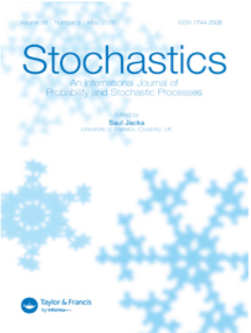

Stochastics

An International Journal of Probability and Stochastic Processes

ISSN: 1744-2508 (Print) 1744-2516 (Online) Journal homepage: www.tandfonline.com/journals/gssr20

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STOCHASTICS
https://doi.org/10.1080/17442508.2026.2628880

Applications of the perturbation formula for Poisson processes to elementary and geometric probability

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ABSTRACT

We present a unified approach to deriving integral representations for the binomial, negative binomial, Poisson, compound Poisson, and Erlang distributions with respect to their continuous parameters. This is achieved using Margulis-Russo-type formulas for Bernoulli and Poisson processes, which also provide a natural probabilistic interpretation of their derivatives. Extending these variational methods, we derive new integro-differential identities that characterize the densities of strictly α-stable multivariate distributions. We further generalize Crofton's derivative formula from integral geometry to the case of Poisson processes. This extension allows us to establish a new probabilistic proof of the formula for binomial point processes, highlighting the underlying geometric structure in a probabilistic framework.

ARTICLE HISTORY
Received 8 June 2025
Accepted 2 February 2026

KEYWORDS
Margulis-Russo formula; binomial distribution; negative binomial distribution; Poisson distribution; compound Poisson distribution; Erlang distribution; multivariate strictly stable distribution; Poisson process; binomial process; Crofton's derivative formula

2020 MATHEMATICS SUBJECT CLASSIFICATIONS
60E05; 60G55

1. Introduction

The purpose of this article is to demonstrate how fundamental derivative formulas, based on the principle of pivotality, can be used to derive a wide range of distributional identities. Some of these identities, particularly for classical univariate distributions, are well-known and can be obtained by standard methods or direct computation. Others, involving multivariate strictly stable distributions and Crofton's formulas, appear here in a generality that, to the best of our knowledge, has not been previously established. Our method not only shows that all these formulas follow from the same principle but also provides a clear probabilistic interpretation: the derivatives represent the expected number, or, more generally, the measure, of certain *pivotal elements*. This interpretation gives new insight into the structure of these identities. The key elements of our approach are two main formulas: one for Bernoulli systems and one for Poisson point processes. These serve as the basis for deriving and understanding the distributional identities presented below.

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Consider a vector $X = (X_1, \dots, X_m), m \in \mathbb{N}$, of independent Bernoulli random variables, defined on its canonical probability space $\Omega = \{0,1\}^m$ with X_i having the success probability $p_i \in [0,1],\ i = 1, \dots, m$. For an $i \in \{1, \dots, m\}$, let $X_{(i)}$ (resp. $X^{(i)}$) be a vector whose entries coincide with those of X except at the i-th coordinate, where the entry is 0 (resp. 1). A coordinate i is called *pivotal* for an event $A \subset \Omega$ in configuration X, if $\mathbf{1}_A(X^{(i)}) \neq \mathbf{1}_A(X_{(i)})$. Note that

$$\mathbf{1}_A(X) = X_i \mathbf{1}_A(X^{(i)}) + (1 - X_i)\mathbf{1}_A(X_{(i)}),$$

and because X_i is independent of $\mathbf{1}_A(X^{(i)})$ and of $\mathbf{1}_A(X_{(i)})$,

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{E}\,\mathbf{1}_A(X) = p_i\,\mathbb{E}\,\mathbf{1}_A(X^{(i)}) + (1-p_i)\,\mathbb{E}\,\mathbf{1}_A(X_{(i)}),\\ \frac{\partial}{\partial p_i}\mathbb{P}(A) &= \mathbb{E}[\mathbf{1}_A(X^{(i)}) - \mathbf{1}_A(X_{(i)})].\end{aligned} \tag{1}$$

The term in the right side of (1) is known as the *influence* of the coordinate i on the event A, see, e.g. [3, p. 46].

Introduce an operator D_i acting on a function $g : \Omega \to \mathbb{R}$ as

$$D_i g(x) := g(x^{(i)}) - g(x_{(i)}), \quad x \in \Omega.$$

Consider the case when $p_i = p$ for all i and add an index p to the probability and expectation notation to show its value. Now, (1) yields

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(A) = \sum_{i=1}^m \frac{\partial}{\partial p_i}\bigg|_{p_i=p} \mathbb{P}(A) = \mathbb{E}_p \sum_{i=1}^m D_i \mathbf{1}_A(X). \tag{2}$$

Because only the terms corresponding to pivotal coordinates are non-zero in the above sum, the derivative of the probability $\mathbb{P}_p(A)$ equals the expected number of pivotal coordinates for the event A. This is the so-called *Margulis–Russo formula* which was first proved in [5], where the authors call pivotal coordinates *essential*. The formula was independently rediscovered in [13,20], with a focus on increasing events, i.e. such events where $\mathbf{1}_A$ is non-decreasing: $D_i \mathbf{1}_A(x) \geq 0$ for all all i and $x \in \Omega$.

Since any function on Ω is a linear combination of indicator functions, (2) extends to a *perturbation formula* for a general function:

$$\frac{\mathrm{d}}{\mathrm{d}p}\,\mathbb{E}_p\, g(X) = \mathbb{E}_p \sum_{i=1}^m D_i g(X). \tag{3}$$

An analogue of D_i is the difference operator D_z defined in (9) as it appears in the perturbation formula for Poisson processes in Section 2.

In practice, it is often more convenient to use a modified form of the Margulis–Russo formula proposed in [14]. Assuming that $p \in (0,1)$ and using again independence of X_i

from $D_i\mathbf{1}_A$, continue (2) as follows:

$$\mathbb{E}_p \sum_{i=1}^m D_i\mathbf{1}_A(X)\mathbf{1}\{D_i\mathbf{1}_A(X)\neq 0\} = \frac{1}{p}\,\mathbb{E}_p \sum_{i=1}^m \mathbf{1}_A(X^{(i)})\mathbf{1}\{D_i\mathbf{1}_A(X)\neq 0\}\mathbf{1}\{X_i=1\}$$
$$-\frac{1}{1-p}\,\mathbb{E}_p \sum_{i=1}^m \mathbf{1}_A(X_{(i)})\mathbf{1}\{D_i\mathbf{1}_A(X)\neq 0\}\mathbf{1}\{X_i=0\}.$$

Introducing a *flip operator* T_i acting on $x \in \mathbf{\Omega}$ as

$$T_i(x_1,\dots,x_i,\dots,x_m) = (x_1,\dots,1-x_i,\dots,x_m),$$

the sums above can be written as

$$N_A^+(X) = \sum_{i=1}^m \mathbf{1}\{X_i=1, X\in A, T_iX\notin A\}, \quad N_A^-(X) = \sum_{i=1}^m \mathbf{1}\{X_i=0, X\in A, T_iX\notin A\}. \tag{4}$$

The coordinates i which contribute non-zero terms to N_A^+ are called $(+)$-*pivotal* for event A in configuration X (respectively, $(-)$-*pivotal* terms in N_A^-). We thus obtain the following form of the Margulis–Russo formula:

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(A) = \frac{1}{p}\,\mathbb{E}_p\,N_A^+ - \frac{1}{1-p}\,\mathbb{E}_p\,N_A^-, \quad p\in(0,1). \tag{5}$$

In the case of an increasing A, there are no (-)-pivotal coordinates so the last term above is zero. Formula (5) was used in [22] to establish an analogous perturbation formula for Poisson point processes which are covered in the next section. It is also easier to apply because it involves counting only pivotal coordinates for $X \in A$ ignoring the ones for $X \notin A$ involved in $D_i\mathbf{1}_A$.

The power of the Margulis–Russo formula lies in the fact that it links the variation of an event's probability to the geometry of configurations comprising the event, i.e. the number of pivotal components for its occurence. Many results in the percolation theory exploit this link, see, e.g. [6] or [3].

To give a flavour, consider the binomial distribution

$$\mathsf{Bin}(n,p;k) := \binom{n}{k} p^k(1-p)^{n-k}, \quad k\in\{0,\dots,n\}$$

with parameters $n\in\mathbb{N}=\{1,2,\dots\}$ and $p\in[0,1]$. Take $k\in\{1,\dots,n\}$, and consider the increasing event $A:=\{S_n\geq k\}$, where $S_n := X_1+\dots+X_n$. We then have

$$N_A^+ = \begin{cases} k, & \text{if } S_n = k,\\ 0, & \text{otherwise.}\end{cases}$$

Invoking (5), leads to

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(A) = \frac{k}{p}\mathsf{Bin}(n,p;k) = \frac{n!}{(k-1)!(n-k)!}p^{k-1}(1-p)^{n-k}.$$

Since $\mathbb{P}_0(A) = 0$, we obtain the following integral representation:

$$\sum_{j=k}^{n} \mathsf{Bin}(n,p;j) = \frac{n!}{(k-1)!(n-k)!} \int_0^p t^{k-1}(1-t)^{n-k}\,\mathrm{d}t, \quad k \in \{1,\dots,n\}. \tag{6}$$

Another example concerns the negative binomial (aka Pascal) distribution

$$\mathsf{NB}(r,p;k) := \binom{k+r-1}{k} p^r(1-p)^k, \quad k \in \mathbb{Z}_+,$$

with parameters $r \in \mathbb{N}$ and $p \in [0,1]$. This is the distribution of the number of failures in a sequence of independent Bernoulli trials until the r-th success. Fix natural numbers r, k. We can take $m{:} = k+r-1$ to see that

$$\sum_{j=0}^{k} \mathsf{NB}(r,p;j) = \mathbb{P}(A),$$

where A denotes the increasing event $\{S_{k+r-1} \geq r\}$. We then have

$$N_A^+ = \begin{cases} r, & \text{if } S_{k+r-1} = r, \\ 0, & \text{otherwise,} \end{cases}$$

and (5) yields that

$$\frac{\mathrm{d}}{\mathrm{d}p}\mathbb{P}_p(A) = \frac{r}{p}\mathsf{Bin}(k+r-1,p;r) = \frac{(k+r-1)!}{(k-1)!(r-1)!}p^{r-1}(1-p)^{k-1}.$$

Since $\mathbb{P}_0(A) = 0$, we get the identity

$$\sum_{j=0}^{k} \mathsf{NB}(r,p;j) = \frac{(k+r-1)!}{(k-1)!(r-1)!} \int_0^p t^{r-1}(1-t)^{k-1}\,\mathrm{d}t, \quad k \in \mathbb{N}. \tag{7}$$

Both (6) and (7) could be checked directly by differentiation, but the variation formula (2) provides a probabilistic insight into the variation of the probability.

The aim of this paper is to demonstrate the power of an analogous to (2) variation formula for Poisson processes. Let λ be a finite measure on some measurable space $\mathbb{X}$ and $\theta \geq 0$. Consider a point process η on $\mathbb{X}$ and a probability measure $\mathbb{P}_\theta$ such that η is (under $\mathbb{P}_\theta$) a Poisson process with intensity measure $\theta\lambda$. It was shown in [22] that, if A is an event defined in terms of η, then (2) holds with θ replacing p, provided that (4) is modified as follows:

$$N_A^+ := \int \mathbf{1}\{\eta + \delta_z \in A, \eta \notin A\}\lambda(\mathrm{d}z), \quad N_A^- := \int \mathbf{1}\{\eta \in A, \eta + \delta_z \notin A\}\lambda(\mathrm{d}z), \tag{8}$$

where, generically, δ_z is the Dirac measure at z. Notice, that by Mecke's formula (see, e.g. [11, Theorem 4.1]),

$$\mathbb{E}_\theta\, N_A^+ = \frac{1}{\theta}\,\mathbb{E}_\theta \int \mathbf{1}\{\eta \in A, \eta - \delta_z \notin A\}\eta(\mathrm{d}z),$$

So, analogously to the Bernoulli case, the process points $z \in \eta$ such that $\eta \in A$, but $\eta - \delta_z \notin A$ maybe called *pivotal points*, whereas $z \in \mathbb{X}$ such that $\eta \in A$, but $\eta + \delta_z \notin A$ are called *pivotal locations*.

In the next section we provide a review of the variation formulas for Poisson processes and to prove with their help distributional identities for some classical one-dimensional distributions: Poisson, Erlang and compound Poisson. We then consider multivariate strictly stable distributions in Section 3. Theorem 3.1 provides formulas which are apparently new and could possibly be used for an effective computation of the stable density which is not known explicitly apart for a very few particular values of its parameters. In Section 4, we extend Crofton's derivative formula. In the final Section 5 we use this extension to give a new probabilistic proof of a version of this formula for binomial point processes.

2. A perturbation formula for Poisson processes

In this section we review a perturbation formula for general Poisson processes. Let $(\mathbb{X}, \mathcal{X})$ be a Borel space and let $\mathbf{N}(\mathbb{X}) \equiv \mathbf{N}$ be the space of integer-valued σ-finite measures φ on $\mathbb{X}$, equipped with the smallest σ-field $\mathcal{N}$ making the mappings $\varphi \mapsto \varphi(B)$ measurable for all $B \in \mathcal{X}$.

For any $g\colon \mathbf{N} \to \mathbb{R}$ and $z \in \mathbb{X}$, introduce a function $D_z g\colon \mathbf{N} \to \mathbb{R}$ by means of

$$D_z g(\varphi) = g(\varphi + \delta_z) - g(\varphi). \tag{9}$$

The mapping $g \mapsto D_z g$ is known as *difference operator*. For $k \in \mathbb{N}$ the k-th iteration $D^k_{z_1,\dots,z_k} g : \mathbf{N} \to \mathbb{R}$, of this operator is inductively defined by $D^k_{z_1,\dots,z_k} g = D_{z_k} D^{k-1}_{z_1,\dots,z_{k-1}} g$ for $(z_1, \dots, z_k) \in \mathbb{X}^k$.

Given any σ-finite measure ρ on $\mathbb{X}$, we let η_ρ denote a Poisson process with this intensity measure. The following perturbation formula is a special case of Theorem 19.3 in [11].

Theorem 2.1: *Let λ be a σ-finite and let ν be a finite measure on $\mathbb{X}$. Let $g\colon \mathbf{N} \to \mathbb{R}$ be a measurable function such that $\mathbb{E}\,|g(\eta_{\lambda+\nu})| < \infty$. Let $\theta \in (-\infty, 1]$ such that $\lambda + \theta\nu$ is a measure. Then*

$$\mathbb{E}\, g(\eta_{\lambda+\theta\nu}) = \mathbb{E}\, g(\eta_\lambda) + \sum_{k=1}^{\infty} \frac{\theta^k}{k!} \int \mathbb{E}\, D^k_{x_1,\dots,x_k} g(\eta_\lambda)\, \nu^k(d(x_1, \dots, x_k)), \tag{10}$$

where the series converges absolutely.

The earliest version of Theorem 2.1 (for a bounded function g) was proved in [22]. Later this was generalized in [15]. For square integrable random variables the result can be extended to certain (signed) σ-finite perturbations; see [10].

For later reference we provide the following consequence of Theorem 2.1 (set $\nu = \lambda$ in (10)).

Proposition 2.2: *Let λ be a finite measure on $\mathbb{X}$. Let $g\colon \mathbf{N} \to \mathbb{R}$ be a measurable function such that $\mathbb{E}\,|g(\eta_{\theta_0\lambda})| < \infty$ for some $\theta_0 > 0$. Then $\theta \mapsto \mathbb{E}\, g(\eta_{\theta\lambda})$ is analytic on $[0, \theta_0]$ and its derivatives are given by*

$$\frac{\mathrm{d}^k}{\mathrm{d}\theta^k} \mathbb{E}\, g(\eta_{\theta\lambda}) = \int \cdots \int \mathbb{E}\, D^k_{z_1,\dots,z_k} g(\eta_{\theta\lambda})\, \lambda(dz_1) \cdots \lambda(dz_k), \quad \theta \le \theta_0. \tag{11}$$

Using the indicator $\mathbf{1}_A$ above as the function g, implies the Poisson process version of the Margulis-Russo formula (2) with notation (8).

The Poisson and the Erlang distribution. We now apply (11) to derive a few distributional identities for classical univariate distributions. The *Poisson distribution* with parameter $\theta \geq 0$ is given by

$$\mathsf{Po}(\theta; k) := \frac{\theta^k}{k!} \mathrm{e}^{-\theta}, \quad k \in \mathbb{Z}_+.$$

The *Erlang distribution* with parameters $n \in \mathbb{N}$ and $\theta > 0$ has density function

$$\mathsf{Er}(n, \theta; x) := \frac{\theta^n}{(n-1)!} x^{n-1} \mathrm{e}^{-\theta x}, \quad x \geq 0.$$

Proposition 2.3: *The Poisson distribution with parameter $\theta \geq 0$ satisfies*

$$\sum_{j=k}^{\infty} \mathsf{Po}(\theta; j) = \int_0^{\theta} \frac{t^{k-1}}{(k-1)!} \mathrm{e}^{-t} \, \mathrm{d}t, \quad k \in \mathbb{N}. \tag{12}$$

The distribution function of the Erlang distribution with parameters $n \in \mathbb{N}$ and $\theta > 0$ may be written as

$$\int_0^x \mathsf{Er}(n, \theta; y) dy = \frac{x^n}{(n-1)!} \int_0^{\theta} t^{n-1} \mathrm{e}^{-tx} \, \mathrm{d}t, \quad x \geq 0. \tag{13}$$

Proof: In the notation of the previous section, set $\mathbb{X} = \{z\}$ to be a singleton and $\lambda\{z\} = 1$. Then $\eta\{z\}$ is just a Poisson random variable with parameter 1. Take $k \in \mathbb{N}$ and consider the event $A := \{\eta\{z\} \geq k\}$. Then $\mathbb{P}_\theta(A)$ is given by the left-hand side of (12) and we have

$$\mathbf{1}_A(\eta + \delta_z) - \mathbf{1}_A(\eta) = \mathbf{1}\{\eta\{z\} = k-1\}.$$

Since $\mathbb{P}_0(A) = 0$, (11) implies (12). Equation (13) follows from (12) and the identity

$$\int_0^x \mathsf{Er}(n, \theta; y) dy = \sum_{j=n}^{\infty} \mathsf{Po}(\theta x; j). \tag{14}$$

■

The Compound Poisson distribution. Let $\mathbb{Q}$ be a probability distribution on $\mathbb{R}$. The *compound Poisson distribution* with parameters $\theta \geq 0$ and $\mathbb{Q}$ is given by

$$\mathsf{CPo}(\theta, \mathbb{Q}) := \sum_{n=0}^{\infty} \frac{\theta^n}{n!} \mathrm{e}^{-\theta} \mathbb{Q}^{*n},$$

so it equals the distribution of Poisson $\mathsf{Po}(\theta)$ number of independent summands each having distribution $\mathbb{Q}$.

Proposition 2.4: *The distribution function* $F(\theta,\mathbb{Q};x) := \mathsf{CPo}(\theta,\mathbb{Q})((-\infty,x])$, $x\in\mathbb{R}$, *of the Compound Poisson distribution satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}\theta}F(\theta,\mathbb{Q};x) = \int F(\theta,\mathbb{Q};x-z)\mathbb{Q}(\mathrm{d}z) - F(\theta,\mathbb{Q};x). \tag{15}$$

When $\mathbb{Q}$ *is concentrated on* $\mathbb{Z}$,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathsf{CPo}(\theta,\mathbb{Q};k) = \sum_{j\neq k} q_{k-j}\mathsf{CPo}(\theta,\mathbb{Q};j) - (1-q_0)\mathsf{CPo}(\theta,\mathbb{Q};k), \quad k\in\mathbb{Z}, \tag{16}$$

where $\mathsf{CPo}(\theta,\mathbb{Q};j) := \mathsf{CPo}(\theta,\mathbb{Q})(\{j\})$, *and* $q_j := \mathbb{Q}(\{j\})$, $j\in\mathbb{Z}$. *Equivalently,*

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathsf{CPo}(\theta,\mathbb{Q};k) = \sum_{j\in\mathbb{Z}} q_{k-j}\mathsf{CPo}(\theta,\mathbb{Q};j) - \mathsf{CPo}(\theta,\mathbb{Q};k), \quad k\in\mathbb{Z}. \tag{17}$$

Proof: To apply (11), take $\mathbb{X} := \mathbb{R}$ and let $\lambda := \mathbb{Q}$. Under $\mathbb{P}_\theta$, the random variable $Z := \int z\eta(\mathrm{d}z)$ has the compound Poisson distribution $\mathsf{CPo}(\theta,\mathbb{Q})$. Consider the event $A := \{Z \le x\}$ for some $x\in\mathbb{R}$. Then, for $z\in\mathbb{R}$,

$$\mathbf{1}_A(\eta+\delta_z) - \mathbf{1}_A(\eta) = \mathbf{1}\{Z > x, Z+z\le x\} - \mathbf{1}\{Z\le x, Z+z > x\},$$

so that (11) writes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{P}_\theta(A) &= \mathbb{E}_\theta\int_{(-\infty,0)} \mathbf{1}\{Z+z\le x\}(1-\mathbf{1}\{Z\le x\})\mathbb{Q}(\mathrm{d}z)\\ &\quad - \mathbb{E}_\theta\int_{(0,\infty)}(1-\mathbf{1}\{Z+z\le x\})\mathbf{1}\{Z\le x\}\mathbb{Q}(\mathrm{d}z)\\ &= \mathbb{E}_\theta\int_{\mathbb{R}\setminus\{0\}}\mathbf{1}\{Z+z\le x\}\mathbb{Q}(\mathrm{d}z) - \mathbb{P}_\theta(Z\le x)\mathbb{Q}(\mathbb{R}\setminus\{0\}).\end{aligned}$$

This yields (15). If $\mathbb{Q}(\mathbb{Z}) = 1$, then (17) (and hence also (16)) follows upon taking suitable differences. ■

Remark 2.1: Identity (16) is equivalent to

$$\mathsf{CPo}(\theta,\mathbb{Q};k) = \mathrm{e}^{-(1-q_0)\theta}\sum_{j\neq k} q_{k-j}\int_0^\theta \mathrm{e}^{(1-q_0)t}\mathsf{CPo}(t,\mathbb{Q};j)\,\mathrm{d}t, \quad k\in\mathbb{Z}. \tag{18}$$

If, in addition, $q_j = 0$ for $j < 0$, then it follows from the definition of $\mathsf{CPo}(\theta,\mathbb{Q})$ (or from $\mathsf{CPo}(\theta,\mathbb{Q};0) = \mathrm{e}^{-(1-q_0)\theta}$ and (18)) that $\mathrm{e}^{(1-q_0)\theta}\mathsf{CPo}(\theta,\mathbb{Q};k)$ is a polynomial in θ of degree k. Equation (18) provide a recursion for the coefficients of these polynomials.

Remark 2.2: The characteristic function of $\mathsf{CPo}(\theta,\mathbb{Q};k)$ is given by

$$G(\theta,\mathbb{Q};s) := \int \mathrm{e}^{\mathbf{i}sz}\mathsf{CPo}(\theta,\mathbb{Q})(\mathrm{d}z) = \exp[\theta(G_{\mathbb{Q}}(s)-1)], \quad s\in\mathbb{R}, \tag{19}$$

where $\mathbf{i}$ is the imaginary unit and $G_{\mathbb{Q}}$ is the characteristic function of $\mathbb{Q}$. The recursion (18) can also be obtained by differentiating (19) with respect to θ. Differentiation of (19) with respect to s and assuming $q_j = 0$ for $j < 0$, yields the widely used Panjer recursion [16]:

$$\mathsf{CPo}(\theta, \mathbb{Q}; k) = \theta \sum_{j=0}^{k-1} \frac{k-j}{k} q_{k-j} \mathsf{CPo}(\theta, \mathbb{Q}; j), \quad k \in \mathbb{N}. \tag{20}$$

Remark 2.3: Consider a compound Poisson process $(X_t)_{t \geq 0}$ driven by a unit rate Poisson process and jump size distribution $\mathbb{Q}$, see e.g. [8, Chapter 12]. Then X_t has distribution $\mathsf{CPo}(t, \mathbb{Q})$ and (17) and (15) are two examples for the Kolmogorov forward equation, see e.g. [8, Chapter 19].

Remark 2.4: Take $\mathbb{X} := [0, \infty)$ and λ as the Lebesgue measure on $\mathbb{X}$. Let $T_1 < T_2 < \dots$ be the atoms of η arranged in increasing order. Let $n \in \mathbb{N}$ and consider the event $A := \{T_n \leq x\}$ for $x \geq 0$. It is well-known that $\mathbb{P}_\theta(A)$ coincides with the left-hand side of (13). On the other hand we have for all $z \geq 0$ (with obvious notation)

$$\mathbf{1}\{T_n(\eta + \delta_z) \leq x\} - \mathbf{1}\{T_n(\eta) \leq x\} = \mathbf{1}\{z \leq x\}\mathbf{1}\{\eta[0, x] = n - 1\},$$

so that (12) yields

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \mathbb{P}_\theta(A) = x\mathbb{P}\left(\eta[0, x] = n - 1\right) = \frac{x^n}{(n-1)!} \theta^{n-1} \,\mathrm{e}^{-\theta x}.$$

Since $\mathbb{P}_0(A) = 0$, we again obtain (13).

3. Strictly α-stable laws

A random vector ξ (or its distribution) is called strictly α-stable (StαS), if the following equality in distribution holds:

$$t^{1/\alpha}\xi' + (1-t)^{1/\alpha}\xi'' \overset{D}{=} \xi, \quad 0 \leq t \leq 1, \tag{21}$$

where ξ', ξ'' are independent distributional copies of ξ. In Euclidean spaces, StαS laws exist only for $0 < \alpha \leq 2$ and $\alpha = 2$ corresponds to the Gaussian distribution centred at the origin. Symmetrical StαS random vectors in $\mathbb{R}^n$ with $\alpha < 2$ and all StαS random vectors with $\alpha < 1$ admit the following *LePage series representation* (see [12]):

$$\xi \overset{D}{=} \sum_{k=1}^{\infty} \Gamma_k^{-1/\alpha} \varepsilon_k, \tag{22}$$

where $\Gamma_1, \Gamma_2, \dots$ are the successive times of jumps of a homogeneous Poisson process on $\mathbb{R}_+$ with intensity θ, and $\varepsilon_1, \varepsilon_2, \dots$ are i.i.d. random vectors on the unit sphere $\mathbb{S}^{n-1}$. Thus their distribution is characterized by two parameters: the Poisson process intensity θ and the probability measure $\hat{\sigma}$ on the sphere–the distribution of ε_k's. By the marking theorem for Poisson processes (see, e.g. [11, Theorem 5.6]), $\sum_{k=1}^{\infty} \delta_{(\Gamma_k, \varepsilon_k)}$ is a Poisson on $\mathbb{R}_+ \times$

$\mathbb{S}^{n-1}$ with intensity measure $\theta\,\mathrm{d}t \times \hat{\sigma}$. Hence we can appeal to the mapping theorem (see, e.g. [11, Theorem 5.1]) to see that

$$\eta_\theta := \sum_{k=1}^{\infty} \delta_{\Gamma_k^{-1/\alpha}\varepsilon_k}$$

is a Poisson process on $\mathbb{R}^n \setminus \{0\}$ with intensity measure

$$\Lambda_\theta := \theta \int_{\mathbb{S}^{n-1}} \int_0^\infty \mathbf{1}\{t^{-1/\alpha}u \in \cdot\}\,\mathrm{d}t\,\hat{\sigma}(\mathrm{d}u) = \theta\Lambda_1. \tag{23}$$

The right-hand side of (22) can be written as a point process integral, so that

$$\xi_\theta := \int u\,\eta_\theta(\mathrm{d}u), \tag{24}$$

is a stable random vector with the given parameters. The integrals here in this section are taken over $\mathbb{R}^n \setminus \{0\}$ unless specified otherwise. The StαS distribution is infinitely divisible with *Lévy measure* Λ_θ given by (23). The measure $\sigma = \theta\hat{\sigma}$ on $\mathbb{S}^{n-1}$ is called the *spectral measure* of ξ_θ (or of Λ_θ). The convergence of the integral (24) for $\alpha < 1$ or for all $\alpha < 2$ in the case of a symmetrical spectral measure is guaranteed, for instance, by [8, Lemma 12.13]. However, the spectral measure need not be symmetric in order for the LePage representation (22) to hold. For instance, when $\alpha \geq 1$ it is sufficient that a non-symmetric spectral measure satisfies $\int s\,\sigma(\mathrm{d}s) = 0$, see [2, Theorem 2].

By definition of Λ_θ, we have for each Borel set $B \subset \mathbb{R}^n \setminus \{0\}$ and each $c > 0$ that

$$\Lambda_\theta(cB) = c^{-\alpha}\Lambda_\theta(B).$$

From (24) and the above scaling property we obtain that $\xi_\theta \overset{D}{=} \theta^{1/\alpha}\xi_1$.

For a $B \subset \mathbb{S}^{n-1}$, introduce the closed positive conical hull

$$\mathrm{cone}(B) = \mathrm{closure}\left\{\sum_{k=1}^{m} \lambda_k x_k : x_k \in B,\ \lambda_k \geq 0,\ m \in \mathbb{N}\right\}.$$

Let S_σ be the support of the spectral measure σ. It follows from (24) that the distribution of ξ_θ is supported by $\mathrm{cone}(S_\sigma)$: the distribution of the sum $\sum_{k=1}^{m} \Gamma_k^{1/\alpha}\varepsilon_k$ is the convolution of distributions with densities. Thus it is non-degenerate if $\mathrm{cone}(S_\sigma)$ has a positive n-volume. Non-degenerate stable laws possess an infinitely differentiable density: the distribution of the first term $\Gamma_1^{-1/\alpha}\varepsilon_1$ in (22) has density with integrable derivatives of all orders implying that its convolution with $\sum_{k=2}^{\infty} \Gamma_k^{-1/\alpha}\varepsilon_k$ can also be differentiated under the integral any number of times leading to expressions for the density derivatives. See also [17, Section 2.3.4] for an alternative proof.

We are now ready to formulate the main result of this section.

Theorem 3.1: *Let ξ_θ be a StαS random vector with LePage representation* (22) *corresponding to the spectral measure $\sigma = \theta\hat{\sigma}$ such that $K := \mathrm{cone}(S_{\hat{\sigma}})$ has a positive n-volume. Then*

(i) *The density f_θ of ξ_θ satisfies*

$$nf_\theta(x) + \langle x, \nabla f_\theta(x)\rangle = \alpha \int [f_\theta(x) - f_\theta(x-z)]\, \Lambda_\theta(\mathrm{d}z), \quad x \in \mathrm{Int}(K), \tag{25}$$

where $\langle \cdot\,, \cdot\rangle$ is the scalar product in $\mathbb{R}^n$.

(ii) *Let $f_{|\xi_\theta|}$ denote the p.d.f. of the radius vector $|\xi_\theta|$. Then for all $r > 0$,*

$$rf_{|\xi_\theta|}(r) = \alpha \int \left[\mathbb{P}\left(|\xi_\theta| \le r\right) - \mathbb{P}\left(|\xi_\theta + z| \le r\right)\right] \Lambda_\theta(\mathrm{d}z). \tag{26}$$

Corollary 3.2: *The c.d.f. F_θ and the p.d.f. f_θ of a positive StαS on $\mathbb{R}_+$ with $0 < \alpha < 1$ are related through*

$$f_\theta(x) + xf_\theta'(x) = \alpha^2\theta \int_0^x [f_\theta(x) - f_\theta(x-z)] z^{-\alpha-1}\,\mathrm{d}z; \tag{27}$$

$$xf_\theta(x) = \theta\alpha^2 \int_0^x [F_\theta(x) - F_\theta(x-z)]\, z^{-\alpha-1}\,\mathrm{d}z \quad \text{for all } x > 0, \tag{28}$$

Proof: For a measurable $B \subset \mathbb{R}^n$ and a counting measure φ, consider the indicator function

$$g_B(\varphi) = \mathbf{1}\left\{\int z\,\varphi(\mathrm{d}z) \in B\right\}.$$

By (24), $\mathbb{E}\, g_B(\eta_\theta) = \mathbb{P}(\xi_\theta \in B)$. Moreover,

$$\mathbb{E}\, g_B(\eta_\theta + \delta_z) = \mathbb{E}\,\mathbf{1}\left\{\int u\,(\eta_\theta + \delta_z)(\mathrm{d}u) \in B\right\} = \mathbb{P}(\xi_\theta + z \in B).$$

Using (11) and noting that $\Lambda_\theta = \theta\Lambda_1$, we obtain that for any measurable $B \subset \mathbb{R}^n$,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{P}(\xi_\theta \in B) = \int \left[\mathbb{P}(\xi_\theta + z \in B) - \mathbb{P}(\xi_\theta \in B)\right]\, \Lambda_1(\mathrm{d}z) \tag{29}$$

$$= \frac{1}{\theta}\int [\mathbb{P}(\xi_\theta \in B - z) - \mathbb{P}(\xi_\theta \in B)]\, \Lambda_\theta(\mathrm{d}z) \tag{30}$$

Since $\xi_\theta \overset{D}{=} \theta^{1/\alpha}\xi_1$, the density and its gradient satisfy

$$f_\theta(x) = \theta^{-n/\alpha} f_1(\theta^{-1/\alpha}x),$$
$$\nabla f_\theta(x) = \theta^{-(n+1)/\alpha}\, \nabla f_1(\theta^{-1/\alpha}x).$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}\theta} f_\theta(x) = -\frac{n}{\alpha}\theta^{-n/\alpha-1} f_1(\theta^{-1/\alpha}x) - \frac{1}{\alpha}\theta^{-n/\alpha}\,\langle \theta^{-1/\alpha-1}x, \nabla f_1(\theta^{-1/\alpha}x)\rangle$$
$$= -\frac{n}{\alpha\theta} f_\theta(x) - \frac{1}{\alpha\theta}\langle x, \nabla f_\theta(x)\rangle,$$

Take a set B such that its closure is in $\mathrm{Int}(K)$. The density f_θ is bounded and the left-hand-side of (29) becomes

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{P}(\xi_\theta \in B) = \frac{\mathrm{d}}{\mathrm{d}\theta}\int_B f_\theta(x)\,\mathrm{d}x = -\frac{1}{\alpha\theta}\int_B \left[nf_\theta(x) + \langle x, \nabla f_\theta(x)\rangle\right]\,\mathrm{d}x \tag{31}$$

The right-hand-side of (30) is

$$\frac{1}{\theta}\int\int_B [f_\theta(x-z) - f_\theta(x)]\,\mathrm{d}x\,\Lambda_\theta(\mathrm{d}z).$$

Equating it to (31), we get the identity which holds for all measurable $B \subset \mathrm{Int}(K)$ which implies the identity (25) for almost all $x \in \mathrm{Int}(K)$. But the density is continuously differentiable there, so it also holds for all $x \in \mathrm{Int}(K)$.

Recall that all one-dimensional StαS laws with $0 < \alpha < 1$ are totally skewed concentrated on either $\mathbb{R}_+$ or $\mathbb{R}_-$. Consider, for definitivness, a positive ξ_θ. The spectral measure σ is then $\theta\delta_1$ and (25) becomes (27).

Now let B in (30) be the ball B_r of radius r centred at the origin. Since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{P}(\xi_\theta \in B_r) &= \frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{P}(|\xi_\theta| \le r) = \frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{P}(|\xi_1| \le \theta^{-1/\alpha}r)\\ &= -\frac{r}{\alpha}\theta^{-1-1/\alpha}\,\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(|\xi_1| \le t)\Big|_{t=\theta^{-1/\alpha}r} = -\frac{r}{\alpha}\theta^{-1-1/\alpha}f_{|\xi_1|}(\theta^{-1/\alpha}r)\end{aligned}$$

and also

$$\begin{aligned}f_{|\xi_\theta|}(r) &= \frac{\mathrm{d}}{\mathrm{d}r}\mathbb{P}(|\xi_\theta| \le r) = \frac{\mathrm{d}}{\mathrm{d}r}\mathbb{P}(|\xi_1| \le \theta^{-1/\alpha}r)\\ &= \theta^{-1/\alpha}\,\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(|\xi_1| \le t\})\Big|_{t=\theta^{-1/\alpha}r} = \theta^{-1/\alpha}f_{|\xi_1|}(\theta^{-1/\alpha}r),\end{aligned}$$

the relation (30) takes the form (26).

Notice that its one-dimensional variant (28), when differentiated, gives (27). ■

4. Crofton's derivative formula for Poisson processes

The classical Crofton formula [4] known in integral and stochastic geometry relates the probability of events and, generally, expectation of a random variable defined by configuration of a fixed number of points uniformly distributed in a domain when the domain is infinitesimally expanded. The property, described by the event or the random variable should depend only on the mutual position of points, so it must be rotation and translation invariant once all the points are still in the domain, see, e.g. [9, Chapter 2]. We will revisit this formula in Section 5, but now we establish its counterpart for Poisson processes.

Let $K \subset \mathbb{R}^n$ be a compact set and define

$$K_t := K + tB^n = \{x + ty : \; x \in K, \; y \in B^n\}, \quad t \ge 0, \tag{32}$$

where B^n is the Euclidean unit ball. This is the so-called *parallel set* of K at distance t. Let $h \colon \mathbb{R}^n \to [0,\infty)$ be a continuous function and let λ be a measure on $\mathbb{R}^n$ with Lebesgue density h. For $t \geq 0$ let λ_t be the restriction of λ to K_t and let η_t be a Poisson process on $\mathbb{R}^n$ with intensity measure λ_t. Let $g\colon \mathbf{N}(\mathbb{R}^n) \to \mathbb{R}$ be measurable. Under certain technical assumptions on K and g, we shall prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\,g(\eta_t)\bigg|_{t=0} = \int_{\partial K} \mathbb{E}\left[g(\eta_0 + \delta_x) - g(\eta_0)\right] h(x)\,\mathcal{H}^{n-1}(\mathrm{d}x), \tag{33}$$

where ∂K is the boundary of K and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on $\mathbb{R}^n$.

Our main technical geometrical tool are the support measures from [7]. We recall here briefly their definition and main properties. We put $p(K,z) := y$ whenever y is a uniquely determined point in K with $d(K,z) := \min\{x - z : x \in K\} = |y - z|$, and we call this point the *metric projection* of z on K. If $0 < d(K,z) < \infty$ and $p(K,z)$ is defined, then $p(K,z)$ lies on the boundary ∂K of K and we put $u(K,z) := (z - p(K,z))/d(K,z)$. The *exoskeleton* $\mathrm{exo}(K)$ of K consists of all points of $\mathbb{R}^n \setminus K$ which do not admit a metric projection on K. The *normal bundle* of K is defined by

$$N(K) := \{(p(K,z), u(K,z)) : z \notin K \cup \mathrm{exo}(K)\}.$$

It is a measurable subset of $\partial K \times \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ is the unit sphere in $\mathbb{R}^n$. The *reach function* $\delta(K,\cdot) : \mathbb{R}^n \times \mathbb{S}^{n-1} \to [0,\infty]$ of K is defined by

$$\delta(K,x,u) := \inf\{t \geq 0 : x + tu \in \mathrm{exo}(K)\}, \quad (x,u) \in N(K),$$

and $\delta(K,x,u) := 0$ for $(x,u) \notin N(K)$. Note that $\delta(K,\cdot) > 0$ on $N(K)$.

We write $x \wedge y$ for $\min\{x,y\}$. By Theorem 2.1 in [7], there exist signed measures $\mu_0(K;\cdot),\ldots,\mu_{n-1}(K;\cdot)$ on $\mathbb{R}^n \times \mathbb{S}^{n-1}$ satisfying

$$\sum_{i=0}^{n-1} \int_{N(K)} (\delta(K,x,u) \wedge r)^{n-i}\,|\mu_i|(K; d(x,u)) < \infty, \quad r > 0, \tag{34}$$

and, for each measurable bounded function $f : \mathbb{R}^n \to \mathbb{R}$ with compact support, we have the following *local Steiner formula*:

$$\int_{\mathbb{R}^n \setminus K} f(x)\,\mathrm{d}x = \sum_{i=0}^{n-1} \omega_{n-i} \int_0^\infty \int_{N(K)} s^{n-1-i}\mathbf{1}\{s < \delta(K,x,u)\} \times f(x + su)\,\mu_i(K; d(x,u))\,\mathrm{d}s, \tag{35}$$

where $\omega_j := j\kappa_j$ and κ_j is the volume of the unit ball in $\mathbb{R}^j$. These measures are called *support measures* of K. They are uniquely defined by (35) and the requirement $|\mu_i|(K; \mathbb{R}^n \times \mathbb{S}^{n-1} \setminus N(K)) = 0$. In general, the total variation measures $|\mu_i|(K;\cdot)$ featuring in (34) are not finite. However, it follows from (35) that

$$\sum_{i=0}^{n-1} \int_{N(K)} \mathbf{1}\{\delta(K,x,u) \geq r\}\,|\mu_i|(K; d(x,u)) < \infty, \quad r > 0. \tag{36}$$

Therefore the integrals on the right-hand side of (35) are well-defined. An important special case is that of a convex set K. Then $\delta(K,x,u) = \infty$ for all $(x,u) \in N(K)$.

We start with the following proposition of independent interest. For $i \in \{1,2\}$ we define $\partial^i K$ as the set of all $x \in \partial K$ such that $\operatorname{card}\{u \in \mathbb{S}^{n-1} : (x,u) \in N(K)\} = i$.

Proposition 4.1: *Let $t_0 > 0$ and let $f\colon \mathbb{R}^n \to \mathbb{R}$ be continuous on K_{t_0}. Then the right and left derivatives of $t \mapsto \int_{K_t \setminus K} f(x)\,\mathrm{d}x$ exist on $(0,t_0)$ and are given by*

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\int_{K_t\setminus K} f(x)\,\mathrm{d}x = \int_{\partial^1 K_t} f(x)\,\mathcal{H}^{n-1}(\mathrm{d}x), \tag{37}$$

$$\frac{\mathrm{d}^-}{\mathrm{d}t}\int_{K_t\setminus K} f(x)\,\mathrm{d}x = \int_{\partial^1 K_t} f(x)\,\mathcal{H}^{n-1}(\mathrm{d}x) + \sum_{i=0}^{n-1}\omega_{n-i}\int \mathbf{1}\{t = \delta(K;x,u)\}t^{n-1-i}f(x+tu)\,\mu_i(K;d(x,u)). \tag{38}$$

Moreover, if

$$\sum_{i=0}^{n-1}\int (\delta(K;x,u)\wedge 1)^{n-i-1}\,|\mu_i|(K;d(x,u)) < \infty, \tag{39}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{K_t\setminus K} f(x)\,\mathrm{d}x\Big|_{t=0} = \int_{\partial^1 K} f(x)\,\mathcal{H}^{n-1}(\mathrm{d}x) + 2\int_{\partial^2 K} f(x)\,\mathcal{H}^{n-1}(\mathrm{d}x). \tag{40}$$

Proof: Let $t \in (0,t_0)$ and $r > 0$ such that $t + r \le t_0$. By the Steiner formula (35)

$$\frac{1}{r}\int_{K_{t+r}\setminus K_t} f(x)\,\mathrm{d}x = \sum_{i=0}^{n-1}\omega_{n-i}\int_{N(K)} r^{-1}\int_t^{t+r} s^{n-1-i}\mathbf{1}\{s < \delta(K,x,u)\} \times f(x+su)\,\mathrm{d}s\,\mu_i(K;d(x,u)).$$

Since f is continuous on K_{t_0}, there exists $c \ge 0$ such that $|f(x+su)| \le c$ for all $(x,u) \in N(K)$ and $s \le t_0$. Moreover we have for each $i \in \{0,\dots,n-1\}$ that

$$\begin{aligned}(n-i)r^{-1}\int_t^{t+r} s^{n-1-i}\mathbf{1}\{s < \delta(K,x,u)\}\,\mathrm{d}s &\le \mathbf{1}\{t < \delta(K,x,u)\}r^{-1}((t+r)^{n-i} - t^{n-i})\\ &\le c_i\mathbf{1}\{t < \delta(K,x,u)\}\end{aligned}$$

for some $c_i \ge 0$ (depending on t but not on r). By (36) and continuity of f we can apply the dominated convergence theorem to conclude that

$$\lim_{r\to 0+}\frac{1}{r}\int_{K_{t+r}\setminus K_t} f(x)\,\mathrm{d}x = \sum_{i=0}^{n-1}\omega_{n-i}\int_{N(K)}\mathbf{1}\{t < \delta(K,x,u)\}t^{n-1-i}f(x+tu)\,\mu_i(K;d(x,u)).$$

By Corollary 4.4 in [7] the above right-hand side equals (note that $\omega_1 = 2$)

$$2\int_{N(K_t)} f(x)\,\mu_{n-1}(K_t;d(x,u)).$$

By Proposition 4.1 in [7] we have for any compact set $A \subset \mathbb{R}^n$, that

$$2\mu_{n-1}(A;\cdot) = \int_{\partial^1 A} \mathbf{1}\{x \in \cdot\}\, \mathcal{H}^{n-1}(\mathrm{d}x) + 2\int_{\partial^2 A} \mathbf{1}\{x \in \cdot\}\, \mathcal{H}^{n-1}(\mathrm{d}x). \tag{41}$$

Since $\partial^2 K_t = \emptyset$ (recall that $t > 0$) we obtain the first assertion (37).

Similarly we obtain for the left derivative

$$\lim_{r\to 0+} \frac{1}{r}\int_{K_t \setminus K_{t-r}} f(x)\,\mathrm{d}x = \sum_{i=0}^{n-1} \omega_{n-i} \int_{N(K)} \mathbf{1}\{t \le \delta(K,x,u)\} t^{n-1-i} f(x+tu)\, \mu_i(K; d(x,u)).$$

Writing $\mathbf{1}\{t \le \delta(K,x,u)\} = \mathbf{1}\{t = \delta(K,x,u)\} + \mathbf{1}\{t < \delta(K,x,u)\}$, we can prove (38) as before.

Assuming (39), the proof of (40) again follows from the Steiner formula, dominated convergence and (41). Details are left to the reader. ■

Let us define I_K as the set of all $t > 0$ such that

$$\sum_{i=0}^{n-1} \int \mathbf{1}\{t = \delta(K;x,u)\}\, |\mu_i|(K; d(x,u)) = 0. \tag{42}$$

In view of (36) the set $(0,\infty) \setminus I_K$ is at most countably infinite.

Theorem 4.2: *Let $g\colon \mathbf{N} \to \mathbb{R}$ be measurable and $t_0 > 0$ such that $\mathbb{E}\,|g(\eta_{t_0})| < \infty$ and $x \mapsto \mathbb{E}\, g(\eta_t + \delta_x)$ is continuous on K_{t_0} for each $t < t_0$. Assume also that there exists $c > 0$ such that*

$$\left|\mathbb{E}\, D^k_{x_1,\dots,x_k} g(\eta_t)\right| \le c^k, \quad x_1,\dots,x_k \in K_{t_0},\ t \le t_0,\ k \in \mathbb{N}. \tag{43}$$

Then $t \mapsto \mathbb{E}\, g(\eta_t)$ is differentiable on $I_K \cap (0,t_0)$ and the derivative is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\, g(\eta_t) = \int_{\partial^1 K_t} \mathbb{E}\left[g(\eta_t + \delta_x) - g(\eta_t)\right] h(x)\, \mathcal{H}^{n-1}(\mathrm{d}x). \tag{44}$$

Moreover, if (39) holds, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\, g(\eta_t)\Big|_{t=0} = \sum_{j=1}^{2} j \int_{\partial^j K} \mathbb{E}\left[g(\eta_0 + \delta_x) - g(\eta_0)\right] h(x)\, \mathcal{H}^{n-1}(\mathrm{d}x). \tag{45}$$

Proof: Let $t \in [0,t_0)$ and let $r > 0$ be such that $t + r \le t_0$. The intensity measure of the process η_{t+r} equals the sum of λ_t and the restriction ν_r of λ to $K_{t+r} \setminus K_t$. Thus, by (10) (for

$\nu = \nu_r$ and $\theta = 1$)

$$\mathbb{E}\, g(\eta_{t+r}) = \mathbb{E}\, g(\eta_t) + \int_{K_{t+r}\setminus K_t} \mathbb{E}\, D_x g(\eta_t) h(x)\, \mathrm{d}x + R(t,r),$$

where

$$R(t,r) := \sum_{k=2}^{\infty} \frac{1}{k!} \int_{(K_{t+r}\setminus K_t)^k} \mathbb{E}\, D^k_{x_1,\dots,x_k} g(\eta_t)\, h(x_1)\cdots h(x_k)\, d(x_1,\dots,x_k).$$

We have that

$$|R(t,r)| \leq \sum_{k=2}^{\infty} \frac{c^k}{k!} \int_{(K_{t+r}\setminus K_t)^k} h(x_1)\cdots h(x_k)\, d(x_1,\dots,x_k) = \exp(c(t,r)) - c(t,r) - 1,$$

where $c(t,r) := c\int_{K_{t+h}\setminus K_t} h(x)\,\mathrm{d}x$. If $t > 0$ then Proposition 4.1 shows the convergence $\lim_{r\to 0+} r^{-1} c(t,r) = c(t)$ for some $c(t) \in \mathbb{R}$. Therefore

$$\limsup_{r\to 0+} r^{-1}|R(t,r)| \leq \lim_{r\to 0+} \frac{c(t,r)^2}{r} \frac{\exp(c(t,r)) - c(t,r) - 1}{c(t,r)^2} = 0.$$

Under assumption (39), we have (40) so that the above remains true for $t = 0$.

Again by Proposition 4.1 we have that

$$\lim_{r\to 0+} r^{-1} \int_{K_{t+r}\setminus K_t} \mathbb{E}\, D_x g(\eta_t) h(x)\,\mathrm{d}x = \sum_{j=1}^{2} j \int_{\partial^j K} \mathbb{E}\left[g(\eta_t + \delta_x) - g(\eta_t)\right] h(x)\, \mathcal{H}^{n-1}(\mathrm{d}x),$$

first for $t > 0$ (then the second term can be skipped) and then under the assumption (39), also for $t = 0$.

Let us now assume that $t - r > 0$. Then it follows as above that

$$-\lim_{r\to 0+} r^{-1}(\mathbb{E}\, g(\eta_{t-r}) - \mathbb{E}\, g(\eta_t)) = \lim_{r\to 0+} r^{-1} \int_{K_t\setminus K_{t-r}} \mathbb{E}\, D_x g(\eta_t) h(x)\,\mathrm{d}x,$$

so that Proposition 4.1 shows that

$$\begin{aligned} &\frac{\mathrm{d}^-}{\mathrm{d}t} \mathbb{E}\, g(\eta_t) \\ &= \int_{\partial^1 K_t} \mathbb{E}\, D_x g(\eta_t) h(x)\, \mathcal{H}^{n-1}(\mathrm{d}x) \\ &+ \sum_{i=0}^{n-1} \omega_{n-i} \int \mathbf{1}\{t = \delta(K;x,u)\} t^{n-1-i}\, \mathbb{E}\, D_{x+tu} g(\eta_t) h(x+tu)\, \mu_i(K; d(x,u)). \end{aligned} \tag{46}$$

Choosing now $t \in I_K$, concludes the proof. ■

A bounded function g satisfies the integrability assumptions of Theorem 4.2 for all $t_0 > 0$, so that (44) holds under a rather weak continuity assumption for each compact K. Equation (45) requires (39), constituting a non-trivial assumption on K. This assumption is certainly satisfied if $|\mu_i|(K; \mathbb{R}^n \times \mathbb{S}^{n-1}) < \infty$ for each $i \in \{0,\dots,n\}$. This is the case, for instance, if K has a *positive reach* or is a finite union of convex sets, see [7].

5. Crofton's derivative formula for binomial processes

For $t \geq 0$ we let K_t and λ_t be as defined in the beginning of Section 4. We assume that $\lambda(K) > 0$. In this section we consider a *binomial process* $\xi_t^{(m)}$ of size $m \in \mathbb{N}$ with sample distribution $\lambda_t/\lambda_t(K_t)$. This is a point process of the form

$$\xi_t^{(m)} = \delta_{X_1} + \cdots + \delta_{X_m},$$

where $X_1, \ldots, X_m$ are independent random vectors in $\mathbb{R}^n$ with distribution $\lambda_t/\lambda_t(K_t)$. It is convenient to let $\xi_t^{(0)} := 0$ be the null measure (a point process with no point.) Let $g\colon \mathbf{N} \to \mathbb{R}$ be a measurable and bounded function. Under certain assumptions on K and g, we wish to prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\,g(\xi_t^{(m)}) = \frac{m}{\lambda(K_t)}\int_{\partial K_t} \mathbb{E}\left[g(\xi_t^{(m-1)} + \delta_x) - g(\xi_t^{(m)})\right] h(x)\,\mathcal{H}^{n-1}(\mathrm{d}x). \tag{47}$$

The heuristic and historic background of this formula is explained in [21]. If the boundaries of the sets K_t are smooth, then (47) follows from more general results in [1]. Our proof is very different and relies on the Poisson version from Section 4.

Recall the definition of the set I_K at (42).

Theorem 5.1: *Let $g\colon \mathbf{N} \to \mathbb{R}$ be measurable and bounded and let $m \in \mathbb{N}$ and $t_0 > 0$. Suppose that $x \mapsto \mathbb{E}\,g(\xi_t^{(m-1)} + \delta_x)$ is continuous on K_{t_0} for each $t < t_0$. Then $t \mapsto \mathbb{E}\,g(\xi_t^{(m)})$ is differentiable on $I_K \cap (0, t_0)$ and the derivative is given by*

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\,g(\xi_t^{(m)}) = \frac{m}{\lambda(K_t)}\int_{\partial^1 K_t} \mathbb{E}\left[g(\xi_t^{(m-1)} + \delta_x) - g(\xi_t^{(m)})\right] h(x)\,\mathcal{H}^{n-1}(\mathrm{d}x). \tag{48}$$

Moreover, if (39) holds, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\,g(\xi_t^{(m)})\bigg|_{t=0} = \sum_{j=1}^{2} j\int_{\partial^j K} \mathbb{E}\left[g(\xi_0^{(m-1)} + \delta_x) - g(\xi_0^{(m)})\right] h(x)\,\mathcal{H}^{n-1}(\mathrm{d}x). \tag{49}$$

Proof: We are using the Poisson process η_t introduced in the beginning of the previous section, and the well-known distributional identity (see e.g. [11, Proposition 3.8])

$$\mathbb{P}(\xi_t^{(m)} \in \cdot\,) = \mathbb{P}(\eta_t \in \cdot \mid \eta_t(K_t) = m) = h_m(\lambda(K_t))\mathbb{P}(\eta_t(K_t) = m, \eta_t \in \cdot\,),$$

where the function $h_m\colon [0,\infty) \to \mathbb{R}$ is defined by $h_m(u) := m!\,\mathrm{e}^u u^{-m}$. Note that the derivative of h_m is given by

$$h_m'(u) = h_m(u) - \frac{m}{u} h_m(u).$$

Let $t \in I_K \cap (0, t_0)$. We apply (44) to the function $\tilde{g}(\varphi) := \mathbf{1}\{\varphi(\mathbb{R}^n) = m\}g(\varphi)$. Since $\mathbb{E}\,g(\xi_t^{(m)}) = h_m(\lambda_t(K_t))\,\mathbb{E}\,\tilde{g}(\eta_t)$ and $\tilde{g}(\eta_t + \delta_x) = \mathbf{1}\{\eta_t(\mathbb{R}^n) = m - 1\}g(\eta_t + \delta_x)$, $x \in \mathbb{R}^n$, this gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\,g(\xi_t^{(m)}) = \left[\frac{\mathrm{d}}{\mathrm{d}t}h_m(\lambda_t(K_t))\right]\mathbb{E}\,\mathbf{1}\{\eta_t(K_t)=m\}g(\eta_t)$$
$$+ h_m(\lambda_t(K_t))\left[\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}\,\mathbf{1}\{\eta_t(K_t)=m\}g(\eta_t)\right].$$

Taking into account (37), we obtain that the first summand equals

$$\mathbb{E}\,g(\xi_t^{(m)})\int_{\partial^1 K_t} h(x)\,\mathcal{H}^{n-1}(\mathrm{d}x) - \frac{m}{\lambda_t(K_t)}\,\mathbb{E}\,g(\xi_t^{(m)})\int_{\partial^1 K_t} h(x)\,\mathcal{H}^{n-1}(\mathrm{d}x).$$

By (44) the second summand equals

$$h_m(\lambda_t(K_t))\int_{\partial^1 K_t}\mathbb{E}\,\mathbf{1}\{\eta_t(K_t)=m-1\}g(\eta_t+\delta_x)h(x)\,\mathcal{H}^{n-1}(\mathrm{d}x)$$
$$- h_m(\lambda_t(K_t))\int_{\partial^1 K_t}\mathbb{E}\,\mathbf{1}\{\eta_t(K_t)=m\}g(\eta_t)h(x)\,\mathcal{H}^{n-1}(\mathrm{d}x)$$
$$= \frac{m}{\lambda_t(K_t)}\int_{\partial^1 K_t}\mathbb{E}\,g(\xi_t^{(m-1)}+\delta_x)h(x)\,\mathcal{H}^{n-1}(\mathrm{d}x) - \mathbb{E}\,g(\xi_t^{(m)})\int_{\partial^1 K_t} h(x)\,\mathcal{H}^{n-1}(\mathrm{d}x).$$

Hence (48) follows. The proof of (49) is similar. ■

6. Conclusion

In this paper, we have shown how a perturbation method can both clarify known results and generate new ones. Our approach is based on applying infinitesimal changes to the parameters of the underlying probabilistic model and using Margulis–Russo type perturbation formulas to control the resulting variations. It would be interesting to know whether our Theorems 4.2 and 5.1 can be used to establish certain monotonicity properties of convex hulls, such as those studied in [18,19].

Disclosure statement

No potential conflict of interest was reported by the author(s).

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