On an Operator-Theoretic Approach to Strichartz Estimates for Rough Wave Equations

Zur Erlangung des akademischen Grades eines

DOKTORS DER NATURWISSENSCHAFTEN

von der KIT-Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)

genehmigte

DISSERTATION

von

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Tag der mündlichen Prüfung:	5. März 2026
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Acknowledgment

First and foremost, I would like to thank my advisor, Prof. Dr. Dorothee Frey, for her guidance, support and immense patience during my doctoral studies. She was always very encouraging and always had an open door for questions and discussions. Furthermore, I benefitted a lot from her mathematical experience, insights and intuition.
Second, I would like to thank Prof. Dr. Roland Schnaubelt for co-examining this thesis, and of course, for very helpful discussions, for sharing his fascinating knowledge about mathematical ideas and results, and for being a very friendly co-head of our working group.
Moreover, I would like to thank all my current colleagues of my working group — Christopher, David, Luca, Henning, Maximilian, Patrick, Richard, and Siliang — and, of course, all my former colleagues, who made this journey much more enjoyable. From discussions about mathematics, to playing chess during our daily coffee break at 4 pm, to attending TULKKA events together, I enjoyed our time together. In particular, I would like to thank Patrick for always taking the time to answer my mathematical questions, and also (former) colleagues from other working groups, such as Alexander, who frequently organized fun activities outside of academia, and Marvin, who has become a good friend.
Last but not least, I would like to extend my heartfelt thanks to my family and friends for their unwavering support, understanding, and for enriching my life.

Notation

In the following table, we summarize the notation used in this thesis.

General

a.e.	almost everywhere
w.r.t.	with respect to
δ_{jk}	Kronecker delta
$A \lesssim B$	$A \le CB$ for some immaterial constant C
$A \lesssim_p B$	$A \le CB$ for some constant C_p depending on p
$A \simeq B$	$A \lesssim B$ and $B \lesssim A$
$I \Subset J$	I being compactly contained in J
$a \wedge b$	the minimum of two reals a and b
a_+	$\max\{a, 0\}$ for a real number a
p'	the Hölder conjugate exponent of p
ξ'	the first $d-1$ variables of $\xi \in \mathbb{R}^d$
$\lvert \cdot \rvert$	the Euclidean norm on $\mathbb{R}^d$ or $\mathbb{C}^d$
$\operatorname{supp}(f)$	the support of a function f
$\langle \lambda \rangle, \langle \xi \rangle$	$(1+\lambda^2)^{1/2}$, $(1+\lvert\xi\rvert^2)^{1/2}$ for $\lambda > 0$, $\xi \in \mathbb{R}^d$
$X^{\sim}$	the completion of a normed space X
$(\cdot\vert\cdot)$	the scalar product in a Hilbert space
$\langle \cdot\vert\cdot \rangle_{X \times X'}$	duality bracket in $X \times X'$
$y \cdot \xi$	$\sum_{j=1}^d y_j \xi_j$ for $y, \xi \in \mathbb{R}^d$
$\operatorname{sinc}(z)$	$\frac{\sin(z)}{z}$ for $z \neq 0$

Sets

$\mathbb{N}, \mathbb{N}_0$	the natural numbers, the natural numbers including zero
$\mathbb{Z}, \mathbb{R}, \mathbb{C}$	the integers, the reals, the complex numbers
$\mathbb{C}_-, \mathbb{C}_+$	the left open half-plane, the right open half-plane
$2^{\mathbb{Z}}$	the set $\{2^k \mid k \in \mathbb{Z}\}$
S_ω	the open sector of angle ω
$\overline{M}$	the topological closure of a set M

$B(x,r)$	the open ball in $\mathbb{R}^d$, centered at x with radius $r>0$
$\overline{B}(x,r)$	the closed ball in $\mathbb{R}^d$, centered at x with radius $r>0$
$A(r_1,r_2)$	the open annulus in $\mathbb{R}^d$ with radii $0\leq r_1<r_2\leq\infty$.

Calculus and Fourier Analysis

∂_x^α	partial derivative $\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}$ for $\alpha\in\mathbb{N}_0^d$ and $x\in\mathbb{R}^d$
∂_j	∂_{x_j}
∇_x	the gradient $(\partial_{x_1},\ldots,\partial_{x_d})$
Δ_x	the Laplacian $\partial_{x_1}^2+\cdots+\partial_{x_d}^2$
D_x^α	$i^{-\lvert\alpha\rvert}\partial_x^\alpha$
∂_t^n	partial derivative of order $n\in\mathbb{N}_0$ w.r.t. time
D_t^n	$i^{-\lvert\alpha\rvert}\partial_t^n$
M_m	the multiplication operator associated to $m\colon\mathbb{R}^d\to\mathbb{C}$
$\mathcal{F}$	the Fourier transform
$m(D_x)$	the Fourier multiplier operator $\mathcal{F}^{-1}M_m\mathcal{F}$
$\lvert D_x\rvert^\alpha$, $\langle D_x\rangle^\alpha$	the Fourier multiplier operators associated to $\lvert\cdot\rvert^\alpha$, $\langle\cdot\rangle^\alpha$, $\alpha\in\mathbb{R}$

Function Spaces

$C^k(V)$	the space of k-times differentiable functions on V
$C^{0,1}(\mathbb{R})$	the space of bounded Lipschitz functions
$C_c^\infty(\mathbb{R}^d)$	the space of smooth, compactly supported functions
$\mathcal{S}(\mathbb{R}^d)$	the Schwartz space
$\mathcal{S}'(\mathbb{R}^d)$	the space of tempered distributions
$\mathrm{L}^p(\mathbb{R}^d)$	the Lebesgue space
$\mathrm{H}^\alpha(\mathbb{R}^d)$	the L^2-based Sobolev space of order $\alpha\in\mathbb{R}$
$\mathrm{H}_L^\alpha(\mathbb{R}^d)$	L-adapted Sobolev space of order $\alpha\in\mathbb{R}$
$W^{\alpha,p}(\mathbb{R}^d)$	the L^p-based Sobolev space of order $\alpha\in\mathbb{R}$
$\dot{W}^{\alpha,p}(\mathbb{R}^d)$	the L^p-based homogeneous Sobolev space of order $\alpha\in\mathbb{R}$
$\mathrm{L}^p(\mathbb{R};X)$	the Bochner space of p-integrable functions $f\colon\mathbb{R}\to X$
$W^{k,p}(\mathbb{R};X)$	the $\mathrm{L}^p(\mathbb{R};X)$-based vector-valued Sobolev space of order $k\in\mathbb{N}_0$
$C^k(\mathbb{R};X)$	the space of k-times differentiable functions with values in X

$C_b^k(\mathbb{R};X)$	the space of functions f in $C^k(\mathbb{R};X)$ with $\partial_t^\ell f$ bounded for all $0 \le \ell \le k$
$C_b(\mathbb{R};X)$	$C_b^0(\mathbb{R};X)$
$\dot{C}_b^1(\mathbb{R};X)$	the space of functions u in $C^1(\mathbb{R};X)$ with $u' \in C_b(\mathbb{R};X)$
$\mathcal{S}(\mathbb{R}^d;X)$	the space of Schwartz functions with values in X
$\mathcal{F}\mathrm{L}^1$	the space $\mathcal{F}(\mathrm{L}^1(\mathbb{R}^d))$
$\mathcal{F}\mathbf{M}$	the space $\mathcal{F}(\mathbf{M}(\mathbb{R}^d))$, $\mathbf{M}(\mathbb{R}^d)$ the space of complex Borel measures with finite variation norm

Operator Theory

$\mathsf{Dom}(L)$	the domain of a linear operator L
$\mathsf{R}(L)$	the range of a linear operator L
L^*	the adjoint operator of L in a Hilbert space
L'	the adjoint operator of L in a Banach space
$\sigma(L)$	the spectrum of a linear operator L
$\rho(L)$	the resolvent set of a linear operator L
$R(\lambda, L)$	the resolvent of L at $\lambda \in \mathbb{C}$
$\overline{L}$	the closure of a linear operator
L^α	fractional power of L
$(\mathrm{e}^{iy\cdot D})_{y\in\mathbb{R}^d}$	the d-parameter C_0-group generated by iD
$\langle D_L\rangle^\alpha$	the operator $(\mathrm{Id}+L)^{\frac{\alpha}{2}}$ or its extension $(\mathrm{Id}+\mathscr{L})^{\frac{\alpha}{2}}$
$\mathcal{L}(X,Y)$	the space of bounded linear operators from X to Y
$\mathcal{L}(X)$	the space $\mathcal{L}(X,X)$
$\mathcal{C}(X)$	the space of closed operators in X

Basic Differential Geometry

$T_\omega\mathbb{S}$	the tangent space at the point $\omega \in \mathbb{S}$
$N_\omega\mathbb{S}$	the normal space at the point $\omega \in \mathbb{S}$
$\mathrm{d}f_\omega$	the differential of f in ω
$\mathrm{d}_\mathbb{S}$	distance function on $\mathbb{S}$

Further Remarks

We preserve the letter d to denote the dimension of the Euclidean space $\mathbb{R}^d$. We use the convention

$$(\mathcal{F}f)(\xi) := \int_{\mathbb{R}^d} \mathrm{e}^{-iy\cdot\xi} f(y)\,\mathrm{d}\xi \quad (f \in \mathcal{S}(\mathbb{R}^d))$$

for the Fourier transform with $(\mathcal{F}^{-1}f)(y) = (2\pi)^{-d}(\mathcal{F}f)(-y)$. Up to some exceptions (which should be clear from the context), we preserve the letter ψ to denote a generic $C_c^\infty(\mathbb{R}^d)$-function of the frequency variable $\xi \in \mathbb{R}^d$, where ψ is supported away from the origin. In contrast, we use the letter φ to denote $C_c^\infty(\mathbb{R}^d)$-functions supported in a neighborhood of the origin. From time to time, we also use φ to define mollifiers or, more importantly, a phase function. We use the letter $\lambda \in 2^{\mathbb{Z}}$ to denote the 'size' of the frequency in the Littlewood–Paley decomposition, and we use the notation ψ_λ to denote the rescaled function $\psi_\lambda(\xi) = \psi(\frac{\xi}{\lambda})$ (again, up to some exceptions that should be clear from the context). The letter α will always denote a regularity parameter in $\mathbb{R}$ or a multiindex in $\mathbb{N}_0^d$. We also want to warn the reader that D does not refer to a derivative, but to a more general generator of a C_0-group $(\mathrm{e}^{iy\cdot D})_{y\in\mathbb{R}^d}$. We use ∇_x and $D_x = -i\nabla_x$ for spatial as well as ∂_t and $D_t = -i\partial_t$ for temporal derivatives instead.
Very frequently (and especially in Chapters 3 and 4), functions are $\mathbb{C}^2$-valued. In this case, we often suppress the target space in the notation and write $\mathrm{L}^p(\mathbb{R})$, $\mathrm{H}^1(\mathbb{R})$, etc. instead of $\mathrm{L}^p(\mathbb{R};\mathbb{C}^2)$, $\mathrm{H}^1(\mathbb{R};\mathbb{C}^2)$ to ease notation. Concerning mixed-norm spaces, we frequently write $\mathrm{L}^p_t(\mathbb{R};\mathrm{L}^q_x(\mathbb{R}^d))$ and $\|\cdot\|_{\mathrm{L}^p_t(\mathbb{R};\mathrm{L}^q_x(\mathbb{R}^d))}$ in place of $\mathrm{L}^p(\mathbb{R};\mathrm{L}^q(\mathbb{R}^d))$ and $\|\cdot\|_{\mathrm{L}^p(\mathbb{R};\mathrm{L}^q(\mathbb{R}^d))}$, respectively, in order to emphasize with respect to which variables the L^p- and L^q-norms are taken. For the theory of Bochner spaces, we refer the reader to [30]. We only assume familiarity with just basic facts about vector-valued Sobolev spaces, as can be found in e.g. [10].

1. Introduction

Since their early formulation in the 1740s by d'Alembert in the context of the vibrating string [14], wave equations have remained an active area of research to this day, owing to the ubiquity of wave phenomena both in nature and applications. In this thesis, we investigate wave equations with rough coefficients, specifically those that exhibit regularity weaker than C^2. Let us try to motivate the research question physically. Consider a wave equation of the form

$$\Big(D_t^2 - \sum_{j,k=1}^{d} c_{jk} D_{x_j} D_{x_k}\Big) u(t,x) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \tag{1.1}$$

where the coefficients c_{jk} are allowed to depend on space and time. The case $c_{jk} = \delta_{jk}$ corresponds to the well-known *classical wave equation*, which describes wave propagation in a homogeneous medium. On the other hand, one motivation for studying wave equations with low-regularity coefficients (aside from quasilinear problems (see Section 1.2)) is to understand the propagation of waves in inhomogeneous, rough media. In dimension $d \geq 2$, a key feature of the behavior of waves is *dispersion*, i.e., the phenomenon by which waves spread out over time, and one might ask:

Under which conditions do waves in inhomogeneous, rough media disperse in the same way as corresponding ones in homogeneous media?

One way to measure dispersion mathematically is via so-called *Strichartz estimates*. Thus, one could ask the following related question:

Under which regularity assumptions on the coefficients c_{jk} do solutions u to (1.1) *satisfy the same Strichartz estimates as solutions to the classical wave equation?*

We address this question, and in particular our contribution to it, in the next section.

1.1. Background on Strichartz Estimates and Main Results

Strichartz estimates refer to a family of space-time estimates for the solutions of linear dispersive equations and play a fundamental role in the well-posedness theory of nonlinear dispersive equations. For a solution $u \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ of the classical wave equation in dimension $d \geq 2$

$$(D_t^2 + \Delta_x)u = F \quad \text{in } \mathbb{R} \times \mathbb{R}^d, \quad u(0,\cdot), \quad D_t u(0,\cdot) = h, \tag{1.2}$$

they read in the homogeneous case $F = 0$

$$\||D_x|^{1-\alpha}u\|_{\mathrm{L}_t^p(\mathbb{R};\mathrm{L}_x^q(\mathbb{R}^d))} \lesssim \|g\|_{\dot{\mathrm{H}}^1(\mathbb{R}^d)} + \|h\|_{\mathrm{L}^2(\mathbb{R}^d)}. \tag{1.3}$$

Here, $(p, q, \alpha) \in [0, \infty]^3$ satisfy the admissibility conditions

$$2 \leq p \leq \infty,\ 2 \leq q < \infty, \quad \frac{1}{p} + \frac{d}{q} = \frac{d}{2} - \alpha, \quad \frac{2}{p} + \frac{d-1}{q} \leq \frac{d-1}{2}. \tag{1.4}$$

Triples satisfying (1.4) are called *(wave-admissible) Strichartz triples*, and they are referred to as *strict* if the fourth condition in (1.4) holds with an equality. Scaling symmetry and the Knapp example show that the third and fourth condition in (1.4) are in fact necessary for (1.3) to hold (see [58]). The conditions $p, q \geq 2$ can also be shown to be necessary. We assume $q < \infty$ in (1.4) because in general, (1.3) fails to hold when $q = \infty$ (see [18], [26] ; however, we note that this failure can be remedied if one replaces the space $\mathrm{L}_x^\infty(\mathbb{R}^d)$ by the homogeneous Besov space $\dot{B}^0_{\infty,2}$, see e.g. [7, Corollary 8.25], provided that one excludes the forbidden triple $(2, \infty, 1)$ in dimension $d = 3$). On the other hand, $q = 2$ is permitted. In fact, $(\infty, 2, 0)$ is the trivial Strichartz triple as in this case, (1.3) is just the energy inequality, which is easy to prove. At the other extreme, strict Strichartz triples with $p = 2$, i.e., $(2, \frac{2(d-1)}{d-3}, \frac{d+1}{2(d-1)})$ for $d \geq 4$, are called *endpoint* Strichartz triples, and (1.3) for such triples are much harder to prove.
Strichartz estimates for the wave equation have a long history. In his seminal work, Strichartz [57] proved (1.3) in the case $p = q$. Later, Ginibre–Velo [23] and Lindblad–Sogge [37] independently established (1.3) for all non-endpoint Strichartz triples. Finally, Keel–Tao [34] settled the endpoint case. There also exist Strichartz estimates for (1.2) in the inhomogeneous setting in (1.2) where $F \neq 0$. They read

$$\||D_x|^{1-\alpha}u\|_{\mathrm{L}_t^p(\mathbb{R};\mathrm{L}_x^q(\mathbb{R}^d))} \lesssim \|g\|_{\dot{\mathrm{H}}^1(\mathbb{R}^d)} + \|h\|_{\mathrm{L}^2(\mathbb{R}^d)} + \||D_x|^{\tilde{\alpha}}F\|_{\mathrm{L}_t^{\tilde{p}'}(\mathbb{R};\mathrm{L}_x^{\tilde{q}'}(\mathbb{R}^d))} \tag{1.5}$$

for all Schwartz solutions u to (1.2) and all Strichartz triples (p, q, α), $(\tilde{p}, \tilde{q}, \tilde{\alpha})$ (see [58, Theorem 2.6]). The fruitfulness of Strichartz estimates in the study of nonlinear problems was already conjectured by Segal [46] in the 1970s and has since been proven in a plethora of works (see e.g. [53], [58], [7] for an account of nonlinear wave equations).
In this thesis, we are concerned with an analog of (1.3) for variable-coefficient linear wave equations. More precisely, we consider

$$\begin{cases} \big(D_t^2 - P(t, x, D_x)\big)u(t, x) = F(t, x), & (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0, x) = g(x), & x \in \mathbb{R}^d, \\ D_t u(0, x) = h(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.6}$$

where $D_t := \frac{1}{i}\partial_t$, $D_x := \frac{1}{i}\nabla_x$ and $P(t,x,D_x)$ is an elliptic second order differential operator

$$P(t,x,D_x) := \sum_{j,k=1}^{d} D_{x_j} c_{jk}(t,x) D_{x_k} \quad \text{or} \quad P(t,x,D_x) := \sum_{j,k=1}^{d} c_{jk}(t,x) D_{x_j} D_{x_k}$$

in divergence or standard form, respectively. Variable-coefficient wave equations arise in the description of many physical phenomena (e.g., electromagnetism, general relativity, acoustics, etc.). Although these are oftentimes of quasilinear structure, establishing local well-posedness of quasilinear wave equations with *rough* initial data (i.e., initial data belonging to $\mathrm{H}^{\alpha}(\mathbb{R}^d) \times \mathrm{H}^{\alpha-1}(\mathbb{R}^d)$ for low values $\alpha > \frac{d}{2}$) often involves an approximation scheme in which Strichartz estimates are used for a linearized wave equation (see e.g. [60], [6], [50] and the references therein). Here, one difficulty is that the linearized wave equation is *rough* in the sense that its coefficients c_{jk} are of low regularity. Therefore, it is essential to understand how low regularity of the coefficients affects the availability of Strichartz estimates. In the following, we state the regularity assumptions that we impose on our coefficients c_{jk}. Note that we also have to make a structural assumption, which we need to obtain better results than the ones in the literature (see Section 1.2 for a more thorough discussion).

Assumption 1.1.1. *We have $c_{jk} = 0$ if $j \neq k$ and*

$$c_{jj}(t,x) := b_j(t) a_j(x_j), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d,\ j \in \{1,\dots,d\}, \tag{1.7}$$

where $a_1,\dots,a_d$ and $b_1,\dots,b_d$ are functions from $\mathbb{R}$ to $\mathbb{R}$ with the following properties.

(A_a) *There exist constants $0 < m_1 \le m_2 < \infty$ such that*

$$m_1 \le a_j(x) \le m_2 \quad \text{for all } x \in \mathbb{R} \text{ and } j \in \{1,\dots,d\}. \tag{1.8}$$

The functions $a_1,\dots,a_d$ are Lipschitz continuous and we assume that

$$m_3 := \max_{1\le j\le d} \left\| \tfrac{\mathrm{d}}{\mathrm{d}x} \log(a_j) \right\|_{\mathrm{L}^1(\mathbb{R})} < 4. \tag{1.9}$$

(A_b) *The functions $b_1,\dots,b_d$ are continuously differentiable and for some sufficiently small $\varepsilon_0 \in (0,\frac{1}{2})$ we have*

$$1-\varepsilon_0 \le b_j(t) \le 1+\varepsilon_0 \quad \text{for all } t \in \mathbb{R} \text{ and } j \in \{1,\dots,d\}. \tag{1.10}$$

We set $m_4 := \max_{1\le j\le d} \|b_j'\|_\infty < \infty$. Moreover, we assume that there exists some sufficiently small $\varepsilon_1 = \varepsilon_1(m_1,m_2,m_4) > 0$ such that

$$\max_{1\le j\le d} \|b_j'\|_{\mathrm{L}^1(\mathbb{R})} \le \varepsilon_1. \tag{1.11}$$

Thus, $P(t,x,D_x)$ is of the form

$$P(t,x,D_x) = \sum_{j=1}^{d} D_{x_j} b_j(t) a_j(x_j) D_{x_j} \quad \text{or} \quad P(t,x,D_x) = \sum_{j=1}^{d} b_j(t) a_j(x_j) D_{x_j}^2.$$

To ease notation, we will write $P(t) \coloneqq P(t,x,D_x)$ (and occasionally just P) in the following. In order to state our main result, we interpret (1.6) as an abstract Cauchy problem, for which we use the following notion of a *weak solution*.

Definition 1.1.2 (Weak Solutions in $\mathrm{H}^\alpha(\mathbb{R}^d)$)**.** Let $\alpha \in \mathbb{R}$ and suppose that $g \in \mathrm{H}^\alpha(\mathbb{R}^d), h \in \mathrm{H}^{\alpha-1}(\mathbb{R}^d)$, and $F \in \mathrm{L}^1(\mathbb{R};\mathrm{H}^{\alpha-1}(\mathbb{R}^d))$. Then, a function $u \in C(\mathbb{R};\mathrm{H}^\alpha(\mathbb{R}^d)) \cap C^1(\mathbb{R};\mathrm{H}^{\alpha-1}(\mathbb{R}^d)) \cap W^{2,1}_{\mathrm{loc}}(\mathbb{R};\mathrm{H}^{\alpha-2}(\mathbb{R}^d))$ is called a *weak solution* to (1.6) in $\mathrm{H}^\alpha(\mathbb{R}^d)$ if

$$\begin{cases} D_t^2 u(t) = P(t)u(t) + F(t) \quad \text{in } \mathrm{H}^{\alpha-2}(\mathbb{R}^d) \quad \text{for a.e. } t \in \mathbb{R}, \\ \quad u(0) = g, \\ D_t u(0) = h. \end{cases}$$

The following theorems are then the main results of this thesis.

Theorem 1.1.3 (Existence and Uniqueness of Weak Solutions in $\mathrm{H}^\alpha(\mathbb{R}^d)$)**.** *Let $\alpha \in [-1,2]$ and suppose that $g \in \mathrm{H}^\alpha(\mathbb{R}^d)$, $h \in \mathrm{H}^{\alpha-1}(\mathbb{R}^d)$, and $F \in \mathrm{L}^1(\mathbb{R};\mathrm{H}^{\alpha-1}(\mathbb{R}^d))$. Then, there exists a unique weak solution u to* (1.6)*.*

Regarding Theorem 1.1.3, we can weaken the assumptions on the coefficients b_j a bit, see Remark 3.3.9. For the following theorems, we suppose $d \geq 2$.

Theorem 1.1.4 (Global-In-Time Strichartz Estimates for Weak Solutions in $\mathrm{H}^1(\mathbb{R}^d)$)**.** *Let (p,q,α) be a wave-admissible Strichartz triple and $\alpha \in [0,2]$. Suppose that $g \in \mathrm{H}^1(\mathbb{R}^d)$, $h \in \mathrm{L}^2(\mathbb{R}^d)$, and $F \in \mathrm{L}^1(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))$. Then, the weak solution to the wave equation* (1.6) *satisfies the global-in-time Strichartz estimate*

$$\||D_x|^{1-\alpha}u\|_{\mathrm{L}^p_t(\mathbb{R};\mathrm{L}^q_x(\mathbb{R}^d))} \lesssim \|g\|_{\mathrm{H}^1(\mathbb{R}^d)} + \|h\|_{\mathrm{L}^2(\mathbb{R}^d)} + \|F\|_{\mathrm{L}^1(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))}. \tag{1.12}$$

In Section 4.4, we demonstrate how the assumption of continuous differentiability of the coefficients b_j can be weakened to Lipschitz continuity:

Theorem 1.1.5 (Global-In-Time Strichartz Estimates for Weak Solutions in $\mathrm{H}^1(\mathbb{R}^d)$)**.** *Let (p,q,α) be a wave-admissible Strichartz triple and $\alpha \in [0,2]$. Suppose further that $b_1,\dots,b_d$ as in* (A_b) *are only Lipschitz continuous and that $g \in \mathrm{H}^1(\mathbb{R}^d)$, $h \in \mathrm{L}^2(\mathbb{R}^d)$, and $F \in \mathrm{L}^1(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))$. Then, the weak solution to the wave equation* (1.6) *satisfies the global-in-time Strichartz estimate*

$$\||D_x|^{1-\alpha}u\|_{\mathrm{L}^p_t(\mathbb{R};\mathrm{L}^q_x(\mathbb{R}^d))} \lesssim \|g\|_{\mathrm{H}^1(\mathbb{R}^d)} + \|h\|_{\mathrm{L}^2(\mathbb{R}^d)} + \|F\|_{\mathrm{L}^1(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))}. \tag{1.13}$$

Note that these results are global in time. Theorems 1.1.4 and 1.1.5 generalize the corresponding result in [21], in that we can assume for a multiplicative time-dependence of the coefficients (we give a more detailed overview over existing results in the literature and a comparison with our result in Section 1.2 below).
The main challenge in this setting arises from the lack of smoothness in the coefficients, which causes approaches by phase space methods, as used in e.g. [59], [60], [51], to break down. However, the lack of smoothness is compensated by the *crucial* structural assumption (1.7), which opens the door for an operator-theoretic approach. This operator-theoretic approach, which has its roots in Frey-Portal's work on sharp L^p-estimates for wave equations with Lipschitz coefficients [19], is essential for the proofs of the theorems above, as they replace Fourier multipliers by more useful operator-adapted analogs defined by functional calculus. Combining this with a parametrix construction which goes back to Smith [47], we are able to prove Theorem 1.1.3. Despite being interesting in its own right, the key point of Theorem 1.1.3 is that it comes with a useful representation formula for the weak solution that is good enough for the purpose of proving global-in-time Strichartz estimates. In fact, Theorem 1.1.4 (and Theorem 1.1.5) will essentially follow from a dispersive estimate for the Fourier transform of a surface-carried measure.

1.2. Review of Existing Results and Discussion

One of the main difficulties in proving Strichartz estimates revolves around finding an effective way of representing the solution or at least a good enough approximation thereof. The classical wave equation (1.2) is amenable to Fourier analysis which provides an explicit representation of the solution of (1.2) in terms of Fourier multipliers. In this case, the proof of (1.3) essentially relies first and foremost on the crucial dispersive estimates for the $\psi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$-truncated half-wave propagator

$$\|\mathrm{e}^{it|D_x|}\psi(D_x)f\|_{\mathrm{L}^q(\mathbb{R}^d)} \lesssim_\psi (1+|t|)^{-\frac{d-1}{2}(1-\frac{2}{q})}\|f\|_{\mathrm{L}^{q'}(\mathbb{R}^d)} \tag{1.14}$$

for $q \geq 2$, $\frac{1}{q'} = 1 - \frac{1}{q}$. For $q = 2$, this estimate is an immediate consequence of Plancherel's theorem, while for $q = \infty$, it is obtained via the stationary phase method for oscillatory integrals. The estimate for intermediate values of q is then derived by interpolating these two endpoints. Combining (1.14) with Littlewood–Paley theory, a TT^*-argument and the Hardy–Littlewood–Sobolev inequality then gives the desired Strichartz estimate (1.3), provided that $p > 2$ (see [7, Chapter 8] for details). The inapplicability of the Hardy–Littlewood–Sobolev inequality in the endpoint case, however, makes a more sophisticated proof necessary which goes back to Keel–Tao (see [34]). In a number of works since then, local (in-time) Strichartz estimates[1] have also been established for linear variable-coefficient wave equations such as (1.6). We give a brief (non-exhaustive) overview in the following.

(I) Smooth Metrics: If the coefficients c_{jk} are assumed to be smooth, Lax's parametrix construction for the half-wave propagator [36] (see also [25, Chapter 6] or [52, Section 4.1] for a more accessible presentation in the setting of wave equations) yields at least for sufficiently small times t a representation of the solution u of (1.6) given by

$$u(t,\cdot) = \mathscr{F}_0(t)g + \mathscr{F}_1(t)h + R_0(t)g + R_1(t)h,$$

where, roughly speaking, $\mathscr{F}_j(t)$ are Fourier integral operators (FIOs) order $-j$ and $R_j(t)$ are smoothing operators. Local Strichartz estimates (at least in the non-endpoint case) are then a consequence of mapping properties of such FIOs proved by Kapitanski ([32, Theorem 2], [33, Theorem 7.5]), see also the result by Mockenhaupt-Seeger-Sogge ([42, Theorem 3.1]) who studied FIOs in an even more general framework.

(II) Metrics with Limited Regularity (the C^2-Case): In the case, where the coefficients c_{jk} possess only limited regularity, Lax's parametrix construction is unfortunately not available anymore. Nevertheless, under the assumption of $C^{1,1}$-coefficients, local Strichartz estimates were first proved by Smith (in dimensions $d = 2, 3$) [47] and then in full generality by Tataru [59] for C^2-coefficients. The proofs first use Littlewood–Paley theory and ideas originating from paradifferential calculus (see [9], [11]) to reduce the wave equation (1.6) to wave equations of the form $(D_t^2 - P_{\lambda^\delta})u_\lambda = F_\lambda + R_\lambda$, where u_λ, F_λ are localized in frequency at scale $\lambda > 0$ and P_{λ^δ} is obtained from P by truncating the coefficients in frequency at scale λ^δ for some $\delta \in (0,1)$. Here, R_λ is an error term, and the parameter δ is chosen in such way that R_λ is of the same 'strength' as the driving force F_λ. It turns out that the C^2-assumption and the choice $\delta = \frac{1}{2}$ achieves exactly this. In a second step, one has to produce a tractable approximation of u_λ. We remark that, while P_{λ^δ} has smooth coefficients, Lax's parametrix construction does not apply since it involves an asymptotic sum of Hörmander-type symbols which does not yield a convergent expression here[2]. For this reason, Smith and Tataru applied wave packet techniques or phase space methods ([47], [51], [59], [60]) to produce a parametrix for $(D_t^2 - P_{\lambda^\delta})u_\lambda = 0$ in order to prove local Strichartz estimates for u_λ with constants *uniform* in λ. These parametrices are essentially obtained by the pullback of the Hamiltonian flow of the symbol of $\sqrt{P_{\lambda^\delta}}$, conjugated by a wave packet/phase space transform. Here, the C^2-

[1] By this, we mean that (1.3) holds with $\||D_x|^{1-\alpha}u\|_{\mathrm{L}_t^p(\mathbb{R};\mathrm{L}_x^q(\mathbb{R}^d))}$ replaced by $\||D_x|^{1-\alpha}u\|_{\mathrm{L}_t^p(I;\mathrm{L}_x^q(\mathbb{R}^d))}$ or $\|\langle D_x\rangle^{1-\alpha}u\|_{\mathrm{L}_t^p(I;\mathrm{L}_x^q(\mathbb{R}^d))}$ for some bounded interval $0 \in I \subseteq \mathbb{R}$.

[2] One can remedy this using a sharper truncation depending on the time interval but this leads to weaker estimates (see [6]).

assumption is again crucial in at least two ways. First, it allows to control the error between the parametrix and the exact solution operator, and second, it translates to the regularity of the Hamiltonian flow which is needed to prove dispersive estimates for the kernel of the parametrix. However, we mention that, with a more careful analysis, one can slightly weaken the assumption by just assuming $\partial_{t,x}^{\beta} c_{jk} \in \mathrm{L}^1_t(I; \mathrm{L}^\infty_x(\mathbb{R}^d))$ for all $|\beta| \leq 2$, which has important applications for the local wellposedness theory of quasilinear wave equations (see [60]).

(III) Metrics in the Low Regularity Regime (the C^r-Case with $0 \leq r < 2$): Strichartz estimates for wave equation with less regular coefficients, namely coefficients in $C^r(\mathbb{R} \times \mathbb{R}^d)$ ($r \in [0, 2)$), are still available; however in this case, these estimates are weaker in the sense that there occurs a loss of derivatives compared to (1.3) (see [59, Corollary 6]). More precisely, we have for Strichartz triples (p, q, α) and $d \geq 4$, $\sigma_r := \frac{2-r}{2+r}$

$$\|\langle D_x \rangle^{1-\alpha-\frac{\sigma_r}{p}} u\|_{\mathrm{L}^p_t(I;\mathrm{L}^q_x(\mathbb{R}^d))} \lesssim_{|I|} \|g\|_{\mathrm{H}^1(\mathbb{R}^d)} + \|h\|_{\mathrm{L}^2(\mathbb{R}^d)}. \tag{1.15}$$

As counterexamples by Smith–Tataru [49] demonstrate, this loss is necessary in general, at least in the case of Strichartz estimates for inhomogeneous linear wave equations.

(IV) Directionally Separable Time-Independent Lipschitz Metrics: The counterexamples produced by Smith–Tataru do not rule out the existence of low regularity metrics for which the full Strichartz estimates do hold *without* loss of derivatives. In fact, if $a_1, \dots, a_d$ satisfy (A_a) from Assumption 1.1.1 and

$$c_{jk}(t,x) := \delta_{jk} \cdot a_j(x_j) \quad t \in \mathbb{R}, x \in \mathbb{R}^d, \text{ i.e., if}$$
$$P(t,x) = L := \sum_{j=1}^d D_{x_j} a_j(x_j) D_{x_j} \quad \text{or} \quad P(t,x) = L := \sum_{j=1}^d a_j(x_j) D_{x_j}^2$$

then one recovers the full global-in-time Strichartz estimates without loss of derivatives (see [21]).

Comparing with these results, we would like to shed light on strengths and weaknesses of Theorem 1.1.4.

(1) First, we note that in comparison to (IV), the essential new feature in Theorem 1.1.4 is that we can allow for a 'multiplicative' time-dependence of the coefficients. Besides being interesting in its own right, time-dependence of the coefficients is important in order to effectively use the Strichartz estimates in quasilinear problems (as alluded to in the beginning of Section 1.1). However, the 'multiplicative' structure in our case prevents a straightforward application to quasilinear problems.

(2) Concerning the C^1-assumption on the time-dependent coefficients b_j (see (A_b) in Assumption 1.1.1), we remark that one cannot hope to obtain a wellposedness result or global Strichartz estimates without loss assuming mere Hölder-continuity for the coefficients b_j. Indeed, in this case, wave equations do not even necessarily admit distributional solutions for smooth initial data [13, Theorem 10]. However, following the paradifferential smoothing procedure used in [47], we can deduce that Lipschitz-regularity for the coefficients b_j is in fact enough to deduce Theorem 1.1.4 (see Corollary 4.4.1).

(3) We also note that local-in-time Strichartz estimates in the setting of Theorem 1.1.4 can be (in some cases) recovered from the results described in (II) by a change of variables. Indeed, due to our structural assumption (1.7), the coordinate transformation $\varphi\colon \mathbb{R}^d \to \mathbb{R}^d$, $\varphi(x) = (\varphi_1(x_1), \ldots, \varphi_d(x_d))$ with $\varphi_j(x_j) = \int_0^{x_j} \frac{1}{\sqrt{a_j(y)}}\,\mathrm{d}y$ leads to the equation

$$(D_t^2 - \tilde{P}(t, D_x))\tilde{u} = \tilde{F} + R\tilde{u}, \quad \tilde{u}(0) = \tilde{g}, \quad D_t\tilde{u}(0) = \tilde{h} \tag{1.16}$$

with $u(t,x) = \tilde{u}(t, \varphi(x))$, $F(t,x) = \tilde{F}(t, \varphi(x))$, $g(x) = \tilde{g}(\varphi(x))$, $h(x) = \tilde{h}(\varphi(x))$ and $\tilde{P}(t, D_x) = \sum_{j=1}^d b_j(t) D_{x_j}^2$ and $R \coloneqq \sum_{j=1}^d c_j(t, x_j) D_{x_j}$, $c_j \in \mathrm{L}^\infty(\mathbb{R}^2)$. Let $0 \in I \subseteq \mathbb{R}$ be a bounded interval. Observe that Hölder's and the energy inequality imply

$$\|R\tilde{u}\|_{\mathrm{L}^1_t(I;\mathrm{L}^2_x(\mathbb{R}^d))} \lesssim_{|I|} \|u\|_{\mathrm{L}^\infty_t(I;\dot{\mathrm{H}}^1(\mathbb{R}^d))} \lesssim \|g\|_{\dot{\mathrm{H}}^1(\mathbb{R}^d)} + \|h\|_{\mathrm{L}^2(\mathbb{R}^d)}$$

Thus, (at least if $b_j \in C^2(\mathbb{R})$) [59, Corollary 5], implies for any Strichartz triple (p, q, α) and compactly supported smooth solution u

$$\begin{aligned}
&\|\langle D_x\rangle^{1-\alpha} u\|_{\mathrm{L}^p_t(I;\mathrm{L}^q_x(\mathbb{R}^d))} \\
\lesssim &\|\langle D_x\rangle^{1-\alpha} \tilde{u}\|_{\mathrm{L}^p_t(I;\mathrm{L}^q_x(\mathbb{R}^d))} \\
\lesssim_{|I|} &\|\tilde{g}\|_{\mathrm{H}^1(\mathbb{R}^d)} + \|\tilde{h}\|_{\mathrm{L}^2(\mathbb{R}^d)} + \|\tilde{F}\|_{\mathrm{L}^1_t(I;\mathrm{L}^2_x(\mathbb{R}^d))} + \|R\tilde{u}\|_{\mathrm{L}^1_t(I;\mathrm{L}^2_x(\mathbb{R}^d))} \\
\lesssim_{|I|} &\|g\|_{\mathrm{H}^1(\mathbb{R}^d)} + \|h\|_{\mathrm{L}^2(\mathbb{R}^d)} + \|F\|_{\mathrm{L}^1_t(I;\mathrm{L}^2_x(\mathbb{R}^d))}.
\end{aligned}$$

So, the main point of Theorem 1.1.4 is that the Strichartz estimates are *global*-in-time. For more regular coefficients (at least C^2), global Strichartz estimates have already been obtained under the assumption of asymptotic flatness of the coefficients and non-existence of trapped rays (see e.g. [61], [48], [41]). The asymptotic flatness compares to our assumptions (1.9), (1.11), and trapped rays are precluded by the structural assumption (1.7).

(4) Unlike in (1.3), we have inhomogeneous norms on the right-hand side of (1.12). However, this can be remedied, see Remark 4.4.2.

(5) We do not discuss the full inhomogeneous Strichartz estimates as in (1.5), but note that we do have the energy case $(\tilde{p}, \tilde{q}, \tilde{\alpha}) = (\infty, 2, 0)$.

(6) The key reason why we can relax the C^2-regularity of the coefficients to Lipschitz regularity in our setting, is the structural assumption (1.7). In the interest of gaining a little microlocal intuition, let us assume for a moment that $a_j \in C^2(\mathbb{R})$. Then, we find that, due to (1.7), the Hamiltonian ODEs associated with the symbol $p(t,x,\xi) = (\sum_{j=1}^d b_j(t) a_j(x_j) \xi_j^2)^{1/2}$, given by

$$\begin{aligned} \dot{x}(t) &= \quad \partial_\xi p(t, x(t), \xi(t)), \quad x(0) = x_0 \in \mathbb{R}^d, \\ \dot{\xi}(t) &= -\partial_x p(t, x(t), \xi(t)), \quad \xi(0) = \xi_0 \in \mathbb{R}^d \setminus \{0\}, \end{aligned}$$

decouple into d two-dimensional systems for (x_j, ξ_j) given by

$$\frac{\dot{x}_j(t)}{a_j(x_j(t))} = b_j(t) \frac{\xi_j(t)}{p_0}, \quad \dot{\xi}_j(t) = -b_j(t) a_j'(x_j) \frac{\xi_j^2(t)}{2p_0}, \tag{1.17}$$

where $p_0 := p(0, x_0, \xi_0)$. This leads to a decoupled Hamiltonian flow $\chi_t \colon (x_0, \xi_0) \mapsto (x(t), \xi(t))$ which, for the purpose of obtaining Strichartz estimates, is close enough to the case of the flat Laplacian, in which $\chi_t(x_0, \xi_0) = (x_0 + t\frac{\xi_0}{|\xi_0|}, \xi_0)$. As it appears, these heuristics do not break down if a_j is only assumed to be Lipschitz, although the ODE for ξ_j in (1.17) need not be well-defined in this case. However, unfortunately, deviating from this structural assumption by allowing a_j to depend on transverse components x_k, $k \neq j$, turns out to be a very delicate matter.

1.3. Outline of the Thesis

In Chapter 2, we lay the groundwork for proving the main results. The chapter is divided into three sections. The first section briefly recalls the holomorphic functional calculus for sectorial operators and in particular, their fractional powers. This is followed by a more detailed discussion of the Phillips functional calculus for bounded d-parameter C_0-groups, which serves as a foundational tool for the thesis and is also included for the benefit of readers who may not be familiar with it. The second section focuses on half-wave groups in one dimension. Finally, the third section brings together the concepts of operator theory and one-dimensional half-wave groups, thereby setting up the framework as introduced in [19] and which we need to prove the main results in Chapters 3 and 4.

Chapter 3 is devoted to the construction of a parametrix for the wave equation (1.6) under Assumption 1.1.1. Here, we adapt an approach, originally developed by Smith [47], to our specific setting.

In Chapter 4, we use the parametrix constructed in Chapter 3 to prove global-in-time Strichartz estimates. A key ingredient is the analysis of the Fourier transform of a surface-carried measure. To avoid hindering the flow of the text, we have moved the somewhat lengthy proof of the properties of this surface to the appendix.

2. Preliminaries

In this chapter, we provide the operator-theoretic background for Chapter 3 and Chapter 4. In Section 2.1, we briefly recall the holomorphic functional calculus for sectorial operators; we then give a more detailed account of the Phillips functional calculus for generators of d-parameter C_0-groups, as it is one of the main tools used in this thesis and not frequently discussed in the literature. In Section 2.2, we study one-dimensional half-wave equations in $\mathrm{L}^p(\mathbb{R})$ for $p \in (1, \infty)$. These are essentially the building blocks for the type of wave equations we deal with in this thesis.

2.1. Functional Calculus

Functional calculus is concerned with inserting an operator L into (suitable) functions f in order to make sense of expressions like e.g. e^{-tL} or $\sqrt{L}$. One well-known instance of this is the Borel functional calculus for self-adjoint operators in a Hilbert space. In the more general setting of Banach spaces, one can use tools from complex analysis (Cauchy's integral theorem and formula) and from harmonic analysis (namely the Fourier transform) in order to meaningfully define functional calculi. For an extensive exposition of the theory, we refer the reader to [27], [35], and [31, Chapter X].

2.1.1. A Short Motivation

If L is a self-adjoint operator in some Hilbert space X, then by the spectral theorem (see e.g. [27, Theorem D.5.1, Theorem D.6.1]), there is an algebra homomorphism from the space of bounded Borel-measurable functions into the space of bounded operators,

$$\Psi\colon B_b(\sigma(L)) \to \mathcal{L}(X), \quad f \mapsto f(L) := \Psi(f).$$

The map Ψ is uniquely determined by its algebraic and continuity properties and called the *Borel functional calculus for L*. The significance of such a functional calculus lies in the fact that it allows to treat the possibly complicated operator L much *as if it was just* a number. To illustrate this, suppose that $L \neq 0$ is nonnegative (i.e., $\sigma(L) \subseteq [0, \infty)$) and consider the abstract (wave-type) evolution equation

$$D_t^2 u(t) = Lu(t) \quad (t \in \mathbb{R}), \quad u(0) = g \in X, \quad D_t u(0) = h \in X. \tag{2.1}$$

Basic ODE theory tells us that in the scalar case (i.e., $X = \mathbb{C}$ and $L \in [0, \infty)$) the solution of (2.1) is given by

$$u(t) = \cos(t\sqrt{L})g + it\operatorname{sinc}(t\sqrt{L})h \quad (t \in \mathbb{R}), \tag{2.2}$$

where $\operatorname{sinc}(z) := \frac{\sin(z)}{z}$ for $z \neq 0$ and $\operatorname{sinc}(0) = 1$. But in fact, applying the functional calculus Ψ, we can make sense of the expression (2.2) in the general vector-valued case by putting

$$\operatorname{Cos}(t\sqrt{L}) := \Psi(\cos(t\sqrt{\mathbf{z}})) \in \mathcal{L}(X), \quad \operatorname{Sinc}(t\sqrt{L}) := \Psi(\operatorname{sinc}(t\sqrt{\mathbf{z}}) \in \mathcal{L}(X)$$

for $t \in \mathbb{R}$. For instance, it can then be shown that

$$u \colon \mathbb{R} \to X, \quad u(t) = \operatorname{Cos}(t\sqrt{L})g + it\operatorname{Sinc}(t\sqrt{L})h \tag{2.3}$$

is the unique classical solution to (2.1), provided that $g \in \mathsf{Dom}(L)$ and $h \in \mathsf{Dom}(\sqrt{L})$ (see Subsection 2.1.2 for the precise definition of the square root $\sqrt{L}$). The significance of (2.3) is that it provides us with a representation of the solution that might be useful for the purpose of proving estimates that one is interested in. In essence, this example captures much of the approach that we use in Chapter 3.

If L is not self-adjoint or X is not a Hilbert space, the spectral theorem is not available anymore. However, it is possible to construct a functional calculus for so-called *sectorial operators* L, and we will briefly recall this construction in the following subsection. The presentation will closely follow ([27, Chapters 2 and 3]).

2.1.2. Functional Calculus for Sectorial Operators

We denote by $S_\omega := \{z \in \mathbb{C} \mid z \neq 0, \arg(z) < \omega\}$ the open sector of angle $\omega \in [0, \pi)$ in the complex plane.

Definition 2.1.1 (Sectorial Operators)**.** A closed, densely defined, linear operator $L \colon \mathsf{Dom}(L) \subseteq X \to X$ in a Banach space X is called *sectorial of angle* $\omega \in [0, \pi)$ if its spectrum $\sigma(L)$ is contained in $\overline{S_\omega}$ and if for any larger angle $\vartheta \in (\omega, \pi)$ we have the estimate

$$\sup_{z \in \mathbb{C} \setminus \overline{S_\vartheta}} \|zR(z, L)\|_{\mathcal{L}(X)} < \infty. \tag{2.4}$$

We call an operator just *sectorial* if it is sectorial of some angle.

Guided by Cauchy's integral formula from complex analysis, we can define a functional calculus for a sectorial operator L on suitable algebras of bounded holomorphic functions (although holomorphy is way more restrictive compared to mere measurability as in the self-adjoint case, it turns out to be sufficient for many purposes). For $\vartheta \in (0,\pi)$, we let $H(S_\vartheta)$ denote the set of holomorphic functions $f\colon S_\vartheta \to \mathbb{C}$ and consider the subspace

$$H_0^\infty(S_\vartheta) := \Big\{ f \in H(S_\vartheta) \mid \exists \alpha > 0\colon |f(z)| \lesssim \Big(|z| \wedge \tfrac{1}{|z|}\Big)^\alpha \, (z \in S_\vartheta) \Big\}. \qquad (2.5)$$

Given a sectorial operator L of angle $\omega \in [0,\pi)$ and any fixed angle $\vartheta \in (\omega,\pi)$, we can define an *elementary functional calculus* for L on the algebra $H_0^\infty(S_\vartheta)$ by

$$\Psi\colon H_0^\infty(S_\vartheta) \to \mathcal{L}(X), \quad \psi \mapsto \psi(L) := \frac{1}{2\pi i} \int_{\partial S_\nu} \psi(z) R(z,L) \,\mathrm{d}z, \qquad (2.6)$$

where $\nu \in (\omega,\vartheta)$ and ∂S_ν is parametrized by some positively-oriented (piecewise smooth) path γ. The resolvent estimate (2.4) and the decay bounds in (2.5) ensure that the integral which defines $\psi(L)$ converges absolutely in $\mathcal{L}(X)$. Moreover, it follows from Cauchy's integral theorem for vector-valued holomorphic functions that $\psi(L)$ does not depend on the specific choice of the angle $\nu \in (\omega,\vartheta)$ or the path γ. The map Ψ is an algebra homomorphism, but its domain is rather restrictive by demanding decay both at 0 and ∞ for functions in $H_0^\infty(S_\vartheta)$. We therefore extend in a first step Ψ linearly to the algebra

$$\mathcal{E}(S_\vartheta) := H_0^\infty(S_\vartheta) \oplus \langle (1+\mathbf{z})^{-1} \rangle \oplus \langle \mathbf{1} \rangle$$

by putting

$$f(L) := \psi(L) + c(\mathrm{Id} + L)^{-1} + d$$

if $f = \psi + c(1+\mathbf{z})^{-1} + d$ for some $\psi \in H_0^\infty(S_\vartheta)$ and $c, d \in \mathbb{C}$. This gives rise to a well-defined algebra homomorphism $\mathcal{E}(S_\vartheta) \to \mathcal{L}(X)$ that extends Ψ and which we (under abuse of notation) still denote by Ψ. However, $\mathcal{E}(S_\vartheta)$ is still a rather small space, as it, for instance, does not contain the fractional powers $\mathbf{z}^\alpha\colon S_\vartheta \to \mathbb{C}$, $z \mapsto z^\alpha$ ($\mathrm{Re}(\alpha) > 0$) since they lack sufficient decay at ∞. Therefore, we extend Ψ even further by a technique called *regularization*: We introduce the subalgebra of regularizers

$$\mathcal{R}_L(S_\vartheta) := \{ e \in \mathcal{E}(S_\vartheta) \mid e(L) \text{ is injective} \}$$

and the algebra $\mathcal{M}(S_\vartheta) := \{ f\colon S_\vartheta \to \mathbb{C} \text{ is meromorphic} \}$. We then call a function $f \in \mathcal{M}(S_\vartheta)$ *regularizable* and $e \in \mathcal{R}_L(S_\vartheta)$ a *regularizer for f* if $ef \in \mathcal{E}(S_\vartheta)$. Finally, set $\mathcal{M}_L(S_\vartheta) := \{ f \in \mathcal{M}(S_\vartheta) \mid f \text{ is regularizable} \}$ and define

$$\tilde{\Psi}\colon \mathcal{M}_L(S_\vartheta) \to \mathcal{C}(X), \quad f \mapsto f(L) := e(L)^{-1}(ef)(L),$$

where e is any regularizer for f and $\mathcal{C}(X)$ is the space of closed linear operators in X. It turns out that this gives a well-defined map called the *extended functional calculus* for L (see [27, Subsection 1.2.2]).

Example 2.1.2 (Fractional Powers)**.** Let L be a sectorial operator.

(a) Let $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$ with $n > \mathrm{Re}(\alpha) > 0$. Then, the function $e := (\mathbf{1}+\mathbf{z})^{-n}$ is a regularizer for $\mathbf{z}^\alpha$ and $e(L)^{-1} = (\mathrm{Id}+L)^n$. Therefore, we may define
$$L^\alpha := (\mathbf{z}^\alpha)(L) = (I+L)^n \left(\frac{\mathbf{z}^\alpha}{(\mathbf{1}+\mathbf{z})^n} \right)(L) \in \mathcal{C}(X)$$
as the *fractional power of L of order α*. In the case $\alpha = 1/2$, the notation $\sqrt{L} := L^{1/2}$ is also customary and $\sqrt{L}$ is referred to as the *square root of L*. We also set $L^0 := \mathrm{Id}$.

(b) If L is injective, we can define L^α even for all $\alpha \in \mathbb{C}$. Indeed, in this case, we may choose the regularizer $e_n := \mathbf{z}^n(1+\mathbf{z})^{-2n}$, where $n \in \mathbb{N}$ is such that $n > |\mathrm{Re}(\alpha)|$.

The fractional powers of a sectorial operator satisfy many properties (law of exponents, etc.) that are known to hold for complex numbers. We record some of them for later reference in the following proposition.

Proposition 2.1.3 (Properties of Fractional Powers)**.** *Let L be a sectorial operator in X and $\alpha, \beta \in \mathbb{C}$ with $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$.*

(a) We have $L^{\alpha+\beta} = L^\alpha L^\beta$. In particular, $\mathsf{Dom}(L^\alpha) \subseteq \mathsf{Dom}(L^\beta)$, provided that $0 < \mathrm{Re}(\beta) < \mathrm{Re}(\alpha)$.

(b) One has $\mathsf{Dom}((\varepsilon\mathrm{Id} + L)^\alpha) = \mathsf{Dom}(L^\alpha)$ for all $\varepsilon > 0$.

(c) If $T \in \mathcal{L}(X)$ commutes with L, then it also commutes with L^α.

(d) $\mathsf{Dom}(L^\alpha)$ is a core for $\mathsf{Dom}(L^\beta)$ if $0 < \mathrm{Re}(\beta) < \mathrm{Re}(\alpha)$.

(e) Let $\gamma, \delta \in \mathbb{C}$. If L is injective, then

(1) $(L^\gamma)^{-1} = L^{-\gamma} = (L^{-1})^\gamma$,

(2) $L^\gamma L^\delta \subseteq L^{\gamma+\delta}, \quad \mathsf{Dom}(L^\delta) \cap \mathsf{Dom}(L^{\gamma+\delta}) = \mathsf{Dom}(L^\gamma L^\delta)$.

In particular, $(\mathrm{Id}+L)^{\gamma+\delta} = (\mathrm{Id}+L)^\gamma(\mathrm{Id}+L)^\delta$.

Proof. See for instance [27, Sections 3.1, 3.2]. □

If the estimate
$$\|\psi(L)\|_{\mathcal{L}(X)} \lesssim_\vartheta \|\psi\|_\infty \quad \text{for all } \psi \in H_0^\infty(S_\vartheta), \tag{2.7}$$
holds, then L is said to have a *bounded $H^\infty(S_\vartheta)$-calculus* on X. Assuming additionally L to be injective, one can extend the holomorphic functional calculus (2.6) to a bounded homomorphism of Banach algebras by defining
$$H^\infty(S_\vartheta) \to \mathcal{L}(X), \quad \psi(L)x := \lim_{n\to\infty} \psi_n(L)x \quad (\psi \in H^\infty(S_\vartheta), x \in X), \tag{2.8}$$
where $(\psi_n)_{n\in\mathbb{N}} \subseteq H_0^\infty(S_\vartheta)$ is any sequence which is bounded in $H^\infty(S_\vartheta)$ and which converges locally uniformly to ψ. In this case, the estimate (2.7) holds for all $\psi \in H^\infty(S_\vartheta)$ (see e.g. [31, Section 10.2], [1, Section (D), Theorem D]).

2.1.2.1. Extrapolation Scales

Let L be a sectorial operator in X. One useful application of fractional powers is the definition of a scale of Sobolev spaces adapted to the operator L. We recall the definition given in [35].

Definition 2.1.4 ([35, Definition 15.21])**.** Let L be a sectorial operator in X. Then, we define a scale of spaces $(X_L^\alpha)_{\alpha\in\mathbb{R}}$ by letting

$$(X_L^\alpha, \|\cdot\|_{X_L^\alpha}) := \begin{cases} (\mathsf{Dom}(L^\alpha), \|(\mathrm{Id}+L)^\alpha \cdot\|_X), & \alpha \geq 0, \\ (X, \|(\mathrm{Id}+L)^\alpha \cdot\|_X)^\sim, & \alpha < 0. \end{cases}$$

We frequently write $\|\cdot\|_\alpha$ instead of $\|\cdot\|_{X_L^\alpha}$ if the operator L is clear from the context. Note that, if $\beta \leq \alpha$, then $X_L^\alpha \hookrightarrow X_L^\beta$ by Proposition 2.1.3. The definition is motivated by the case, where $L = -\Delta_x$ is minus the Laplacian in $\mathrm{L}^p(\mathbb{R}^d)$, $p \in (1,\infty)$, with domain $\mathsf{Dom}(\Delta_x) = \mathrm{H}^{2,p}(\mathbb{R}^d)$. In this case, X_L^α coincides with the Besselpotential space $\mathrm{H}^{2\alpha,p}(\mathbb{R}^d) := \{u \in \mathcal{S}'(\mathbb{R}^d) | (\mathrm{Id} - \Delta_x)^{2\alpha} u \in \mathrm{L}^p(\mathbb{R}^d)\}$ of order $2\alpha \in \mathbb{R}$.

Proposition 2.1.5 ([35, Proposition 15.23])**.** *Let L be a sectorial operator in X and $\alpha > 0$.*

(a) The operator $(\mathrm{Id}+L)^\alpha \colon \mathsf{Dom}(L^\alpha) \to X$ is an isometry $X^\alpha \to X$ with inverse $(\mathrm{Id}+L)^{-\alpha}$, and the operator $(\mathrm{Id}+L)^{-\alpha} \colon X \to X$ extends to an isometry $J_{-\alpha} \colon X_L^{-\alpha} \to X$ whose inverse $(J_{-\alpha})^{-1}$ is an extension of the operator $(\mathrm{Id}+L)^\alpha \colon \mathsf{Dom}(L^\alpha) \to X$.

(b) If X is reflexive, there is a natural isomorphism $X_L^{-\alpha} = \big((X')_{L'}^\alpha\big)'$.

We also put $J_\alpha := (\mathrm{Id}+L)^\alpha$ for $\alpha \geq 0$. To simplify some arguments, we prefer the family of operators $(J_\alpha)_{\alpha\in\mathbb{R}}$ to be defined on one common space. One way to do this is to mimick the definition of distributions. To this end, we suppose that L is sectorial of angle $\omega < \frac{\pi}{2}$ in a reflexive Banach space X. Since X is reflexive, L' is also sectorial of angle ω (the reflexivity is needed to ensure that L' is densely defined). Since $\omega < \frac{\pi}{2}$, the test function space

$$\mathcal{D}_{L'}^\infty := \bigcap_{\alpha\geq 0} \mathsf{Dom}((L')^\alpha).$$

is dense in X' (as L' is the generator of an analytic semigroup). We define a topology on $\mathcal{D}_{L'}^\infty$ by saying that a sequence $(x_n')_n$ in $\mathcal{D}_{L'}^\infty$ converges to $x' \in \mathcal{D}_{L'}^\infty$ if $(x_n')_n$ converges to x' in $\mathsf{Dom}((L')^\alpha)$ for all $\alpha \geq 0$. This makes $\mathcal{D}_{L'}^\infty$ a Frechét space. By Proposition 2.1.3 (b) and (e), $(\mathrm{Id}+L')^\alpha \colon \mathcal{D}_{L'}^\infty \to \mathcal{D}_{L'}^\infty$ defines an isomorphism for each $\alpha \in \mathbb{R}$ and

$$(\mathrm{Id}+L')^{\alpha+\beta} = (\mathrm{Id}+L')^\alpha (\mathrm{Id}+L')^\beta \quad (\alpha,\beta \in \mathbb{R}). \tag{2.9}$$

We define *the space of distributions*

$$\mathcal{D}'_L \coloneqq (\mathcal{D}^\infty_{L'})' = \{\varphi\colon \mathcal{D}^\infty_{L'} \to \mathbb{C} \text{ is linear and continuous}\}.$$

We equip $\mathcal{D}'_L$ with the weak*-topology, i.e., $\varphi_n \to \varphi$ in $\mathcal{D}'_L$ if $\varphi_n(x') \to \varphi(x')$ for all $x' \in \mathcal{D}^\infty_{L'}$. It is readily checked that if $\varphi\colon \mathcal{D}^\infty_{L'} \to \mathbb{C}$ is a linear functional, then

$$\varphi \in \mathcal{D}'_L \iff \exists C, \alpha \geq 0\colon |\varphi(x')| \leq C\|x'\|_\alpha, \quad x' \in \mathcal{D}^\infty_{L'}. \tag{2.10}$$

By duality, $(\mathrm{Id}+L')^\alpha$ induces an isomorphism $(\mathrm{Id}+\mathscr{L})^\alpha \coloneqq ((\mathrm{Id}+L')^\alpha)'\colon \mathcal{D}'_L \to \mathcal{D}'_L$ for every $\alpha \in \mathbb{R}$.

Proposition 2.1.6. *Suppose that X is reflexive and that L is a sectorial operator of angle $\omega < \frac{\pi}{2}$ in X. Let further $\alpha, \beta \in \mathbb{R}$. Then:*

(a) $(\mathrm{Id} + \mathscr{L})^{\alpha+\beta} = (\mathrm{Id} + \mathscr{L})^\alpha(\mathrm{Id} + \mathscr{L})^\beta$.

(b) $X \hookrightarrow \mathcal{D}'_L$ *and* $(\mathrm{Id} + L)^\alpha x = (\mathrm{Id} + \mathscr{L})^\alpha x$ *for* $x \in \mathsf{Dom}(L^\alpha)$, $\alpha \geq 0$.

(c) $X^\alpha_L = \{u \in \mathcal{D}'_L \mid (\mathrm{Id} + \mathscr{L})^\alpha u \in X\}$ *and* $(\mathrm{Id} + \mathscr{L})^\alpha|_{X^\alpha_L} = J_\alpha$.

(d) $(\mathrm{Id} + \mathscr{L})^\beta\colon X^\alpha_L \to X^{\alpha-\beta}_L$ *is an isomorphism.*

2.1.3. Phillips Functional Calculus

In this section, we introduce the Phillips functional calculus for generators of d-parameter C_0-groups. The theory extends without any pitfalls to the already developed theory in the case of one-parameter C_0-(semi)groups, which, for instance, can be found in [27, Section 3.3].

2.1.3.1. Definition and Basic Properties

Let X be a Banach space and $D \coloneqq (D_1, \dots, D_d)$ be a d-tuple of linear operators in X such that

(a) for each $j \in \{1, \dots, d\}$, the operator iD_j is the generator of a bounded C_0-group $(\mathrm{e}^{iyD_j})_{y\in\mathbb{R}}$ on X, and

(b) for each $y = (y_1, \dots, y_d) \in \mathbb{R}^d$, the operators $\mathrm{e}^{iy_1D_1}, \dots, \mathrm{e}^{iy_dD_d} \in \mathcal{L}(X)$ are commuting.

We then obtain a bounded d-parameter C_0-group $(\mathrm{e}^{iy\cdot D})_{y\in\mathbb{R}^d}$ defined by

$$\mathrm{e}^{iy\cdot D}\colon \mathbb{R}^d \to \mathcal{L}(X), \quad \mathrm{e}^{iy\cdot D} \coloneqq \mathrm{e}^{iy_1D_1} \cdots \mathrm{e}^{iy_dD_d} \tag{2.11}$$

satisfying

(i) $\mathrm{e}^{i0\cdot D} = \mathrm{Id}$,

(ii) $\mathrm{e}^{i(y+y')\cdot D} = \mathrm{e}^{iy\cdot D}\mathrm{e}^{iy'\cdot D}$ for all $y, y' \in \mathbb{R}^d$,

(iii) $\mathrm{e}^{iy\cdot D}x \to x \quad (y \to 0)$ for all $x \in X$.

We call *iD the generator* of the bounded d-parameter C_0-group $(\mathrm{e}^{iy\cdot D})_{y\in\mathbb{R}^d}$ and define

$$M \coloneqq \sup_{y\in\mathbb{R}^d} \|\mathrm{e}^{iy\cdot D}\|_{\mathcal{L}(X)} < \infty. \tag{2.12}$$

With a d-parameter bounded C_0-group at hand, one may define the *Phillips functional calculus* for D. To this end, we denote by $(\mathbf{M}(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}(\mathbb{R}^d)}, +, *)$ the Banach algebra of complex Borel measures with finite variation norm, and push this Banach algebra forward via the Fourier transform.

Definition 2.1.7 (The spaces $\mathcal{F}\mathbf{M}$ and $\mathcal{F}\mathrm{L}^1$)**.** We define $\mathcal{F}\mathbf{M} \coloneqq \mathcal{F}(\mathbf{M}(\mathbb{R}^d)) \subseteq BUC(\mathbb{R}^d)$ and equip this space with the norm $\|\varphi\|_{\mathcal{F}\mathbf{M}} \coloneqq \|\mathcal{F}^{-1}\varphi\|_{\mathbf{M}(\mathbb{R}^d)}$. We also define the subalgebra $\mathcal{F}\mathrm{L}^1 \coloneqq \mathcal{F}(\mathrm{L}^1(\mathbb{R}^d)) \subseteq \mathcal{F}\mathbf{M}$, where we identify $\mathrm{L}^1(\mathbb{R}^d)$ with a subspace of $\mathbf{M}(\mathbb{R}^d)$ via the isometric embedding $\iota\colon \mathrm{L}^1(\mathbb{R}^d) \to \mathbf{M}(\mathbb{R}^d)$, $f \mapsto f\,\mathrm{d}x$.

The spaces $(\mathcal{F}\mathbf{M}, \|\cdot\|_{\mathcal{F}\mathbf{M}}, +, \cdot)$ and its closed subspace $(\mathcal{F}L^1, \|\cdot\|_{\mathcal{F}\mathbf{M}}, +, \cdot)$ are Banach algebras. We will frequently use that $\|\cdot\|_{\mathcal{F}\mathbf{M}}$ is scaling-invariant on $\mathcal{F}\mathrm{L}^1$, i.e., $\|\varphi_t\|_{\mathcal{F}\mathbf{M}} = \|\varphi\|_{\mathcal{F}\mathbf{M}}$ for all $t > 0$ and $\varphi \in \mathcal{F}\mathrm{L}^1$, where $\varphi_t(\xi) \coloneqq \varphi(t\xi)$ for $\xi \in \mathbb{R}^d$ (this follows from $\mathcal{F}^{-1}\varphi_t = \frac{1}{t^d}(\mathcal{F}^{-1}\varphi)(\frac{\cdot}{t})$ for $\varphi \in \mathcal{F}L^1$). Also note that we clearly have $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathcal{F}\mathrm{L}^1$.

Definition 2.1.8 (Phillips Functional Calculus)**.** For $\varphi \in \mathcal{F}\mathbf{M}$, we define $\varphi(D) \in \mathcal{L}(X)$ by

$$\varphi(D)\colon X \to X, \quad \varphi(D)x \coloneqq \int_{\mathbb{R}^d} \mathrm{e}^{-iy\cdot D}x\,\mathrm{d}\mu_\varphi(y), \tag{2.13}$$

where $\mu_\varphi \coloneqq \mathcal{F}^{-1}\varphi \in \mathbf{M}(\mathbb{R}^d)$.

It is immediate from the definition that we have

$$\|\varphi(D)\|_{\mathcal{L}(X)} \leq M\|\varphi\|_{\mathcal{F}\mathbf{M}} \quad \text{for all } \varphi \in \mathcal{F}\mathbf{M}.$$

We will mostly work with $\varphi \in \mathcal{F}\mathrm{L}^1$. In this case, $\mathcal{F}^{-1}\varphi \in \mathrm{L}^1(\mathbb{R})$ and thus (2.13) is

$$\varphi(D)x = \int_{\mathbb{R}^d} (\mathcal{F}^{-1}\varphi)(y)\mathrm{e}^{-iy\cdot D}x\,\mathrm{d}y \quad (x \in X).$$

Example 2.1.9. Let $p \in [1,\infty)$ and $X = \mathrm{L}^p(\mathbb{R}^d)$.

(a) Consider the unbounded operator $D_x := \frac{1}{i}(\partial_{x_1}, \dots, \partial_{x_d})\colon \mathsf{Dom}(D_x) \subseteq X \to X^d$, $f \mapsto D_x f$ with $\mathsf{Dom}(D_x) := W^{1,p}(\mathbb{R}^d)$. Then, iD_x generates the d-parameter translation group $(\mathrm{e}^{iy\cdot D_x})_{y\in\mathbb{R}^d}$ defined by

$$(\mathrm{e}^{iy\cdot D_x} f)(z) := f(z+y) \quad \text{for all } f \in X, y \in \mathbb{R}^d \text{ and a.e. } z \in \mathbb{R}^d$$

and

$$\varphi(D_x)f = (\mathcal{F}^{-1}\varphi) * f = \mathcal{F}^{-1}(\varphi\widehat{f}) \quad \text{for } \varphi \in \mathcal{F}\mathrm{L}^1, f \in \mathcal{S}(\mathbb{R}^d).$$

Thus, in this case, the operator $\varphi(D_x)$ coincides with the Fourier multiplier operator associated with the symbol φ (see e.g. [24, Section 2.5]).

(b) Similarly, if $a = (a_1, \dots, a_d)\colon \mathbb{R}^d \to \mathbb{R}^d$ is measurable, then the multiplication operator $M_a = (M_{a_1}, \dots, M_{a_d})\colon \mathsf{Dom}(M_a) \subseteq X \to X^d$, $f \mapsto fa$, with domain $\mathsf{Dom}(M_a) = \{f \in X \mid fa_j \in X \text{ for all } j\}$, generates the bounded d-parameter C_0-group defined by $\mathrm{e}^{iy\cdot M_a} f := \mathrm{e}^{iy\cdot a} f$. Thus, for $\varphi \in \mathcal{F}\mathrm{L}^1$ and $f \in X$, we have by Fourier inversion

$$\varphi(M_a)f = \int_{\mathbb{R}^d} (\mathcal{F}^{-1}\varphi)(y)\mathrm{e}^{-iy\cdot a} f \,\mathrm{d}y = (\varphi\circ a)f = M_{\varphi\circ a} f.$$

Hence, $\varphi(M_a)$ is just the multiplication operator $M_{\varphi\circ a}$.

(c) The most important example in this thesis lies somewhat in between (a) and (b), where a bounded d-parameter C_0-group is constructed from d commuting one-dimensional half-wave groups: Suppose that $p \in (1,\infty)$. We will then show in Section 2.2 that $i\sqrt{\mathbf{L}} = i(\sqrt{L_1}, \dots, \sqrt{L_d})$ generates a bounded C_0-group on X, where $L_j := \sqrt{D_j a_j(x_j) D_j}$ or $L_j := \sqrt{a_j(x_j) D_j^2}$ for functions $a_j\colon \mathbb{R} \to \mathbb{R}$ with suitable properties (see Corollary 2.2.10). The associated Phillips functional calculus for $\sqrt{\mathbf{L}}$ will then be extensively used in Chapter 3 and Chapter 4.

We collect some basic properties of the Phillips functional calculus for D in the following proposition.

Proposition 2.1.10 (Properties of the Phillips Functional Calculus)**.** *The following statements hold true.*

(a) The map $\Phi\colon \mathcal{F}\mathbf{M} \to \mathcal{L}(X)$, $\varphi \mapsto \varphi(D)$, defines a unit-preserving, bounded homomorphism of Banach algebras:

$$(\alpha\varphi + \psi)(D) = \alpha\varphi(D) + \psi(D), \quad \mathbb{1}(D) = \mathrm{Id},$$
$$(\varphi\psi)(D) = \varphi(D)\psi(D), \quad \|\varphi(D)\|_{\mathcal{L}(X)} \le M\|\varphi\|_{\mathcal{F}\mathbf{M}}$$

for all $\varphi, \psi \in \mathcal{F}\mathbf{M}$ and $\alpha \in \mathbb{C}$. Moreover, $(\mathrm{e}^{ite_j\cdot})(D) = \mathrm{e}^{itD_j}$ for $t \in \mathbb{R}$ and $j \in \{1, \dots, d\}$.

(b) *If X is reflexive and $\varphi \in \mathcal{F}\mathbf{M}$, then $(\varphi(D))' = \varphi(D')$. Similarly, $(\varphi(D))^* = \overline{\varphi}(D^*)$, if X is a Hilbert space.*

(c) *Let $\varphi \in \mathcal{F}\mathrm{L}^1$, $x \in X$ and $k \in \mathbb{N}$. Suppose that $\varphi_\alpha \in \mathcal{F}\mathrm{L}^1$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$, where $\varphi_\alpha(\xi) := \xi^\alpha \varphi(\xi)$, $\xi \in \mathbb{R}^d$. Then for any $|\alpha| \le k$, we have $\varphi(D)x \in \mathsf{Dom}(D^\alpha)$ and*

$$D^\alpha \varphi(D)x = \varphi_\alpha(D)x. \tag{2.14}$$

Moreover, $D^\alpha \varphi(D)x = \varphi(D)D^\alpha x$ if $x \in \mathsf{Dom}(D^\alpha)$.

(d) *Let $\varphi \in \mathcal{F}\mathrm{L}^1$. If $p \in (1,\infty)$ and $X = \mathrm{L}^p(\mathbb{R}^d)$, we have*

$$\|\varphi(D)\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R}^d))} \lesssim \|\varphi(D_x)\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R}^d))},$$

where $\varphi(D_x) := \mathcal{F}^{-1}M_\varphi\mathcal{F}$ denotes the Fourier multiplier operator associated to the symbol φ. In particular, we have

$$\begin{aligned}
&\|\varphi(D)\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{R}^d))} \lesssim_M \|\varphi\|_{\mathrm{L}^\infty(\mathbb{R}^d)},\\
&\|\varphi(D)\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R}^d))} \lesssim_{M,p,d} \sup_{|\alpha|\le[d/2]+1} \Big\| |\cdot|^{|\alpha|}\partial^\alpha\varphi(\cdot)\Big\|_{\mathrm{L}^\infty(\mathbb{R}^d)}, \quad p \in (1,\infty).
\end{aligned}$$

Proof. Let $x \in X$.

(a) The linearity and boundedness of Φ is clear. Let $t \in \mathbb{R}$ and $j \in \{1,\dots,d\}$ and define $\mathrm{e}_{t,j}(\xi) := \mathrm{e}^{it\xi_j}$ for $\xi \in \mathbb{R}^d$. Observe that $\mathrm{e}_{t,j} \in \mathcal{F}\mathbf{M}$ since $\mathcal{F}^{-1}(\mathrm{e}_{t,j}) = \delta_{-te_j} \in \mathbf{M}(\mathbb{R}^d)$ and thus

$$\mathrm{e}_{t,j}(D)x = \int_{\mathbb{R}^d} \mathrm{e}^{-iy\cdot D}x \,\mathrm{d}\delta_{-te_j}(y) = \mathrm{e}^{ite_j\cdot D}x = \mathrm{e}^{itD_j}x.$$

In particular, if $t = 0$, we get $\mathbb{1}(D)x = \mathrm{e}_{0,j}(D) = \mathrm{e}^{0D_j}x = x$. It remains to show the multiplicativity of Φ. To this end, let $\varphi, \psi \in \mathcal{F}\mathbf{M}$. By the basic properties of the Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$, we have $\mathcal{F}^{-1}(\varphi\psi) = (\mathcal{F}^{-1}\varphi) * (\mathcal{F}^{-1}\psi) \in \mathbf{M}(\mathbb{R}^d)$. Thus, using Fubini's theorem and writing $\mu_\varphi := \mathcal{F}^{-1}\varphi$, $\mu_\psi := \mathcal{F}^{-1}\psi$ and $\mu_{\varphi\psi} := \mathcal{F}^{-1}(\varphi\psi)$, we compute

$$\begin{aligned}
\varphi(D)\psi(D)x &= \int_{\mathbb{R}^d} \mathrm{e}^{-iu\cdot D}\left(\int_{\mathbb{R}^d} \mathrm{e}^{-iv\cdot D}x \,\mathrm{d}\mu_\psi(v)\right)\mathrm{d}\mu_\varphi(u)\\
&= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{e}^{-i(u+v)\cdot D}x \,\mathrm{d}(\mu_\varphi \otimes \mu_\psi)(u,v)\\
&= \int_{\mathbb{R}^d} \mathrm{e}^{-iy\cdot D}x \,\mathrm{d}(\mu_\varphi * \mu_\psi)(y)\\
&= \int_{\mathbb{R}^d} \mathrm{e}^{-iy\cdot D}x \,\mathrm{d}\mu_{\varphi\psi}(y) = (\varphi\psi)(D)x.
\end{aligned}$$

(b) Let X be reflexive and $\varphi \in \mathcal{F}\mathbf{M}$. Then $(\mathrm{e}^{-iy\cdot D})' = \mathrm{e}^{-iy\cdot D'}$ (see e.g. [16, Subsection I.5.14]) and therefore

$$\varphi(D)'x = \int_{\mathbb{R}^d} \left(\mathrm{e}^{-iy\cdot D}\right)' x \,\mathrm{d}\mu_\varphi(y) = \int_{\mathbb{R}^d} \mathrm{e}^{-iy\cdot D'} x \,\mathrm{d}\mu_\varphi(y) = \varphi(D')x.$$

Now, if X is a Hilbert space, we similarly have $(\mathrm{e}^{-iy\cdot D})^* = \mathrm{e}^{iy\cdot D^*}$ and $\overline{\mu_\varphi} = \mu_{\overline{\varphi^-}}$ (where $\varphi^-(y) = \varphi(-y)$). This implies

$$\varphi(D)^*x = \int_{\mathbb{R}^d} \mathrm{e}^{iy\cdot D^*} x \,\mathrm{d}\overline{\mu_\varphi}(y) = \int_{\mathbb{R}^d} \mathrm{e}^{iy\cdot D^*} x \,\mathrm{d}\mu_{\overline{\varphi}}(-y) = \overline{\varphi}(D^*)x.$$

(c) Let $k \in \mathbb{N}, \varphi \in \mathcal{F}\mathrm{L}^1$ and $\varphi_\alpha \in \mathcal{F}\mathrm{L}^1$ for all $|\alpha| \leq k$. This implies $\mathcal{F}^{-1}\varphi \in W^{k,1}(\mathbb{R}^d)$ and by a standard density argument, we may in fact assume that $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Suppose first $k = 1$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = 1$. Then, $\alpha = e_j$ for some $j \in \{1, \dots, d\}$ and we have for any $h \neq 0$

$$\begin{aligned}\frac{\mathrm{e}^{ihD_j} - \mathrm{Id}}{ih}\varphi(D)x &= \int_{\mathbb{R}^d} (\mathcal{F}^{-1}\varphi)(y) \frac{\mathrm{e}^{-i(y-he_j)\cdot D} - \mathrm{e}^{-iy\cdot D}}{ih} x \,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \frac{(\mathcal{F}^{-1}\varphi)(y+he_j) - (\mathcal{F}^{-1}\varphi)(y)}{ih} \mathrm{e}^{-iy\cdot D} x \,\mathrm{d}y\end{aligned} \tag{2.15}$$

By dominated convergence, the last integral converges for $h \to 0$ to

$$\int_{\mathbb{R}^d} \tfrac{1}{i}\partial_j(\mathcal{F}^{-1}\varphi)(y)\mathrm{e}^{-iy\cdot D} f \,\mathrm{d}y = \int_{\mathbb{R}^d} (\mathcal{F}^{-1}\varphi_{e_j})(y)\mathrm{e}^{-iy\cdot D} f \,\mathrm{d}y = \varphi_{e_j}(D)f,$$

where $\varphi_{e_j}(\xi) = \xi_j \varphi(\xi)$, $\xi \in \mathbb{R}^d$. This shows that $\varphi(D)x$ belongs to $\mathsf{Dom}(D_j)$ with

$$D_j\varphi(D)x = \varphi_{e_j}(D)x.$$

If x belongs to $\mathsf{Dom}(D_j)$, then it follows from the first line of (2.15) and dominated convergence that

$$\frac{\mathrm{e}^{ihD_j} - \mathrm{Id}}{ih}\varphi(D)x \to \int_{\mathbb{R}^d} (\mathcal{F}^{-1}\varphi)(y)\mathrm{e}^{-iy\cdot D} D_j x \,\mathrm{d}y = \varphi(D)D_j x \quad (h \to 0)$$

and thus by uniqueness of the limit, $D_j\varphi(D)f = \varphi_{e_j}(D)f = \varphi(D)D_j f$.

This proves the assertion in the case $k = 1$. The general case follows from a straightforward induction on the order of α.

(d) The assertion immediately follows from Coifman-Weiss's transference principle (see [12, Theorem 2.4]) and Mihlin's multiplier theorem (see e.g. [24, Theorem 6.2.7]).

□

Lemma 2.1.11 (Interchange of Phillips Calculus and Bochner Integration)**.** *Let $I \subseteq \mathbb{R}$ be an interval and $g \in L^1(I; \mathcal{F}\mathrm{L}^1)$. Then, for $\varphi := \int_I g(t)\,\mathrm{d}t \in \mathcal{F}\mathrm{L}^1$ and $x \in X$, we have*

$$\left(\int_I g(t)\,\mathrm{d}t\right)(D)x = \int_I g_t(D)x\,\mathrm{d}t.$$

Proof. This is essentially a consequence of Fubini's Theorem. Let $g \in L^1(I; \mathcal{F}\mathrm{L}^1)$ and $\varphi := \int_I g(t)\,\mathrm{d}t \in \mathcal{F}\mathrm{L}^1$. Since $\mathcal{F}^{-1} \colon \mathcal{F}\mathrm{L}^1 \to L^1(\mathbb{R}^d)$ is bounded (in fact even an isometry), we have $h := \mathcal{F}^{-1} \circ g \in L^1(I; L^1(\mathbb{R}^d))$ and $\mathcal{F}^{-1}\varphi = \int_I \mathcal{F}^{-1}g(t)\,\mathrm{d}t \in L^1(\mathbb{R}^d)$. By [30, Proposition 1.2.24 and 1.2.25], there exists $f \in L^1(I \times \mathbb{R}^d)$ with $\|f\|_{L^1(I\times\mathbb{R}^d)} = \|g\|_{L^1(I;\mathcal{F}\mathrm{L}^1)} < \infty$ satisfying $(\mathcal{F}^{-1}\varphi)(y) = \int_I f(t,y)\,\mathrm{d}t$ for a.e. $y \in \mathbb{R}^d$ and $f(t,\cdot) = \mathcal{F}^{-1}g(t)$ for a.e. $t \in I$. Let $x \in X$. By the boundedness of $(\mathrm{e}^{iy\cdot D})_{y\in\mathbb{R}^d}$, we clearly have $(t,y) \mapsto f(t,y)\mathrm{e}^{-iy\cdot D}x \in L^1(I \times \mathbb{R}^d; X)$, so Fubini's theorem implies

$$\begin{aligned}\varphi(D)x &= \int_{\mathbb{R}^d} (\mathcal{F}^{-1}\varphi)(y)\mathrm{e}^{-iy\cdot D}x\,\mathrm{d}y = \int_{\mathbb{R}^d}\int_I f(t,y)\mathrm{e}^{-iy\cdot D}x\,\mathrm{d}t\,\mathrm{d}y \\ &= \int_I\int_{\mathbb{R}^d} f(t,y)\mathrm{e}^{-iy\cdot D}x\,\mathrm{d}y\,\mathrm{d}t = \int_I\int_{\mathbb{R}^d} \mathcal{F}^{-1}g(t)\mathrm{e}^{-iy\cdot D}x\,\mathrm{d}y\,\mathrm{d}t = \int_I g_t(D)x\,\mathrm{d}t,\end{aligned}$$

as desired. □

With the Phillips functional calculus, we are able to define approximations of the identity adapted to D.

Lemma 2.1.12 (Approximation of the Identity)**.** *Let $\varphi \in \mathcal{F}\mathrm{L}^1$ with $\varphi(0) = 1$. For $t > 0$, set $\varphi_t(\xi) := \varphi(t\xi)$, $\xi \in \mathbb{R}^d$. Then*

$$\varphi(tD)x := \varphi_t(D)x \to x \quad (t \to 0) \quad \text{for all } x \in X.$$

Proof. Let $\varphi \in \mathcal{F}\mathrm{L}^1$ with $\varphi(0) = 1$ and set $\rho := \mathcal{F}^{-1}\varphi \in \mathrm{L}^1(\mathbb{R}^d)$. Then, for $t > 0$ we have $\mathcal{F}^{-1}\varphi_t = \rho^t$, where $\rho^t(\cdot) := \frac{1}{t^d}\rho(\frac{\cdot}{t})$. Note also that the assumption $\varphi(0) = 1$ implies $\int_{\mathbb{R}^d} \rho^t(y)\,\mathrm{d}y = 1$. Now let $x \in X$. Then we have for all $\delta > 0$

$$\begin{aligned}&\|\varphi_t(D)x - x\| \\ &= \left\|\int_{\mathbb{R}^d} \rho^t(y)(\mathrm{e}^{-iy\cdot D}x - x)\,\mathrm{d}y\right\| \\ &\le \int_{B_\delta(0)} \left|\rho^t(y)\right| \left\|\mathrm{e}^{-iy\cdot D}x - x\right\|\,\mathrm{d}y + \int_{\mathbb{R}^d\setminus B_\delta(0)} \left|\rho^t(y)\right| (1+M)\|x\|\,\mathrm{d}y \\ &\le \|\rho\|_{\mathrm{L}^1(\mathbb{R}^d)} \cdot \sup_{|y|\le\delta} \left\|\mathrm{e}^{-iy\cdot D}x - x\right\| + (1+M)\,\|x\|\,\|\rho\|_{\mathrm{L}^1(\mathbb{R}^d\setminus B_{\frac{\delta}{t}}(0))}.\end{aligned}$$

Thus, by Lebesgue's theorem of dominated convergence,

$$\limsup_{t\to 0} \|\varphi_t(D)x - x\|_{\mathrm{L}^p(\mathbb{R}^d)} \le \|\rho\|_{\mathrm{L}^1(\mathbb{R}^d)} \cdot \sup_{|y|\le\delta} \left\|\mathrm{e}^{-iy\cdot D}x - x\right\| \quad \text{for all } \delta > 0.$$

Letting $\delta \to 0$ and using the strong continuity of $y \mapsto \mathrm{e}^{-iy\cdot D}$, we infer the claim. □

Lemma 2.1.12 shows that x can be approximated by $\varphi(tD)x$ as $t \to 0$. The rate of convergence can be linked to the 'regularity' assumptions made on x. The more regular x is, the faster the guaranteed rate of convergence, which is the content of the next lemma.
Next, we construct $L_0 \coloneqq D_1^2 + \cdots + D_d^2$ in a similar fashion as one constructs (minus) the Laplacian from the partial derivatives.

Lemma 2.1.13 (Sectoriality of L)**.** *The operator $L_0 \coloneqq D_1^2 + \cdots + D_d^2$ is closable. Moreover, $L \coloneqq \overline{L_0}$ is sectorial of some angle $\omega \leq \frac{\pi}{2}$ and*

$$(\lambda \mathrm{Id} + L)^{-1} = \Big[(\lambda + |\cdot|^2)^{-1}\Big](D) \quad \textit{for } \mathrm{Re}(\lambda) > 0. \tag{2.16}$$

Proof. For $\lambda \in \mathbb{C}_+ \coloneqq \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ we define $r_\lambda(\xi) \coloneqq (\lambda + |\xi|^2)^{-1}$, $\xi \in \mathbb{R}^d$. Then, $r_\lambda \in \mathcal{F}\mathrm{L}^1$ and in fact,

$$G_\lambda(y) \coloneqq (\mathcal{F}^{-1} r_\lambda)(y) = \frac{1}{(4\pi)^{\frac{d}{2}}} \int_0^\infty t^{-\frac{d}{2}} \mathrm{e}^{-t\lambda} \mathrm{e}^{-\frac{|y|^2}{4t}} \,\mathrm{d}t \quad (y \in \mathbb{R}^d) \tag{2.17}$$

(see e.g. [54, Subsection 5.3.1]). Thus, $R_\lambda \coloneqq r_\lambda(D) \in \mathcal{L}(X)$ is well-defined. The identity $r_\lambda - r_\mu = (\mu - \lambda) r_\lambda r_\mu$ translates to

$$R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu \quad \text{for all } \lambda, \mu \in \mathbb{C}_+ \tag{2.18}$$

by Proposition 2.1.10 (a). Therefore, $\{R_\lambda \in \mathcal{L}(X) \mid \lambda \in \mathbb{C}_+\}$ is a pseudoresolvent on X. Moreover, since $r_1(0) = 1$ and $\lambda^2 r_{\lambda^2} = r_1(\frac{\cdot}{\lambda})$ for $\lambda > 0$, Lemma 2.1.12 implies for all $x \in X$

$$\tfrac{1}{t^2} R_{\frac{1}{t^2}} x = r_1(tD)x \to x \quad (t \to 0+). \tag{2.19}$$

Observe that the operators R_λ all have the same kernel and range by (2.18). Combining this with (2.19), we find that the operators R_λ are injective and have dense range. Therefore, we may invoke [16, Proposition III.4.6] to see that the operator $L \coloneqq R_1^{-1} - \mathrm{Id}$ with domain $\mathsf{Dom}(L) = \mathsf{R}(R_1)$ is a densely defined, closed operator in X with $(\lambda \mathrm{Id} + L)^{-1} = R_\lambda$ for all $\lambda \in \mathbb{C}_+$. Moreover, we have for all $\lambda \in \mathbb{C}_+$ the estimate

$$\|(\lambda \mathrm{Id} + L)^{-1}\|_{\mathcal{L}(X)} \leq M \|r_\lambda\|_{\mathcal{F}\mathbf{M}} \leq \frac{M \|r_1\|_{\mathcal{F}\mathbf{M}}}{\mathrm{Re}(\lambda)}. \tag{2.20}$$

Here, for $\lambda \in (0, \infty)$, the second inequality in (2.20) follows from $r_\lambda = \lambda^{-1} r_1(\lambda^{-\frac{1}{2}} \cdot)$ and the scaling invariance of $\|\cdot\|_{\mathcal{F}\mathbf{M}}$ on $\mathcal{F}\mathrm{L}^1$. For general $\lambda \in \mathbb{C}_+$, we obtain from (2.17) the pointwise bound $|G_\lambda| \leq (\mathrm{Re}(\lambda))^{-1+\frac{d}{2}} |G_1(\mathrm{Re}(\lambda)^{\frac{1}{2}} \cdot)|$ which implies $\|r_\lambda\|_{\mathcal{F}\mathbf{M}} = \|G_\lambda\|_1 \leq (\mathrm{Re}(\lambda))^{-1} \|r_1\|_{\mathcal{F}\mathbf{M}}$. Now, (2.20) is enough to conclude that L is sectorial of angle $\frac{\pi}{2}$. Indeed, if $\vartheta \in (\frac{\pi}{2}, \pi)$ and $\mu \in \mathbb{C}\backslash\overline{S_\vartheta}$, then $\mathrm{Re}(-\mu) = |\cos(\vartheta)||\mu| > 0$ and thus

$$\|(\mu\mathrm{Id} - L)^{-1}\|_{\mathcal{L}(X)} = \|(-\mu\mathrm{Id} + L)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M\|r_1\|_{\mathcal{F}\mathbf{M}}}{\mathrm{Re}(-\mu)} = \frac{C_\vartheta}{|\mu|}$$

with $C_\vartheta := M\|r_1\|_{\mathcal{F}\mathbf{M}}|\cos(\vartheta)|^{-1} > 0$. It remains to show that $L = \overline{L_0}$. To this end, let further $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\varphi(0) = 1$ and set $\varphi_t(\xi) := \varphi(t\xi)$ for $t > 0$. For a vector $x \in X$ and $t > 0$, we write $x_t := \varphi(tD)x$. Then, the following assertions hold true for all $x \in X$ and $t > 0$:

(i) $x_t \in \mathsf{Dom}(L_0) \cap \mathsf{Dom}(L)$ and $(\mathrm{Id} + L_0)x_t = (\mathrm{Id} + L)x_t$,

(ii) $[(\mathrm{Id} + L)x]_t = (\mathrm{Id} + L_0)x_t$ if $x \in \mathsf{Dom}(L)$,

(iii) $(\mathrm{Id} + L_0)x_t = [(\mathrm{Id} + L_0)x]_t$ if $x \in \mathsf{Dom}(L_0)$,

(iv) $x_t \to x$ as $t \to 0$.

Assertions (iii) and (iv) follow immediately from Proposition 2.1.10 (c) and Lemma 2.1.12, respectively. To prove (i), observe first that $x_t \in \mathsf{Dom}(L_0)$ by Proposition 2.1.10 (c). Then, we use Proposition 2.1.10 (a) to write $x_t = [r_1(1+|\cdot|^2)\varphi_t](D)x = R_1(\mathrm{Id}+L_0)x_t$, which implies (i) since $R_1 = (\mathrm{Id}+L)^{-1}$. Finally, if $x \in \mathsf{Dom}(L)$, then $[(\mathrm{Id} + L)x]_t = [(1 + |\cdot|^2)\varphi_t r_1](D)(\mathrm{Id} + L)x = [(1 + |\cdot|^2)\varphi_t](D)x = (\mathrm{Id} + L_0)x_t$, proving (ii).
From these properties, it follows easily that $\overline{L_0} = L$. Indeed, if $x \in \mathsf{Dom}(L_0)$, then $(x_t)_{t>0}$ belongs to $\mathsf{Dom}(L)$ by (i) and

$$\begin{pmatrix} x_t \\ (\mathrm{Id} + L)x_t \end{pmatrix} = \begin{pmatrix} x_t \\ (\mathrm{Id} + L_0)x_t \end{pmatrix} = \begin{pmatrix} x_t \\ [(\mathrm{Id} + L_0)x]_t \end{pmatrix} \to \begin{pmatrix} x \\ (\mathrm{Id} + L_0)x \end{pmatrix},$$

where we used (i), (iii) and then (iv). Since L is closed, it follows that $x \in \mathsf{Dom}(L)$ and $(\mathrm{Id}+L)x = (\mathrm{Id}+L_0)x$. In other words, $L_0 \subseteq L$ and thus $\overline{L_0} \subseteq L$. Conversely, if $x \in \mathsf{Dom}(L)$, then we similarly see that $(x_t, (\mathrm{Id} + L_0)x_t)_{t>0}$ converges to $(x, (\mathrm{Id}+L)x)$ which shows that $L \subseteq \overline{L_0}$. It follows that $\overline{L_0} = L$ as desired. □

Remark 2.1.14. Suppose that X is a UMD-space (see (2.52) for a definition). Then [31, Theorems 10.6.7 and 10.7.10] imply that the operators D_j^2 have a bounded H^∞-calculus of angle 0. It now follows from a version of the famous Dore-Venni Theorem [45] that L_0 is closed and thus $L = L_0$.

We have seen in the lemma above that $L = \overline{D_1^2 + \cdots + D_d^2}$ is a sectorial operator. In addition to the Phillips functional calculus for $D = (D_1, \ldots, D_d)$, there is then the holomorphic functional calculus for L available as introduced in Subsection 2.1.2, and the natural question arises how these functional calculi are related. It is natural to expect that the composition rule $\psi(L) = (\psi \circ |\cdot|^2)(D)$ holds, at least for suitable functions ψ. In the next lemma we show that under reasonable assumptions, this is indeed the case.

Proposition 2.1.15 (Some Composition Rules). *Let* $L = \overline{D_1^2 + \cdots + D_d^2}$. *For* $z \in S_{\frac{\pi}{2}} \cup \{0\}$ *set* $g_z\colon \mathbb{R}^d \to \mathbb{C}$, $g_z(\xi) := \mathrm{e}^{-z|\xi|^2}$ *and for* $t \in \mathbb{R}$ *define* $w_t\colon \mathbb{R}^d \to \mathbb{C}$, $w_t(\xi) := \mathrm{e}^{it|\xi|}$. *Let further* $\psi \in C_c^\infty(\mathbb{R}^d)$ *with* $\operatorname{supp}(\psi) \subseteq \mathbb{R}^d \setminus \{0\}$. *Then the following statements hold true.*

(a) The operator $-L$ *generates the analytic semigroup* $(\mathrm{e}^{-zL})_{z \in S_{\frac{\pi}{2}} \cup \{0\}}$ *on* X *defined by* $\mathrm{e}^{-zL} := g_z(D) \in \mathcal{L}(X)$. *In particular,* L *is sectorial of angle* 0.

(b) Let $\alpha \geq 0$. *Then for all* $x \in X$ *we have*

$$L^\alpha \psi(D)x = (|\cdot|^{2\alpha}\psi)(D)x. \tag{2.21}$$

If $x \in \mathsf{Dom}(L^\alpha)$, *then* $L^\alpha\psi(D)x = \psi(D)L^\alpha x$. *If* L *is injective, then these identities even hold for all* $\alpha \in \mathbb{R}$.

(c) Let $\alpha \in \mathbb{R}$. *Then for all* $x \in X$, *we have*

$$(\mathrm{Id} + L)^\alpha \psi(D)x = ((1 + |\cdot|^2)^\alpha \psi)(D)x.$$

If $x \in \mathsf{Dom}(L^\alpha)$, *then* $(\mathrm{Id} + L)^\alpha\psi(D)x = \psi(D)(\mathrm{Id} + L)^\alpha x$.

(d) If $i\sqrt{L}$ *generates a* C_0*-group* $(\mathrm{e}^{it\sqrt{L}})_{t\in\mathbb{R}}$ *on* X, *then*

$$\Big(w_t\psi\Big)(D)f = \mathrm{e}^{it\sqrt{L}}\psi(D)x \quad \text{for all } x \in X \text{ and } t \in \mathbb{R}. \tag{2.22}$$

(e) Let $\vartheta \in (0, \pi)$ *and suppose that* $\psi \in H_0^\infty(S_\vartheta)$ *with* $|\psi(z)| \lesssim |z|^\alpha \wedge |z|^{-1}$ *for some* $\alpha > \frac{d+1}{2}$. *Define* $h\colon \mathbb{R}^d \to \mathbb{C}$, $h(\xi) = \psi(|\xi|^2)$ *for* $\xi \neq 0$ *and* $h(0) = 0$. *If* h *belongs to* $\mathcal{F}\mathrm{L}^1$, *then*

$$\psi(L)x = h(D)x \quad (x \in X). \tag{2.23}$$

Proof. (a) For $z \in S_{\frac{\pi}{2}}$ we clearly have $g_z \in \mathcal{S}(\mathbb{R}^d) \subseteq \mathcal{F}\mathrm{L}^1$ and

$$g_z(D)x = \int_{\mathbb{R}^d} (4\pi z)^{-\frac{d}{2}} \mathrm{e}^{-\frac{|y|^2}{4z}} \mathrm{e}^{-iy\cdot D} x \,\mathrm{d}y \quad (x \in X). \tag{2.24}$$

It follows from this representation and dominated convergence that $S_{\frac{\pi}{2}} \to \mathcal{L}(X)$, $z \mapsto g_z(D)$ is strongly analytic (which is equivalent to analyticity w.r.t. the uniform operator topology by Cauchy's integral formula and the uniform boundedness principle). Since $g_0 = 1$, $g_0(D) = \mathrm{Id}$ is clear. Now, Proposition 2.1.10 (a) and a similar reasoning as in Lemma 2.1.12 reveal that $(g_z(D))_{z\in S_{\frac{\pi}{2}} \cup \{0\}}$ is a bounded analytic semigroup on X with generator A, say. By [16, Theorem II.4.6], this means that A is sectorial of angle 0. It remains to show that $-L = A$. A moment's thought reveals that for each fixed $\varphi \in \mathcal{S}(\mathbb{R}^d)$

the map $(0,\infty) \to \mathcal{S}(\mathbb{R}^d)$, $t \mapsto \varphi(t\cdot)$, is differentiable with derivative $(\delta_\varphi)'(t) = \frac{1}{t}\delta_\psi(t)$, $\psi(\xi) \coloneqq \nabla_\xi \varphi(\xi) \cdot \xi$.

$$\frac{g_{t+h} - g_t}{h} \to \partial_t g_t \quad (h \to 0) \quad \text{ in } \mathcal{S}(\mathbb{R}^d),$$

it follows from Proposition 2.1.10 (a) that $(0,\infty) \to \mathcal{L}(X)$, $t \mapsto g_t(D)$, is differentiable with derivative $\frac{\mathrm{d}}{\mathrm{d}t} g_t(D) = (\partial_t g_t)(D)$ for all $t > 0$. Since $\partial_t g_t(\xi) = -|\xi|^2 g_t(\xi)$ for all $\xi \in \mathbb{R}^d$, we conclude from Proposition 2.1.10 (c) for all $x \in X$ that

$$\frac{\mathrm{d}}{\mathrm{d}t} g_t(D)x = -\sum_{j=1}^{d} (\xi_j^2 g_t(\xi))(D)x = -\sum_{j=1}^{d} D_j^2 g_t(D)x = -L g_t(D)x.$$

This shows that $-L \subseteq A$. Since L is sectorial by Lemma 2.1.13, $\rho(L) \neq \emptyset$, thus $L = A$.

(b) Let n be an integer larger than α. Letting $\varepsilon \to 0$ in [27, Proposition 3.3.5] gives the representation formula

$$L^\alpha y = c_{n,\alpha} \int_0^\infty t^{n-\alpha} \mathrm{e}^{-tL} L^n y \frac{\mathrm{d}t}{t} \quad (y \in \mathsf{Dom}(L^n)) \tag{2.25}$$

(with $c_{n,\alpha} \coloneqq (\Gamma(n-\alpha))^{-1}$). Let $x \in X$. By Proposition 2.1.10 (c), $\psi(D)x$ belongs to $\mathsf{Dom}(L^n)$, so (2.25) together with (a) leads to

$$\begin{aligned} L^\alpha \psi(D)x &= c_{n,\alpha} \int_0^\infty t^{n-\alpha} \Big(|\cdot|^{2n} g_t \psi \Big)(D)x \frac{\mathrm{d}t}{t} \\ &= \lim_{\substack{\varepsilon \to 0,\\ R \to \infty}} c_{n,\alpha} \int_\varepsilon^R t^{n-\alpha} \Big(|\cdot|^{2n} g_t \psi \Big)(D)x \frac{\mathrm{d}t}{t}. \end{aligned} \tag{2.26}$$

By Lemma 2.1.11, $c_{n,\alpha}$ times the integrand on the right-hand side is equal to

$$\left(c_{n,\alpha} \int_\varepsilon^R t^{n-\alpha} |\cdot|^{2n} g_t(\cdot)\psi(\cdot) \frac{\mathrm{d}t}{t} \right)(D) \eqqcolon h_{\varepsilon,R}(D) \tag{2.27}$$

and the change of variables $\tau = |\xi|^2 t$ shows that

$$h_{\varepsilon,R}(\xi) = \left(c_{n,\alpha} \int_{\varepsilon|\xi|^2}^{R|\xi|^2} \tau^{n-\alpha} \mathrm{e}^{-\tau} \frac{\mathrm{d}\tau}{\tau} \right) \psi(\xi)|\xi|^{2\alpha}, \quad \xi \in \mathbb{R}^d.$$

Using the compact support of ψ away from the origin, it is readily checked that $h_{\varepsilon,R} \to \psi(\cdot)|\cdot|^{2\alpha}$ in $\mathcal{S}(\mathbb{R}^d)$ as $\varepsilon \to 0$ and $R \to \infty$. Thus inserting (2.27) into (2.26) we conclude

$$L^\alpha \psi(D)x = (|\cdot|^{2\alpha}\psi)(D)x.$$

This shows (2.21). If $x \in \mathsf{Dom}(L^\alpha)$, then $L^\alpha \psi(D)x = \psi(D)L^\alpha x$ follows immediately from Proposition 2.1.3 (c) (using that L_0 commutes with $\psi(D)$ by Proposition 2.1.10 (c) and therefore so does $L = \overline{L_0}$). If L is injective, then $L^{-\alpha}$ is well-defined and by the identities already established, we have for $x \in X$

$$L^\alpha\Big(|\cdot|^{-2\alpha}\psi\Big)(D)x = \Big(|\cdot|^{2\alpha}|\cdot|^{-2\alpha}\psi\Big)(D)f = \psi(D)x.$$

Applying $L^{-\alpha} = (L^{-1})^\alpha$ on both sides gives $(|\cdot|^{-2\alpha}\psi)(D)x = L^{-\alpha}\psi(D)x$ as desired. Finally, $L^{-\alpha}x \in \mathsf{Dom}(L^\alpha)$, so $L^\alpha\psi(D)L^{-\alpha}x = \psi(D)L^{-\alpha}L^\alpha x = \psi(D)x$. But this implies $\psi(D)L^{-\alpha}x = L^{-\alpha}\psi(D)x$. We have proved (b).

(c) This is proved in a similar fashion as (b).

(d) Let $x \in X$. Note that using (b) and arguing in a similar fashion as in (a), one obtains for all $t \in \mathbb{R}$ that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(w_t\psi)(D)x &= (i|\cdot|w_t\psi)(D)x = i\sqrt{L}(w_t\psi)(D)x \quad (t \in \mathbb{R}), \\ (w_0\psi)(D)x &= \psi(D)x, \end{aligned}$$

Since classical solution of abstract Cauchy problems are unique (see e.g. [2, Theorem 3.1.12]), we deduce

$$(w_t\psi)(D)x = \mathrm{e}^{it\sqrt{L}}\psi(D)x \quad \text{for all } t \in \mathbb{R}$$

as desired.

(e) Let $\psi \in H_0^\infty(S_\vartheta)$ for some $\vartheta \in (\omega, \pi)$ and suppose that $g := \psi \circ |\cdot|^2$ lies in $\mathcal{F}\mathrm{L}^1$. Fix an angle $\vartheta' \in (\omega, \vartheta)$, a function $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\varphi(0) = 1$ and define the sequence $(\varphi_t)_{t\in(0,1)}$ as in Lemma 2.1.12. Let $x \in X$. Then, for all $z \in \mathbb{C}\backslash\overline{S_\omega}$ and $t \in (0,1)$

$$(\varphi_t r_z)(D)x = \varphi_t(D)(z - L)^{-1}x, \tag{2.28}$$

where $r_z(\xi) := (z - |\xi|^2)^{-1}$, $\xi \in \mathbb{R}^d$. Indeed, for $z \in \mathbb{C}_- := \{w \in \mathbb{C} \mid \mathrm{Re}(w) < 0\}$, this follows immediately from (2.16). The identity then extends to all $z \in \mathbb{C} \setminus \overline{S_\omega}$ by the identity theorem since both sides of (2.28) are holomorphic functions of z. Invoking Lemma 2.1.12 and using (2.28), we obtain

$$\begin{aligned} \psi(L)x &= \lim_{t\to 0}\varphi_t(D)\psi(L)x \\ &= \lim_{t\to 0}\frac{1}{2\pi i}\varphi_t(D)\int_{\partial S_{\vartheta'}} z\psi(z)(z\mathrm{Id} - L)^{-1}x\frac{\mathrm{d}z}{z} \\ &= \lim_{t\to 0}\frac{1}{2\pi i}\int_{\partial S_{\vartheta'}} z\psi(z)(\varphi_t r_z)(D)x\frac{\mathrm{d}z}{z} \\ &= \lim_{t\to 0}\frac{1}{2\pi i}\int_{\partial S_{\vartheta'}}\int_{\mathbb{R}^d} z\psi(z)\Big[\mathcal{F}^{-1}(\varphi_t r_z)\Big](y)\mathrm{e}^{-iy\cdot D}x\,\mathrm{d}y\frac{\mathrm{d}z}{z}. \end{aligned} \tag{2.29}$$

We want to apply Fubini's theorem. Observe that for $t \in (0,1)$, $y \in \mathbb{R}^d$ and $z \in \partial S_{\vartheta'} \setminus \{0\}$ we have

$$(1+|y|)^{d+1}|[\mathcal{F}^{-1}(\varphi_t r_z)](y)| \simeq_d \max_{|\alpha|\le d+1} |y^\alpha[\mathcal{F}^{-1}(\varphi_t r_z)](y)|$$
$$\le (2\pi)^{-d} \max_{|\alpha|\le d+1} \|\partial_\xi^\alpha(\varphi_t r_z)\|_1 \lesssim_{\varphi,d} t^{-d} \max_{|\alpha|\le d+1} \|\partial_\xi^\alpha r_z\|_\infty \lesssim (t^d|z|^{1+\frac{d+1}{2}})^{-1}.$$

Thus, the norm of the integrand in (2.29) is bounded by

$$M|z\psi(z)|(t^d|z|^{1+\frac{d+1}{2}})^{-1}(1+|y|)^{-(d+1)}\|x\| \lesssim \rho(z)(1+|y|)^{-(d+1)}t^{-d}\|x\|,$$

with $\rho(z) \coloneqq (|z| \wedge \frac{1}{|z|})^\beta$, $\beta = \alpha - \frac{d+1}{2} > 0$. Therefore, we may use Fubini's theorem to conclude that the right-hand side of (2.29) is equal to

$$\lim_{t\to 0} \frac{1}{2\pi i} \int_{\mathbb{R}^d} \int_{\partial S_{\vartheta'}} \psi(z)\Big[\mathcal{F}^{-1}(\varphi_t r_z)\Big](y)\,\mathrm{d}z\,\mathrm{e}^{-iy\cdot D}x\,\mathrm{d}y$$
$$= \lim_{t\to 0} \frac{1}{2\pi i} \int_{\mathbb{R}^d} \mathcal{F}^{-1}\Bigg(\int_{\partial S_{\vartheta'}} \frac{\psi(z)}{z-|\cdot|^2}\,\mathrm{d}z \cdot \varphi_t\Bigg)(y)\mathrm{e}^{-iy\cdot D}x\,\mathrm{d}y$$
$$= \lim_{t\to 0} \int_{\mathbb{R}^d} \mathcal{F}^{-1}\Big(\psi(|\cdot|^2)\varphi_t\Big)(y)\mathrm{e}^{-iy\cdot D}x\,\mathrm{d}y = \lim_{t\to 0} h(D)\varphi_t(D)x = h(D)x,$$

where we used Cauchy's integral formula and Lemma 2.1.12 in the second and forth equality, respectively.

□

Remark 2.1.16. By the chosen notation it is tempting to view the C_0-group $(\mathrm{e}^{it\sqrt{L}})_{t\in\mathbb{R}}$ generated by $i\sqrt{L}$ as in Proposition 2.1.15 (d) through the lense of the holomorphic functional calculus for L by setting $\mathrm{e}^{it\sqrt{L}} \coloneqq (e^{it\sqrt{z}})(L)$ for $t \in \mathbb{R}$. However, the function $f_t\colon z \mapsto \mathrm{e}^{it\sqrt{z}}$ does *not* lie in $H_0^\infty(S_\omega)$ and even worse is *not* regularizable. Therefore, f_t is not a admissible functions for the sectorial functional calculus for L and thus, when we write $\mathrm{e}^{it\sqrt{L}}$, we simply mean the C_0-group $i\sqrt{L}$, defined by semigroup theory (see e.g. [44] for rigorous definitions).

2.1.3.2. A Littlewood–Paley Theory Adapted to D

Proposition 2.1.17 (Calderón Reproducing Formula)**.** *Suppose that $\psi \in C_c^\infty(\mathbb{R}^d)$ is supported away from the origin and satisfies*

$$\int_0^\infty \psi(t\xi)\frac{\mathrm{d}t}{t} = 1 \quad \textit{for all} \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

Then,

$$\int_0^\infty \psi(tD)x\frac{\mathrm{d}t}{t} = x \quad \textit{for all } x \in \bigcup_{j=1}^d \overline{\mathsf{R}(D_j)},$$

where the above integral is understood to be an improper integral in X.

Proof. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be as in the statement of the proposition. Now define for $0 < \varepsilon < r < \infty$

$$\psi_{\varepsilon,r}(\xi) \coloneqq \int_\varepsilon^r \psi(t\xi)\frac{\mathrm{d}t}{t}, \quad \xi \in \mathbb{R}^d,$$

and for $r > 0$

$$\psi_{r,\infty}(\xi) \coloneqq \int_r^\infty \psi(t\xi)\frac{\mathrm{d}t}{t}, \quad \xi \in \mathbb{R}^d \setminus \{0\} \text{ and } \psi_{r,\infty}(0) \coloneqq 1.$$

Then $\psi_{\varepsilon,r}$ and $\psi_{r,\infty}$ are smooth functions, $\psi_{\varepsilon,r}$ being supported away from the origin, $\psi_{r,\infty}$ being supported in a bounded neighborhood of the origin. We have to show that

$$\psi_{\varepsilon,R}(D)x \to x \quad (\varepsilon \to 0, r \to \infty) \quad \text{for all } x \in \overline{\bigcup_{j=1}^d \mathsf{R}(D_j)}. \tag{2.30}$$

By a standard density argument, it suffices to show (2.30) for x in $\bigcup_{j=1}^d \mathsf{R}(D_j)$ and the uniform boundedness of the operator family $(\psi_{\varepsilon,r}(D))_{0<\varepsilon<r<\infty}$.

Step 1: Convergence on $\bigcup_{j=1}^d \mathsf{R}(D_j)$.

We show (2.30) first for $x \in \bigcup_{j=1}^d \mathsf{R}(D_j)$. To this end, let $x \in \mathsf{R}(D_j)$ for some $j \in \{1, \dots, d\}$. We observe that $\psi_{\varepsilon,\infty} = \psi_{\varepsilon,r} + \psi_{r,\infty}$ and thus

$$\psi_{\varepsilon,r}(D)x - x = \Big(\psi_{\varepsilon,\infty}(D)x - x\Big) - \psi_{r,\infty}(D)x. \tag{2.31}$$

We estimate the two terms on the right-hand side of (2.31) separately. Note that $\psi_{\varepsilon,\infty} = \psi_{1,\infty}(\varepsilon\cdot)$ and that $\psi_{1,\infty}(0) = 1$. Using Lemma 2.1.12 we obtain for the first term on the right-hand side of (2.31)

$$\psi_{\varepsilon,\infty}(D)x - x \to 0 \quad (\varepsilon \to 0).$$

So it remains to show that

$$\psi_{r,\infty}(D)x \to 0 \quad (r \to \infty). \tag{2.32}$$

As $x \in \mathsf{R}(D_j)$ by assumption, we can write $x = D_j y$ for some $y \in \mathsf{Dom}(D_j)$. Set $\tilde{\psi}(\xi) \coloneqq \xi_j \psi(\xi)$ for $\xi \in \mathbb{R}^d$ and observe that

$$\xi_j \psi_{r,\infty}(\xi) = \xi_j \psi_{1,\infty}(r\xi) = r^{-1}\tilde{\psi}_{1,\infty}(r\xi) \quad \text{for } r > 0.$$

Now Proposition 2.1.10 (c) yields the identity

$$\psi_{r,\infty}(D)x = (\xi_j \psi_{r,\infty})(D)y = \frac{1}{r}(\tilde{\psi}_{1,\infty})(rD)y$$

which, by Proposition 2.1.10 (a), implies

$$\|\psi_{r,\infty}(D)f\| \le \Big(M\|\tilde{\psi}_{1,\infty}\|_{\mathcal{F}\mathbf{M}}\|g\|\Big)\frac{1}{r} \to 0 \tag{2.33}$$

as $r \to \infty$. This shows (2.32).

Step 2: Uniform boundedness of $\Big(\psi_{\varepsilon,r}(D)\Big)_{0<\varepsilon<r<\infty}$

The family of operators $(\psi_{\delta,\infty}(D))_{\delta>0}$ are uniformly bounded as

$$\|\psi_{\delta,\infty}(D)\|_{\mathcal{L}(X)} \le M\|\psi_{\delta,\infty}\|_{\mathcal{F}\mathbf{M}} = M\|\psi_{1,\infty}\|_{\mathcal{F}\mathbf{M}} \quad \text{for all } \delta > 0.$$

This in turn implies for all $0 < \varepsilon < r < \infty$

$$\|\psi_{\varepsilon,r}(D)\|_{\mathcal{L}(X)} = \|\psi_{\varepsilon,\infty}(D) - \psi_{r,\infty}(D)\|_{\mathcal{L}(X)} \le 2M\|\psi_{1,\infty}\|_{\mathcal{F}\mathbf{M}},$$

which shows the uniform boundedness of $(\psi_{\varepsilon,r}(D))_{0<\varepsilon<r<\infty}$. □

We also have a discrete version of Proposition 2.1.17 which can be proved in similar fashion.

Proposition 2.1.18 (Discrete Calderón Reproducing Formula)**.** *Suppose that $\psi \in C_c^\infty(\mathbb{R}^d)$ is supported away from the origin and satisfies*

$$\sum_{\lambda\in 2^{\mathbb{Z}}} \psi_\lambda(\xi) = 1 \quad \textit{for all} \quad \xi \ne 0, \quad \textit{where} \quad \psi_\lambda(\xi) := \psi\left(\frac{\xi}{\lambda}\right).$$

Then,

$$\sum_{\lambda\in 2^{\mathbb{Z}}} \psi_\lambda(D)x = x \quad \textit{for all } x \in \bigcup_{j=1}^{d} \overline{\mathsf{R}(D_j)}.$$

Remark 2.1.19. Let $L := \overline{D_1^2 + \cdots + D_d^2}$. Then Propositions 2.1.17 and 2.1.18 are also true for $f \in \overline{\mathsf{R}(L^\alpha)}$ if $\alpha > 0$. The proof of Proposition 2.1.17 extends verbatim to this case with one minor modification: one only has to replace $\tilde{\psi}$ by $\psi^{(\alpha)} := |\cdot|^\alpha\psi$ in the first step of the proof above and apply Lemma 2.1.15 (b). Then (2.33) becomes

$$\|\psi_{r,\infty}(D)f\| = \|(\psi^{(\alpha)})_{1,\infty}(rD)g\|r^{-\alpha} \le (M\||\psi^{(\alpha)}\|_{\mathcal{F}\mathbf{M}})r^{-\alpha} \to 0 \quad (r \to \infty).$$

For Proposition 2.1.18, one argues similarly.

Proposition 2.1.20 (Littlewood–Paley Inequality)**.** *Suppose that X is a Hilbert space.*

(a) Suppose that there exists some $C_D > 0$ such that

$$\|\varphi(D)\|_{\mathcal{L}(X)} \le C_D\|\varphi\|_\infty \quad \textit{for all } \varphi \in C_c^\infty(\mathbb{R}^d). \tag{2.34}$$

Then for all $\psi \in C_c^\infty(\mathbb{R}^d)$ supported away from the origin, we have

$$\int_0^\infty \|\psi(tD)x\|^2 \frac{\mathrm{d}t}{t} \lesssim_\psi C_D^2\|x\|^2 \quad \textit{for all } x \in X.$$

(b) *Suppose that there exists some* $C_{D^*} > 0$ *such that*

$$\|\varphi(D^*)\|_{\mathcal{L}(X)} \le C_{D^*}\|\varphi\|_\infty \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^d). \tag{2.35}$$

Suppose further that $\bigcup_{j=1}^d \mathsf{R}(D_j)$ *is dense in* X *and that* $\psi \in C_c^\infty(\mathbb{R}^d)$ *is a function as described in Proposition 2.1.17. Then*

$$\frac{1}{C_{D^*}^2}\|x\|^2 \lesssim_\psi \int_0^\infty \|\psi(tD)x\|^2 \frac{\mathrm{d}t}{t} \quad \text{for all } x \in X.$$

Proof. The proof is similar to the proof of the equivalence of the bounded H^∞-calculus and quadratic estimates (see [1, Section (F), Theorem F]). However, for the sake of completeness, we provide a proof.
To prove (a), let $\psi \in C_c^\infty(\mathbb{R}^d)$ be supported away from the origin and $x \in X$. Then, monotone convergence and the change of variables $t = 2^j s$ ($j \in \mathbb{Z}$) show that

$$\begin{aligned}\int_0^\infty \|\psi(tD)x\|^2 \frac{\mathrm{d}t}{t} &= \sum_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \|\psi(tD)x\|^2 \frac{\mathrm{d}t}{t} \\ = \sum_{j\in\mathbb{Z}} \int_1^2 \|\psi(2^j sD)x\|^2 \frac{\mathrm{d}s}{s} &= \lim_{N\to\infty} \int_1^2 \sum_{|j|\le N} \|\psi(2^j sD)x\|^2 \frac{\mathrm{d}s}{s}.\end{aligned}$$

Let $(\varepsilon_j)_{j\in\mathbb{Z}}$ be a Rademacher sequence on some probability space $(\Omega, \mathcal{B}, \mathbb{P})$ (i.e., a sequence of i.i.d. random variables with $\mathbb{P}(\varepsilon_j = \pm 1) = \frac{1}{2}$ for all $j \in \mathbb{Z}$). Then we have for the integrand on the right-hand side of the above display

$$\sum_{|j|\le N} \|\psi(2^j sD)x\|^2 = \mathbb{E}\left\| \sum_{|j|\le N} \varepsilon_j \psi(2^j sD)x \right\|^2. \tag{2.36}$$

Since ψ has compact support away from the origin, we may pick $0 < r_1 < r_2$ such that $\operatorname{supp}(\psi) \subseteq \{\xi \in \mathbb{R}^d \colon r_1 < |\xi| < r_2\}$ and it is readily checked that for all $s \in (1,2)$ the functions $\xi \mapsto \psi(2^j s\xi)$ and $\xi \mapsto \psi(2^k s\xi)$ are disjointly supported whenever $|j-k| > n_\psi := \lfloor \log_2(\frac{r_2}{r_1}) \rfloor$. This implies

$$\left\| \xi \mapsto \sum_{|j|\le N} \varepsilon_j(\omega)\psi(2^j s\xi) \right\|_\infty \le (2n_\psi + 1)\|\psi\|_\infty$$

uniformly in $s \in (1,2)$, $\omega \in \Omega$ and $N \in \mathbb{N}$. Using this and (2.37), we can estimate the right-hand side of (2.36) by

$$\mathbb{E}\bigg(C_D^2 (2n_\psi + 1)^2 \|\psi\|_\infty^2 \|x\|_2^2 \bigg) = C_D^2 (2n_\psi + 1)^2 \|\psi\|_\infty^2 \|x\|_2^2,$$

which finally gives

$$\int_0^\infty \|\psi(tD)x\|^2 \frac{\mathrm{d}t}{t} \le C_D^2 (2n_\psi + 1)^2 \|\psi\|_\infty^2 \log(2) \|x\|^2$$

as desired.
To prove (b), let $\psi \in C_c^\infty(\mathbb{R}^d)$ be as in Proposition 2.1.17 and $x \in X$. Without loss of generality, we may assume $x \neq 0$. Pick another real-valued function $\tilde{\psi} \in C_c^\infty(\mathbb{R}^d)$ supported away from the origin such that $\tilde{\psi} = 1$ on the support of ψ. By Proposition 2.1.17, we have

$$x = \int_0^\infty \psi(tD)x \frac{\mathrm{d}t}{t},$$

which implies

$$\|x\|^2 = (x|x) = \int_0^\infty (x|\psi(tD)x) \frac{\mathrm{d}t}{t}. \tag{2.37}$$

Now, by construction, we have $\tilde{\psi}\psi = \psi$ and thus,

$$(x|\psi(tD)x) = (x|\tilde{\psi}(tD)\psi(tD)x) = (\tilde{\psi}(tD^*)x|\psi(tD)x) \quad \text{for all } t > 0$$

by Proposition 2.1.10 (a) and (b). So the right-hand side of (2.37) is equal to

$$\begin{aligned}\int_0^\infty (\tilde{\psi}(tD^*)x|\psi(D)x) \frac{\mathrm{d}t}{t} &\leq \left(\int_0^\infty \|\tilde{\psi}(tD^*)x\|^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \left(\int_0^\infty \|\psi(tD)x\|^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}} \\ &\lesssim_{\tilde{\psi}} C_{D^*} \|x\| \left(\int_0^\infty \|\psi(tD)x\|^2 \frac{\mathrm{d}t}{t} \right)^{\frac{1}{2}},\end{aligned}$$

where we used the Cauchy–Schwarz inequality and applied part (a) to D^*. Dividing by $C_{D^*}\|x\|$ gives the claim. □

In view of Proposition 2.1.20, we consider the following set of assumptions.

Assumption 2.1.21. *There exist constants $C_D, C_{D^*} > 0$ such that for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ the following holds.*

(HX) *X is a Hilbert space and $\bigcup_{j=1}^d \mathsf{R}(D_j)$ is dense in X,*

(D_∞) *$\|\varphi(D)\|_{\mathcal{L}(X)} \leq C_D \|\varphi\|_\infty$,*

(D_∞^*) *$\|\varphi(D^*)\|_{\mathcal{L}(X')} \leq C_D^* \|\varphi\|_\infty$.*

Remark 2.1.22. (a) By Remark 2.1.19, we can replace the set $\bigcup_{j=1}^d \mathsf{R}(D_j)$ in (HX) by $\mathsf{R}(L)$, where $L = \overline{D_1^2 + \cdots + D_d^2}$. But for any reflexive Banach space X and sectorial operator A in X, we have the direct sum decomposition $X = \mathsf{N}(A) \oplus \overline{\mathsf{R}(A)}$ [31, Proposition 10.1.9]. So (HX) is fulfilled if X is a Hilbert space and L is injective.

(b) If $X = \mathrm{L}^2(\mathbb{R}^d)$, then ($\mathrm{D}_\infty$) and ($\mathrm{D}_\infty^*$) are always satisfied by the Coifman–Weiss' transference principle stated in Proposition 2.1.10 (d).

As an immediate consequence of Proposition 2.1.20 we have the following.

Corollary 2.1.23 (Plancherel)**.** *Suppose that* (HX)*,* (D_∞) *and* (D_∞^*) *are fulfilled. Then,*

$$\int_0^\infty \|\psi(tD)x\|^2 \frac{\mathrm{d}t}{t} \simeq_\psi \|x\|^2 \quad \textit{for all } x \in X,$$

provided that $\psi \in C_c^\infty(\mathbb{R}^d)$ *is supported away from the origin and satisfies* $\int_0^\infty \psi(t\xi)\frac{\mathrm{d}t}{t} = 1$ *for all* $\xi \neq 0$*.*

We also have a discrete version of Corollary 2.1.23. The proof is similar to one of Corollary 2.1.23 and is therefore omitted.

Corollary 2.1.24 (Discrete Plancherel)**.** *Suppose that* (HX)*,* (D_∞)*, and* (D_∞^*) *are fulfilled. Let further* $\psi \in C_c^\infty(\mathbb{R}^d)$ *be supported in the annulus* $A(\frac{1}{2}, 2) := \{\xi \in \mathbb{R}^d \mid \frac{1}{2} < |\xi| < 2\}$ *and satisfy*

$$\sum_{\lambda \in 2^{\mathbb{Z}}} \psi_\lambda(\xi) = 1 \quad \textit{for all} \quad \xi \neq 0, \quad \textit{where} \quad \psi_\lambda(\xi) := \psi\left(\frac{\xi}{\lambda}\right).$$

Then,

$$\sum_{\lambda \in 2^{\mathbb{Z}}} \left\|\psi_\lambda(D)x\right\|^2 \simeq_\psi \|x\|^2 \quad \textit{for all } x \in X.$$

Corollary 2.1.25 (X-Boundedness of Almost-Orthogonal Operators)**.** *Let* (HX)*,* (D_∞)*, and* (D_∞^*) *be satisfied. Let further* $\psi \in C_c^\infty(\mathbb{R}^d)$ *be as in Corollary 2.1.24 and assume that* $h := (h_\lambda)_{\lambda \in 2^{\mathbb{Z}}}$ *is a sequence of functions on* $\mathbb{R}^d$ *such that* h_λ *is smooth on an open neighborhood of* $K_\lambda := \operatorname{supp}(\psi_\lambda)$ *for all* $\lambda \in 2^{\mathbb{Z}}$ *and* $\|h\|_\infty := \sup_{\lambda \in 2^{\mathbb{Z}}} \|h_\lambda\|_{\mathrm{L}^\infty(K_\lambda)} < \infty$*. Then,*

$$T \colon X \to X, \quad Tx := \sum_{\lambda \in 2^{\mathbb{Z}}} (h_\lambda \psi_\lambda)(D)x$$

is a well-defined linear bounded operator with $\|T\| \lesssim \|h\|_\infty$*. More precisely, for* $x \in X$*, there holds*

$$\|Tx\|^2 \lesssim \sum_{\lambda \in 2^{\mathbb{Z}}} \|(h_\lambda \psi_\lambda)(D)x\|^2 \lesssim \|h\|_\infty^2 \sum_{\lambda \in 2^{\mathbb{Z}}} \|\psi_\lambda(D)x\|^2 \simeq \|h\|_\infty^2 \|x\|^2. \tag{2.38}$$

Proof. Let $x \in X$. Given a finite subset $I \subseteq 2^{\mathbb{Z}}$, we define

$$T_I x := \sum_{\lambda \in I} (h_\lambda \psi_\lambda)(D)x \in X.$$

By Corollary 2.1.24, we have

$$\|T_I x\|^2 \simeq \sum_{\mu \in 2^{\mathbb{Z}}} \|\psi_\mu(D) T_I x\|^2. \tag{2.39}$$

But by the support properties of the ψ_λ and (D_∞), we see that

$$\|\psi_\mu(D)T_I x\|^2 = \Big\| \sum_{\lambda\in I_\mu\cap I} (\psi_\mu h_\lambda \psi_\lambda)(D)x \Big\|^2 \lesssim \sum_{\lambda\in I_\mu\cap I} \|(h_\lambda\psi_\lambda)(D)x\|^2,$$

where $I_\mu \coloneqq \{\frac{\mu}{2}, \mu, 2\mu\}$. Inserting this into (2.39) gives

$$\begin{aligned} \|T_I x\|^2 &\lesssim \sum_{\mu\in 2^{\mathbb{Z}}} \sum_{\lambda\in I_\mu\cap I} \|(h_\lambda\psi_\lambda)(D)x\|^2 \\ &= \sum_{\lambda\in I} \sum_{\mu\in I_\lambda} \|(h_\lambda\psi_\lambda)(D)x\|^2 = 3\sum_{\lambda\in I} \|(h_\lambda\psi_\lambda)(D)x\|^2. \end{aligned} \tag{2.40}$$

Choose another function $\tilde{\psi} \in C_c^\infty(\mathbb{R}^d)$ supported in $A(\frac{1}{2}, 2)$ with $\tilde{\psi} = 1$ on the support of ψ. Then, $\tilde{\psi}_\lambda \psi_\lambda = \psi_\lambda$ for all $\lambda \in I$, and invoking (D_∞) again, we can estimate each term on the right-hand side of (2.40) according to

$$\|(h_\lambda\psi_\lambda)(D)x\|^2 = \|(h_\lambda\tilde{\psi}_\lambda)(D)\psi_\lambda(D)x\|^2 \lesssim \|h_\lambda\tilde{\psi}_\lambda\|_\infty^2 \|\psi_\lambda(D)x\|^2$$

Taking the infimum over all such $\tilde{\psi}$, we are able to replace $\|h_\lambda\tilde{\psi}_\lambda\|_\infty^2$ on the right-hand side by $\|h_\lambda\|_{L^\infty(K_\lambda)}$. Hence, the right-hand side of (2.40) is estimated by

$$3\|h\|_\infty^2 \sum_{\lambda\in I} \|\psi_\lambda(D)x\|^2 \simeq \|h\|_\infty^2 \|x\|^2,$$

where we used Corollary 2.1.24 for the last estimate. From this, we conclude that $(T_I x)$ is Cauchy w.r.t. I. Thus, the limit $T_I x = \lim_{I\to 2^{\mathbb{Z}}} T_I x = \sum_{\lambda\in 2^{\mathbb{Z}}} (h_\lambda\psi_\lambda)(D)x$ exists in X and $\|Tx\|_2^2 \lesssim \|h\|_\infty^2 \|x\|_2^2$. Moreover, (2.38) immediately follows from the estimates obtained for $(T_I x)_I$ by taking the limit $I \to 2^{\mathbb{Z}}$. The proof is complete.

$\square$

Let $L = \overline{D_1^2 + \cdots + D_d^2}$. Then, L is sectorial of angle 0 by Lemma 2.1.15 (a). We want to extend Corollary 2.1.25 to the extrapolation spaces X_L^α for $\alpha \in \mathbb{R}$ as defined in Subsection 2.1.2.1. To this end, we will need to extend the definition of the Phillips functional calculus, at least for suitable functions. It is convenient to introduce the notation $\langle D_L \rangle \coloneqq (\mathrm{Id} + \mathscr{L})^{\frac{1}{2}}$, where $\mathscr{L}$ is the extension of L to $\mathcal{D}_L'$ as defined in Subsection 2.1.2.1. Recall that by Proposition 2.1.6 (c),

$$X_L^\alpha = \{u \in \mathcal{D}_L' \mid \langle D_L \rangle^{2\alpha} u \in X\}, \quad \alpha \in \mathbb{R}.$$

Definition 2.1.26 (Extension of the Phillips Functional Calculus)**.** Suppose that X is reflexive and let $\varphi \in C_c^\infty(\mathbb{R}^d)$. If $x \in X_L^\alpha$ for some $\alpha \in \mathbb{R}$, we define

$$\overline{\varphi(D)}x \coloneqq (\langle \xi \rangle^{-2\alpha}\varphi)(D)y \in X$$

where $y \coloneqq \langle D_L \rangle^{2\alpha} x \in X$.

This definition does not depend on $\alpha \in \mathbb{R}$. Indeed, let $x \in X_L^\alpha \cap X_L^\beta$ with $\beta \le \alpha$, say. Then, $\langle D_L\rangle^{2\beta} x = \langle D_L\rangle^{-2(\alpha-\beta)}\langle D_L\rangle^{2\alpha} x = (\mathrm{Id} + L)^{-(\alpha-\beta)}\langle D_L\rangle^{2\alpha} x$ by Proposition 2.1.6 and thus by Proposition 2.1.15 (c)

$$\begin{aligned}&(\varphi\langle\,\cdot\,\rangle^{-2\beta})(D)\langle D_L\rangle^{2\beta}x = (\varphi\langle\,\cdot\,\rangle^{-2\beta})(D)(\mathrm{Id}+L)^{-(\alpha-\beta)}\langle D_L\rangle^{2\alpha}x\\ &= (\varphi\langle\,\cdot\,\rangle^{-2\beta}\langle\,\cdot\,\rangle^{-2(\alpha-\beta)})(D)\langle D_L\rangle^{2\alpha}x = (\varphi\langle\,\cdot\,\rangle^{-2\alpha})(D)\langle D_L\rangle^{2\alpha}x.\end{aligned}$$

Lemma 2.1.15 (c) shows that $\overline{\varphi(D)}x = \varphi(D)x$ if $x \in X_L^\alpha$ and $\alpha \ge 0$. Thus, we have a reasonable extension. To ease notation, we will just write $\varphi(D)$ instead of $\overline{\varphi(D)}$.

Corollary 2.1.27. *Let* (HX), (D_∞) *and* (D_∞^*) *be satisfied,* $\alpha \in \mathbb{R}$. *Assume further that* X_0 *is a dense subset of* X. *Suppose further that* $\psi \in C_c^\infty(\mathbb{R}^d)$ *is supported away from the origin and satisfies*

$$\sum_{\lambda\in 2^{\mathbb{Z}}} \psi_\lambda(\xi) = 1 \quad \text{for all} \quad \xi \neq 0, \quad \text{where} \quad \psi_\lambda(\xi) \coloneqq \psi\left(\frac{\xi}{\lambda}\right).$$

Then,

$$\sum_{\lambda\in 2^{\mathbb{Z}}} \psi_\lambda(D)x = x \quad \text{for all } x \in X_L^\alpha(\mathbb{R}^d).$$

In particular,

$$\mathcal{S}_D^{X_0} \coloneqq \mathrm{span}\{\psi(D)x \mid \psi \in C_c^\infty(\mathbb{R}^d), \mathrm{supp}(\psi) \in \mathbb{R}^d \setminus \{0\}, x \in X_0\}$$

belongs to $\bigcap_{\beta\in\mathbb{R}} \mathsf{Dom}(L^\beta)$ *and is dense in* X_L^α.

Proof. The inclusion $\mathcal{S}_D^{X_0} \subseteq \bigcap_{\beta\in\mathbb{R}} \mathsf{Dom}(L^\beta)$ follows immediately from Proposition 2.1.15 (b). The assertion concerning the density follows from a 2ε-argument, using Proposition 2.1.18, the boundedness of $\psi(D)$ for compactly supported smooth ψ and the density of X_0 in X. □

Theorem 2.1.28 (X_L^α-Boundedness of Almost-Orthogonal Operators)**.** *Suppose that* (HX), (D_∞) *and* (D_∞^*) *are fulfilled. Let* $\alpha \in \mathbb{R}$ *and* $\gamma \in \mathbb{R}$. *Let further* $\psi \in C_c^\infty(\mathbb{R}^d)$ *be as in Corollary 2.1.24 and assume that* $h \coloneqq (h_\lambda)_{\lambda\in 2^{\mathbb{Z}}}$ *is a sequence of functions on* $\mathbb{R}^d$ *such that* h_λ *is smooth on an open neighborhood of* $K_\lambda \coloneqq \mathrm{supp}(\psi_\lambda)$ *for all* $\lambda \in 2^{\mathbb{Z}}$ *and* $\|h\|_\infty \coloneqq \sup_{\lambda\in 2^{\mathbb{Z}}} \|h_\lambda\|_{\mathrm{L}^\infty(K_\lambda)} < \infty$. *Then,*

$$T\colon X_L^\alpha \to X_L^{\alpha-\gamma}, \quad Tx \coloneqq \sum_{\lambda\in 2^{\mathbb{Z}}} \langle\lambda\rangle^{2\gamma}(h_\lambda\psi_\lambda)(D)x$$

is a well-defined linear bounded operator with $\|T\| \lesssim \|h\|_\infty$. *Moreover, for* $x \in X_L^\alpha$, *there holds*

$$\|Tx\|^2_{X_L^{\alpha-\gamma}} \lesssim \sum_{\lambda\in 2^{\mathbb{Z}}} \|(h_\lambda\psi_\lambda)(D)y\|^2 \lesssim \|h\|_\infty^2 \sum_{\lambda\in 2^{\mathbb{Z}}} \|\psi_\lambda(D)y\|^2 \simeq \|h\|_\infty^2\|x\|^2_{X_L^\alpha},$$

where $y \coloneqq \langle D_L\rangle^\alpha x \in X$.

Proof. Let $x \in X_L^\alpha$ and set $y := \langle D_L\rangle^{2\alpha}x \in X$. Choose some compactly supported smooth function with $\tilde\psi \in C_c^\infty(\mathbb{R}^d)$ with $\tilde\psi = 1$ on the support of ψ. For finite $I \subseteq 2^{\mathbb{Z}}$, we define $T_I x := \sum_{\lambda\in I}\langle\lambda\rangle^{2\gamma}(h_\lambda\psi_\lambda)(D)x$. By definition and Proposition 2.1.15 (c),

$$\langle D_L\rangle^{2(\alpha-\gamma)}T_I x = \sum_{\lambda\in I}\Big(\langle\lambda\rangle^{2\gamma}\langle\,\cdot\,\rangle^{-2\gamma}h_\lambda\psi_\lambda\Big)(D)y = \sum_{\lambda\in I}\Big(\tilde h_\lambda^\gamma h_\lambda\psi_\lambda\Big)(D)y,$$

where $\tilde h_\lambda^\gamma := \frac{\langle\lambda\rangle^{2\gamma}}{\langle\,\cdot\,\rangle^{2\gamma}}\tilde\psi_\lambda$. Applying Corollary 2.1.25 and then Proposition 2.1.10 (d), we obtain

$$\begin{aligned}\|T_I x\|^2_{X_L^{\alpha-\gamma}} &= \Big\|\sum_{\lambda\in I}(\tilde h_\lambda^\gamma h_\lambda\psi_\lambda)(D)y\Big\|^2\\ &\lesssim \sum_{\lambda\in I}\|(\tilde h_\lambda^\gamma h_\lambda\psi_\lambda)(D)y\|^2 \lesssim \sum_{\lambda\in I}\|(h_\lambda\psi_\lambda)(D)y\|^2\end{aligned}$$

and further (2.38)

$$\sum_{\lambda\in I}\|(h_\lambda\psi_\lambda)(D)y\|^2 \lesssim \|h\|_\infty\sum_{\lambda\in I}\|\psi_\lambda(D)y\|^2 \simeq \|y\|^2 = \|f\|^2_{X_L^\alpha}.$$

Now, the assertion follows by taking the limit $I \to 2^{\mathbb{Z}}$ just as in Corollary 2.1.25. □

Proposition 2.1.29 (Squarefunction Characterization of the L^p-norm)**.** *Suppose that for each $1 < p < \infty$, the operator iD_p is the generator of a bounded d-parameter C_0-group $(\mathrm{e}^{iy\cdot D_p})$ on $X_p = \mathrm{L}^p(\mathbb{R}^d)$ with $(\mathrm{e}^{iy\cdot D_p})' = \mathrm{e}^{iy\cdot D_{p'}}$ and $\mathrm{e}^{iy\cdot D_p}f = \mathrm{e}^{iy\cdot D_2}f$ for $f \in X_p\cap X_2$. Suppose further that $L_p := D_{1,p}^2+\cdots+D_{d,p}^2$ is injective for all $p \in (1,\infty)$. If $\psi \in C_c^\infty(\mathbb{R}^d)$ is as in Proposition 2.1.17, then*

$$\left\|\left(\int_0^\infty |\psi(tD_p)f(x)|^2\frac{\mathrm{d}t}{t}\right)^{\frac12}\right\|_p \simeq \|f\|_p \quad \textit{for all } f \in X_p. \tag{2.41}$$

Similarly, if $(\psi_\lambda)_{\lambda\in 2^{\mathbb{Z}}}$ is defined as in Corollary 2.1.18, then

$$\left\|\left(\sum_{\lambda\in 2^{\mathbb{Z}}}|\psi_\lambda(D_p)f(x)|^2\right)^{\frac12}\right\|_p \simeq \|f\|_p \quad \textit{for all } f \in X_p. \tag{2.42}$$

Proof. We use the Kahane-Khintchine inequality and Coifman–Weiss' transference principle. Let $f \in X_p$. For finite $I \subseteq 2^{\mathbb{Z}}$, we define $S_I f := (\sum_{\lambda\in I}|\psi_\lambda(D_p)f|^2)^{1/2} \in X_p$. Let $(\varepsilon_\lambda)_{\lambda\in 2^{\mathbb{Z}}}$ be a Rademacher sequence on some probability space $(\Omega,\mathcal{B},\mathbb{P})$. By the Kahane–Khintchine inequality, we have for a.e. $x \in \mathbb{R}^d$

$$\Big(S_I f(x)\Big)^p \simeq \mathbb{E}\Big\|\sum_{\lambda\in I}\varepsilon_j\Big[\psi_\lambda(D_p)f\Big](x)\Big\|^p.$$

Integrating w.r.t. $x \in \mathbb{R}^d$, invoking Fubini's theorem and Proposition 2.1.10 (d) gives

$$\|S_I f\|_p^p = \int_{\mathbb{R}^d} \mathbb{E}\Big\| \sum_{\lambda \in I} \varepsilon_\lambda \Big[\psi_\lambda(D_p) f\Big](x)\Big\|^p \, \mathrm{d}x$$
$$= \mathbb{E}\Big\| \sum_{\lambda \in I} \varepsilon_\lambda \psi_\lambda(D_p) f\Big\|^p \lesssim \|f\|_p^p,$$

where, in the last step, we used that the 'Mihlin norm' of $\varphi_\varepsilon := \sum_{\lambda \in I} \varepsilon_\lambda(\omega)\psi_\lambda$ does not depend on $\omega \in \Omega$. By monotone convergence, we infer

$$\Big\| \Big(\sum_{\lambda \in 2^{\mathbb{Z}}} |\psi_\lambda(D_p) f(x)|^2 \Big)^{\frac{1}{2}} \Big\|_p \lesssim \|f\|_p.$$

The reverse inequality is proved by a duality argument (in a similar fashion as Proposition 2.1.20 (b); note that the injectivity of L_p implies the density of its range, so that the Calderón reproducing formula is available on all of X_p by Remarks 2.1.19 and 2.1.14). $\square$

2.2. One-Dimensional Half-Wave Equations

In this section, we consider half-wave equations as abstract Cauchy problems in $\mathrm{L}^p(\mathbb{R})$, $p \in (1, \infty)$. The main goal is to show that (under suitable conditions) the corresponding C_0-groups are bounded on $\mathrm{L}^p(\mathbb{R})$ for all $p \in (1, \infty)$ and that they satisfy a $\mathrm{L}^\infty_x \mathrm{L}^1_t$-estimate. These results lay the foundation for the results derived in Chapter 3 and Chapter 4.

2.2.1. L^p-Boundedness and $\mathrm{L}^\infty_x \mathrm{L}^1_t$-Estimates

We assume that $a\colon \mathbb{R} \to \mathbb{R}$ is a function satisfying the following assumptions.

Assumption 2.2.1.

(H1) *There are* $0 < m_1 \le m_2 < \infty$ *such that*

$$m_1 \le a(x) \le m_2 \quad \textit{for all } x \in \mathbb{R}.$$

(H2) *The function* a *is Lipschitz continuous.*

(H3) *We have* $\frac{\mathrm{d}}{\mathrm{d}x} \log(a) \in \mathrm{L}^1(\mathbb{R})$ *with*

$$\Big\| \tfrac{\mathrm{d}}{\mathrm{d}x} \log(a) \Big\|_{\mathrm{L}^1(\mathbb{R})} < 4.$$

Let $p \in [1,\infty]$. Since we want to deal with second-order differential operators in divergence and standard form simultaneously, we will work in $X_p \coloneqq \mathrm{L}^p(\mathbb{R};\mathbb{C}^2)$, where we endow $\mathbb{C}^2$ with the Euclidean norm. We consider the differential operator

$$L_p\colon D(L_p) \subseteq X_p \to X_p, \quad L_p f = \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}x} a \frac{\mathrm{d}}{\mathrm{d}x} & 0 \\ 0 & -a\frac{\mathrm{d}^2}{\mathrm{d}x^2} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -(af_1')' \\ -af_2'' \end{pmatrix}$$

with domain $D(L_p) \coloneqq W^{2,p}(\mathbb{R};\mathbb{C}^2)$. If it is clear from the context, we will supress the target space $\mathbb{C}^2$ and often simply write $\mathrm{L}^p(\mathbb{R})$ instead of $\mathrm{L}^p(\mathbb{R};\mathbb{C}^2)$, etc. Our goal in this section is to show that $i\sqrt{L_p}$ generates a bounded C_0-group $(\mathrm{e}^{it\sqrt{L_p}})_{t\in\mathbb{R}}$ on $\mathrm{L}^p(\mathbb{R})$ for $p \in (1,\infty)$ and that the corresponding cosine function satisfies the

$$\|[\mathrm{Cos}(t\sqrt{L_p})f](x)\|_{\mathrm{L}^\infty_x(\mathbb{R};\mathrm{L}^1_t(\mathbb{R}))} \lesssim \|f\|_1 \quad \text{for } f \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{L}^p(\mathbb{R}).$$

To begin with, we show that L_p is sectorial which in particular guarantees that its square root $\sqrt{L_p}$ is well-defined.

Proposition 2.2.2 (Sectoriality of L_p)**.** *Let $p \in (1,\infty)$ and suppose that a satisfies (H1) and (H2). Then, L_p is sectorial with $\sigma(L_p) \subseteq [0,\infty)$. Moreover, the resolvents are consistent in the sense that $R(\lambda, L_p)f = R(\lambda, L_2)f$ for $f \in X_p \cap X_2$ and* $\mathrm{Re}(\lambda) < 0$.

Proof. Let $p \in (1,\infty)$. Define the operators $L_{1,p} \coloneqq -\frac{\mathrm{d}}{\mathrm{d}x} a \frac{\mathrm{d}}{\mathrm{d}x}$ and $L_{2,p} \coloneqq -a\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ in $\mathrm{L}^p(\mathbb{R})$ with domains $\mathsf{Dom}(L_{1,p}) = \mathsf{Dom}(L_{2,p}) = W^{2,p}(\mathbb{R})$. Clearly, $L_p = L_{1,p} \oplus L_{2,p}$, so it suffices to show the assertions for $L_{1,p}$ and $L_{2,p}$ separately. For $L_{2,p}$, the assertions follow from [40, Theorem 3.3] (even for all $p \in [1,\infty]$ and for more general complex-valued $a \in \mathrm{L}^\infty(\mathbb{R})$). To show the sectoriality of $L_{1,p}$, we first note that if $p = 2$, then form methods and regularity theory for elliptic equations imply that $L_{1,2}$ is nonnegative and self-adjoint (just consider the associated form $\mathfrak{a}\colon W^{1,2}(\mathbb{R}) \times W^{1,2}(\mathbb{R}) \to \mathbb{C}$, $\mathfrak{a}(u,v) \coloneqq (au'|v')_{\mathrm{L}^2(\mathbb{R})}$ and apply e.g. [43, Propositions 1.22 and 1.24], [22, Theorem 8.8]). In particular, $L_{1,2}$ is sectorial of angle 0. To extrapolate to general $p \in (1,\infty) \setminus \{0\}$, one could argue by the beautiful theory of heat kernel estimates (see e.g. [43, Chapter 7]), but since this approach does not spare us the proof of $\overline{L_{1,2}}^{\mathrm{L}^p} = L_{2,p}$, we sketch a more direct approach.
We first observe the a priori estimate

$$\|u\|_{2,p} \lesssim \|L_{1,p}u\|_p + \|u\|_p \quad (u \in W^{2,p}(\mathbb{R})). \tag{2.43}$$

For elliptic operators in $\mathrm{L}^p(\mathbb{R}^d)$, this estimate follows from Mihlin's theorem from harmonic analysis and a perturbation argument using (H2). However, since $d = 1$, this estimate is significantly easier. Indeed, using (H1) we have for $u \in W^{2,p}(\mathbb{R})$

$$m_1\|u''\|_p \leq \|au''\|_p = \|L_{1,p}u + a'u'\|_p \leq \|L_{1,p}u\|_p + \|a'\|_\infty \|u'\|_p$$

and thus

$$\begin{aligned}\|u\|_{2,p} &\simeq \|u''\|_p + \|u'\|_p + \|u\|_p \\ &\leq \frac{1}{m_1}\|L_{1,p}u\|_p + \left(1 + \frac{\|a'\|_\infty}{m_1}\right)\|u'\|_p + \|u\|_p \\ &\leq \frac{1}{m_1}\|L_{1,p}u\|_p + \left(1 + \frac{\|a'\|_\infty}{m_1}\right)\left(\frac{\varepsilon^p}{p}\|u''\|_p + \frac{\varepsilon^{-p'}}{p'}\|u\|_p\right) + \|u\|_p\end{aligned}$$

for any $\varepsilon > 0$ (in the last step, we used the inequalities of Gagliardo–Nirenberg and Young). Choosing ε such that $\varepsilon^p < 1 + \frac{\|a'\|_\infty}{m_1}$, we infer (2.43). Note that (2.43) immediately implies that $L_{1,p}$ is closed. Proceeding as in the proof of [44, Theorem 3.6], one shows that the numerical range $W(L_{1,p}) := \{\langle L_{1,p}u, u^*\rangle_{\mathrm{L}^p\times\mathrm{L}^{p'}} \mid u \in W^{2,p}(\mathbb{R}), \|u\|_p = 1, u^* := \overline{u}\,|u|^{p-2}\}$ satisfies

$$\overline{W(L_{1,p})} \subseteq \overline{S_{\omega_p}} \quad \text{with} \quad \omega_p := \arctan\left(\frac{m_2|p-2|}{2m_1\sqrt{p-1}}\right) \tag{2.44}$$

(there are subtleties to keep in mind, however: first, one may assume $u \in C_c^\infty(\mathbb{R})$ to establish (2.44) as the latter space is dense in $W^{2,p}(\mathbb{R})$; if $p \in (1,2)$, one needs to be a little bit more cautious because of possible singularities of $(u^*)'$. To circumvent this, one replaces u^* by $u_\varepsilon^* := (\varepsilon^2 + |u|^2)^{\frac{p-2}{2}}\overline{u}$ and then takes the limit $\varepsilon \to 0$). Now it suffices to show that $-1 \in \rho(L_{1,p})$, for then Theorem [44, 1.3.9] would imply that $L_{1,p}$ is sectorial of angle ω_p. First, note that (2.44) and Hölder's inequality in particular yield

$$\|(\mathrm{Id} + L_{1,p})u\|_p \geq \|u\|_p \quad (u \in W^{2,p}(\mathbb{R})), \tag{2.45}$$

which implies that $\mathrm{Id} + L_{1,p}$ is injective and has closed range. Thus, to prove $-1 \in \rho(L_{1,p})$, it remains to show that $\mathsf{R}(\mathrm{Id} + L_{1,p})$ is dense in $\mathrm{L}^p(\mathbb{R})$. To this end, we will show that $D := \mathrm{L}^2(\mathbb{R}) \cap \mathrm{L}^p(\mathbb{R})$ is contained in $\mathsf{R}(\mathrm{Id} + L_{1,p})$ and distinguish the cases $p \in (1,2)$ and $p \in (2,\infty)$.

Case 1: $p \in (1,2)$. Let $f \in D$. Then, $u := (\mathrm{Id} + L_{1,2})^{-1}f \in W^{2,2}(\mathbb{R})$ and in fact, u also belongs to $W^{1,p}(\mathbb{R})$ by [5, Theorem 2.4]. Now choose some cutoff $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi = 1$ on $B(0,1)$ and put $\varphi_n(\cdot) := \varphi(\frac{\cdot}{n}), u_n := u\varphi_n$ and $f_n := f\varphi_n$ for $n \in \mathbb{N}$. Then, Hölder's inequality implies that $(u_n)_n$ belongs to $W^{2,p}(\mathbb{R})$ since $p < 2$. Thus, (2.43) gives

$$\|u_n - u_m\|_{2,p} \lesssim \|L_{1,p}(u_n - u_m)\|_p + \|u_n - u_m\|_p \tag{2.46}$$

for all $m, n \in \mathbb{N}$. By the product rule, we have

$$\begin{aligned}&\|L_{1,p}(u_n - u_m)\|_p \\ &= \|(L_{1,2}u)(\varphi_n - \varphi_m) - 2au'(\varphi_n - \varphi_m)' + uL_{1,p}(\varphi_n - \varphi_m)\|_p \\ &\leq \|(L_{1,2}u)(\varphi_n - \varphi_m)\|_p + 2\|a\|_\infty\|\varphi\|_\infty\left(\tfrac{1}{n} + \tfrac{1}{m}\right)\|u'\|_p + \|u\|_p\|L_{1,p}(\varphi_n - \varphi_m)\|_\infty \\ &\lesssim \|f_n - f_m\|_p + \left(\tfrac{1}{n} + \tfrac{1}{m}\right)\|u\|_{1,p} + \|u_n - u_m\|_p\end{aligned}$$

and plugging this into (2.46), we infer

$$\|u_n - u_m\|_{2,p} \lesssim \|f_n - f_m\|_p + \Big(\tfrac{1}{n} + \tfrac{1}{m}\Big)\|u\|_{1,p} + \|u_n - u_m\|_p \tag{2.47}$$

for all $m, n \in \mathbb{N}$. It follows that $(u_n)_n$ is a Cauchy sequence in $W^{2,p}(\mathbb{R})$, thus there exists $v \in W^{2,p}(\mathbb{R})$ with $u_n \to v$ in $W^{2,p}(\mathbb{R})$ as $n \to \infty$. On the other hand, dominated convergence implies $u_n \to u$ in $W^{2,2}(\mathbb{R})$ as $n \to \infty$. Since L^q-convergence implies a.e. convergence along subsequences, we deduce $u = v \in W^{2,p}(\mathbb{R})$ with

$$(\mathrm{Id} + L_{1,p})u = (\mathrm{Id} + L_{1,2})u = f. \tag{2.48}$$

This shows that $D \subseteq \mathsf{R}(\lambda\mathrm{Id} + L_{1,p})$ as desired.
Case 2: $p \in (2, \infty)$. One proceeds exactly as in Case 1, with the difference that one uses mollifiers instead of cutoffs. Let $f \in D$. Once again, put $u := (\mathrm{Id}+L_{1,2})^{-1}f \in W^{2,2}(\mathbb{R}) \cap W^{1,p}(\mathbb{R})$. Choose $\varphi \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(\varphi) \subseteq B(0,1)$, $\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}x = 1$ and put $\varphi_n := n\varphi(n\cdot)$, $u_n = u * \varphi_n$ and $f_n * \varphi_n$ for $n \in \mathbb{N}$. Then, Young's convolution inequality implies $(u_n)_n \subseteq W^{2,p}(\mathbb{R})$ since $p > 2$. A short computation gives $Lu_n = f_n - u_n + r_n$ with

$$r_n(x) = \int_{\mathbb{R}} \frac{(a(x) - a(x-y))}{y} u'(x-y)\rho_n(y)\,\mathrm{d}y - (a'u_n')(x) \tag{2.49}$$

with $\rho_n(y) = n\rho(ny)$, $\rho(y) = -y\varphi'(y)$. Integration by parts, one checks that $\int \rho(y)\,\mathrm{d}y = 1$ and thus, (H2) implies that the integral in (2.49) converges to $a'u'$ in $\mathrm{L}^p(\mathbb{R})$ as $n \to \infty$. On the other hand, we also have $a'u_n' \to a'u'$ in $\mathrm{L}^p(\mathbb{R})$ as $u \in W^{1,p}(\mathbb{R})$. We conclude $r_n \to 0$ in $\mathrm{L}^p(\mathbb{R})$. Hence, (2.43) yields

$$\begin{aligned}\|u_n - u_m\|_{2,p} &\lesssim \|L_{1,p}u_n - L_{1,p}u_m\|_p + \|u_n - u_m\|_p \\ &\le \|f_n - f_m\|_p + \|u_n - u_m\|_p + \|r_n\|_p + \|r_m\|_p \to 0\end{aligned}$$

as $m, n \to \infty$. Now, we may argue exactly as in Case 1 to conclude that $u_n \to u \in W^{2,p}(\mathbb{R})$ with $(\mathrm{Id} + L_{1,p})u = f$. Finally, to prove consistency of resolvents, note that we actually proved $R(\lambda, L_{1,p})f = R(\lambda, L_{1,2})f$ for $f \in D$ and $\lambda = -1$. One can check that the proof extends verbatim to the case $\mathrm{Re}(\lambda) < 0$. □

Remark 2.2.3. With more work, one can even show that $\sigma(L_{1,p}) = \sigma(L_{2,p})$ and that the resolvents are consistent on $\mathbb{C} \setminus \sigma(L_{1,2})$ (see e.g. [43, Theorem 7.10]), but we do not need this here.

Theorem 2.2.4 (Kato-Property of L_p)**.** *Let* $p \in (1, \infty)$. *Then,* $\mathsf{Dom}(\sqrt{L_p}) = W^{1,p}(\mathbb{R})$ *and*

$$\|\sqrt{L_p}u\|_p \simeq \|\tfrac{\mathrm{d}}{\mathrm{d}x}u\|_p \quad u \in W^{1,p}(\mathbb{R}). \tag{2.50}$$

Proof. [4, Theorem 5.1] shows that

$$\begin{aligned} \{u \in W^{1,2}(\mathbb{R}) \mid \sqrt{L_2}u \in X_p\} &= \{u \in W^{1,2}(\mathbb{R}) \mid \tfrac{\mathrm{d}}{\mathrm{d}x}u \in X_p\}, \\ \|L_2 u\|_p &\simeq \|\tfrac{\mathrm{d}}{\mathrm{d}x}u\|_p \end{aligned} \tag{2.51}$$

Now, let $f \in \mathsf{Dom}(\sqrt{L_p})$. Since $\mathsf{Dom}(L_p)$ is a core for $\mathsf{Dom}(\sqrt{L_p})$ and $C_c^\infty(\mathbb{R})$ is dense in $\mathsf{Dom}(L_p) = W^{2,p}(\mathbb{R})$, we find a sequence $(f_n)_n$ in $C_c^\infty(\mathbb{R})$ with $f_n \to f$ in $\mathsf{Dom}(\sqrt{L_p})$ as $n \to \infty$. Since $C_c^\infty(\mathbb{R}) \subseteq X_2 \cap X_p$ and the resolvents are consistent by Proposition 2.2.2, we have $R(\lambda, L_p)f_n = R(\lambda, L_2)f_n$ for $\lambda \in \mathbb{C}_-$ and thus $\sqrt{L_p}f_n = \sqrt{L_2}f_n$. Now it follows from (2.51) that $(\sqrt{L_p}f_n)_n$ is a Cauchy sequence in $W^{1,p}(\mathbb{R})$. Thus, there exists $g \in W^{1,p}(\mathbb{R})$ with $f_n \to g$ in $W^{1,p}(\mathbb{R})$. In particular, $f_n \to f$ and $f_n \to g$ in $\mathrm{L}^p(\mathbb{R})$ so that $f = g$ almost everywhere. Hence, $f = g \in W^{1,p}(\mathbb{R})$. The converse inclusion is proved similarly. ☐

Since L_2 is self-adjoint w.r.t. the equivalent scalar product

$$\langle u, v\rangle_A := \langle A^{-1}u, v\rangle_{\mathrm{L}^2(\mathbb{R};\mathbb{C}^2)}, \quad \text{with} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix},$$

it follows from the Borel functional calculus for L_2 that $i\sqrt{L_2}$ generates a bounded C_0-group on $\mathrm{L}^2(\mathbb{R})$. For general $p \in (1, \infty)$, this is a more subtle issue. In order to prove that $i\sqrt{L_p}$ generates a bounded C_0-group, we use the following result, which is a corollary from the theory of abstract cosine functions on Banach spaces (see e.g. [2, Sections 3.14-3.16]). To state it, we need the notion of a *UMD-space*. One of the equivalent definitions is the following: A Banach space X is called a *UMD-space* if the X-valued Hilbert transform

$$(H_X f)(t) := \frac{1}{\pi}\, p.v. \int_{\mathbb{R}} \frac{f(s)}{t-s}\,\mathrm{d}s \quad (f \in \mathcal{S}(\mathbb{R}; X), t \in \mathbb{R}) \tag{2.52}$$

extends to a bounded operator on $\mathrm{L}^p(\mathbb{R}; X)$ for all (or equivalently one) $p \in (1, \infty)$.

Theorem 2.2.5 (Fattorini's Square Root Reduction)**.** *Let X be a UMD-space and L be a sectorial operator in X. If iD is the generator of a C_0-group $(\mathrm{e}^{itD})_{t\in\mathbb{R}}$ on X and if $L = D^2$, then $i\sqrt{L}$ generates a C_0-group $(\mathrm{e}^{it\sqrt{L}})_{t\in\mathbb{R}}$. Moreover, if we set*

$$\mathrm{Cos}(tD) := \frac{1}{2}\Big(\mathrm{e}^{itD} + \mathrm{e}^{-itD}\Big), \quad t\,\mathrm{Sinc}(tD)x := \int_0^t \mathrm{Cos}(sD)x\,\mathrm{d}s \quad (x \in X),$$

then $t \mapsto \mathrm{Sinc}(tD) \in C(\mathbb{R}; \mathsf{Dom}(\sqrt{L}))f$ and

$$\mathrm{e}^{it\sqrt{L}}f = \mathrm{Cos}(tD)f + it\sqrt{L}\mathrm{Sinc}(tD)f \tag{2.53}$$

for all $f \in X$ and $t \in \mathbb{R}$.

Proof. By assumption, $-L = D^2$ is the generator of the cosine function $(\mathrm{Cos}(tD))_{t\in\mathbb{R}}$. Now, the assertion follows immediately from [2, Theorem 3.16.7 and Proposition 3.16.3]. $\square$

Let $p \in [1,\infty)$ and recall that $X_p = \mathrm{L}^p(\mathbb{R};\mathbb{C}^2)$. We define the Dirac operator D_p by

$$D_p := \begin{pmatrix} 0 & -\frac{\mathrm{d}}{\mathrm{d}x} \\ a\frac{\mathrm{d}}{\mathrm{d}x} & 0 \end{pmatrix} \tag{2.54}$$

which we view as the closed unbounded operator

$$D_p \colon \mathsf{Dom}(D_p) \subseteq X_p \to X_p, \quad D_p f = \begin{pmatrix} -f_2' \\ a_j f_1' \end{pmatrix}.$$

with domain $\mathsf{Dom}(D_p) = W^{1,p}(\mathbb{R};\mathbb{C}^2)$. We observe that $D_p^2 = L_p$. Indeed, for $f \in \mathsf{Dom}(D_p^2) = W^{2,p}(\mathbb{R};\mathbb{C}^2) = \mathsf{Dom}(L_p)$ we have

$$D_p^2 f = \begin{pmatrix} 0 & -\frac{\mathrm{d}}{\mathrm{d}x} \\ a\frac{\mathrm{d}}{\mathrm{d}x} & 0 \end{pmatrix}\begin{pmatrix} 0 & -\frac{\mathrm{d}}{\mathrm{d}x} \\ a\frac{\mathrm{d}}{\mathrm{d}x} & 0 \end{pmatrix} f = \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}x}a\frac{\mathrm{d}}{\mathrm{d}x} & 0 \\ 0 & -a\frac{\mathrm{d}^2}{\mathrm{d}x^2} \end{pmatrix} = L_p f.$$

Thus, in order to use Theorem 2.2.5, we need to show that iD_p generates a C_0-group. To this end, we need the following lemma.

Lemma 2.2.6. *Let $p \in [1,\infty)$. Let $b \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R})$ be real-valued with $\|b\|_{\mathrm{L}^1(\mathbb{R})} < 1$. We consider the linear operators A and B in $X_p = \mathrm{L}^p(\mathbb{R};\mathbb{C}^2)$ given by*

$$A := \begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix} \quad \text{and} \quad B := b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with domains $\mathsf{Dom}(A) = W^{1,p}(\mathbb{R};\mathbb{C}^2)$ and $\mathsf{Dom}(B) = X_p$. Then, $C := A+B$ generates a bounded C_0-group $(\mathrm{e}^{tC})_{t\in\mathbb{R}}$ on X_p. Moreover, if $f \in \mathrm{L}^1(\mathbb{R};\mathbb{C}^2) \cap \mathrm{L}^p(\mathbb{R};\mathbb{C}^2)$, there is a measurable version of $\mathbb{R}^2 \to \mathbb{C}^2$, $(t,x) \mapsto (\mathrm{e}^{tC} f)(x)$ with

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}} |(\mathrm{e}^{tC} f)(x)|\,\mathrm{d}t \lesssim \|f\|_1. \tag{2.55}$$

Proof. Let $p \in [1,\infty)$. It is clear that A generates the bounded C_0-group $(e^{tA})_{t\in\mathbb{R}}$ on X_p given by

$$(\mathrm{e}^{tA} f)(x) := \begin{pmatrix} f_1(x+t) \\ f_2(x-t) \end{pmatrix} \quad \text{for a.e. } x \in \mathbb{R} \text{ and all } t \in \mathbb{R}, f = (f_1,f_2) \in X_p.$$

Moreover, we have $B \in \mathcal{L}(X_p)$ as $b \in \mathrm{L}^\infty(\mathbb{R})$ by assumption. Now, it follows from standard perturbation theory of C_0-(semi)groups that $C = A + B$ is the generator of a C_0-group $(\mathrm{e}^{tC})_{t\in\mathbb{R}}$ (see e.g. [16, Theorem III.1.3]).

However, the boundedness of $(\mathrm{e}^{tC})_{t\in\mathbb{R}}$ is a more subtle issue. It would be an immediate consequence of the Kato-Trotter product formula (see e.g. [16, Corollary III.5.8]) if both C_0-groups $(\mathrm{e}^{tA})_{t\in\mathbb{R}}$ and $(\mathrm{e}^{tB})_{t\in\mathbb{R}}$ were contractive. Unfortunately, contractivity even fails for $(\mathrm{e}^{tA})_{t\in\mathbb{R}}$ if $p \neq 2$. In fact, we have

$$\|\mathrm{e}^{tA}\|_{\mathcal{L}(X_p)} = 2^{|1/p-1/2|} \quad \text{for all } t \neq 0. \tag{2.56}$$

Therefore, we adopt a different approach and show the boundedness of $(\mathrm{e}^{tC})_{t\in\mathbb{R}}$ using the Dyson-Phillips series: We define the strongly continuous operator family $(S_n(t))_{t\in\mathbb{R}}$ in $\mathcal{L}(X_p)$ by

$$S_0(t)f := \mathrm{e}^{tA}f, \quad S_{n+1}(t)f := \int_0^t \mathrm{e}^{(t-s)A}BS_n(s)f\,\mathrm{d}s \tag{2.57}$$

for all $t \in \mathbb{R}$, $n \in \mathbb{N}_0$, and $f \in X_p$. As for each $t \in \mathbb{R}$, we have

$$\sum_{n=0}^{N} S_n(t) \to \mathrm{e}^{tC} \quad (N \to \infty) \quad \text{in } \mathcal{L}(X_p)$$

(see e.g. [16, Theorem III.1.10]), it suffices to show for some $M_p > 0$ that

$$\sum_{n=0}^{\infty} \|S_n(t)\|_{\mathcal{L}(X_p)} \leq M_p \quad \text{for all } t \in \mathbb{R}. \tag{2.58}$$

We consider first the case $p = 1$ and show (2.58) by finding first an explicit representation of $(S_n(t))_{n\in\mathbb{N}}$. To this end, we define $b_t \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R})$ by $b_t(x) = b(x+t)$ for $t \in \mathbb{R}$ and a.e. $x \in \mathbb{R}$. Let $f = (f_1, f_2) \in X_1$. Then, induction and a tedious calculation shows that

$$\Big(S_n(t)f\Big)(x) = E^n \begin{pmatrix} \frac{1}{2}\int_{x-t}^{x+t} K_n^+(t,r,x)f_1(r)\,\mathrm{d}r \\ \frac{1}{2}\int_{x-t}^{x+t} K_n^-(t,r,x)f_2(r)\,\mathrm{d}r \end{pmatrix} \tag{2.59}$$

for all $t \in \mathbb{R}, n \in \mathbb{N}$ and $x \in \mathbb{R}$, where $E \in \mathcal{L}(X_1)$ is the isometry defined by

$$Ef := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} f = \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix}$$

and the kernels $K_n^\pm \colon \mathbb{R}^3 \to \mathbb{R}$ are functions of the form

$$K_n^+(t,r,x) = k_n^+\Big(\tfrac{r+t-x}{2}, \tfrac{r-t-x}{2}, x\Big), \quad K_n^-(t,r,x) = k_n^-\Big(\tfrac{x+t-r}{2}, \tfrac{x-t-r}{2}, x\Big)$$

with $k_n(u,v,\cdot) := k_n^+(u,v,\cdot)$ given by

$$k_1(u,v,\cdot) := b_v, \quad k_2(u,v,\cdot) := \int_0^u b_{v+s_1}b_{s_1}\,\mathrm{d}s_1,$$

$$k_3(u,v,\cdot) := \int_v^0 \int_0^u b_{v+s_1}b_{s_1+s_2}b_{s_2}\,\mathrm{d}s_1\,\mathrm{d}s_2,$$

$$k_4(u,v,\cdot) := \int_0^u \int_v^0 \int_{s_3}^u b_{v+s_1}b_{s_1+s_2}b_{s_2+s_3}\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}s_3$$

and for odd $n \geq 5$

$$k_n(u,v,\cdot) := \int_v^0 \int_0^u \int_v^{s_{n-1}} \int_{s_{n-2}}^u \cdots \int_v^{s_4} \int_{s_3}^u b_{v+s_1} b_{s_1+s_2} b_{s_2+s_3} \cdots \\ b_{s_{n-1}+s_{n-2}} b_{s_{n-1}} \, \mathrm{d}s_1 \cdots \mathrm{d}s_{n-1}.$$

and similarly for even $n \geq 6$,

$$k_n(u,v,\cdot) := \int_0^u \int_v^0 \int_{s_{n-1}}^u \int_v^{s_{n-2}} \cdots \int_v^{s_4} \int_{s_3}^u b_{v+s_1} b_{s_1+s_2} b_{s_2+s_3} \cdots \\ b_{s_{n-1}+s_{n-2}} b_{s_{n-1}} \, \mathrm{d}s_1 \cdots \mathrm{d}s_{n-1}.$$

The function $k_n^-(u,v,\cdot)$ is defined by the same formula as $k_n(u,v,\cdot)$, with every occurrence of b_τ replaced by $b_{-\tau}$ in the above integrals. Now, observe that for each $t, r \in \mathbb{R}$,

$$\|K_n^{\pm}(t,r,\cdot)\|_{\mathrm{L}^1(\mathbb{R})} \leq 2\|b\|_1^n \quad (n \in \mathbb{N}). \tag{2.60}$$

Indeed, for instance, in the case where $n \geq 6$ is even, we have for $K_n = K_n^+$ and every $t, r \in \mathbb{R}$

$$\begin{aligned} &\int_{\mathbb{R}} |K_n(t,r,x)| \, \mathrm{d}x \\ &\leq \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} |b_{\frac{r-t-x}{2}+s_1} b_{s_1+s_2} \cdots b_{s_{n-1}+s_{n-2}} b_{s_{n-1}}|(x) \, \mathrm{d}x \, \mathrm{d}s_1 \cdots \mathrm{d}s_{n-1} \\ &= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} |b(\tfrac{r-t+x}{2} + s_1) b(s_1+s_2) \cdots b(s_{n-1}+s_{n-2}) b(s_{n-1})| \, \mathrm{d}x \, \mathrm{d}s_1 \cdots \mathrm{d}s_{n-1} \\ &= 2\|b\|_1 \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} b(s_1+s_2) \cdots b(s_{n-1}+s_{n-2}) b(s_{n-1})| \, \mathrm{d}s_1 \cdots \mathrm{d}s_{n-1} = 2\|b\|_1^n, \end{aligned}$$

where we used the changes of variables $s_1 \mapsto s_1 + x$, $s_3 \mapsto s_3 + x, \ldots, s_{n-1} \mapsto s_{n-1} + x$ in the third line. The other cases can be proved similarly. Therefore, we obtain from (2.59) and (2.60)

$$\begin{aligned} &\|S_n(t)f\|_{X_1} \\ &\leq \frac{1}{2}\left(\left\| \int_{\mathbb{R}} K_n^+(t,r,\cdot) f_1(r) \, \mathrm{d}r \right\|_1 + \left\| \int_{\mathbb{R}} K_n^-(t,r,\cdot) f_2(r) \, \mathrm{d}r \right\|_1 \right) \\ &\leq \frac{1}{2}\left(\int_{\mathbb{R}} \|K_n^+(t,r,\cdot)\|_1 |f_1(r)| \, \mathrm{d}r + \left\| \int_{\mathbb{R}} \|K_n^-(t,r,\cdot)\|_1 |f_2(r)| \, \mathrm{d}r \right) \\ &\leq \|b\|_1^n (\|f_1\|_1 + \|f_2\|_1) \leq 2^{\frac{1}{2}} \|b\|_1^n \|f\|_{X_1} \end{aligned} \tag{2.61}$$

for all $n \in \mathbb{N}$. This proves $\|S_n(t)\|_{\mathcal{L}(X_1)} \leq 2^{\frac{1}{2}} \|b\|_1^n$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$. For $n = 0$, this estimate is also true by (2.56). Thus, since $\|b\|_1 < 1$,

$$\|\mathrm{e}^{tC}\|_{\mathcal{L}(X_1)} \leq \sum_{n=0}^{\infty} \|S_n(t)\|_{\mathcal{L}(X_1)} \leq \frac{2^{1/2}}{1 - \|b\|_{\mathrm{L}^1(\mathbb{R})}} =: M_1 \quad (t \in \mathbb{R}), \tag{2.62}$$

which proves the boundedness of $(\mathrm{e}^{tC})_{t\in\mathbb{R}}$ in the case $p = 1$. To prove the assertion for general $p \in (1,\infty)$, let us be more precise and write C_p for the X_p-realization of the operator C. For $p = 2$, it is readily checked that $\tilde{C} := -iC_2$ is a self-adjoint operator in the Hilbert space X_2 and thus $\|\mathrm{e}^{tC_2}\|_{\mathcal{L}(X_2)} = \|\mathrm{e}^{it\tilde{C}}\|_{\mathcal{L}(X_2)} = 1$ for all $t \in \mathbb{R}$ by Stone's theorem (here, $\mathrm{e}^{it\tilde{C}}$ is defined by the Borel functional calculus for $\tilde{C}$). Now, for general $p \in (1,\infty)$, note that it follows from (2.57) and (2.62) that

$$\mathrm{e}^{tC_p} f = \mathrm{e}^{tC_q} f \quad \text{for all } 1 \le p \le q < \infty,\ f \in X_p \cap X_q.$$

Hence, the boundedness of $(\mathrm{e}^{tC_p})_{t\in\mathbb{R}}$ for $p \in (1,2)$ follows by interpolating the cases $p_0 = 1$ and $p_1 = 2$. The case $p \in (2,\infty)$ follows from a duality argument, noting that $(C_p)' = -C_{p'}$ and thus $(\mathrm{e}^{tC_p})' = \mathrm{e}^{-tC_{p'}}$ for $t \in \mathbb{R}$ and $p' \in (1,2)$ being the Hölder conjugate of p.
Finally, we prove (2.55). Suppose first that $f \in \mathrm{L}^1(\mathbb{R};\mathbb{C}^2) \cap W^{1,p}(\mathbb{R};\mathbb{C}^2)$. Since $\mathsf{Dom}(C) = \mathsf{Dom}(A) = W^{1,p}(\mathbb{R};\mathbb{C}^2)$, we have that $t \mapsto \mathrm{e}^{tC} f$ belongs to $C(\mathbb{R}; W^{1,p}(\mathbb{R};\mathbb{C}^2))$ and therefore also to $C(\mathbb{R}; C_b(\mathbb{R};\mathbb{C}^2))$ by Sobolev embedding. In particular, $\mathbb{R}^2 \to \mathbb{C}^2$, $(t,x) \mapsto (\mathrm{e}^{tC} f)(x)$ is well-defined and continuous. Now, note that a similar reasoning as in the proof of (2.60) yields for $n \in \mathbb{N}$ the bounds

$$|K_n^{\pm}(t,r,x)| \le 2\|b\|_1^{n-1}\|b\|_\infty, \tag{2.63}$$

$$\int_{\mathbb{R}} |K_n^{\pm}(t,r,x)| \,\mathrm{d}t \le 2\|b\|_1^{n}. \tag{2.64}$$

The bound (2.63) together with (2.62) imply that $\sum_{k=0}^{n} S_k(t) f$ converges uniformly to $\mathrm{e}^{tC} f$ as $n \to \infty$. The bound (2.64) implies $\int_{\mathbb{R}} |S_n(t) f(x)| \,\mathrm{d}t \lesssim \|b\|_1^n \|f\|_1$ and thus, we conclude

$$\begin{aligned} \int_{\mathbb{R}} |\mathrm{e}^{tC} f(x)| \,\mathrm{d}t &\le \int_{\mathbb{R}} |S_0(t) f(x)| \,\mathrm{d}t + \sum_{k=0}^{\infty} \int_{\mathbb{R}} |(S_k(t) f(x)| \,\mathrm{d}t \\ &\lesssim \|f\|_1 \sum_{k=1}^{\infty} \|b\|_1^{k-1}\|b\| \lesssim \|f\|_1 \lesssim \|f\|_1. \end{aligned}$$

uniformly in $x \in \mathbb{R}^d$. For general $f \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{L}^p(\mathbb{R};\mathbb{C}^2)$, one argues by approximation using [30, Propositions 1.2.24, 1.2.25]The proof is complete. □

Armed with Lemma 2.2.6, we are now in the position to prove that iD_p generates a bounded C_0-group on X_p for all $p \in [1,\infty)$.

Proposition 2.2.7 (Boundedness of $(\mathrm{e}^{itD_p})_{t\in\mathbb{R}}$ on L^p and $\mathrm{L}_x^\infty L_t^1$-Estimates)**.** *Let $p \in [1,\infty)$ and $X_p = \mathrm{L}^p(\mathbb{R};\mathbb{C}^2)$. Then, the operator iD_p generates a bounded C_0-group $(\mathrm{e}^{itD_p})_{t\in\mathbb{R}}$ on X_p with $\mathrm{e}^{itD_p} f = \mathrm{e}^{itD_2} f$ for $f \in X_p \cap X_2$.*

Moreover, for all $f \in X_1 \cap X_p$, *there is a measurable version of* $\mathbb{R}^2 \to \mathbb{C}^2$, $(t,x) \mapsto (\mathrm{e}^{itD_p} f)(x)$ *with*

$$\operatorname*{ess\,sup}_{x\in\mathbb{R}} \int_{\mathbb{R}} |(\mathrm{e}^{itD_p} f)(x)| \,\mathrm{d}t \lesssim \|f\|_1. \tag{2.65}$$

Proof. Let $p \in [1,\infty)$. We show that iD_p is isomorphically equivalent to a generator of a bounded C_0-group. To ease notation, we just write D in place of D_p in the following. We first observe that $iD = M\frac{\mathrm{d}}{\mathrm{d}x}$, where

$$M = M(x) = i\begin{pmatrix} 0 & -1 \\ a(x) & 0 \end{pmatrix} \quad (x \in \mathbb{R})$$

has eigenvalues $\lambda_{1/2} = \pm i\sqrt{a(x)}$. We may therefore diagonalize M according to

$$U^{-1}MU = \sqrt{a(x)}\,\Lambda, \quad U = U(x) = \begin{pmatrix} 1 & 1 \\ i\sqrt{a(x)} & -i\sqrt{a(x)} \end{pmatrix},$$

where $\Lambda = \operatorname{diag}(1,-1)$. By (H1), the linear operator $T_U \colon X_p \to X_p$, $T_U f = Uf$, associated to U defines an isomorphism and using (H2), we may conjugate iD by T_U, which yields for $f \in \mathsf{Dom}(D)$

$$\begin{aligned} C_1 f \coloneqq T_U^{-1} iDT_U f = U^{-1}M\tfrac{\mathrm{d}}{\mathrm{d}x}Uf &= U^{-1}MU\tfrac{\mathrm{d}}{\mathrm{d}x}f + U^{-1}M(\tfrac{\mathrm{d}}{\mathrm{d}x}U)f \\ &= \Lambda\Big(\sqrt{a}\tfrac{\mathrm{d}}{\mathrm{d}x}\Big)f + U^{-1}M(\tfrac{\mathrm{d}}{\mathrm{d}x}U)f \\ &= \Lambda\Big(\sqrt{a}\tfrac{\mathrm{d}}{\mathrm{d}x}\Big)f + \tilde{b}(x)Vf \end{aligned} \tag{2.66}$$

with

$$V = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad \tilde{b}(x) = \tfrac{(\sqrt{a})'(x)}{2} \quad (x \in \mathbb{R}).$$

Next, we define a transformation that maps the vector field $\sqrt{a(x)}\frac{\mathrm{d}}{\mathrm{d}x}$ to $\frac{\mathrm{d}}{\mathrm{d}x}$. To this end, put

$$T_\varphi f \coloneqq f \circ \varphi, \text{ where } \varphi(x) \coloneqq \int_0^x a^{-\frac{1}{2}}(y)\,\mathrm{d}y, \quad x \in \mathbb{R}.$$

It follows again from (H1) that T_φ defines an isomorphism on X_p. Then, since $\sqrt{a(x)}\frac{\mathrm{d}}{\mathrm{d}x}T_\varphi f = T_\varphi \frac{\mathrm{d}}{\mathrm{d}x} f$, we get from (2.66)

$$\begin{aligned} C_2 f \coloneqq T_{\varphi^{-1}} C_1 T_\varphi f &= \Lambda f + b(x)Vf \\ &= \Lambda(\tfrac{\mathrm{d}}{\mathrm{d}x} + b(x)) + b(x)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad b \coloneqq \tilde{b}\circ\varphi^{-1}. \end{aligned} \tag{2.67}$$

Finally, we define a transformation that maps $\frac{\mathrm{d}}{\mathrm{d}x} + b(x)$ to $\frac{\mathrm{d}}{\mathrm{d}x}$. To this end, we let

$$T_m f \coloneqq mf, \quad m(x) \coloneqq a^{-\frac{1}{4}}(\varphi^{-1}(x)) \quad (x \in \mathbb{R}).$$

Again, (H1) implies that the multiplication operator T_m is an isomorphism on X_p. Then, we finally obtain

$$Cf \coloneqq T_{m^{-1}} C_2 T_m f = \Lambda \tfrac{\mathrm{d}}{\mathrm{d}x} + b(x) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$= A + B,$$

where

$$A \coloneqq \Lambda \tfrac{\mathrm{d}}{\mathrm{d}x} = \begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix} \quad \text{and} \quad B \coloneqq b \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with domains $\mathsf{Dom}(A) = W^{1,p}(\mathbb{R};\mathbb{C}^2)$ and $\mathsf{Dom}(B) = X_p$. Changing variables $x = \varphi(y)$, we have

$$\|b\|_{\mathrm{L}^1(\mathbb{R})} = \frac{1}{4} \int_{\mathbb{R}} \frac{|a'(y)|}{|a(y)|} \,\mathrm{d}y = \frac{1}{4} \Big\| \tfrac{\mathrm{d}}{\mathrm{d}x} \log(a) \Big\|_{\mathrm{L}^1(\mathbb{R})} < 1$$

by (H3). It now follows from Lemma 2.2.6 that C is the generator of a bounded C_0-group $(\mathrm{e}^{tC})_{t\in\mathbb{R}}$. But this means that iD is the generator of the C_0-group given by

$$\mathrm{e}^{itD} = T \mathrm{e}^{tC} T^{-1} \quad \text{for all } t \in \mathbb{R} \tag{2.68}$$

with $T \coloneqq T_m T_\varphi T_U$. Finally, in view of (H1) and (H2), it is clear that T is also an isomorphism on $\mathrm{L}^\infty(\mathbb{R};\mathbb{C}^2)$. Thus, (2.65) follows from (2.55) and (2.68). □

Theorem 2.2.8 (Boundedness of $(\mathrm{e}^{it\sqrt{L_p}})_{t\in\mathbb{R}}$ on L^p and $\mathrm{L}^\infty_x L^1_t$-Estimates)**.** *Let $p \in (1,\infty)$ and $X_p = \mathrm{L}^p(\mathbb{R};\mathbb{C}^2)$. Then, $i\sqrt{L_p}$ has domain $W^{1,p}(\mathbb{R};\mathbb{C}^2)$ and generates a bounded C_0-group $(\mathrm{e}^{it\sqrt{L_p}})_{t\in\mathbb{R}}$ on X_p with $\mathrm{e}^{it\sqrt{L_p}} f = \mathrm{e}^{it\sqrt{L_2}} f$ for all $f \in X_p \cap X_2$. Moreover, for all $f \in \mathrm{L}^1(\mathbb{R};\mathbb{C}^2) \cap \mathrm{L}^p(\mathbb{R};\mathbb{C}^2)$, there is a measurable version of $\mathbb{R}^2 \to \mathbb{C}^2$, $(t,x) \mapsto (\mathrm{e}^{it\sqrt{L_p}} f)(x)$ with*

$$\|[\mathrm{Cos}(t\sqrt{L_p})f](x)\|_{\mathrm{L}^\infty_x(\mathbb{R};\mathrm{L}^1_t(\mathbb{R}))} \lesssim \|f\|_1. \tag{2.69}$$

Proof. Let $p \in (1,\infty)$. We already proved $\mathsf{Dom}(\sqrt{L_p}) = W^{1,p}(\mathbb{R};\mathbb{C}^2)$ in Theorem 2.2.4. It is a classical fact in harmonic analysis that X_p is a UMD-space (since $p \in (1,\infty)$). Therefore, Proposition 2.2.7 and Theorem 2.2.5 imply that $i\sqrt{L_p}$ generates a C_0-group $(\mathrm{e}^{it\sqrt{L_p}})_{t\in\mathbb{R}}$. To prove that $(\mathrm{e}^{it\sqrt{L_p}})_{t\in\mathbb{R}}$ is bounded, let $f \in X_p$ and $t \in \mathbb{R}$. We use (2.53) to estimate

$$\|\mathrm{e}^{it\sqrt{L_p}} f\|_p \le \|\mathrm{Cos}(tD_p)f\|_p + \|\sqrt{L_p}\mathrm{Sinc}(tD_p)f\|_p.$$

Since $(\mathrm{e}^{isD_p})_{s\in\mathbb{R}}$ is bounded, we have

$$\|\mathrm{Cos}(tD_p)f\|_p \le \frac{1}{2}\Big(\|\mathrm{e}^{itD_p}f\|_p + \|\mathrm{e}^{-itD_p}f\|_p\Big) \lesssim_p \frac{1}{2}(\|f\|_p + \|f\|_p) = \|f\|_p.$$

On the other hand, we have by Theorem 2.2.4 and (H1),

$$\|\sqrt{L_p}\mathrm{Sinc}(tD_p)f\|_p \simeq \|\tfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{Sinc}(tD_p)f\|_p \simeq \|D_p\mathrm{Sinc}(tD_p)f\|_p$$

and further by the theory of C_0-(semi)groups

$$\begin{aligned}\|D_p\mathrm{Sinc}(tD_p)f\|_p &= \frac{1}{2}\left\|D_p\int_0^t \mathrm{e}^{isD_p}f\,\mathrm{d}s + D_p\int_0^t \mathrm{e}^{-isD_p}f\,\mathrm{d}s\right\|_p \\ &= \frac{1}{2}\left\|\mathrm{e}^{itD_p}f - f + \mathrm{e}^{-itD_p}f - f\right\|_p \\ &\leq \frac{1}{2}\Big(\|\mathrm{e}^{itD_p}f\|_p + \|f\|_p + \|\mathrm{e}^{-itD_p}f\|_p + \|f\|_p\Big) \lesssim \|f\|_p\end{aligned}$$

by the boundedness of $(\mathrm{e}^{isD_p})_{s\in\mathbb{R}}$. Finally, (2.69) follows immediately from (2.65) in view of the identity $\mathrm{Cos}(t\sqrt{L_p}) = \mathrm{Cos}(tD_p) = \frac{1}{2}(\mathrm{e}^{itD_p} + \mathrm{e}^{-itD_p})$ for all $t \in \mathbb{R}$. The proof is complete. □

Remark 2.2.9. (1) Let $p \in \{1, \infty\}$. Then, we cannot expect $i\sqrt{L_p}$ to generate a C_0-group on X_p. In the simplest case $a = 1$, the operator $L_2 = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ generates the bounded C_0-group $\mathrm{e}^{it\sqrt{L_2}} = \mathcal{F}^{-1}M_{m_t}\mathcal{F}$ on $\mathrm{L}^2(\mathbb{R})$, where m_t is the Fourier multiplier given by $m_t(\xi) := \mathrm{e}^{it|\xi|}$ ($\xi \in \mathbb{R}$). However, m_t is not a Fourier multiplier on $\mathrm{L}^p(\mathbb{R})$ for each fixed $t \neq 0$. Indeed, by dilation invariance, m_t is a L^p-Fourier multiplier if and only if m_1 is a L^p-Fourier multiplier. Approximating m_1 by $m_{1,\varepsilon}(\xi) := \mathrm{e}^{-(\varepsilon-i)|\xi|}$ ($\varepsilon > 0$), we arrive at

$$\mathcal{F}^{-1}(m_1) = \frac{1}{2}\Big(\delta_1 + \delta_{-1}\Big) + i\left(p.v.\frac{1}{x+1} + p.v.\frac{1}{x-1}\right) \quad \text{in} \quad \mathcal{S}'(\mathbb{R}^d)$$

which is *not* a finite Borel measure. By [24, Theorem 2.5.8], $\mathrm{e}^{i\sqrt{-\partial_x^2}}$ does not extend to a bounded operator on $\mathrm{L}^1(\mathbb{R})$. By duality, the same holds for $\mathrm{L}^\infty(\mathbb{R})$.

(2) We would also like to stress that the presented approach via Fattorini's theorem is limited to dimension one. It hinges only on cf. Lemma 2.2.6 which uses that wave equations in dimension one are non-dispersive and essentially transport equations; and in fact for $p \in (1, \infty)$, it is well-known that $\mathrm{e}^{it\sqrt{-\Delta_x}}$ is a bounded operator on L^p if and only if $p = 2$ or $d = 1$.

(3) In [21, Proposition 2] it was already proven under (H1) and (H2) that for each $p \in (1, \infty)$, the operator $i\sqrt{L_p}$ is the generator of a C_0-group. We made an effort to prove that this C_0-group is indeed *bounded*, and this is why we needed ((H3)). The boundedness of $(\mathrm{e}^{it\sqrt{L_p}})_{t\in\mathbb{R}}$ is important to us because a growth bound of type $\|\mathrm{e}^{it\sqrt{L_p}}\| \leq M\mathrm{e}^{\omega|t|}$ would

restrict the class of functions φ for which $\varphi(\sqrt{L_p})$ in the sense of the Phillips functional calculus (see Definition 2.1.8) renders a well-defined bounded operator. On the other hand, (H3) also allows us to prove the estimate (2.69) for $\mathrm{Cos}(t\sqrt{L_p})$, which was already known for its first component $\mathrm{Cos}(t\sqrt{L_{1,p}})$ for more general a (see e.g. [8, Theorem 1.1]) but (at least to the author) unclear for its second component $\mathrm{Cos}(t\sqrt{L_{2,p}})$.

2.2.2. A Construction of a d-Parameter C_0-Group

Suppose now that $\tilde{a}_1, \ldots, \tilde{a}_d \colon \mathbb{R} \to \mathbb{R}$ are functions which satisfy (H1), (H2), and (H3). Let $p \in (1, \infty)$. By Theorem 2.2.8, we obtain for each a_j a corresponding bounded half-wave C_0-group on $X_p = \mathrm{L}^p(\mathbb{R}; \mathbb{C}^2)$. From these d (one-parameter) bounded C_0-groups, we want to construct a bounded d-parameter C_0-group on $Z_p := \mathrm{L}^p(\mathbb{R}^d; \mathbb{C}^2)$. To this end, we lift $\tilde{a}_1, \ldots, \tilde{a}_d$ to functions $a_1, \ldots, a_d$ on $\mathbb{R}^d$ by setting

$$a_j(x) = \tilde{a}_j(x_j) \quad x = (x_1, \ldots, x_d), j \in \{1, \ldots, d\}.$$

For $j \in \{1, \ldots, d\}$, we define the operator $L_j := L_{j,p}$ by

$$L_j = \begin{pmatrix} D_j a_j D_j & 0 \\ 0 & a_j D_j^2 \end{pmatrix} \quad \text{with domain}$$
$$\mathsf{Dom}(L_j) = \{u \in Z_p \mid D_j^k u \in Z_p \text{ for } k \in \{0,1,2\}\}.$$

As in the case $d = 1$, one can show that L_j is sectorial and that

$$\mathsf{Dom}(\sqrt{L_j}) = \{u \in Z_p \mid \partial_j u \in Z_p\},$$
$$\|\sqrt{L_j} u\|_p \simeq \|\partial_j u\|_p.$$

Corollary 2.2.10. *Let $p \in (1, \infty)$, $d \geq 2$ and $Z_p = \mathrm{L}^p(\mathbb{R}^d; \mathbb{C}^2)$. Then, the operator $i\sqrt{\mathbf{L}_p} := i(\sqrt{L_{1,p}}, \ldots, \sqrt{L_{d,p}})$ generates a bounded d-parameter C_0-group $(\mathrm{e}^{iy\cdot\sqrt{\mathbf{L}_p}})_{t\in\mathbb{R}}$ on Z_p with $\mathrm{e}^{iy\cdot\sqrt{\mathbf{L}_p}} f = \mathrm{e}^{iy\cdot\sqrt{\mathbf{L}_2}} f$ for $f \in X_2 \cap X_p$. Moreover, for all $f \in \mathrm{L}^1(\mathbb{R}^d; \mathbb{C}^2) \cap \mathrm{L}^p(\mathbb{R}^d; \mathbb{C}^2)$, we have*

$$\|\mathrm{Cos}(y\sqrt{\mathbf{L}_p} f)(x)\|_{\mathrm{L}^\infty_x(\mathbb{R}^d; \mathrm{L}^1_y(\mathbb{R}^d))} \lesssim \|f\|_1, \tag{2.70}$$

where $\mathrm{Cos}(y\sqrt{\mathbf{L}_p}) := \prod_{j=1}^d \mathrm{Cos}(y_j\sqrt{L_{j,p}})$.

Proof. That L_j generates a bounded C_0-group $(\mathrm{e}^{it\sqrt{L_j}})_{t\in\mathbb{R}}$ is a consequence of Theorem 2.2.8 and Lemma 2.2.11 below. Note that $L_j L_k = L_k L_j$ for all $j, k \in \{1, \ldots, d\}$ and thus by functional calculus, the same holds for their square roots. This implies that their C_0-groups are commuting, too (by the Euler exponential formula, for instance). Finally, (2.70) follows from Lemma 2.2.11, (2.69), and Fubini's theorem. □

Lemma 2.2.11 (Lifting of C_0-Groups on L^p-spaces)**.** *Let $p \in [1,\infty)$, $d \geq 2$ and $X := \mathrm{L}^p(\mathbb{R};\mathbb{C}^2)$, $Y := \mathrm{L}^p(\mathbb{R}^{d-1};\mathbb{C}^2)$ and $Z = \mathrm{L}^p(\mathbb{R}^d;\mathbb{C}^2)$. Suppose that iA is the generator of a C_0-group $(T(t))_{t\in\mathbb{R}}$ on X. Then, the operator $\overline{iA \otimes 0_Y}$ generates a C_0-group $(S(t))_{t\in\mathbb{R}}$ on Z with $\|S(t)\|_{\mathcal{L}(Z)} = \|T(t)\|_{\mathcal{L}(X)}$ for all $t \in \mathbb{R}$.*

Proof. Let $\mathcal{D} := C_c^\infty(\mathbb{R})$, $Y_0 := \{\mathbb{1}_M | M \in \mathcal{B}(\mathbb{R}^{d-1}), |M| < \infty\} \subseteq Y$ and put

$$Z_0 := \Big\{ \sum_{k=1}^n \varphi_k(A) f_k \otimes g_k \mid n \in \mathbb{N}, \varphi_k \in \mathcal{D}, f_k \in X, g_k \in Y_0 \Big\} \subseteq Z,$$

where $\varphi_k(A) \in \mathcal{L}(X)$ is defined by the Phillips functional calculus for A. Note that Z_0 is dense in Z. Indeed, $Z = \overline{X \otimes Y_0}$ by standard integration theory and $X \otimes Y_0 = \overline{Z_0}$ by Lemma 2.1.12. Let $t \in \mathbb{R}$ and $h \in Z_0$. Then, $h = \sum_{k=1}^n \varphi_k(A) f_k \otimes g_k$ for some $n \in \mathbb{N}$, $\varphi_k \in \mathcal{D}$, $f_k \in X$ and $g_k \in Y_0$. Writing $\mathrm{e}^{itA} := T(t)$, we set

$$S_0(t)h := \sum_{k=1}^n (\mathrm{e}^{itA} \varphi_k(A) f_k) \otimes g_k.$$

Proceeding as in the proof of [15, Section 4.1, Lemma 1.1], one shows that $S(t)f$ does not depend on the representation of h. By Proposition 2.1.10 (a), we have for $\varphi \in \mathcal{D}$ that $\mathrm{e}^{itA}\varphi(A) = \varphi_t(A)$ with $\varphi_t := \mathrm{e}^{it\cdot}\varphi \in \mathcal{D}$. This implies that $S_0(t)$ leaves Z_0 invariant and thus, $S_0(t)\colon Z_0 \to Z_0$ is a well-defined linear operator. To estimate $\|S_0(t)h\|_Z$, we first note that we may assume without restriction that the supports of the g_k are disjoint. Then, by Fubini's theorem,

$$\begin{aligned} \|S_0(t)h\|_Z^p &= \sum_{k=1}^n \|\mathrm{e}^{itA}\varphi_k(A) f_k\|_X^p \|g_k\|_Y^p \\ &\leq \|\mathrm{e}^{itA}\|_{\mathcal{L}(X)}^p \sum_{k=1}^n \|\varphi_k(A) f_k\|_X^p \|g_k\|_Y^p = \|\mathrm{e}^{itA}\|_{\mathcal{L}(X)}^p \|h\|_Z^p, \end{aligned}$$

which shows that $S_0(t)$ is bounded with $\|S(t)\|_{\mathcal{L}(Z)} \leq \|\mathrm{e}^{itA}\|_{\mathcal{L}(X)}$. Since Z_0 is dense in Z, there is a unique extension $S(t) \in \mathcal{L}(Z)$ of $S_0(t)$ with $\|S(t)\|_{\mathcal{L}(Z)} \leq \|\mathrm{e}^{itA}\|_{\mathcal{L}(X)}$. Note that the properties of a C_0-group translate from $(\mathrm{e}^{itA})_{t\in\mathbb{R}}$ to $(S_0(t))_{t\in\mathbb{R}}$ in a straightforward fashion. By a standard density argument and the fact that $\|S(t)\|_{\mathcal{L}(Z)} \leq \|\mathrm{e}^{itA}\|_{\mathcal{L}(X)}$, these properties extend to $(S(t))_{t\in\mathbb{R}}$. We conclude that $(S(t))_{t\in\mathbb{R}}$ defines a C_0-group on Z. Choose $\varphi \in \mathcal{D}$ with $\varphi(0) = 1$. Then, Lemma 2.1.12 and the density of $\operatorname{span}(Y_0)$ in Y imply for $h = \sum_{k=1}^n f_k \otimes g_k \in X \otimes Y$

$$S(t)h = \lim_{t\to 0} \sum_{k=1}^n (\mathrm{e}^{itA}\varphi(tD) f_k) \otimes g_k = \sum_{k=1}^n (\mathrm{e}^{itA} f_k) \otimes g_k. \tag{2.71}$$

Choosing $h = f \otimes g \in X \otimes Y$ with $\|g\|_Y = 1$, we easily deduce from (2.71) that $\|S(t)\|_{\mathcal{L}(Z)} \geq \|e^{itA}\|_{\mathcal{L}(X)}$ and thus $\|S(t)\|_{\mathcal{L}(Z)} = \|e^{itA}\|_{\mathcal{L}(X)}$ for all $t \in \mathbb{R}$. It remains to identify the generator of $(S(t))_{t\in\mathbb{R}}$, which we denote by B. We deduce from Proposition 2.1.10 (c) and (2.71) that $B_0 \subseteq iA \otimes 0_Y \subseteq B$, where $B_0 \coloneqq i(A \otimes 0_Y)|_{Z_0}$. On the other hand, since Z_0 is dense in Z and $S(t)$-invariant, it is in fact a core for $\mathsf{Dom}(B)$ by [16, Proposition II.1.7]. But this means precisely that $\overline{B}_0 = B$ and since B is closed (being a generator of a C_0-group), it follows that $\overline{B_0} = \overline{A \otimes 0_Y} = B$. The proof is complete. $\square$

2.3. Preparation of the Proof of Theorem 1.1.3 and Theorem 1.1.4

In this section, we collect the tools developed in the previous sections as a preparation for the proofs of Theorem 1.1.3 and Theorem 1.1.4. The coefficients $b_1, \dots, b_d$ play no role here; we only recall from Assumption 1.1.1:

(A_a) There exist constants $0 < m_1 \leq m_2 < \infty$ such that

$$m_1 \leq a_j(x) \leq m_2 \quad \text{for all } x \in \mathbb{R} \text{ and } j \in \{1, \dots, d\}.$$

The functions $a_1, \dots, a_d$ are Lipschitz continuous, and we assume that

$$m_3 \coloneqq \max_{1\leq j\leq d} \left\| \tfrac{\mathrm{d}}{\mathrm{d}x} \log(a_j) \right\|_{\mathrm{L}^1(\mathbb{R})} < 4.$$

For each $j \in \{1, \dots, d\}$ and $p \in (1, \infty)$ we recall the operator $L_{j,p}$ in $\mathrm{L}^p(\mathbb{R}^d; \mathbb{C}^2)$ defined by

$$L_{j,p} \coloneqq \begin{pmatrix} D_j a_j D_j & 0 \\ 0 & a_j D_j^2 \end{pmatrix},$$

$$\mathsf{Dom}(L_{j,p}) \coloneqq \{f \in \mathrm{L}^p(\mathbb{R}^d; \mathbb{C}^2) \mid D_j^k u \in \mathrm{L}^p(\mathbb{R}^d; \mathbb{C}^2) \text{ for } k \in \{0, 1, 2\}\},$$

as introduced in Subsection 2.2.2. We have seen in Corollary 2.2.10 that $i\sqrt{\mathbf{L}_p} \coloneqq i(\sqrt{L_{1,p}}, \dots, \sqrt{L_{d,p}})$ generates a bounded C_0-group $(e^{iy\cdot\sqrt{\mathbf{L}_p}})_{y\in\mathbb{R}^d}$ on $\mathrm{L}^p(\mathbb{R}^d) \coloneqq \mathrm{L}^p(\mathbb{R}^d; \mathbb{C}^2)$. Since $e^{iy\cdot\sqrt{\mathbf{L}_p}} f = e^{iy\cdot\sqrt{\mathbf{L}_2}} f$ for $f \in X_p \cap X_2$, there is no danger of ambiguity, and we will just write $e^{iy\cdot\sqrt{\mathbf{L}}}$ instead of $e^{iy\cdot\sqrt{\mathbf{L}_p}}$.
This d-parameter bounded C_0-group gives rise to the Phillips functional calculus for $\sqrt{\mathbf{L}}$ as introduced in Subsection 2.1.3.

Definition 2.3.1 (Phillips Functional Calculus For $\sqrt{\mathbf{L}}$)**.** Let $p \in (1, \infty)$. For $\varphi \in \mathcal{F}\mathrm{L}^1$, we define $\varphi(\sqrt{\mathbf{L}}) \in \mathcal{L}(\mathrm{L}^p(\mathbb{R}^d))$ by

$$\varphi(\sqrt{\mathbf{L}})\colon \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^p(\mathbb{R}^d), \quad \varphi(\sqrt{\mathbf{L}})f \coloneqq \int_{\mathbb{R}^d} (\mathcal{F}^{-1}\varphi)(y) e^{-iy\cdot\sqrt{\mathbf{L}}} f \, \mathrm{d}y.$$

Remark 2.3.2. If $\varphi \in \mathcal{F}\mathrm{L}^1$ is even in every component, i.e., if

$$\varphi(\xi) = \varphi^{(\mathrm{e})}(\xi) \coloneqq \frac{1}{2^d} \sum_{(\varepsilon_j)_{j=1}^d \in \{-1,1\}^d} \varphi(\varepsilon_1 \xi_1, \dots, \varepsilon_d \xi_d) \quad (\xi \in \mathbb{R}^d),$$

then the same is true for $\psi \coloneqq \mathcal{F}^{-1}\varphi$. In this case, we have by Fubini's theorem

$$\begin{aligned}
\varphi(\sqrt{\mathbf{L}})f &= \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} \psi(y) \frac{1}{2}\Big(\mathrm{e}^{-iy_1\sqrt{L_1}} + \mathrm{e}^{-iy_1\sqrt{L_1}}\Big)\,\mathrm{d}y_1 \right) \mathrm{e}^{-iy'\cdot\sqrt{\mathbf{L}'}} f\,\mathrm{d}y' \\
&= \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} \psi(y) \frac{1}{2}\Big(\mathrm{e}^{iy_1\sqrt{L_1}} + \mathrm{e}^{-iy_1\sqrt{L_1}}\Big)\,\mathrm{d}y_1 \right) \mathrm{e}^{-iy'\cdot\sqrt{\mathbf{L}'}} f\,\mathrm{d}y' \\
&= \int_{\mathbb{R}^{d-1}} \left(\int_{\mathbb{R}} \psi(y) \mathrm{Cos}(y_1\sqrt{L_1})\Big)\,\mathrm{d}y_1 \right) \mathrm{e}^{-iy'\cdot\sqrt{\mathbf{L}'}} f\,\mathrm{d}y' \\
&= \cdots = \int_{\mathbb{R}^d} \psi(y) \mathrm{Cos}(y\sqrt{\mathbf{L}}) f\,\mathrm{d}y.
\end{aligned}$$

Thus, for such functions φ, we are free to replace $\mathrm{e}^{-iy\cdot\sqrt{\mathbf{L}}}$ by $\mathrm{Cos}(y\sqrt{\mathbf{L}})$ in the integral which defines $\varphi(\sqrt{\mathbf{L}})$. This observation turns out to be useful in Chapter 4.

We also consider the operator L in $\mathrm{L}^2(\mathbb{R}^d) \coloneqq \mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^2)$ defined by

$$L \coloneqq L_{1,2} + \cdots + L_{d,2} = \begin{pmatrix} \sum_{j=1}^d D_j a_j D_j & 0 \\ 0 & \sum_{j=1}^d a_j D_j^2 \end{pmatrix} \tag{2.72}$$

with domain $\mathrm{H}^2(\mathbb{R}^d)$. Note that L is self-adjoint w.r.t. the equivalent scalar product

$$\langle u, v\rangle_A \coloneqq \langle A^{-1}u, v\rangle_{\mathrm{L}^2(\mathbb{R}^d;\mathbb{C}^2)}, \quad \text{with} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \quad a \coloneqq a_1 \cdots a_d. \tag{2.73}$$

Moreover, L is injective with spectrum contained in $[0,\infty)$. In particular, L is sectorial, and we may therefore define an L-adapted $\mathrm{L}^2(\mathbb{R}^d)$-based Sobolev scale of spaces as in Subsection 2.1.2.1. For $\alpha \in \mathbb{R}$, we set

$$(\mathrm{H}^\alpha_L(\mathbb{R}^d), \|\cdot\|_{\mathrm{H}^\alpha_L(\mathbb{R}^d)}) \coloneqq (\mathrm{L}^2(\mathbb{R}^d))_L^{\frac{\alpha}{2}} = \begin{cases} (\mathsf{Dom}(L^{\frac{\alpha}{2}}), \|(\mathrm{Id}+L)^{\frac{\alpha}{2}}\cdot\|_2), & \alpha \geq 0, \\ (\mathrm{L}^2(\mathbb{R}^d), \|(\mathrm{Id}+L)^{\frac{\alpha}{2}}\cdot\|_2)^\sim, & \alpha < 0. \end{cases}$$

We also set $\mathrm{H}^\infty_L(\mathbb{R}^d) \coloneqq \bigcap_{\alpha\in\mathbb{R}} \mathrm{H}^\alpha_L(\mathbb{R}^d)$, $\mathrm{H}^{-\infty}_L(\mathbb{R}^d) \coloneqq \bigcup_{\alpha\in\mathbb{R}} \mathrm{H}^\alpha_L(\mathbb{R}^d)$ and we will frequently write $\|\cdot\|_\alpha$ instead of $\|\cdot\|_{\mathrm{H}^\alpha_L(\mathbb{R}^d)}$. Let $\beta \in \mathbb{R}$. We recall from Proposition 2.1.6 that the fractional power $(\mathrm{Id}+L)^{\frac{\beta}{2}}$ extends to an operator $\langle D_L\rangle^\beta \coloneqq (\mathrm{Id}+\mathscr{L})^{\frac{\beta}{2}}$ on $\mathrm{H}^{-\infty}_L(\mathbb{R}^d)$ such that $\langle D_L\rangle^\beta \colon \mathrm{H}^\alpha_L(\mathbb{R}^d) \to \mathrm{H}^{\alpha-\beta}_L(\mathbb{R}^d)$ is a bounded isomorphism for all $\alpha \in \mathbb{R}$. For $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $f \in \mathrm{H}_L^\alpha(\mathbb{R}^d)$, the Phillips functional calculus for $\sqrt{\mathbf{L}}$ extends to $\mathrm{H}_L^\alpha(\mathbb{R}^d)$ by letting

$$\varphi(\sqrt{\mathbf{L}})f \coloneqq (\langle \cdot \rangle^{-\alpha}\varphi)(\sqrt{\mathbf{L}})g, \quad g \coloneqq \langle D_L \rangle^\alpha f \in \mathrm{L}^2(\mathbb{R}^d)$$

(see Definition 2.1.26). Moreover, as L is injective, the fractional powers L^β are defined for all $\beta \in \mathbb{R}$ by Proposition 2.1.3 (e), and thus Proposition 2.1.15 (b) implies that

$$L^\beta \psi(\sqrt{\mathbf{L}})f = (|\cdot|^{2\beta}\psi)(\sqrt{\mathbf{L}})f \quad (f \in \mathrm{H}_L^\alpha(\mathbb{R}^d)) \tag{2.74}$$

for all $\alpha, \beta \in \mathbb{R}$ and all smooth $\psi \in C_c^\infty(\mathbb{R}^d)$ supported away from the origin. It follows similarly from Proposition 2.1.15 (b) that

$$\langle D_L \rangle^\beta \varphi(\sqrt{\mathbf{L}})f = (\langle \cdot \rangle^\beta \varphi)(\sqrt{\mathbf{L}})f \quad (f \in \mathrm{H}_L^\alpha(\mathbb{R}^d)) \tag{2.75}$$

for all $\alpha, \beta \in \mathbb{R}$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$. We use the following result in order to switch from the L-adapted scale to the classical one.

Proposition 2.3.3 (Kato). *(a) For $\alpha \in [-2, 2]$, we have $\mathrm{H}_L^\alpha(\mathbb{R}^d) = \mathrm{H}^\alpha(\mathbb{R}^d)$ and*

$$\|(\mathrm{Id} + L)^{\frac{\alpha}{2}} u\|_2 \simeq \|\langle D_x \rangle^\alpha u\|_2 \quad (u \in \mathrm{H}^\alpha(\mathbb{R}^d)).$$

(b) For all $\beta \in [-1, 1]$ and $q \in (1, \infty)$, we have $\dot{W}^{\beta,q}(\mathbb{R}^d) = \dot{W}_L^{\beta,q}(\mathbb{R}^d)$ with equivalent norms $\|L^{\frac{\beta}{2}} u\|_q \simeq \||D_x|^\beta u\|_q$. Here, the L-adapted L^q-based homogeneous Sobolev space $\dot{W}_L^{\beta,q}(\mathbb{R}^d)$ is defined as in [35, Definition 15.21] with A replaced by the closure of the L^q-part of $\sqrt{L}$ and α replaced by β therein.

Proof. Consider first the case $\alpha = 2$. Then, $\mathrm{H}_L^2(\mathbb{R}^d) = \mathsf{Dom}(L) = \mathrm{H}^2(\mathbb{R}^d)$ by definition. The estimate $\|u\|_{\mathrm{H}_L^2(\mathbb{R}^d)} \lesssim \|u\|_{\mathrm{H}^2(\mathbb{R}^d)}$ is trivial (by the Lipschitz continuity of the a_j), and the reverse one then follows from the open mapping theorem. Thus, $\mathrm{H}_L^2(\mathbb{R}^d) = \mathrm{H}^2(\mathbb{R}^d)$ with equivalent norms. For $\alpha = 0$, there is nothing to prove. For intermediate value $\alpha \in (0, 2)$, one argues by interpolation, as both $(\mathrm{H}_L^\beta(\mathbb{R}^d))_{\beta\in\mathbb{R}}$ and $(\mathrm{H}^\beta(\mathbb{R}^d))_{\beta\in\mathbb{R}}$ form complex interpolation scales [39, Theorem 4.17]. The case $\alpha \in [-2, 0)$ follows from duality. The second assertion is a very deep result, known as the Kato square root problem for $q = 2$ and $\beta = 1$, which was resolved in [3] (for operators in divergence form with even complex L^∞-coefficients). By virtue of the structural assumption (1.7), we may extrapolate this to general $q \in (1, \infty)$. Indeed, condition (1.7) in particular implies that $L = L_1 + \cdots + L_d$ is a sum of *commuting* (essentially one-dimensional) operators so that the generated semigroup

$$\mathrm{e}^{-tL} = \mathrm{e}^{-tL_1} \cdots \mathrm{e}^{-tL_d} \quad (t \in \mathbb{R})$$

factors. Now, [5, Theorem 2.36] shows that $(\mathrm{e}^{-tL})_{t>0}$ as well as $(\sqrt{t}\nabla_x \mathrm{e}^{-tL})_{t>0}$ satisfy Gaussian estimates. Combining this with [20], [35, Theorem 15.28] and a duality argument, we infer the claim. □

In order to prove Theorem 1.1.4, we will heavily make use of the Phillips functional calculus and the Littlewood–Paley type results as developed in Subsection 2.1.3. For the convenience of the reader, we restate them here in the setting of the concrete operator $\sqrt{\mathbf{L}}$. To this end, we fix some standard (homogeneous) Littlewood–Paley partition of unity, i.e., some radially symmetric $\psi \in C_c^\infty(\mathbb{R}^d)$ supported in $\{\xi \in \mathbb{R}^d \colon \frac{1}{2} < |\xi| < 2\}$ such that we have

$$\sum_{\lambda \in 2^{\mathbb{Z}}} \psi_\lambda(\xi) = 1 \quad \text{for all} \quad \xi \neq 0, \quad \text{where } \psi_\lambda(\xi) := \psi\left(\frac{\xi}{\lambda}\right). \tag{2.76}$$

Proposition 2.3.4 (Calderón Reproducing Formula on $\mathrm{H}^\alpha_L(\mathbb{R}^d)$)**.** *Let $\alpha \in \mathbb{R}$ and $f \in \mathrm{H}^\alpha_L(\mathbb{R}^d)$. Then,*

$$\sum_{\lambda \in 2^{\mathbb{Z}}} \psi_\lambda(\sqrt{\mathbf{L}}) f = f.$$

In particular, for $\mathcal{D} := C_c^\infty(\mathbb{R}^d)$,

$$\mathcal{S}_{\sqrt{\mathbf{L}}} := \mathcal{S}^{\mathcal{D}}_{\sqrt{\mathbf{L}}} = \operatorname{span}\{\psi(\sqrt{\mathbf{L}}) f \mid \psi, f \in \mathcal{D}, \operatorname{supp}(\psi) \Subset \mathbb{R}^d \setminus \{0\}\} \tag{2.77}$$

belongs to $\bigcap_{\beta \in \mathbb{R}} \mathsf{Dom}(L^\beta) \cap \bigcap_{p \in (1,\infty)} \mathrm{L}^p(\mathbb{R}^d)$ and is dense in $\mathrm{H}^\alpha_L(\mathbb{R}^d)$.

Proof. Just use Corollary 2.1.27, $\mathcal{D} \subseteq \mathrm{L}^p(\mathbb{R}^d)$, and $\psi(\sqrt{\mathbf{L}}) \in \mathcal{L}(\mathrm{L}^p(\mathbb{R}^d))$ for all $p \in (1, \infty)$ and $\psi \in \mathcal{D}$. □

Proposition 2.3.5 (L^2-Boundedness of Almost Orthogonal Operators)**.** *The following statements hold true.*

(a) Let $h \in \mathcal{F}\mathrm{L}^1$. Then $\|h(\sqrt{\mathbf{L}})\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{R}^d))} \lesssim \|h\|_\infty$.

(b) Let $\alpha \in \mathbb{R}$ and $\gamma \in \mathbb{R}$. Assume that $h := (h_\lambda)_{\lambda \in 2^{\mathbb{Z}}}$ is a sequence of functions on $\mathbb{R}^d$ such that h_λ is smooth on an open neighborhood of $K_\lambda := \operatorname{supp}(\psi_\lambda)$ for all $\lambda \in 2^{\mathbb{Z}}$ and $\|h\|_\infty := \sup_{\lambda \in 2^{\mathbb{Z}}} \|h_\lambda\|_{\mathrm{L}^\infty(K_\lambda)} < \infty$. Then,

$$T \colon \mathrm{H}^\alpha_L(\mathbb{R}^d) \to \mathrm{H}^{\alpha-\gamma}_L(\mathbb{R}^d), \quad Tf := \sum_{\lambda \in 2^{\mathbb{Z}}} \langle \lambda \rangle^\gamma (h_\lambda \psi_\lambda)(\sqrt{\mathbf{L}}) f$$

is a well-defined linear bounded operator with $\|T\| \lesssim \|h\|_\infty$. Moreover, for $f \in \mathrm{H}^\alpha_L(\mathbb{R}^d)$, there holds

$$\|Tf\|^2_{\mathrm{H}^{\alpha-\gamma}_L(\mathbb{R}^d)} \lesssim \sum_{\lambda \in 2^{\mathbb{Z}}} \|(h_\lambda \psi_\lambda)(\sqrt{\mathbf{L}}) g\|_2^2 \lesssim \sum_{\lambda \in 2^{\mathbb{Z}}} \|\psi_\lambda(\sqrt{\mathbf{L}}) g\|_2^2 \simeq \|f\|^2_{\mathrm{H}^\alpha_L(\mathbb{R}^d)},$$

where $g := \langle D_L \rangle^\alpha f \in \mathrm{L}^2(\mathbb{R}^d)$.

Proof. See Proposition 2.1.10 (d) and Theorem 2.1.28. □

Proposition 2.3.6 (Squarefunction Characterization of the L^p-norm)**.** *Let* $1 < p < \infty$. *Then,*

$$\Big\| \Big(\sum_{\lambda \in 2^{\mathbb{Z}}} \big| \psi_\lambda(\sqrt{\mathbf{L}}) f(x) \big|^2 \Big)^{1/2} \Big\|_p \simeq \|f\|_p \quad \text{for all } f \in \mathrm{L}^p(\mathbb{R}^d).$$

Proof. See Proposition 2.1.29. □

Lemma 2.3.7 (Differentiation under the Sum)**.** *Let* $\alpha \in \mathbb{R}$, $f \in \mathrm{H}^\alpha_L(\mathbb{R}^d)$, *and* $n \in \mathbb{N}_0$. *Suppose that* $\{h_{t,\lambda} \mid \lambda \in 2^{\mathbb{Z}}, t \in \mathbb{R}\}$ *is a set of smooth functions on* $\mathbb{R}^d \setminus \{0\}$ *which satisfy the following assumptions:*

(i) For all $\lambda \in 2^{\mathbb{Z}}$ *and each* $\xi \in K_\lambda := \operatorname{supp}(\psi_\lambda)$, *the map* $t \mapsto h_{t,\lambda}(\xi)$ *belongs to* $C^n(\mathbb{R};\mathbb{C})$ *and*

$$|\partial_t^k h_{t,\lambda}(\xi)| \le C(t) \langle \lambda \rangle^k \quad (0 \le k \le n)$$

with some locally bounded function $t \mapsto C(t)$.

(ii) For each fixed $t_0 \in \mathbb{R}, \lambda \in 2^{\mathbb{Z}}$, *we have*

$$\big\| \partial_t^n h_{t,\lambda} - \partial_t^n h_{t_0,\lambda} \big\|_{L^\infty_\xi(K_\lambda)} \to 0 \quad (t \to t_0).$$

Then, for all $k \in \{0, \dots, n\}$, *the map*

$$u \colon t \mapsto T(t) f := \sum_{\lambda \in 2^{\mathbb{Z}}} (h_{t,\lambda} \psi_\lambda)(\sqrt{\mathbf{L}}) f$$

belongs to $C^k(\mathbb{R}; \mathrm{H}^{\alpha-k}_L(\mathbb{R}^d))$ *and it holds*

$$u^{(k)}(t) = \sum_{\lambda \in 2^{\mathbb{Z}}} (\partial_t^k h_{t,\lambda} \psi_\lambda)(\sqrt{\mathbf{L}}) f, \quad \|u^{(k)}(t)\|_{\alpha-k} \lesssim C(t) \|f\|_\alpha \quad (t \in \mathbb{R}).$$

Proof. Fix $\alpha \in \mathbb{R}, f \in \mathrm{H}^\alpha_L(\mathbb{R}^d)$, and let $n \in \mathbb{N}_0$. First of all, we observe that assumption (i) with $k = 0$ implies that $\|h_{t,\lambda}\|_{L^\infty_\xi(K_\lambda)} \le C(t)$ for all $\lambda \in 2^{\mathbb{Z}}$ and $t \in \mathbb{R}$. Thus, Proposition 2.3.5 (b) shows that the operator $T(t) \colon \mathrm{H}^{\alpha-k}_L(\mathbb{R}^d) \to \mathrm{H}^{\alpha-k}_L(\mathbb{R}^d)$ is a well-defined and bounded with $\|T(t)\| \lesssim C(t)$ for each $t \in \mathbb{R}$ and $k \in \{0, \dots, n\}$. We set $g := \langle D_L \rangle^\alpha f \in \mathrm{L}^2(\mathbb{R}^d)$ and prove that $u \colon t \mapsto T(t) f$ belongs to $C^k(\mathbb{R}; \mathrm{H}^{\alpha-k}_L(\mathbb{R}^d))$ for all $k \in \{0, \dots, n\}$ by induction on n.

- $n = 0$: By the opening remarks, we have $\|u(t)\|_\alpha \le \|T(t)\| \|f\|_\alpha \lesssim C(t) \|f\|_\alpha$ for all $t \in \mathbb{R}$. To prove that u is continuous, fix $t_0 \in \mathbb{R}$. We have to show that

$$T(t) f - T(t_0) f \to 0 \quad (t \to t_0) \quad \text{in } \mathrm{H}^\alpha_L(\mathbb{R}^d).$$

First, we use (i) to see that $\|h\|_\infty := \sup_{\lambda\in 2^{\mathbb{Z}},|t-t_0|\le 1} \|h_{t,\lambda}\|_{L^\infty_\xi(K_\lambda)}$ is finite. By Definition 2.1.26 and Proposition 2.1.15 (c), we have

$$\langle D_L\rangle^\alpha \psi_\lambda(D) f = \langle D_L\rangle^\alpha(\langle\,\cdot\,\rangle^{-\alpha}\psi_\lambda)(D)g = \psi_\lambda(D)g.$$

Let $t\in\mathbb{R}$ with $|t-t_0|\le\delta$. Then, Proposition 2.3.5 (b) yields

$$\begin{aligned}&\|T(t)f - T(t_0)f\|^2_{\mathrm{H}^\alpha_L(\mathbb{R}^d)}\\ &\lesssim \sum_{\lambda\in 2^{\mathbb{Z}}} \|\Big((h_{t,\lambda}-h_{t_0,\lambda})\psi_\lambda\Big)(\sqrt{\mathbf{L}})g\|_2^2\\ &\lesssim \|h\|^2_\infty \sum_{\lambda\in 2^{\mathbb{Z}}} \|\psi_\lambda(\sqrt{\mathbf{L}})g\|_2^2 \lesssim \|g\|_2^2 = \|f\|^2_{\mathrm{H}^\alpha_L(\mathbb{R}^d)}.\end{aligned}$$

Moreover, for each $\lambda\in 2^{\mathbb{Z}}$ we have

$$\|\Big((h_{t,\lambda}-h_{t_0,\lambda}\Big)\psi_\lambda)(\sqrt{\mathbf{L}})g\|_2^2 \lesssim \|(h_{t,\lambda}-h_{t_0,\lambda})\psi_\lambda\|^2_\infty\|g\|_2^2 \to 0$$

as $t\to t_0$ by Proposition 2.3.5 (a) and assumption (ii). Thus, it follows from dominated convergence in $\ell^1(2^{\mathbb{Z}})$ that

$$\|T(t)f - T(t_0)f\|_{\mathrm{H}^\alpha_L(\mathbb{R}^d)} \lesssim \sum_{\lambda\in 2^{\mathbb{Z}}} \|\Big((h_{t,\lambda}-h_{t_0,\lambda})\psi_\lambda\Big)(\sqrt{\mathbf{L}})f\|_2^2 \to 0$$

as $t\to t_0$. This shows that u belongs to $C(\mathbb{R};\mathrm{H}^\alpha_L(\mathbb{R}^d))$.

- $n\ge 1$: By the induction hypothesis, we may suppose that for any $k\in\{0,\dots,n-1\}$ the map $u\colon t\mapsto T(t)f$ belongs to $C^k(\mathbb{R};H^{\alpha-k}_L(\mathbb{R}^d))$ with

$$u^{(k)}(t) = \sum_{\lambda\in 2^{\mathbb{Z}}} (h^{(k)}_{t,\lambda}\psi_\lambda)(\sqrt{\mathbf{L}})f, \quad \|u^{(k)}(t)\|_{\alpha-k}\lesssim C(t_0)\|f\|_\alpha \quad (t\in\mathbb{R}),$$

where $h^{(k)}_{t,\lambda} := \partial_t^k h_{t,\lambda}$. We have to show that $u\in C^n(\mathbb{R};\mathrm{H}^{\alpha-n}_L(\mathbb{R}^d))$. But $u\in C^{n-1}(\mathbb{R};\mathrm{H}^{\alpha-(n-1)}_L(\mathbb{R}^d))$ by induction hypothesis and since $\mathrm{H}^{\alpha-(n-1)}_L(\mathbb{R}^d)\hookrightarrow \mathrm{H}^{\alpha-n}_L(\mathbb{R}^d)$, we infer $u\in C^{n-1}(\mathbb{R};\mathrm{H}^{\alpha-n}_L(\mathbb{R}^d))$. Hence, it remains to show that $u^{(n-1)}$ is continuously differentiable with values in $\mathrm{H}^{\alpha-n}_L(\mathbb{R}^d)$ and that $\|u^{(n)}(t)\|_{\alpha-n}\lesssim C(t)\|f\|_\alpha$. To this end, fix again $t_0\in\mathbb{R}$.

First, note that by Proposition 2.3.5 (b),

$$v(t_0) := \sum_{\lambda\in 2^{\mathbb{Z}}} (\partial_t^n h_{t_0,\lambda}\psi_\lambda)(\sqrt{\mathbf{L}})f \in \mathrm{H}^{\alpha-n}_L(\mathbb{R}^d), \quad \|v(t_0)\|_{\alpha-n}\lesssim C(t)\|f\|_\alpha,$$

since $|\partial_t^n h_{t_0,\lambda}(\xi)|\le C(t_0)\langle\lambda\rangle^n$ on K_λ by assumption (i). Let us set $C := \sup_{|t-t_0|\le 1} C(t) < \infty$ and put

$$\rho_{t,\lambda}(\xi) := \left(\frac{h^{(n-1)}_{t,\lambda}(\xi) - h^{(n-1)}_{t_0,\lambda}(\xi)}{t-t_0} - \partial_t^n h_{t_0,\lambda}(\xi)\right)\langle\xi\rangle^{-n} \quad (\xi\ne 0)$$

for $|t-t_0| \leq 1$, $\lambda \in 2^{\mathbb{Z}}$. An application of the fundamental theorem of calculus and (i) shows that for $\xi \in K_\lambda$

$$\begin{aligned} |\rho_{t,\lambda}(\xi)| &\leq \langle \xi \rangle^{-n} \int_0^1 |\partial_t^n h_{t_0+\tau(t-t_0),\lambda}(\xi) - \partial_t^n h_{t_0,\lambda}(\xi)| \,\mathrm{d}\tau \\ &\leq C(t) \langle \xi \rangle^{-n} \int_0^1 2 \langle \lambda \rangle^n \,\mathrm{d}\tau \lesssim C. \end{aligned} \tag{2.78}$$

Thus, for $0 < |t-t_0| \leq 1$, it follows once again from Proposition 2.3.5 (b) that

$$\begin{aligned} &\left\| \frac{u^{(n-1)}(t) - u^{(n-1)}(t_0)}{t-t_0} - v(t_0) \right\|^2_{\mathrm{H}_L^{\alpha-n}(\mathbb{R}^d)} \\ &= \Big\| \sum_{\lambda \in 2^{\mathbb{Z}}} \big(\rho_{t,\lambda} \psi_\lambda \big)(\sqrt{\mathbf{L}}) g \Big\|_2^2 \\ &\lesssim \sum_{\lambda \in 2^{\mathbb{Z}}} \big\| \big(\rho_{t,\lambda} \psi_\lambda \big)(\sqrt{\mathbf{L}}) g \big\|_2^2 \\ &\lesssim_C \sum_{\lambda \in 2^{\mathbb{Z}}} \|\psi_\lambda(\sqrt{\mathbf{L}}) g\|_2^2 \lesssim \|g\|_2^2 = \|f\|^2_{\mathrm{H}_L^{\alpha}(\mathbb{R}^d)}. \end{aligned}$$

Moreover, if $\delta_t \coloneqq |t-t_0|$, then Proposition 2.3.5 (a), the first line of (2.78), and assumption (ii) yield for each $\lambda \in 2^{\mathbb{Z}}$

$$\begin{aligned} \big\| \big(\rho_{t,\lambda} \psi_\lambda \big)(\sqrt{\mathbf{L}}) g \big\|_2 &\lesssim \|\rho_{t,\lambda} \psi_\lambda\|_\infty \|g\|_2 \\ &\lesssim \langle \lambda \rangle^{-n} \|g\|_2 \sup_{|s-t_0| \leq \delta_t} \|\partial_t^n h_{s,\lambda} - \partial_t^n h_{t_0,\lambda}\|_{L^\infty_\xi(K_\lambda)} \to 0 \end{aligned}$$

as $t \to t_0$. Applying dominated convergence in $\ell^1(2^{\mathbb{Z}})$ once again, we infer that $u^{(n)}(t_0) = v(t_0)$ in $\mathrm{H}_L^{\alpha-n}(\mathbb{R}^d)$. Using that

$$\langle D_L \rangle^{\alpha-n} u^{(n)}(t) = \sum_{\lambda \in 2^{\mathbb{Z}}} (\tilde{h}_{t,\lambda} \psi_\lambda)(\sqrt{\mathbf{L}}) g, \quad \tilde{h}_{t,\lambda} \coloneqq \langle \cdot \rangle^{-n} \partial_t^n h_{t,\lambda},$$

the continuity of $u^{(n)} \colon \mathbb{R} \to \mathrm{H}_L^{\alpha-n}(\mathbb{R}^d)$ is shown exactly as the already shown continuity of $u \colon \mathbb{R} \to \mathrm{H}_L^{\alpha}(\mathbb{R}^d)$.

The proof is complete. □

Lemma 2.3.8 (Extension of the Operator L_j to $\mathrm{H}_L^{\alpha}(\mathbb{R}^d)$)**.** *Let $\alpha \in \mathbb{R}$, $n \in \mathbb{N}_0$ and $j \in \{1, \dots, d\}$. Then, the operator $(L_j)^n \colon \mathcal{S}_{\sqrt{\mathbf{L}}} \to \mathcal{S}_{\sqrt{\mathbf{L}}}$ extends uniquely to a bounded operator from $\mathrm{H}_L^{\alpha}(\mathbb{R}^d)$ to $\mathrm{H}_L^{\alpha-n}(\mathbb{R}^d)$. In particular, for each $t \in \mathbb{R}$, we have $P(t) \colon \mathrm{H}_L^{\alpha}(\mathbb{R}^d) \to \mathrm{H}_L^{\alpha-2}(\mathbb{R}^d)$.*

Proof. Uniqueness is clear, since $\mathcal{S}_{\sqrt{\mathbf{L}}}$ is dense in $\mathrm{H}_L^{\alpha}(\mathbb{R}^d)$ by Corollary 2.1.27. Now, just note that the operator

$$T \colon \mathrm{H}_L^{\alpha}(\mathbb{R}^d) \to \mathrm{H}_L^{\alpha-n}(\mathbb{R}^d), \quad Tf \coloneqq \sum_{\lambda \in 2^{\mathbb{Z}}} \lambda^n \Big(\Big(\tfrac{\xi_j}{\lambda} \Big)^n \psi_\lambda \Big)(\sqrt{\mathbf{L}}) f$$

is bounded by Proposition 2.3.5 (b) and that by Proposition 2.1.10 (c) and Proposition 2.3.4, we have $Tf = (L_j)^{\frac{n}{2}} f$ for $f \in \mathcal{S}_{\sqrt{\mathbf{L}}}$. The second assertion is proved analogously. The second assertion follows from $P(t) = \sum_{j=1}^{d} b_j(t) L_j$ and the first assertion with $n = 2$. $\square$

3. A Parametrix Construction and Weak Solutions

Equipped with the tools from Chapter 2, we now proceed to prove Theorem 1.1.3 in this chapter, which serves as a bridge to the proof of the global-in-time Strichartz estimates (Theorem 1.1.4) in Chapter 4. The chapter is structured as follows: we begin in Section 3.1 with an outline of the proof strategy, in which we also briefly explain how Theorem 1.1.3 is used in the proof of Theorem 1.1.4 in Chapter 4. The central part of this chapter is the parametrix construction in Section 3.2, which is subsequently utilized in Section 3.3 to establish the existence of weak solutions.
We deal with operators in divergence and standard form simultaneously by considering the (decoupled) two-dimensional system

$$\Big(D_t^2 - P(t)\Big)u(t) = F(t), \quad u(0) = g, \quad D_t u(0) = h, \tag{3.1}$$

where $P(t)$ is given by

$$P(t) \coloneqq \sum_{j=1}^{d} b_j(t) L_j = \begin{pmatrix} \sum_{j=1}^{d} b_j(t) D_{x_j} a_j(x_j) D_{x_j} & 0 \\ 0 & \sum_{j=1}^{d} b_j(t) a_j(x_j) D_{x_j}^2 \end{pmatrix}$$

and the functions $g, h \colon \mathbb{R}^d \to \mathbb{C}^2$ and $F(t) = F(t, \cdot) \colon \mathbb{R}^d \to \mathbb{C}^2$ are fixed. In this section, we aim to find a parametrix for (3.1). It will turn out to be convenient to use the notation

$$B(t) \coloneqq \operatorname{diag}(b_1(t), \dots, b_d(t)) \quad (t \in \mathbb{R})$$

and we write $\|A\|$ for the operator norm of a matrix $A \in \mathbb{R}^{d\times d}$. In particular, $\|B^{(k)}(t)\| = \max_{1\le j\le d} |b^{(k)}(t)|$ for $k \in \{0,1\}$ and $t \in \mathbb{R}$. We will also frequently write

$$\Box_P \coloneqq D_t^2 - P(t) \quad (t \in \mathbb{R}).$$

Moreover, for the rest of this thesis, we fix some standard Littlewood–Paley sequence $(\psi_\lambda)_{\lambda\in 2^{\mathbb{Z}}}$ as introduced in (2.76).

3.1. Strategy of Proof

In this section, we want to briefly explain the main ideas of the proof of Theorem 1.1.3 and how the latter is linked to the proof of Theorem 1.1.4

in Chapter 4. We use a parametrix construction for (3.1) that goes back to Smith [47] (see also [33, Section 4] in the context of smooth wave equations). The idea is to construct families of operators $(C(t,s))_{(t,s)\in\mathbb{R}^2}$ in $\mathcal{L}(\mathrm{H}^\alpha_L(\mathbb{R}^d))$ and $(S(t,s))_{(t,s)\in\mathbb{R}^2}$ in $\mathcal{L}(\mathrm{H}^\alpha_L(\mathbb{R}^d);\mathrm{H}^{\alpha+1}_L(\mathbb{R}^d))$ which are approximate solution operators to the equation $(D_t^2 - P(t))u = 0$, in the sense that

$$(D_t^2 - P(t))C(t,s)\colon \mathrm{H}^\alpha_L(\mathbb{R}^d) \to \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d),$$
$$(D_t^2 - P(t))S(t,s)\colon \mathrm{H}^\alpha_L(\mathbb{R}^d) \to \mathrm{H}^{\alpha}_L(\mathbb{R}^d)$$

are bounded operators. This then allows us to prove the existence of a weak solution that can be represented by

$$u(t) = C(t,0)g + S(t,0)h + \int_0^t S(t,s)G(s)\,\mathrm{d}s.$$

Here, $G \in \mathrm{L}^1(\mathbb{R};\mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))$ is the solution to a Volterra equation and depends on g, h, and F. Uniqueness is shown by energy estimates.
The key point of the structural assumption on the coefficients of $P(t)$ (see (1.7)) is that it provides us with a Phillips functional calculus for $\sqrt{\mathbf{L}}$, which in turn allows to construct the operators $C(t,s)$ and $S(t,s)$ in a fairly explicit way. In fact, $C(t,s)$ and $S(t,s)$ will be constructed in terms of operators of the form

$$T(t,s) = (\mathrm{e}^{i\varphi_{t,s}}\psi)(\sqrt{\mathbf{L}}) \tag{3.2}$$

with an explicit phase function $\varphi_{t,s}\colon \mathbb{R}^d \to \mathbb{R}$ that is positively homogeneous of degree one and smooth away from the origin, and some amplitude $\psi \in C_c^\infty(\mathbb{R}^d)$ supported away from the origin. At this point, we also want to highlight that the Borel functional calculus for the self-adjoint operator $\sqrt{L} = \sqrt{L_1 + \cdots + L_d}$ is not enough to define the operator in (3.2) in a meaningful way. Indeed, for (suitable) radial functions $\rho\colon [0,\infty) \to \mathbb{R}$, we have $\rho(\sqrt{L}) = (\rho \circ |\cdot|)(\sqrt{\mathbf{L}})$ (cf. Proposition 2.1.15 (e)); however, in general, the phase function $\varphi_{t,s}$ is *not* radial.
To prove the global-in-time Strichartz estimates (Theorem 1.1.4) in Chapter 4 later, we appeal to the abstract framework developed by Keel–Tao [34] for which we need dispersive estimates for the operator in (3.2). Using the properties of the Phillips functional calculus for $\sqrt{\mathbf{L}}$ and the $\mathrm{L}^\infty_x\mathrm{L}^1_y$-bounds for $\mathrm{Cos}(y\sqrt{\mathbf{L}})$ as proved in Corollary 2.2.10, proving these dispersive estimates essentially boils down to pointwise estimates for the Fourier transform of the surface measure over the hypersurface $\mathbb{S}^{d-1}_{t,s} = \{\xi \in \mathbb{R}^d \mid \varphi_{t,s}(\xi) = t - s\}$. This is the heart of the proof. Finally, we only need to use Proposition 2.3.3 to relate the spaces $\mathrm{H}^\alpha_L(\mathbb{R}^d)$ and $\mathrm{H}^\alpha(\mathbb{R}^d)$, as well as the homogeneous norms $\|L^\beta u\|_q$ and $\||D_x|^\beta u\|_q$ to deduce Theorem 1.1.4.

3.2. The Parametrix Construction

In this section, we construct a parametrix for $\Box_P u = 0$.

3.2.1. Derivation of a Parametrix

To motivate our ansatz for the parametrix, we first recall the easier time-independent case, where $B(t) = \mathrm{Id}$ for all $t \in \mathbb{R}$. Then $P(t) = L$ and since $i\sqrt{L}$ generates a bounded C_0-group on $\mathrm{L}^2(\mathbb{R}^d)$ (by the Borel functional calculus for L), we can express the solution of (3.1) in terms of the C_0-group $(\mathrm{e}^{it\sqrt{L}})_{t\in\mathbb{R}}$. For the latter, we have by Proposition 2.1.15 (d) for any $f \in \mathrm{L}^2(\mathbb{R}^d)$, $s, t \in \mathbb{R}$ and $\lambda \in 2^{\mathbb{Z}}$ the representation

$$\mathrm{e}^{i(t-s)\sqrt{L}}\psi_\lambda(\sqrt{\mathbf{L}})f = \left(\mathrm{e}^{i(t-s)|\,\cdot\,|}\psi_\lambda\right)(\sqrt{\mathbf{L}})f = \int_{\mathbb{R}^d} K_\lambda(t-s, y)\mathrm{e}^{-iy\cdot\sqrt{\mathbf{L}}}f\,\mathrm{d}y,$$

where $K_\lambda(\tau, y) = \mathcal{F}^{-1}(\mathrm{e}^{i\tau|\,\cdot\,|}\psi_\lambda)(y) = (2\pi)^{-d}\int_{\mathbb{R}^d}\mathrm{e}^{i(y\cdot\xi+\tau|\xi|)}\psi_\lambda(\xi)\,\mathrm{d}\xi$ is the ψ_λ-truncated kernel of the classical half-wave group. Summing over the frequencies $\lambda \in 2^{\mathbb{Z}}$, we obtain from Proposition 2.3.4 the representation

$$\mathrm{e}^{i(t-s)\sqrt{L}}f = \sum_{\lambda\in 2^{\mathbb{Z}}}\int_{\mathbb{R}^d} K_\lambda(t-s, y)\mathrm{e}^{-iy\cdot\sqrt{\mathbf{L}}}f\,\mathrm{d}y \quad \text{in } \mathrm{L}^2(\mathbb{R}^d).$$

Guided by this observation, in the time-dependent case, we seek a parametrix of the form

$$T(t,s)f \coloneqq \sum_{\lambda\in 2^{\mathbb{Z}}} T_\lambda(t,s)f \coloneqq \sum_{\lambda\in 2^{\mathbb{Z}}}\int_{\mathbb{R}^d} K_\lambda(t,s,y)\mathrm{e}^{-iy\cdot\sqrt{\mathbf{L}}}f\,\mathrm{d}y \tag{3.3}$$

with a kernel $K_\lambda\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$. Let $\lambda \in 2^{\mathbb{Z}}$ and fix $s \in \mathbb{R}$. If we formally integrate by parts, we find for any $f \in \mathrm{H}^2(\mathbb{R}^d)$

$$\begin{aligned} P(t)T_\lambda(t,s)f &= \sum_{j=1}^d b_j(t)\int_{\mathbb{R}^d} K_\lambda(t,s,y)L_j\mathrm{e}^{-iy\cdot\sqrt{\mathbf{L}}}f\,\mathrm{d}y \\ &= \sum_{j=1}^d b_j(t)\int_{\mathbb{R}^d} K_\lambda(t,s,y)(D_{y_j})^2\mathrm{e}^{-iy\cdot\sqrt{\mathbf{L}}}f\,\mathrm{d}y \\ &= \sum_{j=1}^d b_j(t)\int_{\mathbb{R}^d} D_{y_j}^2K_\lambda(t,s,y)\mathrm{e}^{-iy\cdot\sqrt{\mathbf{L}}}f\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d}(p(t,D_y)K_\lambda(t,s,y))\mathrm{e}^{-iy\cdot\sqrt{\mathbf{L}}}f\,\mathrm{d}y \end{aligned}$$

with symbol

$$p(t,\xi) \coloneqq \sum_{j=1}^d b_j(t)\xi_j^2 = (B(t)\xi|\xi), \quad (t,\xi) \in \mathbb{R}\times\mathbb{R}^d.$$

So formally, we have

$$\left(D_t^2 - P(t)\right)T_\lambda(t,s)f = \int_{\mathbb{R}^d}\left[\left(D_t^2 - p(t,D_y)\right)K_\lambda(t,s,y)\right]\mathrm{e}^{-iy\cdot\sqrt{\mathbf{L}}}f\,\mathrm{d}y.$$

Therefore, it is natural to require K_λ to solve the wave equation

$$(D_t^2 - p(t, D_y))K_\lambda(t, s, y) = 0, \quad (t, y) \in \mathbb{R} \times \mathbb{R}^d,$$

and to this end, we follow the idea of Lax's parametrix construction (see [36], [52, Section 4.1] or [25, Chapter 6]): We are looking for a solution of the form

$$K_\lambda^\pm(t, s, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{i\Phi^\pm(t,s,y,\xi)} \psi_\lambda(\xi) \, \mathrm{d}\xi, \tag{3.4}$$

where the phase functions $\Phi^\pm$ solve the eikonal equations

$$\partial_t \Phi^\pm = \pm q(t, \nabla_y \Phi^\pm), \quad \Phi^\pm(s, s, y, \xi_s) = y_s \cdot \xi_s$$

with symbol $q(t, \xi) \coloneqq \sqrt{p(t, \xi)} = \sqrt{(B(t)\xi|\xi)}$ and fixed $y_s, \xi_s \in \mathbb{R}^d$. The theory on Hamilton–Jacobi equations implies that equations of this type are solvable (at least locally in time) (see e.g. [62, Chapter I.15]), and in fact, in our particular case, we can construct the solutions $\Phi^\pm$ explicitly. Let us focus on Φ^+ first. We recall that the full symbol $q_0^+(t, y, \eta, \xi) = \eta - q(t, \xi)$ generates a Hamiltonian flow $(y_s, \xi_s) \mapsto (y(t), \xi(t))$ on the phase space $\mathbb{R}_y^d \times \mathbb{R}_\xi^d$, which is defined by the ODE

$$\begin{aligned} \dot{y}(t) = \quad & \nabla_\xi q_0^+(t, y(t), \eta(t), \xi(t)) = -\frac{B(t)\xi(t)}{(B(t)\xi(t)|\xi(t))^{1/2}}, \quad && y(s) = y_s, \\ \dot{\xi}(t) = & -\nabla_y q_0^+(t, y(t), \eta(t), \xi(t)) = 0, && \xi(s) = \xi_s, \end{aligned}$$

which can be easily integrated:

$$\xi(t) = \xi_s, \quad y(t) = y_s - \int_s^t \frac{B(\tau)\xi_s}{(B(\tau)\xi_s|\xi_s)^{1/2}} \, \mathrm{d}\tau \quad (t \in \mathbb{R}).$$

Now it can be shown that (see [17, Subsection VIII.64.2])

$$\Phi^+(t, s, y, \xi_s) = y_s \cdot \xi_s = \left(y + \int_s^t \frac{B(\tau)\xi_s}{(B(\tau)\xi_s|\xi_s)^{1/2}} \, \mathrm{d}\tau \right) \cdot \xi_s = y \cdot \xi_s + \int_s^t (B(\tau)\xi_s|\xi_s)^{\frac{1}{2}} \, \mathrm{d}\tau.$$

Turning to Φ^-, we note that the full symbol is then given by $q_0^-(t, y, \eta, \xi) = \eta + q(t, \xi)$ and we similarly obtain

$$\Phi^-(t, s, y, \xi_s) = y_s \cdot \xi_s = \left(y - \int_s^t \frac{B(\tau)\xi_s}{(B(\tau)\xi_s|\xi_s)^{1/2}} \, \mathrm{d}\tau \right) \cdot \xi_s = y \cdot \xi_s - \int_s^t (B(\tau)\xi_s|\xi_s)^{\frac{1}{2}} \, \mathrm{d}\tau.$$

Letting $\xi \coloneqq \xi_s$ vary over $\mathbb{R}^d$, we can write $\Phi^\pm(t, s, y, \xi) = y \cdot \xi \pm \varphi_{t,s}(\xi)$ with $\varphi_{t,s}(\xi) \coloneqq \int_s^t (B(\tau)\xi|\xi)^{1/2} \, \mathrm{d}\tau$, and plugging this expression back into (3.4), we get

$$K_\lambda^\pm(t, s, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{i(y \cdot \xi \pm \varphi_{t,s}(\xi))} \psi_\lambda(\eta) \, \mathrm{d}\eta = \mathcal{F}^{-1}(\mathrm{e}^{\pm i\varphi_{t,s}} \psi_\lambda)(y)$$

and therefore finally the expression

$$T_\lambda^\pm(t,s)f = (\mathrm{e}^{\pm i\varphi_{t,s}}\psi_\lambda)(\sqrt{\mathbf{L}})f.$$

In the next subsection, we verify that the operators

$$T^\pm(t,s) = \sum_{\lambda\in 2^{\mathbb{Z}}} T_\lambda^\pm(t,s) = \sum_{\lambda\in 2^{\mathbb{Z}}} (\mathrm{e}^{\pm i\varphi_{t,s}}\psi_\lambda)(\sqrt{\mathbf{L}}) \tag{3.5}$$

indeed define parametrices for the wave operator $\square_P = D_t^2 - P(t)$ in the scale of L-adapted Sobolev spaces $\mathrm{H}_L^\alpha(\mathbb{R}^d), \alpha \in \mathbb{R}$.

3.2.2. The Parametrix and its Properties

Let $s, t \in \mathbb{R}$. Recall the phase function

$$\varphi_{t,s} \colon \mathbb{R}^d \to \mathbb{R}, \quad \varphi_{t,s}(\xi) \coloneqq \int_s^t (B(\tau)\xi|\xi)^{\frac{1}{2}}\,\mathrm{d}\tau \tag{3.6}$$

constructed in the previous subsection.

Lemma 3.2.1 (Pointwise Estimates for Derivatives of the Phase Function)**.** *For $s, t, t_0 \in \mathbb{R}$, $\xi \in \mathbb{R}^d \setminus \{0\}$ and $\alpha \in \mathbb{N}_0^d$, we have the bounds*

$$|\partial_\xi^\alpha \varphi_{t,s}(\xi)| \lesssim_\alpha |\xi|^{1-|\alpha|}|t-s|, \tag{3.7}$$

$$|\partial_t \varphi_{t,s}(\xi)| \simeq |\xi|, \quad |\partial_t^2 \varphi_{t,s}(\xi)| \lesssim \|B'(t)\|\,|\xi|, \tag{3.8}$$

$$|\partial_t^2 \varphi_{t,s}(\xi) - \partial_t^2 \varphi_{t_0,s}(\xi)| \lesssim_{m_4} |\xi|(|t-t_0| + \|B'(t) - B'(t_0)\|), \tag{3.9}$$

where we wrote $\partial_t^2\varphi_{t_0,s}(\xi)$ for $\partial_t^2\varphi_{t,s}(\xi)|_{t=t_0}$. Moreover, $\varphi_{t,s} = -\varphi_{s,t}$ and thus $\partial_s^k \varphi_{t,s} = -\partial_\tau^k \varphi_{\tau,t}\,|_{\tau=s}$ for all $k \in \{0,1,2\}$, and $\varphi_{t,s}$ is positively homogeneous of degree one.

We first show that the operators defined in (3.5) are bounded operators on $\mathrm{H}_L^\alpha(\mathbb{R}^d), \alpha \in \mathbb{R}$.

Proposition 3.2.2 (Boundedness of Parametrices I)**.** *Let $\alpha \in \mathbb{R}$. Then, the linear operators*

$$T^\pm(t,s) \colon \mathrm{H}_L^\alpha(\mathbb{R}^d) \to \mathrm{H}_L^\alpha(\mathbb{R}^d), \quad T^\pm(t,s)f = \sum_{\lambda\in 2^{\mathbb{Z}}} (\mathrm{e}^{\pm i\varphi_{t,s}}\psi_\lambda)(\sqrt{\mathbf{L}})f$$

are bounded with operator norms uniformly bounded in $s, t \in \mathbb{R}$.

Proof. Let $s, t, \alpha \in \mathbb{R}$. As observed in Lemma 3.2.1, we have $\varphi_{t,s} = -\varphi_{s,t}$, which implies the identity $T^+(t,s) = T^-(s,t)$. Thus, it is enough to show the boundedness of $T(t,s) \coloneqq T^+(t,s)$ on $\mathrm{H}_L^\alpha(\mathbb{R}^d)$. Put $h \coloneqq \mathrm{e}^{i\varphi_{t,s}}$. Clearly, $\|h\|_\infty = 1$. Now, Proposition 2.3.5 (b) shows that $T(t,s)$ is bounded on $\mathrm{H}_L^\alpha(\mathbb{R}^d)$ with operator norm $\|T\| \lesssim \|h\|_\infty = 1$. □

Clearly, the wave operator $\Box_P = D_t^2 - P(t)$ is a differential operator of order two. Therefore, if $(R(t,s))_{t,s}$ is a nontrivial family of operators in $\mathcal{L}(\mathrm{H}^\alpha_L(\mathbb{R}^d))$ which is unrelated to $\Box_P$, one would expect that the action of $\Box_P$ on $R(t,s)$ loses two derivatives in the scale of L-adapted Sobolev spaces $\mathrm{H}^\alpha_L(\mathbb{R}^d)$. More precisely, we would expect $\Box_P R(t,s)\colon \mathrm{H}^\alpha_L(\mathbb{R}^d) \to \mathrm{H}^{\alpha-2}_L(\mathbb{R}^d)$ boundedly. However, as we shall see in the following, $\Box_P T^\pm(t,s)$ maps $\mathrm{H}^\alpha_L(\mathbb{R}^d)$ to $\mathrm{H}^{\alpha-1}_L(\mathbb{R}^d)$ boundedly. Consequently, $\Box_P T^\pm(t,s)$ behaves *one* order better than expected, and it is in this sense that the operators $T^\pm(t,s)$ can be thought of as parametrices, i.e., approximate solution operators to the wave equation $\Box_P u = 0$. This better behavior allows us to prove a representation of the solution of (3.1) (see Theorem 3.3.7), which we will subsequently use to show global-in-time Strichartz estimates. We begin with some preparatory lemmas.

Lemma 3.2.3 (Strong Differentiability of Parametrices I)**.** *Let $s_0, t_0 \in \mathbb{R}$ and assume $f \in \mathrm{H}^\alpha_L(\mathbb{R}^d)$ for some $\alpha \in \mathbb{R}$. Then, for all $k \in \{0,1,2\}$, the maps*

$$t \mapsto T^\pm(t,s_0)f \quad \textit{and} \quad s \mapsto T^\pm(t_0,s)f$$

belong to $C^k_b(\mathbb{R};\mathrm{H}^{\alpha-k}_L(\mathbb{R}^d))$ with

$$\begin{aligned}
D_t^k T^\pm(t,s_0)f &= \sum_{\lambda\in 2^{\mathbb{Z}}} \Big(D_t^k \mathrm{e}^{\pm i\varphi_{t,s_0}}\psi_\lambda\Big)(\sqrt{\mathbf{L}})f,\\
D_s^k T^\pm(t_0,s)f &= \sum_{\lambda\in 2^{\mathbb{Z}}} \Big(D_s^k \mathrm{e}^{\pm i\varphi_{t_0,s}}\psi_\lambda\Big)(\sqrt{\mathbf{L}})f \qquad \textit{and}\\
\|D_t^k T^\pm(t,s_0)f\|_{\alpha-k} &+ \|D_s^k T^\pm(t_0,s)f\|_{\alpha-k} \lesssim \|f\|_\alpha.
\end{aligned}$$

Proof. Let $s_0, t_0, \alpha \in \mathbb{R}$ and $f \in \mathrm{H}^\alpha_L(\mathbb{R}^d)$. Since $T^\pm(t,s) = T^\mp(s,t)$ for $s,t \in \mathbb{R}$, it suffices to consider the maps $u^\pm\colon t \mapsto T^\pm(t,s_0)f$. But in this case, the corresponding statement is just a straightforward consequence of Lemma 2.3.7 and Lemma 3.2.1. Indeed, put $h_{t,\lambda} \coloneqq \mathrm{e}^{\pm i\varphi_{t,s_0}}$ for all $t \in \mathbb{R}, \lambda \in 2^{\mathbb{Z}}$. Then, $h_{t,\lambda}$ is smooth away from the origin, and by Lemma 3.2.1, we have the bounds

$$\begin{aligned}
&|h_{t,\lambda}(\xi)| = 1,\\
&|D_t h_{t,\lambda}(\xi)| = |D_t\varphi_{t,s_0}(\xi)| \lesssim |\xi| \lesssim \langle\xi\rangle,\\
&|D_t^2 h_{t,\lambda}(\xi)| = \Big|\Big(D_t\varphi_{t,s_0}(\xi)\Big)^2 + D_t^2\varphi_{t,s_0}(\xi)\Big| \lesssim_{m_4} |\xi|^2 + |\xi| \lesssim \langle\xi\rangle^2.
\end{aligned} \tag{3.10}$$

Using (3.9) and (3.10), one also checks that

$$|D_t^2 h_{t,\lambda}(\xi) - D_t^2 h_{t_0,\lambda}(\xi)| \lesssim_{m_4} \langle\xi\rangle^3\Big(|t-t_0| + \|B'(t) - B'(t_0)\|\Big).$$

Therefore, Lemma 2.3.7 implies $u^\pm \in C^k_b(\mathbb{R};\mathrm{H}^{\alpha-k}_L(\mathbb{R}^d))$ for $k \in \{0,1,2\}$ as desired.

□

Lemma 3.2.4 (Derivative Gain on Dyadic Frequencies)**.** *Let $s, t \in \mathbb{R}$, $\lambda \in 2^{\mathbb{Z}}$, and $f \in \mathrm{H}^{\alpha}_{L}(\mathbb{R}^d)$ for some $\alpha \in \mathbb{R}$. Then,*

$$\Big(D_t^2 - P(t)\Big)\Big(\mathrm{e}^{i\varphi_{t,s}}\psi_\lambda\Big)(\sqrt{\mathbf{L}})f = \lambda\Big(\mathrm{e}^{i\varphi_{t,s}} \cdot r_{t,\lambda} \cdot \psi_\lambda\Big)(\sqrt{\mathbf{L}})f,$$

where $r_{t,\lambda}$ is smooth away from the origin with $\|r_{t,\lambda}\|_{\mathrm{L}^\infty(K_\lambda)} \lesssim \|B'(t)\|$ and $K_\lambda := \operatorname{supp}(\psi_\lambda)$.

Proof. On the one hand, dominated convergence and Lemma 3.2.1 imply

$$D_t^2(\mathrm{e}^{i\varphi_{t,s}}\psi_\lambda)(\sqrt{\mathbf{L}})f = (D_t^2\mathrm{e}^{i\varphi_{t,s}}\psi_\lambda)(\sqrt{\mathbf{L}})f. \tag{3.11}$$

Recalling that $\varphi_{t,s}(\xi) = \int_s^t (B(\tau)\xi|\xi)^{\frac{1}{2}}\,\mathrm{d}\tau$, we compute

$$D_t^2\mathrm{e}^{i\varphi_{t,s}(\xi)} = D_t((B(t)\xi|\xi)^{\frac{1}{2}} \cdot \mathrm{e}^{i\varphi_{t,s}(\xi)}) = \Big((B(t)\xi|\xi) + D_t(B(t)\xi|\xi)^{\frac{1}{2}}\Big)\mathrm{e}^{i\varphi_{t,s}(\xi)}.$$

Plugging this into (3.11), we get

$$\begin{aligned} &D_t^2(\mathrm{e}^{i\varphi_{t,s}}\psi_\lambda)(\sqrt{\mathbf{L}})f \\ &= ((B(t)\xi|\xi)\mathrm{e}^{i\varphi_{t,s}}\psi_\lambda)(\sqrt{\mathbf{L}})f + \lambda\Big(\mathrm{e}^{i\varphi_{t,s}}D_t(B(t)\tfrac{\xi}{\lambda}|\tfrac{\xi}{\lambda})^{\frac{1}{2}}\psi_\lambda\Big)(\sqrt{\mathbf{L}})f. \end{aligned}$$

By Proposition 2.1.10 (c), the first term on the right-hand side is equal to

$$\sum_{j=1}^d b_j(t)(\xi_j^2\mathrm{e}^{i\varphi_{t,s}}\psi_\lambda)(\sqrt{\mathbf{L}})f = \sum_{j=1}^d b_j(t)L_j(\mathrm{e}^{i\varphi_{t,s}}\psi_\lambda)(\sqrt{\mathbf{L}})f = P(t)(\mathrm{e}^{i\varphi_{t,s}}\psi_\lambda)(\sqrt{\mathbf{L}})f.$$

We therefore conclude

$$\Big(D_t^2 - P(t)\Big)\Big(\mathrm{e}^{i\varphi_{t,s}}\psi_\lambda\Big)(\sqrt{\mathbf{L}})f = \lambda\Big(\mathrm{e}^{i\varphi_{t,s}}D_t(B(t)\tfrac{\xi}{\lambda}|\tfrac{\xi}{\lambda})^{\frac{1}{2}}\psi_\lambda\Big)(\sqrt{\mathbf{L}})f.$$

But by (3.8), we have

$$|(D_t(B(t)\tfrac{\xi}{\lambda}|\tfrac{\xi}{\lambda})^{\frac{1}{2}}| = |D_t^2\varphi_{t,s}(\tfrac{\xi}{\lambda})| \lesssim \|B'(t)\|\,|\tfrac{\xi}{\lambda}| \simeq \|B'(t)\| \quad (\xi \in K_\lambda),$$

so the assertion follows setting $r_{t,\lambda}(\xi) := (D_t(B(t)\frac{\xi}{\lambda}|\frac{\xi}{\lambda})^{\frac{1}{2}}$. □

Theorem 3.2.5 (Derivative Gain I)**.** *Let $s, t, \alpha \in \mathbb{R}$. Then, the operator*

$$(D_t^2 - P(t))T^{\pm}(t,s)\colon \mathrm{H}^{\alpha}_{L}(\mathbb{R}^d) \to \mathrm{H}^{\alpha-1}_{L}(\mathbb{R}^d)$$

is bounded with an operator norm uniform in s and t. Moreover, we have the following estimates,

$$\begin{aligned} &(i) \quad \|(D_t^2 - P(t))T^{\pm}(t,s)f\|_{\mathrm{H}^{\alpha-1}_{L}(\mathbb{R}^d)} \lesssim \|B'(t)\| \cdot \|f\|_{\mathrm{H}^{\alpha}_{L}(\mathbb{R}^d)}, \\ &(ii) \quad \|(D_t^2 - P(t))T^{\pm}(t,s)f\|_{\mathrm{L}^1_t(\mathbb{R};\mathrm{H}^{\alpha-1}_{L}(\mathbb{R}^d))} \lesssim \|B'\|_1\|f\|_{\mathrm{H}^{\alpha}_{L}(\mathbb{R}^d)}. \end{aligned}$$

Proof. It is enough to prove the estimates (i) and (ii). Let $s,t,\alpha \in \mathbb{R}$ and $f \in \mathrm{H}^{\alpha}_{L}(\mathbb{R}^d)$. To ease notation, we just give the proof for $T \coloneqq T^+$ (the proof for T^- is analogous). By Lemma 3.2.3, we have $D_t^2 T(t,s)f \in \mathrm{H}^{\alpha-2}_{L}(\mathbb{R}^d)$. On the other hand, Proposition 3.2.2 and Lemma 2.3.8 yield $P(t)T(t,s)f \in \mathrm{H}^{\alpha-2}_{L}(\mathbb{R}^d)$. Thus, $(D_t^2 - P(t))T(t,s)f$ is a well-defined element in $\mathrm{H}^{\alpha-2}_{L}(\mathbb{R}^d)$. Now, Lemma 3.2.3 and Lemma 3.2.4 imply

$$\begin{aligned}(D_t^2 - P(t))T(t,s)f &= \sum_{\lambda\in 2^{\mathbb{Z}}} \Big(D_t^2 - P(t)\Big)\Big(\mathrm{e}^{i\varphi_{t,s}}\psi_\lambda\Big)(\sqrt{\mathbf{L}})f \\ &= \sum_{\lambda\in 2^{\mathbb{Z}}} \lambda\Big(\mathrm{e}^{i\varphi_{t,s}} r_{t,\lambda}\cdot\psi_\lambda\Big)(\sqrt{\mathbf{L}})f \\ &= \sum_{\lambda\in 2^{\mathbb{Z}}} \lambda(h_{t,s,\lambda}\psi_\lambda)(\sqrt{\mathbf{L}})f\end{aligned}$$

with

$$h_{t,s,\lambda}\colon \mathbb{R}^d\setminus\{0\}\to\mathbb{C},\quad h_{t,s,\lambda}(\xi) = \mathrm{e}^{i\varphi_{t,s}(\xi)} r_{t,\lambda}(\xi)$$

satisfying

$$\|h_{t,s,\lambda}\|_{\mathrm{L}^\infty(K_\lambda)} = \|r_{t,\lambda}\|_{\mathrm{L}^\infty(K_\lambda)} \lesssim \|B'(t)\|$$

uniformly in $\lambda \in 2^{\mathbb{Z}}$ and $s \in \mathbb{R}$. It therefore follows from Proposition 2.3.5 (b) that $\Big(D_t^2 - P(t)\Big)T(t,s)f$ in fact belongs to $\mathrm{H}^{\alpha-1}_{L}(\mathbb{R}^d)$ and that

$$\|(D_t^2 - P(t))T(t,s)f\|_{\mathrm{H}^{\alpha-1}_{L}(\mathbb{R}^d)} \lesssim \|B'(t)\|\,\|f\|_{\mathrm{H}^{\alpha}_{L}(\mathbb{R}^d)}.$$

This proves (i), and (ii) follows by integrating (i) with respect to $t \in \mathbb{R}$. □

Based on the operators $T^{\pm}(t,s)$, we define the following two families of operators $C(t,s)$ and $S(t,s)$, which play a similar role as $\mathrm{Cos}((t-s)\sqrt{L})$ and $i(t-s)\mathrm{Sinc}((t-s)\sqrt{L})$ do in the time-independent case.

Definition 3.2.6. Let $s,t,\alpha \in \mathbb{R}$. For $f \in \mathrm{H}^{\alpha}_{L}(\mathbb{R}^d)$, we define

$$\begin{aligned}C(t,s)f &\coloneqq \sum_{\lambda\in 2^{\mathbb{Z}}} \Big((\cos\circ\varphi_{t,s})\cdot\psi_\lambda\Big)(\sqrt{\mathbf{L}})f, \\ S(t,s)f &\coloneqq i\sum_{\lambda\in 2^{\mathbb{Z}}} \left(\frac{\sin\circ\varphi_{t,s}}{\partial_s\varphi_{t,s}}\cdot\psi_\lambda\right)(\sqrt{\mathbf{L}})f. \qquad (3.12)\end{aligned}$$

The properties established for $T^{\pm}(t,s)$ extend to $C(t,s)$ in a natural way. The situation for $S(t,s)$ is slightly different. In view of the factor $|\partial_s\varphi_{t,s}|^{-1} \simeq |\xi|^{-1}$ in (3.12), the mapping properties of $S(t,s)$ are even improved by one order in the $\mathrm{H}^{\alpha}_{L}(\mathbb{R}^d)$-scale (see Lemma 3.2.7 (b) (iv) below). This will be crucial in Subsection 3.3.2. Another difference is that the very same factor reduces the strong differentiability of $s \mapsto S(t,s)$ by one order, which fortunately turns out to be irrelevant for our purposes.

Lemma 3.2.7 (Relation between Parametrices)**.** *Let $s,t,\alpha \in \mathbb{R}$.*

(a) We have

$$C(t,s)f = \frac{1}{2}\Big(T^+(t,s) + T^-(t,s)\Big)f \quad (f \in \mathrm{H}^\alpha_L(\mathbb{R}^d)). \tag{3.13}$$

(b) Define for $f \in \mathrm{H}^\alpha_L(\mathbb{R}^d)$

$$\tilde{T}(t,s)f \coloneqq \sum_{\lambda\in 2^{\mathbb{Z}}} \Big(\tilde{h}_{t,s}\psi_\lambda\Big)(\sqrt{\mathbf{L}})f, \quad \tilde{h}_{t,s}(\xi) \coloneqq \langle\xi\rangle \frac{\sin(\varphi_{t,s}(\xi))}{\partial_s\varphi_{t,s}(\xi)}.$$

Then:

(i) $\tilde{T}(t,s) \in \mathcal{L}(\mathrm{H}^\alpha_L(\mathbb{R}^d))$ *with* $\|\tilde{T}(t,s)\| \lesssim \langle t-s\rangle$.

(ii) The map $\tau \mapsto \tilde{T}(\tau,s)f$ *belongs to* $C^k(\mathbb{R};\mathrm{H}^{\alpha-k}_L(\mathbb{R}^d))$ *for all* $k \in \{0,1,2\}$.

(iii) The map $\tau \mapsto \tilde{T}(t,\tau)f$ *belongs to* $C^k(\mathbb{R};\mathrm{H}^{\alpha-k}_L(\mathbb{R}^d))$ *for all* $k \in \{0,1\}$.

(iv) We have the identity

$$S(t,s)f = i\langle D_L\rangle^{-1}\,\tilde{T}(t,s)f \quad (f \in \mathrm{H}^\alpha_L(\mathbb{R}^d)). \tag{3.14}$$

(v) We have

$$\Big(D_t^2 - P(t)\Big)(\tilde{h}_{t,s}\psi_\lambda)(\sqrt{\mathbf{L}})f = \lambda(\tilde{r}_{t,s,\lambda}\psi_\lambda)(\sqrt{\mathbf{L}})f \quad (\lambda \in 2^{\mathbb{Z}}),$$

where $\tilde{r}_{t,s,\lambda}\colon \mathbb{R}^d\setminus\{0\} \to \mathbb{C}$ *is smooth and satisfies the estimate* $\|\tilde{r}_{t,s,\lambda}\|_{\mathrm{L}^\infty_\xi(K_\lambda)} \lesssim \|B'(t)\|$, $K_\lambda \coloneqq \operatorname{supp}(\psi_\lambda)$.

Proof. Assertion (a) is obvious. To prove (b), we first observe the estimates

$$\begin{aligned} |\tilde{h}_{t,s}(\xi)| &\lesssim \langle t-s\rangle, & |\partial_t\tilde{h}_{t,s}(\xi)| &\lesssim \langle\xi\rangle, \\ |\partial_s\tilde{h}_{t,s}(\xi)| &\lesssim_{m_4} \langle t-s\rangle\langle\xi\rangle, & |\partial_t^2\tilde{h}_{t,s}(\xi)| &\lesssim \langle t-s\rangle\langle\xi\rangle^2 \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} |D_s\tilde{h}_{t,s}(\xi) - D_s\tilde{h}_{t,s_0}(\xi)| &\lesssim_{m_4} \langle\xi\rangle^2\Big(|s-s_0| + \|B'(s)-B'(s_0)\|\Big), \\ |D_t^2\tilde{h}_{t,s}(\xi) - D_t^2\tilde{h}_{t_0,s}(\xi)| &\lesssim_{m_4} \langle\xi\rangle^3\Big(|t-t_0| + \|B'(t)-B'(t_0)\|\Big). \end{aligned} \tag{3.16}$$

for $s,t,s_0,t_0 \in \mathbb{R}$, $\xi \in \mathbb{R}^d\setminus\{0\}$. Indeed, (3.8) implies

$$|\tilde{h}_{t,s}(\xi)| \le \frac{\langle\xi\rangle}{|\partial_s\varphi_{t,s}(\xi)|} \simeq \frac{\langle\xi\rangle}{|\xi|} \lesssim 1 \quad \text{for } |\xi| \ge 1,$$

while for $|\xi| \le 1$, (3.7) gives

$$|\tilde{h}_{t,s}(\xi)| \le 2\frac{|\sin(\varphi_{t,s}(\xi))|}{|\partial_s\varphi_{t,s}(\xi)|} \lesssim \frac{|\varphi_{t,s}(\xi)|}{|\xi|} \lesssim |t-s|.$$

This shows $|\tilde{h}_{t,s}(\xi)| \lesssim \langle t-s\rangle$. Using this estimate and (3.8), we deduce

$$\begin{aligned} |\partial_t \tilde{h}_{t,s}(\xi)| &= \frac{|\cos(\varphi_{t,s}(\xi))|\,|\partial_t\varphi_{t,s}(\xi)|}{|\partial_s\varphi_{t,s}(\xi)|}\langle\xi\rangle \lesssim \langle\xi\rangle, \\ |\partial_s \tilde{h}_{t,s}(\xi)| &= \left|\cos(\varphi_{t,s}(\xi))\langle\xi\rangle - \frac{\partial_s^2\varphi_{t,s}(\xi)}{\partial_s\varphi_{t,s}(\xi)}\tilde{h}_{t,s}(\xi)\right| \\ &\lesssim \langle\xi\rangle + \|B'(s)\|\,|\tilde{h}_{t,s}(\xi)| \\ &\lesssim_{m_4} \langle\xi\rangle + \langle t-s\rangle \le 2\langle t-s\rangle\langle\xi\rangle \end{aligned}$$

and finally

$$\begin{aligned} |\partial_t^2 \tilde{h}_{t,s}(\xi)| &= \left|\tilde{h}_{t,s}(\xi)|\partial_t\varphi_{t,s}(\xi)|^2 - \frac{\partial_t^2\varphi_{t,s}(\xi)}{\partial_s\varphi_{t,s}(\xi)}\cos(\varphi_{t,s}(\xi))\langle\xi\rangle\right| \\ &\lesssim \langle t-s\rangle|\xi|^2 + \|B'(t)\|\langle\xi\rangle \lesssim_{m_4} \langle t-s\rangle\langle\xi\rangle^2. \end{aligned}$$

The bounds (3.16) are shown similarly. Using (3.15) and (3.16), one may argue as in the proofs of Proposition 3.2.2 and Lemma 3.2.3 to show assertions (i), (ii), and (iii). Furthermore, the boundedness of $\langle D_L\rangle^{-1}\colon \mathrm{H}^\alpha_L(\mathbb{R}^d) \to \mathrm{H}^{\alpha+1}_L(\mathbb{R}^d)$ (see Proposition 2.1.6 (d)) and Proposition 2.1.15 (c) shows for each $f \in \mathrm{H}^\alpha_L(\mathbb{R}^d)$

$$\begin{aligned} i\langle D_L\rangle^{-1}\tilde{T}(t,s)f &= i\sum_{\lambda\in 2^{\mathbb{Z}}} \langle D_L\rangle^{-1}(\tilde{h}_{t,s}\psi_\lambda)(\sqrt{\mathbf{L}})f \\ &= i\sum_{\lambda\in 2^{\mathbb{Z}}} (\langle\xi\rangle^{-1}\tilde{h}_{t,s}\psi_\lambda)(\sqrt{\mathbf{L}})f = S(t,s)f, \end{aligned}$$

proving (iv). Finally, (v) is exactly shown as Lemma 3.2.4. □

The following two theorems are the main results of this section.

Theorem 3.2.8 (Boundedness and Strong Differentiability of Parametrices II)**.** *Let $\alpha \in \mathbb{R}$.*

(a) Let $s, t \in \mathbb{R}$. Then, the linear operators

$$C(t,s)\colon \mathrm{H}^\alpha_L(\mathbb{R}^d) \to \mathrm{H}^\alpha_L(\mathbb{R}^d), \tag{3.17}$$
$$S(t,s)\colon \mathrm{H}^\alpha_L(\mathbb{R}^d) \to \mathrm{H}^{\alpha+1}_L(\mathbb{R}^d) \tag{3.18}$$

are bounded with $\|C(t,s)\| \lesssim 1$ and $\|S(t,s)\| \lesssim \langle t-s\rangle$.

(b) Let $f \in \mathrm{H}^\alpha_L(\mathbb{R}^d)$ and $s_0, t_0 \in \mathbb{R}$ and $k \in \{0,1,2\}$. Then, the map $u_{s_0}\colon t \mapsto C(t,s_0)f$ belongs to $C^k_b(\mathbb{R};\mathrm{H}^{\alpha-k}_L(\mathbb{R}^d))$ with $\|u^{(k)}_{s_0}(t)\|_{\alpha-k} \lesssim \|f\|_\alpha$. The map $v_{s_0}\colon t \mapsto S(t,s_0)f$ belongs to $C^k(\mathbb{R};\mathrm{H}^{\alpha+1-k}_L(\mathbb{R}^d))$ with

$\|v_{s_0}^{(k)}(t)\|_{\alpha+1-k} \lesssim \langle t - s_0\rangle^{1-(1\wedge k)}\|f\|_\alpha$. The map $w_{t_0}\colon s \mapsto S(t_0, s)f$ belongs to $\dot{C}_b^1(\mathbb{R}; \mathrm{H}_L^\alpha(\mathbb{R}^d))$. Moreover,

$$C(t,t) = \mathrm{Id}, \qquad D_t C(t,s)|_{t=s} = 0, \tag{3.19}$$

$$S(t,t) = 0, \qquad D_t S(t,s)|_{t=s} = \mathrm{Id}, \tag{3.20}$$

where the time derivative is to be understood in the strong sense.

Proof. In view of the identity (3.13), the boundedness of $C(t,s)$ and its strong differentiability as stated in (b) immediately follow from Proposition 3.2.2 and Lemma 3.2.3, respectively. Combining the strong differentiability of $C(t,s)$ with Proposition 2.3.4 then also yields (3.19). To prove the corresponding statements for $S(t,s)$, we use that $S(t,s) = i\langle D_L\rangle^{-1}\tilde{T}(t,s)$ by Lemma 3.2.7 (b) (iv). Thus, the assertions for $S(t,s)$ follow from the properties of $\tilde{T}(t,s)$ stated in Lemma 3.2.7 (b) (i)-(iii), and the fact that $\langle D_L\rangle^{-1}$ gains one derivative in the $\mathrm{H}_L^\alpha(\mathbb{R}^d)$-scale, i.e., $\langle D_L\rangle^{-1}\colon \mathrm{H}_L^\alpha(\mathbb{R}^d) \to \mathrm{H}_L^{\alpha+1}$. □

Theorem 3.2.9 (Derivative Gain II)**.** *Let $s, t, \alpha \in \mathbb{R}$. Then,*

$$\Big(D_t^2 - P(t)\Big)C(t,s)\colon \mathrm{H}_L^\alpha(\mathbb{R}^d) \to \mathrm{H}_L^{\alpha-1}(\mathbb{R}^d),$$
$$\Big(D_t^2 - P(t)\Big)S(t,s)\colon \mathrm{H}_L^\alpha(\mathbb{R}^d) \to \mathrm{H}_L^{\alpha}(\mathbb{R}^d)$$

are bounded. More precisely, the following estimates hold true, uniformly in $s \in \mathbb{R}$.

$$\|\Big(D_t^2 - P(t)\Big)C(t,s)f\|_{\mathrm{H}_L^{\alpha-1}(\mathbb{R}^d)} \lesssim \|B'(t)\| \cdot \|f\|_{\mathrm{H}_L^\alpha(\mathbb{R}^d)},$$
$$\|\Big(D_t^2 - P(t)\Big)S(t,s)f\|_{H_L^{\alpha}(\mathbb{R}^d)} \lesssim \|B'(t)\| \cdot \|f\|_{\mathrm{H}_L^\alpha(\mathbb{R}^d)},$$

and

$$\|\Big(D_t^2 - P(t)\Big)C(t,s)f\|_{\mathrm{L}_t^1(\mathbb{R};\mathrm{H}_L^{\alpha-1}(\mathbb{R}^d))} \lesssim \|B'\|_1\|f\|_{\mathrm{H}_L^\alpha(\mathbb{R}^d)},$$
$$\|\Big(D_t^2 - P(t)\Big)S(t,s)f\|_{\mathrm{L}_t^1(\mathbb{R};\mathrm{H}_L^{\alpha}(\mathbb{R}^d))} \lesssim \|B'\|_1\|f\|_{\mathrm{H}_L^\alpha(\mathbb{R}^d)}.$$

Proof. Once again, the properties for $C(t,s)$ follow from Theorem 3.2.5 and the identity (3.13). To prove the assertions for $S(t,s)$, we once again use $S(t,s) = i\langle D_L\rangle^{-1}\tilde{T}(t,s)$ as stated in Lemma 3.2.7 (b) (iv). Using (b) (v) of the very same lemma in place of Lemma 3.2.4, we find that Theorem 3.2.5 holds true with $\tilde{T}(t,s)$ in place of $T^\pm(t,s)$. Thus, the assertions for $S(t,s)$ follow again from the fact that $\langle D_L\rangle^{-1}\colon \mathrm{H}_L^\alpha(\mathbb{R}^d) \to \mathrm{H}_L^{\alpha+1}$. □

3.3. Weak Solutions in $\mathrm{H}_L^\alpha(\mathbb{R}^d)$

In this section, we establish the existence and uniqueness of weak solutions to (3.1) in $\mathrm{H}_L^\alpha(\mathbb{R}^d)$, using the parametrices constructed in the previous section. First, we define weak solutions in $\mathrm{H}_L^\alpha(\mathbb{R}^d)$ in the same manner as in

Definition 1.1.2, except that the classical Sobolev spaces $\mathrm{H}^\alpha(\mathbb{R}^d)$ are replaced by the L-adapted ones. For later reference, we give the definition.

Definition 3.3.1 (Weak Solution in $\mathrm{H}^\alpha_L(\mathbb{R}^d)$)**.** Let $\alpha \in \mathbb{R}$ and suppose that $g \in \mathrm{H}^\alpha_L(\mathbb{R}^d), h \in \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d)$, and $F \in \mathrm{L}^1(\mathbb{R};\mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))$. Then, a function $u \in C(\mathbb{R};\mathrm{H}^\alpha_L(\mathbb{R}^d)) \cap C^1(\mathbb{R};\mathrm{H}^{\alpha-1}_L(\mathbb{R}^d)) \cap W^{2,1}_{\mathrm{loc}}(\mathbb{R};\mathrm{H}^{\alpha-2}_L(\mathbb{R}^d))$ is called a *weak solution* of (3.1) in $\mathrm{H}^\alpha_L(\mathbb{R}^d)$ if

$$\begin{cases} D_t^2 u(t) = P(t)u(t) + F(t) \quad \text{in } \mathrm{H}^{\alpha-2}_L(\mathbb{R}^d) \quad \text{for a.e. } t \in \mathbb{R}, \\ \quad u(0) = g, \\ D_t u(0) = h. \end{cases}$$

We will use energy methods to ensure that weak solutions are unique. Existence will then be established using an iterative procedure involving the parametrices $C(t,s)$ and $S(t,s)$.

3.3.1. Uniqueness of Weak Solutions in $\mathrm{H}^\alpha_L(\mathbb{R}^d)$

The classical energy inequalities yield for linear wave equations with $C_b^\infty(\mathbb{R}^d)$-coefficients the uniqueness of weak solutions in the scale of the standard Sobolev spaces $\mathrm{H}^\alpha(\mathbb{R}^d)$, $\alpha \in \mathbb{R}$. As the coefficients of our wave operator $\Box_P$ possess only limited regularity, these energy inequalities are applicable only in a restricted range of exponents α. However, combining these with Proposition 2.3.3 gives a first result.

Lemma 3.3.2 (Classical Energy Inequalities)**.** *Let $J \subseteq \mathbb{R}$ be a bounded open interval with $0 \in J$. If*

$$u \in C(\overline{J};\mathrm{H}^1_L(\mathbb{R}^d)) \cap C^1(\overline{J};\mathrm{L}^2(\mathbb{R}^d)) \cap W^{2,1}(J;\mathrm{H}^{-1}_L(\mathbb{R}^d))$$

with $\Box_P u = (D_t^2 - P(t))u \in \mathrm{L}^1(J;\mathrm{L}^2(\mathbb{R}^d))$, then

$$\|u\|_{C(\overline{J};\mathrm{H}^1_L(\mathbb{R}^d))} \lesssim_{|J|} \|u(0)\|_{\mathrm{H}^1_L(\mathbb{R}^d)} + \|D_t u(0)\|_{\mathrm{L}^2(\mathbb{R}^d)} + \|\Box_P u\|_{\mathrm{L}^1(J;\mathrm{L}^2(\mathbb{R}^d))}. \tag{3.21}$$

Proof. We first observe that (3.21) holds with $\mathrm{H}^1_L(\mathbb{R}^d)$ replaced by the classical Sobolev space $\mathrm{H}^1(\mathbb{R}^d)$. Indeed, a careful inspection of the proof of [53, Chapter I, Theorem 3.1] shows that the latter is applicable with $s = 0$ in the assumptions of that theorem (the coefficients $c_j(t,x) = b_j(t)a_j(x_j)$ of $P(t)$ are C^1 with respect to time and Lipschitz w.r.t. x, so taking one time derivative and integrating by parts once w.r.t. x is justified to make the energy inequality work). But, $\mathrm{H}^{\pm 1}_L(\mathbb{R}^d) = \mathrm{H}^{\pm 1}(\mathbb{R}^d)$ by Proposition 2.3.3, so the claim follows. ☐

Note that Lemma 3.3.2 in particular implies the uniqueness of weak solutions in $\mathrm{H}^1_L(\mathbb{R}^d)$. In the following, we lift this result to general exponents $\alpha \in \mathbb{R}$.

The reason why we can do this is that, while $\langle D_x \rangle^\alpha$ does not commute with $P(t) = \sum_{j=1}^d b_j(t) L_j$ for large α (by the limited regularity of the coefficients a_j), the fractional power $\langle D_L \rangle^\alpha$ in fact *does* (on appropriate subsets of functions). In order to exploit this fact, we need the following approximation lemma. To state it, we introduce some notation first. Let $\alpha \in \mathbb{R}$, $p \in [1, \infty]$, $m \in \mathbb{N}_0$, and $J \subseteq \mathbb{R}$ be a bounded open interval. Put

$$\mathcal{W}^{m,p}_{J,\alpha} := \begin{cases} W^{m,p}(J; \mathrm{H}^\alpha_L(\mathbb{R}^d)) & \text{for } p \in [1, \infty), \\ C^m(\overline{J}; \mathrm{H}^\alpha_L(\mathbb{R}^d)) & \text{for } p = \infty. \end{cases}$$

Let $u \in \mathcal{W}^{m,p}_{J,\alpha}$ and $n \in \mathbb{N}_0$. Then, we set $\mathbb{Z}_n := \{-n, \dots, n\}$ and define

$$(\Pi_n u)(t) := \sum_{\lambda \in 2^{\mathbb{Z}_n}} \psi_\lambda(\sqrt{\mathbf{L}}) u(t) \quad \text{for all } t \in \overline{J}$$

if $p = \infty$. If $p \in [1, \infty)$, we define $\Pi_n u$ by the same expression in an almost everywhere sense. If $I \Subset J$ is an open subinterval and $4^{-n} < \operatorname{dist}(I, J^c)$, we define

$$(S_n u)(t) := (\varphi_n * u)(t) = \int_{\mathbb{R}} \varphi_n(t - s) u(s) \, \mathrm{d}s \quad \text{for } t \in \overline{I},$$

where $\varphi_n(t) := 4^n \varphi(4^n t)$ and $\varphi \in C_c^\infty(\mathbb{R})$ satisfies $\operatorname{supp}(\varphi) \subseteq (-1, 1)$ and $\int_{\mathbb{R}} \varphi(t) \, \mathrm{d}t = 1$.

Lemma 3.3.3 (Approximation Lemma)**.** *Let $I \Subset J$ be bounded open intervals in $\mathbb{R}$ and $m, \ell \in \mathbb{N}_0, p \in [1, \infty], \alpha, \gamma \in \mathbb{R}$.*

(a) For all $n \in \mathbb{N}_0$, we have $\Pi_n \in \mathcal{L}(\mathcal{W}^{m,p}_{J,\alpha}, \mathcal{W}^{m,p}_{J,\alpha+\gamma})$ with $\|\Pi_n\| \lesssim_\gamma 2^{(n+1)\gamma_+}$. Moreover, $\Pi_n \to \mathrm{Id}$ strongly on $\mathcal{W}^{m,p}_{J,\alpha}$ as $n \to \infty$.

(b) For all $n \geq n_0$, we have $S_n \in \mathcal{L}(\mathcal{W}^{m,p}_{J,\alpha}, \mathcal{W}^{m+\ell,p}_{I,\alpha})$ with $\|S_n\| \lesssim 4^{n\ell_+}$, where n_0 is such that $4^{-n_0} < \operatorname{dist}(I, J^c)$. Moreover, if $u \in \mathcal{W}^{m,p}_{J,\alpha}$, then $(S_n u)^{(k)} = S_n u^{(k)}$ for $k \leq m$ and $S_n u \to u|_I$ in $\mathcal{W}^{m,p}_{I,\alpha}$ as $n \to \infty$.

Proof. (*a*) Suppose first that $p \in [1, \infty)$. If $\gamma \leq 0$, then $\|\Pi_n\| \leq 1$ since $\mathrm{H}^\alpha_L(\mathbb{R}^d) \hookrightarrow \mathrm{H}^{\alpha+\gamma}_L(\mathbb{R}^d)$ in this case. Therefore, we may suppose $\gamma > 0$. By Proposition 2.1.15 (c) and Proposition 2.1.10 (d), we then have

$$\|\Pi_n u(t)\|_{\mathrm{H}^{\alpha+\gamma}_L(\mathbb{R}^d)} \lesssim \sum_{\lambda \in 2^{\mathbb{Z}_n}} \langle \lambda \rangle^\gamma \|u(t)\|_{\mathrm{H}^\alpha_L(\mathbb{R}^d)} \lesssim_\gamma 2^{(n+1)\gamma} \|u(t)\|_{\mathrm{H}^\alpha_L(\mathbb{R}^d)}$$

for a.e. $t \in J$. Raising this inequality to the p-th power and integrating it with respect to $t \in J$, we deduce the bound $\|\Pi_n\| \lesssim 2^{(n+1)\gamma}$. This proves the first claim. To prove the second claim, note that we have proved that $(\Pi_n)_n$ is in particular uniformly bounded on $\mathcal{W}^{m,p}_{J,\alpha}$. Let $v \in D := C^\infty(\overline{J}; \mathrm{H}^\alpha_L(\mathbb{R}^d))$. Then, $\Pi_n v \to v$ in $\mathcal{W}^{m,p}_{J,\alpha}$ by Proposition 2.3.4 and the uniform continuity

of v. Now, since D is dense in $\mathcal{W}^{m,p}_{J,\alpha}$ (see e.g. [10, Corollary 1.4.37]), the strong convergence of $(\Pi_n)_n$ to Id on $\mathcal{W}^{m,p}_{J,\alpha}$ follows from a standard density argument using uniform boundedness. The case $p=\infty$ is proved similarly.
(*b*) These assertions are proved exactly as in the case of scalar-valued functions (see e.g. [29, Section 1.3]). □

Proposition 3.3.4 (Energy Estimates in $\mathrm{H}_L^\alpha(\mathbb{R}^d)$)**.** *Let* $J \subseteq \mathbb{R}$ *be a bounded open interval with* $0 \in J$. *Suppose further that for some* $\alpha \in \mathbb{R}$, *the function* u *belongs to* $C(\overline{J};\mathrm{H}_L^\alpha(\mathbb{R}^d)) \cap C^1(\overline{J};\mathrm{H}_L^{\alpha-1}(\mathbb{R}^d)) \cap W^{2,1}(J;\mathrm{H}_L^{\alpha-2}(\mathbb{R}^d))$. *If* $\Box_P u \in \mathrm{L}^1(J;\mathrm{H}_L^{\alpha-1}(\mathbb{R}^d))$, *then*

$$\|u\|_{C(\overline{J};\mathrm{H}_L^\alpha(\mathbb{R}^d))} \lesssim_{|J|} \|u(0)\|_{\mathrm{H}_L^\alpha(\mathbb{R}^d)} + \|D_t u(0)\|_{\mathrm{H}_L^{\alpha-1}(\mathbb{R}^d)} + \|\Box_P u\|_{\mathrm{L}^1(J;\mathrm{H}_L^{\alpha-1}(\mathbb{R}^d))}.$$

Proof. Let $J \subseteq \mathbb{R}, \alpha \in \mathbb{R}$ and u be as in the statement of the proposition. Fix some open interval I with $0 \in \overline{I} \subseteq J$. To ease notation, we simply write $\mathrm{H}_L^\alpha := \mathrm{H}_L^\alpha(\mathbb{R}^d)$, $\mathrm{L}^2 := \mathrm{L}^2(\mathbb{R}^d)$, etc. in the following. Now, for $n \in \mathbb{N}$ with $4^n \geq (\mathrm{dist}(I,J^c))^{-1}$, we define $u_n \colon \overline{I} \to \mathrm{H}_L^\alpha$ by

$$u_n(t) := (S_n \Pi_n u)(t) = \int_{\mathbb{R}} \varphi_n(t-s)\Pi_n u(s)\,\mathrm{d}s \quad (t \in \overline{I}).$$

Observe that u_n belongs to $C^\infty(\overline{I};\mathrm{H}_L^\infty)$. Indeed, since $u \in C(\overline{J};\mathrm{H}_L^\alpha)$, it follows that $\Pi_n u \in C(\overline{J};\mathrm{H}_L^\beta)$ for all $\beta \in \mathbb{R}$ by Lemma 3.3.3 (a) and therefore $u_n \in C^k(\overline{I};\mathrm{H}_L^\beta)$ by Lemma 3.3.3 (b) for all $\beta \in \mathbb{R}$ and $k \in \mathbb{N}_0$. This proves $u_n \in C^\infty(\overline{I};\mathrm{H}_L^\infty)$. But then also $v_n := \langle D_L\rangle^{\alpha-1}u_n \in C^\infty(\overline{I};\mathrm{H}_L^\infty)$ and $\Box_P v_n \in C^1(\overline{I};\mathrm{H}_L^\infty) \subseteq \mathrm{L}^1(I;\mathrm{L}^2)$ (note that the loss of regularity in t arises from the fact that the b_j are only C^1). Applying Lemma 3.3.2 to v_n, we obtain

$$\begin{aligned}\|u_n\|_{C(\overline{I};\mathrm{H}_L^\alpha)} &= \|v_n\|_{C(\overline{I};\mathrm{H}_L^1)} \\ &\lesssim_{|I|} \|v_n(0)\|_{\mathrm{H}_L^1} + \|D_t v_n(0)\|_{\mathrm{L}^2} + \|\Box_P v_n\|_{\mathrm{L}^1(I;\mathrm{L}^2)} \\ &= \|u_n(0)\|_{\mathrm{H}_L^\alpha} + \|D_t u_n(0)\|_{\mathrm{H}_L^{\alpha-1}} + \|\Box_P u_n\|_{\mathrm{L}^1(I;\mathrm{H}_L^{\alpha-1})},\end{aligned} \tag{3.22}$$

where, for the last equality, we used that $\Box_P$ and $\langle D_L\rangle^{\alpha-1}$ commute when applied to u_n. Now, we want to pass to the limit $n \to \infty$ in (3.22). By Lemma 3.3.3, we infer $u_n \to u$ in $C(\overline{I};\mathrm{H}_L^\alpha)$ and $D_t u_n \to D_t u$ in $C^1(\overline{I};\mathrm{H}_L^{\alpha-1})$. We claim that

$$\|\Box_P u_n\|_{\mathrm{L}^1(I;\mathrm{H}_L^{\alpha-1}} \lesssim_{|I|} \|\Box_P u\|_{\mathrm{L}^1(J;\mathrm{H}_L^{\alpha-1})} + 2^{-n}\|u\|_{C(\overline{J};\mathrm{H}_L^\alpha)}. \tag{3.23}$$

Taking (3.23) for granted, we may let $n \to \infty$ in (3.22) to obtain

$$\|u\|_{C(\overline{I};\mathrm{H}_L^\alpha)} \lesssim_{|I|} \|u(0)\|_{\mathrm{H}_L^\alpha} + \|D_t u(0)\|_{\mathrm{H}_L^{\alpha-1}} + \|\Box_P u\|_{\mathrm{L}^1(J;\mathrm{H}_L^{\alpha-1})}, \tag{3.24}$$

and we infer the claim from (3.24) by simply exhausting J by open, compactly contained subintervals $I \Subset J$. So it remains to prove (3.23). First, note that since $u \in W^{2,1}(J;\mathrm{H}_L^{\alpha-2})$ by assumption, Lemma 3.3.3 (b) implies

$$D_t^2 u_n = D_t^2(\varphi_n * \Pi_n u) = (\varphi_n * \Pi_n D_t^2 u) = S_n \Pi_n D_t^2 u \tag{3.25}$$

in $\mathrm{L}^1(I;\mathrm{H}_L^{\alpha-2})$. On the other hand, since $P(t) = \sum_{j=1}^d b_j(t)L_j$, one may verify that

$$(Pu_n)(t) = (S_n\Pi_n Pu)(t) - Ru(t) \quad \text{for } t \in \overline{I} \tag{3.26}$$

with $Ru = \sum_{j=1}^d R_j u$ and

$$R_j u(t) = \int_{\mathbb{R}} \varphi_n(t-s)(b_j(t) - b_j(s))(L_j\Pi_n u)(s)\,\mathrm{d}s.$$

Note that by Lemma 2.3.8 and Lemma 3.3.3 (a)

$$\begin{aligned}\|R_j u(t)\|_{\mathrm{H}_L^{\alpha-1}} &\le \|b_j'\|_\infty \|\Pi_n u\|_{C(\overline{J};\mathrm{H}_L^{\alpha+1})} \int_{\mathbb{R}} |\varphi_n(t-s)||t-s|\,\mathrm{d}s \\ &\lesssim m_4 \cdot \left(2^{(n+1)}\|\Pi_n u\|_{C(\overline{J};\mathrm{H}_L^{\alpha})}\right) \cdot 4^{-n}\|\varphi\|_1 \\ &\lesssim 2^{-n}\|u\|_{C(\overline{J};\mathrm{H}_L^{\alpha})}\end{aligned}$$

for each $t \in \overline{I}$ and thus

$$\|Ru\|_{\mathrm{L}^1(I;\mathrm{H}_L^{\alpha-1})} \lesssim_{|I|} 2^{-n}\|u\|_{C(\overline{J},\mathrm{H}_L^{\alpha})}. \tag{3.27}$$

Combining (3.25) and (3.26), we conclude

$$\Box_P u_n = (D_t^2 - P)u_n = S_n\Pi_n\Box_P u - Ru \quad \text{in } \mathrm{L}^1(I;\mathrm{H}_L^{\alpha-1}),$$

and applying Lemma 3.3.3 and (3.27) gives

$$\begin{aligned}\|\Box_P u_n\|_{\mathrm{L}^1(I;\mathrm{H}_L^{\alpha-1})} &\le \|S_n\Pi_n\Box_P u_n\|_{\mathrm{L}^1(I;\mathrm{H}_L^{\alpha-1})} + \|Ru\|_{\mathrm{L}^1(I;\mathrm{H}_L^{\alpha-1})} \\ &\lesssim_{|I|} \|\Box_P u_n\|_{\mathrm{L}^1(I;\mathrm{H}_L^{\alpha-1})} + 2^{-n}\|u\|_{C(\overline{J},\mathrm{H}_L^{\alpha})}\end{aligned}$$

as desired. This completes the proof. □

Corollary 3.3.5 (Uniqueness of Weak Solutions in $\mathrm{H}_L^\alpha(\mathbb{R}^d)$)**.** *Let $\alpha \in \mathbb{R}$ and suppose that $g \in \mathrm{H}_L^\alpha(\mathbb{R}^d)$, $h \in \mathrm{H}_L^{\alpha-1}(\mathbb{R}^d)$, and $F \in \mathrm{L}^1(\mathbb{R};\mathrm{H}_L^{\alpha-1}(\mathbb{R}^d))$. If u and v are weak solutions to* (3.1) *in $\mathrm{H}_L^\alpha(\mathbb{R}^d)$, then $u = v$.*

Proof. Define $J_n := (-n, n)$ for $n \in \mathbb{N}$. Applying Proposition 3.3.4 to $w_n := (u-v)|_{\overline{J_n}}$ gives $w_n = 0$. Letting $n \to \infty$ yields $w = 0$ and thus $u = v$ as desired. □

3.3.2. Existence of Weak Solutions in $\mathrm{H}_L^\alpha(\mathbb{R}^d)$

In this subsection, we aim to establish the existence of weak solutions to (3.1) in $\mathrm{H}_L^\alpha(\mathbb{R}^d)$. We prove existence using an idea by Smith [47], which was subsequently also used by Hassell–Rozendaal in their L^p-theory for rough wave equations [28]. To motivate the idea, we once again consider first the easier time-independent case $B(t) = \mathrm{Id}$ $(t \in \mathbb{R})$ as in the beginning of Subsection 3.2.1. Then $P(t) = L$ and since $i\sqrt{L}$ generates a C_0-group $(\mathrm{e}^{it\sqrt{L}})_{t\in\mathbb{R}}$ on $\mathrm{L}^2(\mathbb{R}^d)$, we have the representation (cf. (2.3))

$$u(t) = \mathrm{Cos}(t\sqrt{L})g + it\,\mathrm{Sinc}(t\sqrt{L})h + \int_0^t i(t-s)\mathrm{Sinc}((t-s)\sqrt{L})F(s)\,\mathrm{d}s,$$

where

$$\mathrm{Cos}(t\sqrt{L}) \coloneqq \frac{1}{2}\Big(\mathrm{e}^{it\sqrt{L}} + \mathrm{e}^{-it\sqrt{L}}\Big), \quad t\,\mathrm{Sinc}(t\sqrt{L}) \coloneqq \int_0^t \mathrm{Cos}(s\sqrt{L})\,\mathrm{d}s.$$

Now, the idea in the time-dependent case is to replace the operators $\mathrm{Cos}((t-s)\sqrt{L})$ and $i(t-s)\,\mathrm{Sinc}((t-s)\sqrt{L})$ by the operators $C(t,s)$ and $S(t,s)$ constructed in Subsection 3.2.2. So we make the ansatz

$$u(t) = C(t,0)g + S(t,0)h + \int_0^t S(t,s)G(s)\,\mathrm{d}s$$

with a suitable function $G \in \mathrm{L}^1(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))$. This function will turn out to be the solution to a Volterra equation. Recall that we know from Theorem 3.2.9 that

$$\begin{aligned} R_1(t,s) &\coloneqq (D_t^2 - P(t))C(t,s)\colon \mathrm{H}^\alpha_L(\mathbb{R}^d) \to \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d), \\ R_2(t,s) &\coloneqq (D_t^2 - P(t))S(t,s)\colon \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d) \to \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d) \end{aligned}$$

boundedly.

Lemma 3.3.6 (Volterra-Operator)**.** *Let* $\alpha \in \mathbb{R}$ *and* $G \in \mathrm{L}^1(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))$. *Put*

$$V(t) \coloneqq \int_0^t S(t,s)G(s)\,\mathrm{d}s \quad (t \in \mathbb{R}).$$

Then, $V \in C(\mathbb{R}; \mathrm{H}^\alpha_L(\mathbb{R}^d)) \cap C^1(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d)) \cap W^{2,1}_{\mathrm{loc}}(\mathbb{R}; \mathrm{H}^{\alpha-2}_L(\mathbb{R}^d))$. *Moreover,*

$$(D_t^2 - P(t))V(t) = \tfrac{1}{i}[(\mathbf{Id} + i\mathbf{R})G](t)$$

for a.e. $t \in \mathbb{R}$*, where*

$$\mathbf{R}\colon \mathrm{L}^1(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d)) \to \mathrm{L}^1(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d)), \quad (\mathbf{R}G)(t) = \int_0^t R_2(t,s)G(s)\,\mathrm{d}s$$

has operator norm $\|\mathbf{R}\| \le C\|B'\|_1$ *with a constant* C *depending only on* m_1, m_2 *and* m_4.

Proof. Let $\alpha \in \mathbb{R}$ and $G \in \mathrm{L}^1(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))$. We write

$$V(t) = \int_0^t v(t,s)\,\mathrm{d}s \quad (t \in \mathbb{R}),$$

where we have set $v(t,s) := S(t,s)G(s)$ for $t \in \mathbb{R}$ and a.e. $s \in \mathbb{R}$. Then, by Theorem 3.2.8, we have for a.e. $s \in \mathbb{R}$ and all $k \in \{0,1,2\}$

$$v(\cdot,s) \in C^k(\mathbb{R}; \mathrm{H}_L^{\alpha-k}(\mathbb{R}^d)) \quad \text{with}$$
$$\|D_t^k v(t,s)\|_{\alpha-k} \lesssim \langle t-s\rangle^{1-(k\wedge 1)} \|G(s)\|_{\alpha-1}. \tag{3.28}$$

First Step: $\mathbf{V} \in \mathbf{C}(\mathbb{R}; \mathrm{H}_{\mathbf{L}}^{\alpha}(\mathbb{R}^{\mathbf{d}}))$

Fix $t_0 \in \mathbb{R}$ and let $t \in \mathbb{R}$. We have to show $V(t) \to V(t_0)$ in $\mathrm{H}_L^\alpha(\mathbb{R}^d)$ as $t \to t_0$. Observe that

$$\|V(t) - V(t_0)\|_\alpha \leq \int_{t_0}^t \|v(t,s)\|_\alpha \,\mathrm{d}s + \int_0^{t_0} \|v(t,s) - v(t_0,s)\|_\alpha \,\mathrm{d}s. \tag{3.29}$$

Now for the first integral on the right-hand side of (3.29), we have

$$\int_{t_0}^t \|v(t,s)\|_\alpha \,\mathrm{d}s \lesssim \langle t - t_0\rangle \int_{t_0}^t \|G(s)\|_{\alpha-1} \,\mathrm{d}s \to 0 \quad (t \to t_0),$$

where we used (3.28) with $k = 0$ and then dominated convergence. To estimate the second integral on the right-hand side of (3.29), note that once again by (3.28) (with $k = 0$), the integrand converges to zero as $t \to t_0$ for a.e. s and is controlled by $\langle t_0 - s\rangle \|G(s)\|_{\alpha-1} \lesssim \langle t_0\rangle \|G(s)\|_{\alpha-1}$ if $|t - t_0| \leq 1$. Thus, we conclude once again that

$$\int_0^{t_0} \|v(t,s) - v(t_0,s)\|_\alpha \,\mathrm{d}s \to 0 \quad (t \to t_0)$$

by dominated convergence. This shows that $V \in C(\mathbb{R}; \mathrm{H}_L^\alpha(\mathbb{R}^d))$.

Second Step: $\mathbf{V} \in \mathbf{C}^{\mathbf{1}}(\mathbb{R}; \mathbf{H}_{\mathbf{L}}^{\alpha-\mathbf{1}}(\mathbb{R}^{\mathbf{d}}))$

Let $t \in \mathbb{R}$. Then, for $h \neq 0$,

$$\begin{aligned}\frac{V(t+h) - V(t)}{ih} &= \int_0^{t+h} \frac{v(t+h,s) - v(t,s)}{ih} \,\mathrm{d}s + \frac{1}{ih} \int_t^{t+h} v(t,s) \,\mathrm{d}s \\ &=: \Delta_{1,h} + \Delta_{2,h}.\end{aligned}$$

Now, it follows again from (3.28) with $k = 1$ and dominated convergence that

$$\Delta_{1,h} \to \int_0^t D_t v(t,s) \,\mathrm{d}s \quad (h \to 0) \quad \text{in } \mathrm{H}_L^{\alpha-1}(\mathbb{R}^d).$$

Turning to $\Delta_{2,h}$, we have to be a bit more careful since G need not be continuous and thus we cannot use the fundamental theorem. But using $S(t,t) = 0$ and Theorem 3.2.8 (b), we have

$$\begin{aligned}\|\Delta_{2,h}\|_{\alpha-1} &= \left\| \int_{\mathbb{R}} \mathbb{1}_{(t,t+h)}(s) \left(\frac{s-t}{h}\right) \frac{S(t,s) - S(t,t)}{s-t} G(s) \,\mathrm{d}s \right\|_{\alpha-1} \\ &\leq \int_{\mathbb{R}} \mathbb{1}_{(t,t+h)}(s) \int_0^1 \|\partial_s S(t, t + (s-t)\tau) G(s)\|_{\alpha-1} \,\mathrm{d}\tau \,\mathrm{d}s \\ &\lesssim \int_{\mathbb{R}} \mathbb{1}_{(t,t+h)}(s) \|G(s)\|_{\alpha-1} \,\mathrm{d}s \to 0 \quad (h \downarrow 0).\end{aligned}$$

One argues similarly for $h \uparrow 0$. This proves that $V \in C^1(\mathbb{R}; \mathrm{H}_L^{\alpha-1}(\mathbb{R}^d))$ with $D_t V(t) = \int_0^t D_t v(t,s)\,\mathrm{d}s$.

Third Step: $\mathbf{D_t V \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}; H_L^{\alpha-2}(\mathbb{R}^d))}$

Let $J \coloneqq (-T,T)$, $T>0$. We have to show that $D_t V \in W^{1,1}(J; \mathrm{H}_L^{\alpha-2}(\mathbb{R}^d))$. Exactly as in the second step, one writes

$$\begin{aligned}&\frac{D_t V(t+h) - D_t V(t)}{ih}\\ &= \int_0^{t+h} \frac{D_t v(t+h,s) - D_t v(t,s)}{ih}\,\mathrm{d}s + \frac{1}{ih}\int_t^{t+h} D_t v(t,s)\,\mathrm{d}s\\ &=: \Delta_{1,h}(t) + \Delta_{2,h}(t).\end{aligned}$$

Using (3.28) with $k=2$ and dominated convergence, we deduce $\Delta_{1,h} \to \int_0^t D_t^2 v(t,s)\,\mathrm{d}s$ in $C(\overline{J}; \mathrm{H}_L^{\alpha-2}(\mathbb{R}^d)) \hookrightarrow \mathrm{L}^1(J; \mathrm{H}_L^{\alpha-2}(\mathbb{R}^d))$ as $h \to 0$. Turning to $\Delta_{2,h}(t)$, we further split for $h > 0$

$$\begin{aligned}&\|\Delta_{2,h}(t) - \tfrac{1}{i}G(t)\|_{\alpha-2}\\ &\le \frac{1}{h}\int_t^{t+h} \|D_t S(t,s)G(s) - G(s)\|_{\alpha-2}\,\mathrm{d}s + \frac{1}{h}\int_t^{t+h}\|G(s)-G(t)\|_{\alpha-2}\,\mathrm{d}s\\ &=: \Delta_{2,h}^1(t) + \Delta_{2,h}^2(t),\end{aligned}$$

so that

$$\|\Delta_{2,h}(t) - \tfrac{1}{i}G(t)\|_{L^1_t(J;\mathrm{H}_L^{\alpha-2}(\mathbb{R}^d))} \le \int_J \Delta_{2,h}^1(t)\,\mathrm{d}t + \int_J \Delta_{2,h}^2(t)\,\mathrm{d}t.$$

Arguing just as in the proof of [10, Proposition 1.4.29]), we find $\int_J \Delta_{2,h}^2(t)\,\mathrm{d}t \to 0$ as $h \downarrow 0$. On the other hand, using (3.20) and (3.28), we estimate

$$\begin{aligned}\|D_t S(t,s)G(s) - G(s)\|_{\alpha-2} &= \|D_t S(t,s)G(s) - D_t S(s,s)G(s)\|_{\alpha-2}\\ &\le \int_0^1 \|D_t^2 S(s+(t-s)\tau, s)G(s)\|_{\alpha-2}\,\mathrm{d}\tau\,|t-s|\\ &\lesssim \|G(s)\|_{\alpha-1}|t-s|,\end{aligned}$$

which implies

$$\begin{aligned}\int_J \Delta_{2,h}^1(t)\,\mathrm{d}t &\le \int_J \frac{1}{h}\int_t^{t+h}\|G(s)\|_{\alpha-1}|t-s|\,\mathrm{d}s\,\mathrm{d}t\\ &\le \int_{-T}^{T+h}\left(\frac{1}{h}\int_{s-h}^{s}|t-s|\,\mathrm{d}t\right)\|G(s)\|_{\alpha-1}\,\mathrm{d}s \le \|G\|_{\mathrm{L}^1(\mathbb{R};\mathrm{H}_L^{\alpha-1}(\mathbb{R}^d))}h \to 0\end{aligned}$$

for $h \downarrow 0$. One argues similarly for the limit $h \uparrow 0$. Thus, we have shown that

$$\frac{D_t V(t+h) - D_t V(t)}{ih} \to \int_0^t D_t^2 v(t,s)\,\mathrm{d}s + \tfrac{1}{i}G(t) \quad (h \to 0)$$

in $\mathrm{L}^1_t(J;\mathrm{H}^{\alpha-2}_L(\mathbb{R}^d))$. It now follows (see e.g. [10]) that $D_tV \in W^{1,1}(J;\mathrm{H}^{\alpha-2}_L(\mathbb{R}^d))$ with

$$D_t^2V(t) = \int_0^t D_t^2 v(t,s)\,\mathrm{d}s + \tfrac{1}{i}G(t) \quad \text{in } \mathrm{H}^{\alpha-2}_L(\mathbb{R}^d) \tag{3.30}$$

for a.e. $t \in \mathbb{R}$.

Fourth Step: **Conclusion**

By Theorem 3.2.8 and Lemma 2.3.8, we have

$$P(t)V(t) = \int_0^t P(t)v(t,s)\,\mathrm{d}s \quad \text{in } \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d).$$

Recalling that $(D_t^2 - P(t))v(t,s) = R_2(t,s)G(s)$, it follows that

$$\begin{aligned}(D_t^2 - P(t))V(t) &= \tfrac{1}{i}G(t) + \int_0^t R_2(t,s)G(s)\,\mathrm{d}s \\ &= \tfrac{1}{i}[(\mathbf{Id} + i\mathbf{R})G](t).\end{aligned}$$

Now by Theorem 3.2.9, we have for a.e. $t \in \mathbb{R}$

$$\begin{aligned}&\|(\mathbf{R}G)(t)\|_{\mathrm{H}^{\alpha-1}_L(\mathbb{R}^d)} \\ \le &\int_0^t \|R_2(t,s)G(s)\|_{\mathrm{H}^{\alpha-1}_L(\mathbb{R}^d)}\,\mathrm{d}s \\ \lesssim &\int_0^t \|B'(t)\|\|G(s)\|_{\mathrm{H}^{\alpha-1}_L(\mathbb{R}^d)}\,\mathrm{d}s \le \|B'(t)\| \cdot \|G\|_{\mathrm{L}^1(\mathbb{R};\mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))},\end{aligned}$$

which yields

$$\|\mathbf{R}G\|_{\mathrm{L}^1(\mathbb{R};\mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))} \lesssim \|B'\|_1 \cdot \|G\|_{\mathrm{L}^1(\mathbb{R};\mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))}$$

as desired. Finally, tracing back all constants arising in the preceding proofs, one checks that the implicit constant only depends on m_4 and on the L^2-bounds for the Phillips functional calculus for $\sqrt{\mathbf{L}}$, the latter depending in turn on $M := \sup_{y\in\mathbb{R}^d} \|\mathrm{e}^{-iy\cdot\sqrt{\mathbf{L}}}\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{R}^d))}$. But $M \simeq_{m_1,m_2} 1$ by the self-adjointness of $L_1,\dots,L_d$ with respect to (2.73). □

Theorem 3.3.7 (Existence and Uniqueness of Weak Solutions in $\mathrm{H}^\alpha_L(\mathbb{R}^d)$)**.** *Let ε_1 be as in* (1.11) *and suppose that $\varepsilon_1 \in (0, \frac{1}{C})$, where C is the constant from Lemma 3.3.6. Let $\alpha \in \mathbb{R}$ and suppose that $g \in \mathrm{H}^\alpha_L(\mathbb{R}^d)$, $h \in \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d)$, and $F \in \mathrm{L}^1(\mathbb{R};\mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))$. Then, there exists a unique weak solution u to* (3.1) *in $\mathrm{H}^\alpha_L(\mathbb{R}^d)$. Moreover, there exists $G \in \mathrm{L}^1(\mathbb{R};\mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))$ such that*

$$u(t) = C(t,0)g + S(t,0)h + \int_0^t S(t,s)G(s)\,\mathrm{d}s \quad (t \in \mathbb{R}) \tag{3.31}$$

and we have the estimate

$$\|G\|_{\mathrm{L}^1(\mathbb{R};\mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))} \lesssim \|g\|_{\mathrm{H}^\alpha_L(\mathbb{R}^d)} + \|h\|_{\mathrm{H}^{\alpha-1}_L(\mathbb{R}^d)} + \|F\|_{\mathrm{L}^1(\mathbb{R};\mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))}.$$

Proof. Let $\alpha \in \mathbb{R}$ and suppose that $g \in \mathrm{H}^\alpha_L(\mathbb{R}^d)$, $h \in \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d)$, and $F \in \mathrm{L}^1(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))$. Uniqueness of weak solutions was already proved in Corollary 3.3.5. To prove existence, define for a given $G \in \mathrm{L}^1(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))$ the function u by the right-hand side of (3.31). Then, by Theorem 3.2.8 and Lemma 3.3.6, $u \in C(\mathbb{R}; \mathrm{H}^\alpha_L(\mathbb{R}^d)) \cap C^1(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d)) \cap W^{2,1}_{\mathrm{loc}}(\mathbb{R}; \mathrm{H}^{\alpha-2}_L(\mathbb{R}^d))$ with $u(0) = g$ and $D_t u(0) = h$. Now Theorem 3.2.9 and Lemma 3.3.6 imply that u is a weak solution to (3.1) in $H^\alpha_L(\mathbb{R}^d)$ if and only if for a.e. $t \in \mathbb{R}$

$$F(t) = ((D_t^2 - P(t))u(t) = R_1(t,0)g + R_2(t,0)h + \tfrac{1}{i}\Big[(\mathbf{Id} + i\mathbf{R})G\Big](t),$$

which is equivalent to

$$\tfrac{1}{i}\Big(\mathbf{Id} + i\mathbf{R}\Big)G(t) = F(t) - R_1(t,0)g - R_2(t,0)h.$$

Since $\|\mathbf{R}\|_{\mathcal{L}(L^1(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d)))} \le C\|B'\|_1 \le C\varepsilon_1 < 1$ by Lemma 3.3.6 and the assumption, the operator $\mathbf{Id} + i\mathbf{R}$ is invertible. On the other hand, $R_1(t,0)g$ and $R_2(t,0)h$ belong to $\mathrm{L}^1_t(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))$ with

$$\begin{aligned}
\|R_1(t,0)g\|_{\mathrm{L}^1_t(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))} &\lesssim \|B'\|_1 \|g\|_{\mathrm{H}^\alpha_L(\mathbb{R}^d)}, \\
\|R_2(t,0)h\|_{\mathrm{L}^1_t(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))} &\lesssim \|B'\|_1 \|h\|_{\mathrm{H}^{\alpha-1}_L(\mathbb{R}^d)}
\end{aligned}$$

by Theorem 3.2.9. Therefore, if we choose

$$G = i\Big(\mathbf{Id} + i\mathbf{R}\Big)^{-1}\Big(F - R_1(t,0)g - R_2(t,0)h\Big) \in \mathrm{L}^1_t(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d)),$$

it follows that u given by (3.31) indeed defines a weak solution to (3.1). Moreover, we have the estimate

$$\begin{aligned}
&\|G\|_{\mathrm{L}^1_t(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))} \\
&\lesssim \|R_1(t,0)g\|_{\mathrm{L}^1_t(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))} + \|R_2(t,0)h\|_{\mathrm{L}^1_t(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))} + \|F\|_{\mathrm{L}^1_t(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))} \\
&\lesssim \|g\|_{\mathrm{H}^\alpha_L(\mathbb{R}^d)} + \|h\|_{\mathrm{H}^{\alpha-1}_L(\mathbb{R}^d)} + \|F\|_{\mathrm{L}^1(\mathbb{R}; \mathrm{H}^{\alpha-1}_L(\mathbb{R}^d))}.
\end{aligned}$$

as desired. The proof is complete. □

By Proposition 2.3.3, we can identify the spaces $\mathrm{H}^\alpha(\mathbb{R}^d)$ and $\mathrm{H}^\alpha_L(\mathbb{R}^d)$ for $\alpha \in [-2, 2]$, so Theorem 3.3.7 immediately gives the proof of Theorem 1.1.3:

Corollary 3.3.8 (Existence and Uniqueness of Weak Solutions in $\mathrm{H}^\alpha(\mathbb{R}^d)$)**.** *Let $\varepsilon_1 > 0$ be as in Theorem 3.3.7 and suppose additionally $-1 \le \alpha \le 2$. Assume that $g \in \mathrm{H}^\alpha(\mathbb{R}^d)$, $h \in \mathrm{H}^{\alpha-1}(\mathbb{R}^d)$, and $F \in \mathrm{L}^1(\mathbb{R}; \mathrm{H}^{\alpha-1}(\mathbb{R}^d))$. Then, the function u as in Theorem 3.3.7 defines the unique weak solution to* (3.1) *in* $\mathrm{H}^\alpha(\mathbb{R}^d)$.

Remark 3.3.9. Theorem 3.3.7 and hence Corollary 3.3.8 are also true under milder conditions on the coefficients $b_1, \dots, b_d$. First, observe that up to now, we did not use the smallness of ε_0 as in (A_b) of Assumption 1.1.1. Thus, it would be enough if the coefficients b_j are bounded from above and below by a positive constant, just as the coefficients a_j are. Second, the smallness of $\varepsilon_1 > 0$ was only needed in the proof of Theorem 3.3.7 in order to invert the operator $\mathbf{Id} + i\mathbf{R}$ on $\mathrm{L}^1(\mathbb{R}; \mathrm{H}_L^{\alpha-1}(\mathbb{R}^d))$ (using a Neumann-series argument). However, *without* assuming the smallness of ε_1, we can invert this operator *locally* in time, i.e., on the space $\mathrm{L}^1(J; \mathrm{H}_L^{\alpha-1}(\mathbb{R}^d))$, where $J := (-T, T)$ for some fixed $T \in (0, \infty)$. Indeed, a straightforward induction on $k \in \mathbb{N}_0$ yields

$$\|\mathbf{R}^k G\|_{\mathrm{L}^1(J;\mathrm{H}_L^{\alpha-1}(\mathbb{R}^d))} \lesssim \frac{(C_J T)^k}{k!} \|G\|_{\mathrm{L}^1(J;\mathrm{H}_L^{\alpha-1}(\mathbb{R}^d))} \quad (k \in \mathbb{N}_0) \tag{3.32}$$

with

$$C_J := \sup_{t,s \in J} \|R_2(t,s)\|_{\mathcal{L}(\mathrm{H}_L^{\alpha-1}(\mathbb{R}^d))} \lesssim \sup_{t,s \in J} \|B'(t)\| \leq m_4,$$

where we used Theorem 3.2.9 and (A_b). Thus, $A := \sum_{k=0}^{\infty} (-1)^k i^k \mathbf{R}^k$ converges in $\mathcal{L}(\mathrm{L}^1(J; \mathrm{H}_L^{\alpha-1}(\mathbb{R}^d)))$ (with operator norm bounded by a constant times $\mathrm{e}^{m_4 T}$) and defines the inverse of $\mathbf{Id} + i\mathbf{R}$. Following the proof of Theorem 3.3.7, we obtain a unique weak solution on J, i.e., a unique function $u_J \in C(J; \mathrm{H}_L^{\alpha}(\mathbb{R}^d)) \cap C^1(J; \mathrm{H}_L^{\alpha-1}(\mathbb{R}^d)) \cap W^{2,1}(J; \mathrm{H}_L^{\alpha-2}(\mathbb{R}^d))$ with $u(0) = g$, $D_t u(0) = h$ and $D_t^2 u = P(t)u(t) + F(t)$ for a.e. $t \in J$. Exhausting $\mathbb{R}$ by bounded open intervals J then gives the claim. Observe that this gives only $G \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathrm{H}^{\alpha-1}(\mathbb{R}^d))$ in the representation formula for u. Nevertheless, we have shown that, for the purpose of proving Theorem 1.1.3, we can replace (A_b) by the condition

(A_b') The functions $b_1, \dots, b_d$ are continuous differentiable (or only Lipschitz continuous, see Section 4.4) and we have

$$m_1 \leq b_j(t) \leq m_2$$

for all $t \in \mathbb{R}$ and $j \in \{1, \dots, d\}$.

Thus, we have seen: The smallness of ε_1 ensures that $G \in \mathrm{L}^1(\mathbb{R}; \mathrm{H}_L^{\alpha-1}(\mathbb{R}^d))$, which in turn means that the energy stays *globally* bounded, which is generally nontrivial (see e.g. [61]). However, this is a minimum requirement in order to prove global-in-time Strichartz estimates, which we turn to next and which is our prior interest in this thesis.

4. Global-In-Time Strichartz Estimates

In this chapter, we prove global-in-time Strichartz estimates for (3.1). The key ingredients are dispersive estimates for the operator $(\mathrm{e}^{\pm i\varphi_{t,s}}\psi_\lambda)(\sqrt{\mathbf{L}})$ that was used to construct the parametrices $C(t,s)$ and $S(t,s)$. Once these dispersive estimates are established, Strichartz estimates will follow from the representation formula (3.31) for the weak solution and from the abstract machinery developed by Keel–Tao [34], which converts dispersive estimates into Strichartz estimates. We assume $d \geq 2$ in this chapter.

4.1. Decay Estimates for Truncated Half-Wave Kernel

Let $s, t \in \mathbb{R}$. We are looking for dispersive estimates of the form

$$\|(\mathrm{e}^{\pm i\varphi_{t,s}}\psi)(\sqrt{\mathbf{L}})f\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \lesssim (1+|t-s|)^{-\frac{d-1}{2}}\|f\|_{\mathrm{L}^1(\mathbb{R}^d)} \tag{4.1}$$

for $f \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^2(\mathbb{R}^d)$. Recall that

$$(\mathrm{e}^{\pm i\varphi_{t,s}}\psi)(\sqrt{\mathbf{L}})f = \int_{\mathbb{R}^d} K_{t,s}^{\pm}(y)\mathrm{e}^{-iy\cdot\sqrt{\mathbf{L}}}f \,\mathrm{d}y$$

with

$$K_{t,s}^{\pm}(y) \coloneqq \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{i(y\cdot\xi\pm\varphi_{t,s}(\xi))}\psi(\xi)\,\mathrm{d}\xi \quad (y \in \mathbb{R}^d).$$

As we shall see, (4.1) will be a consequence of the L^∞-estimate

$$\|K_{t,s}^{\pm}\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \lesssim (1+|t-s|)^{-\frac{d-1}{2}}. \tag{4.2}$$

In view of the identity $K_{t,s}^{+} = K_{s,t}^{-}$, we may restrict our attention to $K_{t,s} \coloneqq K_{t,s}^{-}$. The kernel $K_{t,s}$ is a so-called *oscillatory integral* (see [55, Chapter VIII] for a well-written introduction) and as such, its asymptotic behavior is largely governed by the *critical points* of its phase, i.e., by those points in the support of ψ in which the ξ-gradient of the full phase function

$$\Phi(y,\xi) \coloneqq \Phi_{t,s}(y,\xi) \coloneqq y\cdot\xi - \varphi_{t,s}(\xi) \quad (y,\xi \in \mathbb{R}^d)$$

vanishes. This happens exactly when $y = \nabla_\xi \varphi_{t,s}(\xi)$. We are therefore led to consider the singular set

$$\Sigma_{t,s} \coloneqq \nabla_\xi \varphi_{t,s}(\mathbb{R}^d \setminus \{0\}) = \left\{ \int_s^t \frac{B(\tau)\xi}{(B(\tau)\xi|\xi)^{1/2}} \,\mathrm{d}\tau \mid \xi \in \mathbb{R}^d \setminus \{0\} \right\}. \tag{4.3}$$

Note that $\Sigma_{t,s} = \nabla_\xi \varphi_{t,s}(S^{d-1})$ as $\nabla_\xi \varphi_{t,s}$ is positively homogeneous of degree zero, and hence $\Sigma_{t,s}$ is compact by the continuity of $\nabla_\xi \varphi_{t,s}$ and the compactness of S^{d-1}. One can think of $\Sigma_{t,s}$ being the t-section of the *light cone* $\mathcal{C}_{s,0} \coloneqq \bigcup_{t\in\mathbb{R}} \left(\{t\} \times \Sigma_{t,s}\right)$ emanating from the point $(s,0) \in \mathbb{R} \times \mathbb{R}^d$ in Minkowski space. The light cone $\mathcal{C}_{s,0}$ arises as the singular support of the distribution

$$\mathcal{K}_s(t,y) \coloneqq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{i(y\cdot\xi - \varphi_{t,s}(\xi))} \,\mathrm{d}\xi \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$$

(see e.g. [29, Section 8.1] for a glimpse of the microlocal analysis of oscillatory integrals). The term 'light cone' is motivated by the classical, time-independent case $B(t) = \mathrm{Id}$ $(t \in \mathbb{R})$, where $\mathcal{C}_{s,0} = \{(t,y) \in \mathbb{R}^{d+1} \colon |y| = |t-s|\}$ is the standard double cone centered at $(s,0)$ in Minkowski space (and where $\Sigma_{t,s}$ is a sphere of radius $|t-s|$, accordingly).
We first establish L^∞-estimates away from the light cone and subsequently derive L^∞-bounds in a neighbourhood of the light cone by a more refined analysis. To this end, it turns out to be convenient to introduce the normalizations

$$\tilde{\varphi}_{t,s} \coloneqq \tfrac{1}{t-s} \varphi_{t,s}, \quad \tilde{\Sigma}_{t,s} \coloneqq \tfrac{1}{t-s} \Sigma_{t,s} = \tilde{\varphi}_{t,s}(S^{d-1}). \tag{4.4}$$

for $(t,s) \in \Delta^c \coloneqq \{(\tau,\sigma) \in \mathbb{R}^2 \mid \tau \neq \sigma\}$.

4.1.1. Estimates away from the Light Cone

By the principle of nonstationary phase, we expect rapid decay if y is far away from $\Sigma_{t,s}$. The following proposition makes this heuristic precise and serves as a first step towards the proof of (4.2).

Proposition 4.1.1 (Rapid Decay away from the Light Cone)**.** *Let $N \in \mathbb{N}_0$ and $\delta > 0$. Then,*

$$|K^\pm_{t,s}(y)| \lesssim_{N,\delta} (1 + |t-s|)^{-N} \quad \textit{if} \quad \mathrm{dist}(y, \Sigma_{t,s}) \geq \delta |t-s|.$$

Proof. This follows from the principle of nonstationary phase. For the sake of completeness, we provide a proof. Since $K^+_{t,s} = K^-_{s,t}$ and $\Sigma_{t,s} = \Sigma_{s,t}$, we may just consider $K_{t,s} \coloneqq K^-_{t,s}$ with full phase function $\Phi_{t,s}$. Now, let $N \in \mathbb{N}_0$ and $\delta > 0$. Let further $s,t \in \mathbb{R}$ and $y \in \mathbb{R}^d$ with $\mathrm{dist}(y, \Sigma_{t,s}) \geq \delta|t-s|$. The estimate is trivial if $t = s$, for then $|K_{t,s}(y)| \leq (2\pi)^{-d} \|\psi\|_1$ by the triangle inequality. So without restriction, we may suppose $t \neq s$. Put $\tilde{\Phi}_{t,s} := \frac{1}{t-s}\Phi_{t,s}$. Then,

$$K_{t,s}(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{i(t-s)\tilde{\Phi}(y,\xi)} \psi(\xi)\,\mathrm{d}\xi.$$

Now, observe that

$$|\nabla_\xi \tilde{\Phi}_{t,s}(y,\xi)| = \tfrac{1}{|t-s|}|y - \nabla_\xi \varphi_{t,s}(\xi)| \geq \tfrac{1}{|t-s|}\mathrm{dist}(y, \Sigma_{t,s}) \geq \delta \tag{4.5}$$

and that by (3.7), we have for any $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \geq 2$

$$|\partial_\xi^\alpha \tilde{\Phi}(y,\xi)| = |\partial_\xi^\alpha \tilde{\varphi}_{t,s}(\xi)| \lesssim_\alpha |\xi|^{1-|\alpha|} \lesssim 1 \quad \text{on } \mathrm{supp}(\psi). \tag{4.6}$$

For $v(y,\xi) := \frac{\nabla_\xi \tilde{\Phi}_{t,s}(y,\xi)}{|\nabla_\xi \tilde{\Phi}_{t,s}(y,\xi)|^2}$ $(y \in \mathbb{R}^d, \xi \neq 0)$, the estimates (4.5), (4.6) imply for all $\alpha \in \mathbb{N}_0^d$ the bounds

$$|\partial_\xi^\alpha v(y,\xi)| \lesssim_{\alpha,\delta} 1 \qquad \text{on } \mathrm{supp}(\psi). \tag{4.7}$$

Now let $\mathcal{L}$ be the differential operator defined by $\mathcal{L} := \varepsilon \mathrm{Id} + v(y,\xi)\cdot D_\xi$ with $\varepsilon := \mathrm{sgn}(t-s)$. Then, $\mathcal{L}^N \mathrm{e}^{i(t-s)\tilde{\Phi}(y,\xi)} = \varepsilon^N (1+|t-s|)^N \mathrm{e}^{i(t-s)\tilde{\Phi}(y,\xi)}$ and we infer from (4.7) that $\|(\mathcal{L}^T)^N \psi\|_\infty \lesssim_{N,\delta} 1$. Thus, integrating by parts, we obtain

$$\begin{aligned}(2\pi)^d (1+|t-s|)^N |K_{t,s}(y)| &= \left| \int_{\mathbb{R}^d} \left(\mathcal{L}^N \mathrm{e}^{i(t-s)\tilde{\Phi}(y,\xi)}\right) \psi(\xi)\,\mathrm{d}\xi \right| \\ &= \left| \int_{\mathbb{R}^d} \mathrm{e}^{i(t-s)\tilde{\Phi}(y,\xi)} (\mathcal{L}^T)^N \psi(\xi)\,\mathrm{d}\xi \right| \\ &\lesssim_d \|(\mathcal{L}^T)^N \psi\|_\infty \lesssim_{N,\delta} 1\end{aligned}$$

as desired. □

4.1.2. Estimates in a Neighborhood of the Light Cone

For $\delta > 0$, let $\Sigma_{t,s}^\delta := \{y \in \mathbb{R}^d \mid \mathrm{dist}(y, \Sigma_{t,s}) < \delta|t-s|\}$ denote the open $\delta|t-s|$-neighborhood of $\Sigma_{t,s}$. Then, Proposition 4.1.1 tells us that the significant contributions to $\|K_{t,s}\|_\infty$ come from those y lying in $\Sigma_{t,s}^\delta$. In order to obtain the required L^∞-bounds for $K_{t,s}$ on $\Sigma_{t,s}^\delta$, we shall invoke the stationary phase theorem for nondegenerate critical points (see e.g. [29, Theorem 7.7.5]). Recall that critical points of $\Phi(y,\xi)$ are those $\xi \in \mathrm{supp}(\psi)$ such that

$$y = \nabla_\xi \varphi_{t,s}(\xi).$$

Since $\nabla_\xi \varphi_{t,s}$ is positively homogeneous of degree zero, critical points come in radial rays: if ξ is a critical point, so is $r\xi$ for any $r > 0$. In particular, critical points of Φ are degenerate in radial direction. We address this by performing a suitable change of variables. More precisely, we will see in Proposition 4.1.2 that for $t \neq s$

$$K_{t,s}(y) = \frac{1}{(2\pi)^d} \int_0^\infty \mathrm{e}^{-irt} \left(\int_{\mathbb{S}^{d-1}_{t,s}} \mathrm{e}^{iry\cdot\omega} \psi(r\omega) \,\mathrm{d}\omega \right) \mathrm{d}r,$$

where $\mathbb{S}^{d-1}_{t,s}$ is the hypersurface defined by

$$\mathbb{S}^{d-1}_{t,s} \coloneqq \{\xi \in \mathbb{R}^d \mid \tilde{\varphi}_{t,s}(\xi) = 1\} \tag{4.8}$$

and $\mathrm{d}\omega$ is the surface measure on $\mathbb{S}^{d-1}_{t,s}$ normalized by $|\nabla_\xi \tilde{\varphi}_{t,s}|$. The L^∞-bounds for $K_{t,s}$ are then reduced to corresponding bounds for the Fourier transform of the surface-carried measure $\mathrm{d}\sigma \coloneqq \psi(r\omega)\,\mathrm{d}\omega$. This is exactly where assumption (1.10) enters the picture: it guarantees that $\mathbb{S}^{d-1}_{t,s}$ is a small perturbation of the unit sphere, with still nonvanishing Gaussian curvature. By stationary phase on $\mathbb{S}^{d-1}_{t,s}$, we then obtain the desired decay of order $(1+|t-s|)^{-\frac{d-1}{2}}$ for $y \in \Sigma^\delta_{t,s}$.
Recall that $\Delta^c = \{(t,s) \in \mathbb{R}^2 \mid t \neq s\}$. It follows immediately from (1.10) and (4.8) that there exists some annulus in which all hypersurfaces $\mathbb{S}^{d-1}_{t,s}$, $(t,s) \in \Delta^c$, are contained. More precisely, for all $(t,s) \in \Delta^c$

$$\mathbb{S}^{d-1}_{t,s} \subseteq \left\{ \xi \in \mathbb{R}^d \colon \tfrac{1}{c_1} \leq |\xi| \leq \tfrac{1}{c_2} \right\} \tag{4.9}$$

with $c_1 \coloneqq c_{1,\varepsilon_0} \coloneqq \sqrt{1+\varepsilon_0}$ and $c_2 \coloneqq c_{2,\varepsilon_0} \coloneqq \sqrt{1-\varepsilon_0}$. Similarly, we deduce from (1.10) and (4.3) that

$$\tilde{\Sigma}_{t,s} \subseteq \left\{ \xi \in \mathbb{R}^d \colon \tfrac{c_2^2}{c_1} \leq |\xi| \leq \tfrac{c_1^2}{c_2} \right\}. \tag{4.10}$$

We summarize important properties of the hypersurfaces $\mathbb{S}^{d-1}_{t,s}$ in the following proposition, for which we provide a proof in the appendix (see Proposition A.0.1).

Proposition 4.1.2 (Properties of $\mathbb{S}^{d-1}_{t,s}$)**.** *Let $(t,s) \in \Delta^c$.*

(i) $\mathbb{S}^{d-1}_{t,s}$ is a smooth, compact hypersurface in $\mathbb{R}^d$, with normal space at each $\omega \in \mathbb{S}^{d-1}_{t,s}$ given by

$$N_\omega(\mathbb{S}^{d-1}_{t,s}) = \operatorname{span}\left\{ \nabla_\xi \tilde{\varphi}_{t,s}(\omega) \right\}. \tag{4.11}$$

Moreover, $\mathbb{S}^{d-1}_{t,s}$ is the boundary of the compact, strictly convex set

$$C_{t,s} \coloneqq \{\xi \in \mathbb{R}^d \mid \tilde{\varphi}_{t,s}(\xi) \leq 1\}.$$

(ii) The Gauss map on $\mathbb{S}_{t,s}^{d-1}$ *given by*

$$n\colon \mathbb{S}_{t,s}^{d-1} \to S^{d-1}, \quad n(\omega) = \frac{\nabla_\xi \tilde{\varphi}_{t,s}(\omega)}{|\nabla_\xi \tilde{\varphi}_{t,s}(\omega)|}$$

is a diffeomorphism with $n(-\omega) = -n(\omega)$*. In particular, for each* $\nu \in S^{d-1}$ *there exists exactly one* $\omega_0 \in \mathbb{S}_{t,s}^{d-1}$ *such that* $\pm\nu = n(\pm\omega_0)$*.*

(iii) Let n *be the Gauss map from (ii) and* ε_0 *from* (1.10) *sufficiently small. Then, there exists some* $\kappa = \kappa(\varepsilon_0) > 0$ *independent of* $(t,s) \in \Delta^c$ *such that in each point* $\omega \in \mathbb{S}_{t,s}^{d-1}$ *the principal curvatures* $\kappa_1(\omega), \dots, \kappa_{d-1}(\omega)$ *with respect to* $-n$ *satisfy* $\kappa_j(\omega) \geq \kappa$*.*

(iv) Let $\mathrm{d}\omega := |\nabla_\xi \tilde{\varphi}_{t,s}|^{-1}\,\mathrm{d}\mathcal{H}^{d-1}$ *be the* $(d-1)$*-dimensional Hausdorff measure on* $\mathbb{S}_{t,s}^{d-1}$*, normalized by* $|\nabla_\xi \tilde{\varphi}_{t,s}|$*. Then,*

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \int_0^\infty \left(\int_{\mathbb{S}_{t,s}^{d-1}} f(r\omega)\,\mathrm{d}\omega \right) r^{d-1}\,\mathrm{d}r$$

for any integrable function $f\colon \mathbb{R}^d \to \mathbb{C}$*.*

Remark 4.1.3. We want to emphasize that the nonvanishing Gaussian curvature of $\mathbb{S}_{t,s}^{d-1}$ as implied by Proposition 4.1.2 (iii) is the crucial property that we will need to establish the L^∞-bounds for $K_{t,s}^{\pm}$. In fact, our proof of this property is the only point where we need ε_0 to be small.

We want to treat the family of hypersurfaces $(\mathbb{S}_{t,s}^{d-1})_{(t,s)\in\Delta^c}$ essentially as *one* hypersurface, and to this end, we need to make sure that implicit constants arising in our estimates are uniform in the parameter $(t,s) \in \Delta^c$. This is why we need the following two lemmas. The first one, roughly speaking, asserts that the unit normals on $\mathbb{S}$ are uniformly separated in angle, provided that $\mathbb{S}$ is a hypersurface satisfying the properties stated in Proposition 4.1.2 (ii) and (iii).

Lemma 4.1.4. *Let* $\mathbb{S}$ *be a smooth hypersurface in* $\mathbb{R}^d$ *with bijective, smooth Gauss map* $n\colon \mathbb{S} \to S^{d-1}$ *satisfying* $n(-\omega) = -n(\omega)$ *for all* $\omega \in \mathbb{S}$*. Suppose that there exists* $\kappa > 0$ *such that in each point* $\omega \in \mathbb{S}$*, the principal curvatures* $\kappa_1(\omega), \dots, \kappa_{d-1}(\omega)$ *with respect to* $-n$ *satisfy* $\kappa_j(\omega) \geq \kappa$*. Put* $c_\kappa := \sqrt{2}\pi^{-1}\kappa > 0$ *and* $c_{d,\kappa} := (d-1)^{-\frac{1}{2}} c_\kappa > 0$*. Then, for all* $\delta \in (0, c_\kappa^{-1})$ *and* $\omega, \omega_0 \in \mathbb{S}$ *with* $|\omega \pm \omega_0| \geq \delta$*, there exists some* $v \in T_\omega(\mathbb{S})$ *with* $|v| = 1$ *and*

$$|(v|n(\omega_0))| \geq c_{d,\kappa}\delta.$$

Proof. Let $\omega \in \mathbb{S}$. Recall that the principal curvatures $\kappa_1(\omega), \dots, \kappa_{d-1}(\omega)$ with respect to $-n$ are the eigenvalues of the self-adjoint shape operator $L_\omega\colon T_\omega(\mathbb{S}) \to T_\omega(\mathbb{S})$, $L_\omega = D_v n$. Under the identification $T_\omega(\mathbb{S}) \simeq$

$T_{n(\omega)}(S^{d-1})$, we have $\mathrm{d}n_\omega = L_\omega$ and thus by the inverse function theorem that n^{-1} is smooth with $\|\mathrm{d}(n^{-1})_\nu\| = \|(\mathrm{d}n_{n^{-1}(\nu)})^{-1}\| \le \kappa^{-1}$ for all $\nu \in S^{d-1}$. Thus,

$$|n^{-1}(\nu) - n^{-1}(\nu_0)| \le \|\mathrm{d}n^{-1}\|_\infty \,\mathrm{d}_{S^{d-1}}(\nu,\nu_0) \le \tfrac{\pi}{2\kappa}|\nu-\nu_0|$$

for $\nu,\nu_0 \in S^{d-1}$. In particular, n^{-1} is Lipschitz continuous with Lipschitz constant $\frac{\pi}{2\kappa}$. Thus, if $\delta \in (0,\frac{1}{c_\kappa})$ and $\omega,\omega_0 \in \mathbb{S}$ with $|\omega \pm \omega_0| \ge \delta$, then

$$\begin{aligned}\Big(\tfrac{2\kappa\delta}{\pi}\Big)^2 \le (\tfrac{2\kappa}{\pi}|\omega-\omega_0|)^2 \le |n(\omega)-n(\omega_0)|^2 \\ &= |n(\omega)|^2 + |n(\omega_0)|^2 - 2(n(\omega)|n(\omega_0)) \\ &= 2\Big(1-(n(\omega)|n(\omega_0))\Big),\end{aligned}$$

which implies $(n(\omega)|n(\omega_0)) \le 1 - \frac{1}{2}(\frac{2\kappa\delta}{\pi})^2$. Replacing ω_0 by $-\omega_0$ and using that $n(-\omega_0) = -n(\omega_0)$, we also get $-(n(\omega)|n(\omega_0)) \le 1-\frac{1}{2}(\frac{2\kappa\delta}{\pi})^2$ and thus

$$|(n(\omega)|n(\omega_0))| \le 1 - \tfrac{1}{2}\Big(\tfrac{2\kappa\delta}{\pi}\Big)^2 =: r$$

(note that $r \in (0,1)$ by the assumption on δ). Now choose an orthonormal basis $(v_1,\dots,v_{d-1})$ of $T_\omega(\mathbb{S})$. Then,

$$\sum_{j=1}^{d-1} |(v_j|n(\omega_0))|^2 = |n(\omega_0)|^2 - |(n(\omega)|n(\omega_0)|^2 \ge 1-r^2 \ge 1-r.$$

Thus, there must exist some $j \in \{1,\dots,d-1\}$ such that

$$|(v_j|n(\omega_0))|^2 \ge \tfrac{1}{d-1}(1-r) = \tfrac{1}{2(d-1)}\Big(\tfrac{2\kappa\delta}{\pi}\Big)^2 = c_{d,\kappa}^2\delta^2,$$

proving the claim. □

The next lemma guarantees that we can find a Δ^c-independent number of parametrizations of $\mathbb{S}^{d-1}_{t,s}$ which satisfy bounds which are uniform in $(t,s) \in \Delta^c$. We provide a proof in Lemma A.0.2.

Lemma 4.1.5. *Let $\delta_0 := (2d^{\frac{1}{2}}c_1)^{-1} \in (0,1)$. Then, for each $(t,s) \in \Delta^c$ there exists an open cover $\mathcal{U}^{t,s} := \{\mathbb{U}^{t,s}_j \mid j \in \{1,\dots,2d\}\}$ of $\mathbb{S}^{d-1}_{t,s}$ such that the following holds for all $(t,s) \in \Delta^c$ and $j \in \{1,\dots,2d\}$:*

(i) For each $\omega \in \mathbb{S}^{d-1}_{t,s}$ we have $B(\omega,\delta_0) \subseteq \mathbb{U}^{t,s}_i$ for some $i \in \{1,\dots,2d\}$.

(ii) There exists an open, convex 0-neighborhood $V^{t,s}_j \subseteq \mathbb{R}^{d-1}$ and a smooth parametrization $g^{t,s}_j \colon V^{t,s}_j \to \mathbb{U}^{t,s}_j$ which is a graph of a smooth function and which satisfies the following bounds,

$$\begin{aligned} &\|g^{t,s}_j\|_{C^M(V^{t,s}_j)} \lesssim_M 1 && (M \in \mathbb{N}_0),\\ &(G^{t,s}_j(\xi)x|x) \ge 1 && (\xi \in V^{t,s}_j, |x|=1),\\ &|g^{t,s}_j(\xi) - g^{t,s}_j(\eta)| \ge |\xi-\eta| && (\xi,\eta \in V^{t,s}_j)\end{aligned}$$

uniformly w.r.t. $(t,s) \in \Delta^c$. *Here,* $G_j^{t,s}(\xi) = (\partial_k g_j^{t,s}(\xi)|\partial_\ell g_j^{t,s}(\xi))_{k,\ell}$ *denotes the Gram matrix of* $g_j^{t,s}$.

The following two theorems are the main tools we need to establish the L^∞-bounds for $K_{t,s}^{\pm}$, and are a consequence of the principles of stationary and nonstationary phase.

Theorem 4.1.6 (Stationary Phase)**.** *Let* $\mathbb{S}$ *be a compact smooth hypersurface in* $\mathbb{R}^d$ *with a Gauss map* $n\colon \mathbb{S} \to S^{d-1}$. *Suppose there exists* $\kappa \in (0,1)$ *such that at each point* $\omega \in \mathbb{S}$ *the principal curvatures* $\kappa_1(\omega), \dots, \kappa_{d-1}(\omega)$ *satisfy* $\kappa_j(\omega) \geq \kappa$. *Let* $\mathrm{d}\omega$ *denote the surface measure on* $\mathbb{S}$ *and suppose that* $\beta \in C^\infty(\mathbb{S})$ *and* $\rho \in C_c^\infty(\mathbb{R})$ *with* $\operatorname{supp}(\rho) \subseteq (-1,1)$. *Then, there exists* $\delta > 0$ *such that for all* $N \in \mathbb{N}_0$, *one has the estimate*

$$\left| \left(\frac{\mathrm{d}}{\mathrm{d}\mu} \right)^N \mathrm{e}^{-i\mu\nu_0\cdot\omega_0} \int_{\mathbb{S}} \mathrm{e}^{i\mu\nu_0\cdot\omega} \rho\Big(\tfrac{\omega-\omega_0}{\delta}\Big)\beta(\omega)\,\mathrm{d}\omega \right| \lesssim_N (1+|\mu|)^{-\frac{d-1}{2}-N}$$

uniformly in $\mu \in \mathbb{R}, \omega_0 \in \mathbb{S}$, *and* $\nu_0 = n(\omega_0)$, *with an implicit constant* $A_N = A_N(\kappa,\rho,\beta,\delta,\mathbb{S})$.

Proof. We want to apply [29, Theorem 7.7.5] but we shall proceed with a little caution as we want to track how implicit constants really depend on the hypersurface $\mathbb{S}$.

For each $\omega \in \mathbb{S}$ there exist open neighborhoods $\mathbb{U}_\omega \subseteq \mathbb{S}$ of ω and $V_\omega \subseteq \mathbb{R}^{d-1}$ of 0, along with a smooth parametrization $g_\omega\colon V_\omega \to \mathbb{U}_\omega$ with $g_\omega(0) = \omega$. By the implicit function theorem, we may assume without restriction that g_ω is the graph of a smooth function. By possibly shrinking V_ω, we may also assume that V_ω is convex and that

$$C_{M,\omega} := \|g_\omega\|_{C^M(V_\omega)} < \infty \qquad (M \in \mathbb{N}_0), \tag{4.12}$$

$$(G_\omega(\xi)x|x) \geq 1 \qquad (\xi \in V_\omega, |x| = 1), \tag{4.13}$$

$$|g_\omega(\xi) - g_\omega(\eta)| \geq |\xi - \eta| \qquad (\xi,\eta \in V_\omega), \tag{4.14}$$

where $G_\omega(\xi) := (\partial_k g_\omega(\xi)|\partial_\ell g_\omega(\xi))_{k,\ell}$ denotes the Gramian of g_ω (note that (4.13), (4.14) follow from g_ω being of graph-type). As the hypersurface $\mathbb{S}$ is compact, we find finitely many sets $\mathbb{U}_{\omega_1}, \dots, \mathbb{U}_{\omega_m}$ ($m \in \mathbb{N}$) which cover $\mathbb{S}$. We abbreviate $V_j := V_{\omega_j}$, $\mathbb{U}_j := \mathbb{U}_{\omega_j}$ and $g_j := g_{\omega_j}$ for $j \in \{1,\dots,m\}$ and let us also set

$$C_M := \max_{j\in\{1,\dots,m\}} C_{M,\omega_j} \quad (M \in \mathbb{N}_0). \tag{4.15}$$

By the Lebesgue number lemma, there exists some $\delta_0 > 0$ such that for each $\omega_0 \in \mathbb{S}$ the closed δ_0-ball $\overline{B}(\omega_0,\delta_0)$ is fully contained in one of the $\mathbb{U}_j$. Now, let $\delta \in (0,\delta_0)$ be fixed later and $N \in \mathbb{N}$. Suppose that $\beta \in C^\infty(\mathbb{S})$ and $\rho \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}(\rho) \subseteq (-1,1)$. Given $\omega_0 \in \mathbb{S}$ and $\nu_0 = n(\omega_0) \in N_{\omega_0}(\mathbb{S})$, we consider the oscillatory integral

$$J_\delta(\mu) := \int_{\mathbb{S}} \mathrm{e}^{i\mu\nu_0\cdot\omega} \rho\Big(\tfrac{\omega-\omega_0}{\delta}\Big)\beta(\omega)\,\mathrm{d}\omega \quad (\mu \in \mathbb{R}).$$

Clearly, the integrand has its support in $B(\omega_0,\delta) \subseteq B(\omega_0,\delta_0)$. Thus, there exists some $j = j(\omega_0)$ such that $B(\omega_0,\delta) \subseteq \mathbb{U}_j$ and therefore

$$J_\delta(\mu) = \int_{V_j} \mathrm{e}^{i\mu f_j(\xi)} \tilde{\rho}_{j,\delta}(\xi)\,\mathrm{d}\xi,$$

where $\xi_0 := g_j^{-1}(\omega_0)$, $\tilde{\rho}_{j,\delta}(\xi) := \rho\Big(\frac{g_j(\xi)-g_j(\xi_0)}{\delta}\Big)\beta(g_j(\xi))\sqrt{\det G_j(\xi)}$ and

$$f_j \colon V_j \to \mathbb{R}^d, \quad f_j(\xi) = \nu_0 \cdot g_j(\xi).$$

We claim that ξ_0 is a critical point of f_j. Indeed, since $\mathcal{T} := \{t_1,\dots,t_{d-1}\} := \{\partial_1 g_j(\xi_0),\dots,\partial_{d-1} g_j(\xi_0)\} \subseteq T_{\omega_0}(\mathbb{S})$ and $\nu_0 \in N_{\omega_0}(\mathbb{S})$, we have

$$\partial_k f_j(\xi_0) = \nu \cdot \partial_k g_j(\xi_0) = 0 \quad \text{for all } k \in \{1,\dots,d-1\}.$$

Moreover, ξ_0 is nondegenerate. Indeed, the Hessian of f_j is given by

$$Hf_j(\xi_0) = (\nu_0|\partial_{k\ell} g_j(\xi_0))_{k,\ell} = -\Big(\mathrm{II}(\partial_k g_j(\xi_0), \partial_\ell g_j(\xi_0))\Big)_{k,\ell} = -(\mathrm{II}(t_k,t_\ell))_{k,\ell},$$

where II denotes the second fundamental form w.r.t. $-n$ (see e.g. [63, Chapter 1, Proposition 9.1]). Let $\mathcal{V} := \{v_1,\dots,v_{d-1}\}$ be an orthonormal basis of $T_{\omega_0}(\mathbb{S})$ corresponding to the principal curvatures $\kappa_1(\omega_0),\dots,\kappa_{d-1}(\omega_0)$ and let $V := ((t_\ell|v_k))_{k,\ell} \in \mathbb{R}^{(d-1)\times(d-1)}$ be the transition matrix from $\mathcal{T}$ to $\mathcal{V}$. From the identity in the above display, we then obtain $Hf_j(\xi_0) = -V^T\Lambda V$, where $\Lambda := \operatorname{diag}(\kappa_1(\omega_0),\dots,\kappa_{d-1}(\omega_0))$. Thus, for all $x \in \mathbb{R}^{d-1}$ with $|x| = 1$ we conclude

$$|(Hf_j(\xi_0)x|x)| = (\Lambda Vx|Vx) \geq \kappa(Vx|Vx) \geq \kappa,$$

where we used $V^TV = G_j(\xi_0)$ together with (4.13) in the last step. In particular, this implies the nondegeneracy of ξ_0.
Finally, we claim that for some $\delta = \delta(\delta_0,\kappa,C_4) > 0$, we have that $\tilde{\rho}_{j,\delta}$ is compactly supported in $B(\xi_0,\delta) \subseteq V_j$ and that

$$|\nabla f_j(\xi)| \geq \tfrac{\kappa}{2}|\xi-\xi_0| \quad \text{for all } \xi \in B(\xi_0,\delta). \tag{4.16}$$

First of all, observe that $\tilde{\rho}_{j,\delta}$ is compactly supported in $\tilde{B}_\delta := g_j^{-1}(B(\omega_0,\delta))$. Now, on the one hand, (4.14) implies that $\tilde{B}_\delta \subseteq B(\xi_0,\delta)$. On the other hand, we infer from (4.12) and the convexity of V_j that

$$|g_j(\xi) - g_j(\eta)| \leq C_1|\xi-\eta| \quad (\xi,\eta \in V_j)$$

which (together with the continuity of g_j^{-1} and $\overline{B}(\omega_0,\delta_0) \subseteq \mathbb{U}_j$) implies that $B(\xi_0,\frac{\delta_0}{C_1}) \subseteq V_j$. So we have the chain of inclusions $\tilde{B}_\delta \subseteq B(\xi_0,\delta) \subseteq B(\xi_0,\frac{\delta_0}{C_1}) \subseteq V_j$ provided that $\delta \leq \frac{\delta_0}{C_1}$, which we may assume. Then, for all

$\xi \in B(\xi_0, \delta)$, we infer from Taylor's theorem (with $c_d > 0$ a constant only depending on the dimension d and $C := c_d(1 + C_4)$)

$$\begin{aligned} |\nabla_\xi f_j(\xi)| &\geq |Hf_j(\xi_0)(\xi - \xi_0)| - C|\xi - \xi_0|^2 \\ &\geq \kappa|\xi - \xi_0| - C|\xi - \xi_0|^2 \\ &\geq \tfrac{\kappa}{2}|\xi - \xi_0| \end{aligned}$$

provided that $\delta \leq \frac{\kappa}{2C}$. So

$$\delta := \min\{\tfrac{\delta_0}{C}, \tfrac{\kappa}{2C}\} \tag{4.17}$$

does the job (here, we used $C_1 \leq C_4 \leq C$). Now, Theorem [29, Theorem 7.7.5] implies that

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}\mu}\right)^N \mathrm{e}^{-i\mu\nu_0\cdot\omega_0} J_\delta(\mu)\right| \leq A(1 + |\mu|)^{-\frac{d-1}{2}-N} \quad (\mu \in \mathbb{R}),$$

where

$$A := A(M, C_{3M+1}, \kappa, d, (\|\tilde{\rho}_\delta\|_{C^{2M}(V_j)})_{j=1}^m)$$

and $M \in \mathbb{N}$ is such that $M > N + \frac{d-1}{2}$. Note that this estimate does not depend on $\omega_0 \in \mathbb{S}$ (in view of (4.15)). Therefore, the proof is complete. □

Theorem 4.1.7 (Nonstationary Phase)**.** *Let $\mathbb{S}$ be a compact smooth hypersurface in $\mathbb{R}^d$ with surface measure* $\mathrm{d}\omega$ *and $\beta \in C^\infty(\mathbb{S})$, $f \in C^\infty(\mathbb{S})$. Suppose further that for some $\tilde{\delta} > 0$ we have $\|\mathrm{d}f_\omega\| \geq \tilde{\delta}$ for all ω in the support of β. Then, for any $N, \ell \in \mathbb{N}_0$ we have*

$$\left|D_\mu^\ell \int_{\mathbb{S}} \mathrm{e}^{i\mu f(\omega)} \beta(\omega)\,\mathrm{d}\omega\right| \leq C(1 + |\mu|)^{-N} \quad (\mu \in \mathbb{R})$$

with $C = C(N, \ell, \tilde{\delta}, \beta, f, \mathbb{S})$.

Proof. By dominated convergence,

$$D_\mu^\ell \int_{\mathbb{S}} \mathrm{e}^{i\mu f(\omega)} \beta(\omega)\,\mathrm{d}\omega = \int_{\mathbb{S}} \mathrm{e}^{i\mu f(\omega)} (f(\omega))^\ell \beta(\omega)\,\mathrm{d}\omega.$$

As in the proof of Theorem 4.1.6, we may choose a finite open cover $\mathcal{U} := \{\mathbb{U}_j \mid j \in \{1, \ldots, m\}\}$ ($m \in \mathbb{N}$) with smooth parametrizations $g_j \colon V_j \to \mathbb{U}_j$ (where $V_j \subseteq \mathbb{R}^{d-1}$ is open). As in (4.13), we may assume $(G_j(\xi)x|x) \geq 1$ for all $j \in \{1, \ldots, m\}$, $\xi \in V_j$, and $|x| = 1$. Now, choose a smooth partition of unity $(\varphi_j)_{j=1}^m$ of $\mathbb{S}$ subordinate to the cover $\mathcal{U}$. Then,

$$\int_{\mathbb{S}} \mathrm{e}^{i\mu f(\omega)} (f(\omega))^\ell \beta(\omega)\,\mathrm{d}\omega = \sum_{j=1}^m \int_{\mathbb{S}} \mathrm{e}^{i\mu f(\omega)} (f(\omega))^\ell \beta(\omega)\varphi_j(\omega)\,\mathrm{d}\omega.$$

Using the parametrization g_j, the j-th term in the above sum is equal to

$$I_j(\mu) := \int_{V_j} \mathrm{e}^{i\mu f(g_j(\xi))} h_j(\xi)\, \mathrm{d}\xi$$

with $h_j(\xi) := (f^\ell \beta\varphi_j)(g_j(\xi))\sqrt{\det G_j(\xi)}$. A moment's thought reveals that the assumption $\|\,\mathrm{d}_\omega f\| \geq \tilde{\delta}$ implies

$$|\nabla_\xi (f \circ g_j)(\xi)| \geq \tfrac{\tilde{\delta}}{2} \quad (\xi \in V_j). \tag{4.18}$$

where $c_j := \min_{|\lambda|=1, \xi\in K_j} |\sum_{k=1}^{d-1} \lambda_k \partial_k g_j(\xi)| = \min_{|\lambda|=1,\xi\in K_j} |(G_j(\xi)\lambda|\lambda)| > 0$. Indeed, if $\xi \in K_j$, then by assumption there exists $v \in T_{g_j(\xi)}(\mathbb{S})$ with $|v| = 1$ and $|\,\mathrm{d}_\omega f(v)| \geq \frac{\tilde{\delta}}{2}$. But we can write $v = \sum_{k=1}^{d-1} \lambda_k \partial_k g_j(\xi)$ for some $\lambda \in \mathbb{R}^{d-1}$ and hence, setting $w(\xi) := \nabla_\xi(f \circ g_j)(\xi)$, we conclude

$$\tfrac{\tilde{\delta}}{2} \leq |\,\mathrm{d}_\omega f(v)| = \Big|\sum_{k=1}^{d-1} \lambda_k w_k(\xi)\Big| \leq |\lambda||w(\xi)| \leq |w(\xi)|,$$

where we used that

$$1 = |v| = (\sum_{k=1}^{d-1} \lambda_k \partial_k g_j(\xi) | \sum_{k=1}^{d-1} \lambda_k \partial_k g_j(\xi)) = (G_j(\xi)\lambda|\lambda) \geq |\lambda|.$$

This proves (4.18). Now, we may apply [29, Theorem 7.7.1] to obtain the estimate

$$|I_j(\mu)| \lesssim C_{N,\ell,j,\tilde{\delta}} (1 + |\mu|)^{-N} \quad (\mu \in \mathbb{R})$$

with

$$C_{N,\ell,j,\tilde{\delta}} := \frac{\|f \circ g_j\|_{C^{N+1}(K_j)} \|h_j\|_{C^N(K_j)}}{(c_j \tilde{\delta})^{2N}}. \tag{4.19}$$

Therefore, setting $C_{N,\ell,j,\tilde{\delta}} := \sum_{j=1}^m C_{N,\ell,\tilde{\delta}}$ gives the claim. □

Remark 4.1.8. Two remarks concerning the validity of Theorems 4.1.6 and Theorems 4.1.7 are in order.

(i) Theorems 4.1.6 and 4.1.7 remain true if one multiplies the surface measure $\mathrm{d}\omega$ by some nonzero smooth function b on $\mathbb{S}$ since we can always replace β by βb. This is a minor detail but worth mentioning since we want to use these theorems after applying Proposition 4.1.2 (iv), where the surface measure is normalized by $|\nabla_\xi \tilde{\varphi}_{t,s}|$.

(ii) Fourier transforms of surface-carried measures have already been studied a long time ago, and the asymptotic decay estimates as in Theorems 4.1.6 and 4.1.7 are classical by now (see e.g. [38], [56], [52]). However, little emphasis is typically placed on the exact quantities that the implicit constants depend on. We considered this to be an issue because we want to apply Theorems 4.1.6 and Theorems 4.1.7 to the *family* of hypersurfaces $(\mathbb{S}^{d-1}_{t,s})_{(t,s)\in\Delta^c}$. This is the reason why we really attempted to shed light on the dependence of implicit constants. Indeed, a careful inspection of the proofs (see in particular (4.17)) and Lemma 4.1.5 show that $\delta > 0$ as in Theorem 4.1.6 can be chosen *independently* from $(t,s) \in \Delta^c$ and that the bounds in Theorem 4.1.6 and Theorem 4.1.7 hold *uniformly* w.r.t. $(t,s) \in \Delta^c$.

We are finally in the position to also treat the case where y is near $\Sigma_{t,s}$, and to prove the desired L^∞-bounds for $K^{\pm}_{t,s}$.

Theorem 4.1.9 (Decay Estimate for $K^{\pm}_{t,s}$)**.** *There are constants $C_N > 0$ ($N \in \mathbb{N}_0$) such that the following estimates hold true for all $s,t \in \mathbb{R}, y \in \mathbb{R}^d$, and all $N \in \mathbb{N}_0$:*

$$\left|K^{\pm}_{t,s}(y)\right| \leq C_N(1+|y|)^{-\frac{d-1}{2}}(1+\operatorname{dist}(y,\Sigma_{t,s}))^{-N}. \tag{4.20}$$

In particular,

$$\|K^{\pm}_{t,s}\|_\infty \lesssim (1+|t-s|)^{-\frac{d-1}{2}} \quad (s,t \in \mathbb{R}). \tag{4.21}$$

Proof. Let $s,t \in \mathbb{R}$. As in the proof of Proposition 4.1.1, we may restrict to $K_{t,s} := K^-_{t,s}$ with full phase function Φ. Note that the claim is trivial if $s = t$, for then $K_{t,t} = \mathcal{F}^{-1}\psi \in \mathcal{S}(\mathbb{R}^d)$ and $\Sigma_{t,t} = \{0\}$. We may therefore suppose $(t,s) \in \Delta^c$ in the following. We divide the proof into three steps.

Step 1: Rescaling

Recall the normalizations $\tilde{\varphi}_{t,s} = \frac{1}{t-s}\varphi_{t,s}$ and $\tilde{\Sigma}_{t,s} = \frac{1}{t-s}\Sigma_{t,s} = \tilde{\varphi}_{t,s}(\mathbb{S}^{d-1})$ from (4.4) and define $\tilde{\Phi}(z,\xi) := z\cdot\xi - \tilde{\varphi}_{t,s}(\xi)$ for $(z,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$. Then,

$$K_{t,s}(y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{i(t-s)\tilde{\Phi}\left(\frac{y}{t-s},\xi\right)}\psi(\xi)\,\mathrm{d}\xi =: \frac{1}{(2\pi)^d} I\Big(t-s, \Big(\tfrac{y}{t-s}\Big)\Big), \tag{4.22}$$

where

$$I(\lambda,z) := I_{t,s}(\lambda,z) := \int_{\mathbb{R}^d} \mathrm{e}^{i\lambda\tilde{\Phi}(z,\xi)}\psi(\xi)\,\mathrm{d}\xi \quad (z \in \mathbb{R}^d, \lambda \in \mathbb{R}),$$

and (4.20) would follow from the uniform estimates

$$\left|I(\lambda,z)\right| \lesssim_N (1+|\lambda z|)^{-\frac{d-1}{2}}(1+|\lambda|\operatorname{dist}(z,\tilde{\Sigma}_{t,s}))^{-N} \quad (z \in \mathbb{R}^d, \lambda \in \mathbb{R}). \tag{4.23}$$

This is what we will show next.

Step 2 : The Proof of Estimate (4.23)

We deal first with the easier case $z = 0$ and then proceed with the case $z \neq 0$.

Case 1 : $z = 0$.

Put $\delta := \operatorname{dist}(0, \tilde{\Sigma}_{t,s}) > 0$. Then, $\delta \simeq 1$ by (4.10) and therefore, Proposition 4.1.1 immediately implies

$$|I(\lambda, 0)| \lesssim_N (1 + |\lambda|)^{-N} \lesssim (1 + |\lambda|\, \delta)^{-N} \quad (\lambda \in \mathbb{R})$$

which is exactly (4.23).

Case 2 : $z \neq 0$.

Let $z \in \mathbb{R}^d \setminus \{0\}$ and set $\nu_z := \frac{z}{|z|} \in S^{d-1}$. Let $\chi \in C_c^\infty((0,\infty))$ with $\chi = 1$ on $(\frac{1}{2}c_2, 2c_1)$, where c_1 and c_2 are the constants from (4.9). By Proposition 4.1.2 (iv), we then have

$$\begin{aligned} I(\lambda, z) &= \int_0^\infty \mathrm{e}^{-ir\lambda} \bigg(\int_{\mathbb{S}^{d-1}_{t,s}} \mathrm{e}^{ir\lambda z\cdot\omega} \psi(r\omega)\,\mathrm{d}\omega \bigg) \chi(r) r^{d-1}\,\mathrm{d}r \\ &= \int_0^\infty \mathrm{e}^{-ir\lambda} J(r|\lambda z|, \nu_z) \tilde{\chi}(r)\,\mathrm{d}r, \end{aligned}$$

where $\tilde{\chi}(r) := r^{d-1}\chi(r)$ and for fixed $r > 0$,

$$J(\mu, \nu) := J^r_{t,s}(\mu, \nu) := \int_{\mathbb{S}^{d-1}_{t,s}} \mathrm{e}^{i\mu\nu\cdot\omega} \psi(r\omega)\,\mathrm{d}\omega \quad (\mu \in \mathbb{R}, \nu \in S^{d-1}).$$

By Proposition 4.1.2 (iii), there is exactly one $\omega_z \in \mathbb{S}^{d-1}_{t,s}$ such that $\pm\nu_z = n(\pm\omega_z)$. We localize $J(\mu, \nu_z)$ around $\pm\omega_z$: To this end, we choose $\rho \in C_c^\infty(\mathbb{R})$ such that $\rho = 1$ on $(-\frac{1}{2}, \frac{1}{2})$, $\operatorname{supp}(\rho) \subseteq (-1, 1)$ and split $J(\mu, \nu_z) = J_1(\mu, \nu_z) + J_2(\mu, \nu_z) + J_3(\mu, \nu_z)$, where

$$\begin{aligned} J_1(\mu, \nu_z) &:= \int_{\mathbb{S}^{d-1}_{t,s}} \mathrm{e}^{i\mu\nu_z\cdot\omega} \rho\Big(\tfrac{|\omega-\omega_z|}{\delta}\Big) \psi(r\omega)\,\mathrm{d}\omega, \\ J_2(\mu, \nu_z) &:= \int_{\mathbb{S}^{d-1}_{t,s}} \mathrm{e}^{i\mu\nu_z\cdot\omega} \rho\Big(\tfrac{|\omega+\omega_z|}{\delta}\Big) \psi(r\omega)\,\mathrm{d}\omega, \\ J_3(\mu, \nu_z) &:= \int_{\mathbb{S}^{d-1}_{t,s}} \mathrm{e}^{i\mu\nu_z\cdot\omega} \psi(r\omega) \bigg(1 - \rho\Big(\tfrac{|\omega-\omega_z|}{\delta}\Big) + \rho\Big(\tfrac{|\omega+\omega_z|}{\delta}\Big)\bigg)\,\mathrm{d}\omega. \end{aligned}$$

Here, the parameter $\delta > 0$ is the same as in Theorem 4.1.6, the hypothesis of which are satisfied by Proposition 4.1.2 and Remark 4.1.8 (ii). Then, we obtain the splitting $I(\lambda, z) = I_1(\lambda, z) + I_2(\lambda, z) + I_3(\lambda, z)$ with

$$I_k(\lambda, z) = \int_0^\infty \mathrm{e}^{-ir\lambda} J_k(r|\lambda z|, \nu_z) \tilde{\chi}(r)\,\mathrm{d}r$$

for $k = \{1, 2, 3\}$. We estimate each of these terms separately. Let $k \in \{1, 2\}$. Note that by Euler's relation for $\tilde{\varphi}_{t,s}$,

$$\nu_z \cdot \omega_z = n(\omega_z) \cdot \omega_z = \frac{\nabla_\xi \tilde{\varphi}_{t,s}(\omega_z) \cdot \omega_z}{|\nabla_\xi \tilde{\varphi}_{t,s}(\omega_z)|} = \frac{\tilde{\varphi}_{t,s}(\omega_z)}{|\nabla_\xi \tilde{\varphi}_{t,s}(\omega_z)|} = \frac{1}{|\nabla_\xi \tilde{\varphi}_{t,s}(\omega_z)|} =: \frac{1}{\sigma}$$

and therefore

$$I_k(\lambda, z) = \int_0^\infty \mathrm{e}^{-ir\lambda(1-\frac{\varepsilon|z|}{\sigma})} \cdot \mathrm{e}^{-ir\frac{|\lambda z|}{\sigma}} J_k(r|\lambda z|, \nu_z)\tilde{\chi}(r)\,\mathrm{d}r,$$

where $\varepsilon := \operatorname{sgn}(\lambda)$. Hence, integrating by parts and applying Theorem 4.1.6, we obtain

$$\begin{aligned}
&\Big|\Big(\lambda\Big(1-\tfrac{\varepsilon|z|}{\sigma}\Big)\Big)^N I_k(\lambda, z)\Big| \\
&= \left|\int_0^\infty \left((-D_r)^N \mathrm{e}^{-ir\lambda(1-\frac{\varepsilon|z|}{\sigma})}\right)\mathrm{e}^{-ir\frac{|\lambda z|}{\sigma}} J_k(r|\lambda z|, \nu_z)\tilde{\chi}(r)\,\mathrm{d}r\right| \\
&= \left|\int_0^\infty \mathrm{e}^{-ir\lambda(1-\frac{\varepsilon|z|}{\sigma})} D_r^N\left(\mathrm{e}^{-ir\frac{|\lambda z|}{\sigma}} J_k(r|\lambda z|, \nu_z)\tilde{\chi}(r)\right)\mathrm{d}r\right| \\
&\le \int_0^\infty \left|D_r^N\left(\mathrm{e}^{-ir\frac{|\lambda z|}{\sigma}} J_k(r|\lambda z|, \nu_z)\tilde{\chi}(r)\right)\right|\mathrm{d}r \\
&\lesssim_N \sum_{\ell=0}^N \int_0^\infty (1+r|\lambda z|)^{-\frac{d-1}{2}} |\partial_r^{N-\ell}\tilde{\chi}(r)|\,\mathrm{d}r \lesssim_N (1+|\lambda z|)^{-\frac{d-1}{2}}.
\end{aligned}$$

This yields

$$|I_k(\lambda, z)| \lesssim_N (1+|\lambda z|)^{-\frac{d-1}{2}}\Big(1+\Big|\lambda\Big(1-\frac{\varepsilon|z|}{\sigma}\Big)\Big|\Big)^{-N}$$

for $k \in \{1, 2\}$. For $k = 3$, we use the principle of nonstationary phase. Indeed, Lemma 4.1.4 implies (for $\delta < c_\kappa^{-1}$, which we may assume without restriction) that $f\colon \mathbb{S}_{t,s}^{d-1} \to \mathbb{R}$, $f(\omega) = \nu_z \cdot \omega$ satisfies

$$\|\,\mathrm{d}f_\omega\| \ge c_{d,\kappa}\,\delta$$

uniformly $z \in \mathbb{R}^d \setminus \{0\}$, $(t, s) \in \Delta^c$, $\omega_z \in \mathbb{S}_{t,s}^{d-1}$ and $\omega \in \mathbb{S}_{t,s}^{d-1} \setminus (B_\delta(\omega_z) \cup B_\delta(-\omega_z))$. Thus, by Theorem 4.1.7,

$$|D_r^\ell J_3(\mu, \nu_z)| \lesssim_{\ell,M} (1+|\mu|)^{-M} \quad (N, \ell \in \mathbb{N}_0).$$

Following the above argument in the case $k \in \{1, 2\}$, we therefore obtain

$$\begin{aligned}
&\Big|\Big(\lambda\Big(1-\tfrac{\varepsilon|z|}{\sigma}\Big)\Big)^N I_3(\lambda, z)\Big| \\
&\le \int_0^\infty \left|D_r^N\left(\mathrm{e}^{-ir\frac{|\lambda z|}{\sigma}} J_3(r|\lambda z|, \nu_z)\tilde{\chi}(r)\right)\right|\mathrm{d}r \\
&\lesssim_{N,d} \sum_{\substack{\ell\in\mathbb{N}_0^3,\\ |\ell|=N}} \int_0^\infty \left|D_r^{\ell_1}\mathrm{e}^{-ir\frac{|\lambda z|}{\sigma}}\right| \cdot \left|D_r^{\ell_2} J_3(r|\lambda z|, \nu_z)\right| \cdot \left|D_r^{\ell_3}\tilde{\chi}(r)\right|\mathrm{d}r \\
&\lesssim_{N,M} \sum_{\substack{\ell\in\mathbb{N}_0^3,\\ |\ell|=N}} \int_0^\infty |\lambda z|^{\ell_1+\ell_2}(1+r|\lambda z|)^{-M} \cdot \left|D_r^{\ell_3}(\tilde{\chi}(r))\right|\mathrm{d}r \\
&\lesssim (1+|\lambda z|)^{-(M-N)}.
\end{aligned}$$

Choosing $M = 2N$ gives

$$|I_3(\lambda, z)| \lesssim_N (1 + |\lambda z|)^{-N}\Big(1 + \lambda\Big|1 - \tfrac{\varepsilon|z|}{\sigma}\Big|\Big)^{-N}.$$

Since $|I(\lambda, z)| \le |I_1(\lambda, z)| + |I_2(\lambda, z)| + |I_3(\lambda, z)|$, we have proved that

$$|I(\lambda, z)| \lesssim_N (1 + |\lambda z|)^{-\frac{d-1}{2}}\Big(1 + \Big|\lambda\Big(1 - \tfrac{\varepsilon|z|}{\sigma}\Big)\Big|\Big)^{-N}.$$

Now it just remains to recall that $\sigma = |\nabla_\xi \tilde{\varphi}_{t,s}(\varepsilon\omega_z)| \simeq 1$ by (4.10) as well as $z = \frac{|z|}{\sigma}\nabla_\xi \tilde{\varphi}_{t,s}(\omega_z)$, which gives

$$\operatorname{dist}(z, \tilde{\Sigma}_{t,s}) \le \Big|z - \nabla_\xi \tilde{\varphi}_{t,s}(\varepsilon\omega_z)\Big| = \sigma\Big|1 - \tfrac{\varepsilon|z|}{\sigma}\Big| \simeq \Big|1 - \tfrac{\varepsilon|z|}{\sigma}\Big|.$$

This proves (4.23) as desired.

Step 3 : The Proof of (4.20)

Recall from (4.10) that

$$\tilde{\Sigma}_{t,s} \subseteq \{\xi \in \mathbb{R}^d \colon \tilde{c}_1 \le |\xi| \le \tilde{c}_2\} \tag{4.24}$$

with $\tilde{c}_1 \coloneqq \frac{c_2^2}{c_1}, \tilde{c}_2 \coloneqq \frac{c_1^2}{c_2}$. Therefore, we deduce from (4.23) the uniform bounds

$$|I(\lambda, z)| \lesssim (1 + |\lambda|)^{-\frac{d-1}{2}} \quad (\lambda \in \mathbb{R}, z \in \mathbb{R}^d). \tag{4.25}$$

Indeed, if $\operatorname{dist}(z, \tilde{\Sigma}_{t,s}) \le \frac{\tilde{c}_1}{2}$, then (4.24) implies $|z| \ge \frac{\tilde{c}_1}{2}$, so (4.23) with $N = 0$ yields

$$|I(\lambda, z)| \lesssim (1 + |\lambda z|)^{-\frac{d-1}{2}} \lesssim_{\tilde{c}_1, d} (1 + |\lambda|)^{-\frac{d-1}{2}} \quad (\lambda \in \mathbb{R}, z \in \mathbb{R}^d).$$

On the other hand, if $\operatorname{dist}(z, \tilde{\Sigma}_{t,s}) \ge \frac{\tilde{c}_1}{2}$, then choosing some $N \in \mathbb{N}$ with $N \ge \frac{d-1}{2}$ in (4.23) yields

$$|I(\lambda, z)| \lesssim_N (1 + |\lambda| \operatorname{dist}(z, \tilde{\Sigma}_{t,s}))^{-N} \lesssim_{\delta_1, N} (1 + |\lambda|)^{-\frac{d-1}{2}} \quad (\lambda \in \mathbb{R}, z \in \mathbb{R}^d).$$

This proves (4.25). Now, (4.20) is an immediate consequence of (4.25) in view of (4.22). The proof is complete. □

Remark 4.1.10. We want to remark a useful observation concerning the validity of Theorem 4.1.9. To this end, let $\beta \in C_c^\infty(\mathbb{R}^d)$ be supported away from the origin and let $f \colon \mathbb{R} \to \mathbb{R}$ be a function. We set $\overline{B}(\tau) \coloneqq B(f(\tau))$ for $\tau \in \mathbb{R}$ and consider

$$K^\beta_{\overline{\varphi}_{t,s}}(y) \coloneqq \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{i(y\cdot\xi - \overline{\varphi}_{t,s}(\xi))}\beta(\xi)\,\mathrm{d}\xi, \quad \overline{\varphi}_{t,s}(\xi) \coloneqq \int_s^t (\overline{B}(\tau)\xi|\xi)^{\frac{1}{2}}\,\mathrm{d}\tau.$$

Observe that all results presented in this section are preserved under the transformation $B \mapsto \overline{B}$ (and $\psi \mapsto \beta$) as they solely rely on (1.10), which is invariant under this transformation. Therefore, we are allowed to apply Theorem 4.1.9 to $K^\beta_{\overline{\varphi}_{t,s}}$ to see that

$$\Big|K^\beta_{\overline{\varphi}_{t,s}}(y)\Big| \lesssim_\beta (1 + |t - s|)^{-\frac{d-1}{2}}.$$

We will use this later with the affine linear transformation $f(\tau) \coloneqq \frac{\tau}{\lambda} + s$ for fixed $\lambda > 0$ and $s \in \mathbb{R}$.

4.2. Strichartz Estimates for Frequency-Localized Approximate Solutions

In Subsection 3.2.2, we constructed the parametrices $C(t,s)$ and $S(t,s)$ on the basis of the parametrices $T^{\pm}(t,s)$, where

$$T^{\pm}(t,s) = \sum_{\lambda \in 2^{\mathbb{Z}}} T^{\pm}_{\lambda}(t,s), \quad T^{\pm}_{\lambda}(t,s) = \Big(\mathrm{e}^{\pm i\varphi_{t,s}}\psi_{\lambda}\Big)(\sqrt{\mathbf{L}}). \tag{4.26}$$

In this section, we prove Strichartz estimates for the λ-frequency-localized operators $T^{\pm}_{\lambda}(t,s)$ with a constant which is *uniform* in $\lambda \in 2^{\mathbb{Z}}$. It essentially suffices to consider $T^{\pm}_{1}(t,s)$, as the estimates for general λ can then be recovered by a scaling argument. The Strichartz estimates for $T^{\pm}_{1}(t,s)$ are in turn obtained via the famous Keel–Tao argument [34] and in order to apply the latter, we use the kernel bounds established in Theorem 4.1.9 and the $\mathrm{L}^{\infty}_{x}\mathrm{L}^{1}_{y}$-estimates for $\mathrm{Cos}(y\sqrt{\mathbf{L}})$ from Corollary 2.2.10 (see Lemma 4.2.2).
To prepare for the scaling argument, we define for $\lambda > 0$ the dilation operator

$$\delta_{\lambda} \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \quad (\delta_{\lambda} f)(x) \coloneqq f_{\lambda}(x) = f\Big(\frac{x}{\lambda}\Big).$$

and the rescaled operator $\mathbf{L}_{\lambda} \coloneqq (L_{1,\lambda}, \dots, L_{d,\lambda})$ by

$$L_{j,\lambda} \coloneqq \begin{pmatrix} D_j a_{j,\lambda} D_j & \\ 0 & a_{j,\lambda} D_j^2 \end{pmatrix}, \qquad a_{j,\lambda}(\cdot) \coloneqq a_j\Big(\frac{\cdot}{\lambda}\Big).$$

Lemma 4.2.1 (Scaling Symmetry of 1D-Half-Wave Groups)**.** *Let* $\lambda > 0$ *and* $y \in \mathbb{R}^d$*. Then,*

$$\delta_{\lambda} \mathrm{e}^{i\frac{y}{\lambda}\sqrt{\mathbf{L}}} = \mathrm{e}^{iy\sqrt{\mathbf{L}_{\lambda}}}\delta_{\lambda}. \tag{4.27}$$

In particular,

$$\sup_{\lambda>0, y\in\mathbb{R}^d} \|\mathrm{e}^{iy\sqrt{\mathbf{L}_{\lambda}}}\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{R}^d))} \leq 1. \tag{4.28}$$

Proof. Let $f \in \mathrm{H}^2_L(\mathbb{R}^d)$. Then a straightforward computation reveals the identity

$$\mathbf{L}_{\lambda} f_{\lambda} = \lambda^{-2}(\mathbf{L}f)_{\lambda},$$

which, by functional calculus, entails

$$\sqrt{\mathbf{L}_{\lambda}} f_{\lambda} = \lambda^{-1}(\sqrt{\mathbf{L}}f)_{\lambda}.$$

But this implies that for each $j \in \{1, \dots, d\}$, the function $u \colon \mathbb{R} \to \mathrm{L}^2(\mathbb{R}^d)$, $u(t) \coloneqq \Big(\mathrm{e}^{i\frac{t}{\lambda}\sqrt{L_j}} f\Big)_{\lambda}$ is a classical solution to the abstract Cauchy problem

$$u'(t) = i\sqrt{L_{j,\lambda}}u(t) \quad (t \in \mathbb{R}), \quad u(0) = f_{\lambda}.$$

Therefore, the uniqueness of classical solutions (see e.g. [2, Theorem 3.1.12]) gives

$$\left(\mathrm{e}^{i\frac{t}{\lambda}\sqrt{L_j}}f\right)_\lambda = \mathrm{e}^{it\sqrt{L_{j,\lambda}}}f_\lambda \quad (t \in \mathbb{R}). \tag{4.29}$$

This identity extends to all $f \in \mathrm{L}^2(\mathbb{R}^d)$ by density of $\mathrm{H}^2_L(\mathbb{R}^d)$ in $\mathrm{L}^2(\mathbb{R}^d)$. Since $\mathrm{e}^{iy\cdot\sqrt{\mathbf{L}_\lambda}} = \mathrm{e}^{iy_1\sqrt{L_{1,\lambda}}}\cdots\mathrm{e}^{iy_d\sqrt{L_{d,\lambda}}}$ for $y \in \mathbb{R}^d$, we may apply (4.29) iteratively for $j = 1,\dots,d$ to obtain

$$\left(\mathrm{e}^{i\frac{y}{\lambda}\sqrt{\mathbf{L}}}f\right)_\lambda = \mathrm{e}^{iy\sqrt{\mathbf{L}_\lambda}}f_\lambda \quad (y \in \mathbb{R}^d,\ f \in \mathrm{L}^2(\mathbb{R}^d)),$$

which is exactly (4.27). The second claim is now a straightforward consequence, since

$$\left\|\mathrm{e}^{iy\cdot\sqrt{\mathbf{L}_\lambda}}\right\| = \left\|\delta_\lambda\left(\mathrm{e}^{i\frac{y}{\lambda}\cdot\sqrt{\mathbf{L}}}\right)\delta_{\lambda^{-1}}\right\| \le \left\|\delta_\lambda\right\|\left\|\left(\mathrm{e}^{i\frac{y}{\lambda}\cdot\sqrt{\mathbf{L}}}\right)\right\|\left\|\delta_{\lambda^{-1}}\right\| \le \lambda^{\frac{d}{2}}\cdot 1\cdot\lambda^{-\frac{d}{2}} = 1,$$

where in the above display $\|\cdot\|$ denotes the norm in $\mathcal{L}(\mathrm{L}^2(\mathbb{R}^d))$. □

Lemma 4.2.2. *We have*

$$\|\mathrm{Cos}(y\sqrt{\mathbf{L}_\lambda}f)(x)\|_{\mathrm{L}^\infty_x(\mathbb{R}^d;\mathrm{L}^1_y(\mathbb{R}^d))} \lesssim \|f\|_1 \tag{4.30}$$

for all $f \in \mathrm{L}^2(\mathbb{R}^d)\cap\mathrm{L}^1(\mathbb{R}^d)$ *and all* $\lambda > 0$.

Proof. This is an immediate consequence of Corollary 2.2.10, noting that the implicit constant in (2.70) just depends on m_1, m_2, and m_3, which are invariant under the transformation $a_j \mapsto a_{j,\lambda}$. □

Lemma 4.2.3 (Dispersive Estimates for $T^\pm_\lambda$)**.** *Let* $s,t \in \mathbb{R}, \lambda > 0$ *and suppose that* $\beta \in C^\infty_c(\mathbb{R}^d)$ *is supported away from the origin. We define*

$$\bar{\varphi}_{u,v}(\xi) := \int_u^v (\bar{B}(\tau)\xi|\xi)^{\frac{1}{2}}\,\mathrm{d}\tau, \quad \bar{B}(\tau) := B\left(\tfrac{\tau}{\lambda}+s\right) \quad (u,v \in \mathbb{R}).$$

(i) We have

$$\left[\left(\mathrm{e}^{\pm i\varphi_{t,s}}\beta_\lambda\right)(\sqrt{\mathbf{L}})f\right]_\lambda = \left(\mathrm{e}^{\pm i\bar{\varphi}_{\lambda(t-s),0}}\beta\right)(\sqrt{\mathbf{L}_\lambda})f_\lambda \quad (f \in \mathrm{L}^2(\mathbb{R}^d)).$$

(ii) For all $\sigma,\tau \in \mathbb{R}$*, there holds the estimate*

$$\left\|\left|\left(\mathrm{e}^{\pm i\bar{\varphi}_{\tau,\sigma}}\beta\right)(\sqrt{\mathbf{L}_\lambda})g\right\|_\infty \lesssim (1+|\tau-\sigma|)^{-\frac{d-1}{2}}\|g\|_1 \quad (g \in \mathrm{L}^1(\mathbb{R}^d)\cap\mathrm{L}^2(\mathbb{R}^d)).$$

Proof. Let $s,t \in \mathbb{R}, \lambda > 0$ and $f \in \mathrm{L}^2(\mathbb{R}^d)$. By Remark 2.3.2, we have

$$\left[(\mathrm{e}^{\pm i\varphi_{t,s}}\beta_\lambda)(\sqrt{\mathbf{L}})f\right]_\lambda = \int_{\mathbb{R}^d} K^{\beta_\lambda,\pm}_{\varphi_{t,s}}(y)\left(\mathrm{Cos}(y\sqrt{\mathbf{L}})f\right)_\lambda \mathrm{d}y \tag{4.31}$$

with

$$K^{\beta_\lambda,\pm}_{\varphi_{t,s}}(y) := \mathcal{F}^{-1}(\mathrm{e}^{\pm i\varphi_{t,s}}\beta_\lambda)(y) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{i(y\cdot\xi\pm\varphi_{t,s}(\xi))}\beta_\lambda(\xi)\,\mathrm{d}\xi.$$

Applying the changes of variables $\xi \mapsto \lambda\xi$ and $\tau \mapsto \frac{\tau}{\lambda} + s$, we obtain

$$\begin{aligned} K^{\beta_\lambda,\pm}_{\varphi_{t,s}}(y) &= \frac{\lambda^d}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{i(\lambda y\cdot\xi\pm\lambda\varphi_{t,s}(\xi))}\beta(\xi)\,\mathrm{d}\xi \\ &= \frac{\lambda^d}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{i(\lambda y\cdot\xi\pm\bar{\varphi}_{\lambda(t-s),0}(\xi))}\beta(\xi)\,\mathrm{d}\xi = \lambda^d K^{\beta,\pm}_{\bar{\varphi}_{\lambda(t-s),0}}(\lambda y). \end{aligned}$$

Substituting this into (4.31), applying the change of variables $y \mapsto \frac{y}{\lambda}$ and invoking Lemma 4.2.1, we arrive at

$$\begin{aligned} \Big[(\mathrm{e}^{\pm i\varphi_{t,s}}\beta_\lambda)(\sqrt{\mathbf{L}})f\Big]_\lambda &= \int_{\mathbb{R}^d} \lambda^d K^{\beta,\pm}_{\bar{\varphi}_{\lambda(t-s),0}}(\lambda y)\Big[\mathrm{Cos}(y\sqrt{\mathbf{L}})f\Big]_\lambda\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} K^{\beta,\pm}_{\bar{\varphi}_{\lambda(t-s),0}}(y)\Big[\mathrm{Cos}\Big(\tfrac{y}{\lambda}\sqrt{\mathbf{L}}\Big)f\Big]_\lambda\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} K^{\beta,\pm}_{\bar{\varphi}_{\lambda(t-s),0}}(y)\mathrm{Cos}\Big(y\sqrt{\mathbf{L}_\lambda}\Big)f_\lambda\,\mathrm{d}y \\ &= (\mathrm{e}^{\pm i\bar{\varphi}_{\lambda(t-s),0}}\beta)(\sqrt{\mathbf{L}_\lambda})f_\lambda. \end{aligned}$$

This proves (i). Similarly, if $g \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^2(\mathbb{R}^d)$, it follows from Theorem 4.1.9, Remark 4.1.10 and Lemma 4.2.2 that for all $\tau, \sigma \in \mathbb{R}$

$$\begin{aligned} \operatorname*{ess\,sup}_{x\in\mathbb{R}^d}\Big|(\mathrm{e}^{\pm i\bar{\varphi}_{\tau,\sigma}}\beta)(\sqrt{\mathbf{L}_\lambda})g(x)\Big| &= \Big|\int_{\mathbb{R}^d} K^{\beta,\pm}_{\bar{\varphi}_{\tau,\sigma}}(y)\Big[\mathrm{Cos}\Big(y\sqrt{\mathbf{L}_\lambda}\Big)g\Big](x)\,\mathrm{d}y\Big| \\ &\le \Big\|K^{\beta,\pm}_{\bar{\varphi}_{\tau,\sigma}}\Big\|_\infty \int_{\mathbb{R}^d}\Big|\Big[\mathrm{Cos}\Big(y\sqrt{\mathbf{L}_\lambda}\Big)g\Big](x)\Big|\,\mathrm{d}y \\ &\lesssim (1+|\tau-\sigma|)^{-\frac{d-1}{2}}\|g\|_1, \end{aligned}$$

which proves (ii). □

Definition 4.2.4 (γ-Admissibility)**.** Let $\gamma > 0$. Then, an exponent pair $(p, q) \in [2, \infty]^2$ is called γ-*admissible* if $(p, q, \sigma) \neq (2, \infty, 1)$ and $\frac{1}{p} + \frac{\gamma}{q} \le \frac{\gamma}{2}$.

Note that if (p, q, α) is a wave-admissible Strichartz triple, then (p, q) is $\frac{d-1}{2}$-admissible (cf. (1.4)). The following is the famous result by Keel–Tao.

Theorem 4.2.5 ([34, Theorem 1.2])**.** *Let (Ω, μ) be a measure space and H be a Hilbert space. Suppose that $U(t)\colon H \to \mathrm{L}^2(\Omega)$ is a linear operator for each $t \in \mathbb{R}$ satisfying the following properties:*

(i) Uniform Boundedness: For all $t \in \mathbb{R}$ and $f \in H$

$$\|U(t)f\|_{\mathrm{L}^2_x(\Omega)} \lesssim \|f\|_H.$$

(ii) Truncated Decay: For some $\gamma > 0$, we have

$$\|U(s)U(t)^*g\|_{L^\infty_x(\Omega)} \lesssim (1+|t-s|)^{-\gamma}\|g\|_{L^1_x(\Omega)} \quad \textit{for all } s,t \in \mathbb{R}, g \in S,$$

where $S \subseteq \mathrm{L}^1(\Omega) \cap \mathrm{L}^2(\Omega)$ denotes a dense subset in $\mathrm{L}^1(\Omega)$.

Then, the estimates

$$\|U(t)f\|_{\mathrm{L}^p_t(\mathbb{R};\mathrm{L}^q_x(\Omega))} \lesssim \|f\|_H \quad (f \in H)$$

and

$$\left\|\int_0^t U(t)U(s)^*F(s)\,\mathrm{d}s\right\|_{\mathrm{L}^p_t(\mathbb{R};\mathrm{L}^q_x(\Omega))} \lesssim \|f\|_H \quad (F \in L^{\tilde{p}'}_t(\mathbb{R};L^{\tilde{q}'}_x(\Omega))),$$

hold true for any γ-admissible exponent pairs $(p,q),(\tilde{p},\tilde{q})$.

Theorem 4.2.6 (Global Strichartz Estimates for $T^{\pm}_\lambda$). *Suppose that (p,q,α) is a wave-admissible Strichartz triple. Let $\lambda \in 2^{\mathbb{Z}}$ and $s \in \mathbb{R}$. Then,*

$$\|L^{-\frac{\alpha}{2}}(\mathrm{e}^{\pm i\varphi_{t,s}}\psi_\lambda)(\sqrt{\mathbf{L}})f\|_{\mathrm{L}^p_t(\mathbb{R};L^q_x(\mathbb{R}^d))} \lesssim \|f\|_2 \quad (f \in \mathrm{L}^2(\mathbb{R}^d)), \tag{4.32}$$

with an implicit constant independent of s and λ.

Proof. Let (p,q,α) be a wave-admissible Strichartz triple. For $s,t \in \mathbb{R}$ and $\lambda > 0$ we let

$$\bar{\varphi}_{t,0}(\xi) = \int_0^t (\bar{B}(\tau)\xi|\xi)^{\frac{1}{2}}\,\mathrm{d}\tau \quad (\xi \in \mathbb{R}^d), \quad \bar{B}(\tau) = B\Big(\frac{\tau}{\lambda}+s\Big) \quad (\tau \in \mathbb{R})$$

as in Lemma 4.2.3. Let $\beta := |\cdot|^{-\alpha}\psi$ and let $f \in \mathrm{L}^2(\mathbb{R}^d)$. By Lemma 2.1.15 (b) and Lemma 4.2.3 (a), we have

$$\begin{aligned} L^{-\frac{\alpha}{2}}(\mathrm{e}^{\pm i\varphi_{t,s}}\psi_\lambda)(\sqrt{\mathbf{L}})f(x) &= \lambda^{-\alpha}\Big(\mathrm{e}^{\pm i\varphi_{t,s}}\beta_\lambda\Big)(\sqrt{\mathbf{L}})f(x) \\ &= \lambda^{-\alpha}\Big(\mathrm{e}^{\pm i\bar{\varphi}_{\lambda(t-s),0}}\beta\Big)(\sqrt{\mathbf{L}_\lambda})f_\lambda(\lambda x) \\ &= \lambda^{-\alpha}[U_\pm(\lambda(t-s))f_\lambda](\lambda x) \end{aligned}$$

for a.e. $x \in \mathbb{R}^d$, where we define for $\tau \in \mathbb{R}$

$$U_\pm(\tau)\colon \mathrm{L}^2(\mathbb{R}^d) \to \mathrm{L}^2(\mathbb{R}^d), \quad U_\pm(\tau)f := \Big(\mathrm{e}^{\pm i\bar{\varphi}_{\tau,0}}\beta\Big)(\sqrt{\mathbf{L}_\lambda})f.$$

Let us suppose for a moment that

$$\|U_\pm(\tau)f\|_{\mathrm{L}^p_\tau(\mathbb{R};\mathrm{L}^q_x(\mathbb{R}^d))} \lesssim \|f\|_2 \quad (f \in \mathrm{L}^2(\mathbb{R}^d)), \tag{4.33}$$

whose proof we postpone to the end. Now, applying the change of variables $x \mapsto \frac{x}{\lambda}$, $t \mapsto \frac{t}{\lambda}$ and invoking (4.33), we obtain

$$\begin{aligned}\|L^{-\frac{\alpha}{2}}(\mathrm{e}^{\pm i\varphi_{t,s}}\psi_\lambda)(\sqrt{\mathbf{L}})f\|_{\mathrm{L}^p_t(\mathbb{R};L^q_x(\mathbb{R}^d))} &= \lambda^{-\alpha}\|(U_\pm(\lambda(t-s))f_\lambda)(\lambda x)\|_{\mathrm{L}^p_t(\mathbb{R};L^q_x(\mathbb{R}^d))}\\ &\lesssim \lambda^{-\left(\alpha+\frac{1}{p}+\frac{d}{q}\right)}\|(U(t)f_\lambda)(x)\|_{\mathrm{L}^p_t(\mathbb{R};L^q_x(\mathbb{R}^d))}\\ &\lesssim \lambda^{-\left(\alpha+\frac{1}{p}+\frac{d}{q}\right)}\|f_\lambda\|_{L^2_x(\mathbb{R}^d)}\\ &\lesssim \lambda^{-\left(\alpha-\frac{d}{2}+\frac{1}{p}+\frac{d}{p}\right)}\|f\|_{L^2_x(\mathbb{R}^d)} = \|f\|_{L^2_x(\mathbb{R}^d)},\end{aligned}$$

where the last equality follows from the fact (p, q, α) is a wave-admissible Strichartz triple. Thus, it remains to prove (4.33). To ease notation, we provide the proof for $U \coloneqq U_+$ (the proof for U_- is analogous). To this end, we want to verify properties (i) and (ii) in Theorem 4.2.5. Property (i) is easy since

$$\|U(\tau)f\|_2 \lesssim \|\mathrm{e}^{i\bar{\varphi}_{\tau,0}}\beta\|_\infty\|f\|_2 \lesssim_\beta \|f\|_2 \quad (f \in \mathrm{L}^2(\mathbb{R}^d), \tau \in \mathbb{R})$$

by Proposition 2.3.5 (a) (note that the bounds are independent of λ by (4.28)). As $\mathbf{L}_\lambda$ and therefore also $\sqrt{\mathbf{L}_\lambda}$ is self-adjoint w.r.t. (2.73), we have by Proposition 2.1.10 (b)

$$(U(\sigma))^* = \left(\overline{\mathrm{e}^{i\bar{\varphi}_{\sigma,0}}\beta}\right)(\sqrt{\mathbf{L}_\lambda}) = \left(\mathrm{e}^{i\bar{\varphi}_{0,\sigma}}\beta\right)(\sqrt{\mathbf{L}_\lambda})$$

and therefore

$$U(\tau)(U(\sigma))^* = \left(\mathrm{e}^{i(\bar{\varphi}_{\tau,0}+\bar{\varphi}_{0,\sigma})}\beta^2\right)(\sqrt{\mathbf{L}_\lambda}) = \left(\mathrm{e}^{i\bar{\varphi}_{\tau,\sigma}}\beta^2\right)(\sqrt{\mathbf{L}_\lambda}).$$

Thus, by Lemma 4.2.3 (b)

$$\|U(\tau)(U(\sigma))^*g\|_\infty = \left\|\left(\mathrm{e}^{i\tilde{\varphi}_{\tau,\sigma}}\beta^2\right)(\sqrt{\mathbf{L}_\lambda})g\right\|_\infty \lesssim (1+|\tau-\sigma|)^{-\frac{d-1}{2}}\|g\|_1 \quad (\tau, \sigma \in \mathbb{R})$$

for $g \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^2(\mathbb{R}^d)$, which shows (ii). Applying Theorem 4.2.5 (separately in both components of $\mathrm{L}^2(\mathbb{R}^d) = \mathrm{L}^2(\mathbb{R}^d; \mathbb{C}^2)$), we obtain (4.33) as desired.

□

4.3. Global Strichartz Estimates

In Theorem 4.2.6, we proved global-in-time Strichartz estimates for the operators $T^\pm_\lambda(t, s)$ with a constant *uniform* in $\lambda \in 2^{\mathbb{Z}}$. The following is a roadmap that will lead us from these uniform estimates for the operators $T^\pm_\lambda(t, s)$ to the global-in-time Strichartz estimates for the weak solution to (3.1):

(1) We obtain global-in-time Strichartz estimates for $T^{\pm}(t,s)$ by patching the estimates for $(T_\lambda^{\pm})_{\lambda\in 2^{\mathbb{Z}}}$ together with the help of Proposition 2.1.29. This is done in Theorem 4.3.2.

(2) In Corollary 4.3.3, we carry over the estimates for $T^{\pm}(t,s)$ to corresponding ones for $C(t,s)$ and $S(t,s)$ using Lemma 3.2.7.

(3) We use the estimates for $C(t,s)$ and $S(t,s)$ obtained in (2) and the representation formula (3.31) in Theorem 3.3.7 to prove global-in-time Strichartz estimates for the weak solution to (3.1) in $\mathrm{H}^1_L(\mathbb{R}^d)$ (see Theorem 4.3.4).

(4) Finally, in order to prove Theorem 1.1.4, it remains to employ Proposition 2.3.3 to identify the L-adapted spaces with the standard Sobolev spaces. This is the point that leads to additional restrictions for the regularity parameter α of the Strichartz triple.

Before we go on this journey, we recall the Christ-Kiselev lemma, which will allow us to estimate the Duhamel term in the representation formula (3.31) later on.

Lemma 4.3.1 ([58, Lemma 2.4]). *Let X, Y be Banach spaces and let $K \in C(\mathbb{R}\times\mathbb{R};\mathcal{L}(X,Y))$. Suppose that $1 \le p < q \le \infty$ is such that*

$$\Big\| \int_{\mathbb{R}} K(t,s)F(s)\,\mathrm{d}s \Big\|_{\mathrm{L}^q_t(\mathbb{R};Y)} \lesssim \|F\|_{\mathrm{L}^p_t(\mathbb{R};X)}$$

for all $F \in L^p_t(\mathbb{R};X)$. Then, one also has

$$\Big\| \int_{-\infty}^{t} K(t,s)F(s)\,\mathrm{d}s \Big\|_{\mathrm{L}^q_t(\mathbb{R};Y)} \lesssim \|F\|_{\mathrm{L}^p_t(\mathbb{R};X)}.$$

Theorem 4.3.2 (Strichartz Estimates for Parametrices I). *Suppose that (p,q,α) is a wave-admissible Strichartz triple.*

(i) The estimate

$$\|L^{-\frac{\alpha}{2}}T^{\pm}(t,s)f\|_{\mathrm{L}^p_t(\mathbb{R};\mathrm{L}^q_x(\mathbb{R}^d))} \lesssim \|f\|_2 \quad (f \in \mathcal{S}_{\sqrt{\mathbf{L}}})$$

holds uniformly in $s \in \mathbb{R}$.

(ii) Moreover, if $(\tilde p, \tilde q, \tilde\alpha)$ is another wave-admissible Strichartz triple with $\tilde p' < p$, then

$$\Big\| \int_0^t L^{-\frac{\alpha+\tilde\alpha}{2}} T^{\pm}(t,s)G(s)\,\mathrm{d}s \Big\|_{\mathrm{L}^p_t(\mathbb{R};(\mathrm{L}^q_x(\mathbb{R}^d)))} \lesssim \|G\|_{\mathrm{L}^{\tilde p'}_t(\mathbb{R};\mathrm{L}^{\tilde q'}_x(\mathbb{R}^d))}$$

for all measurable functions $G\colon \mathbb{R} \to \mathcal{S}_{\sqrt{\mathbf{L}}}$ of compact support.

Proof. Let (p,q,α) be a wave-admissible Strichartz triple.

(i) Let $f \in \mathcal{S}_{\sqrt{\mathbf{L}}}$ and $s \in \mathbb{R}$. Then, Proposition 2.1.29 followed by an application of Minkowski's inequality ($p, q \geq 2$) gives

$$\begin{aligned}\|L^{-\frac{\alpha}{2}}T^{\pm}(t,s)f\|^2_{\mathrm{L}^p_t(\mathbb{R};\mathrm{L}^q_x(\mathbb{R}^d))} &\simeq \left\| \left(\sum_{\mu \in 2^{\mathbb{Z}}} \left| \psi_\mu(\sqrt{\mathbf{L}}) L^{-\frac{\alpha}{2}} T^{\pm}(t,s) f \right|^2 \right)^{\frac{1}{2}} \right\|^2_{\mathrm{L}^p_t(\mathbb{R};\mathrm{L}^q_x(\mathbb{R}^d))} \\ &\lesssim \sum_{\mu \in 2^{\mathbb{Z}}} \left\| \psi_\mu(\sqrt{\mathbf{L}}) L^{-\frac{\alpha}{2}} T^{\pm}(t,s) f \right\|^2_{\mathrm{L}^p_t(\mathbb{R};\mathrm{L}^q_x(\mathbb{R}^d))}.\end{aligned} \tag{4.34}$$

Note that by the Phillips functional calculus and the fact that the ψ_λ are almost disjointly supported, we have

$$\begin{aligned}\psi_\mu(\sqrt{\mathbf{L}}) L^{-\frac{\alpha}{2}} T^{\pm}(t,s) f &= \sum_{\lambda \in 2^{\mathbb{Z}}} (|\cdot|^{-\alpha} \mathrm{e}^{\pm i \varphi_{t,s}} \psi_\lambda \psi_\mu)(\sqrt{\mathbf{L}}) f \\ &= \sum_{\lambda \in I_\mu} L^{-\alpha} (\mathrm{e}^{\pm i \varphi_{t,s}} \psi_\lambda)(\sqrt{\mathbf{L}}) \psi_\mu(\sqrt{\mathbf{L}}) f,\end{aligned}$$

where $I_\mu \coloneqq \{\frac{\mu}{2}, \mu, 2\mu\}$. So the sum on the right-hand side of (4.34) can be estimated by

$$\begin{aligned}&\sum_{\mu \in 2^{\mathbb{Z}}} \sum_{\lambda \in I_\mu} \left\| L^{-\frac{\alpha}{2}} (\mathrm{e}^{\pm i \varphi_{t,s}} \psi_\lambda)(\sqrt{\mathbf{L}}) \psi_\mu(\sqrt{\mathbf{L}}) f \right\|^2_{\mathrm{L}^p_t(\mathbb{R};\mathrm{L}^q_x(\mathbb{R}^d))} \\ &= \sum_{\lambda \in 2^{\mathbb{Z}}} \sum_{\mu \in I_\lambda} \left\| L^{-\frac{\alpha}{2}} (\mathrm{e}^{\pm i \varphi_{t,s}} \psi_\lambda)(\sqrt{\mathbf{L}}) \psi_\mu(\sqrt{\mathbf{L}}) f \right\|^2_{\mathrm{L}^p_t(\mathbb{R};\mathrm{L}^q_x(\mathbb{R}^d))} \\ &\lesssim \sum_{\lambda \in 2^{\mathbb{Z}}} \sum_{\mu \in I_\lambda} \left\| \psi_\mu(\sqrt{\mathbf{L}}) f \right\|_2^2 = 3 \sum_{\mu \in 2^{\mathbb{Z}}} \left\| \psi_\mu(\sqrt{\mathbf{L}}) f \right\|_2^2 \simeq \|f\|_2^2.\end{aligned}$$

where we used Theorem 4.2.6 and Corollary 2.1.24. This proves (i).

(ii) Let $(\tilde{p}, \tilde{q}, \tilde{\alpha})$ be another wave-admissible Strichartz triple with $\tilde{p}' < p$. Fix $s \in \mathbb{R}$. Since $\mathcal{S}_{\sqrt{\mathbf{L}}}$ is dense in $\mathrm{L}^2(\mathbb{R}^d)$, it follows from (i) that

$$\mathcal{T}_s \colon \mathrm{L}^2(\mathbb{R}^d) \to L^{\tilde{p}}_t(\mathbb{R}; L^{\tilde{q}}_x(\mathbb{R}^d)), \quad (\mathcal{T}_s f)(t) = L^{-\frac{\tilde{\alpha}}{2}} T^{\pm}(t,s) f$$

is a bounded linear operator. Recalling the scalar product (2.73) on $\mathrm{L}^2(\mathbb{R}^d)$, we may define the equivalent dual pairing

$$\langle F, G \rangle \coloneqq \int_{\mathbb{R}} \langle F(\tau), G(\tau) \rangle_A \, \mathrm{d}\tau$$

on $L^{\tilde{p}}_\tau(\mathbb{R}; L^{\tilde{q}}_x(\mathbb{R}^d)) \times L^{\tilde{p}'}_\tau(\mathbb{R}; L^{\tilde{q}'}_x(\mathbb{R}^d))$. Since $\mathcal{T}_s$ is bounded, the adjoint operator $\mathcal{T}_s^* \colon L^{\tilde{p}'}_t(\mathbb{R}; L^{\tilde{q}'}_x(\mathbb{R}^d)) \to \mathrm{L}^2(\mathbb{R}^d)$ and thus the composition $\mathcal{T}_s \mathcal{T}_s^*$ are bounded, too (here, we wrote $\mathcal{T}_s^*$ instead of $\mathcal{T}_s'$ for the adjoint operator to align our notation with the classical TT^*-argument as in [23]). Let $G \colon \mathbb{R} \to \mathcal{S}_{\sqrt{\mathbf{L}}}$ be measurable and of compact support. Then, a straightforward computation reveals

$$\mathcal{T}_s^* G = \int_{\mathbb{R}} \mathcal{T}_\tau(s) G(\tau) \, \mathrm{d}\tau,$$

and using that

$$T^\pm(t,s)T^\pm(s,\tau) = T^\pm(t,\tau), \quad T^\pm(t,s)^* = T^\pm(s,t) \quad \text{for} \quad t,s,\tau \in \mathbb{R}$$

as identities on $\mathcal{L}(\mathrm{L}^2(\mathbb{R}^d))$, we conclude

$$(\mathcal{T}_s \mathcal{T}_s^* G)(t) = \int_{\mathbb{R}} L^{-\frac{\alpha+\tilde{\alpha}}{2}} T^\pm(t,\tau) G(\tau) \, \mathrm{d}\tau$$

for a.e. $t \in \mathbb{R}$. Now, the claim follows from the boundedness of $\mathcal{T}_s \mathcal{T}_s^*$ and Lemma 4.3.1. □

Corollary 4.3.3 (Strichartz Estimates for Parametrices II)**.** *Suppose that (p,q,α) is a wave-admissible Strichartz triple.*

(i) We have

$$\|L^{\frac{1-\alpha}{2}} C(t,s) f\|_{\mathrm{L}_t^p(\mathbb{R};L_x^q(\mathbb{R}^d))} \lesssim \|L^{\frac{1}{2}} f\|_2 \quad (f \in \mathrm{H}_L^1(\mathbb{R}^d)) \tag{4.35}$$

and

$$\|L^{\frac{1-\alpha}{2}} S(t,s) f\|_{\mathrm{L}_t^p(\mathbb{R};L_x^q(\mathbb{R}^d))} \lesssim \|f\|_2 \quad (f \in \mathrm{L}^2(\mathbb{R}^d)) \tag{4.36}$$

uniformly in $s \in \mathbb{R}$.

(ii) Suppose that $(\tilde{p},\tilde{q},\tilde{\alpha})$ is another Strichartz triple with $\tilde{p}' < p$. Then, there holds the inhomogeneous estimate

$$\Big\| \int_0^t L^{\frac{1-\alpha-\tilde{\alpha}}{2}} S(t,s) G(s) \, \mathrm{d}s \Big\|_{\mathrm{L}_t^p(\mathbb{R};(\mathrm{L}_x^q(\mathbb{R}^d)))} \lesssim \|G\|_{\mathrm{L}_t^{\tilde{p}'}(\mathbb{R};\mathrm{L}_x^{\tilde{q}'}(\mathbb{R}^d))}$$

for $G \in \mathrm{L}_t^{\tilde{p}'}(\mathrm{L}_x^{\tilde{q}'}(\mathbb{R}^d))$.

Proof. Let (p,q,α) be wave-admissible and let $f \in \mathcal{S}_{\sqrt{\mathbf{L}}}$. By Lemma 3.2.7 (a) and Proposition 2.1.15 (b), we may write

$$L^{\frac{1-\alpha}{2}} C(t,s) f = \tfrac{1}{2} L^{-\frac{\alpha}{2}} T^+(t,s) L^{\frac{1}{2}} f + \tfrac{1}{2} L^{-\frac{\alpha}{2}} T^-(t,s) L^{\frac{1}{2}} f,$$

so Theorem 4.3.2 gives $\|L^{\frac{1-\alpha}{2}} C(t,s) f\|_{\mathrm{L}_t^p(\mathbb{R};L_x^q(\mathbb{R}^d))} \lesssim \|L^{\frac{1}{2}} f\|_2$ uniformly in $s \in \mathbb{R}$. Now, (4.35) follows as $\mathcal{S}_{\sqrt{\mathbf{L}}}$ is dense in $\mathrm{H}_L^1(\mathbb{R}^d)$ (in particular, the left-hand side $L^{\frac{1-\alpha}{2}} C(t,s) f$ is to be understood as a limit in $\mathrm{L}_t^p(\mathbb{R};\mathrm{L}_x^q(\mathbb{R}^d))$). This proves (4.35). By the definition of $\mathcal{S}_{\sqrt{\mathbf{L}}}$, we find $\chi \in C_c^\infty(\mathbb{R}^d)$ supported away from the origin such that $\chi(\sqrt{\mathbf{L}})f = f$. In view of (3.2.6), we may then write

$$L^{\frac{1-\alpha}{2}} S(t,s)f = \tfrac{1}{2} L^{-\frac{\alpha}{2}} T^{+}(t,s) m_s(\sqrt{\mathbf{L}})f - \tfrac{1}{2} L^{-\frac{\alpha}{2}} T^{-}(t,s) m_s(\sqrt{\mathbf{L}})f$$

with $m_s(\xi) := \frac{|\xi|}{(B(s)\xi|\xi)^{1/2}} \chi(\xi)$. Since $\|m_s(\sqrt{\mathbf{L}})f\|_2 \lesssim \|f\|_2$, we argue as above to infer (4.36). Finally, (ii) is proved similarly using $\|m_s(\sqrt{\mathbf{L}})\|_{\mathcal{L}(\mathrm{L}^{\tilde{q}'}_x(\mathbb{R}^d))} \lesssim 1$ by Proposition 2.1.10 (d). ☐

From the preceding corollary and Theorem 3.3.7, we obtain Strichartz estimates for weak solutions in $\mathrm{H}^1_L(\mathbb{R}^d)$.

Theorem 4.3.4 (Global-In-Time Strichartz Estimates for Weak Solutions in $\mathrm{H}^1_L(\mathbb{R}^d)$)**.** *Let (p,q,α) be a wave-admissible Strichartz triple. Suppose that $g \in \mathrm{H}^1_L(\mathbb{R}^d)$, $h \in \mathrm{L}^2(\mathbb{R}^d)$, and $F \in \mathrm{L}^1(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))$. Then, the weak solution to the wave equation* (3.1) *satisfies the global-in-time Strichartz estimate*

$$\|L^{\frac{1-\alpha}{2}} u\|_{\mathrm{L}^p_t(\mathbb{R};\mathrm{L}^q_x(\mathbb{R}^d))} \lesssim \|g\|_{\mathrm{H}^1_L(\mathbb{R}^d)} + \|h\|_{\mathrm{L}^2(\mathbb{R}^d)} + \|F\|_{\mathrm{L}^1(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))}.$$

Proof. By Theorem 3.3.7, we have the representation

$$u(t) = C(t,0)g + S(t,0)h + \int_0^t S(t,s)G(s)\,\mathrm{d}s \quad (t \in \mathbb{R})$$

with

$$\|G\|_{\mathrm{L}^1(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))} \lesssim \|g\|_{\mathrm{H}^1_L(\mathbb{R}^d)} + \|h\|_{\mathrm{L}^2(\mathbb{R}^d)} + \|F\|_{\mathrm{L}^1(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))}. \tag{4.37}$$

By Corollary 4.3.3 (i), we have

$$\begin{aligned}\|L^{\frac{1-\alpha}{2}} C(t,0)g(x)\|_{\mathrm{L}^p_t(\mathrm{L}^q_x(\mathbb{R}^d))} &\lesssim \|L^{\frac{1}{2}} g\|_2 \lesssim \|g\|_{\mathrm{H}^1_L(\mathbb{R}^d)},\\ \|L^{\frac{1-\alpha}{2}} S(t,0)g(x)\|_{\mathrm{L}^p_t(\mathrm{L}^q_x(\mathbb{R}^d))} &\lesssim \|h\|_2.\end{aligned}$$

Moreover, applying Corollary 4.3.3 (ii) with $(\tilde{p},\tilde{q},\tilde{\alpha}) = (\infty,2,0)$ (note that then $\tilde{p}' = 1 < 2 \le p$) and (4.37), we find

$$\begin{aligned}\left\| \int_0^t L^{\frac{1-\alpha}{2}} S(t,s)G(s)\,\mathrm{d}s \right\|_{\mathrm{L}^p_t(\mathrm{L}^q_x(\mathbb{R}^d))} &\lesssim \|G\|_{\mathrm{L}^1(\mathrm{L}^2(\mathbb{R}^d))}\\ &\lesssim \|g\|_{\mathrm{H}^1_L(\mathbb{R}^d)} + \|h\|_{\mathrm{L}^2(\mathbb{R}^d)} + \|F\|_{\mathrm{L}^1(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))}.\end{aligned}$$

☐

Combining Theorem 4.3.4 with Proposition 2.3.3, we arrive at the proof of Theorem 1.1.4.

Corollary 4.3.5 (Global-In-Time Strichartz Estimates for Weak Solutions in $\mathrm{H}^1(\mathbb{R}^d)$)**.** *Let (p,q,α) be a wave-admissible Strichartz triple and $\alpha \in [0,2]$. Suppose further that $g \in \mathrm{H}^1(\mathbb{R}^d)$, $h \in \mathrm{L}^2(\mathbb{R}^d)$, and $F \in \mathrm{L}^1(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))$. Then, the weak solution to the wave equation* (3.1) *satisfies the global-in-time Strichartz estimate*

$$\||D_x|^{1-\alpha} u\|_{\mathrm{L}^p_t(\mathbb{R};\mathrm{L}^q_x(\mathbb{R}^d))} \lesssim \|g\|_{\mathrm{H}^1(\mathbb{R}^d)} + \|h\|_{\mathrm{L}^2(\mathbb{R}^d)} + \|F\|_{\mathrm{L}^1(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))}. \tag{4.38}$$

4.4. Paradifferential Smoothing and the Proof of Theorem 1.1.5

In this last section, we want to indicate how to relax the regularity assumption on the coefficients b_j from C^1 to Lipschitz. One can use a paradifferential smoothing procedure as described in [47] to essentially reduce to the C^1-case. We only sketch the ideas. So we suppose that (A_a) holds and that $b_1, \dots, b_d$ satisfy

$(\mathrm{A}_b)_{\mathrm{Lip}}$ The functions $b_1, \dots, b_d$ are Lipschitz continuous and for some sufficiently small $\varepsilon_0 \in (0, \frac{1}{2})$ we have

$$1 - \varepsilon_0 \le b_j(t) \le 1 + \varepsilon_0 \quad \text{for all } t \in \mathbb{R} \text{ and } j \in \{1, \dots, d\}. \tag{4.39}$$

We set $m_4 \coloneqq \max_{1 \le j \le d} \|b_j'\|_\infty < \infty$. Moreover, we assume that there exists some sufficiently small $\varepsilon_1 = \varepsilon_1(m_1, m_2, m_4) > 0$ such that

$$\max_{1 \le j \le d} \|b_j'\|_{\mathrm{L}^1(\mathbb{R})} \le \varepsilon_1. \tag{4.40}$$

Recall that, by the Rademacher Theorem, Lipschitz functions are differentiable a.e., so (4.40) makes sense. Now, we smooth out the coefficients using a mollifier. More precisely, let $\rho \in C_c^\infty(\mathbb{R})$ be nonnegative with $\int_{\mathbb{R}} \rho(t)\,\mathrm{d}t = 1$ and $\operatorname{supp}(\rho) \subseteq (-1, 1)$. For $\lambda \in 2^{\mathbb{Z}}$, we set $\rho^\lambda(t) = \lambda\rho(\lambda t)$,

$$B^\lambda(t) \coloneqq \operatorname{diag}(b_1^\lambda(t), \dots, b_d^\lambda(t)), \quad b_j^\lambda \coloneqq \rho^\lambda * b_j.$$

and $P^\lambda(t) = \sum_{j=1}^d b_j^\lambda(t) L_j$. Observe that the conditions (4.39), (4.40) are invariant under convolution with ρ^λ. Now, the idea is to replace the parametrices $T^\pm(t,s)$ as defined in Subsection 3.2.2 by

$$T_\#^\pm(t,s)f \coloneqq \sum_{\lambda \in 2^{\mathbb{Z}}} \left(\mathrm{e}^{i\varphi_{t,s}^\lambda}\psi_\lambda\right)(\sqrt{\mathbf{L}})f, \quad \varphi_{t,s}^\lambda(\xi) \coloneqq \int_s^t (B^\lambda(\tau)\xi|\xi)^{1/2}\,\mathrm{d}\tau,$$

i.e., $T_\#^\pm(t,s)$ is obtained from $T^\pm(t,s)$ by replacing $B(t)$ with $B^\lambda(t)$ at frequency scale λ. We may argue as we already did for $T^\pm(t,s)$ to see that

$$\begin{aligned} &\left(D_t^2 - P(t)\right)T_\#^\pm(t,s)f \\ &= \sum_{\lambda \in 2^{\mathbb{Z}}} (D_t^2 - P^\lambda(t))(\mathrm{e}^{i\varphi_{t,s}^\lambda}\psi_\lambda)(\sqrt{\mathbf{L}})f + \sum_{\lambda \in 2^{\mathbb{Z}}} (P^\lambda(t) - P(t))(\mathrm{e}^{i\varphi_{t,s}^\lambda}\psi_\lambda)(\sqrt{\mathbf{L}})f \\ &=: R_1(t,s)f + R_2(t,s)f. \end{aligned}$$

Now, observe that both operators $R_1(t,s)$ and $R_2(t,s)$ are of order *one*, i.e., bounded operators from $\mathrm{H}_L^\alpha(\mathbb{R}^d) \to \mathrm{H}_L^{\alpha-1}(\mathbb{R}^d)$. Indeed, $R_1(t,s)f$ is treated exactly as in Lemma 3.2.4, and thus, $R_1(t,s)$ is of order one. To handle $R_2(t,s)f$, note that each term in the sum is

$$
\begin{aligned}
&(P^\lambda(t) - P(t))(\mathrm{e}^{i\varphi^\lambda_{t,s}}\psi_\lambda)(\sqrt{\mathbf{L}})f \\
&= \sum_{j=1}^{d}(b_j^\lambda(t) - b_j(t))(\xi_j^2\mathrm{e}^{i\varphi^\lambda_{t,s}}\psi_\lambda)(\sqrt{\mathbf{L}})f = \lambda(h_{t,s,\lambda}\psi_\lambda)(\sqrt{\mathbf{L}})f
\end{aligned}
$$

with

$$
h_{t,s,\lambda} \coloneqq \sum_{j=1}^{d}\Big[\lambda(b_j^\lambda(t) - b_j(t))\Big]\Big(\tfrac{\xi_j}{\lambda}\Big)^2\mathrm{e}^{i\varphi^\lambda_{t,s}}
$$

Since b_j is Lipschitz, we have

$$
\lambda\|b_j^\lambda - b_j\|_\infty \lesssim \|b_j'\|_\infty \le m_4,
$$

so $|h_{t,s,\lambda}| \lesssim 1$ on the support of ψ_λ. It follows that $T^\pm_\#(t,s)$ define parametrices for the wave operator $\Box_P$, from which we may deduce all the previous results, at least locally in time. To get global results, however, it is more convenient to work with the parametrices $T^\pm(t,s)$ directly. Indeed, repeating the proofs in Chapter 3 with minor modifications shows that Theorem 3.3.7 remains valid under $(\mathrm{A}_b)_{\mathrm{Lip}}$. Since the arguments in this chapter do not use the regularity of the b_j, we deduce Theorem 1.1.5, which we record as a corollary:

Corollary 4.4.1 (Global-In-Time Strichartz Estimates for Lipschitz-coefficients)**.** *Let* (p,q,α) *be a wave-admissible Strichartz triple and* $\alpha \in [0,2]$. *Suppose further that the conditions* (A_a) *and* $(\mathrm{A}_b)_{\mathrm{Lip}}$ *hold true and that* $g \in \mathrm{H}^1(\mathbb{R}^d)$, $h \in \mathrm{L}^2(\mathbb{R}^d)$, *and* $F \in \mathrm{L}^1(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))$. *Then, the weak solution to the wave equation* (3.1) *satisfies the global-in-time Strichartz estimate*

$$
\||D_x|^{1-\alpha}u\|_{\mathrm{L}^p_t(\mathbb{R};\mathrm{L}^q_x(\mathbb{R}^d))} \lesssim \|g\|_{\mathrm{H}^1(\mathbb{R}^d)} + \|h\|_{\mathrm{L}^2(\mathbb{R}^d)} + \|F\|_{\mathrm{L}^1(\mathbb{R};\mathrm{L}^2(\mathbb{R}^d))}. \tag{4.41}
$$

Remark 4.4.2. In contrast to the Strichartz estimates for the classical wave equation, we have the inhomogeneous norm $\|g\|_{\mathrm{H}^1(\mathbb{R}^d)}$ in place of the homogeneous norm $\|g\|_{\dot{\mathrm{H}}^1_L(\mathbb{R}^d)}$ on the right-hand side of the Strichartz estimates. This is because we used the inhomogeneous scale $\mathrm{H}^\alpha_L(\mathbb{R}^d), \alpha \in \mathbb{R}$, for the parametrix construction. However, using the homogeneous scale of spaces $\dot{\mathrm{H}}^\alpha_L(\mathbb{R}^d)$, $\alpha \in \mathbb{R}$, (see [35, Appendix E] for a precise definition) for the parametrix construction, one should also be able to obtain (4.38) with $\|g\|_{\dot{\mathrm{H}}^1(\mathbb{R}^d)}$ on the right-hand side.

A. Properties of $\mathbb{S}_{t,s}^{d-1}$

In this section, we provide proofs of Proposition 4.1.2 and Lemma 4.1.5, which summarized some important properties of the hypersurfaces $\mathbb{S}_{t,s}^{d-1}$. Recall that for $(t,s) \in \Delta^c$

$$\mathbb{S}_{t,s}^{d-1} := \{\xi \in \mathbb{R}^d \colon \tilde{\varphi}_{t,s}(\xi) = 1\}, \quad \tilde{\Sigma}_{t,s} := \{\nabla_\xi \tilde{\varphi}_{t,s}(\xi) \in \mathbb{R}^d \mid \xi \neq 0\},$$

where $\tilde{\varphi}_{t,s} \colon \mathbb{R}^d \to \mathbb{R}$, $\tilde{\varphi}_{t,s}(\xi) := \frac{1}{t-s} \int_s^t (B(\tau)\xi|\xi)^{\frac{1}{2}} \,\mathrm{d}\tau$. As was already observed in (4.9) and (4.10), we have

$$\mathbb{S}_{t,s}^{d-1} \subseteq \Big\{\xi \in \mathbb{R}^d \colon \tfrac{1}{c_1} \leq |\xi| \leq \tfrac{1}{c_2}\Big\}, \quad \tilde{\Sigma}_{t,s} \subseteq \Big\{\xi \in \mathbb{R}^d \colon \tfrac{c_2^2}{c_1} \leq |\xi| \leq \tfrac{c_1^2}{c_2}\Big\}. \tag{A.1}$$

with $c_1 := c_{1,\varepsilon_0} := \sqrt{1+\varepsilon_0}$ and $c_2 := c_{2,\varepsilon_0} := \sqrt{1-\varepsilon_0}$. We will frequently use that $\nabla_\xi \tilde{\varphi}_{t,s}(\xi) = \mathsf{B}(\xi)\xi$ for $\xi \neq 0$, where

$$\mathsf{B}(\xi) := \operatorname{diag}(\mathsf{B}_1(\xi), \ldots, \mathsf{B}_d(\xi)), \quad \mathsf{B}_j(\xi) := \frac{1}{t-s} \int_s^t \frac{b_j(\tau)}{(B(\tau)\xi|\xi)^{1/2}} \,\mathrm{d}\tau.$$

Note that by (A.1) we have

$$c \leq \mathsf{B}_j(\omega) \leq \tilde{c} \quad (\omega \in \mathbb{S}_{t,s}^{d-1}, j \in \{1, \ldots, d\}) \tag{A.2}$$

with $c := \frac{c_2^3}{c_1}$ and $\tilde{c} := \frac{c_1^3}{c_2}$.

Proposition A.0.1 (Properties of $\mathbb{S}_{t,s}^{d-1}$)**.** *Let* $(t,s) \in \Delta^c$.

(i) $\mathbb{S}_{t,s}^{d-1}$ *is a smooth, compact hypersurface in* $\mathbb{R}^d$*, with normal space at each* $\omega \in \mathbb{S}_{t,s}^{d-1}$ *given by*

$$N_\omega(\mathbb{S}_{t,s}^{d-1}) = \operatorname{span}\Big\{\nabla_\xi \tilde{\varphi}_{t,s}(\omega)\Big\}. \tag{A.3}$$

Moreover, $\mathbb{S}_{t,s}^{d-1}$ *is the boundary of the compact, strictly convex set*

$$C_{t,s} := \{\xi \in \mathbb{R}^d \mid \tilde{\varphi}_{t,s}(\xi) \leq 1\}.$$

(ii) The Gauss map on $\mathbb{S}_{t,s}^{d-1}$ *given by*

$$n \colon \mathbb{S}_{t,s}^{d-1} \to S^{d-1}, \quad n(\omega) := \frac{\nabla_\xi \tilde{\varphi}_{t,s}(\omega)}{|\nabla_\xi \tilde{\varphi}_{t,s}(\omega)|}$$

is a diffeomorphism with $n(-\omega) = -n(\omega)$*. In particular, for each* $\nu \in S^{d-1}$ *there exists exactly one* $\omega_0 \in \mathbb{S}_{t,s}^{d-1}$ *such that* $\pm\nu = n(\pm\omega_0)$.

(iii) *Let n be the Gauss map from (ii) and ε_0 from* (1.10) *sufficiently small. Then, there exists some $\kappa = \kappa(\varepsilon_0) > 0$ independent of $(t,s) \in \Delta^c$ such that in each point $\omega \in \mathbb{S}_{t,s}^{d-1}$, the principal curvatures $\kappa_1(\omega), \ldots, \kappa_{d-1}(\omega)$ with respect to $-n$ satisfy $\kappa_j(\omega) \geq \kappa$.*

(iv) *Let $\mathrm{d}\omega := |\nabla_\xi \tilde{\varphi}_{t,s}|^{-1}\,\mathrm{d}\mathcal{H}^{d-1}$ be the $(d-1)$-dimensional Hausdorff measure on $\mathbb{S}_{t,s}^{d-1}$ normalized by $|\nabla_\xi \tilde{\varphi}_{t,s}|$. Then*

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \int_0^\infty \left(\int_{\mathbb{S}_{t,s}^{d-1}} f(r\omega)\,\mathrm{d}\omega \right) r^{d-1}\,\mathrm{d}r$$

for any integrable function $f\colon \mathbb{R}^d \to \mathbb{C}$.

Proof. Let $(t,s) \in \Delta^c$.
(i) Clearly, $\tilde{\varphi}_{t,s}\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ is smooth with $\nabla_\xi \tilde{\varphi}_{t,s}(\xi) = \mathsf{B}(\xi)\xi \neq 0$ for all $\xi \neq 0$. As 0 does not belong to $\mathbb{S}_{t,s}^{d-1} = (\tilde{\varphi}_{t,s})^{-1}(1)$, we infer that 1 is a regular value of $\tilde{\varphi}_{t,s}$. This implies that $\mathbb{S}_{t,s}^{d-1} = (\tilde{\varphi}_{t,s})^{-1}(1)$ is a smooth hypersurface in $\mathbb{R}^d$ satisfying (A.3).
The set $C_{t,s}$ is closed by the continuity of $\tilde{\varphi}_{t,s}$ and bounded as $C_{t,s} \subseteq \overline{B_{c_2}(0)}$ by (A.1) and the positive homogeneity of $\tilde{\varphi}_{t,s}$. Hence, $C_{t,s}$ and therefore also $\mathbb{S}_{t,s}^{d-1}$ (being a closed subset of $C_{t,s}$) are compact. To prove that $C_{t,s}$ is strictly convex, we just note that $\tilde{\varphi}_{t,s}$ defines a strictly convex norm on $\mathbb{R}^d$. Indeed, in view of (1.10), for each $\tau \in \mathbb{R}$ the function

$$\mathbb{R}^d \to \mathbb{R}, \quad \xi \mapsto (B(\tau)\xi|\xi)^{\frac{1}{2}}$$

defines a strictly convex norm and the norm axioms as well as strict convexity are preserved under integration (with respect to τ).
By definition, $U := \{\xi \in \mathbb{R}^d \colon \tilde{\varphi}_{t,s}(\xi) < 1\}$ belongs to $C_{t,s}$ and is open by the continuity of $\tilde{\varphi}_{t,s}$. We infer that U belongs to the interior of $C_{t,s}$ and thus $\partial C_{t,s} \subseteq C_{t,s} \setminus U = \mathbb{S}_{t,s}^{d-1}$. Conversely, if $\omega \in \mathbb{S}_{t,s}^{d-1}$, we consider $h\colon \mathbb{R} \to \mathbb{R}$, $h(\tau) = \tilde{\varphi}_{t,s}(\omega + \tau\nu)$, with $\nu := \nabla_\xi \tilde{\varphi}_{t,s}(\omega) \neq 0$. Then h is differentiable in 0 with $h'(0) = |\nu|^2 > 0$ which implies $\tilde{\varphi}(\omega + \tau\nu) = h(\tau) > h(0) = 1$ and similarly $\tilde{\varphi}(\omega - \tau\nu) = h(-\tau) < h(0) = 1$ for sufficiently small $\tau > 0$. For those τ, we conclude $\omega + \tau\nu \notin C_{t,s}$ and $\omega - \tau\nu \in C_{t,s}$, showing that $\omega \in \partial C_{t,s}$. We have proved that $\mathbb{S}_{t,s}^{d-1} = \partial C_{t,s}$.

(iii) Let n be the Gauss map as defined in (ii) and $\omega \in \mathbb{S}_{t,s}^{d-1}$. Recall that the principal curvatures $\kappa_1(\omega), \ldots, \kappa_{d-1}(\omega)$ with respect to $-n$ are the eigenvalues of the self-adjoint shape operator

$$L_\omega\colon T_\omega(\mathbb{S}_{t,s}^{d-1}) \to T_\omega(\mathbb{S}_{t,s}^{d-1}), \quad L_\omega(v) = D_v n(\omega)$$

(here, D_v denotes the directional derivative along the tangent vector v). Thus, we have to show that there exists some $\kappa \in (0,1)$ such that $\kappa_j(\omega) \geq \kappa$ for all $(s,t) \in \Delta^c$, $j \in \{1, \ldots, d-1\}$ and $\omega \in \mathbb{S}_{t,s}^{d-1}$. Equivalently, we need to show that the bilinear form associated with L_ω, namely the second fundamental form in ω defined by

$$\mathrm{II}_\omega\colon T_\omega(\mathbb{S}^{d-1}_{t,s}) \times T_\omega(\mathbb{S}^{d-1}_{t,s}) \to \mathbb{R}, \quad \mathrm{II}_\omega(v,\tilde{v}) = (L_\omega v|\tilde{v})$$

is positive definite, uniformly in $(s,t) \in \Delta^c$ and $\omega \in \mathbb{S}^{d-1}_{t,s}$. Unfortunately, the strict convexity of $C_{t,s}$, as shown in (i), only implies that II_ω is positive *semi*-definite. To prove strict positive definiteness, we employ a perturbation argument. Fix $\omega = (\omega_1, \dots, \omega_d) \in \mathbb{S}^{d-1}_{t,s}$. A computation yields for any $v \in T_\omega(\mathbb{S}^{d-1}_{t,s})$

$$|\nabla_\xi \tilde{\varphi}_{t,s}(\omega)| \cdot \mathrm{II}_\omega(v,v) = (\nabla^2_\xi \tilde{\varphi}_{t,s}(\omega)v|v) = (\mathsf{B}(\omega)v|v) - (\mathsf{R}(\omega)v|v) \tag{A.4}$$

with

$$\mathsf{R}(\omega) = (r_{jk}(\omega)\omega_j\omega_k)^d_{j,k=1}, \quad r_{jk}(\omega) := \frac{1}{t-s}\int_s^t \frac{b_j(\tau)b_k(\tau)}{(B(\tau)\omega|\omega)^{3/2}}\,\mathrm{d}\tau. \tag{A.5}$$

Therefore, in view of (A.1), (A.2),

$$\mathrm{II}_\omega(v,v) \gtrsim (\mathsf{B}(\omega)v|v) - (\mathsf{R}(\omega)v|v) \gtrsim |v|^2 - (\mathsf{R}(\omega)v|v).$$

So it is enough to show that

$$|(R(\omega)v|v)| = \mathcal{O}(\varepsilon_0) \cdot |v|^2 \quad (v \in T_\omega(\mathbb{S}^{d-1}_{t,s})). \tag{A.6}$$

Put $\nu := \nabla_\xi \tilde{\varphi}_{t,s}(\omega) = \mathsf{B}(\omega)\omega$. Without restriction, we may assume $\nu_d \neq 0$. Then, $T_\omega(\mathbb{S}^{d-1}_{t,s}) = \operatorname{span}\{v_1, \dots, v_{d-1}\}$ with $v_k := \nu_d e_k - \nu_k e_d$. Now, if $v \in T_\omega(\mathbb{S}^{d-1}_{t,s})$, then there exists $\lambda \in \mathbb{R}^{d-1}$ such that $v = \sum_{k=1}^{d-1} \lambda_k v_k$ and thus

$$(R(\omega)v|v) = \sum_{j,k=1}^{d-1} \lambda_j\lambda_k (R(\omega)v_k|v_j)$$

with

$$\begin{aligned}(R(\omega)v_k|v_j) &= \Big[\Big(r_{jk}\mathsf{B}_d - r_{jd}\mathsf{B}_k\Big)\mathsf{B}_d - \Big(r_{dk}\mathsf{B}_d - r_{dd}\mathsf{B}_k\Big)\mathsf{B}_j\Big]\omega_j\omega_k\omega_d^2 \\ &= \Big[\Big(r_{jk} - r_{jd}\Big)\mathsf{B}_d^2 + r_{jd}\Big(\mathsf{B}_d - \mathsf{B}_k\Big)\mathsf{B}_d\Big]\omega_j\omega_k\omega_d^2 \\ &\quad + \Big[\Big(r_{dd} - r_{dk}\Big)\mathsf{B}_j\mathsf{B}_k + r_{dk}\Big(\mathsf{B}_k - \mathsf{B}_d\Big)\mathsf{B}_j\Big]\omega_j\omega_k\omega_d^2.\end{aligned}$$

Now, by (1.10) we have $\|b_k - b_j\|_\infty \le 2\varepsilon_0$ for all $j,k \in \{1,\dots,d\}$. Hence,

$$|(r_{jk} - r_{jd})(\omega)| \le \left(\frac{1}{|t-s|}\int_s^t \frac{b_j(\tau)}{(B(\tau)\omega|\omega)^{3/2}}\,\mathrm{d}\tau\right) \cdot \|b_k - b_d\|_\infty \lesssim \varepsilon_0.$$

and similarly

$$|\mathsf{B}_d(\omega) - \mathsf{B}_k(\omega)| \le \left(\frac{1}{|t-s|}\int_s^t \frac{1}{(B(\tau)\omega|\omega)^{1/2}}\,\mathrm{d}\tau\right) \cdot \|b_k - b_d\|_\infty \lesssim \varepsilon_0.$$

We conclude that $|(R(\omega)v_k|v_j)| \lesssim \varepsilon_0|\omega_j\omega_k|\omega_d^2$ and since $|\omega_d| \simeq |\nu_d|$ by (A.2), we obtain

$$|(R(\omega)v|v)| \lesssim \varepsilon_0\nu_d^2 \sum_{j,k=1}^{d-1} \lambda_j\lambda_k|\omega_j||\omega_k| \lesssim \varepsilon_0\nu_d^2|\lambda|^2 \leq \varepsilon_0|v|^2$$

(the last inequality follows from the identity $|v|^2 = (\lambda|\nu')^2 + \nu_d^2|\lambda|^2$, which is readily verified). This proves (A.6) and therefore the claim.

(ii) Note that by (iii), $\mathrm{d}n_\omega \colon T_\omega(\mathbb{S}_{t,s}^{d-1}) \to S^{d-1}$ has nonvanishing determinant for each $\omega \in \mathbb{S}_{t,s}^{d-1}$, so n is a local diffeomorphism. Also, $n(-\omega) = -n(\omega)$ for $\omega \in \mathbb{S}_{t,s}^{d-1}$ follows immediately from the definition of n. Therefore, it suffices to show that n is bijective.

Injectivity: Suppose that $\nu := n(\omega_1) = n(\omega_2)$ for some $\omega_1, \omega_2 \in \mathbb{S}_{t,s}^{d-1}$. By (i), the set $C_{t,s}$ is strictly convex and n is the outer unit normal vector field on the boundary of $C_{t,s}$. Therefore, we must have $C_{t,s}\backslash\{\omega_j\} \subseteq H_j$, where H_j is the open supporting hyperplane at ω_j defined by

$$H_j := \{\xi \in \mathbb{R}^d \colon (\xi - \omega_j|\nu) < 0\} \quad (j \in \{1, 2\}).$$

But this is only possible if $\omega_1 = \omega_2$.

Surjectivity: Let $\nu \in S^{d-1}$. We consider the linear function

$$f \colon \mathbb{R}^d \to \mathbb{R}, \quad f(\omega) = \nu \cdot \omega.$$

By the extreme value theorem and the compactness of $C_{t,s}$, there are extremal points $\omega_1, \omega_2 \in C_{t,s}$ in which f attains a maximum and a minimum. The function f, being linear, cannot attain extremal values in the interior of $C_{t,s}$ and thus ω_1 and ω_2 have to lie on $\partial C_{t,s} = \mathbb{S}_{t,s}^{d-1}$. In particular, $f(\omega_1)$ and $f(\omega_2)$ are local extremas on $\mathbb{S}_{t,s}^{d-1}$. Therefore, $\nu = \nabla_\omega f(\omega_j)$ must belong to $N_{\omega_j}(\mathbb{S}_{t,s}^{d-1})$ by the Lagrange multiplier theorem. Combining this with (i), we conclude $\pm n(\omega_1) = \nu = \pm n(\omega_2)$. If $n(\omega_1) = n(\omega_2)$, then $\omega_1 = \omega_2$ by the already shown injectivity. But then f would have to be constant (since minimum and maximum of f would be equal), contradicting the fact that $f \neq 0$ is linear. So either $\nu = n(\omega_1)$ or $\nu = n(\omega_2)$, which shows that n is surjective.

(iv) The idea is to foliate $\mathbb{R}^d \setminus \{0\}$ by the level sets of $\tilde{\varphi}_{t,s}$ and to observe that the level sets scale according to $\{\tilde{\varphi}_{t,s} = r\} = r\mathbb{S}_{t,s}^{d-1}$ $(r > 0)$ by the positive homogeneity of $\tilde{\varphi}_{t,s}$. To make this idea precise, we let $\tilde{\varphi}'_{t,s}(\xi') := \tilde{\varphi}_{t,s}(\xi', 0)$ for $\xi' \in \mathbb{R}^{d-1}$ and define the $\tilde{\varphi}'_{t,s}$-adapted open unit ball in $\mathbb{R}^{d-1}$ by

$$\mathcal{B} := \{\xi' \in \mathbb{R}^{d-1} \mid \tilde{\varphi}'_{t,s}(\xi') < 1\}.$$

For fixed $\xi' \in \mathcal{B}$, the continuous function

$$f_{\xi'} \colon [0,\infty) \to \mathbb{R}, \quad h \mapsto \tilde{\varphi}_{t,s}(\xi', h)$$

satisfies $f_{\xi'}(0) = \tilde{\varphi}'_{t,s}(\xi') < 1$ and $\lim_{h\to\infty} f(h) \geq \lim_{h\to\infty} c_2 h = \infty$. Invoking the intermediate value theorem and the strict injectivity of f, we deduce that there exists a unique point $h = h(\xi') > 0$ such that $f_{\xi'}(h(\xi')) = 1$. By allowing ξ' to vary over $\mathcal{B}$, this defines a function $h\colon \mathcal{B} \to \mathbb{R}$ and it is straightforward to show that

$$\mathbb{S}^{d-1}_{t,s,\pm} \coloneqq \mathbb{S}^{d-1}_{t,s} \cap \mathbb{R}^d_{\pm} = \{(\xi', \pm h(\xi'))\colon \xi' \in \mathcal{B}\}.$$

Moreover, h is smooth by the implicit function theorem with

$$\nabla_{\xi'} h(\xi') = -\frac{\nabla_{\xi'}\tilde{\varphi}_{t,s}(g(\xi'))}{\partial_d \tilde{\varphi}_{t,s}(g(\xi'))} \quad (\xi' \in \mathcal{B}), \tag{A.7}$$

where $g(\xi') \coloneqq (\xi', h(\xi'))$ for $\xi' \in \mathcal{B}$. Now, it is readily checked that

$$\mathcal{G}\colon (0,\infty) \times \mathcal{B} \to \mathbb{R}^d_+, \quad (r, \xi') \mapsto r g(\xi')$$

is a smooth diffeomorphism with inverse given by

$$\mathcal{G}^{-1}(\xi) = \Big(\tilde{\varphi}_{t,s}(\xi), \tfrac{\xi'}{\tilde{\varphi}_{t,s}(\xi)}\Big), \quad \xi \in \mathbb{R}^d_+.$$

We compute the Jacobian determinant of $\mathcal{G}$. We have

$$\begin{aligned}
\Big|\mathcal{G}'(r,\xi')\Big| &= \left| \begin{pmatrix} \xi' & r\mathrm{I}_{d-1} \\ h(\xi') & r(\nabla_\xi h(\xi'))^T \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} \xi' & r\mathrm{I}_{d-1} \\ h(\xi') - \Big(\nabla_{\xi'} h(\xi')|\xi'\Big) & 0 \end{pmatrix} \right| = r^{d-1}\Big|h(\xi') - (\nabla_{\xi'} h(\xi')|\xi')\Big|.
\end{aligned}$$

The latter expression can be simplified. Indeed, by (A.7),

$$\begin{aligned}
&h(\xi') - \Big(\nabla_{\xi'} h(\xi')|\xi'\Big) \\
&= h(\xi') + \frac{\Big(\nabla_{\xi'}\tilde{\varphi}_{t,s}(g(\xi'))|\xi'\Big)}{\partial_d\tilde{\varphi}_{t,s}(g(\xi'))} = \frac{\Big(\nabla_{\xi}\tilde{\varphi}_{t,s}(g(\xi'))|g(\xi')\Big)}{\partial_d\tilde{\varphi}_{t,s}(g(\xi'))} = \frac{1}{\partial_d\tilde{\varphi}_{t,s}(g(\xi'))},
\end{aligned}$$

where we used Euler's relation $\Big(\nabla_\xi \tilde{\varphi}_{t,s}(\xi)|\xi\Big) = \tilde{\varphi}_{t,s}(\xi)$ for 1-homogeneous functions and that $g(\xi') \in \mathbb{S}^{d-1}_{t,s}$. Therefore, we conclude

$$|\mathcal{G}'(r,\xi')| = r^{d-1}\frac{1}{|\partial_d \tilde{\varphi}_{t,s}(g(\xi'))|}.$$

On the other hand,

$$\frac{\mathrm{d}\mathcal{H}^{d-1}(g(\xi'))}{\mathrm{d}\xi'} = (1+|\nabla_{\xi'} h(\xi')|^2)^{\frac{1}{2}} = \left(1+\frac{|\nabla_{\xi'}\tilde{\varphi}_{t,s}(\xi')|^2}{|\partial_d\tilde{\varphi}_{t,s}(g(\xi'))|^2}\right)^{\frac{1}{2}} = \frac{|\nabla_{\xi}\tilde{\varphi}_{t,s}(g(\xi'))|}{|\partial_d\tilde{\varphi}_{t,s}(g(\xi'))|}.$$

Comparing expressions yields

$$|\mathcal{G}'(r,\xi')| = r^{d-1}\frac{\mathrm{d}\omega(g(\xi'))}{\mathrm{d}\xi'}.$$

Now, let $f\colon \mathbb{R}^d \to \mathbb{C}$ be integrable. Changing variables $x = rg(\xi')$ and applying Fubini's theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d_+} f(x)\,\mathrm{d}x &= \int_0^\infty \left(\int_{\mathcal{B}} f(rg(\xi'))\frac{\mathrm{d}\omega(g(\xi'))}{\mathrm{d}\xi'}\,\mathrm{d}\xi'\right) r^{d-1}\,\mathrm{d}r \\ &= \int_0^\infty \left(\int_{\mathbb{S}^{d-1}_{t,s,+}} f(r\omega)\,\mathrm{d}\omega\right) r^{d-1}\,\mathrm{d}r,\end{aligned}$$

and changing variables $x = (x', x_d) \mapsto (x', -x_d)$, we obtain from the above identity similarly

$$\int_{\mathbb{R}^d_-} f(x)\,\mathrm{d}x = \int_0^\infty \left(\int_{\mathbb{S}^{d-1}_{t,s,-}} f(r\omega)\,\mathrm{d}\omega\right) r^{d-1}\,\mathrm{d}r.$$

Summing these equations yields the claim. □

Lemma A.0.2. *Let* $\delta_0 \coloneqq (2d^{\frac{1}{2}}c_1)^{-1} \in (0,1)$. *Then, for each* $(t,s) \in \Delta^c$ *there exists an open cover* $\mathcal{U}^{t,s} \coloneqq \{\mathbb{U}_j^{t,s} \mid j \in \{1,\dots,2d\}\}$ *of* $\mathbb{S}^{d-1}_{t,s}$ *such that the following holds for all* $(t,s) \in \Delta^c$ *and* $j \in \{1,\dots,2d\}$*:*

(i) For each $\omega \in \mathbb{S}^{d-1}_{t,s}$ *we have* $B(\omega,\delta_0) \subseteq \mathbb{U}_i^{t,s}$ *for some* $i \in \{1,\dots,2d\}$.

(ii) There exists an open, convex 0*-neighborhood* $V_j^{t,s} \subseteq \mathbb{R}^{d-1}$ *and a smooth parametrization* $g_j^{t,s}\colon V_j^{t,s} \to \mathbb{U}_j^{t,s}$ *which is a graph of a smooth function and which satisfies the following bounds,*

$$\begin{aligned}&\|g_j^{t,s}\|_{C^M(V_j^{t,s})} \lesssim_M 1 && (M \in \mathbb{N}_0),\\ &(G_j^{t,s}(\xi)x|x) \geq 1 && (\xi \in V_j^{t,s}, |x| = 1),\\ &|g_j^{t,s}(\xi) - g_j^{t,s}(\eta)| \geq |\xi - \eta| && (\xi,\eta \in V_j^{t,s})\end{aligned}$$

uniformly w.r.t. $(t,s) \in \Delta^c$. *Here,* $G_j^{t,s}(\xi) = (\partial_k g_j^{t,s}(\xi)|\partial_\ell g_j^{t,s}(\xi))_{k,\ell}$ *denotes the Gram matrix of* $g_j^{t,s}$.

Proof. The idea is to use the parametrization from the proof of Proposition A.0.1 (iv) in every coordinate direction. To be precise, let $(t,s) \in \Delta^c$ and let $\mathcal{B} \subseteq \mathbb{R}^{d-1}$, $h = h^{t,s} \colon \mathcal{B} \to \mathbb{R}$ be as in the proof of Proposition A.0.1 (iv). Now for $(t,s) \in \Delta^c$ and $j \in \{1, \ldots, d\}$, we put

$$\mathbb{U}^{t,s}_{j,\pm} \coloneqq \{\omega \in \mathbb{S}^{d-1}_{t,s} \mid \pm\omega_j > \tfrac{\delta_0}{2}\}, \quad V^{t,s}_{j,\pm} \coloneqq (h^{t,s})^{-1}((\tfrac{\delta_0}{2}, \infty))$$

and define

$$g^{t,s}_{j,\pm} \colon V^{t,s}_{j,\pm} \to \mathbb{U}^{t,s}_{j,\pm}, \quad g^{t,s}_{j,\pm}(\xi) = (\xi_1, \ldots, \xi_{j-1}, \pm h^{t,s}(\xi), \xi_{j+1}, \ldots, \xi_{d-1})$$

(to ensure better readability in the following, we deviated slightly from the notation used in Proposition A.0.1 (iv) and wrote ξ instead of ξ' for a variable in $\mathbb{R}^{d-1}$). To prove (i), let $\omega \in \mathbb{S}^{d-1}_{t,s}$. Then, there exists some $j \in \{1, \ldots, d\}$ with $|\omega_j| = |\omega|_\infty$. But then (A.1) implies

$$|\omega_j| \geq d^{-\frac{1}{2}}|\omega| \geq d^{-\frac{1}{2}} c_1^{-1} = 2\delta_0.$$

So $\overline{B}(\omega, \delta_0) \subseteq \mathbb{U}^{t,s}_{j,+}$ if $\omega_j > 0$ and $\overline{B}(\omega, \delta_0) \subseteq \mathbb{U}^{t,s}_{j,-}$ if $\omega_j < 0$, proving (i). In particular, $\{\mathbb{U}^{t,s}_{j,\pm} \colon j \in \{1, \ldots, 2d\}$ is an open cover of $\mathbb{S}^{t,s}_{d-1}$. To prove (ii), we first note that $0 \in V^{t,s}_{j,\pm}$ since $h^{t,s}(0) = 1 > \delta_0$. Moreover, $V^{t,s}_{j,\pm}$ is open since it is the preimage of $(\frac{\delta_0}{2}, \infty)$ under the continuous function $h^{t,s}$, and it is convex since $h_{t,s}$ is a convex function (the latter follows from the definition of $h^{t,s}$ and the fact that $\mathbb{S}^{d-1}_{t,s}$ is the boundary of the (even strictly) convex set $C_{t,s}$, see Proposition A.0.1 (i) and (iv)). Now, we observe that it follows from (A.1), (A.7) and induction that for all $\alpha \in \mathbb{N}_0^{d-1}$, $(t,s) \in \Delta^c$, $j \in \{1, \ldots, d\}$, $\xi \in V^{t,s}_j$ and $g^{t,s}_j = g^{t,s}_{j,\pm}$

$$|\partial^\alpha_\xi g^{t,s}_j(\xi)| \lesssim_\alpha |\partial^\alpha_\xi h^{t,s}(\xi)| \lesssim_\alpha |\partial_j \tilde{\varphi}_{t,s}(g^{t,s}_j(\xi))|^{-(2|\alpha|+1)} \simeq |h^{t,s}_j(\xi)|^{-(2|\alpha|+1)} \lesssim_{\alpha,\delta_0} 1.$$

Similarly, since $G^{t,s}_j(\xi) = \operatorname{diag}(1, \ldots, 1, 1 + |\nabla_\xi h^{t,s}_j(\xi)|^2)$, it is clear that $(G^{t,s}_j(\xi)x|x) \geq 1$ for all $|x| = 1$. Finally, the inequality $|g^{t,s}_j(\xi) - g^{t,s}_j(\eta)| \geq |\xi - \eta|$ is trivial as $g^{t,s}_j$ is a graph. The proof is complete (after an obvious relabeling of the $\mathbb{U}^{t,s}_{j,\pm}$). $\square$

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