\mathcal{B}_1 f_\alpha](\alpha^{-1} x) \qquad (f \in \mathrm{L}^2_0(C_\alpha),\ x \in C_\alpha).$$

Clearly, $\mathcal{B}_\alpha$ is bounded from $\mathrm{L}^2_0(C_\alpha)$ onto $\mathrm{H}^1_0(C_\alpha; \mathbb{C}^d)$ and satisfies $\operatorname{div}(\mathcal{B}_\alpha f) = f$. Furthermore, by rescaling, the following inequalities holds

$$\|\nabla \mathcal{B}_\alpha f\|_{\mathrm{L}^2(C_\alpha)} \leq C_{Bog} \|f\|_{\mathrm{L}^2(C_\alpha)} \qquad (f \in \mathrm{L}^2_0(C_\alpha)) \tag{2.6}$$

and

$$\|\mathcal{B}_\alpha f\|_{\mathrm{L}^2(C_\alpha)} \leq \alpha C_{Bog} \|f\|_{\mathrm{L}^2(C_\alpha)} \qquad (f \in \mathrm{L}^2_0(C_\alpha)). \tag{2.7}$$

Finally, let use mention that - with a slight abuse of notation - we will denote the Bogovskiĭ operator on annuli not centered in the origin, i.e., on $B(x_0,\alpha) \setminus \overline{B(x_0,\alpha/2)}$ by $\mathcal{B}_\alpha$ as well and notice that (2.6) and (2.7) hold with the same constant.

We are in the position to prove a lemma on non-local pressure estimates.

Lemma 2.1 *Let the coefficients μ satisfy (1.4) and (1.5) with constants $\mu^\bullet, \mu_\bullet > 0$. Let $\lambda \in \mathbb{C}$ and let for $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$ and $F \in \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^{d \times d})$ the functions $u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ and $\phi \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$ solve*

$$\begin{cases} \lambda u - \operatorname{div}(\mu \nabla u) + \nabla \phi = f + \operatorname{div}(F) & \text{in } \mathbb{R}^d, \\ \operatorname{div}(u) = 0 & \text{in } \mathbb{R}^d \end{cases}$$

in the sense of distributions. Let $x_0 \in \mathbb{R}^d$ *and* $r > 0$ *and define for* $k \in \mathbb{N}$ *the annulus* $\mathcal{C}_k := C_{2^k r} = B(x_0, 2^k r) \setminus \overline{B(x_0, 2^{k-1} r)}$. *Let* $\mathcal{C}_0$ *denote the ball* $B(x_0, r)$. *Then there exists a constant* $C > 0$ *depending only on* $\mu^\bullet$ *and* d *such that for all* $k \in \mathbb{N}$ *we have*

$$\begin{aligned}\Big(\int_{\mathcal{C}_k} |\phi - \phi_{\mathcal{C}_k}|^2 \, \mathrm{d}x\Big)^{\frac{1}{2}} &\\ \leq C\bigg(\sum_{\ell=0}^{k-2} 2^{\frac{d}{2}(\ell-k)}\big(\|\nabla u\|_{\mathrm{L}^2(\mathcal{C}_\ell)} + \|F\|_{\mathrm{L}^2(\mathcal{C}_\ell)}\big) &+ \sum_{\substack{\ell\in\mathbb{N}_0\\ |\ell-k|\leq 1}} \big(\|\nabla u\|_{\mathrm{L}^2(\mathcal{C}_\ell)} + \|F\|_{\mathrm{L}^2(\mathcal{C}_\ell)}\big)\\ + \sum_{\ell=k+2}^{\infty} 2^{(\frac{d}{2}+1)(k-\ell)}\big(\|\nabla u\|_{\mathrm{L}^2(\mathcal{C}_\ell)} + \|F\|_{\mathrm{L}^2(\mathcal{C}_\ell)}\big)\bigg).\end{aligned}$$

Proof Let $k \in \mathbb{N}$ and let $\mathcal{B}_{\mathcal{C}_k} := \mathcal{B}_{2^k r}$ denote the Bogovskiĭ operator on $\mathcal{C}_k$. By an extension by zero, we regard the function $\mathcal{B}_{\mathcal{C}_k}((\phi - \phi_{\mathcal{C}_k})|_{\mathcal{C}_k})$ as a function in $\mathrm{H}^1(\mathbb{R}^d; \mathbb{C}^d)$ whose support is contained in $\overline{\mathcal{C}_k}$. Define the test function

$$v_k := (\mathrm{Id} - \mathbb{P})\mathcal{B}_{\mathcal{C}_k}((\phi - \phi_{\mathcal{C}_k})|_{\mathcal{C}_k})$$

which is the difference of a function in $\mathrm{H}^1(\mathbb{R}^d; \mathbb{C}^d)$ with support in $\overline{\mathcal{C}_k}$ and of a function in $\mathrm{H}^1_\sigma(\mathbb{R}^d)$. Since $u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$, $\phi \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$, and since $\mathrm{C}^\infty_{c,\sigma}(\mathbb{R}^d) := \{\varphi \in \mathrm{C}^\infty_c(\mathbb{R}^d; \mathbb{C}^d) : \operatorname{div}(\varphi) = 0\}$ is dense in $\mathrm{H}^1_\sigma(\mathbb{R}^d)$, an approximation argument allows us to test the resolvent equation by this test function. Because $v_k \in \mathrm{L}^2_\sigma(\mathbb{R}^d)^\perp$ and $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$ this yields

$$\begin{aligned}\int_{\mathbb{R}^d} \phi \, \operatorname{div}(\mathcal{B}_{\mathcal{C}_k}((\overline{\phi - \phi_{\mathcal{C}_k}})|_{\mathcal{C}_k})) \, \mathrm{d}x &= \int_{\mathbb{R}^d} \mu^{ij}_{\alpha\beta} \partial_\alpha u_j \partial_\beta [(\mathrm{Id} - \mathbb{P})\mathcal{B}_{\mathcal{C}_k}((\overline{\phi - \phi_{\mathcal{C}_k}})|_{\mathcal{C}_k})]_i \, \mathrm{d}x\\ &\quad + \int_{\mathbb{R}^d} F_{\alpha\beta} \partial_\alpha [(\mathrm{Id} - \mathbb{P})\mathcal{B}_{\mathcal{C}_k}((\overline{\phi - \phi_{\mathcal{C}_k}})|_{\mathcal{C}_k})]_\beta \, \mathrm{d}x.\end{aligned}$$

Notice that we can subtract an arbitrary constant from the left-most pressure since integration is only performed on $\mathcal{C}_k$. By our technical preparation in front of this lemma, we derive the identity

$$\begin{aligned}\int_{\mathcal{C}_k} |\phi - \phi_{\mathcal{C}_k}|^2 \, \mathrm{d}x &= \int_{\mathbb{R}^d} \mu^{ij}_{\alpha\beta} \partial_\alpha u_j [(\mathrm{Id} - \mathbb{P})\partial_\beta \mathcal{B}_{\mathcal{C}_k}((\overline{\phi - \phi_{\mathcal{C}_k}})|_{\mathcal{C}_k})]_i \, \mathrm{d}x\\ &\quad + \int_{\mathbb{R}^d} F_{\alpha\beta} [(\mathrm{Id} - \mathbb{P})\partial_\alpha \mathcal{B}_{\mathcal{C}_k}((\overline{\phi - \phi_{\mathcal{C}_k}})|_{\mathcal{C}_k})]_\beta \, \mathrm{d}x.\end{aligned}$$

With a constant $C_{d,\mu^\bullet} > 0$ depending only on d and $\mu^\bullet$, we then find

$$\begin{aligned}\int_{\mathcal{C}_k} |\phi - \phi_{\mathcal{C}_k}|^2 \, \mathrm{d}x \leq C_{d,\mu^\bullet} \sum_{\ell=0}^{\infty} \Big(&\big(\|\nabla u\|_{\mathrm{L}^2(\mathcal{C}_\ell)} + \|F\|_{\mathrm{L}^2(\mathcal{C}_\ell)}\big)\\ &\cdot \sum_{\beta=1}^{d} \|(\mathrm{Id} - \mathbb{P})\partial_\beta \mathcal{B}_{\mathcal{C}_k}((\overline{\phi - \phi_{\mathcal{C}_k}})|_{\mathcal{C}_k})\|_{\mathrm{L}^2(\mathcal{C}_\ell)}\Big).\end{aligned}$$

We proceed as follows: if $\operatorname{dist}(\mathcal{C}_\ell, \mathcal{C}_k) = 0$, we use the fact that $\mathrm{Id} - \mathbb{P}$ is an orthogonal projection and thus, that its L^2-operator norm is 1. If $\operatorname{dist}(\mathcal{C}_\ell, \mathcal{C}_k) > 0$, we use the kernel representation of $\mathrm{Id} - \mathbb{P}$ stated in (2.4) and (2.5). In any case, the last inequality in the following estimates follows by employing either (2.6) or (2.7) depending on the particular situation.

If $\operatorname{dist}(\mathcal{C}_\ell, \mathcal{C}_k) > 0$, $k \geq 2$, and $\ell \leq k-2$, then there is a constant $C_d > 0$ such that

$$\begin{aligned}
&\sum_{\beta=1}^{d} \|(\mathrm{Id} - \mathbb{P})\partial_\beta \mathcal{B}_{\mathcal{C}_k}((\overline{\phi - \phi_{\mathcal{C}_k}})|_{\mathcal{C}_k})\|_{\mathrm{L}^2(\mathcal{C}_\ell)} \\
&\leq C_d \bigg(\int_{\mathcal{C}_\ell} \bigg(\int_{\mathcal{C}_k} \frac{|\nabla \mathcal{B}_{\mathcal{C}_k}((\overline{\phi - \phi_{\mathcal{C}_k}})|_{\mathcal{C}_k})(y)|}{|x-y|^d} \, \mathrm{d}y \bigg)^2 \, \mathrm{d}x \bigg)^{\frac{1}{2}} \\
&\leq C_d \frac{|\mathcal{C}_\ell|^{\frac{1}{2}} |\mathcal{C}_k|^{\frac{1}{2}}}{\operatorname{dist}(\mathcal{C}_\ell, \mathcal{C}_k)^d} \|\nabla \mathcal{B}_{\mathcal{C}_k}((\overline{\phi - \phi_{\mathcal{C}_k}})|_{\mathcal{C}_k})\|_{\mathrm{L}^2(\mathcal{C}_k)} \\
&\leq C_d C_{Bog} \frac{|\mathcal{C}_\ell|^{\frac{1}{2}} |\mathcal{C}_k|^{\frac{1}{2}}}{\operatorname{dist}(\mathcal{C}_\ell, \mathcal{C}_k)^d} \|\phi - \phi_{\mathcal{C}_k}\|_{\mathrm{L}^2(\mathcal{C}_k)}.
\end{aligned}$$

If $\operatorname{dist}(\mathcal{C}_\ell, \mathcal{C}_k) > 0$, $k \geq 1$, and $\ell \geq k+2$, we use the observation on $(\mathrm{Id} - \mathbb{P})\partial_\beta$ below (2.5) to estimate

$$\begin{aligned}
&\sum_{\beta=1}^{d} \|(\mathrm{Id} - \mathbb{P})\partial_\beta \mathcal{B}_{\mathcal{C}_k}((\overline{\phi - \phi_{\mathcal{C}_k}})|_{\mathcal{C}_k})\|_{\mathrm{L}^2(\mathcal{C}_\ell)} \\
&\leq C_d \bigg(\int_{\mathcal{C}_\ell} \bigg(\int_{\mathcal{C}_k} \frac{|\mathcal{B}_{\mathcal{C}_k}((\overline{\phi - \phi_{\mathcal{C}_k}})|_{\mathcal{C}_k})(y)|}{|x-y|^{d+1}} \, \mathrm{d}y \bigg)^2 \, \mathrm{d}x \bigg)^{\frac{1}{2}} \\
&\leq C_d \frac{|\mathcal{C}_\ell|^{\frac{1}{2}} |\mathcal{C}_k|^{\frac{1}{2}}}{\operatorname{dist}(\mathcal{C}_\ell, \mathcal{C}_k)^{d+1}} \|\mathcal{B}_{\mathcal{C}_k}((\overline{\phi - \phi_{\mathcal{C}_k}})|_{\mathcal{C}_k})\|_{\mathrm{L}^2(\mathcal{C}_k)} \\
&\leq C_d C_{Bog} \frac{|\mathcal{C}_\ell|^{\frac{1}{2}} |\mathcal{C}_k|^{\frac{1}{2}} 2^k r}{\operatorname{dist}(\mathcal{C}_\ell, \mathcal{C}_k)^{d+1}} \|\phi - \phi_{\mathcal{C}_k}\|_{\mathrm{L}^2(\mathcal{C}_k)}.
\end{aligned}$$

Moreover, we find

$$\operatorname{dist}(\mathcal{C}_\ell, \mathcal{C}_k) \geq 2^{k-1} r - 2^\ell r \geq (2^{k-1} - 2^{k-2}) r = 2^{k-2} r \quad \text{if} \quad k \geq 2 \quad \text{and} \quad \ell \leq k-2$$

and

$$\operatorname{dist}(\mathcal{C}_\ell, \mathcal{C}_k) \geq 2^{\ell-1} r - 2^k r = (2^{\ell-1} - 2^{\ell-2}) r = 2^{\ell-2} r \quad \text{if} \quad k \geq 1 \quad \text{and} \quad \ell \geq k+2.$$

Combining all the previous estimates delivers in the case $k \geq 2$ and $\ell \leq k-2$ that

$$\sum_{\beta=1}^{d} \|(\mathrm{Id} - \mathbb{P})\partial_\beta \mathcal{B}_{\mathcal{C}_k}((\overline{\phi - \phi_{\mathcal{C}_k}})|_{\mathcal{C}_k})\|_{\mathrm{L}^2(\mathcal{C}_\ell)} \leq C_d C_{Bog} 2^{\frac{d}{2}(\ell - k)} \|\phi - \phi_{\mathcal{C}_k}\|_{\mathrm{L}^2(\mathcal{C}_k)}$$

and in the case $k \geq 1$ and $\ell \geq k+2$ that

$$\sum_{\beta=1}^{d} \|(\mathrm{Id} - \mathbb{P})\partial_\beta \mathcal{B}_{\mathcal{C}_k}((\overline{\phi - \phi_{\mathcal{C}_k}})|_{\mathcal{C}_k})\|_{\mathrm{L}^2(\mathcal{C}_\ell)} \leq C_d C_{Bog} 2^{(\frac{d}{2}+1)(k-\ell)} \|\phi - \phi_{\mathcal{C}_k}\|_{\mathrm{L}^2(\mathcal{C}_k)}.$$

As C_{Bog} also only depends on d, we altogether get a constant $C_{d,\mu^\bullet} > 0$ that depends only on d and $\mu^\bullet$ such that

$$\begin{aligned}&\int_{\mathcal{C}_k}|\phi-\phi_{\mathcal{C}_k}|^2\,\mathrm{d}x\\&\quad\leq C_{d,\mu^\bullet}\Big(\sum_{\ell=0}^{k-2}2^{\frac{d}{2}(\ell-k)}\big(\|\nabla u\|_{\mathrm{L}^2(\mathcal{C}_\ell)}+\|F\|_{\mathrm{L}^2(\mathcal{C}_\ell)}\big)+\sum_{\substack{\ell\in\mathbb{N}_0\\|\ell-k|\leq 1}}\big(\|\nabla u\|_{\mathrm{L}^2(\mathcal{C}_\ell)}+\|F\|_{\mathrm{L}^2(\mathcal{C}_\ell)}\big)\\&\qquad+\sum_{\ell=k+2}^{\infty}2^{(\frac{d}{2}+1)(k-\ell)}\big(\|\nabla u\|_{\mathrm{L}^2(\mathcal{C}_\ell)}+\|F\|_{\mathrm{L}^2(\mathcal{C}_\ell)}\big)\Big)\|\phi-\phi_{\mathcal{C}_k}\|_{\mathrm{L}^2(\mathcal{C}_k)}.\end{aligned}$$

Division by $\|\phi-\phi_{\mathcal{C}_k}\|_{\mathrm{L}^2(\mathcal{C}_k)}$ finally delivers the desired estimate. □

The last lemma gave a control of the pressure by the gradient of u and some parts of the right-hand side of the resolvent equation. The next lemma will contain the standard proof of Caccioppoli's inequality and will provide an estimate of $|\lambda|^{1/2}u$ and ∇u by u, $|\lambda|^{-1/2}f$, and F and also by an arbitrary small pressure term. As in Lemma 2.1 we adopt the notation $\mathcal{C}_k := B(x_0, 2^k r)\setminus\overline{B(x_0, 2^{k-1}r)}$ for $k\in\mathbb{N}$.

Lemma 2.2 *Let μ satisfy Assumption 1.1 with constants $\mu^\bullet, \mu_\bullet > 0$ and let $\omega_0\in(0,\frac{\pi}{2})$ be given by (1.7). Then for all $\theta\in(0,\pi-\omega_0)$ there exists $C>0$ such that for all $\lambda\in\mathrm{S}_\theta$, $\delta>0$, $c\in\mathbb{C}^d$, $f\in\mathrm{L}^2_\sigma(\mathbb{R}^d)$, and $F\in\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^{d\times d})$ the unique solutions $u\in\mathrm{H}^1_\sigma(\mathbb{R}^d)$ and $\phi\in\mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$ to*

$$\begin{cases}\lambda u-\operatorname{div}(\mu\nabla u)+\nabla\phi=f+\operatorname{div}(F) & \textit{in } \mathbb{R}^d,\\ \operatorname{div}(u)=0 & \textit{in } \mathbb{R}^d\end{cases}$$

satisfy

$$\begin{aligned}&|\lambda|\int_{B(x_0,2^k r)}|u|^2\,\mathrm{d}x+\int_{B(x_0,2^k r)}|\nabla u|^2\,\mathrm{d}x\\&\quad\leq\delta\int_{\mathcal{C}_{k+1}}|\phi-\phi_{\mathcal{C}_{k+1}}|^2\,\mathrm{d}x+C\Big(1+\frac{1}{\delta}\Big)\frac{2^{-2k}}{r^2}\int_{B(x_0,2^{k+1}r)}|u+c|^2\,\mathrm{d}x\\&\qquad+|\lambda||c|C\int_{B(x_0,2^{k+1}r)}|u|\,\mathrm{d}x+\frac{C}{|\lambda|}\int_{B(x_0,2^{k+1}r)}|f|^2\,\mathrm{d}x+C\int_{B(x_0,2^{k+1}r)}|F|^2\,\mathrm{d}x\\&\qquad+C|B(x_0,2^{k+1}r)||c|^2.\end{aligned}$$

The constant $C>0$ depends only on $\mu_\bullet$, $\mu^\bullet$, d, and θ.

Proof Let $\eta\in\mathrm{C}^\infty_c(B(x_0,2^{k+1}r))$ with $\eta\equiv 1$ in $B(x_0,2^k r)$, $0\leq\eta\leq 1$, and $\|\nabla\eta\|_{\mathrm{L}^\infty}\leq C_d/(2^k r)$ for some dimensional constant $C_d>0$. Using $v:=\eta^2(u+c)$ as a test function then delivers

$$\begin{aligned}&\lambda\int_{\mathbb{R}^d}|u|^2\eta^2\,\mathrm{d}x+\lambda\overline{c}\int_{\mathbb{R}^d}u\eta^2\,\mathrm{d}x+\int_{\mathbb{R}^d}\mu^{ij}_{\alpha\beta}\partial_\beta u_j\partial_\alpha[(\overline{u_i}+\overline{c_i})\eta^2]\,\mathrm{d}x-\int_{\mathbb{R}^d}\phi\,\operatorname{div}(\eta^2(\overline{u}+\overline{c}))\,\mathrm{d}x\\&=\int_{\mathbb{R}^d}f\cdot(\overline{u}+\overline{c})\eta^2\,\mathrm{d}x-\int_{\mathbb{R}^d}F_{\alpha\beta}\partial_\alpha[(\overline{u_\beta}+\overline{c_\beta})\eta^2]\,\mathrm{d}x.\end{aligned}\tag{2.8}$$

First of all, the pressure term can be rewritten as

$$\int_{\mathbb{R}^d} \phi \operatorname{div}(\eta^2(\overline{u}+\overline{c}))\,\mathrm{d}x = \int_{\mathbb{R}^d} (\phi - \phi_{\mathcal{C}_{k+1}}) \operatorname{div}(\eta^2(\overline{u}+\overline{c}))\,\mathrm{d}x$$
$$= 2\int_{\mathbb{R}^d} (\phi - \phi_{\mathcal{C}_{k+1}})\eta\nabla\eta\cdot(\overline{u}+\overline{c})\,\mathrm{d}x.$$

Next, the second-order term can be rewritten as

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mu^{ij}_{\alpha\beta}\partial_\beta u_j \partial_\alpha[(\overline{u_i}+\overline{c_i})\eta^2]\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \mu^{ij}_{\alpha\beta}\partial_\beta(u_j+c_j)\partial_\alpha[(\overline{u_i}+\overline{c_i})\eta]\eta\,\mathrm{d}x + \int_{\mathbb{R}^d} \mu^{ij}_{\alpha\beta}\partial_\beta(u_j+c_j)[\partial_\alpha\eta](\overline{u_i}+\overline{c_i})\eta\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \mu^{ij}_{\alpha\beta}\partial_\beta[(u_j+c_j)\eta]\partial_\alpha[(\overline{u_i}+\overline{c_i})\eta]\,\mathrm{d}x - \int_{\mathbb{R}^d} \mu^{ij}_{\alpha\beta}\partial_\beta\eta\partial_\alpha[(\overline{u_i}+\overline{c_i})\eta](u_j+c_j)\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^d} \mu^{ij}_{\alpha\beta}\partial_\beta[(u_j+c_j)\eta][\partial_\alpha\eta](\overline{u_i}+\overline{c_i})\,\mathrm{d}x - \int_{\mathbb{R}^d} \mu^{ij}_{\alpha\beta}\partial_\beta\eta[\partial_\alpha\eta](\overline{u_i}+\overline{c_i})(u_j+c_j)\,\mathrm{d}x \\
&=: \mathrm{I} - \mathrm{II} + \mathrm{III} - \mathrm{IV}.
\end{aligned}$$

Rearrange (2.8) in terms of the just derived identities to conclude that

$$\begin{aligned}
\lambda\int_{\mathbb{R}^d}|u|^2\eta^2\,\mathrm{d}x + \mathrm{I} = {}& 2\int_{\mathbb{R}^d}(\phi-\phi_{\mathcal{C}_{k+1}})\eta\nabla\eta\cdot(\overline{u}+\overline{c})\,\mathrm{d}x + \mathrm{II} - \mathrm{III} + \mathrm{IV} + \int_{\mathbb{R}^d} f\cdot\overline{u}\eta^2\,\mathrm{d}x \\
&+ \overline{c}\cdot\int_{\mathbb{R}^d} f\eta^2\,\mathrm{d}x - \int_{\mathbb{R}^d} F_{\alpha\beta}\partial_\alpha[(\overline{u_\beta}+\overline{c_\beta})\eta]\eta\,\mathrm{d}x \\
&- \int_{\mathbb{R}^d} F_{\alpha\beta}[\partial_\alpha\eta](\overline{u_\beta}+\overline{c_\beta})\eta\,\mathrm{d}x - \lambda\overline{c}\cdot\int_{\mathbb{R}^d} u\eta^2\,\mathrm{d}x.
\end{aligned}$$

As in (1.8), the ellipticity and boundedness conditions (1.4) and (1.5) imply that

$$\mathrm{I} \in \overline{\mathrm{S}_{\omega_0}}.$$

Since $\lambda \in \mathrm{S}_\theta$ and since $\theta + \omega_0 < \pi$, there exists $C > 0$ depending only on θ and ω_0 (and thus only on θ, $\mu_\bullet$, $\mu^\bullet$, and d) such that

$$|\lambda|\int_{\mathbb{R}^d}|u|^2\eta^2\,\mathrm{d}x + |\mathrm{I}| \leq C\Big|\lambda\int_{\mathbb{R}^d}|u|^2\eta^2\,\mathrm{d}x + \mathrm{I}\Big|.$$

Employing the ellipticity condition (1.4) once again, shows that

$$|\lambda|\int_{\mathbb{R}^d}|u|^2\eta^2\,\mathrm{d}x + \int_{\mathbb{R}^d}|\nabla[(u+c)\eta]|^2\,\mathrm{d}x \leq C_{\theta,\mu_\bullet,\mu^\bullet,d}\Big|\lambda\int_{\mathbb{R}^d}|u|^2\eta^2\,\mathrm{d}x + \mathrm{I}\Big|, \tag{2.9}$$

where $C_{\theta,\mu_\bullet,\mu^\bullet,d} > 0$ still only depends on θ, $\mu_\bullet$, $\mu^\bullet$, and d.

Let $\delta > 0$. The remaining terms are estimated by Young's inequality as

$$\begin{aligned}
&2\Big|\int_{\mathbb{R}^d}(\phi-\phi_{\mathcal{C}_{k+1}})\eta\nabla\eta\cdot(\overline{u}+\overline{c})\,\mathrm{d}x\Big| \\
&\quad\leq \frac{2C_d\cdot 2^{-k}}{r}\int_{\mathcal{C}_{k+1}}|\phi-\phi_{\mathcal{C}_{k+1}}||(u+c)\eta|\,\mathrm{d}x \qquad (2.10)\\
&\quad\leq \frac{\delta}{2C_{\theta,\mu_\bullet,\mu^\bullet,d}}\int_{\mathcal{C}_{k+1}}|\phi-\phi_{\mathcal{C}_{k+1}}|^2\,\mathrm{d}x + \frac{2C_d^2C_{\theta,\mu_\bullet,\mu^\bullet,d}2^{-2k}}{\delta r^2}\int_{\mathcal{C}_{k+1}}|u+c|^2\,\mathrm{d}x
\end{aligned}$$

and

$$
\begin{aligned}
|\mathrm{II}-\mathrm{III}| &\le \frac{2C_d\cdot 2^{-k}d^4\mu^\bullet}{r}\int_{\mathcal{C}_{k+1}}|\nabla[(u+c)\eta]||u+c|\,\mathrm{d}x\\
&\le \frac{1}{4C_{\theta,\mu_\bullet,\mu^\bullet,d}}\int_{\mathbb{R}^d}|\nabla[(u+c)\eta]|^2\,\mathrm{d}x\\
&\quad+\frac{4C_d^2\cdot 2^{-2k}d^8(\mu^\bullet)^2C_{\theta,\mu_\bullet,\mu^\bullet,d}}{r^2}\int_{\mathcal{C}_{k+1}}|u+c|^2\,\mathrm{d}x
\end{aligned}
\tag{2.11}
$$

and

$$
|\mathrm{IV}|\le\frac{4\cdot 2^{-2k}d^4\mu^\bullet}{r^2}\int_{\mathcal{C}_{k+1}}|u+c|^2\,\mathrm{d}x \tag{2.12}
$$

and

$$
\Big|\int_{\mathbb{R}^d}f\cdot\overline{u}\eta^2\,\mathrm{d}x\Big|\le\frac{C_{\theta,\mu_\bullet,\mu^\bullet,d}}{2|\lambda|}\int_{B(x_0,2^{k+1}r)}|f|^2\,\mathrm{d}x+\frac{|\lambda|}{2C_{\theta,\mu_\bullet,\mu^\bullet,d}}\int_{\mathbb{R}^d}|u\eta|^2\,\mathrm{d}x \tag{2.13}
$$

and

$$
\begin{aligned}
\Big|\overline{c}\cdot\int_{\mathbb{R}^d}f\eta^2\,\mathrm{d}x\Big| &\le |c||B(x_0,2^{k+1}r)|^{\frac12}\Big(\int_{B(x_0,2^{k+1}r)}|f|^2\,\mathrm{d}x\Big)^{\frac12}\\
&\le\frac{|B(x_0,2^{k+1}r)|}{2}|c|^2+\int_{B(x_0,2^{k+1}r)}|f|^2\,\mathrm{d}x
\end{aligned}
\tag{2.14}
$$

and

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^d}F_{\alpha\beta}\partial_\alpha[(\overline{u_\beta}+\overline{c_\beta})\eta]\eta\,\mathrm{d}x\Big|\\
&\quad\le\frac{1}{4C_{\theta,\mu_\bullet,\mu^\bullet,d}}\int_{\mathbb{R}^d}|\nabla[(u+c)\eta]|^2\,\mathrm{d}x+C_{\theta,\mu_\bullet,\mu^\bullet,d}\int_{B(x_0,2^{k+1}r)}|F|^2\,\mathrm{d}x
\end{aligned}
\tag{2.15}
$$

and

$$
\Big|\int_{\mathbb{R}^d}F_{\alpha\beta}[\partial_\alpha\eta](\overline{u_\beta}+\overline{c_\beta})\eta\,\mathrm{d}x\Big|\le\frac{C_d^2\cdot 2^{-2k}}{r^2}\int_{\mathcal{C}_{k+1}}|u+c|^2\,\mathrm{d}x+\int_{\mathcal{C}_{k+1}}|F|^2\,\mathrm{d}x. \tag{2.16}
$$

Putting all the estimates (2.9), (2.10), (2.11), (2.12), (2.13), (2.14), (2.15), and (2.16) together and rearranging the terms finally delivers

$$
\begin{aligned}
&|\lambda|\int_{\mathbb{R}^d}|u\eta|^2\,\mathrm{d}x+\int_{\mathbb{R}^d}|\nabla[(u+c)\eta]|^2\,\mathrm{d}x\\
&\le\delta\int_{\mathcal{C}_{k+1}}|\phi-\phi_{\mathcal{C}_{k+1}}|^2\,\mathrm{d}x+|\lambda||c|C\int_{B(x_0,2^{k+1}r)}|u|\,\mathrm{d}x\\
&\quad+\Big(1+\frac{1}{\delta}\Big)\frac{C\cdot 2^{-2k}}{r^2}\int_{\mathcal{C}_{k+1}}|u+c|^2\,\mathrm{d}x\\
&\quad+\frac{C}{|\lambda|}\int_{B(x_0,2^{k+1}r)}|f|^2\,\mathrm{d}x+C\int_{B(x_0,2^{k+1}r)}|F|^2\,\mathrm{d}x\\
&\quad+C|B(x_0,2^{k+1}r)||c|^2,
\end{aligned}
$$

where $C>0$ depends only on θ, $\mu_\bullet$, $\mu^\bullet$, and d. Finally, use that $\eta\equiv 1$ in $B(x_0,2^kr)$ and conclude the desired estimate. □

Before we come to the proof of Theorem 1.2, we state and prove the following elementary lemma.

Lemma 2.3 *Let $0 < \nu < d + 2$ and $(a_\ell)_{\ell \in \mathbb{N}_0} \in \ell^\infty$. Then there exists a constant $C > 0$ depending only on d and ν such that*

$$\sum_{k=1}^\infty 2^{-\nu k} \Big(\sum_{\ell=0}^{k-2} 2^{\frac{d}{2}(\ell - k)} a_\ell + \sum_{\substack{\ell \in \mathbb{N}_0 \\ |\ell - k| \leq 1}} a_\ell + \sum_{\ell = k+2}^\infty 2^{(\frac{d}{2} + 1)(k - \ell)} a_\ell \Big)^2 \leq C \sum_{\ell = 0}^\infty 2^{-\nu \ell} a_\ell^2.$$

Proof Let $\vartheta \in (0, 1)$ satisfy $\vartheta(d + 2) = (d + 2 + \nu)/2$. By the Cauchy–Schwarz inequality the series is then estimated by

$$\begin{aligned}
&\sum_{k=1}^\infty 2^{-\nu k} \Big(\sum_{\ell=0}^{k-2} 2^{\frac{d}{2}(\ell - k)} a_\ell + \sum_{\substack{\ell \in \mathbb{N}_0 \\ |\ell - k| \leq 1}} a_\ell + \sum_{\ell = k+2}^\infty 2^{(\frac{d}{2} + 1)(k - \ell)} a_\ell \Big)^2 \\
&\leq 3 \sum_{k=1}^\infty 2^{-\nu k} \Big\{ \Big(\sum_{\ell=0}^{k-2} 2^{(1-\vartheta)\frac{d}{2}(\ell - k)} 2^{\vartheta \frac{d}{2}(\ell - k)} a_\ell \Big)^2 + 3 \sum_{\substack{\ell \in \mathbb{N}_0 \\ |\ell - k| \leq 1}} a_\ell^2 \\
&\quad + \Big(\sum_{\ell = k+2}^\infty 2^{(1-\vartheta)(\frac{d}{2} + 1)(k - \ell)} 2^{\vartheta(\frac{d}{2} + 1)(k - \ell)} a_\ell \Big)^2 \Big\} \\
&\leq 3 \sum_{k=1}^\infty 2^{-\nu k} \Big\{ \sum_{\ell=0}^{k-2} 2^{(1-\vartheta) d (\ell - k)} \cdot \sum_{\ell=0}^{k-2} 2^{\vartheta d(\ell - k)} a_\ell^2 + 3 \sum_{\substack{\ell \in \mathbb{N}_0 \\ |\ell - k| \leq 1}} a_\ell^2 \\
&\quad + \sum_{\ell = k+2}^\infty 2^{(1-\vartheta)(d+2)(k - \ell)} \cdot \sum_{\ell = k+2}^\infty 2^{\vartheta(d+2)(k - \ell)} a_\ell^2 \Big\}.
\end{aligned}$$

Consequently, there exists a constant $C > 0$ depending only on d and ν such that

$$\begin{aligned}
&\sum_{k=1}^\infty 2^{-\nu k} \Big(\sum_{\ell=0}^{k-2} 2^{\frac{d}{2}(\ell - k)} a_\ell + \sum_{\substack{\ell \in \mathbb{N}_0 \\ |\ell - k| \leq 1}} a_\ell + \sum_{\ell = k+2}^\infty 2^{(\frac{d}{2} + 1)(k - \ell)} a_\ell \Big)^2 \\
&\leq C \sum_{k=1}^\infty 2^{-\nu k} \Big\{ \sum_{\ell=0}^{k-2} 2^{\vartheta d(\ell - k)} a_\ell^2 + \sum_{\substack{\ell \in \mathbb{N}_0 \\ |\ell - k| \leq 1}} a_\ell^2 + \sum_{\ell = k+2}^\infty 2^{\vartheta(d+2)(k - \ell)} a_\ell^2 \Big\}.
\end{aligned}$$

The only terms that are of interest right now are the first and the third series. After applying Fubini's theorem to each of the series we derive by virtue of $\vartheta(d + 2) - \nu > 0$ with a different constant $C > 0$ still depending only on d and ν that

$$\begin{aligned}
&\sum_{k=1}^\infty 2^{-\nu k} \Big\{ \sum_{\ell=0}^{k-2} 2^{\vartheta d(\ell - k)} a_\ell^2 + \sum_{\ell = k+2}^\infty 2^{\vartheta(d+2)(k - \ell)} a_\ell^2 \Big\} \\
&= \sum_{\ell=0}^\infty 2^{\ell \vartheta d} a_\ell^2 \sum_{k = \ell + 2}^\infty 2^{-(\vartheta d + \nu) k} + \sum_{\ell = 3}^\infty 2^{-\vartheta(d+2)\ell} a_\ell^2 \sum_{k=1}^{\ell - 2} 2^{(\vartheta(d+2) - \nu) k}
\end{aligned}$$

$$\leq C\bigg\{\sum_{\ell=0}^{\infty} 2^{-\nu\ell} a_\ell^2 + \sum_{\ell=3}^{\infty} 2^{-\nu\ell} a_\ell^2\bigg\}.$$

□

Proof of Theorem 1.2 Let $\delta > 0$ be a constant to be fixed during the proof. By virtue of Lemma 2.2 applied with constant $c = c_k \in \mathbb{C}^d$, we estimate with a constant $C > 0$ depending only on $\mu_\bullet$, $\mu^\bullet$, d, and θ

$$\begin{aligned}
&|\lambda| \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^k r)} |u|^2 \,\mathrm{d}x + \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^k r)} |\nabla u|^2 \,\mathrm{d}x \\
&\quad\leq \delta 2^\nu \sum_{k=1}^{\infty} 2^{-\nu k} \int_{\mathcal{C}_k} |\phi - \phi_{\mathcal{C}_k}|^2 \,\mathrm{d}x + \Big(1 + \frac{1}{\delta}\Big) \frac{C}{r^2} \sum_{k=0}^{\infty} 2^{-(\nu+2)k} \int_{\mathcal{C}_{k+1}} |u + c_k|^2 \,\mathrm{d}x \\
&\qquad + |\lambda| \sum_{k=0}^{\infty} |c_k| 2^{-\nu k} \int_{B(x_0,2^{k+1}r)} |u| \,\mathrm{d}x + \frac{C}{|\lambda|} \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^{k+1}r)} |f|^2 \,\mathrm{d}x \\
&\qquad + C \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^{k+1}r)} |F|^2 \,\mathrm{d}x + C \sum_{k=0}^{\infty} 2^{-\nu k} |B(x_0, 2^{k+1} r)| |c_k|^2.
\end{aligned} \tag{2.17}$$

Now, we employ Lemma 2.1 first, followed by Lemma 2.3 with $a_\ell := \|\nabla u\|_{\mathrm{L}^2(\mathcal{C}_\ell)} + \|F\|_{\mathrm{L}^2(\mathcal{C}_\ell)}$ for $\ell \in \mathbb{N}_0$ to establish for some constant $C_{d,\nu} > 0$ that

$$\begin{aligned}
&\sum_{k=1}^{\infty} 2^{-\nu k} \int_{\mathcal{C}_k} |\phi - \phi_{\mathcal{C}_k}|^2 \,\mathrm{d}x \\
&\leq \sum_{k=1}^{\infty} 2^{-\nu k} \bigg(\sum_{\ell=0}^{k-2} 2^{\frac{d}{2}(\ell-k)} a_\ell + \sum_{\substack{\ell \in \mathbb{N}_0 \\ |\ell-k| \leq 1}} a_\ell + \sum_{\ell=k+2}^{\infty} 2^{(\frac{d}{2}+1)(k-\ell)} a_\ell \bigg)^2 \\
&\leq C_{d,\nu} \sum_{\ell=0}^{\infty} 2^{-\nu\ell} \bigg(\int_{\mathcal{C}_\ell} |\nabla u|^2 \,\mathrm{d}x + \int_{\mathcal{C}_\ell} |F|^2 \,\mathrm{d}x \bigg).
\end{aligned}$$

Plugging this estimate into (2.17) and using that $\mathcal{C}_k \subset B(x_0, 2^k r)$ then delivers

$$\begin{aligned}
&|\lambda| \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^k r)} |u|^2 \,\mathrm{d}x + \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^k r)} |\nabla u|^2 \,\mathrm{d}x \\
&\quad\leq \delta 2^\nu C_{d,\nu} \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^k r)} |\nabla u|^2 \,\mathrm{d}x + \Big(1 + \frac{1}{\delta}\Big) \frac{C}{r^2} \sum_{k=0}^{\infty} 2^{-(\nu+2)k} \int_{\mathcal{C}_{k+1}} |u + c_k|^2 \,\mathrm{d}x \\
&\qquad + |\lambda| \sum_{k=0}^{\infty} |c_k| 2^{-\nu k} \int_{B(x_0,2^{k+1}r)} |u| \,\mathrm{d}x + \frac{C}{|\lambda|} \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^{k+1}r)} |f|^2 \,\mathrm{d}x \\
&\qquad + (C + \delta 2^\nu C_{d,\nu}) \sum_{k=0}^{\infty} 2^{-\nu k} \int_{B(x_0,2^{k+1}r)} |F|^2 \,\mathrm{d}x + C \sum_{k=0}^{\infty} 2^{-\nu k} |B(x_0, 2^{k+1} r)| |c_k|^2.
\end{aligned}$$

Now, choose δ such that $\delta 2^\nu C_{d,\nu} = 1/2$. Absorbing the first term on the right-hand side into the corresponding term on the left-hand side finally yields the desired estimate. □

3 A digression on sectorial and $\mathcal{R}$-sectorial operators

We start by introducing the concepts of sectorial and $\mathcal{R}$-sectorial operators.

Definition 3.1 Let X and Y denote Banach spaces over the complex field and let $B : \mathcal{D}(B) \subset X \to X$ be a linear operator.

1. The operator B is said to be *sectorial* of angle $\omega \in (0, \pi)$ if
$$\sigma(B) \subset \overline{\mathrm{S}_\omega}$$
and if for all $\theta \in (0, \pi - \omega)$ the family $\{\lambda(\lambda + B)^{-1}\}_{\lambda \in \mathrm{S}_\theta} \subset \mathcal{L}(X)$ is bounded.
2. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is said to be *$\mathcal{R}$-bounded* if there exists a positive constant $C > 0$ such that for any $k_0 \in \mathbb{N}$, $(T_k)_{k=1}^{k_0} \subset \mathcal{T}$, and $(x_k)_{k=1}^{k_0} \subset X$ the inequality
$$\Big\| \sum_{k=1}^{k_0} r_k(\cdot) T_k x_k \Big\|_{\mathrm{L}^2(0,1;Y)} \leq C \Big\| \sum_{k=1}^{k_0} r_k(\cdot) x_k \Big\|_{\mathrm{L}^2(0,1;X)}$$
holds. Here, $r_k(t) := \operatorname{sgn}(\sin(2^k \pi t))$ are the *Rademacher functions*. The smallest such constant C is called *$\mathcal{R}$-bound* of $\mathcal{T}$ and denoted by $\mathcal{R}(\mathcal{T})$.
3. The operator B is said to be *$\mathcal{R}$-sectorial* of angle $\omega \in [0, \pi)$ if
$$\sigma(B) \subset \overline{\mathrm{S}_\omega}$$
and if for all $\theta \in (0, \pi - \omega)$ the family $\{\lambda(\lambda + B)^{-1}\}_{\lambda \in \mathrm{S}_\theta} \subset \mathcal{L}(X)$ is $\mathcal{R}$-bounded.

Remark 3.2
1. By taking $k_0 = 1$ one sees that $\mathcal{R}$-boundedness implies boundedness of a family of operators. If X and Y are isomorphic to a Hilbert space, then $\mathcal{R}$-boundedness is equivalent to the boundedness of the family of operators, see [7, Rem. 3.2].
2. If X is a subspace of $\mathrm{L}^p(\Omega; \mathbb{C}^m)$ for some $1 < p < \infty$, $m \in \mathbb{N}$, and $\Omega \subset \mathbb{R}^d$ Lebesgue measurable, then there exists $C > 0$ such that for all $k_0 \in \mathbb{N}$ and $(f_k)_{k=1}^{k_0}$ it holds
$$\frac{1}{C} \Big\| \sum_{k=1}^{k_0} r_k(\cdot) f_k \Big\|_{\mathrm{L}^2(0,1;X)} \leq \Big\| \Big[\sum_{k=1}^{k_0} |f_k|^2 \Big]^{1/2} \Big\|_{\mathrm{L}^p(\Omega)} \leq C \Big\| \sum_{k=1}^{k_0} r_k(\cdot) f_k \Big\|_{\mathrm{L}^2(0,1;X)}.$$
This means, that $\mathcal{R}$-boundedness in L^p-spaces is equivalent to so-called *square function estimates* [7, Rem. 3.2].
3. The operator $-B$ generates a strongly continuous bounded analytic semigroup on X if and only if B is densely defined and sectorial of angle $\omega \in [0, \pi/2)$, see [11, Thm. II.4.6].

It is well-known from classical form theory [22], that operators associated to bounded, accretive and elliptic sesquilinear forms are sectorial. In particular, the generalized Stokes operator A is sectorial in $\mathrm{L}^2_\sigma(\mathbb{R}^d)$ and the angle can be computed to be at least ω_0 defined in (1.7). The following proposition provides a more detailed statement. As it relies on a standard application of the Lax–Milgram lemma and of using the solution u as a test function to the resolvent equation, we omit the details.

Proposition 3.3 *Let μ satisfy Assumption 1.1 for some constants $\mu_\bullet, \mu^\bullet > 0$ and let $\omega_0 \in (0, \frac{\pi}{2})$ be given by (1.7). Then the spectrum of A is contained in $\overline{\mathrm{S}_{\omega_0}}$ and for all $\theta \in (0, \pi - \omega_0)$*

there exists $C > 0$ *such that for all* $\lambda \in \mathrm{S}_\theta$ *and all* $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$ *we have with* $u := (\lambda + A)^{-1} f$ *that*

$$|\lambda| \|u\|_{\mathrm{L}^2_\sigma} + |\lambda|^{\frac{1}{2}} \|\nabla u\|_{\mathrm{L}^2} \leq C \|f\|_{\mathrm{L}^2_\sigma}.$$

Moreover, for all $\lambda \in \mathrm{S}_\theta$ *and all* $F \in \mathrm{C}^\infty_c(\mathbb{R}^d; \mathbb{C}^{d \times d})$ *there exists* $C > 0$ *such that with* $u := (\lambda + A)^{-1} \mathbb{P} \operatorname{div}(F)$ *it holds*

$$|\lambda|^{\frac{1}{2}} \|u\|_{\mathrm{L}^2_\sigma} + \|\nabla u\|_{\mathrm{L}^2} \leq C \|F\|_{\mathrm{L}^2}.$$

The constants C *only depend on* d, $\mu_\bullet$, $\mu^\bullet$, *and* θ.

In the course of this article, we will show that A gives rise to an $\mathcal{R}$-sectorial operator on $\mathrm{L}^p_\sigma(\mathbb{R}^d)$ for all $p \in (1, \infty)$ satisfying (1.11). In particular, this will imply Theorems 1.5 and 1.3. Eventually, the following observation will be crucial.

Observation 3.4 Proposition 3.3 implies that $\{\lambda(\lambda + A)^{-1}\}_{\lambda \in \mathrm{S}_\theta} \subset \mathcal{L}(\mathrm{L}^2_\sigma(\mathbb{R}^d))$ is bounded for all $\theta \in (0, \pi - \omega_0)$, so that A is sectorial in $\mathrm{L}^2_\sigma(\mathbb{R}^d)$ of angle ω_0. Moreover, Remark 3.2 (1) yields that A is also $\mathcal{R}$-sectorial of angle ω_0.

A combination of Definition 3.1 (2) and Remark 3.2 (2) reveals that the following statement is immediate:

For $1 < p < \infty$, the family $\{\lambda(\lambda + A)^{-1}\}_{\lambda \in \mathrm{S}_\theta}$ extends from $\mathrm{C}^\infty_{c,\sigma}(\mathbb{R}^d)$ to an $\mathcal{R}$-bounded family in $\mathcal{L}(\mathrm{L}^p_\sigma(\mathbb{R}^d))$ if and only if there exists $C > 0$ such that for all $k_0 \in \mathbb{N}$, all $(\lambda_k)_{k=1}^{k_0} \subset \mathrm{S}_\theta$, and all $f := (f_1, \dots, f_{k_0}, 0, \dots)$ with $f_k \in \mathrm{C}^\infty_{c,\sigma}(\mathbb{R}^d)$, $1 \leq k \leq k_0$, the operator $T_{(\lambda_k)_{k=1}^{k_0}}$ given by

$$T_{(\lambda_k)_{k=1}^{k_0}} f \mapsto \begin{pmatrix} \lambda_1(\lambda_1 + A)^{-1} f_1 \\ \vdots \\ \lambda_{k_0}(\lambda_{k_0} + A)^{-1} f_{k_0} \\ 0 \\ \vdots \end{pmatrix}$$

satisfies

$$\|T_{(\lambda_k)_{k=1}^{k_0}} f\|_{\mathrm{L}^p(\mathbb{R}^d; \ell^2(\mathbb{C}^d))} \leq C \|f\|_{\mathrm{L}^p(\mathbb{R}^d; \ell^2(\mathbb{C}^d))}. \tag{3.1}$$

In other words, the family of all operators that can be formed by the procedure above extends to a bounded family in $\mathrm{L}^p(\mathbb{R}^d; \ell^2(\mathbb{C}^d))$. In particular, from the first part of this observation, we know that this statement is valid in the case $p = 2$.

4 A glimpse onto a non-local L^p-extrapolation theorem

To extrapolate the $\mathcal{R}$-bounded family $\{\lambda(\lambda + A)^{-1}\}_{\lambda \in \mathrm{S}_\theta}$ for $\theta \in (0, \pi - \omega_0)$ in $\mathcal{L}(\mathrm{L}^2_\sigma(\mathbb{R}^d))$ to an $\mathcal{R}$-bounded family in $\mathcal{L}(\mathrm{L}^p_\sigma(\mathbb{R}^d))$ for p satisfying (1.11) we want to employ the following vector valued and non-local analogue of Shen's L^p-extrapolation theorem [35, Thm. 1.2]. See [28, Thm. 3.1] for Shen's original theorem.

Theorem 4.1 *Let* X, Y, *and* Z *be Banach spaces,* $\mathcal{M}, \mathcal{N} > 0$, *and let*

$$T \in \mathcal{L}(\mathrm{L}^2(\mathbb{R}^d; X), \mathrm{L}^2(\mathbb{R}^d; Y)) \quad \textit{with} \quad \|T\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{R}^d; X), \mathrm{L}^2(\mathbb{R}^d; Y))} \leq \mathcal{M}$$

$$\textit{and}\quad \mathcal{C} \in \mathcal{L}(\mathrm{L}^2(\mathbb{R}^d; X), \mathrm{L}^2(\mathbb{R}^d; Z)) \quad \textit{with} \quad \|\mathcal{C}\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{R}^d;X),\mathrm{L}^2(\mathbb{R}^d;Z))} \leq \mathcal{N}.$$

Suppose that there exist constants $p_0 > 2$, $\iota > 1$, *and* $C > 0$ *such that for all balls* $B \subset \mathbb{R}^d$ *and all compactly supported* $f \in \mathrm{L}^\infty(\mathbb{R}^d; X)$ *with* $f = 0$ *in* ιB *the estimate*

$$\Big(\fint_B \|Tf\|_Y^{p_0} \,\mathrm{d}x \Big)^{\frac{1}{p_0}} \leq C \sup_{B' \supset B} \Big(\fint_{B'} \big(\|Tf\|_Y^2 + \|\mathcal{C}f\|_Z^2\big) \,\mathrm{d}x \Big)^{\frac{1}{2}} \tag{4.1}$$

holds. Here the supremum runs over all balls B' *containing* B.

Then for each $2 < p < p_0$ *there exists a constant* $K > 0$ *such that for all* $f \in \mathrm{L}^\infty(\mathbb{R}^d; X)$ *with compact support it holds*

$$\|Tf\|_{\mathrm{L}^p(\mathbb{R}^d;Y)} \leq K\big(\|f\|_{\mathrm{L}^p(\mathbb{R}^d;X)} + \|\mathcal{C}f\|_{\mathrm{L}^p(\mathbb{R}^d;Z)}\big).$$

In particular, if $\mathcal{C}$ *is bounded from* $\mathrm{L}^p(\mathbb{R}^d; X)$ *into* $\mathrm{L}^p(\mathbb{R}^d; Z)$, *then the restriction of* T *onto* $\mathrm{L}^2(\mathbb{R}^d; X) \cap \mathrm{L}^p(\mathbb{R}^d; X)$ *extends to a bounded linear operator from* $\mathrm{L}^p(\mathbb{R}^d; X)$ *into* $\mathrm{L}^p(\mathbb{R}^d; Y)$. *The constant* K *depends only on* d, p_0, p, ι, C, $\mathcal{M}$, *and* $\mathcal{N}$.

In our situation, we would like to choose $X = Y = Z := \ell^2(\mathbb{C}^d)$, $\mathcal{C} := \mathrm{Id}$, and the operator T as one of the operators defined in Observation 3.4. If all these operators would satisfy the assumptions of Theorem 4.1 in a uniform manner, we could conclude the $\mathcal{R}$-boundedness of this family in L^p-spaces. However, there is one issue, namely, the resolvent operators $(\lambda + A)^{-1}$ are only defined on $\mathrm{L}^2_\sigma(\mathbb{R}^d)$ and not on $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^d)$. Clearly, one could try to replace $(\lambda + A)^{-1}$ by the operator $(\lambda + A)^{-1}\mathbb{P}$, which is a bounded operator defined on all of $\mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^d)$. However, as it was mentioned earlier, to verify (4.1) one needs Caccioppoli's inequality, cf. Theorem 1.2, and this inequality requires the right-hand side f to be solenoidal. More precisely, the solenoidality was essentially used in the proof of Lemma 2.1. Being bound to right-hand sides in $\mathrm{L}^2_\sigma(\mathbb{R}^d)$ we present in the following, how to adapt the proof of Theorem 4.1 to our needs. We advise the reader to have a copy of the proof at hand. Notice that a similar analysis was performed in [34, Sect. 5] for X and Y being finite dimensional.

Throughout the proof, a function f is fixed and for exactly this function the boundedness estimate

$$\|Tf\|_{\mathrm{L}^p(\mathbb{R}^d;Y)} \leq K\big(\|f\|_{\mathrm{L}^p(\mathbb{R}^d;X)} + \|\mathcal{C}f\|_{\mathrm{L}^p(\mathbb{R}^d;Z)}\big)$$

is proved. To establish this estimate a good-λ argument is used. An analysis of this good-λ argument reveals that the L^2-boundedness of T and $\mathcal{C}$ as well as (4.1) are used exactly once, namely, in order to deduce an inequality of the form

$$\begin{aligned} |\{x \in Q : M_{2Q^*}(\|Tf\|_Y^2)(x) > \alpha\}| &\leq \frac{C}{\alpha} \int_{2\iota Q^*} \big(\|f\|_X^2 + \|\mathcal{C}f\|_Z^2\big) \,\mathrm{d}x \\ &\quad + \frac{C|Q|}{\alpha^{p_0/2}} \Big\{ \sup_{Q' \supset 2Q^*} \Big(\frac{1}{|Q'|} \int_{Q'} \big(\|Tf\|_Y^2 + \|f\|_X^2 + \|\mathcal{C}f\|_Z^2\big) \,\mathrm{d}x \Big)^{\frac{1}{2}} \Big\}^{p_0}, \end{aligned} \tag{4.2}$$

cf. the proof of Claim 3 in [35, Thm. 1.2]. Here, $\alpha > 0$ is arbitrary, Q is a cube in $\mathbb{R}^d$, Q^* is its dyadic "parent", i.e., Q arises from Q^* by bisecting its sides, and M_{2Q^*} is the localized maximal operator

$$M_{2Q^*}g(x) := \sup_{\substack{x \in R \\ R \subset 2Q^*}} \frac{1}{|R|} \int_R |g| \,\mathrm{d}y \qquad (x \in 2Q^*),$$

where in the supremum R denotes a cube in $\mathbb{R}^d$. Following the proof of [35, Thm. 1.2], to conclude (4.2) from (4.1) and the L^2-boundedness of T and $\mathcal{C}$, it first has to be noticed that (4.1) can equivalently be formulated with cubes instead of balls. Then, f is decomposed as $f = f\chi_{2\iota Q^*} + f\chi_{\mathbb{R}^d\setminus 2\iota Q^*}$, where χ denotes the characteristic function of a set. This decomposition is used on the left-hand side of (4.2) to estimate

$$\begin{aligned} |\{x \in Q : M_{2Q^*}(\|Tf\|_Y^2)(x) > \alpha\}| \leq |\{x \in Q &: M_{2Q^*}(\|Tf\chi_{2\iota Q^*}\|_Y^2)(x) > \alpha/4\}| \\ &+ |\{x \in Q : M_{2Q^*}(\|Tf\chi_{\mathbb{R}^d\setminus 2\iota Q^*}\|_Y^2)(x) > \alpha/4\}|. \end{aligned} \tag{4.3}$$

The first term on the right-hand side is controlled by the weak type-(1, 1) estimate of the localized maximal operator and the L^2-boundedness of T, yielding the first term on the right-hand side of (4.2). The second term on the right-hand side is controlled by the embedding $\mathrm{L}^{p_0/2} \hookrightarrow \mathrm{L}^{p_0/2,\infty}$ and the $\mathrm{L}^{p_0/2}$-boundedness of the localized maximal operator followed by (4.1) and the L^2-boundedness of T and $\mathcal{C}$ yielding the remaining terms on the right-hand side of (4.2), cf. the proof of Claim 3 in [34, Thm. 1.2].

Essentially, the only thing that happened in (4.3) was that Tf was decomposed by means of

$$Tf = Tf\chi_{2\iota Q^*} + Tf\chi_{\mathbb{R}^d\setminus 2\iota Q^*}. \tag{4.4}$$

We would like to emphasize here that this decomposition of Tf is induced by the linearity of T and a decomposition of f. Clearly, one could imagine that other suitable decompositions of Tf into a sum of two functions exist. For example, in the next section T will incorporate operators $T_j f := \lambda_j(\lambda_j + A)^{-1} f$ and we will decompose $T_j f$ as $\lambda_j u_1 + \lambda_j u_2$, where u_1 solves an appropriate resolvent problem in $2\iota Q^*$ with right-hand side $f|_{2\iota Q^*}$. Since u_1 is undefined outside of $2\iota Q^*$ this yields, in general, a different decomposition as in (4.4).

Taking this into account in the formulation of the L^p-extrapolation theorem might yield a more flexible result. This could be an advantage if a certain structure of f (such as solenoidality) is eminent and which is destroyed by multiplication by characteristic functions. This indicates the need of a formulation of Shen's L^p-extrapolation theorem that does not rely on a particular decomposition of Tf and is presented in the following without delving further into its proof.

To this end, we say that Q^* is the parent of a cube $Q \subset \mathbb{R}^d$ if Q arises from Q^* by bisecting its sides. Moreover, for $x_0 \in \mathbb{R}^d$ and $r > 0$ let $Q(x_0, r)$ denote the cube in $\mathbb{R}^d$ with center x_0 and $\mathrm{diam}(Q(x_0, r)) = r$. Finally, for a number $\alpha > 0$ denote by αQ the cube $Q(x_0, \alpha r)$. In the following formulation of the L^p-extrapolation theorem, we simply replace the L^2-boundedness of T and $\mathcal{C}$ together with (4.1) by the assumption that (4.2) is valid.

Theorem 4.2 *Let X, Y, and Z be Banach spaces. Let further $2 < p < p_0$, $f \in \mathrm{L}^2(\mathbb{R}^d; X) \cap \mathrm{L}^p(\mathbb{R}^d; X)$, and let T be an operator (not necessarily linear) such that $T(f)$ is defined and contained in $\mathrm{L}^2(\mathbb{R}^d; Y)$.*

Suppose that there exist constants $\iota > 1$ and $C > 0$ and an operator $\mathcal{C}$ (not necessarily linear) such that $\mathcal{C}(f)$ lies in $\mathrm{L}^2(\mathbb{R}^d; Z)$ and such that for all $\alpha > 0$, all $Q = Q(x_0, r)$ with $r > 0$ and $x_0 \in \mathbb{R}^d$, and all parents Q^* of Q the estimate

$$|\{x \in Q : M_{2Q^*}(\|T(f)\|_Y^2)(x) > \alpha\}| \leq \frac{C}{\alpha}\int_{2\iota Q^*} \big(\|f\|_X^2 + \|\mathcal{C}(f)\|_Z^2\big)\,\mathrm{d}x + \frac{C|Q|}{\alpha^{p_0/2}}\bigg\{\sup_{Q' \supset 2Q^*}\bigg(\frac{1}{|Q'|}\int_{Q'}\big(\|T(f)\|_Y^2 + \|f\|_X^2 + \|\mathcal{C}(f)\|_Z^2\big)\,\mathrm{d}x\bigg)^{\frac{1}{2}}\bigg\}^{p_0}, \tag{4.5}$$

holds. Here the supremum runs over all cubes Q' containing $2Q^*$.

Then there exists a constant $K > 0$ depending on d, p, p_0, ι, and C such that

$$\|T(f)\|_{\mathrm{L}^p(\mathbb{R}^d;Y)} \leq K\big(\|f\|_{\mathrm{L}^p(\mathbb{R}^d;X)} + \|\mathcal{C}(f)\|_{\mathrm{L}^p(\mathbb{R}^d;Z)}\big).$$

5 Proofs of Theorems 1.3, 1.4, and 1.5

This section is dedicated to the proofs of Theorems 1.3, 1.4, and 1.5. To begin with, we have another look onto a Caccioppoli inequality which is similar to the one in Lemma 2.2.

Lemma 5.1 *Let μ satisfy Assumption 1.1 for some constants $\mu_\bullet, \mu^\bullet > 0$ and let $\omega_0 \in (0, \frac{\pi}{2})$ be given by (1.7). Then for all $\theta \in (0, \pi - \omega_0)$ there exists $C > 0$ such that for all $\lambda \in \mathrm{S}_\theta$ and all solutions $u \in \mathrm{H}^1(B(x_0, 2r); \mathbb{C}^d)$ and $\phi \in \mathrm{L}^2(B(x_0, 2r))$ (in the sense of distributions) to*

$$\begin{cases} \lambda u - \operatorname{div}(\mu \nabla u) + \nabla \phi = 0 & \text{in } B(x_0, 2r), \\ \operatorname{div}(u) = 0 & \text{in } B(x_0, 2r) \end{cases}$$

satisfy for all $\eta \in \mathrm{C}_c^\infty(B(x_0, 2r))$ with $\eta \equiv 1$ in $B(x_0, r)$, $0 \leq \eta \leq 1$, and $\|\nabla \eta\|_{\mathrm{L}^\infty} \leq 2/r$ and for all $c_1 \in \mathbb{C}$ and $c_2 \in \mathbb{C}^d$ the estimate

$$\begin{aligned} |\lambda| \int_{B(x_0,2r)} |u\eta|^2\,\mathrm{d}x + \int_{B(x_0,2r)} |\nabla[(u + c_2)\eta]|^2\,\mathrm{d}x \\ \leq C\bigg\{\frac{1}{r^2}\int_{B(x_0,2r)} |u + c_2|^2\,\mathrm{d}x + |c_2||\lambda|\int_{B(x_0,2r)} |u|\eta^2\,\mathrm{d}x \\ + \frac{1}{r}\bigg(\int_{B(x_0,2r)\setminus B(x_0,r)} |\phi - c_1|^2\,\mathrm{d}x\bigg)^{\frac{1}{2}}\bigg(\int_{B(x_0,2r)} |(u + c_2)\eta|^2\,\mathrm{d}x\bigg)^{\frac{1}{2}}\bigg\}. \end{aligned}$$

The constant $C > 0$ depends only on $\mu_\bullet$, $\mu^\bullet$, d, and θ.

Proof The proof is literally the same as the proof of Lemma 2.2 in the case $k = 0$. The only difference is the estimate of the pressure term in (2.10). Notice that in order to derive (2.10) we subtracted the constant $\phi_{\mathcal{C}_{k+1}}$ of ϕ but that it was possible to subtract any other constant as well. Thus, it is no problem to replace $\phi_{\mathcal{C}_{k+1}}$ by c_1 in (2.10). This term then reads

$$2\bigg|\int_{\mathbb{R}^d} (\phi - c_1)\eta \nabla \eta \cdot (\overline{u} + \overline{c_2})\,\mathrm{d}x\bigg|.$$

Now, by the properties of η and by Hölder's inequality we find that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^d}(\phi-c_1)\eta\nabla\eta\cdot(\overline{u}+\overline{c_2})\,\mathrm{d}x\Big|\\&\quad\leq\frac{2}{r}\bigg(\int_{B(x_0,2r)\setminus B(x_0,r)}|\phi-c_1|^2\,\mathrm{d}x\bigg)^{\frac{1}{2}}\bigg(\int_{B(x_0,2r)}|(u+c_2)\eta|^2\,\mathrm{d}x\bigg)^{\frac{1}{2}}.\end{aligned}$$

This readily concludes the proof. □

To proceed we introduce another sesquilinear form, which is associated to the Stokes problem in a ball but with Neumann boundary conditions. For this purpose, let $B\subset\mathbb{R}^d$ denote a ball, let

$$\mathcal{L}^2_\sigma(B):=\{f\in\mathrm{L}^2(B;\mathbb{C}^d):\mathrm{div}(f)=0\text{ in the sense of distributions}\},$$

and let

$$\mathcal{H}^1_\sigma(B):=\{f\in\mathrm{H}^1(B;\mathbb{C}^d):\mathrm{div}(f)=0\}.$$

Now, define the sesquilinear form

$$\mathfrak{b}_B:\mathcal{H}^1_\sigma(B)\times\mathcal{H}^1_\sigma(B)\to\mathbb{C},\quad(u,v)\mapsto\int_{\mathbb{R}^d}\mu^{ij}_{\alpha\beta}\partial_\beta u_j\overline{\partial_\alpha v_i}\,\mathrm{d}x.$$

We abuse the notation and denote the same sesquilinear form but with domain $\mathrm{H}^1(B;\mathbb{C}^d)\times\mathrm{H}^1(B;\mathbb{C}^d)$ again by $\mathfrak{b}_B$. The operator associated to $\mathfrak{b}_B$ is a generalized Stokes operator on B subject to Neumann-type boundary conditions. The following lemma shows how to reconstruct the pressure associated to its resolvent problem. Its proof is an adaption of the constant coefficients case carried out in [25, Pf. of Thm. 6.8].

Lemma 5.2 *Let $\theta\in(0,\pi-\omega_0)$ and $B\subset\mathbb{R}^d$ be a ball. There exists a constant $C>0$ such that for all $\lambda\in\mathrm{S}_\theta$, $f\in\mathcal{L}^2_\sigma(B)$ and $F\in\mathrm{L}^2(B;\mathbb{C}^{d\times d})$ there exists a unique solution $u\in\mathcal{H}^1_\sigma(B)$ to*

$$\lambda\int_B u\cdot\overline{v}\,\mathrm{d}x+\mathfrak{b}_B(u,v)=\int_B f\cdot\overline{v}\,\mathrm{d}x-\int_B F_{\alpha\beta}\overline{\partial_\alpha v_\beta}\,\mathrm{d}x\quad(v\in\mathcal{H}^1_\sigma(B))\tag{5.1}$$

and an associated $\phi\in\mathrm{L}^2(B)$ solving

$$\begin{aligned}&\lambda\int_B u\cdot\overline{v}\,\mathrm{d}x+\mathfrak{b}_B(u,v)-\int_B\phi\,\overline{\mathrm{div}(v)}\,\mathrm{d}x\\&\quad=\int_B f\cdot\overline{v}\,\mathrm{d}x-\int_B F_{\alpha\beta}\overline{\partial_\alpha v_\beta}\,\mathrm{d}x\quad(v\in\mathrm{H}^1(B;\mathbb{C}^d))\end{aligned}\tag{5.2}$$

which satisfy the estimate

$$\begin{aligned}&|\lambda|\|u\|_{\mathcal{L}^2_\sigma(B)}+|\lambda|^{\frac{1}{2}}\|\nabla u\|_{\mathrm{L}^2(B)}+|\lambda|^{\frac{1}{2}}\|\phi\|_{\mathrm{L}^2(B)}\\&\quad\leq C\big(\|f\|_{\mathcal{L}^2_\sigma(B)}+|\lambda|^{\frac{1}{2}}\|F\|_{\mathrm{L}^2(B)}\big).\end{aligned}\tag{5.3}$$

The constant $C>0$ only depends on d, θ, $\mu_\bullet$, and $\mu^\bullet$.

Proof Observe that the arguments between (1.6) and (1.9) carry over to $\mathfrak{b}_B$ with the same angle. Thus, the existence of u follows by the Lax–Milgram lemma. Testing (5.1) by u gives the estimates of u and ∇u in (5.3). From now on, we focus on the construction of the pressure ϕ.

Since $\mathrm{C}_{c,\sigma}^\infty(B) \subset \mathcal{H}_\sigma^1(B)$ there exists, by virtue of [30, Lem. II.2.2.1], a function $\vartheta \in \mathrm{L}^2(B)$, which is unique up to the addition of constants, such that

$$\begin{aligned} &\lambda \int_B u \cdot \overline{v} \,\mathrm{d}x + \mathfrak{b}_B(u, v) - \int_B \vartheta \,\overline{\operatorname{div}(v)} \,\mathrm{d}x \\ &\quad = \int_B f \cdot \overline{v} \,\mathrm{d}x - \int_B F_{\alpha\beta} \overline{\partial_\alpha v_\beta} \,\mathrm{d}x \quad (v \in \mathrm{H}_0^1(B; \mathbb{C}^d)). \end{aligned} \tag{5.4}$$

The task is now to find a suitable constant $c \in \mathbb{C}$ such that $\phi := \vartheta + c$ satisfies (5.2). To this end, let $\varphi_0 \in \mathrm{H}^{1/2}(\partial B; \mathbb{C}^d)$ with

$$\int_{\partial B} \frac{x - x_0}{|x - x_0|} \cdot \varphi_0(x) \,\mathrm{d}\sigma(x) = 1,$$

where x_0 denotes the center of B and σ its surface measure. Let $E\varphi_0 \in \mathrm{H}^1(B; \mathbb{C}^d)$ denote an extension of φ_0 and define

$$\begin{aligned} c := &- \int_B f \cdot \overline{E\varphi_0} \,\mathrm{d}x + \int_B F_{\alpha\beta} \overline{\partial_\alpha (E\varphi_0)_\beta} \,\mathrm{d}x + \lambda \int_B u \cdot \overline{E\varphi_0} \,\mathrm{d}x \\ &+ \mathfrak{b}_B(u, E\varphi_0) - \int_B \vartheta \,\overline{\operatorname{div}(E\varphi_0)} \,\mathrm{d}x. \end{aligned}$$

Now, for $v \in \mathrm{H}^1(B; \mathbb{C}^d)$ we find with

$$\eta := \int_{\partial B} \frac{x - x_0}{|x - x_0|} \cdot v|_{\partial B}(x) \,\mathrm{d}\sigma(x)$$

and the divergence theorem that

$$\begin{aligned} &\lambda \int_B u \cdot \overline{v} \,\mathrm{d}x + \mathfrak{b}_B(u, v) - \int_B (\vartheta + c) \,\overline{\operatorname{div}(v)} \,\mathrm{d}x \\ &\quad = \lambda \int_B u \cdot \overline{v} \,\mathrm{d}x + \mathfrak{b}_B(u, v) - \int_B \vartheta \,\overline{\operatorname{div}(v)} \,\mathrm{d}x - c\overline{\eta} \\ &\quad = \lambda \int_B u \cdot \overline{(v - \eta E\varphi_0)} \,\mathrm{d}x + \mathfrak{b}_B(u, v - \eta E\varphi_0) - \int_B \vartheta \,\overline{\operatorname{div}(v - \eta E\varphi_0)} \,\mathrm{d}x \\ &\qquad - \int_B f \cdot \overline{(v - \eta E\varphi_0)} \,\mathrm{d}x + \int_B F_{\alpha\beta} \overline{\partial_\alpha (v - \eta E\varphi_0)_\beta} \,\mathrm{d}x \\ &\qquad + \int_B f \cdot \overline{v} \,\mathrm{d}x - \int_B F_{\alpha\beta} \overline{\partial_\alpha v_\beta} \,\mathrm{d}x. \end{aligned}$$

By virtue of [25, Lem. 2.3] the trace operator

$$\operatorname{tr} : \mathcal{H}_\sigma^1(B) \to \left\{ g \in \mathrm{H}^{\frac{1}{2}}(\partial B; \mathbb{C}^d) : \int_{\partial B} \frac{x - x_0}{|x - x_0|} \cdot g \,\mathrm{d}\sigma(x) = 0 \right\} =: \mathrm{H}_n^{\frac{1}{2}}(\partial B)$$

is onto. By construction, we have that $v - \eta E\varphi_0 \in \mathrm{H}_n^{1/2}(\partial B)$. Thus, there exists $\psi \in \mathcal{H}_\sigma^1(B)$ with $\operatorname{tr}(\psi) = v - \eta E\varphi_0$. In particular, we have that $v - \eta E\varphi_0 - \psi \in \mathrm{H}_0^1(B; \mathbb{C}^d)$. Thus, employing first (5.4) and then (5.1) delivers

$$\begin{aligned} &\lambda \int_B u \cdot \overline{(v - \eta E\varphi_0)} \,\mathrm{d}x + \mathfrak{b}_B(u, v - \eta E\varphi_0) - \int_B \vartheta \,\overline{\operatorname{div}(v - \eta E\varphi_0)} \,\mathrm{d}x \\ &\quad - \int_B f \cdot \overline{(v - \eta E\varphi_0)} \,\mathrm{d}x + \int_B F_{\alpha\beta} \overline{\partial_\alpha (v - \eta E\varphi_0)_\beta} \,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \lambda \int_B u \cdot \overline{\psi} \, \mathrm{d}x + \mathfrak{b}_B(u, \psi) - \int_B f \cdot \overline{\psi} \, \mathrm{d}x + \int_B F_{\alpha\beta} \overline{\partial_\alpha \psi_\beta} \, \mathrm{d}x \\
&= 0.
\end{aligned}
$$

This establishes (5.2). Having (5.2) at our disposal, we can also derive a bound on the pressure function ϕ. Indeed, testing (5.2) with $v := \nabla \Delta_D^{-1} \phi$ (where Δ_D denotes the Dirichlet Laplacian on B), using that u and f are orthogonal to v, and using (5.3) delivers with a constant $C > 0$ depending only on d, θ, $\mu_\bullet$, and $\mu^\bullet$ that

$$
|\lambda|^{\frac{1}{2}} \|\phi\|_{\mathrm{L}^2(B)} \leq C\big(\|f\|_{\mathcal{L}^2_\sigma(B)} + |\lambda|^{\frac{1}{2}} \|F\|_{\mathrm{L}^2(B)}\big). \qquad \square
$$

As described in Section 4 we want to study $u := (\lambda + A)^{-1} f$ and - in order to verify (4.5) - we want to find suitable decompositions of u into $u = u_1 + u_2$ which should be valid in $2Q^*$ for a given cube Q. This is done in the following lemma. The argument to arrive at the desired estimate is subtle. We will apply Lemma 5.1 with $c_2 = 0$ and use that we left the term on the right-hand side involving the pressure, i.e.,

$$
\frac{1}{r}\bigg(\int_{2B \setminus B} |\phi - c_1|^2 \, \mathrm{d}x\bigg)^{\frac{1}{2}} \bigg(\int_{2B} |u\eta|^2 \, \mathrm{d}x\bigg)^{\frac{1}{2}}
$$

in a product structure. In this situation, one can still decide whether one estimates the term by Young's inequality as

$$
\frac{1}{r}\bigg(\int_{2B \setminus B} |\phi - c_1|^2 \, \mathrm{d}x\bigg)^{\frac{1}{2}} \bigg(\int_{2B} |u\eta|^2 \, \mathrm{d}x\bigg)^{\frac{1}{2}} \leq \frac{1}{2} \int_{2B \setminus B} |\phi - c_1|^2 \, \mathrm{d}x + \frac{1}{2r^2} \int_{2B} |u\eta|^2 \, \mathrm{d}x
$$

or for some suitable $\varepsilon > 0$ as

$$
\begin{aligned}
\frac{1}{r}\bigg(\int_{2B \setminus B} |\phi - c_1|^2 \, \mathrm{d}x\bigg)^{\frac{1}{2}} \bigg(\int_{2B} |u\eta|^2 \, \mathrm{d}x\bigg)^{\frac{1}{2}} &\leq \frac{1}{2\varepsilon r^2 |\lambda|} \int_{2B \setminus B} |\phi - c_1|^2 \, \mathrm{d}x \\
&\quad + \frac{|\lambda| \varepsilon}{2} \int_{2B} |u\eta|^2 \, \mathrm{d}x.
\end{aligned}
$$

In the first situation, one leaves the term involving u on the right-hand side and in the second situation, one can absorb this term onto the left-hand side. Depending on the particular situation, we will need to decide differently.

Lemma 5.3 *Let μ satisfy Assumption 1.1 with constants $\mu^\bullet, \mu_\bullet > 0$ and let $\omega_0 \in (0, \frac{\pi}{2})$ be given by (1.7). Then for any $\theta \in (0, \pi - \omega_0)$ the following holds:*

Let $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$, $F \in \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^{d \times d})$, and let $\lambda \in \mathrm{S}_\theta$. For $u \in \mathrm{H}^1_\sigma(\mathbb{R}^d)$ defined by $u := (\lambda + A)^{-1}(f + \mathbb{P}\operatorname{div}(F))$ and $x_0 \in \mathbb{R}^d$ and $r_0 > 0$ there exists a decomposition of u of the form $u = u_1 + u_2$ with $u_1 \in \mathcal{H}^1_\sigma(B(x_0, r_0))$ and $u_2 \equiv u$ in $\mathbb{R}^d \setminus B(x_0, r_0)$ and there exists $\phi_1 \in \mathrm{L}^2(B(x_0, r_0))$ and $C > 0$ such that for any ball $B \subset \mathbb{R}^d$ of radius $r > 0$ with $2B \subset B(x_0, r_0)$ we have

$$
\begin{aligned}
&|\lambda|^3 r^2 \int_B |u_2|^2 \, \mathrm{d}x + |\lambda|^2 r^2 \int_B |\nabla u_2|^2 \, \mathrm{d}x \\
&\quad \leq C\bigg\{\sum_{\ell=0}^{\infty} 2^{-\ell d - \ell} \int_{2^\ell B} \big(|\lambda u|^2 + |f|^2 + ||\lambda|^{\frac{1}{2}} F|^2\big) \, \mathrm{d}x + \int_{2B} |\lambda u_1|^2 \, \mathrm{d}x + \int_{2B} ||\lambda|^{\frac{1}{2}} \phi_1|^2 \, \mathrm{d}x\bigg\}.
\end{aligned}
\tag{5.5}
$$

Moreover, u_1 and ϕ_1 satisfy for some $C > 0$

$$\begin{aligned} &|\lambda| \|u_1\|_{\mathrm{L}^2(B(x_0,r_0))} + |\lambda|^{\frac{1}{2}} \|\nabla u_1\|_{\mathrm{L}^2(B(x_0,r_0))} + |\lambda|^{\frac{1}{2}} \|\phi_1\|_{\mathrm{L}^2(B(x_0,r_0))} \\ &\quad \leq C\big(\|f\|_{\mathrm{L}^2(B(x_0,r_0))} + |\lambda|^{\frac{1}{2}} \|F\|_{\mathrm{L}^2(B(x_0,r_0))}\big). \end{aligned} \tag{5.6}$$

In both inequalities, the constant C does only depend on d, θ, $\mu_\bullet$, and $\mu^\bullet$.

Proof Fix $f \in \mathrm{L}^2_\sigma(\mathbb{R}^d)$, $F \in \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^{d\times d})$, and $\lambda \in \mathrm{S}_\theta$. Define $u := (\lambda + A)^{-1}(f + \mathbb{P}\operatorname{div}(F))$ and let $\phi \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$ be the associated pressure. Let $B \subset \mathbb{R}^d$ be a ball of radius $r > 0$ with $2B \subset B(x_0, r_0)$ and let $g := f|_{B(x_0,r_0)}$. The definition of $\mathcal{L}^2_\sigma(B(x_0, r_0))$ implies that $g \in \mathcal{L}^2_\sigma(B(x_0, r_0))$. Let further $G := F|_{B(x_0,r_0)}$. Then, there exists $u_1 \in \mathcal{H}^1_\sigma(B(x_0, r_0))$ such that for all $v \in \mathcal{H}^1_\sigma(B(x_0, r_0))$ we have

$$\lambda \int_{B(x_0,r_0)} u_1 \cdot \overline{v} \,\mathrm{d}x + \mathfrak{b}_{B(x_0,r_0)}(u_1, v) = \int_{B(x_0,r_0)} g \cdot \overline{v} \,\mathrm{d}x - \int_{B(x_0,r_0)} G_{\alpha\beta} \cdot \overline{\partial_\alpha v_\beta} \,\mathrm{d}x.$$

Let $\phi_1 \in \mathrm{L}^2(B(x_0, r_0))$ denote the associated pressure. By (5.3) we find that

$$\begin{aligned} &|\lambda| \|u_1\|_{\mathrm{L}^2(B(x_0,r_0))} + |\lambda|^{\frac{1}{2}} \|\nabla u_1\|_{\mathrm{L}^2(B(x_0,r_0))} + |\lambda|^{\frac{1}{2}} \|\phi_1\|_{\mathrm{L}^2(B(x_0,r_0))} \\ &\quad \leq C\big(\|f\|_{\mathrm{L}^2(B(x_0,r_0))} + |\lambda|^{\frac{1}{2}} \|F\|_{\mathrm{L}^2(B(x_0,r_0))}\big). \end{aligned}$$

Notice that the constant $C > 0$ depends only on d, θ, $\mu_\bullet$, and $\mu^\bullet$. In particular, it does not depend on r_0.

Now, define $u_2 := u - \widetilde{u_1}$ and $\phi_2 := \phi - \widetilde{\phi_1}$. Here, $\widetilde{u_1}$ and $\widetilde{\phi_1}$ denote the extensions by zero to all of $\mathbb{R}^d$ of u_1 and ϕ_1. By the above definitions, we find that

$$\begin{aligned} &\lambda \int_{B(x_0,r_0)} u_2 \cdot \overline{v} \,\mathrm{d}x + \mathfrak{b}_{B(x_0,r_0)}(u_2, v) - \int_{B(x_0,r_0)} \phi_2 \,\overline{\operatorname{div}(v)} \,\mathrm{d}x \\ &\quad = 0 \quad (v \in \mathrm{H}^1_0(B(x_0, r_0); \mathbb{C}^d)). \end{aligned}$$

Let $\eta \in \mathrm{C}^\infty_c(2B)$ with $\eta \equiv 1$ in B, $0 \leq \eta \leq 1$, and $\|\nabla\eta\|_{\mathrm{L}^\infty} \leq 2/r$. We apply Lemma 5.1 with $c_1 \in \mathbb{C}$ and $c_2 = 0$ to u_2 and ϕ_2 leading to the estimate

$$\begin{aligned} &|\lambda|^3 r^2 \int_{2B} |u_2\eta|^2 \,\mathrm{d}x + |\lambda|^2 r^2 \int_{2B} |\nabla[u_2\eta]|^2 \,\mathrm{d}x \\ &\quad \leq C|\lambda|^2 r \Big(\int_{2B\setminus B} |\phi_2 - c_1|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_{2B} |u_2\eta|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} + C|\lambda|^2 \int_{2B} |u_2|^2 \,\mathrm{d}x \\ &\quad \leq C|\lambda|^2 r \Big(\int_{2B\setminus B} |\phi - c_1|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_{2B} |u_2\eta|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} + C|\lambda|^2 \int_{2B} |u_2|^2 \,\mathrm{d}x \\ &\qquad + C|\lambda|^2 r \Big(\int_{2B\setminus B} |\phi_1|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} \Big(\int_{2B} |u_2\eta|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}}. \end{aligned}$$

Set $c_1 := \phi_{2B\setminus B}$ and apply Lemma 2.1 with $k = 1$ to estimate $\phi - \phi_{2B\setminus B}$ in the first inequality. In the second, use Hölder's inequality for series and in the third, employ Theorem 1.2 with $\nu = d + 1$ and $c_\ell = 0$. This yields

$$
\begin{aligned}
&|\lambda|^3 r^2 \int_{2B} |u_2\eta|^2 \, \mathrm{d}x + |\lambda|^2 r^2 \int_{2B} |\nabla[u_2\eta]|^2 \, \mathrm{d}x \\
&\quad \leq C|\lambda|^2 r \biggl\{ \biggl(\sum_{\substack{\ell \in \mathbb{N}_0 \\ |\ell-1| \leq 1}} (\|\nabla u\|_{\mathrm{L}^2(\mathcal{C}_\ell)} + \|F\|_{\mathrm{L}^2(\mathcal{C}_\ell)}) + \sum_{\ell=3}^\infty 2^{(\frac{d}{2}+1)(1-\ell)} (\|\nabla u\|_{\mathrm{L}^2(\mathcal{C}_\ell)} + \|F\|_{\mathrm{L}^2(\mathcal{C}_\ell)}) \biggr) \\
&\qquad \cdot \biggl(\int_{2B} |u_2\eta|^2 \, \mathrm{d}x \biggr)^{\frac{1}{2}} \biggr\} + C|\lambda|^2 \int_{2B} |u_2|^2 \, \mathrm{d}x + C|\lambda|^2 r \biggl(\int_{2B\setminus B} |\phi_1|^2 \, \mathrm{d}x \biggr)^{\frac{1}{2}} \biggl(\int_{2B} |u_2\eta|^2 \, \mathrm{d}x \biggr)^{\frac{1}{2}} \\
&\quad \leq C_d |\lambda|^2 r \biggl(\sum_{\ell=0}^\infty 2^{-\ell d - \ell} \int_{2^\ell B} |\nabla u|^2 \, \mathrm{d}x \biggr)^{\frac{1}{2}} \biggl(\int_{2B} |u_2\eta|^2 \, \mathrm{d}x \biggr)^{\frac{1}{2}} \\
&\qquad + C_d |\lambda|^2 r \biggl(\sum_{\ell=0}^\infty 2^{-\ell d - \ell} \int_{2^\ell B} |F|^2 \, \mathrm{d}x \biggr)^{\frac{1}{2}} \biggl(\int_{2B} |u_2\eta|^2 \, \mathrm{d}x \biggr)^{\frac{1}{2}} \\
&\qquad + C|\lambda|^2 \int_{2B} |u_2|^2 \, \mathrm{d}x + C|\lambda|^2 r \biggl(\int_{2B\setminus B} |\phi_1|^2 \, \mathrm{d}x \biggr)^{\frac{1}{2}} \biggl(\int_{2B} |u_2\eta|^2 \, \mathrm{d}x \biggr)^{\frac{1}{2}} \\
&\quad \leq C_{d,\theta,\mu^\bullet,\mu_\bullet} \biggl(\sum_{\ell=0}^\infty 2^{-\ell d - 3\ell} \int_{2^\ell B} |\lambda u|^2 \, \mathrm{d}x \biggr)^{\frac{1}{2}} \biggl(\int_{2B} ||\lambda| u_2|^2 \, \mathrm{d}x \biggr)^{\frac{1}{2}} \\
&\qquad + C_{d,\theta,\mu^\bullet,\mu_\bullet} \biggl(\sum_{\ell=0}^\infty 2^{-\ell d - \ell} \int_{2^\ell B} |f|^2 \, \mathrm{d}x \biggr)^{\frac{1}{2}} \biggl(|\lambda|^3 r^2 \int_{2B} |u_2\eta|^2 \, \mathrm{d}x \biggr)^{\frac{1}{2}} \\
&\qquad + C_{d,\theta,\mu^\bullet,\mu_\bullet} \biggl(\sum_{\ell=0}^\infty 2^{-\ell d - \ell} \int_{2^\ell B} ||\lambda|^{\frac{1}{2}} F|^2 \, \mathrm{d}x \biggr)^{\frac{1}{2}} \biggl(|\lambda|^3 r^2 \int_{2B} |u_2\eta|^2 \, \mathrm{d}x \biggr)^{\frac{1}{2}} \\
&\qquad + C \int_{2B} ||\lambda| u_2|^2 \, \mathrm{d}x + C \biggl(\int_{2B\setminus B} ||\lambda|^{\frac{1}{2}} \phi_1|^2 \, \mathrm{d}x \biggr)^{\frac{1}{2}} \biggl(|\lambda|^3 r^2 \int_{2B} |u_2\eta|^2 \, \mathrm{d}x \biggr)^{\frac{1}{2}}.
\end{aligned}
$$

Consequently, by Young's inequality, we find that

$$
\begin{aligned}
&|\lambda|^3 r^2 \int_{2B} |u_2\eta|^2 \, \mathrm{d}x + |\lambda|^2 r^2 \int_{2B} |\nabla[u_2\eta]|^2 \, \mathrm{d}x \\
&\quad \leq \frac{C_{d,\theta,\mu^\bullet,\mu_\bullet}}{2} \sum_{\ell=0}^\infty 2^{-\ell d - 3\ell} \int_{2^\ell B} |\lambda u|^2 \, \mathrm{d}x + C \int_{2B} ||\lambda| u_2|^2 \, \mathrm{d}x \\
&\quad + C^2_{d,\theta,\mu^\bullet,\mu_\bullet} \sum_{\ell=0}^\infty 2^{-\ell d - \ell} \int_{2^\ell B} |f|^2 \, \mathrm{d}x \\
&\qquad + C^2_{d,\theta,\mu^\bullet,\mu_\bullet} \sum_{\ell=0}^\infty 2^{-\ell d - \ell} \int_{2^\ell B} ||\lambda|^{\frac{1}{2}} F|^2 \, \mathrm{d}x \\
&\quad + \frac{C^2}{3} \int_{2B\setminus B} ||\lambda|^{\frac{1}{2}} \phi_1|^2 \, \mathrm{d}x + \frac{3}{4} |\lambda|^3 r^2 \int_{2B} |u_2\eta|^2 \, \mathrm{d}x.
\end{aligned}
$$

Now, the last term on the right-hand side can be absorbed onto the left-hand side. Moreover, we use that $u_2 = u - \widetilde{u_1}$ and we estimate $2^{-\ell d - 3\ell} \leq 2^{-\ell d - \ell}$, so that we find a constant $C > 0$ depending on $d, \theta, \mu^\bullet$, and $\mu_\bullet$ such that

$$\begin{aligned}|\lambda|^3 r^2 \int_{2B} |u_2\eta|^2 \, \mathrm{d}x + |\lambda|^2 r^2 \int_{2B} |\nabla[u_2\eta]|^2 \, \mathrm{d}x \\ \leq C\bigg\{ \sum_{\ell=0}^{\infty} 2^{-\ell d - \ell} \int_{2^\ell B} \big(|\lambda u|^2 + |f|^2 + ||\lambda|^{\frac{1}{2}} F|^2 \big) \, \mathrm{d}x \\ + \int_{2B} |\lambda u_1|^2 \, \mathrm{d}x + \int_{2B} ||\lambda|^{\frac{1}{2}} \phi_1|^2 \, \mathrm{d}x \bigg\}.\end{aligned}$$

Since $\eta \equiv 1$ in B we conclude the estimate (5.5). □

We are now in the position to present the proofs of Theorems 1.3 and 1.5.

Proofs of Theorems 1.3 and 1.5 It is well-known, see [36, Thm. 4.2], that the maximal regularity statement in Theorem 1.5 follows from the $\mathcal{R}$-sectoriality of A_p of angle $\omega_0 \in (0, \frac{\pi}{2})$. Moreover, since $\mathcal{R}$-sectoriality implies sectoriality by Remark 3.2 (1), it is our task to prove that A_p is $\mathcal{R}$-sectorial on $\mathrm{L}^p_\sigma(\mathbb{R}^d)$ of angle ω_0.

The case $p > 2$

Let $k_0 \in \mathbb{N}$. For $\theta \in (0, \pi - \omega_0)$, let $(\lambda_k)_{k=1}^{k_0} \subset \mathrm{S}_\theta$ and let $(f_k)_{k=1}^{k_0} \subset \mathrm{C}^\infty_{c,\sigma}(\mathbb{R}^d)$. For $f = (f_1, \dots, f_{k_0}, 0, \dots)$, we saw in Observation 3.4 that we need to prove the estimate

$$\| T_{(\lambda_k)_{k=1}^{k_0}} f \|_{\mathrm{L}^p(\mathbb{R}^d; \ell^2(\mathbb{C}^d))} \leq C \| f \|_{\mathrm{L}^p(\mathbb{R}^d; \ell^2(\mathbb{C}^d))},$$

with a constant being uniform with respect to all choices above. This will be done by verifying the assumptions of Theorem 4.2 uniformly with respect to the choices of parameters above. For the application of Theorem 4.2 we set $X = Y = Z = \ell^2(\mathbb{C}^d)$ and let $\mathcal{C} = \mathrm{Id}$.

For this purpose, define for $1 \leq k \leq k_0$ the functions $u_k := (\lambda_k + A)^{-1} f_k$. Let $x_0 \in \mathbb{R}^d$, $r > 0$, and let $B := B(x_0, r)$. Let further $u_{k,1}$, $u_{k,2}$, and $\phi_{k,1}$ denote the functions provided by Lemma 5.3 with $r_0 := 2r$. Notice that

$$\| T_{(\lambda_k)_{k=1}^{k_0}} f \|_{\ell^2} = \Big[\sum_{k=1}^{k_0} |\lambda_k u_k|^2 \Big]^{\frac{1}{2}} \leq \Big[\sum_{k=1}^{k_0} |\lambda_k u_{k,1}|^2 \Big]^{\frac{1}{2}} + \Big[\sum_{k=1}^{k_0} |\lambda_k u_{k,2}|^2 \Big]^{\frac{1}{2}}. \tag{5.7}$$

Moreover, in B we have for $1 \leq j \leq d$ by the chain rule and the Cauchy–Schwarz inequality that

$$\Big| \partial_j \Big[\sum_{k=1}^{k_0} |\lambda_k u_{k,2}|^2 \Big]^{\frac{1}{2}} \Big| = \Big| \Big[\sum_{k=1}^{k_0} |\lambda_k u_{k,2}|^2 \Big]^{-\frac{1}{2}} \sum_{k=1}^{k_0} |\lambda_k|^2 \Re(\overline{u_{k,2}} \partial_j u_{k,2}) \Big| \leq \Big[\sum_{k=1}^{k_0} |\lambda_k \partial_j u_{k,2}|^2 \Big]^{\frac{1}{2}}. \tag{5.8}$$

We start by deriving some kind of non-local weak reverse Hölder inequality for the second term of the right-hand side of (5.7). If $d = 2$ let $p_0 > 2$ and if $d \geq 3$ let $p_0 := 2d/(d-2)$. By Sobolev's embedding theorem together with (5.8), there exists a constant $C_{d,p_0} > 0$ depending only on d and p_0 such that

$$\begin{aligned}&\Big(\fint_B \Big[\sum_{k=1}^{k_0} |\lambda_k u_{k,2}|^2 \Big]^{\frac{p_0}{2}} \, \mathrm{d}x \Big)^{\frac{1}{p_0}} \\ &\leq C \bigg\{ \Big(\fint_B \Big[\sum_{k=1}^{k_0} |\lambda_k u_{k,2}|^2 \Big]^{\frac{2}{2}} \, \mathrm{d}x \Big)^{\frac{1}{2}} + \Big(r^2 \fint_B \Big| \nabla \Big[\sum_{k=1}^{k_0} |\lambda_k|^2 |u_{k,2}|^2 \Big]^{\frac{1}{2}} \Big|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}} \bigg\}\end{aligned}$$

$$\leq C\bigg\{\Big(\fint_B \sum_{k=1}^{k_0} |\lambda_k u_{k,2}|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} + \Big(\sum_{k=1}^{k_0} |\lambda_k|^2 r^2 \fint_B |\nabla u_{k,2}|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}}\bigg\}.$$

To estimate the second term on the right-hand side, let $\eta \in \mathrm{C}_c^\infty(2B)$ with $\eta \equiv 1$ in B, $0 \leq \eta \leq 1$, and $\|\nabla \eta\|_{\mathrm{L}^\infty} \leq 2/r$. Observe that

$$\fint_B |\nabla u_{k,2}|^2 \,\mathrm{d}x \leq 2^d \fint_{2B} |\nabla [u_{k,2}\eta]|^2 \,\mathrm{d}x.$$

Thus, employing (5.5) with $F = 0$ in the first inequality and the decomposition $u_k = u_{k,1} + u_{k,2}$ together with (5.6) in the second, delivers for some constant $C > 0$ depending only on d, θ, $\mu_\bullet$, $\mu^\bullet$, and p_0 that

$$\begin{aligned}
&\Big(\fint_B \Big[\sum_{k=1}^{k_0} |\lambda_k u_{k,2}|^2\Big]^{\frac{p_0}{2}} \,\mathrm{d}x\Big)^{\frac{1}{p_0}} \\
&\quad \leq C\bigg\{\Big(\fint_B \sum_{k=1}^{k_0} |\lambda_k u_{k,2}|^2 \,\mathrm{d}x\Big)^{\frac{1}{2}} + \Big(\sum_{\ell=0}^{\infty} \frac{2^{-d\ell-\ell}}{r^d} \int_{2^\ell B} \sum_{k=1}^{k_0} \big(|\lambda_k u_k|^2 + |f_k|^2\big) \,\mathrm{d}x \\
&\qquad + \fint_{2B} \sum_{k=1}^{k_0} \big(||\lambda_k|^{\frac{1}{2}} \phi_{k,1}|^2 + |\lambda_k u_{k,1}|^2\big) \,\mathrm{d}x\Big)^{\frac{1}{2}}\bigg\} \\
&\quad \leq C\Big(\sum_{\ell=0}^{\infty} 2^{-\ell} \fint_{2^\ell B} \sum_{k=1}^{k_0} \big(|\lambda_k u_k|^2 + |f_k|^2\big) \,\mathrm{d}x\Big)^{\frac{1}{2}}.
\end{aligned} \tag{5.9}$$

We verify the assumptions of Theorem 4.2: Let $Q = Q(x_0, r/18)$ be a cube in $\mathbb{R}^d$ with center x_0 and $\operatorname{diam}(Q) = r/18$ and notice that $2Q^* \subset B(x_0, r/6)$, where Q^* denotes a parent of Q. Then, for any $\alpha > 0$, we find by virtue of (5.7) that

$$\begin{aligned}
|\{x \in Q : M_{2Q^*}(\|T_{(\lambda_k)_{k=1}^{k_0}} f\|_{\ell^2}^2)(x) > \alpha\}| &\leq \Big|\Big\{x \in Q : M_{2Q^*}\Big(\sum_{k=1}^{k_0} |\lambda_k u_{k,1}|^2\Big)(x) > \frac{\alpha}{4}\Big\}\Big| \\
&\quad + \Big|\Big\{x \in Q : M_{2Q^*}\Big(\sum_{k=1}^{k_0} |\lambda_k u_{k,2}|^2\Big)(x) > \frac{\alpha}{4}\Big\}\Big|.
\end{aligned} \tag{5.10}$$

The weak-$(1, 1)$ estimate of the localized maximal operator followed by (5.6) directly yields with some constant depending only on d, θ, $\mu_\bullet$, and $\mu^\bullet$ that

$$\Big|\Big\{x \in Q : M_{2Q^*}\Big(\sum_{k=1}^{k_0} |\lambda_k u_{k,1}|^2\Big)(x) > \frac{\alpha}{4}\Big\}\Big| \leq \frac{C}{\alpha} \int_{Q(x_0, 4\sqrt{d} r)} \Big[\sum_{k=1}^{k_0} |f_k|^2\Big]^{\frac{2}{2}} \,\mathrm{d}x.$$

For the second term in (5.10) use the weak-$(p_0/2, p_0/2)$ inequality of the localized maximal operator followed by (5.9). This gives, with a constant $C > 0$ depending only on d, θ, $\mu_\bullet$, $\mu^\bullet$, and p_0, that

$$\Big|\Big\{x \in Q : M_{2Q^*}\Big(\sum_{k=1}^{k_0} |\lambda_k u_{k,2}|^2\Big)(x) > \frac{\alpha}{4}\Big\}\Big|$$

$$
\begin{aligned}
&\leq \frac{C|B(x_0,r/6)|}{\alpha^{p_0/2}} \fint_{B(x_0,r/6)} \Big[\sum_{k=1}^{k_0} |\lambda_k u_{k,2}|^2\Big]^{\frac{p_0}{2}} \, \mathrm{d}x \\
&\leq \frac{C|B(x_0,r/6)|}{\alpha^{p_0/2}} \Big(\sum_{\ell=0}^{\infty} 2^{-\ell} \fint_{B(x_0,2^\ell r)} \Big[\sum_{k=1}^{k_0} |\lambda_k u_k|^2\Big]^{\frac{2}{2}} + \Big[\sum_{k=1}^{k_0} |f_k|^2\Big]^{\frac{2}{2}} \, \mathrm{d}x\Big)^{\frac{p_0}{2}} \\
&\leq \frac{C|Q|}{\alpha^{p_0/2}} \sup_{Q' \supset 2Q^*} \Big(\fint_{Q'} \Big[\sum_{k=1}^{k_0} |\lambda_k u_k|^2\Big]^{\frac{2}{2}} + \Big[\sum_{k=1}^{k_0} |f_k|^2\Big]^{\frac{2}{2}} \, \mathrm{d}x\Big)^{\frac{p_0}{2}}.
\end{aligned}
$$

This concludes the proof of this case.

The case $p < 2$

This case follows directly by the duality principle as described by Kalton and Weis in [16, Lem. 3.1] since $\mathrm{L}^p_\sigma(\mathbb{R}^d)$ is of non-trivial Rademacher type if $1 < p < \infty$. □

Proof of Theorem 1.4 For $\theta \in (0, \pi - \omega_0)$ let $\lambda \in \mathrm{S}_\theta$. We argue by duality and prove the L^p-boundedness of

$$
T := |\lambda|^{1/2} (\lambda + A)^{-1} \mathbb{P} \operatorname{div}
$$

for $p \geq 2$ satisfying (1.11). The uniform bound follows by verifying the assumptions of Theorem 4.2 uniformly with respect to λ.

We choose $X = Y = Z = \mathbb{C}^{d \times d}$ and $\mathcal{C} = \mathrm{Id}$. Let $F \in \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^{d \times d})$ and define $u := (\lambda + A)^{-1} \mathbb{P} \operatorname{div}(F)$. Let further $x_0 \in \mathbb{R}^d$ and $r > 0$ and let u_1, u_2, and ϕ_1 denote the corresponding functions from Lemma 5.3 with $r_0 := 2r$. Let $p_0 > 2$ if $d = 2$ and $p_0 := 2d/(d-2)$ if $d \geq 3$. Then, Sobolev's inequality implies that

$$
\Big(\fint_{B(x_0,r)} ||\lambda|^{\frac{1}{2}} u_2|^{p_0} \, \mathrm{d}x\Big)^{\frac{1}{p_0}} \leq C\Big\{\Big(\fint_{B(x_0,r)} ||\lambda|^{\frac{1}{2}} u_2|^2 \, \mathrm{d}x\Big)^{\frac{1}{2}} + \Big(r^2 \fint_{B(x_0,r)} ||\lambda|^{\frac{1}{2}} \nabla u_2|^2 \, \mathrm{d}x\Big)^{\frac{1}{2}}\Big\}.
$$

Now, use (5.5) with $f = 0$ in the first inequality and then (5.6) with $f = 0$ to get

$$
\begin{aligned}
\Big(\fint_{B(x_0,r)} ||\lambda|^{\frac{1}{2}} u_2|^{p_0} \, \mathrm{d}x\Big)^{\frac{1}{p_0}} &\leq C\Big\{\Big(\fint_{B(x_0,r)} ||\lambda|^{\frac{1}{2}} u_2|^2 \, \mathrm{d}x\Big)^{\frac{1}{2}} \\
&\quad + \Big(\sum_{\ell=0}^{\infty} \frac{2^{-\ell d - \ell}}{r^d} \int_{2^\ell B} \big(||\lambda|^{\frac{1}{2}} u|^2 + |F|^2\big) \, \mathrm{d}x \\
&\quad + \int_{2B} ||\lambda|^{\frac{1}{2}} u_1|^2 \, \mathrm{d}x + \int_{2B} |\phi_1|^2 \, \mathrm{d}x\Big)^{\frac{1}{2}}\Big\} \\
&\leq C\Big(\sum_{\ell=0}^{\infty} 2^{-\ell} \fint_{2^\ell B} \big(||\lambda|^{\frac{1}{2}} u|^2 + |F|^2\big) \, \mathrm{d}x\Big)^{\frac{1}{2}}.
\end{aligned}
$$

The rest of the proof can be finished literally as the proof of Theorem 1.5 starting from (5.10). □

6 Boundedness of the H^∞-calculus

In this section we present the proof of Theorem 1.7. Let us briefly introduce the notion of the H^∞-calculus of a sectorial and densely defined operator B of angle $\omega \in [0, \pi)$ on a Banach space X over the complex field.

Define for $\theta \in (0, \pi)$ the class of all *regularly decaying functions*

$$\mathrm{H}_0^\infty(\mathrm{S}_\theta) := \left\{ f : \mathrm{S}_\theta \to \mathbb{C} : f \text{ holomorphic and } \exists \varepsilon, C > 0 \ \forall z \in \mathrm{S}_\theta \ : \ |f(z)| \leq \frac{C|z|^\varepsilon}{(1+|z|)^{2\varepsilon}} \right\}.$$

Then for any $\theta \in (\omega, \pi)$ and $\vartheta \in (\omega, \theta)$ one defines for $f \in \mathrm{H}_0^\infty(\mathrm{S}_\theta)$

$$f(B) := \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\vartheta} f(\lambda)(\lambda - B)^{-1} \, \mathrm{d}\lambda.$$

Here, the path $\partial \mathrm{S}_\vartheta$ rounds the spectrum $\sigma(B)$ in a counterclockwise manner. If B is injective, then one can define $f(B)$ for

$$f \in \mathrm{H}^\infty(\mathrm{S}_\theta) := \{ g \in \mathrm{S}_\theta \to \mathbb{C} : g \text{ bounded and holomorphic} \}$$

via *regularization* as

$$f(B) := [B(1+B)^{-2}]^{-1}[\mathbf{z}/(1+\mathbf{z})^2 \cdot f](B),$$

where $\mathbf{z}/(1+\mathbf{z})^2$ denotes the function $z \mapsto z/(1+z)^2$. We now say, that the $\mathrm{H}^\infty(\mathrm{S}_\theta)$-*calculus of B is bounded* if $f(B) \in \mathcal{L}(X)$ for all $f \in \mathrm{H}^\infty(\mathrm{S}_\theta)$.

This property has far reaching consequences. For instance, it implies that B has bounded imaginary powers and this in turn implies that fractional power domains of B can be computed via complex interpolation

$$\mathcal{D}(B^{\alpha s}) = \big[X, \mathcal{D}(B^\alpha)\big]_s \qquad (s \in (0,1), \alpha > 0). \tag{6.1}$$

In the following, we use a comparison principle of Kunstmann and Weis [23, Thm. 9] to transfer the boundedness of the H^∞-calculus of the Laplacian to the generalized Stokes operator A. We refer to [12, p. 250–256] for a short summary of their approach.

Proof of Theorem 1.7 Let p satisfy (1.11), A_p denote the generalized Stokes operator on $\mathrm{L}^p_\sigma(\mathbb{R}^d)$, and $B_p := -\Delta$ the Laplacian on $\mathrm{L}^p(\mathbb{R}^d; \mathbb{C}^d)$. Let $R_p : \mathrm{L}^p(\mathbb{R}^d; \mathbb{C}^d) \to \mathrm{L}^p_\sigma(\mathbb{R}^d)$ be given by $R_p f := \mathbb{P} f$ and let $S_p : \mathrm{L}^p_\sigma(\mathbb{R}^d) \to \mathrm{L}^p(\mathbb{R}^d; \mathbb{C}^d)$ be the inclusion map. Define the holomorphic functions $\varphi := \psi := \mathbf{z}(1+\mathbf{z})^{-2} \in \mathrm{H}_0^\infty(\mathrm{S}_\theta)$, $\theta \in (\omega_0, \pi)$. In view of [23, Thm. 9], the statement on the boundedness of the H^∞-calculus of A_p follows from the boundedness of the $\mathrm{H}^\infty(\mathrm{S}_{\omega_0})$-calculus of the Laplacian once the $\mathcal{R}$-bounds

$$\begin{aligned} \sup_{1 \leq s,t \leq 2} \mathcal{R}\Big\{ \varphi(s2^{j+\ell} A_q) R_q \psi(t 2^j B_q) : j \in \mathbb{Z} \Big\} &\leq C 2^{-|\ell|\delta} \\ \sup_{1 \leq s,t \leq 2} \mathcal{R}\Big\{ \varphi(s2^{j+\ell} A_q)' S_q' \psi(t 2^j B_q)' : j \in \mathbb{Z} \Big\} &\leq C 2^{-|\ell|\delta} \end{aligned} \tag{6.2}$$

are proven for $q = p$ and some $C, \delta > 0$ and all $\ell \in \mathbb{Z}$. Here, T' denotes the adjoint of an operator T. In the following, we will establish (6.2) by interpolating corresponding estimates for $q = 2$ and some $\delta = \beta > 0$ and for $q = p_1$ that satisfies (1.11) and $p_1 > p$ (if $p > 2$) or $p_1 < p$ (if $p < 2$) and $\delta = 0$.

The case $q = p_1$

It is well-known that the Laplacian is $\mathcal{R}$-sectorial on $\mathrm{L}^{p_1}(\mathbb{R}^d;\mathbb{C}^d)$ and, by Theorem 1.5, A_{p_1} is $\mathcal{R}$-sectorial on $\mathrm{L}^{p_1}_\sigma(\mathbb{R}^d)$, both with angle at least ω_0. Consequently, the sets

$$\Big\{\varphi(s2^{j+\ell}A_{p_1}) : 1 \leq s \leq 2,\ j \in \mathbb{Z}\Big\} \subset \mathcal{L}(\mathrm{L}^{p_1}_\sigma(\mathbb{R}^d)) \quad \text{and}$$
$$\Big\{\psi(t2^{j}B_{p_1}) : 1 \leq t \leq 2,\ j \in \mathbb{Z}\Big\} \subset \mathcal{L}(\mathrm{L}^{p_1}(\mathbb{R}^d;\mathbb{C}^d))$$

are $\mathcal{R}$-bounded with $\mathcal{R}$-bound independent of ℓ. Since products of $\mathcal{R}$-bounded sets are $\mathcal{R}$-bounded, cf. [7, Prop. 3.4], we find (as a subset of $\mathcal{L}(\mathrm{L}^{p_1}(\mathbb{R}^d;\mathbb{C}^d), \mathrm{L}^{p_1}_\sigma(\mathbb{R}^d))$)

$$\sup_{1\leq s,t\leq 2} \mathcal{R}\Big\{\varphi(s2^{j+\ell}A_{p_1})R_{p_1}\psi(t2^{j}B_{p_1}) : j \in \mathbb{Z}\Big\} \leq C.$$

By duality, see [16, Lem. 3.1], we analogously find

$$\sup_{1\leq s,t\leq 2} \mathcal{R}\Big\{\varphi(s2^{j+\ell}A_{p_1})' S_{p_1}' \psi(t2^{j}B_{p_1})' : j \in \mathbb{Z}\Big\} \leq C,$$

where C is independent of ℓ. Both estimates are exactly (6.2) with $q = p_1$ and $\delta = 0$.

The case $q = 2$

Let $\beta \in (0, \frac{1}{2})$. To establish (6.2) for $q = 2$ and $\delta = \beta$, one can use [23, Prop. 10] and show that

$$\mathbb{P}_2\, \mathcal{D}((-\Delta_2)^\alpha) \subset \mathcal{D}(A_2^\alpha) \quad \text{and} \quad \|A_2^\alpha \mathbb{P}_2 u\|_{\mathrm{L}^2} \leq C\|(-\Delta_2)^\alpha u\|_{\mathrm{L}^2} \quad \text{for} \quad u \in \mathcal{D}((-\Delta_2)^\alpha) \tag{6.3}$$

$$S_2\, \mathcal{D}(A_2^\alpha) \subset \mathcal{D}((-\Delta_2)^\alpha) \quad \text{and} \quad \|(-\Delta_2)^\alpha S_2 u\|_{\mathrm{L}^2} \leq C\|A_2^\alpha u\|_{\mathrm{L}^2} \quad \text{for} \quad u \in \mathcal{D}(A_2^\alpha) \tag{6.4}$$

for $\alpha = \pm\beta$. We start with the case $\alpha = \beta$ and note that the fractional power domains of the Laplacian are Bessel potential spaces

$$\mathcal{D}((-\Delta)^\alpha) = \mathrm{H}^{2\alpha,2}(\mathbb{R}^d;\mathbb{C}^d).$$

To characterize the fractional power domains for A_2, we employ a classical result of Kato which allows to compare $\mathcal{D}(A_2^\alpha)$ with a fractional power domain of the real part of A_2. This real part is the non-negative and self-adjoint operator $H : \mathcal{D}(H) \subset \mathrm{L}^2_\sigma(\mathbb{R}^d) \to \mathrm{L}^2_\sigma(\mathbb{R}^d)$ associated to the symmetric sesquilinear form

$$\mathfrak{h} : \mathrm{H}^1_\sigma(\mathbb{R}^d) \times \mathrm{H}^1_\sigma(\mathbb{R}^d) \to \mathbb{C}, \quad (u, v) \mapsto \sum_{\alpha,\beta,i,j}^{d} \int_{\mathbb{R}^d} \frac{1}{2}\big(\mu_{\alpha\beta}^{ij} + \overline{\mu_{\beta\alpha}^{ji}}\big)\partial_\beta u_j \overline{\partial_\alpha v_i}\, \mathrm{d}x.$$

Kato's second representation theorem [22, Thm. VI.2.23] implies that $\mathcal{D}(H^{1/2}) = \mathrm{H}^1_\sigma(\mathbb{R}^d)$ and the self-adjointness of H, that fractional power domains can be computed via complex interpolation as

$$\mathcal{D}(H^\alpha) = \big[\mathrm{L}^2_\sigma(\mathbb{R}^d), \mathcal{D}(H^{1/2})\big]_\alpha = \big[\mathrm{L}^2_\sigma(\mathbb{R}^d), \mathrm{H}^1_\sigma(\mathbb{R}^d)\big]_\alpha = \mathrm{H}^{2\alpha,2}_\sigma(\mathbb{R}^d).$$

Finally, [21, Thm. 3.1] implies that

$$\mathcal{D}(A_2^\alpha) = \mathcal{D}(H^\alpha) = \mathrm{H}^{2\alpha,2}_\sigma(\mathbb{R}^d).$$

Having these characterizations at our disposal, (6.3) follows since $\mathbb{P}_2$ is bounded from $\mathrm{H}^{2\alpha,2}(\mathbb{R}^d;\mathbb{C}^d)$ to $\mathrm{H}^{2\alpha,2}_\sigma(\mathbb{R}^d)$. Moreover, as S_2 maps $\mathrm{H}^{2\alpha,2}_\sigma(\mathbb{R}^d)$ boundedly into $\mathrm{H}^{2\alpha,2}(\mathbb{R}^d;\mathbb{C}^d)$ we find (6.4).

Observe that the previous argument was independent of the coefficients of A_2, so that (6.3) and (6.4) hold for the Hilbert space adjoint A_2^* of A_2 as well. We use this fact and duality to deduce (6.3) for $\alpha = -\beta$ for A_2 from (6.4) for A_2^* and fractional power $-\alpha$.

We first note some important facts:

(1) we have $\mathbb{P}_2^* = S_2$;
(2) we have $[(A_2^*)^\alpha]^* = A_2^\alpha$;
(3) property (6.4) for A_2^* with fractional power $-\alpha > 0$ yields that $T := (-\Delta_2)^{-\alpha} S_2 (A_2^*)^\alpha$ extends by density to a bounded operator from $\mathrm{L}^2_\sigma(\mathbb{R}^d)$ to $\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^d)$.

For $u \in \mathcal{D}((-\Delta_2)^\alpha)$ and $v \in \mathcal{D}((A_2^*)^\alpha)$ we then find

$$\langle \mathbb{P}_2 u, (A_2^*)^\alpha v\rangle_{\mathrm{L}^2_\sigma} = \langle (-\Delta_2)^\alpha u, Tv\rangle_{\mathrm{L}^2} = \langle T^*(-\Delta_2)^\alpha u, v\rangle_{\mathrm{L}^2_\sigma}.$$

It follows that $\mathbb{P}_2 u \in \mathcal{D}(A_2^\alpha)$ and that

$$\|A_2^\alpha \mathbb{P}_2 u\|_{\mathrm{L}^2_\sigma} = \|T^*(-\Delta_2)^\alpha u\|_{\mathrm{L}^2_\sigma} \leq C\|(-\Delta_2)^\alpha u\|_{\mathrm{L}^2}.$$

Similarly, one deduces (6.4) for $\alpha = -\beta$ for A_2 from (6.3) for A_2^* and fractional power $-\alpha$.

The interpolation argument

Write $1/p = (1-\theta)/2 + \theta/p_1$ for some $\theta \in (0,1)$ and let $\beta \in (0, \frac{1}{2})$. Using [17, Prop. 3.7], we interpolate the estimates (6.2) for $q = 2$ and $\delta = \beta$ as well as for $q = p_1$ and $\delta = 0$ to deduce (6.2) for $q = p$ and $\delta = (1-\theta)\beta$. Now, [23, Thm. 9] yields the boundedness of the H^∞-calculus of A_p. Moreover, the second part of this theorem implies that

$$\mathcal{D}(A_p^s) = \mathrm{H}^{2s,p}_\sigma(\mathbb{R}^d) \quad \text{for} \quad 0 < s < (1-\theta)\beta \tag{6.5}$$

with equivalent norms. Finally, observe that the parameters β and p_1 were arbitrary. Maximizing the condition on s in (6.5) yields

$$\mathcal{D}(A_p^s) = \mathrm{H}^{2s,2}_\sigma(\mathbb{R}^d) \quad \text{for} \quad 0 < s < \frac{1}{2} - \frac{d}{2}\left|\frac{1}{2} - \frac{1}{p}\right|. \qquad \square$$

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References

1. Auscher, P.: On necessary and sufficient conditions for L^p-estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates. Mem. Amer. Math. Soc. **186**(871), (2007)
2. Blunck, S., Kunstmann, P.C.: Calderón-Zygmund theory for non-integral operators and the H^∞ functional calculus. Rev. Mat. Iberoamericana **19**(3), 919–942 (2003)
3. Chang, T., Kang, K.: On Caccioppoli's inequalities of Stokes equations and Navier-Stokes equations near boundary. J. Differential Equations **269**(9), 6732–6757 (2020)
4. Choe, H.J., Kozono, H.: The Stokes problem for Lipschitz domains. Indiana Univ. Math. J. **51**(5), 1235–1260 (2002)
5. Davies, E.B.: Uniformly elliptic operators with measurable coefficients. J. Funct. Anal. **132**(1), 141–169 (1995)
6. Davies, E.B.: Limits on L^p regularity of self-adjoint elliptic operators. J. Differential Equations **135**(1), 83–102 (1997)
7. Denk, R., Hieber, M., Prüss, J.: $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. Mem. Amer. Math. Soc. **166**(788), (2003)
8. Diening, L., Růžička, M., Wolf, J.: Existence of weak solutions for unsteady motions of generalized Newtonian fluids. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **9**(1), 1–46 (2010)
9. Dore, G.: Maximal regularity in L^p spaces for an abstract Cauchy problem. Adv. Differential Equations **5**(1–3), 293–322 (2000)
10. Egert, M.: L^p-estimates for the square root of elliptic systems with mixed boundary conditions. J. Differential Equations **265**(4), 1279–1323 (2018)
11. Engel, K.-J., Nagel, R.: One-parameter semigroups for linear evolution equations. Graduate Texts in Mathematics, vol. 194. Springer, New York, (2000)
12. Gabel, F., Tolksdorf, P.: The Stokes operator in two-dimensional bounded Lipschitz domains. J. Differential Equations **340**, 227–272 (2022)
13. Galdi, G.P.: An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems. Springer Monographs in Mathematics. Springer, New York (2011)
14. Giaquinta, M., Modica, G.: Nonlinear systems of the type of the stationary Navier-Stokes system. J. Reine Angew. Math. **330**, 173–214 (1982)
15. Haase, M.: The functional calculus for sectorial operators. Operator Theory: Advances and Applications, 169. Birkhäuser Verlag, Basel, (2006)
16. Kalton, N.J., Weis, L.: The H^∞-calculus and sums of closed operators. Math. Ann. **312**(2), 319–345 (2001)
17. Kalton, N.J., Weis, L.: Perturbation and interpolation theorems for the H^∞-calculus with applications to differential operators. Math. Ann. **336**(4), 747–801 (2006)
18. Kaplický, P., Málek, J., Stará, J.: $C^{1,\alpha}$-Solutions to a Class of Nonlinear Fluids in the 2D Stationary Dirichlet Problem. J. Math. Sci. **109**(5), 1867–1893 (2002)
19. Kaplický, P., Málek, J., Stará, J.: Global-in-time Hölder continuity of the velocity gradients for fluids with shear-dependent viscosities. NoDEA Nonlinear Differential Equations Appl. **9**(2), 175–195 (2002)
20. Kaplický, P., Wolf, J.: On the higher integrability of weak solutions to the generalized Stokes system with bounded measurable coefficients. Dyn. Partial Differ. Equ. **15**(2), 127–146 (2018)
21. Kato, T.: Fractional powers of dissipative operators. J. Math. Soc. Japan **13**, 246–274 (1961)
22. Kato, T.: Perturbation theory for linear operators. Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, (1995)
23. Kunstmann, P., Weis, L.: New criteria for the H^∞-calculus and the Stokes operator on bounded Lipschitz domains. J. Evol. Equ. **17**(1), 387–409 (2017)
24. Kuusi, T., Mingione, G., Sire, Y.: Nonlocal self-improving properties. Anal. PDE **8**(1), 57–114 (2015)
25. Mitrea, M., Monniaux, S., Wright, M.: The Stokes operator with Neumann boundary conditions in Lipschitz domains. J. Math. Sci. **176**(3), 409–457 (2011)
26. Prüss, J.: H^∞-calculus for generalized Stokes operators. J. Evol. Equ. **18**(3), 1543–1574 (2018)
27. Prüss, J., Simonett, G.: Moving interfaces and quasilinear parabolic evolution equations. Monographs in Mathematics, vol. 105. Birkhäuser/Springer, Cham, (2016)
28. Shen, Z.: Bounds of Riesz transforms on L^p spaces for second order elliptic operators. Ann. Inst. Fourier (Grenoble) **55**(1), 173–197 (2005)
29. Shen, Z.: Resolvent estimates in L^p for the Stokes operator in Lipschitz domains. Arch. Ration. Mech. Anal. **205**(2), 395–424 (2012)
30. Sohr, H.: The Navier-Stokes equations. An elementary functional analytic approach. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, (2001)

31. Solonnikov, V.A.: L_p-estimates for solutions to the initial boundary-value problem for the generalized Stokes system in a bounded domain. Function theory and differential equations J. Math. Sci. (New York) **105**(5), 2448–2484 (2001)
32. Stein, E.M.: Harmonic Analysis: real-variable methods, orthogonality, and oscillatory integrals. Princeton University Press, vol. 43. Princeton, NJ, (1993)
33. Tolksdorf, P.: $\mathbb{R}$-sectoriality of higher-order elliptic systems on general bounded domains. J. Evol. Equ. **18**(2), 323–349 (2018)
34. Tolksdorf, P.: The Stokes resolvent problem: optimal pressure estimates and remarks on resolvent estimates in convex domains. Calc. Var. Partial Differential Equations **59**(5), article no. 154 (2020)
35. Tolksdorf, P.: L^p-extrapolation of non-local operators: Maximal regularity of elliptic integrodifferential operators with measurable coefficients. J. Evol. Equ. **27**(3), 3129–3151 (2021)
36. Weis, L.: Operator-valued Fourier multiplier theorems and maximal L_p-regularity. Math. Ann. **319**(4), 735–758 (2001)