

Field quantizations in Schwarzschild spacetime: Theory versus low-energy experiments

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Nonrelativistic quantum particles in Earth's gravitational field are successfully described by the Schrödinger equation with Newton's gravitational potential. Particularly, quantum mechanics is in agreement with such experiments as free fall and quantum interference induced by gravity. However, quantum mechanics is a low-energy approximation to quantum field theory. The latter is successful by the description of high-energy experiments. Gravity is embedded in quantum field theory through the general-covariance principle. This framework is known in the literature as quantum field theory in curved spacetime, where the concept of a quantum particle is, though, ambiguous. In this article, we study in this framework how a Hawking particle moves in the far-horizon region of Schwarzschild spacetime by computing its propagator. We find this propagator differs from that which follows from the path-integral formalism—the formalism which adequately describes both free fall and quantum interference induced by gravity.

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I. INTRODUCTION

Classical mechanics allows multiple equivalent formulations, one of which is based on the Hamilton-Jacobi equation [1]:

$$\partial_t S(t, \mathbf{x}) + H(t, \mathbf{x}, \mathbf{p}) = 0 \quad \text{with} \quad \mathbf{p} = \partial_{\mathbf{x}} S(t, \mathbf{x}), \quad (1)$$

where $S(t, \mathbf{x})$ is Hamilton's principle function and $H(t, \mathbf{x}, \mathbf{p})$ is Hamiltonian. The elementary system we intend to deal with is a nonrelativistic particle of mass M placed in the vicinity of Earth's surface. Such a particle is accordingly described relative to Earth's surface by

$$S(x, X) = M\Delta t \left(\frac{\Delta \mathbf{x}^2}{2\Delta t^2} - \phi_{\oplus} - \frac{g_{\oplus}(z+Z)}{2} - \frac{(g_{\oplus}\Delta t)^2}{24} \right), \quad (2)$$

where $\Delta x = x - X$, $x = (t, \mathbf{x})$ and $X = (T, \mathbf{X})$ parametrizes initial time and position of the particle, and ϕ_{\oplus} and g_{\oplus} stand for, respectively, Newton's gravitational potential and the free-fall acceleration at Earth's surface.

Classical mechanics is, however, incapable of describing the wavelike aspects of particles, requiring $\hbar > 0$. Quantum mechanics in this respect surpasses classical mechanics in

the nonrelativistic regime. The particle's dynamics is described by the particle's propagator to equal the probability amplitude for $|X\rangle$ to evolve into $|x\rangle$. The path-integral formalism [2,3] gives for a quantum particle of mass M placed in the vicinity of Earth's surface that

$$\langle x|X\rangle = \left(\frac{M/\hbar}{2\pi i\Delta t} \right)^{\frac{3}{2}} \exp(iS(x, X)/\hbar). \quad (3)$$

Such a description is consistent not only with free fall, but also with gravity-induced quantum interference [4,5]. The propagator regarded as a non-normalizable wave function is a solution of Schrödinger's equation which includes the Hamilton-Jacobi equation (1) in the limit $\hbar \rightarrow 0$ [6,7]. Quantum mechanics coherently reduces to classical mechanics in the classical limit.

Quantum mechanics is, however, a low-energy approximation to quantum field theory [8]. The basic concept there is a quantum field, e.g., $\hat{\Phi}(x)$. This is a distribution-valued operator. The theory accordingly needs the consideration of a Hilbert-space representation of the field operator algebra. This is constructed by adopting the concept of quantum vacuum which is a state with no particles present. The choice of quantum vacuum thus determines the concept of quantum particles in theory. This choice is, in general, nonunique [9].

In theoretical particle physics, there exists a unique state, $|\Omega\rangle$, which is invariant under the Minkowski-spacetime isometries—the Poincaré group. Particles' states built on top of $|\Omega\rangle$ are related this way with irreducible unitary representations of the Poincaré group, in accord with the

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Wigner classification [8,9]. This implies

$$\hat{\Phi}(x) = \hat{a}(x) + \hat{a}^\dagger(x), \quad (4a)$$

where $\hat{a}(x)$ annihilates $|\Omega\rangle$, meaning $\hat{a}(x)|\Omega\rangle = 0$, and a one-particle state reads

$$|a(x)\rangle \equiv \hat{a}^\dagger(x)|\Omega\rangle. \quad (4b)$$

The invariance under unitary transformations implementing the Poincaré group gives

$$\langle a(x)|\mathbf{p}\rangle = e^{-ip\cdot x/\hbar} \quad \text{with} \quad p \equiv \left(\sqrt{(Mc)^2 + \mathbf{p}^2}, \mathbf{p} \right), \quad (5)$$

where $|\mathbf{p}\rangle$ is a state with a single particle of momentum \mathbf{p} . This is the defining characteristic of $|a(x)\rangle$, which allows one to interpret it as an asymptotic state appearing in scattering processes in collider physics. Such processes are described in quantum field theory by S -matrix elements which equal probability amplitudes for some initial N -particle states to evolve into some final n -particle states. The S -matrix in turn is linked to time-ordered $n + N$ -point functions via the Lehmann-Symanzik-Zimmermann reduction formula [8]. Asymptotic states have accordingly to satisfy (5) in order to give a nonzero S -matrix element [10]. This assertion follows from the reduction formula and the completeness relation for the one-particle states, namely

$$\int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{c}{2\omega_p} |\mathbf{p}\rangle\langle\mathbf{p}| = \hat{1} \quad \text{with} \quad \omega_p \equiv (c/\hbar)\sqrt{(Mc)^2 + \mathbf{p}^2}, \quad (6)$$

ensuring with (5) that the time-ordered probability amplitude—Feynman's propagator—has a pole on mass shell in momentum space, making a nonzero contribution to the S -matrix. In addition, the completeness relation and the Poincaré invariance provide

$$\langle a(x)|a(X)\rangle \xrightarrow{c \rightarrow \infty} \frac{e^{-iMc^2\Delta t/\hbar}}{2Mc/\hbar} \langle x|X\rangle|_{G \rightarrow 0}. \quad (7)$$

Accordingly, $|a(x)\rangle$ is a one-particle state which reduces to the quantum-mechanics state $|x\rangle$ in the nonrelativistic limit in Minkowski spacetime (Newton's constant $G \rightarrow 0$).

The observable Universe is a non-Minkowski spacetime in general relativity. It is taken into account in quantum mechanics by adding $M\phi(\mathbf{x})$ to Hamiltonian, where $\phi(\mathbf{x})$ is the Newton gravitational potential. This modification is enough to explain the gravity-induced quantum interference of thermal neutrons [4,5]. Another example is the appearance of bound states of neutrons above a reflecting plate held parallelly to Earth's surface [11,12]. These effects were predicted by quantum mechanics before their observations. Quantum field theory adopts gravity through

the principle of general covariance. This framework is known in the literature as quantum field theory in curved spacetime [13,14].

The state $|x\rangle = |t, \mathbf{x}\rangle$ depends on $\phi(\mathbf{x})$ as the gravitational potential alters the Hamiltonian. However, $\phi(\mathbf{x}) \neq 0$ leaves the concept of a quantum particle unaltered in quantum mechanics, because $|x\rangle$ is oblivious to gravity. In contrast, the concept of a quantum particle is generally agreed to depend on the observer's notion of time in quantum field theory in curved spacetime. This hypothesis leads to the quantum particles' ambiguity in theory. In Schwarzschild spacetime, approximately describing Earth's gravitational field, this implies the doubling of particle types:

$$\hat{\Phi}(x) = \hat{n}(x) + \hat{n}^\dagger(x) + \hat{h}(x) + \hat{h}^\dagger(x), \quad (8)$$

as there are two independent types of radial-mode solutions [13]. While particles related with $\hat{n}^\dagger(x)$ are, to our knowledge, nameless, particles associated with $\hat{h}^\dagger(x)$ are known as Hawking particles [15–17]. A few states are defined this way in Schwarzschild spacetime [18–21]. In Earth's case, Boulware's state $|B\rangle$ yields quantum vacuum, i.e., $\hat{n}(x)|B\rangle = 0$ and $\hat{h}(x)|B\rangle = 0$, and, correspondingly, there are two types of one-particle states:

$$|n(x)\rangle \equiv \hat{n}^\dagger(x)|B\rangle, \quad (9a)$$

$$|h(x)\rangle \equiv \hat{h}^\dagger(x)|B\rangle, \quad (9b)$$

with the propagators $\langle n(x)|n(X)\rangle$ and $\langle h(x)|h(X)\rangle$, respectively.

Schwarzschild spacetime turns into Minkowski spacetime at spatial infinity. In Minkowski spacetime, $|a(x)\rangle$ is a one-particle state which corresponds to an asymptotic state in collider-physics experiments. For $|n(x)\rangle$ and $|h(x)\rangle$ to be asymptotic states, these must satisfy

$$\langle n(x)|\mathbf{p}\rangle|_{\text{spatial infinity}} \propto e^{-ip\cdot x/\hbar}, \quad (10a)$$

$$\langle h(x)|\mathbf{p}\rangle|_{\text{spatial infinity}} \propto e^{-ip\cdot x/\hbar}. \quad (10b)$$

These would, however, imply that $\langle n(x)|h(X)\rangle \neq 0$ which contradicts the circumstance that $[\hat{n}(x), \hat{h}^\dagger(X)] = 0$ [17]. This implies in turn that either $|n(x)\rangle$ or $|h(x)\rangle$, or both give only zero S -matrix elements, and, correspondingly, one of $|n(x)\rangle$ and $|h(x)\rangle$, or both are unobservable in collider-physics experiments.

The spatial infinity of Schwarzschild spacetime is a mathematical abstract. The observable Universe is a non-Schwarzschild spacetime. The Schwarzschild geometry is an approximation to local geometry of spherically symmetric compact objects in the observable Universe. The spatial infinity is, in practice, a region starting from the distance being much bigger than the Schwarzschild radius

R_S of a compact object up to the distance at which the gravitational field of rest matter is negligible. In Earth's case, such a region is available at Earth's surface at R_\oplus . Earth's gravitational field is characterized at R_\oplus by the free-fall acceleration g_\oplus . It can, however, be locally eliminated by introducing local Minkowski coordinates, such as, for example, Riemann normal coordinates [22]. It is due to Einstein's equivalence principle built into general relativity [23]. In general, the Minkowski-spacetime structure locally emerges in Schwarzschild spacetime by treating Riemann normal coordinates and neglecting space-time curvature. Hence, instead of asymptotically Minkowski spacetime, we have, in practice, local Minkowski frames at $R_\oplus \leq R < \infty$ with the extent much less than $R\sqrt{R/R_S}$.

In theory, particle physics employs Minkowski spacetime as a basic space-time background in which scattering processes take place. In practice, particle physics employs the Minkowski-spacetime approximation in the vicinity of a given point in Schwarzschild spacetime, which approximates local geometry of the observable Universe in the vicinity of Earth. It means that particle physics deals with $|a(y)\rangle$ with space-time curvature neglected, where y denotes Riemann normal coordinates. In Sec. II, we first show that

$$\langle a(y)|a(Y)\rangle \xrightarrow{c \rightarrow \infty} \frac{e^{-iMc^2\Delta t/\hbar}}{2Mc/\hbar} \langle x|X\rangle, \quad (11)$$

where Riemann normal coordinates y are defined at R_\oplus , such that $y = y(x)$ and $Y = y(X)$, and $y \rightarrow x$ in the $G \rightarrow 0$ limit. This agrees not only with high-energy experiments in colliders in the presence of Earth's gravitational field, but also with free-fall experiments and the gravity-induced quantum interference. This implies that the field quantization in theoretical particle physics is adequate also for quantum effects in gravity whenever space-time curvature plays no role. This field quantization allows thus one to coherently reduce quantum field theory to quantum mechanics if $c \rightarrow \infty$ and to classical mechanics if additionally $\hbar \rightarrow 0$.

In Sec. III, we second study the probability amplitudes $\langle n(x)|n(X)\rangle$ and $\langle h(x)|h(X)\rangle$. The study is based, first, on analytic and numerical computations of the amplitudes at large distances ($|X| \gg R_S$ and $|X| \gg |\Delta x|$) and, second, on comparison of the amplitudes with $\langle a(x)|a(X)\rangle$ afterwards. The latter probability amplitude adequately describes the high- and low-energy phenomena, even in gravity. It is due to the first term on the right-hand side of

$$\langle a(x)|a(X)\rangle = \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \frac{c}{2\omega_p} e^{-ip \cdot (y(x)-y(X))/\hbar} + \text{curvature-dependent corrections.} \quad (12)$$

However, $\langle n(x)|n(X)\rangle$ and $\langle h(x)|h(X)\rangle$ cannot simultaneously have such a representation, as otherwise this would

contradict the canonical commutation relation of the quantum field and its canonical conjugate and $\langle n(x)|h(X)\rangle = 0$. This means that at least one of the amplitudes is irreducible to $\langle x|X\rangle$ given in (3) in the nonrelativistic limit. We find that it is $\langle h(x)|h(X)\rangle$ which differs from (3) in the nonrelativistic limit in the far-horizon region, while $\langle n(x)|n(X)\rangle$ approaches $\langle a(x)|a(X)\rangle$, at least if $G \rightarrow 0$.

In what follows, we use natural units $c = G = \hbar = 1$, unless otherwise stated.

II. QUANTUM MECHANICS IN SCHWARZSCHILD SPACETIME

A. Schrödinger equation with Newton's gravitational potential

The starting point of doing quantum field theory in curved spacetime is to pick a quantum field and a space-time geometry in order to provide a concrete expansion of the quantum field over annihilation and creation operators. This is necessary for a solid physical interpretation of the quantum field as we directly observe only particles in real-world experiments. The basic equation we wish to consider reads

$$\left(\square_x + M^2 - \frac{1}{6}R(x)\right)\hat{\Phi}(x) = 0, \quad (13)$$

where $R(x)$ stands for the Ricci scalar, and $\hat{\Phi}(x)$ is the quantum scalar field satisfying with its conjugate $\hat{\Pi}(x)$ the standard canonical commutation relation. Furthermore, we consider the Schwarzschild geometry which is given in isotropic coordinates by the following line element:

$$ds^2 = \left(1 + 2\frac{\phi(\mathbf{x})}{c^2}\right)(cdt)^2 - \left(1 - 2\frac{\psi(\mathbf{x})}{c^2}\right)d\mathbf{x}^2, \quad (14)$$

where we keep track of the c -factors entering the line element, as we wish to consider the nonrelativistic approximation shortly, and the gravitational potentials read

$$\phi(\mathbf{x}) = +\frac{c^2}{2} \left(\frac{r - \frac{1}{4}R_S}{r + \frac{1}{4}R_S}\right)^2 - \frac{c^2}{2}, \quad (15a)$$

$$\psi(\mathbf{x}) = -\frac{c^2}{2} \left(\frac{r + \frac{1}{4}R_S}{r}\right)^4 + \frac{c^2}{2}, \quad (15b)$$

where $r \equiv \sqrt{\mathbf{x} \cdot \mathbf{x}}$ and R_S is the Schwarzschild radius. The reason of dealing with this geometry consists in the fact that Earth's gravitational field is approximately described by this line element with the Schwarzschild radius

$$R_{S,\oplus} \approx 8.87 \times 10^{-3} \text{ m}, \quad (16)$$

provided Earth's rotation can be neglected, which we assume in what follows.

To model a quantum particle in the framework of quantum field theory, we must define an operator which creates the particle's state by applying that on quantum vacuum, $|\Omega\rangle$, namely

$$|\varphi\rangle \equiv \hat{a}^\dagger(\varphi)|\Omega\rangle, \quad (17a)$$

where $\varphi(x)$ describes the particle's state. For this purpose, we generalize the operator used for the definition of asymptotic states in theoretical particle physics to curved spacetime [24,25]:

$$\hat{a}^\dagger(\varphi) \equiv -i \int_{\Sigma} d\Sigma(x) n^\mu(x) (\varphi(x) \nabla_\mu \hat{\Phi}^\dagger(x) - \hat{\Phi}^\dagger(x) \nabla_\mu \varphi(x)), \quad (17b)$$

where $n^\mu(x)$ is a future-directed unit four-vector orthogonal to a Cauchy surface Σ and $d\Sigma(x)$ is the volume element in Σ , and ∇_μ is the covariant derivative. It follows from

$$\left(\square_x + M^2 - \frac{1}{6} R(x) \right) \varphi(x) = 0 \quad (18a)$$

that (17b) is independent of the choice of a Cauchy surface. We obtain from the normalization condition $\langle \varphi | \varphi \rangle = 1$ that

$$\left(\left(\frac{1}{c^2} - \frac{2\phi(\mathbf{x})}{c^4} \right) \partial_t^2 - \left(1 + \frac{2\psi(\mathbf{x})}{c^2} \right) \partial_x^2 - \partial_x \frac{\phi(\mathbf{x}) - \psi(\mathbf{x})}{c^2} \cdot \partial_x + (Mc)^2 \right) \varphi(x) \approx 0, \quad (20)$$

where we have considered only terms linearly depending on the gravitational potentials, and have taken into consideration the fact that $\psi(\mathbf{x}) \approx \phi(\mathbf{x})$ for $r \gg R_S$. Introducing

$$\tilde{\varphi}(x) \equiv \sqrt{2Mc} e^{+iMc^2 t} \varphi(x), \quad (21)$$

we obtain from (20) in the nonrelativistic limit, i.e., $c \rightarrow \infty$, that

$$i \partial_t \tilde{\varphi}(x) \approx \left(-\frac{1}{2M} \partial_x^2 + M\phi(\mathbf{x}) \right) \tilde{\varphi}(x). \quad (22a)$$

This is the Schrödinger equation with Newton's gravitational potential [7]. The normalization condition (18b) turns in this limit into the standard quantum-mechanics one:

$$\int_t d^3 \mathbf{x} \overline{\tilde{\varphi}(t, \mathbf{x})} \tilde{\varphi}(t, \mathbf{x}) \approx 1. \quad (22b)$$

$$-i \int_{\Sigma} d\Sigma(x) n^\mu(x) (\varphi(x) \nabla_\mu \overline{\varphi(x)} - \overline{\varphi(x)} \nabla_\mu \varphi(x)) = 1, \quad (18b)$$

where bar stands for the complex conjugation. The normalization condition $\langle \varphi | \varphi \rangle = 1$ implies that $\varphi(x)$ must be a zero-rank tensor—scalar—as the right-hand side of (18b) is independent of coordinates utilized. In other words, the quantum state $|\varphi\rangle$ cannot appear or disappear by going from one reference frame to another one. This is in accord with the general principle of relativity—general covariance. Einstein's equations need matter to be modeled covariantly, i.e., through an energy-momentum tensor. Since matter itself is best described by quantum theory, this description should be invariant under general coordinate transformations, at least in the framework of classical gravity. It is achieved in our case if $\varphi(x)$ is a zero-rank tensor [26–30].

Our purpose is to model a quantum particle of mass M which freely moves near Earth's surface. Therefore, taking into account that Earth's radius

$$R_{\oplus} \approx 6.37 \times 10^6 \text{ m} \quad (19)$$

is much bigger than its Schwarzschild radius (16), we find from (18a) that

The approximation signs in (22a) and (22b) turn into equality signs in quantum mechanics.

B. Stationary solution at Earth's surface

We wish to consider a noninertial reference frame at the point

$$\mathbf{X}_{\oplus} \equiv R_{\oplus} \mathbf{e}_z \quad \text{with} \quad \mathbf{e}_z \equiv (0, 0, 1). \quad (23)$$

Considering $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{X}_{\oplus}$ until the end of this section and then assuming $|\mathbf{x}| \ll R_{\oplus}$, we have from (15) at the leading order of approximation in \mathbf{x}/R_{\oplus} that

$$\phi(\mathbf{x}) \approx \phi_{\oplus} + g_{\oplus} z, \quad (24a)$$

$$\psi(\mathbf{x}) \approx \psi_{\oplus} + g_{\oplus} z, \quad (24b)$$

where ϕ_{\oplus} and ψ_{\oplus} are the corresponding gravitational potentials at \mathbf{X}_{\oplus} , and

$$g_{\oplus} \equiv \frac{c^2 R_{S,\oplus}}{2(R_{\oplus})^2} \approx 9.81 \text{ m/s}^2 \quad (25)$$

is the free-fall acceleration at Earth's surface, and z accordingly measures altitude above Earth's surface.

The Schrödinger equation (22a) in the homogeneous gravitational field (24a) can be solved by applying the method of separation of variables, also known as Fourier's method. Namely, following [31], we obtain

$$\begin{aligned} \tilde{\varphi}_{\mathcal{E},\mathcal{K}}(x) &\equiv \frac{\exp\left(-i\left(\mathcal{E} + \frac{\mathcal{K}^2}{2M}\right)t + i\mathcal{K} \cdot \mathbf{x}_{\perp}\right)}{(2g_{\oplus}/M)^{\frac{1}{6}}/\sqrt{2}} \\ &\times \text{Ai}\left((2M^2 g_{\oplus})^{\frac{1}{3}}\left(z - \frac{\mathcal{E} - M\phi_{\oplus}}{Mg_{\oplus}}\right)\right), \quad (26) \end{aligned}$$

where $\text{Ai}(z)$ is Airy's function, $(\mathcal{E}, \mathcal{K}) = (\mathcal{E}, \mathcal{K}_x, \mathcal{K}_y)$ are Fourier parameters, and $\mathbf{x}_{\perp} = (x, y)$. Making use of the completeness relation

$$\int_{\mathbf{R}^3} \frac{d\mathcal{E}d\mathcal{K}}{(2\pi)^2} |\mathcal{E}, \mathcal{K}\rangle \langle \mathcal{E}, \mathcal{K}| = \hat{1} \quad (27)$$

and $\tilde{\varphi}_{\mathcal{E},\mathcal{K}}(x) = \langle x | \mathcal{E}, \mathcal{K} \rangle$, we obtain

$$\begin{aligned} \langle x | X \rangle &= \int_{\mathbf{R}^3} \frac{d\mathcal{E}d^2\mathcal{K}}{(2\pi)^2} \tilde{\varphi}_{\mathcal{E},\mathcal{K}}(x) \overline{\tilde{\varphi}_{\mathcal{E},\mathcal{K}}(X)} \\ &= \left(\frac{M}{2\pi i \Delta t}\right)^{\frac{3}{2}} \exp(iS(x, X)). \quad (28) \end{aligned}$$

This agrees with the propagator (3) following from the path-integral formalism ($\hbar \equiv 1$).

A few remarks are in order. First, the derivation based on the stationary modes $\tilde{\varphi}_{\mathcal{E},\mathcal{K}}(x)$, in particular, needs $\mathcal{E} \in (-\infty, +\infty)$. The Fourier parameter \mathcal{E} cannot, therefore, be interpreted as the particle's energy. Second, the modes $\tilde{\varphi}_{\mathcal{E},\mathcal{K}}(x)$ are implicitly defined relative to X_{\oplus} . The gravitational potential $\phi(\mathbf{x})$ has the form (24a) at Earth's surface only. Thereby, $\tilde{\varphi}_{\mathcal{E},\mathcal{K}}(x)$ approximately solve the scalar-field equation if $|\mathbf{x}| \ll R_{\oplus}$. Although this assumption rules out the orthonormality condition requiring $z \in (-\infty, +\infty)$ in theory, $\tilde{\varphi}_{\mathcal{E},\mathcal{K}}(x)$ adequately model local quantum dynamics in practice if applied to determine the propagator $\langle x | X \rangle$. Third, the Fourier method is also applied in quantum field theory in Schwarzschild spacetime, wherein, however, stationary modes are supposed to be positive-frequency ones, see Sec. III.

C. Covariant solution at Earth's surface

We now intend to gain $\langle x | X \rangle$ by applying the principle of general covariance. In theoretical particle physics, an asymptotic state models a particle moving at a constant velocity. Such a state is characterized by a wave packet being a superposition of plane waves

$$\varphi_{Y,K}(y) \equiv \exp(-iK \cdot (y - Y)), \quad (29)$$

where we have used Riemann normal coordinates y , such those $y = y(x)$ and $Y = y(X)$, and the Fourier parameters K can be interpreted as four-momentum. It, accordingly, satisfies the on-mass-shell condition:

$$K \cdot K = (Mc)^2. \quad (30)$$

The plane waves can be expressed through the general coordinates x as

$$\varphi_{X,K}(x) = \exp(-iK \cdot (y(x) - y(X))), \quad (31)$$

where K is now understood as a four-vector belonging to the cotangent space at X . Deriving geodesic distance [32] and afterwards using the relation between Riemann normal coordinates and geodesic distance [33], we find in the homogeneous gravitational field (24a) that Riemann normal coordinates depend on the isotropic coordinates as

$$y^0(x) \approx y^0(X) + c\Delta t \left(1 + \frac{\phi_{\oplus}}{c^2} + \frac{g_{\oplus}z}{c^2} + \frac{(g_{\oplus}\Delta t)^2}{6c^2}\right), \quad (32a)$$

$$y^1(x) \approx y^1(X) + \Delta x, \quad (32b)$$

$$y^2(x) \approx y^2(X) + \Delta y, \quad (32c)$$

$$y^3(x) \approx y^3(X) + \Delta z + \frac{g_{\oplus}\Delta t^2}{2}, \quad (32d)$$

where we have omitted higher-order terms being negligible in the nonrelativistic limit. Note that terms in $y^a(x)$ linearly depending on g_{\oplus} follow from the same term in geodesic distance. Using this result, we obtain in the nonrelativistic limit that

$$\begin{aligned} K \cdot (y(x) - y(X)) &\approx Mc^2\Delta t + \frac{K^2}{2M}\Delta t - \mathbf{K} \cdot \Delta \mathbf{x} \\ &+ M\Delta t \left(\phi_{\oplus} + g_{\oplus}z + \frac{(g_{\oplus}\Delta t)^2}{6}\right) \\ &- K_z \frac{g_{\oplus}\Delta t^2}{2}, \quad (33) \end{aligned}$$

where we have also taken into account the on-mass-shell condition. Redefining the covariant modes according to the nonrelativistic approximation, we obtain that

$$\tilde{\varphi}_{X,K}(x) \equiv \lim_{c \rightarrow \infty} e^{+iMc^2\Delta t} \varphi_{X,K}(x) \quad (34)$$

exactly solves the Schrödinger equation (22a) with the Newtonian potential (24a).

Such a solution has been previously obtained in [34] by relying, however, on the Einstein principle, stating the

equivalence between uniform acceleration and homogeneous gravity [35]. Specifically, this principle being applied to the motion of a quantum particle assumes that its wave function is invariant up to a phase factor under the coordinate transformation changing a reference frame with uniform gravity to another one with no uniform gravity.

In theoretical particle physics, we have from the relativistic completeness relation (6) and Poincaré symmetry that

$$\begin{aligned} \langle a(y)|a(Y)\rangle &= \int \frac{d^3\mathbf{K}}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{K}}} e^{-i\mathbf{K}\cdot(y-Y)} \\ &\xrightarrow{c\rightarrow\infty} \frac{e^{-iMc^2\Delta t}}{2Mc} \int \frac{d^3\mathbf{K}}{(2\pi)^3} \tilde{\varphi}_{X,K}(x), \end{aligned} \quad (35)$$

where we directly find

$$\int \frac{d^3\mathbf{K}}{(2\pi)^3} \tilde{\varphi}_{X,K}(x) = \langle x|X\rangle, \quad (36)$$

which thus proves (11), assuming $\hbar \equiv 1$.

A few remarks are in order. First, in contrast to $\tilde{\varphi}_{\varepsilon,\mathcal{K}}(x)$, the exact solution $\tilde{\varphi}_{X,K}(x)$ is *not* an eigenfunction of ∂_t , by virtue of the Δt^2 and Δt^3 terms in its phase. This circumstance is irrelevant for its application in physics. Still, $\tilde{\varphi}_{X,K}(x)$ is given by $\exp(-iM\tau)$, where τ stands for proper time for a geodesic connecting X with x , see [36]. Second, the plane waves (29) cannot in general be exact solutions of the scalar-field equation (18a) in curved spacetime. This is due to space-time curvature [26,27,30], which we neglect in this section.

D. Wave function

The particle's dynamics is described by $\langle x|X\rangle$ in quantum mechanics. Namely, assuming the particle has position \mathbf{X} and momentum \mathbf{P} at $t = T$, we consider a Gaussian wave packet of the form

$$\begin{aligned} \tilde{\varphi}_{X,P}(x) &\equiv N \left(\frac{D^2}{\pi}\right)^{\frac{3}{2}} \int d^3\mathbf{Q} \langle t, \mathbf{x}|T, \mathbf{X} + \mathbf{Q}\rangle \\ &\times \exp(i\mathbf{P}\cdot\mathbf{Q} - D^2\mathbf{Q}^2), \end{aligned} \quad (37)$$

where D denotes momentum variance and N is a normalization factor determined from (22b). This wave packet is, accordingly, given via the convolution of the propagator with the initial wave packet [7]. Alternatively, we also have

$$\tilde{\varphi}_{X,P}(x) = N \int \frac{d^3\mathbf{K}}{(2\pi)^3} \tilde{\varphi}_{X,K}(x) \exp\left(-\frac{(\mathbf{P}-\mathbf{K})^2}{4D^2}\right), \quad (38)$$

which follows from a Gaussian superposition of the plane waves in the Riemann frame.

E. Free fall

Direct experiments with thermal neutrons [37–39] showed that neutrons fall down with the free-fall acceleration. The precision of these experiments is many orders of magnitude smaller than those with macroscopic objects. For instance, the MICROSCOPE experiment found no relative acceleration at the 10^{-15} level between macroscopic masses of various compositions [40,41]. Besides, free-fall experiments in atom interferometry achieved the 10^{-12} level for the Eötvös parameter for ^{85}Rb and ^{87}Rb [42], see also [43,44], that measures relative acceleration between the atoms.

According to the quantum-mechanics operator formalism, the position operator applied to the wave function reduces to the coordinate multiplication. Thus, its expected value gives the center-of-mass position of the wave function, following from Born's statistical interpretation. We therefore obtain for the quantum-particle position that

$$\begin{aligned} \langle \hat{x}(t)\rangle &\equiv \int_t d^3\mathbf{x} \overline{\tilde{\varphi}_{X,P}(x)} \mathbf{x} \tilde{\varphi}_{X,P}(x) \\ &= \mathbf{X} + \frac{\mathbf{P}\Delta t}{M} - \frac{g_{\oplus}\Delta t^2}{2} \mathbf{e}_z \end{aligned} \quad (39)$$

is the quantum-particle trajectory in Earth's uniform gravitational field. This result is in agreement with classical physics, where the g_{\oplus} -dependent term in (39) is due to the last term on the right-hand side of (33).

The quantum-particle momentum is given by the momentum-operator expectation value:

$$\begin{aligned} \langle \hat{\mathbf{p}}(t)\rangle &\equiv \int_t d^3\mathbf{x} \overline{\tilde{\varphi}_{X,P}(x)} (-i\nabla) \tilde{\varphi}_{X,P}(x) \\ &= \mathbf{P} - Mg_{\oplus}\Delta t \mathbf{e}_z. \end{aligned} \quad (40)$$

This result also agrees with classical physics, where the g_{\oplus} -dependent term in (40) comes from gravitational time dilation [45], entering the right-hand side of (33) in the form $Mg_{\oplus}z\Delta t$.

Finally, the quantum-particle energy reads

$$\begin{aligned} \langle \hat{H}(t)\rangle &\equiv \int_t d^3\mathbf{x} \overline{\tilde{\varphi}_{X,P}(x)} (+i\partial_t) \tilde{\varphi}_{X,P}(x) \\ &= \frac{\mathbf{P}^2}{2M} + \frac{3D^2}{2M} + M(\phi_{\oplus} + g_{\oplus}Z), \end{aligned} \quad (41)$$

which differs from the classical result by the term depending on the momentum variance. This can, however, be eliminated by shifting the rest energy from Mc^2 to $Mc^2 + 3D^2/2M$, resulting in the redefinition of the (Lagrangian) rest mass. Furthermore, the quantum-particle energy is independent of time, albeit (31) is not an

eigenfunction of t . All Δt -dependent terms entering (33) cancel each other in the integral (41). The energy conservation requires the Δt^3 term in (33), known as the Kennard phase [46,47]. It has been observed in [48], see also [49–51].

F. Quantum interference induced by gravity

An interference experiment proposed in [4] was designed to measure a relative phase shift gained by two beams of thermal neutrons during their propagation at different altitudes with respect to Earth’s surface. The phase shift accordingly reads

$$\delta(\Delta z) = -\frac{M^2 g_{\oplus}}{2\pi\hbar^2} \Delta z \lambda L, \quad (42)$$

where λ is a de Broglie wavelength of the neutrons and L is a horizontal distance covered, see [7,34] for further details. In 1975, this theoretical result was empirically confirmed [5], which is known in the literature as the Colella-Overhauser-Werner experiment.

The wave-function phase depends on the altitude z above Earth’s surface. Specifically, its phase changes by shifting the altitude from z to $z + \Delta z$ as follows:

$$\Delta \text{Arg}(\tilde{\varphi}_{X,P}) = -\frac{M g_{\oplus}}{\hbar} \Delta t \Delta z. \quad (43)$$

This phase shift is due to the term in (33) describing gravitational time dilation. Taking into account that $\Delta t = L/V$ with $V = P/M$ and $P = 2\pi\hbar/\lambda$, we find that (43) agrees with (42).

G. Einstein’s principle

An interference experiment with thermal neutrons was performed in 1983 by making use of an accelerated interferometer [52]. The observed phase shift relative to the accelerated device appeared to be in agreement with Einstein’s principle [35], namely quantum interference cannot be used to distinguish between homogeneous gravity and uniform acceleration [34].

We wish to consider a uniformly accelerated frame parametrized by coordinates x_R , that moves relative to a local Minkowski frame. Specifically, we first consider

$$ds^2|_{\text{Universe}} \approx ds^2|_{\text{Minkowski}} = \eta_{ab} dy^a dy^b, \quad (44)$$

where the approximation sign is to underline that the observable Universe is not globally flat, however, herein we neglect the local universe curvature [22]. A uniformly accelerated frame in Minkowski spacetime is known in the literature as Rindler spacetime [14], such that $y = y(x_R)$ are given by

$$y^0(x_R) = \left(\frac{c^2}{a} + z_R\right) \sinh\left(\frac{at_R}{c}\right), \quad (45a)$$

$$y^1(x_R) = x_R, \quad (45b)$$

$$y^2(x_R) = y_R, \quad (45c)$$

$$y^3(x_R) = \left(\frac{c^2}{a} + z_R\right) \cosh\left(\frac{at_R}{c}\right) - \frac{c^2}{a}, \quad (45d)$$

where a stands for proper acceleration. We then have in terms of x_R that

$$ds^2|_{\text{Universe}} \approx \left(1 + 2\frac{\phi(x_R)}{c^2}\right) (cdt_R)^2 - \left(1 - 2\frac{\psi(x_R)}{c^2}\right) (dx_R)^2, \quad (46)$$

where the “gravitational potentials” read

$$\phi(x_R) = +\frac{c^2}{2} \left(1 + \frac{az_R}{c^2}\right)^2 - \frac{c^2}{2}, \quad (47a)$$

$$\psi(x_R) = 0. \quad (47b)$$

In contrast to Schwarzschild spacetime, there is no analog of gravitational length contraction in Rindler spacetime. This circumstance may have impact on the applicability of the Einstein principle if one takes into account the finiteness of the speed of light, $c < \infty$ [29].

As Rindler spacetime is a patch of Minkowski spacetime, the plane-wave modes (29) must be considered, following from the principle of general covariance, cf. [19,53,54]. We thus have

$$\begin{aligned} K \cdot (y(x_R) - y(X_R)) &\approx Mc^2 t_R + \frac{\mathbf{K}^2}{2M} t_R - \mathbf{K} \cdot \mathbf{x}_R \\ &+ Mt_R \left(az_R + \frac{(at_R)^2}{6} \right) \\ &- K_z \frac{at_R^2}{2} \end{aligned} \quad (48)$$

in the nonrelativistic limit, where $X_R = 0$ has been set for the sake of simplicity. Comparing this result with (33), we find that there is no physical difference between uniform acceleration and homogeneous gravity if $c \rightarrow \infty$, in accord with the Bonse-Wroblewski experiment [52].

III. QUANTUM FIELD THEORY IN SCHWARZSCHILD SPACETIME

A. Field quantizations in Schwarzschild spacetime

The application of quantum field theory to the description of particle physics assumes that the observable

Universe can be approximated by Minkowski spacetime. It is justifiable within the framework of general relativity due to Einstein's equivalence principle. This principle assures the existence of local Minkowski frames. In such local frames, special relativity replaces general relativity, where the Poincaré group plays a pivotal role. Even in Schwarzschild spacetime, approximating Earth's gravitational field, one may consider

$$\hat{\Phi}(x) = \hat{a}(x) + \hat{a}^\dagger(x), \quad (49a)$$

where, in terms of Riemann normal coordinates from the previous section, we have

$$\hat{a}(x) \approx \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{2\omega_p} e^{-ip \cdot y(x)} \hat{a}_p \quad \text{with} \quad \hat{a}_p^\dagger|\Omega\rangle = |\mathbf{p}\rangle. \quad (49b)$$

We use the approximation sign in (49b) to underline that plane waves are nonexact solutions of the scalar-field equation. This approximation appears to be adequate for the description of high-energy phenomena in particle accelerators and the low-energy phenomena, including the effects of Earth's gravitational field, as shown in the previous section.

The field quantization in Minkowski spacetime is global in theory, while local in practice. It is generally agreed, nevertheless, that field quantization in a curved spacetime needs global isometry group of the spacetime to introduce the concept of a quantum particle. Moreover, it is also generally agreed that the concept of observer's time plays a pivotal role by choosing a Hilbert-space representation of a field operator algebra [14,55]. This in turn implies that the concept of a quantum particle is ambiguous due to relativity of time in general relativity. This section is to explore consequences of this hypothesis and their coherence with well-established laws in particle physics.

Schwarzschild spacetime has four Killing vectors—its isometry group is four dimensional. Its global isometry corresponds to invariance under time translations and spatial rotations to leave the coordinate-frame origin unaltered. The Schwarzschild-time translation is generated by the Killing vector ∂_t . It is commonly assumed that positive- and negative-frequency modes defined with respect to ∂_t are relevant to particle physics. Namely, this assumption gives

$$\hat{\Phi}(x) = \hat{n}(x) + \hat{n}^\dagger(x) + \hat{h}(x) + \hat{h}^\dagger(x), \quad (50a)$$

where there are then twice as many independent mode functions in Schwarzschild spacetime than in local Minkowski frames [13,16,17,56–58]:

$$\hat{n}(x) \approx \sum_{l=0}^{\infty} \sum_{m=-l}^{m=+l} \int_M \frac{d\omega}{\sqrt{2\omega}} \left(\frac{\omega}{\sqrt{\omega^2 - M^2}} \right)^{\frac{1}{2}} N_{\omega lm}(x) \hat{n}_{\omega lm}, \quad (50b)$$

$$\hat{h}(x) \approx \sum_{l=0}^{\infty} \sum_{m=-l}^{m=+l} \int_0^{\infty} \frac{d\omega}{\sqrt{2\omega}} H_{\omega lm}(x) \hat{h}_{\omega lm}, \quad (50c)$$

where, from the time-translation and spherical (relative to $|\mathbf{x}| = 0$) symmetries, one assumes

$$N_{\omega lm}(x) = \frac{i}{\sqrt{2\pi}} e^{-i\omega t} \frac{r \mathcal{N}_{\omega l}(r)}{\left(r + \frac{1}{4} R_S\right)^2} Y_{lm}(\Omega_{\mathbf{x}}), \quad (51a)$$

$$H_{\omega lm}(x) = \frac{i}{\sqrt{2\pi}} e^{-i\omega t} \frac{r \mathcal{H}_{\omega l}(r)}{\left(r + \frac{1}{4} R_S\right)^2} Y_{lm}(\Omega_{\mathbf{x}}), \quad (51b)$$

where $Y_{lm}(\Omega_{\mathbf{x}})$ denotes the spherical harmonics, $\mathcal{N}_{\omega l}(r)$ and $\mathcal{H}_{\omega l}(r)$ are radial modes.

The approximation signs in (49b) and (50b) with (50c) serve to emphasize the observable Universe is neither Minkowski nor Schwarzschild spacetime.

B. Radial-mode solutions

In terms of the confluent Heun (Hc) function (as defined in Maple), we have

$$\mathcal{N}_{\omega l}(r_s) = B_l(\omega, M) \frac{Hc\left(2\alpha, -2\beta, 0, \gamma, \delta, 1 - \frac{r_s}{R_S}\right)}{e^{-ikr_s \frac{R_S}{r_s} \left(\frac{r_s}{R_S} - 1\right)^{+\beta}}, \quad (52a)$$

$$\begin{aligned} \mathcal{H}_{\omega l}(r_s) &= \frac{Hc\left(2\alpha, +2\beta, 0, \gamma, \delta, 1 - \frac{r_s}{R_S}\right)}{e^{-ikr_s \frac{R_S}{r_s} \left(\frac{r_s}{R_S} - 1\right)^{-\beta}} \\ &+ \mathcal{A}_l(\omega, M) \frac{Hc\left(2\alpha, -2\beta, 0, \gamma, \delta, 1 - \frac{r_s}{R_S}\right)}{e^{-ikr_s \frac{R_S}{r_s} \left(\frac{r_s}{R_S} - 1\right)^{+\beta}}, \end{aligned} \quad (52b)$$

where r_s is the Schwarzschild radial coordinate ($r = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ is the isotropic one), and

$$\alpha \equiv -ikR_S, \quad (53a)$$

$$\beta \equiv +i\omega R_S, \quad (53b)$$

$$\gamma \equiv \alpha^2 + \beta^2, \quad (53c)$$

$$\delta \equiv -\alpha^2 - \beta^2 - l(l+1), \quad (53d)$$

and where, by definition,

$$k \equiv \begin{cases} \sqrt{\omega^2 - M^2}, & \omega \in [M, \infty), \\ i\sqrt{M^2 - \omega^2}, & \omega \in [0, M). \end{cases} \quad (54)$$

For the computation of the reflection coefficient $\mathcal{A}_l(\omega, M)$ and the transmission coefficient $B_l(\omega, M)$, we need asymptotic forms of the radial modes at spatial infinity [13]. These read

$$\mathcal{N}_{\omega l}(r_s) \xrightarrow{r_s \rightarrow \infty} i^l (e^{-ikr_s - i\eta \ln 2kr_s} + A_l(\omega, M)e^{+ikr_s + i\eta \ln 2kr_s}), \quad (55a)$$

$$\mathcal{H}_{\omega l}(r_s) \xrightarrow{r_s \rightarrow \infty} i^l \mathcal{B}_l(\omega, M)e^{+ikr_s + i\eta \ln 2kr_s}, \quad (55b)$$

where

$$\eta \equiv \frac{R_S}{2k}(\omega^2 + k^2). \quad (55c)$$

Using the constancy of the Wronskian for the radial-field equation for various combinations of $\mathcal{N}_{\omega l}(r)$, $\mathcal{H}_{\omega l}(r)$ and their complex conjugate, we find that the reflection coefficients $A_l(\omega, M)$ and $\mathcal{A}_l(\omega, M)$ and the transmission coefficients $B_l(\omega, M)$ and $\mathcal{B}_l(\omega, M)$ are related for $\omega \geq M$ as follows [13]:

$$|B_l(\omega, M)|^2 = (k/\omega)(1 - |A_l(\omega, M)|^2), \quad (56a)$$

$$B_l(\omega, M) = (k/\omega)\mathcal{B}_l(\omega, M), \quad (56b)$$

$$|A_l(\omega, M)| = |\mathcal{A}_l(\omega, M)|. \quad (56c)$$

Our numerics for $|B_l(\omega, M)|^2$ are shown in Fig. 1. We however leave aside the computation of $|B_l(\omega, M)|^2$ for

$\omega \in [0, M)$. This is because we are interested here in local physics at distances being much bigger than R_S . At such distances, $\mathcal{H}_{\omega l}(r)$ exponentially decays for $\omega \in [0, M)$. In this case, the particle interpretation for the modes $H_{\omega lm}(x)$ is no longer justifiable.

Apart from numerical computations of both $\langle n(x)|n(X) \rangle$ and $\langle h(x)|h(X) \rangle$, we also wish to obtain as much analytic information about these propagators as possible. This is particularly needed for the analysis of numerical results. For this reason, we wish to consider approximate solutions for $\mathcal{N}_{\omega l}(r)$ and $\mathcal{H}_{\omega l}(r)$, which solve the radial-field equation up to the leading order in R_S . These approximate radial-mode solutions are given by

$$\mathcal{N}_{\omega l}^{(1)}(r) = \frac{i^l \Gamma(l+1-i\eta)}{(-1)^{l+1} (2)_{2l}} \frac{e^{+\pi\eta/2} M_{-i\eta, l+\frac{1}{2}}(+2ikr)}{r \left(r - \frac{1}{4}R_S\right)^{1/2} \left(r + \frac{1}{4}R_S\right)^{-3/2}}, \quad (57a)$$

$$\mathcal{H}_{\omega l}^{(1)}(r) = i^l \mathcal{B}_l(\omega, M) \frac{e^{-\pi\eta/2} W_{+i\eta, l+\frac{1}{2}}(-2ikr)}{r \left(r - \frac{1}{4}R_S\right)^{1/2} \left(r + \frac{1}{4}R_S\right)^{-3/2}}, \quad (57b)$$

where $M_{\kappa, \mu}(z)$ and $W_{\kappa, \mu}(z)$ are Whittaker's functions, $\Gamma(z)$ and $(z)_\nu$ are the gamma function and the Pochhammer symbol. The index “(1)” means we deal with the first-order solutions in the Schwarzschild radius R_S . However, (57a) fulfills the asymptotic condition (55a) with

$$A_l^{(1)}(\omega, M) = (-1)^{l+1} \frac{(l)_{1-i\eta}}{(l)_{1+i\eta}}. \quad (58)$$

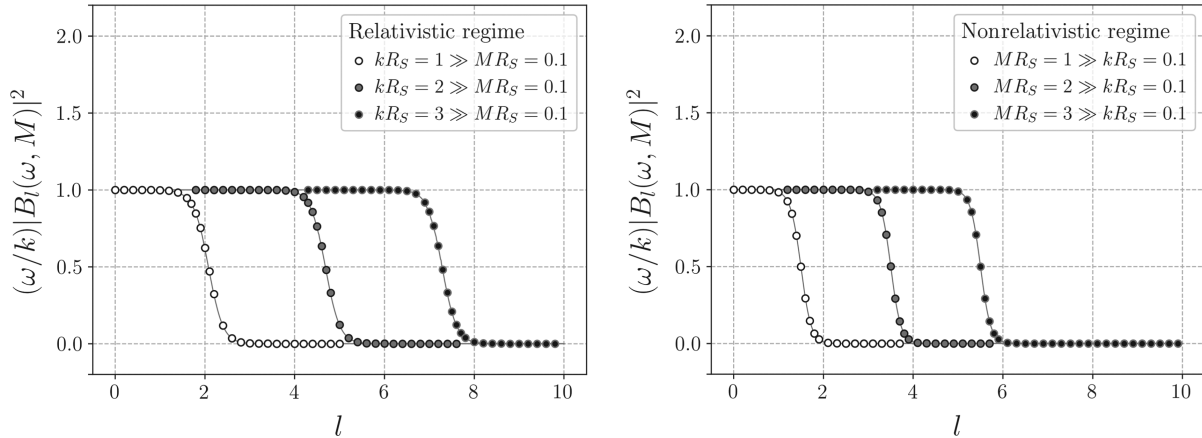


FIG. 1. Numerical computations of $(\omega/k)|B_l(\omega, M)|^2$ as a function of l for various values of ωR_S and $M R_S$. We compute $\mathcal{A}_l(\omega, M)$ by evaluating the confluent Heun functions entering $\mathcal{H}_{\omega l}(r_s)$ and their derivatives with respect to r_s at $r_s = 10^3 R_S$. We next confirm that the values of $\mathcal{A}_l(\omega, M)$ are essentially independent of r_s by computing some of those also at $r_s = 10^4 R_S$. Our numerical results shown here and below are however based on the computations of $\mathcal{A}_l(\omega, M)$ at $r_s = 10^3 R_S$, because it requires less computational resources. Left: we first assume the relativistic regime, i.e., $k \gg M$. Our numerics agree with the DeWitt approximation, see (144) in [13]. Right: we next consider the nonrelativistic regime, i.e., $M \gg k$. In this case, $|B_l(\omega, M)|^2$ approximately equals $(k/\omega)\theta(l_{\max} - l)$ with $l_{\max} \equiv (3\sqrt{2}/2)MR_S$.

This differs from $A_l(\omega, M)$. It is because the radial-mode solution (57a) is oblivious to higher-order terms in R_S , which also contribute to the reflection coefficient, see [59] for more details about scattering theory. In the nonrelativistic limit, though, we have

$$\mathcal{N}_{\omega l}(r) \xrightarrow{c \rightarrow \infty} \mathcal{N}_{\omega l}^{(1)}(r), \quad (59)$$

whereas $\eta \rightarrow (Mc)^2 R_S / 2k$. This follows from the radial-field equation by taking into account the asymptotic condition (55a) and that $R_S \propto 1/c^2 \rightarrow 0$ in the limit $c \rightarrow \infty$.

C. Hawking particles

There are three quantum states which are usually considered in Schwarzschild spacetime [18–21]. The state choice depends on the type of a spherically symmetric compact object. In case of a black hole formed via gravitational collapse—only the future horizon is present—one considers the Unruh state $|U\rangle$ defined by

$$\langle U | (\hat{h}_{\omega l m})^\dagger \hat{h}_{\omega' l' m'} | U \rangle = 0, \quad (60a)$$

$$\langle U | (\hat{h}_{\omega l m})^\dagger \hat{h}_{\omega' l' m'} | U \rangle = \frac{1}{e^{4\pi\omega R_S} - 1} \delta(\omega - \omega') \delta_{ll'} \delta_{mm'}. \quad (60b)$$

This is a thermal state with respect to $\hat{h}(x)$ and $\hat{h}^\dagger(x)$ operators, characterized by Hawking's temperature $1/4\pi R_S$ [15,16]—the Unruh state is therefore a many-Hawking-particle state. It is necessary for an energy-momentum tensor to be nonsingular on the future horizon.

In fact, we obtain for the massive scalar field conformally coupled to gravity that the trace of its energy-momentum tensor reads

$$\begin{aligned} \langle U | \hat{\Theta}_\mu^\mu(x) | U \rangle &= M^2 \langle U | \hat{\Phi}^2(x) | U \rangle \\ &= \frac{M^2}{2\pi^2} \int_0^\infty \frac{k^2 dk}{2\omega} \mathcal{N}_\omega(\mathbf{x}, \mathbf{x}) \\ &\quad + \frac{M^2}{2\pi^2} \int_0^\infty \frac{k d\omega}{2} \mathcal{H}_\omega(\mathbf{x}, \mathbf{x}) \coth(2\pi\omega R_S), \end{aligned} \quad (61)$$

where by definition

$$\mathcal{N}_\omega(\mathbf{x}, \mathbf{x}) \equiv \frac{1}{4k^2 r_s^2} \sum_{l=0}^\infty (2l+1) |\mathcal{N}_{\omega l}(r_s)|^2, \quad (62a)$$

$$\mathcal{H}_\omega(\mathbf{x}, \mathbf{x}) \equiv \frac{1}{4k\omega r_s^2} \sum_{l=0}^\infty (2l+1) |\mathcal{H}_{\omega l}(r_s)|^2. \quad (62b)$$

By use of the asymptotic conditions and numerical computations, we obtain

$$\mathcal{N}_\omega(\mathbf{x}, \mathbf{x}) \rightarrow \begin{cases} 1 + \frac{\eta}{kr_s}, & r_s \rightarrow \infty, \\ \frac{1}{(2kr_s)^2} \Gamma(\omega, M), & r_s \rightarrow R_S, \end{cases} \quad (63a)$$

and

$$\mathcal{H}_\omega(\mathbf{x}, \mathbf{x}) \rightarrow \begin{cases} \frac{\omega/k}{(2kr_s)^2} \Gamma(\omega, M), & r_s \rightarrow \infty \\ \frac{\omega/k}{f(r_s)}, & r_s \rightarrow R_S, \end{cases} \quad (63b)$$

where a lapse function in terms of the Schwarzschild radial coordinate reads

$$f(r_s) \equiv 1 + 2\phi(r(r_s)) = 1 - \frac{R_S}{r_s} \quad (64)$$

and the graybody factor is defined as follows:

$$\Gamma(\omega, M) \equiv \sum_{l=0}^\infty (2l+1) |B_l(\omega, M)|^2. \quad (65)$$

Our numerics for $\Gamma(\omega, M)$ are presented in Fig. 2. Our numerics validating (63a) and (63b) are presented in Figs. 3 and 4.

Both integrals in (61) diverge as ω^2 at $\omega \rightarrow \infty$. It is because quantum fields are operator-valued distributions whose products at the same space-time point are typically singular. The standard approach to deal with this problem is renormalization theory [14]. Since this applies to any state, we treat $|\Omega\rangle$ which locally reduces to the Minkowski quantum vacuum as defined in theoretical particle physics [8]. We find in this state that

$$\begin{aligned} \langle \Omega | \hat{\Theta}_\mu^\mu(x) | \Omega \rangle &= M^2 \lim_{X \rightarrow x} \langle \Omega | \hat{\Phi}(x) \hat{\Phi}(X) | \Omega \rangle \\ &= M^2 \lim_{X \rightarrow x} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2\omega_p} e^{-ip \cdot (y(x) - y(X))} \\ &\quad + \text{finite terms} \\ &= M^2 \lim_{X \rightarrow x} \frac{-1}{8\pi^2 \sigma(x, X)} + \frac{M^2}{16\pi^2} \ln \sigma(x, X) \\ &\quad + \text{finite terms}, \end{aligned} \quad (66)$$

where $\sigma(x, X)$ is geodesic distance [32]. The singular terms in $\langle \Omega | \hat{\Phi}(x) \hat{\Phi}(X) | \Omega \rangle$ at $X \rightarrow x$ are common for locally Minkowski states. These singularities can be eliminated by making use of the Hadamard renormalization [61]. It is based on subtracting the reciprocal and logarithmic divergences with respect to $\sigma(x, X) \rightarrow 0$ in (66). We approximately obtain by use of [62] that

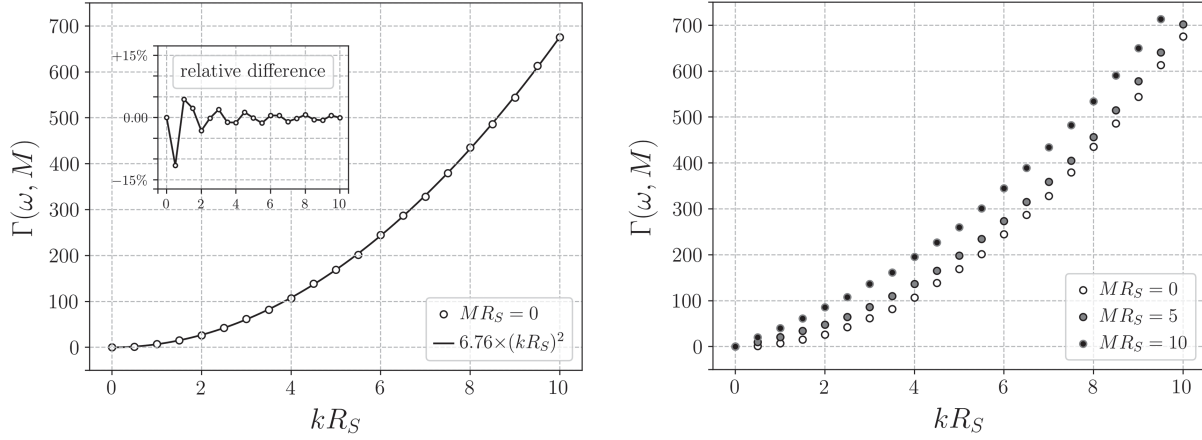


FIG. 2. Numerical computations of the graybody factor $\Gamma(\omega, M)$ as a function of kR_S for various values of MR_S . We compute the transmission probability $|B_l(\omega, M)|^2$ by using the method outlined in the caption of Fig. 1. Left: numerical results for $\Gamma(\omega, M)$ with $MR_S = 0$, in accord with DeWitt's approximation [13]. Right: numerical results for $\Gamma(\omega, M)$ with $MR_S \in \{5, 10\}$, suggesting $\Gamma(\omega, M)$ increases with growing MR_S and vanishes if $kR_S \rightarrow 0$, which agrees with (58) as $kR_S \rightarrow 0$ if $c \rightarrow \infty$.

$$\langle \Omega | \hat{\Theta}_\mu^\mu(x) | \Omega \rangle \xrightarrow[r_s \rightarrow \infty]{r_s \rightarrow R_S} \frac{M^2}{2\pi^2} \int_0^\infty \frac{k d\omega}{2} \Omega_\omega(r_s) \coth(\pi\omega/\kappa(r_s)) + \text{finite terms} \quad (67)$$

in the near- and far-horizon regions, where

$$\Omega_\omega(r_s) \equiv \frac{|K_{i\omega/\kappa(r_s)+1}\left(\frac{f^{\frac{1}{2}}(r_s)M}{\kappa(r_s)}\right)|^2 - |K_{i\omega/\kappa(r_s)}\left(\frac{f^{\frac{1}{2}}(r_s)M}{\kappa(r_s)}\right)|^2}{(k/\omega)f(r_s)|\Gamma(1+i\omega/\kappa(r_s))|^2(\kappa(r_s)/f^{\frac{1}{2}}(r_s)M)^2}, \quad (68)$$

where $K_\nu(z)$ is the modified Bessel function, and

$$\kappa(r_s) \equiv f'(r_s)/2 \quad (69)$$

is surface gravity. In fact, we have from 6.794.3 on p. 751 in [63] that (67) with (68) is equal to (66) with $\sigma(x, X)$ in which $t = T + 0$ and $x = X$. Furthermore, we obtain

$$\Omega_\omega(r_s) \rightarrow \begin{cases} \theta(\omega - M) \left(1 + \frac{\eta}{kr_s}\right), & r_s \rightarrow \infty, \\ \frac{\omega/k}{f(r_s)}, & r_s \rightarrow R_S, \end{cases} \quad (70)$$

where $\theta(z)$ is the Heaviside function. Comparing (70) with (63a) at $r_s \rightarrow \infty$ and with (63b) at $r_s \rightarrow R_S$, we observe that $\Omega_\omega(r_s)$ approximately interpolates between $\mathcal{N}_\omega(x, x)$ in the far-horizon region and $\mathcal{H}_\omega(x, x)$ in the near-horizon region.

The singularities in $\langle U | \hat{\Theta}_\mu^\mu(x) | U \rangle$ and $\langle \Omega | \hat{\Theta}_\mu^\mu(x) | \Omega \rangle$ accordingly match at spatial infinity, at least up to the leading order in $1/r_s$. This is due to the operators $\hat{n}(x)$ and $\hat{n}^\dagger(x)$. In the near-horizon region, these match owing to the

operators $\hat{h}(x)$ and $\hat{h}^\dagger(x)$, at least up to the leading order in $1/f(r_s)$. As a consequence, their difference is finite at the event horizon, at least in the massless limit, $M \rightarrow 0$. The cancellation of $1/f(r_s) \rightarrow \infty$ in the $r_s \rightarrow R_S$ limit needs that the Unruh state is characterized by the Hawking temperature $\kappa(R_S)/2\pi = 1/4\pi R_S$. This is in accord with (60). In the absence of future horizon, however, the Boulware state $|B\rangle$, which is vacuum with respect to both $\hat{n}(x)$ and $\hat{h}(x)$, is admissible as $r_s > R_S$ implies $1/f(r_s) < \infty$.

The line element of Schwarzschild spacetime approximately approaches the line element of Minkowski spacetime far away from the horizon and of Rindler spacetime nearby the horizon. In fact, the gravitational potentials (15) vanish at $r \rightarrow \infty$, while if $x \rightarrow (x + R_S e_z)/4$, where $|x| \ll R_S$, the line element (14) approaches (46), assuming $z_R \rightarrow z_R - 1/a$ and $a = 1/2R_S$. Rindler spacetime is a patch of Minkowski spacetime, in which stationary observers move at a constant proper acceleration, equaling $\kappa(r_s)$ in our case. The Schwarzschild-time coordinate t is then approximated by the Rindler time t_R at the horizon and a local Minkowski time, t_M , at spatial infinity [62]:

$$\Delta t_M \approx \frac{f^{\frac{1}{2}}(r_s)}{\kappa(r_s)} \sinh(\kappa(r_s) \Delta t), \quad (71)$$

where we have neglected terms vanishing as $(f(r_s))^{3/2}$ at $r_s \rightarrow R_S$ and as $(f'(r_s))^2$ at $r_s \rightarrow \infty$. This explains the emergence of the effective temperature parameter $\kappa(r_s)/2\pi$ in (67), see [21]. This effective temperature approaches Hawking's temperature $1/4\pi R_S$ at the future horizon and asymptotically vanishes at spatial infinity. If there is more than one black hole, then the effective temperature in (67) acquires different values depending on the Schwarzschild radius of a given black hole. This illustrates the local character of the state $|\Omega\rangle$.

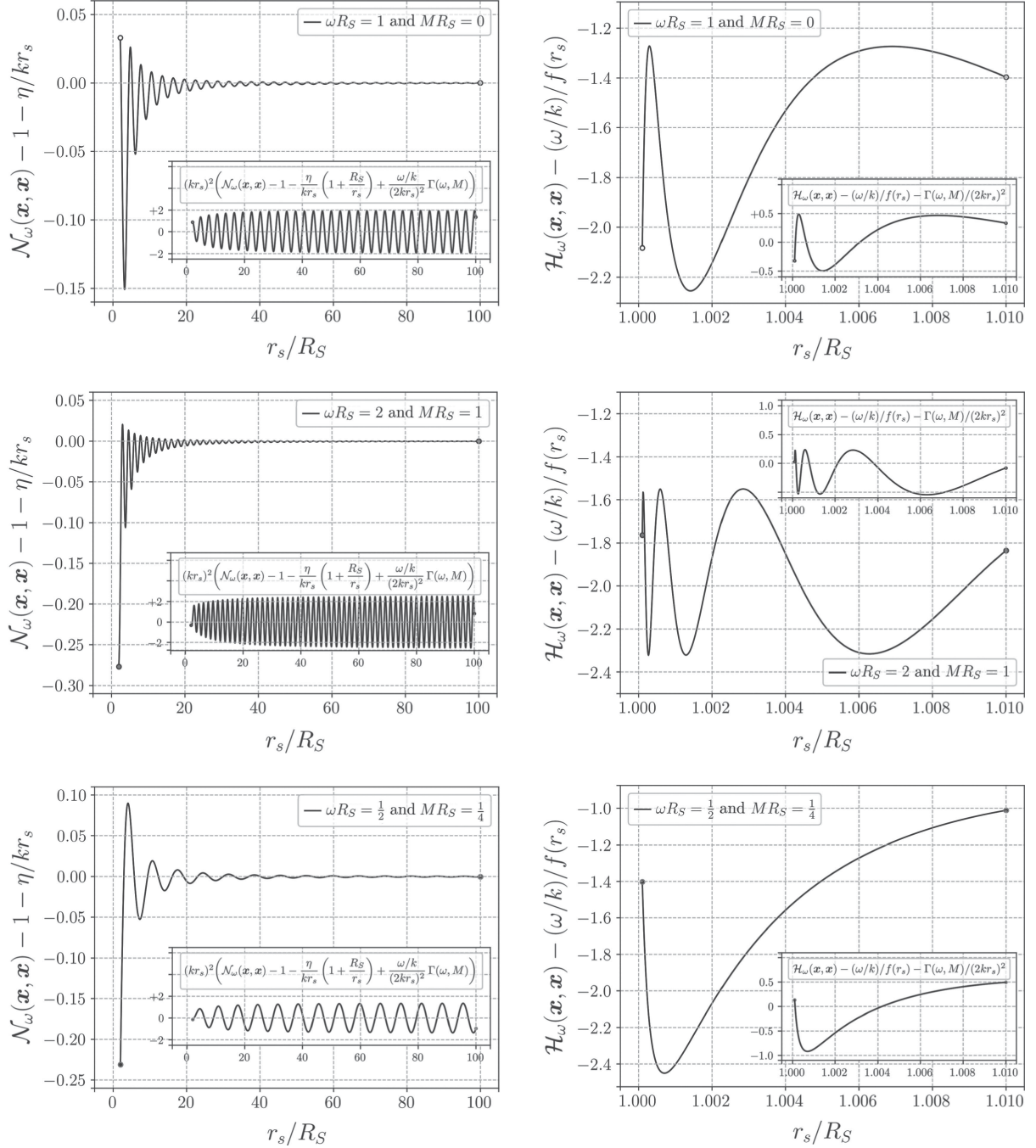


FIG. 3. Left column: numerical computations of $\mathcal{N}_\omega(\mathbf{x}, \mathbf{x})$ for various values of ω and M , while the Schwarzschild radial coordinate $r_s \in [2, 100] \times R_S$. Our numerics support (63) at $r_s \rightarrow \infty$. Shortly we shall derive this spatial-infinity asymptotic of $\mathcal{N}_\omega(\mathbf{x}, \mathbf{x})$ by use of $\mathcal{N}_\omega^{(1)}(r)$ given in (57a). It will also reveal the functional origin of the oscillations shown in the subplots. Right column: numerical computations of $\mathcal{H}_\omega(\mathbf{x}, \mathbf{x})$ for the same ω and M , while $r_s \in [1.0001, 1.01] \times R_S$. Our numerics verify (63b) at $r_s \rightarrow R_S$. This asymptotic of $\mathcal{H}_\omega(\mathbf{x}, \mathbf{x})$ at $r_s \rightarrow R_S$ also agrees with [60] for $M = 0$.

The Unruh state $|U\rangle$ is thermal with respect to $\hat{h}(x)$ and $\hat{h}^\dagger(x)$ operators at the prescribed temperature. It is necessary for the energy-momentum tensor $\langle U|\hat{\Theta}_\nu^\mu(x)|U\rangle$ to be nonsingular on the event horizon of a prescribed black hole. This aspect of $|U\rangle$ must thus manifest itself at spatial infinity. In particular, we obtain by use of (55a)

and (55b) that

$$\langle U|\hat{\Theta}_i^x(x)|U\rangle \xrightarrow{r \rightarrow \infty} + \frac{1}{8\pi^2 r^2} \int_0^\infty dk \omega^2 \Gamma(\omega, M) \frac{x}{e^{4\pi\omega R_S} - 1 |x|}, \quad (72)$$

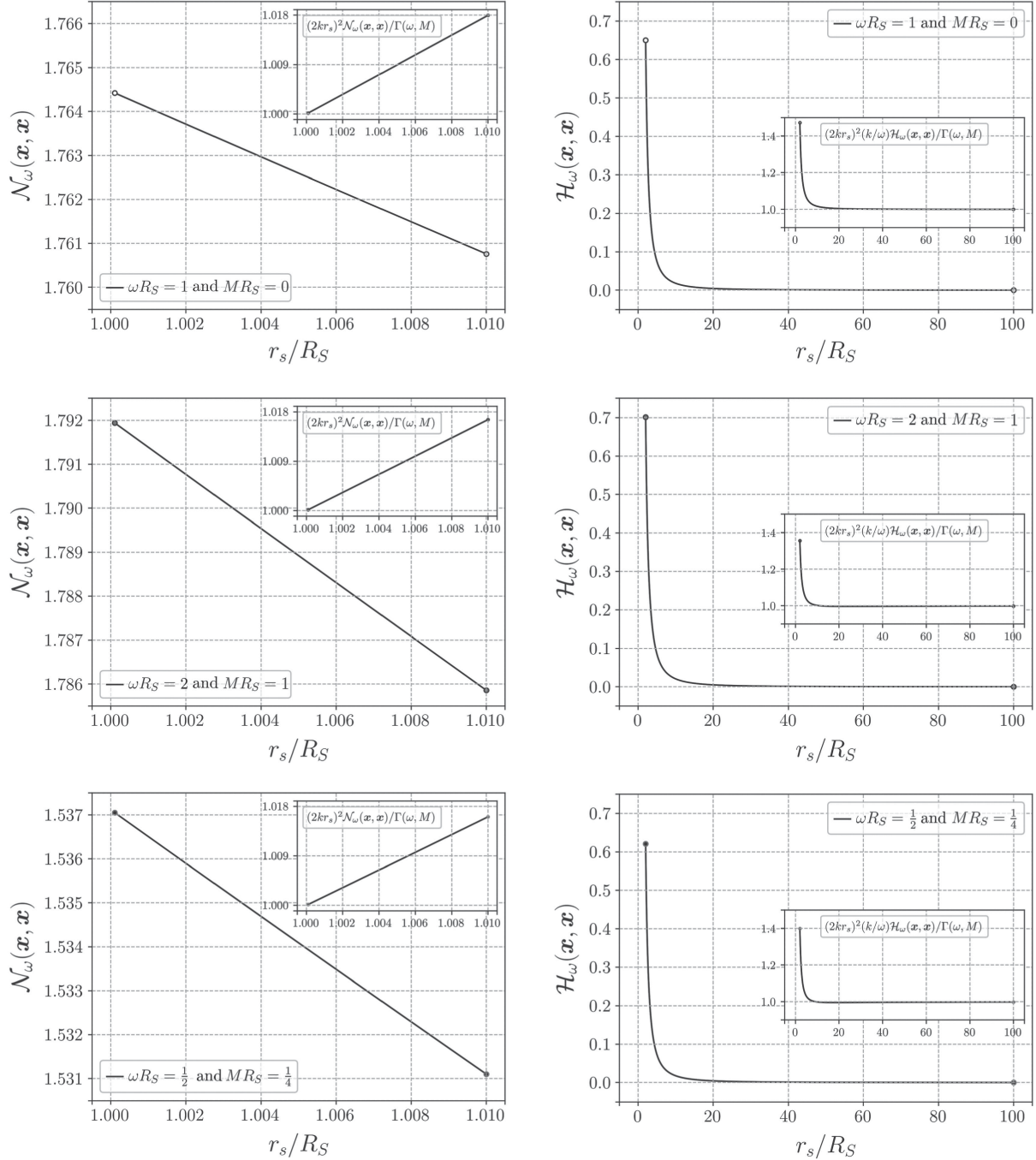


FIG. 4. Left column: numerical computations of $\mathcal{N}_\omega(\mathbf{x}, \mathbf{x})$ for the same ω and M as in Fig. 3, while $r_s \in [1.0001, 1.01] \times R_S$. Our numerics support (63a) at $r_s \rightarrow R_S$. Right column: numerical results for $\mathcal{H}_\omega(\mathbf{x}, \mathbf{x})$ with $r_s \in [2, 100] \times R_S$, which confirm (63b) at $r_s \rightarrow \infty$. In contrast to $\mathcal{N}_\omega(\mathbf{x}, \mathbf{x})$, which approaches unity at $r_s \rightarrow \infty$ as shown in Fig. 3, $\mathcal{H}_\omega(\mathbf{x}, \mathbf{x})$ vanishes as $(1/r_s)^2$ at spatial infinity. This circumstance particularly implies that the radial modes $N_{\omega lm}(x)$ and $H_{\omega lm}(x)$ differently behave in the regime in which Newton's mechanics successfully works. This has impact on how the one-particle states $|n(x)\rangle$ and $|h(x)\rangle$ propagate at $|x| \gg R_S$, as will be shown in the subsequent sections.

where we have taken into account that $H_{\omega lm}(x)$ exponentially vanishes at spatial infinity for $\omega < M$. This quantity describes an outward (spherically symmetric) flux of energy, in accord with the Hawking effect [16], see

also [13]. This energy flux is carried by Hawking particles. In other words, the one-particle state $|h(x)\rangle = \hat{h}^\dagger(x)|B\rangle$ defined in (9b) can be interpreted as a one-Hawking-particle state at the point x .

D. Propagator $\langle n(x)|n(X)\rangle$ in the far-horizon region

It proves useful to get the scalar-field mode $N_{\omega lm}(x)$ in Cartesian coordinates. It suffices to obtain that in perturbation theory up to the leading order in the Schwarzschild radius R_S . We obtain from

$$N_{\omega lm}^{(1)}(x) = \frac{i}{\sqrt{2\pi}} e^{-i\omega t} \frac{r \mathcal{N}_{\omega l}^{(1)}(r)}{\left(r + \frac{1}{4}R_S\right)^2} Y_{lm}(\Omega_{\mathbf{x}}), \quad (73)$$

which follows from (51a) with (57a), by making use of 6.6.3.4 on p. 347 in [64], that

$$\begin{aligned} N_{\mathbf{k}}^{(1)}(x) &\equiv \frac{(2\pi)^{\frac{3}{2}}}{k} \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} i^l N_{\omega lm}^{(1)}(x) \overline{Y_{lm}(\Omega_{\mathbf{k}})}, \\ &= e^{-ik \cdot x + \pi\eta/2} \Gamma(1 - i\eta) \frac{M(i\eta, 1, i(kr - \mathbf{k} \cdot \mathbf{x}))}{(1 - (R_S/4r)^2)^{\frac{1}{2}}}, \end{aligned} \quad (74)$$

where $M(a, b, z)$ is the Kummer function. This nonexact solution matches the exact solution used in quantum mechanics to describe a charged particle elastically scattered by Coulomb's potential, assuming the nonrelativistic limit, i.e., $M \gg k$, the denominator set to unity and the Sommerfeld parameter numerically given by $M^2 R_S / 2k$, see [59] for more details. In this case, $N_{\mathbf{k}}(x)$ can be interpreted at $r \gg R_S$ as modeling a nonrelativistic particle elastically scattered by Newton's potential.

The interpretation of the field quantization (50) in terms of scattering, e.g., see [56], differs from that of the setup we have dealt with in Sec. II. Namely, a quantum particle of mass M at Earth's surface has no asymptotic descriptions of scattering theory. The quantum particle in free fall is not asymptotically free. The field quantization (49) is based on the application of quantum field theory to the description of scattering processes in collider physics. These are, however, owing to nongravitational interactions. In general relativity, nonspinning particles move along geodesics. In particular, geodesics which satisfy appropriate conditions for initial position and momentum may model scattering processes due to gravitational interaction. In terms of Riemann normal coordinates, all geodesics passing through $y = Y$ turn locally into straight world lines. These in turn correspond to trajectories of asymptotic states entering the S -matrix in collider physics. In noninertial coordinates, these trajectories may correspond to free fall or scattering via gravitational interaction, explaining why the field quantization (49) also agrees with particles' motion which is not asymptotically free. The field quantization (50) appears, in contrast, to be designed relying on scattering theory, see [56]. This thus needs the consideration of asymptotic regions in space and time which in practice do not exist, bearing in mind the observable Universe is a non-Schwarzschild spacetime.

The scattering modes $N_{\omega lm}(x)$ may locally give rise to a propagator which still matches (3) in the $c \rightarrow \infty$ limit—this

is what we intend to study here. If affirmative, then the one-particle state $|n(x)\rangle$ defined in (9a) has properties matching those of observable particles, particularly considered in Sec. II. We have from (9a) and (50b) that

$$\langle n(x)|n(X)\rangle = \frac{1}{2\pi^2} \int_0^{\infty} \frac{k^2 dk}{2\omega} e^{-i\omega\Delta t} \mathcal{N}_{\omega}(\mathbf{x}, \mathbf{X}), \quad (75a)$$

where by definition

$$\begin{aligned} \mathcal{N}_{\omega}(\mathbf{x}, \mathbf{X}) &\equiv \frac{1}{4k^2} \sum_{l=0}^{+\infty} (2l+1) \frac{r \mathcal{N}_{\omega l}(r)}{\left(r + \frac{1}{4}R_S\right)^2} \frac{R \overline{\mathcal{N}_{\omega l}(R)}}{\left(R + \frac{1}{4}R_S\right)^2} \\ &\times P_l\left(\frac{\mathbf{x} \cdot \mathbf{X}}{rR}\right), \end{aligned} \quad (75b)$$

where $P_{\nu}(z)$ is the Legendre polynomial and we recall that $r = |\mathbf{x}|$ and, accordingly, $R = |\mathbf{X}|$. Note that $\mathcal{N}_{\omega}(\mathbf{x}, \mathbf{X})$ reduces to $\mathcal{N}_{\omega}(\mathbf{x}, \mathbf{x})$ from (62) if $\mathbf{X} = \mathbf{x}$. We, first, obtain at $R_S \rightarrow 0$ by use of (57a) that

$$\mathcal{N}_{\omega}(\mathbf{x}, \mathbf{X}) \xrightarrow{R_S \rightarrow 0} j_0(k|\Delta\mathbf{x}|), \quad (76)$$

where $j_{\nu}(z)$ is the spherical Bessel function, and we have taken into account 13.18.8, 10.27.6, 10.47.3, and 10.60.2 in [65]. This gives

$$\langle n(x)|n(X)\rangle \xrightarrow{c \rightarrow \infty} \frac{e^{-iMc^2\Delta t}}{2Mc} \langle x|X\rangle_{G \rightarrow 0}. \quad (77)$$

Thus, the one-particle state $|n(x)\rangle$ models a quantum particle of mass M , which freely moves as in classical mechanics in the absence of gravity (Newton's constant $G \rightarrow 0$).

To determine how $|n(x)\rangle$ moves in the presence of gravity, $G > 0$, we, second, approximate $\mathcal{N}_{\omega l}(r)$ entering (75b) by $\mathcal{N}_{\omega l}^{(1)}(r)$ given in (57a). Up to the leading order in the Schwarzschild radius R_S , we have

$$\begin{aligned} \mathcal{N}_{\omega}^{(1)}(\mathbf{x}, \mathbf{X}) &= \sum_{l=0}^{+\infty} \frac{(1)_l |(1 - i\eta)_l|^2}{(-1)^l l! (1)_{2l} (2)_{2l}} \\ &\times \frac{M_{-i\eta, l + \frac{1}{2}}(2ikr) M_{-i\eta, l + \frac{1}{2}}(2ikR)}{e^{-\pi\eta} |\Gamma(1 - i\eta)|^{-2} (2ikr)(2ikR)} P_l\left(\frac{\mathbf{x} \cdot \mathbf{X}}{rR}\right). \end{aligned} \quad (78)$$

Making use of [66], we obtain

$$\begin{aligned} \mathcal{N}_{\omega}^{(1)}(\mathbf{x}, \mathbf{X}) &= \frac{(\partial_{\xi_+} - \partial_{\xi_-}) M_{i\eta, \frac{1}{2}}(-i\xi_+) M_{i\eta, \frac{1}{2}}(-i\xi_-)}{e^{-\pi\eta} |\Gamma(1 - i\eta)|^{-2} (\xi_+ - \xi_-)} \Big|_{\xi_{\pm} = k(r + R \pm |\Delta\mathbf{x}|)}. \end{aligned} \quad (79)$$

In fact, in the limit $\eta \rightarrow 0$ or $R_S \rightarrow 0$, the right-hand sides of (78) and (79) coincide. This can be shown by taking into account 13.6.9 on p. 328 in [65] and 5.10.3.3 on p. 621 in [67]. Moreover, assuming that $\mathbf{x} \cdot \mathbf{X} = -rR$, the sum in (78) can also be exactly evaluated for any $\eta > 0$ with the help of 6.6.2.8 on p. 347 in [64], which matches that of (79) if \mathbf{x} and \mathbf{X} are collinear in opposite directions.

With this result at hand, we next consider (79) with $\mathbf{x} = \Delta\mathbf{x} + \mathbf{X}$ and $|\mathbf{X}| \gg R_S$, whereas $|\Delta\mathbf{x}| \ll |\mathbf{X}|$, and define for $\Delta\mathbf{x} \perp \mathbf{X}$ and $\Delta\mathbf{x} \parallel \mathbf{X}$, respectively,

$$\begin{aligned} \Delta_{\perp} \mathcal{N}_{\omega}(\mathbf{x}, \mathbf{X}) &\equiv \mathcal{N}_{\omega}(\mathbf{x}, \mathbf{X}) - \left(1 + \frac{R_S}{2|\mathbf{X}_s|}\right) j_0(k|\Delta\mathbf{x}_s|) \\ &\quad - \frac{\omega^2 R_S}{2k^2 |\mathbf{X}_s|} \cos k|\Delta\mathbf{x}_s|, \end{aligned} \quad (80a)$$

$$\begin{aligned} \Delta_{\parallel} \mathcal{N}_{\omega}(\mathbf{x}, \mathbf{X}) &\equiv \mathcal{N}_{\omega}(\mathbf{x}, \mathbf{X}) - j_0(k|\Delta\mathbf{x}_s|) - \frac{\eta \cos k|\Delta\mathbf{x}_s|}{k|\mathbf{X}_s|} \\ &\quad - i\omega \Gamma(\omega, M) \frac{\sin k|\Delta\mathbf{x}_s|}{4k^3 |\mathbf{X}_s|^2}, \end{aligned} \quad (80b)$$

where the index ‘‘s’’ refers to the Schwarzschild-Cartesian coordinates. Our numerical results given in Fig. 5 show that (79) properly approximates (75b) up to the leading order in $1/|\mathbf{X}|$. Accordingly, we find

$$\langle n(\mathbf{x}) | n(\mathbf{X}) \rangle \xrightarrow{c \rightarrow \infty} \frac{e^{-iMc^2 \Delta t}}{2Mc} \langle \mathbf{x} | \mathbf{X} \rangle + \mathcal{O}\left(\frac{1}{X^2}\right), \quad (81)$$

where we have taken into account that $R_S \rightarrow 0$ in the limit $c \rightarrow \infty$, whereas $\omega^2 R_S > 0$. Thus, this result agrees with (3) to the leading order in $1/|\mathbf{X}|$.

A few remarks are in order. First, $\mathcal{N}_{\omega}^{(1)}(\mathbf{x}, \mathbf{X})$ given in (79) is insufficient to unambiguously determine terms in $\mathcal{N}_{\omega}(\mathbf{x}, \mathbf{X})$ with $\mathbf{x} \sim \mathbf{X}$, approaching zero faster than $1/|\mathbf{X}|$ at $|\mathbf{X}| \rightarrow \infty$. Among such terms is $g \propto R_S/X^2$. This is because $\mathcal{N}_{\omega}^{(1)}(\mathbf{x}, \mathbf{X})$ is an approximate solution which merely takes the leading-order correction with respect to $R_S/|\mathbf{X}|$ into account. Second, $\mathcal{N}_{\omega}^{(1)}(\mathbf{X}, \mathbf{X})$ at $|\mathbf{X}| \rightarrow \infty$ has the Taylor-series term $\eta \sin(2k|\mathbf{X}|)/2|k\mathbf{X}|^2$. This qualitatively accounts for the oscillations shown in the subplots in Fig. 3, left column, while their phase and amplitude should gain corrections depending on higher-order terms in R_S . Such corrections must vanish at $c \rightarrow \infty$, according to (59). Replacing (75b) by (79) in (75a) at $c \rightarrow \infty$ gives (3) with the term $g \propto R_S/X^2$ included. However, $\eta \sin(2k|\mathbf{X}|)/2|k\mathbf{X}|^2$ represents an extra term which is still regular at $k \rightarrow 0$ in (75a), whereas higher-order Taylor-series terms diverge. This gives an additive correction to (3) of the order of $g_{\oplus} M^2/\hbar^2$ if $\Delta t \ll MR_{\oplus}^2/\hbar \sim 10^{21}(M/M_n)$ s, where M_n is neutron’s mass and the universe age is roughly 10^{17} s. At Earth’s surface, this correction to (3) is negligible if

$\Delta t \ll (\hbar/Mg_{\oplus}^2)^{1/3}$. This is particularly violated in the classical limit. And, finally, (12) implies in terms of isotropic coordinates that

$$\begin{aligned} \langle a(\mathbf{x}) | a(\mathbf{X}) \rangle &= \frac{1}{2\pi^2} \int_0^{\infty} \frac{p^2 dp}{2\omega_p} e^{-i\omega_p \sqrt{1+2\phi(\mathbf{x})} \Delta t} \\ &\quad \times j_0\left(p\sqrt{1-2\psi(\mathbf{x})}|\Delta\mathbf{x}|\right) + \mathcal{O}\left(\frac{1}{X^2}\right), \end{aligned} \quad (82)$$

where we have used $y(\mathbf{x}) \approx y(\mathbf{X}) + (\sqrt{1+2\phi(\mathbf{x})}\Delta t, \sqrt{1-2\psi(\mathbf{x})}\Delta\mathbf{x})$ approximately holding up to $1/|\mathbf{X}|$ in the limit $|\mathbf{X}| \rightarrow \infty$. In contrast to the integral representation of $\langle n(\mathbf{x}) | n(\mathbf{X}) \rangle$, the integral representation of $\langle a(\mathbf{x}) | a(\mathbf{X}) \rangle$ explicitly involves both gravitational time dilation and gravitational length contraction. It is the former general-relativity effect which gives rise to both ϕ_{\oplus} - and g_{\oplus} -dependent terms in (3).

E. Propagator $\langle h(\mathbf{x}) | h(\mathbf{X}) \rangle$ in the far-horizon region

We finally wish to explore how a Hawking particle propagates far away from a spherically symmetric compact object. The probability amplitude for $|h(\mathbf{X})\rangle$ to evolve into $|h(\mathbf{x})\rangle$ reads

$$\langle h(\mathbf{x}) | h(\mathbf{X}) \rangle = \frac{1}{2\pi^2} \int_0^{\infty} \frac{k^2 dk}{2\omega} e^{-i\omega \Delta t} \mathcal{H}_{\omega}(\mathbf{x}, \mathbf{X}), \quad (83a)$$

where by definition

$$\begin{aligned} \mathcal{H}_{\omega}(\mathbf{x}, \mathbf{X}) &\equiv \frac{1}{4k\omega} \sum_{l=0}^{+\infty} (2l+1) \frac{r\mathcal{H}_{\omega l}(r)}{(r+\frac{1}{4}R_S)^2} \frac{\overline{R\mathcal{H}_{\omega l}(R)}}{(R+\frac{1}{4}R_S)^2} \\ &\quad \times P_l\left(\frac{\mathbf{x} \cdot \mathbf{X}}{rR}\right), \end{aligned} \quad (83b)$$

such that $\mathcal{H}_{\omega}(\mathbf{x}, \mathbf{X})$ reduces to $\mathcal{H}_{\omega}(\mathbf{x}, \mathbf{x})$ defined in (62) if $\mathbf{X} = \mathbf{x}$.

Using the approximation (57b) and our numerical results for (83b), we assume $|\Delta\mathbf{x}| \ll |\mathbf{X}|$ and define for $\Delta\mathbf{x} \perp \mathbf{X}$ and $\Delta\mathbf{x} \parallel \mathbf{X}$, respectively,

$$\Delta_{\perp} \mathcal{H}_{\omega}(\mathbf{x}, \mathbf{X}) \equiv \text{Re } \mathcal{H}_{\omega}(\mathbf{x}, \mathbf{X}) - \frac{\omega/k}{|2k\mathbf{X}_s|^2} \Gamma(\omega, M), \quad (84a)$$

$$\begin{aligned} \Delta_{\parallel} \mathcal{H}_{\omega}(\mathbf{x}, \mathbf{X}) &\equiv \text{Re } \mathcal{H}_{\omega}(\mathbf{x}, \mathbf{X}) - \omega \Gamma(\omega, M) \frac{\cos k|\Delta\mathbf{x}_s|}{4k^3 |\mathbf{X}_s|^2} \\ &\quad - \frac{4R_S}{|\mathbf{X}_s|} \text{Re } \Delta_{\parallel} \mathcal{N}_{\omega}(\mathbf{x}, \mathbf{X}), \end{aligned} \quad (84b)$$

where we have taken into account that

$$\text{Im}(\mathcal{N}_{\omega}(\mathbf{x}, \mathbf{X}) + \mathcal{H}_{\omega}(\mathbf{x}, \mathbf{X})) = 0, \quad (85)$$

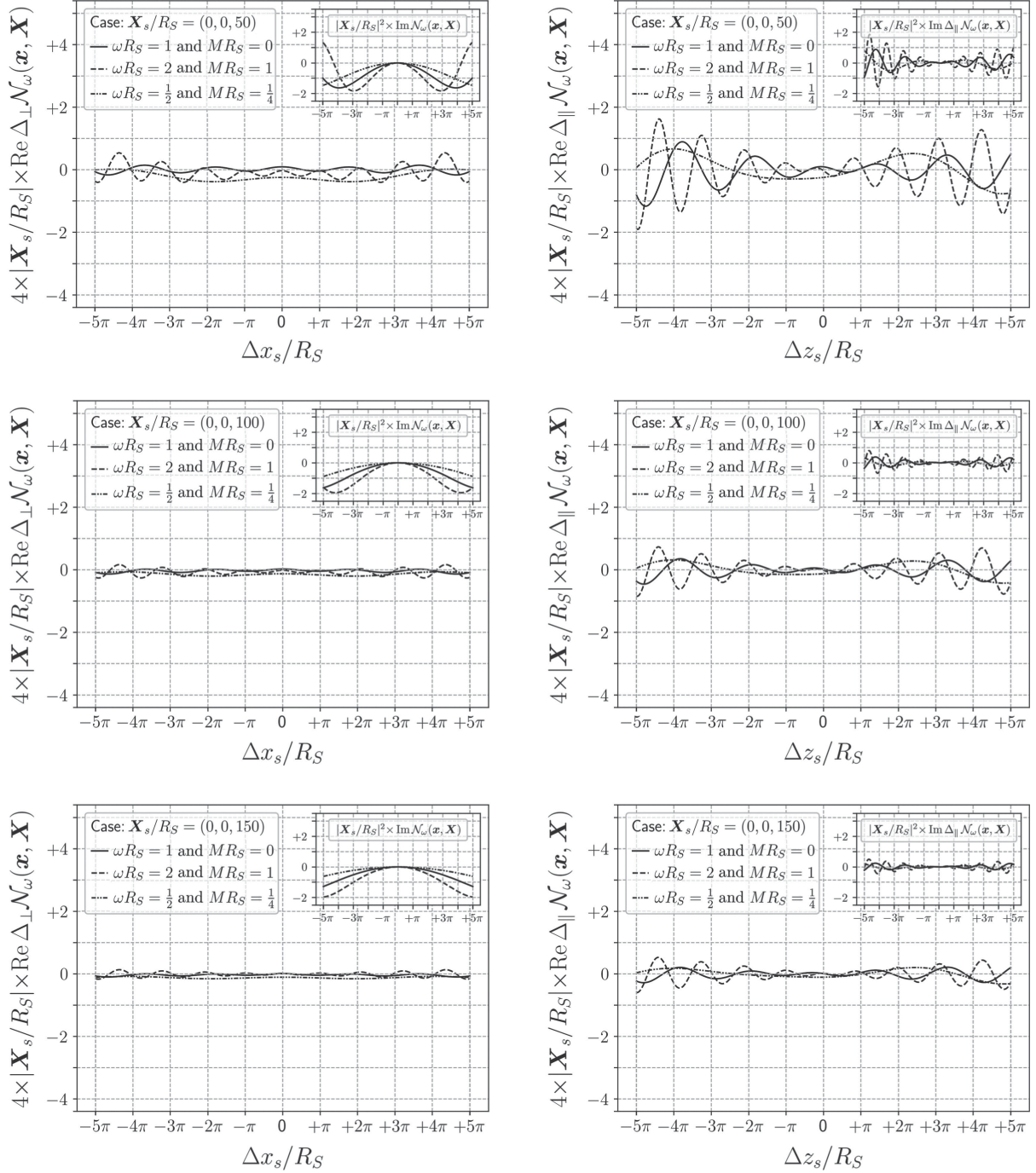


FIG. 5. Numerical computations of $\mathcal{N}_\omega(\mathbf{x}, \mathbf{X})$ for various values of ω and M , whereas $\mathbf{x} = \Delta\mathbf{x} + \mathbf{X}$ with $\mathbf{X} = (0, 0, Z)$ and $\Delta\mathbf{x}$ being either $(\Delta x, 0, 0)$ or $(0, 0, \Delta z)$. For a given value of Z in units of R_S , we thus compute how $\mathcal{N}_\omega(\mathbf{x}, \mathbf{X})$ changes by varying $\Delta\mathbf{x}$ either perpendicularly or parallelly to \mathbf{X} . Left column: $\text{Re } \Delta_\perp \mathcal{N}_\omega(\mathbf{x}, \mathbf{X})$ with $\Delta\mathbf{x} = (\Delta x, 0, 0)$ and $Z_s/R_S \in \{50, 100, 150\}$, while the subplots show $\text{Im } \mathcal{N}_\omega(\mathbf{x}, \mathbf{X})$. Right column: $\text{Re } \Delta_\parallel \mathcal{N}_\omega(\mathbf{x}, \mathbf{X})$ with $\Delta\mathbf{x} = (0, 0, \Delta z)$ and the same values of Z_s and the subplots display $\text{Im } \Delta_\parallel \mathcal{N}_\omega(\mathbf{x}, \mathbf{X})$.

which can be proved by using the way the confluent Heun function transforms under $\omega \rightarrow -\omega$ and $k \rightarrow -k$. Our numerical results for (84) are shown in Fig. 6. We thus find for $\mathbf{x} = \Delta\mathbf{x} + \mathbf{X}$ with $|\Delta\mathbf{x}| \ll |\mathbf{X}|$ and $\mathbf{X} \propto e_z$ —local outward radial direction—that

$$\langle h(\mathbf{x}) | h(\mathbf{X}) \rangle = \frac{1}{16\pi^2} \frac{1}{X^2} \int_0^\infty \frac{dk}{k} \Gamma(\omega, M) e^{-i\omega\Delta t + ik\Delta z} + \mathcal{O}\left(\frac{1}{X^3}\right). \quad (86)$$

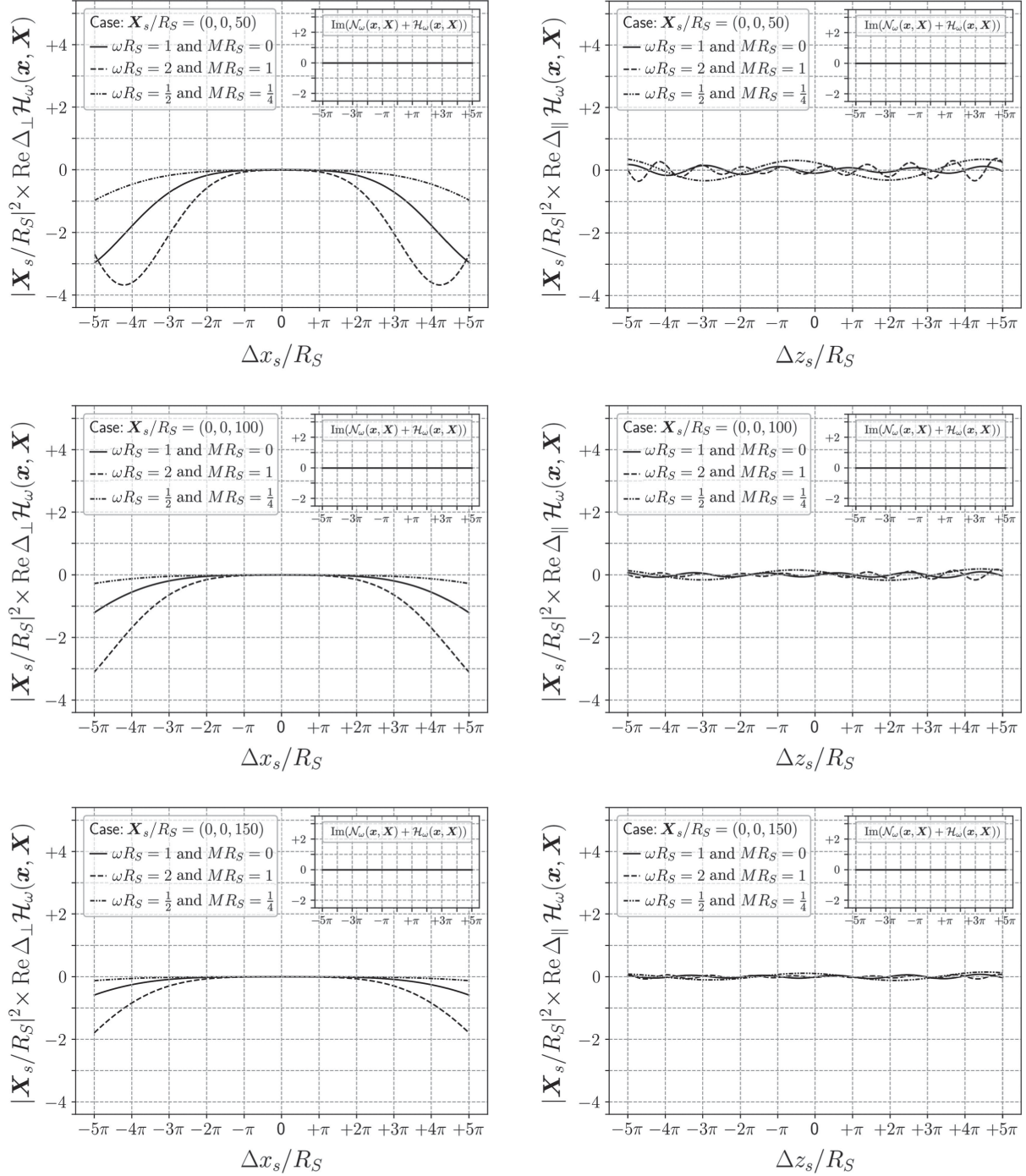


FIG. 6. Numerical computations of $\mathcal{H}_\omega(\mathbf{x}, \mathbf{X})$ for the same values of ω and M , $\mathbf{x} = \Delta\mathbf{x} + \mathbf{X}$ and \mathbf{X} as in Fig. 5. Left column: $\text{Re } \Delta_\perp \mathcal{H}_\omega(\mathbf{x}, \mathbf{X})$ for $\Delta\mathbf{x} \perp \mathbf{X}$. Right column: $\text{Re } \Delta_\parallel \mathcal{H}_\omega(\mathbf{x}, \mathbf{X})$ for $\Delta\mathbf{x} \parallel \mathbf{X}$. Both $\text{Re } \Delta_\perp \mathcal{H}_\omega(\mathbf{x}, \mathbf{X})$ and $\text{Re } \Delta_\parallel \mathcal{H}_\omega(\mathbf{x}, \mathbf{X})$ decrease with increasing values of $|\mathbf{X}|$. In both cases, we find that $\text{Im } \mathcal{H}_\omega(\mathbf{x}, \mathbf{X}) = -\text{Im } \mathcal{N}_\omega(\mathbf{x}, \mathbf{X})$ with accuracy of 15 digits after the comma. It corresponds to the number of digits we have kept by our numerical computations.

In contrast to $\langle n(x)|n(X) \rangle$, which approaches the Minkowski-spacetime propagator at spatial infinity, $\langle h(x)|h(X) \rangle \rightarrow 0$ in the far-horizon region. This agrees with our expectation in Sec. I. Moreover, the Hawking-particle motion is suppressed in directions perpendicular to

e_z . This contradicts to quantum particles' dynamics following from (3), assuming $c \rightarrow \infty$.

A few remarks are in order. First, this form of $\langle h(x)|h(X) \rangle$ agrees with the Hawking effect (72) if $|B\rangle$ is replaced by $|U\rangle$. Second, our numerical computations of

$\Gamma(\omega, M)$ shown in Fig. 2 suggest that $\Gamma(\omega, M) \rightarrow 0$ if $c \rightarrow \infty$ as this assumes both $\omega R_S \rightarrow 0$ and $kR_S \rightarrow 0$, resulting in $\langle h(x)|h(X) \rangle \rightarrow 0$ if $c \rightarrow \infty$. This is consistent with the fact that there is only one type of $|x\rangle$ in quantum mechanics. Third, if the compact object is a black hole, then the computations of $\mathcal{N}_\omega(\mathbf{x}, \mathbf{x})$ and $\mathcal{H}_\omega(\mathbf{x}, \mathbf{x})$ made in Sec. III C suggest that $\langle h(x)|h(X) \rangle$ may approximately turn into the Minkowski-spacetime propagator in the near-horizon region if $|B\rangle$ is replaced by $|U\rangle$. If so, it would validate the application of such concepts as geodesic and classical action to a Hawking particle assumed in [68]. This is certainly the case for $|a(x)\rangle = \hat{a}^\dagger(x)|\Omega\rangle$, although no black-hole evaporation is present in this case. However, in the far-horizon region, $\langle n(x)|n(X) \rangle$ asymptotically approaches the Minkowski-spacetime propagator. Therefore, such concepts as geodesic and classical action are still not applicable to a Hawking particle at spatial infinity, even in the presence of an event horizon. And, finally, the fact that $\langle h(x)|h(X) \rangle$ differs from the Minkowski-spacetime propagator at spatial infinity agrees with the canonical commutation relation. Indeed, $[\hat{\Phi}(x), \hat{\Phi}(X)]$ equals $\langle a(x)|a(X) \rangle - \langle a(X)|a(x) \rangle$, whereas, at spatial infinity, $\langle n(x)|n(X) \rangle \rightarrow \langle a(x)|a(X) \rangle$, as found in Sec. III D, implying then that $\langle h(x)|h(X) \rangle \rightarrow 0$.

IV. CONCLUDING REMARKS

Classical mechanics is successful by the description of particle physics in the regime which is compatible with $\hbar \rightarrow 0$ and $c \rightarrow \infty$. Classical mechanics is replaced by quantum mechanics for particle-physics phenomena which require $\hbar > 0$. Quantum mechanics is in turn replaced by quantum field theory if $\hbar > 0$ and $c < \infty$ need to be taken into account. However, classical mechanics and quantum mechanics deal primarily with particles, while quantum field theory with a field operator algebra. The concept of a particle depends accordingly on the choice of a Hilbert-space representation of such an algebra. In contrast to quantum mechanics, quantum field theory allows multiple Hilbert-space representations which may or may not be unitarily equivalent to each other [9].

The formalism of quantum field theory is successfully used for the description of scattering processes and decay rates in collider physics. The Poincaré group plays a key role by choosing the unique Hilbert-space representation. Particles' states are accordingly linked to irreducible unitary representations of the Poincaré group. This is the isometry group of Minkowski spacetime in theory, while of local Minkowski frames in practice. On Earth's surface, the space-time geometry can approximately be modeled by

Schwarzschild spacetime, provided that Earth's rotation is ignored. Particles' dynamics is accordingly modeled by their propagators which have been established in quantum field theory in Minkowski spacetime [8]. By relying on the principle of general covariance, their dynamics can be treated in terms of a coordinate frame which is at rest with respect to Earth's surface. We have shown in Sec. II that, still, the Minkowski-spacetime propagators properly model particles' dynamics. Particularly, in the regime $\hbar \rightarrow 0$ and $c \rightarrow \infty$, particles' dynamics agrees with free fall, while, in the regime $\hbar > 0$ and $c \rightarrow \infty$, that agrees with quantum interference induced by gravity. The field quantization (49) which is used in theoretical particle physics works not only for high-energy processes in collider physics, but also for the low-energy phenomena in Earth's gravitational field.

The field quantization (50) which relies on the isometry group of Schwarzschild spacetime is generally used in quantum field theory in curved spacetime. This field quantization assumes the doubling of particle types in theory. One of these particle types is known in the literature as a Hawking particle. We have shown in Sec. III that Hawking particles cannot be identified with particles which are coherently described by quantum mechanics and classical mechanics in the weak-gravity regime. This conclusion follows from the observation that the propagator of a Hawking particle in the far-horizon region of Schwarzschild spacetime differs from that following from the path-integral formalism. This field quantization accordingly lacks not only experimental confirmations for the moment, but also coherence with the well-established laws in particle physics. This observation implies Hawking particles obey nonstandard mechanics. Therefore, insisting on the existence of Hawking particles (and, accordingly, of the Hawking effect) should, at least, influence experimental techniques designed for their detection.

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DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

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