STABLE MODULI SPACES OF ODD-DIMENSIONAL MANIFOLD TRIADS

Zur Erlangung des akademischen Grades eines

DOKTORS DER NATURWISSENSCHAFTEN

von der KIT-Fakultät für Mathematik des

Karlsruher Instituts für Technologie (KIT)

genehmigte

DISSERTATION

von

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Tag der mündlichen Prüfung:	6 Mai 2026
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*Dedicado aos meus pais e à minha irmã
por todo o amor e apoio incondicional.*

Abstract

We establish a homotopy-theoretic description of the homology of stable moduli spaces of $(2n+1)$-dimensional manifold triads $(N, \partial^h N, \partial^v N)$ with fixed $\partial^v N$, whenever $n \geq 3$ and $(N, \partial^h N)$ is 1-connected. Stabilization is performed by taking boundary connected sum with $S^n \times D^{n+1}$. This is an analog of earlier work of Galatius and Randal-Williams for even-dimensional manifolds with fixed boundary, and it extends a previous result by Botvinnik and Perlmutter. As a byproduct, we obtain an analog for odd-dimensional triads of Kreck's stable diffeomorphism classification of even-dimensional manifolds.

Acknowlegments

First and foremost, I would like to wholeheartedly thank my advisor Manuel Krannich for his continuous support, encouragement, and guidance. His view on the *what, why*, and *how* of mathematical research has profoundly influenced the mathematician that I have become over the last three years. I am eternally grateful for his energy, patience, openness, and interest in my mathematical path.

I would also like to express my gratitude to Oscar Randal-Williams for his interest, advice, and expertise which led to many enlightening discussions, as well as for his hospitality at the University of Cambridge. As it will become clear in the coming pages, this thesis is heavily influenced by the work of Søren Galatius and Oscar Randal-Williams, to whom I acknowledge an enormous intellectual debt.

In the past three years, I have been fortunate to be surrounded by wonderful fellow young researchers, to which a good part of my development is owed, and that made my mathematical life ever more pleasurable. I would like to highlight three of whom in particular. First, collaborating and discussing all things manifoldy with Samuel Muñoz-Echániz has been both motivating and exciting. Second, I have shared a great part of my daily mathematical life at KIT with Florian Kranhold and Fadi Mezher, to whom I am grateful for their support, openness to numerous sanity checks, and interesting discussions.

A special thanks to Ines Knoch for taking care of me and the rest of the IAG, thus making it possible for me to focus almost solely on research.

Outside of academia, I have felt the support and unconditional love throughout my journey from many sources. First and foremost, I warmly thank my beloved parents and sister for supporting me continuously through every step of the way, and to whom I dedicate this thesis. To my dear Mariana for simply being the beautiful way you are, and always being there for me.

My life in Karlsruhe would not have been so exciting and pleasurable, if it were not for the beautiful friends that I had the pleasure of getting to know. To Benji for always being there for me in the best and the worse times, and always giving me the best advice. To Darya for your contagious energy and humour, and for showing me a new Karlsruhe. To Jerónimo for the long nights in Iuno, Wolfsbräu, Rosa, and Kap talking about life and mathematics. To Julia for your warmth and friendship, which I hold very dear. To Simon for the long breakfasts before work, for your friendship and, of course, for your banana pancakes. Finally, to Hem, Josemi, Patri, and the rest of the Miercoleo, Veranito, Otonito, and of all the numerous Signal groups, for your friendship and all the laughs we shared together.

João Lobo Fernandes
Karlsruhe, February 2026

Contents

1. Introduction.

In this introduction, we will introduce the main objects of study in this thesis, *moduli spaces of manifolds*, and motivate their study from a homotopy-theoretic perspective. The main goal of this introduction is to provide the necessary context for the main problem that this thesis addresses, stated in Section 1.5. This section is structured as follows:

1.1. What a moduli space (sometimes) is.

The main object of study in this thesis is a specific example of a *moduli space*. This term has become ubiquitous in mathematics, especially in the fields of algebra, geometry and topology, since it was first introduced by Riemann in the beginning of the 20-th century. In this introduction, we will not attempt to define *moduli spaces* in a way that would make justice to its many different incarnations in mathematics. Instead, we will present three selection criteria that the *author* finds fit for a space to deservedly be called a moduli space, along with examples that satisfy those criteria.

Criterion 1. *The points of a moduli space are objects of a fixed geometric nature and paths are ways of deforming such objects while preserving this nature.*

The first (and arguably most important) moduli space that was defined was the *moduli space of Riemann surfaces with genus g and n marked points* $\mathscr{M}_{g,n}$. Roughly speaking, the points in $\mathscr{M}_{g,n}$ are isomorphism classes of *Riemann surfaces*, that is, smooth projective connected complex curves, of genus g and a choice of n-many distinct points on the curve. Paths in $\mathscr{M}_{g,n}$ are deformations of the curve together with ways of moving the points avoiding collisions. This space was first introduced and studied by Riemann, and has been a central object of study in algebraic geometry ever since.

Before continuing with the further criteria, let us comment about the word *space* in the previous and forthcoming discussion. In this thesis, the word *space* will mean a topological space or a homotopy type. This definition is, however, not always the most convenient when studying moduli spaces. For example, when attempting to construct the topological space $\mathscr{M}_{g,n}$ with the requirements set above, one obtains a (quite) pathological space that is only a shadow of a more complex but also more natural object (e.g. a *stack*). This topological space will not only be difficult to study but also will not enjoy natural properties that one would hope–for example, instances of the forthcoming criteria. However, such pathologies will not appear in the moduli spaces that we will consider in this thesis. Therefore, from now on, all moduli spaces will, in particular, be spaces.

Criterion 2. *Moduli spaces classify families of objects of a fixed geometric nature indexed by reasonably non-pathological spaces.*

Another important family of moduli spaces are the *Grassmanians*. For non-negative integers $0 \leq d \leq N$, we define the space $\mathrm{Gr}(d,N)$ to be the space of linear subspaces of $\mathbf{R}^N$ of dimension d, and call it the *Grassmanian of d-planes in $\mathbf{R}^N$*. The topology of $\mathrm{Gr}(d,N)$ is an important object of study for several fields in mathematics. For example, the study of its cohomology ring is a useful step towards solving enumeration problems in *enumerative geometry*. More relevantly to this thesis, these spaces completely determine the classification problem of real vector bundles in the following way. Observe that, by seeing a linear subspace of $\mathbf{R}^N$ as a linear subspace of $\mathbf{R}^{N+1}$ by the inclusion $\mathbf{R}^N \subset \mathbf{R}^{N+1}$ in the first N-coordinates, we have an inclusion $\mathrm{Gr}(d,N) \subset \mathrm{Gr}(d,N+1)$. Denote the of all $\mathrm{Gr}(d,N)$ over N as $\mathrm{Gr}(d)$–the points in this space can be seen as linear subspaces of $\mathbf{R}^\infty$ of dimension d–equipped with the weak topology. This space *classifies* d-dimensional real vector bundles in the following sense.

Theorem (Classification of vector bundles)**.** *There exists a d-dimensional real vector bundle γ_d over $\mathrm{Gr}(d)$ which is universal, i.e. there is a natural bijection*

$$\left\{\begin{array}{c}\textit{homotopy classes of}\\ \textit{continuous functions}\\ X \to \mathrm{Gr}(d)\end{array}\right\} \xrightarrow{\cong} \left\{\begin{array}{c}\textit{isomorphism classes of}\\ \textit{d-dimensional real}\\ \textit{vector bundles over } X\end{array}\right\}$$

for any paracompact space X, taking a continuous function $f : X \to \mathrm{Gr}(d)$ to the pullback bundle $f^*\gamma_d$.

This result–usually atributed to Steenrod [Ste51]–is a precise interpretation of Criterion 2 showing that $\mathrm{Gr}(d)$ classifies families of d-dimensional real vector spaces (i.e. vector bundles) indexed by paracompact spaces. Similarly, complex Grassmanians classify complex vector bundles, oriented Grassmanians classify oriented vector bundles, and so on and so forth.

Remark. A formal consequence of Steenrod's theorem above is that the cohomology ring $\mathrm{H}^*(\mathrm{Gr}(d);R)$ with coefficients in a ring R is isomorphic to the ring of *characteristic classes* of d-dimensional real vector bundles with coefficients in R, that is, a natural transformation from the functor $\mathrm{Vect}_d(-)$ taking a paracompact space X to the set $\mathrm{Vect}_d(X)$ of isomorphism classes of d-dimensional vector bundles to the cohomology functor $\mathrm{H}^*(-;R)$. Stiefel-Whitney classes and Pontrjagin classes are examples of characteristic classes, and are important invariants in differential geometry and topology. This is, in fact, a general feature for any moduli space satisfying Criterion 2.

Criterion 3. *Moduli spaces are (unions of) classifying spaces for automorphisms groups of objects of a fixed geometric nature.*

A *classifying space* of a group G is any space $\mathrm{B}G$ which is obtained by taking the quotient of a weakly contractible reasonably nice free G-space by the action of G. Such a space always exists and, in fact, all classifying spaces for a group G are weakly equivalent.

Our first example was already introduced, namely $\mathrm{Gr}(d)$. One way of defining the topology of $\mathrm{Gr}(d)$ is as the quotient of the space $V_d(\mathbf{R}^\infty)$ of injective linear transformations $T : \mathbf{R}^d \to \mathbf{R}^\infty$ by the action of $\mathrm{GL}_d(\mathbf{R})$ given by pre-composition. One can check that $V_d(\mathbf{R}^\infty)$ is weakly contractible and the action of $\mathrm{GL}_d(\mathbf{R})$ is free. Thus, $\mathrm{Gr}(d)$ is a classifying space $\mathrm{BGL}_d(\mathbf{R})$ of the general linear group $\mathrm{GL}_d(\mathbf{R})$.

On another hand, a classifying space of a group G also satisfies Criterion 2 in the sense that it classifies principal G-bundles–in other words, a family of free and transitive G-sets–over paracompact spaces in the same sense as the Grassmanian $\mathrm{Gr}(d)$ above. We will see more examples of classifying spaces in the next section.

To conclude, the above list of criteria for moduli spaces and examples thereof shows that these spaces appear in different forms, and are an important class of objects for different fields. In the next sections, we will define the main object of interest in this thesis, which satisfies Criteria 1 to 3, and present some of the ways that its homotopy type is studied.

1.2. Moduli spaces of manifolds.

The main object of interest in this thesis is the *moduli space of smooth manifolds of type M* for a compact smooth manifold M. This homotopy type can be modelled in various ways. However, the following model will be favourable for our purposes.

Definition 1.2.1. Let M be a compact smooth manifold, the *moduli space of manifolds of type M* is the quotient

$$\mathrm{BDiff}(M) \coloneqq \mathrm{Emb}(M, \mathbf{R}^\infty_+)/\mathrm{Diff}(M)$$

of the action of the diffeomorphism group $\mathrm{Diff}(M)$ on the space of neat[1] smooth

[1]An embedding $e : M \hookrightarrow \mathbf{R}^N_+$ is *neat* if $e^{-1}(\partial\mathbf{R}^N_+) = \partial M$ and M intersects $\partial\mathbf{R}^N_+$ transversely.

embeddings $\mathrm{Emb}(M, \mathbf{R}_+^\infty)$ of M into $\mathbf{R}_+^\infty$–the union of $\mathbf{R}_+^N = [0, +\infty) \times \mathbf{R}^{N-1}$ for all $N \geq 0$–both topologized with *Whitney's C^∞-topology*[2].

Before we consider any examples of this definition, let us briefly justify Criteria 1 to 3 for this definition. First, observe that, $\mathrm{BDiff}(M)$ is in bijection with the set of neat submanifolds of $\mathbf{R}_+^\infty$ that are abstractly diffeomorphic to M by the following reasoning: any such submanifold $W \subset \mathbf{R}_+^\infty$ defines an embedding by choosing a diffeomorphism $\phi : M \xrightarrow{\cong} W$ and post-composing with the inclusion into $\mathbf{R}_+^\infty$, and two such embeddings which have the same image are on the same orbit of the action of $\mathrm{Diff}(M)$. Similarly, paths in this space are given by isotopies of submanifolds, hence we see that $\mathrm{BDiff}(M)$ satisfies Criterion 1.

Remark. This alternative description justifies considering $\mathrm{BDiff}(M)$ as a smooth analog of $\mathrm{Gr}(d)$ by replacing linear subspaces with smooth submanifolds. On the other hand, seeing $\mathrm{Gr}(d)$ as the quotient of $V_d(\mathbf{R}^\infty)$ by the action of $\mathrm{GL}_d(\mathbf{R})$ (as above) can be seen as a linear analog of $\mathrm{BDiff}(M)$ where embeddings are replaced by linear injections, and diffeomorphisms by linear automorphisms.

We move now to Criterion 3, and from there deduce Criterion 2 as a consequence. To do so, observe that the action of $\mathrm{Diff}(M)$ on $\mathrm{Emb}(M, \mathbf{R}_+^\infty)$ is free. On the other hand, the latter space is weakly contractible by *Whitney's embedding theorem*, and thus[3] $\mathrm{BDiff}(M)$ is, as the notation suggests, a classifying space for the topological group $\mathrm{Diff}(M)$, hence establishing Criterion 3. From this fact, it is a formal consequence that $\mathrm{BDiff}(M)$ classifies principal $\mathrm{Diff}(M)$-bundles, in the same sense as the theorem above on vector bundles. Additionally, for every principal $\mathrm{Diff}(M)$-bundle $P \to X$, we can define the fiber bundle $P \times_{\mathrm{Diff}(M)} M \to X$, whose fibers are homeomorphic to M, and whose structure group is $\mathrm{Diff}(M)$, also called *smooth M-bundle* over X. From that, one can deduce that $\mathrm{BDiff}(M)$ classifies smooth M-bundles, and hence satisfies Criterion 2. Let us now consider some examples of Definition 1.2.1.

Example (Dimension 0)**.** A 0-dimensional compact manifold is simply a finite set S, and its diffeomorphism group is simply the group of permutations $\mathfrak{S}_S$ of the elements of S. Moreover, the space of neat embeddings $\mathrm{Emb}(S, \mathbf{R}_+^\infty)$ is homeomorphic to the space of tuples $(x_s)_{s \in S}$ of pairwise distinct points in $\mathbf{R}^\infty$, where $\mathfrak{S}_S$ acts by permuting the labels. Thus, we see that $\mathrm{BDiff}(S)$ is homeomorphic to the space of finite subsets of $\mathbf{R}^\infty$ with the same cardinality as S. The homology groups of this space were first extensively computed by Nakaoka [Nak60, Nak61], leading

[2] *Whitney's C^∞-topology* of a subspace of the space of smooth maps from a compact smooth manifold M to any manifold N is uniquely characterized by the following property: a sequence of smooth functions converges if and only if it converges pointwise and all its derivatives converge pointwise. See e.g. [Hir76] for clarification.

[3] In order to conclude this fact, one needs the extra property that the projection $\mathrm{Emb}(M, \mathbf{R}_+^\infty)$ *has slices*, which holds. See e.g. [BF].

to the discovery of the phenomenon of *homological stability*, which is nowadays known for many families of groups.

Example (Spheres)**.** Let $d \geq 0$ be an integer. Given a linear subspace $V \subset \mathbf{R}^\infty$ of dimension $d+1$ together with an inner product, then the subspace of all points in V of norm 1, denoted by $S(V)$, is a smooth manifold of $\mathbf{R}^\infty_+$ abstractly diffeomorphic to S^d. Thus, it is no surprise that there is a map

$$S(-) : \mathrm{Gr}^o(d+1) \longrightarrow \mathrm{BDiff}(S^d)$$

where the source is the Grassmanian of d-planes of $\mathbf{R}^\infty$ together with an inner product, given by taking V to $S(V)$. By observing that $\mathrm{Gr}^o(d+1)$ models $\mathrm{BO}(d+1)$, the map $S(-)$ can also be modeled as the map induced on classifying spaces of the inclusion $\mathrm{O}(d+1) \to \mathrm{Diff}(S^d)$ given by the action of $\mathrm{O}(d+1)$ on S^d by linear isometries. From the example above, one can see that $S(-)$ is an equivalence for $d = 0$. This phenomenon also holds for $d = 1, 2, 3$ but fails in every other dimension except these. The case $d = 1$ follows from the fact that there is essentially a canonical way to isotope a diffeomorphism of an interval $[0, 1]$ fixing the boundary to the identity. The case $d = 2$ was proved by Smale [Sma59], and the case $d = 3$ by Hatcher [Hat83], solving Smale's conjecture. We will come back to the failure of the analogous phenomenon in higher dimensions in Section 1.4.3.

The study of moduli spaces of surfaces is a point of intersection between many fields in algebra, topology, and geometry. Let us look at two examples outside of topology.

Example (Dimension 2–Group theory)**.** Let Σ_g be an orientable surface of genus $g \geq 2$. First, Earle and Eells showed that the path components of the subgroup $\mathrm{Diff}^+(\Sigma_g)$ of orientation-preserving diffeomorphisms are contractible [EE69, p. 21], and hence $\mathrm{Diff}^+(\Sigma_g)$ is homotopy equivalent to the discrete group $\pi_0(\mathrm{Diff}^+(\Sigma_g))$, which is commonly called the *mapping class group* of Σ_g, and often denoted by Γ_g. Thus, the moduli space $\mathrm{BDiff}^+(\Sigma_g)$ of orientable surfaces of genus g is equivalent to the classifying space of Γ_g. The latter object is heavily studied in group theory (see e.g. [FM11]). For example, Γ_g acts on $\pi_1(\Sigma_g)$ via outer automorphisms, hence producing a group homomorphism $\sigma : \Gamma_g \to \mathrm{Out}(\pi_1(\Sigma_g))$ and the *Dehn–Nielsen–Baer theorem* states that σ is an isomorphism, see [FM11, Thm. 8.1]. Moreover, taking abelianizations, we have a map $(-)^{\mathrm{ab}} : \mathrm{Out}(\pi_1(\Sigma_g)) \to \mathrm{GL}_{2g}(\mathbf{Z})$, as the abelianization of $\pi_1(\Sigma_g)$ is $\mathrm{H}_1(\Sigma_g) \cong \mathbf{Z}^{2g}$. The kernel of the composite $\Gamma_g \to \mathrm{GL}_{2g}(\mathbf{Z})$ is called the *Torelli subgroup*. The cohomology of this group is still a mysterious object, being the source of many conjectures and open questions, see e.g. [Mor99, CF11].

For $g = 1$, observe that $\Sigma_1 = S^1 \times S^1$ is a Lie group, and thus acts on itself by translation. This produces a map $S^1 \times S^1 \to \mathrm{Diff}(S^1 \times S^1)_{\mathrm{id}}$, onto the path component of the identity in the group of diffeomorphisms of $S^1 \times S^1$, which Earle and Eells showed to be a homotopy equivalence [EE69, p. 21]. From this result, Morita [Mor] completely computed the cohomology group $\mathrm{H}^*(\mathrm{BDiff}^+(S^1 \times S^1); \mathbf{Q})$. This computation can be also obtained via the *Eichler-Shimura isomorphism* (see e.g. [BZ25, Thm. 5.15]), thus providing a somewhat unexpected relationship between characteristic classes of manifold bundles and spaces of modular forms.

Example (Dimension 2–Algebraic geometry)**.** The moduli spaces $\mathrm{BDiff}^+(\Sigma_g)$ are of relevance to the field of algebraic geometry for the following reason. There exists a map $\mathscr{M}_g \to \mathrm{BDiff}^+(\Sigma_g)$ roughly taking a Riemann surface and forgetting the complex structure on it. One can show that this map induces an isomorphism on rational cohomology. There are certain *tautological classes* $\kappa_{e^i} \in \mathrm{H}^*(\mathscr{M}_g; \mathbf{Q})$, often called the *Miller–Morita–Mumford classes*, which were first defined by Mumford [Mum83], motivated by the study of enumerative problems in the moduli space $\mathscr{M}_g$. In loc.cit, the author conjectured that, in a certain range of degrees (coming from Harer's stability theorem [Har85]), these classes form an algebraically independent set of generators of $\mathrm{H}^*(\mathscr{M}_g; \mathbf{Q})$. After more than 20 years since Mumford's work, this conjecture was finally proved by Madsen and Weiss [MW07], by purely geometric-topological methods. These methods inspired a wave of new developments in the study of moduli spaces of manifolds, best exemplified in the generalization of the main result of loc.cit. to all even dimensions by Galatius and Randal-Williams [GR14] (to which we will return later, see Section 1.4.2), which in turn heavily inspired the main result of this thesis.

As a final example, we expand on some of the known results about moduli spaces of 3-dimensional manifolds.

Example (Dimension 3)**.** After the resolution of Smale's conjecture by Hatcher [Hat83], as mentioned above, and the proof of *Thurston's geometrization conjecture* by Perelman, a new version of Smale's conjecture emerged stating that for a Riemannian metric g on a 3-dimensional closed manifold M of constant sectional curvature, the inclusion $\mathrm{Isom}(M, g) \hookrightarrow \mathrm{Diff}(M)$ is a homotopy equivalence, see [Gab01, Hon+12]. The first instance of this fact is Hatcher's theorem for $M = S^3$, since $\mathrm{Isom}(S^3, g) \cong \mathrm{O}(4)$ for g the standard round metric. Later, this conjecture was proved for hyperbolic manifolds (that is, constant sectional curvature -1) by Gabai [Gab01]. Using techniques from geometric analysis, namely by exploiting further improvements in the study of *Ricci flows*, Balmer and Kleiner [BK22] proved this conjecture for a large class of prime manifolds, eventually completing the proof of this conjecture. This conjecture has lead to a plethora of outstanding research and lies in the heart of 3-dimensional manifold topology, which this example is too small to contain. We refer to loc.cit. for a historical reference.

As we have seen, the moduli spaces of manifolds $\mathrm{BDiff}(M)$ are objects of interest to many fields. However, their study seems to be (at least in the examples presented) quite case-dependent and case-sensitive. In the next sections, we present systematic approaches to understanding these homotopy types when M is of dimension at least 5. We will start with the *classical* approach, which studies these spaces via *surgery theory*, and later explain a more *modern* approach, which focuses on the study of *embedding spaces* as models for the homotopy theoretic difference between moduli spaces of different manifolds.

1.3. The classical approach to $\mathrm{BDiff}(M)$.

In this subsection, we explain the *surgery-pseudoisotopy program* to study $\mathrm{BDiff}(M)$ for compact manifolds M. This program attempts to study $\mathrm{BDiff}(M)$ in terms of a *larger* moduli space of *block manifolds* $\mathrm{B}\widetilde{\mathrm{Diff}}(M)$. The homotopy type of this space is characterized by the property that it classifies *smooth block bundles* over simplicial complexes with fiber M–a weakening of the notion of smooth fiber bundle–up to concordance, see [ER14, Defn. 2.2] for a model of this space. Since a smooth fiber bundle over a simplicial complex is a block bundle, we have a map $\mathrm{BDiff}(M) \to \mathrm{B}\widetilde{\mathrm{Diff}}(M)$ classifying the universal smooth fiber bundle over $\mathrm{BDiff}(M)$ seen as a block bundle. The *surgery-pseudoisotopy program* studies $\mathrm{BDiff}(M)$ in the following two step procedure:

1. **Study** $\mathrm{B}\widetilde{\mathrm{Diff}}(M)$. The moduli space of block manifolds $\mathrm{B}\widetilde{\mathrm{Diff}}(M)$ can be completely studied using *surgery theory* in the following sense. One can construct a map $\mathrm{B}\widetilde{\mathrm{Diff}}(M) \to \mathrm{BAut}(M, \partial M)$[4], where the target is the classifying space of the group-like monoid of homotopy automorphisms of M which fix ∂M setwise. The homotopy fiber of this map agrees with a collection of components of the *surgery structure space* $\mathcal{S}(M, \partial M)$, which is the central object of investigation in surgery theory. The main theorem of surgery theory (see [WR99, Qui70]) states that, for $d = \dim M \geq 5$, there exists a fiber sequence

$$\mathcal{S}(M, \partial M) \to \mathrm{N}(M) \to L^q(M, \partial M)$$

   where $\mathrm{N}(M) \simeq \mathrm{Map}(M, \mathrm{G}/\mathrm{O})$ is the space of normal invariants and $L^q(M, \partial M)$ is the *quadratic L-theory space*, whose homotopy groups compute *Wall's quadratic L-groups* [WR99]. Both of these spaces are amenable to techniques in (stable) homotopy theory.

2. **Study** $\big(\widetilde{\mathrm{Diff}}/\mathrm{Diff}\big)(M)$. The homotopy fiber of the map from $\mathrm{BDiff}(M)$ to $\mathrm{B}\widetilde{\mathrm{Diff}}(M)$, denoted by $\big(\widetilde{\mathrm{Diff}}/\mathrm{Diff}\big)(M)$, can be described in a range using

[4]This follows from the fact that a block fibration with fiber M admits a *canonical strictification* as a fibration, or more precisely, that the map $\mathrm{BAut}(M, \partial M) \to \mathrm{B}\widetilde{\mathrm{Aut}}(M, \partial M)$ is a weak equivalence. See e.g. [Kra22, Lemma A.3].

Waldhausen's algebraic K-theory of spaces [Wal78] via the seminal work of Weiss and Williams [WW88]. If M is closed, there exists a map

$$\big(\widetilde{\mathrm{Diff}}/\mathrm{Diff}\big)(M) \to \Omega^{\infty+1}\mathrm{Wh}^{\mathrm{Diff},s}(M)_{C_2}$$

where $\mathrm{Wh}^{\mathrm{Diff},s}(M)$ is the 2-connective cover of the cofiber of the map of spectra $\Sigma^\infty_+ M \to A(M)$ and $(-)_{C_2}$ denotes the homotopy orbits of a certain C_2-action on this spectrum. Weiss and Williams proved that this map is roughly $\phi(M)$-connected, where $\phi(M)$ is the *concordance stable range* of M, which is bounded below roughly by $\frac{d}{3}$ by *Igusa's stability theorem* [Igu88]. Once again, the space on the right hand side is amenable to stable homotopy theoretic methods, and thus tractable.

This program has lead to many computations of the homotopy types of moduli spaces of manifolds (see e.g. [FH78, Bur79, BL82]). However, this method is inherently limited by the concordance stable range, which has been proved to be bounded above (roughly) by d for a large class of manifolds (see [GKK23]). In the next section, we will explain a more modern program which does not have this limitation, and which was first put forward by Weiss [Wei22].

1.4. A modern approach.

In this subsection, we expand on an emerging approach to the study of moduli spaces of manifolds, which relies on the *parameterized isotopy extension theorem*, which we now explain. First, let us define a small variation of the moduli spaces studied above. Let M be a compact manifold and denote by $\mathrm{BDiff}_\partial(M)$ the classifying space of the group of diffeomorphisms of M that are the identity on the boundary–which can be modelled similarly as above and classifies smooth fiber bundles with fiber M together with a trivialization of the induced bundle with fiber ∂M.

Let now $M_0, M_1 \subset M$ be two compact codimension 0 submanifolds such that $M = M_0 \cup M_1$ and whose interiors are disjoint. Then a version of the *parameterized isotopy extension theorem* states that the following sequence

$$\mathrm{Emb}_{\partial_1}(M_1, M)_{\iota_1} \to \mathrm{BDiff}_\partial(M_0) \to \mathrm{BDiff}_\partial(M)_{\iota_0}$$

is a fiber sequence. Here, the leftmost space is the space of embeddings of M_1 into M which are fixed on $\partial_1 \coloneqq \partial M \cap M_1$ and are isotopic to the inclusion of M_1 relative to ∂_1. The rightmost space is the classifying space of the subgroup $\mathrm{Diff}_\partial(M)_{\iota_0} \subset \mathrm{Diff}_\partial(M)$ of those diffeomorphisms isotopic to a diffeomorphism which is the identity in M_1. The right map is given by extension along the identity and the left map (roughly) takes an embedding $e : M_1 \hookrightarrow M$ isotopic to the inclusion of M_1 to its complement $M - e(M_1)^\circ$, which is abstractly diffeomorphic to M_0 and thus a point in $\mathrm{BDiff}_\partial(M_0)$. There is also a version for manifolds with boundary (without fixing the boundary), which we will return to later.

This result tells us that embedding spaces encode the *homotopy theoretic difference* between moduli spaces of different manifolds. Observe also that, if we wish to use the fiber sequence above–even if we are only interested in understanding $\mathrm{BDiff}(M)$ of a closed manifold–we are forced to consider the diffeomorphism groups of manifolds with boundary fixing the boundary, since, if M is connected, a compact codimension 0 submanifold of M which is neither empty nor M, will have non-empty boundary. This result also suggests a strategy to understand the moduli space of M (resp. of M_0) by first understanding the moduli space of a submanifold M_0 (resp. of the ambient manifold M), and second study the space of embeddings of the complement M_1 of M_0 on the ambient manifold M. This strategy can then be *iterated* in order to understand moduli spaces of more and more complicated manifolds.

1.4.1. Embedding spaces and Goodwillie-Weiss embedding calculus. We will now address the following question: *why are spaces of smooth embeddings between compact manifolds any easier than diffeomorphism groups?*

The study of embeddings of smooth manifolds can be traced back to the work of Whitney [Whi44], who investigated the problem of embedding a compact manifold into the euclidean space $\mathbf{R}^N$. In loc.cit, Whitney first provided a recipe to embed every d-dimensional compact manifold M into $\mathbf{R}^{2d+1}$, and then showed that, in fact, this embedding can be chosen to completely lie in $\mathbf{R}^{2d}$. To do so, the author provided a method to eliminate generic self-intersection points of maps from M to $\mathbf{R}^{2d}$ up to homotopy, now known as the *Whitney trick*. This was the starting point for the study of obstructions for a map of smooth manifolds $M^d \to N^{d+k}$ to be homotopic to an embedding, see e.g. [HS62, Hud67, Hud72, Dax72, HQ74].

In more homotopy-theoretic terms, the collection of these works describes the homotopy type of the homotopy fiber of the map

$$\mathrm{Emb}(M,N) \to \mathrm{Map}(M,N)$$

in a certain range, in terms of algebraic invariants. Almost half a century after Whitney's original work, the study of this map was revolutionized by the work of Goodwillie and Weiss [Goo90, Wei99, GW99]. One of their main insights is that this map is only the first stage in a tower of approximations, which lies (almost) completely within the realm of homotopy theory, and is faithful in describing the full homotopy type of $\mathrm{Emb}(M,N)$, when the codimension of M in N is at least 3. More precisely, the authors defined the *Goodwillie–Weiss embedding tower*

$$\begin{array}{ccc} & & T_\infty \operatorname{Emb}_K(M,N) \\ & \overset{\eta_\infty}{\nearrow} & \downarrow \\ & & \vdots \\ & & \downarrow \\ & \overset{\eta_2}{\nearrow} & T_2 \operatorname{Emb}_K(M,N) \\ & & \downarrow \\ \operatorname{Emb}_K(M,N) & \xrightarrow{\eta_1} & T_1 \operatorname{Emb}_K(M,N) \end{array}$$

where $K \subseteq M$ is a submanifold and $\operatorname{Emb}_K(M,N)$ is the space of embeddings extending a fixed embedding $K \hookrightarrow N$, which enjoys the following properties:

① If $\dim N - 3 \geq \operatorname{hdim}(M,K)$, then the map η_k is $f(k)$-connected for all $k \geq 1$, for a certain divergent function $f : \mathbf{Z} \to \mathbf{Z}$, where $\operatorname{hdim}(M,K)$ is the handle dimension of M relative to $K \subseteq M$. In particular, the map η_∞ is an equivalence in this case. This was established by Goodwillie–Klein–Weiss [GK15, Wei99].

② The map η_1 is equivalent to the derivative map $\operatorname{Emb}_K(M,N) \to \operatorname{Map}_K(TM,TN)$ to the space of bundle maps from TM to TN.

③ The homotopy fiber of the map $T_k \operatorname{Emb}_K(M,N) \to T_{k-1} \operatorname{Emb}_K(M,N)$ is equivalent to a certain relative section space of a fibration over $\operatorname{Emb}(\underline{k}, M)/\Sigma_k$, whose fiber is equivalent to the total homotopy fiber of the cube $S \subseteq \underline{k} \mapsto \operatorname{Emb}(S,N)$.

Roughly speaking, the properties above reduce the problem of understanding of the homotopy type of $\operatorname{Emb}(M,N)$ to understanding various configuration spaces (and maps between them) of the manifolds M and N. For example, the latter can be approached by inducting on the number of points, and studying the homotopy types of the results of removing finite sets of points to M and N. Hence, this approach successfully reduces a purely geometric problem in terms of homotopy theory of manifolds. This is thus one possible justification for embedding spaces being more amenable to methods in homotopy theory than diffeomorphism groups, which this approach fails to describe, since the dimension assumption above is not satisfied.

1.4.2. The work of Galatius and Randal-Williams. We will now expand on the work of Galatius and Randal-Williams [GR14, GR17b] on *stable* moduli spaces of even-dimensional manifolds, which first provides a starting point for implementing the approach mentioned in this section for even-dimensional manifolds, and second serves as the main inspiration for the results of this thesis. In this subsubsection, we will state the main result of loc.cit. and expand on some immediate consequences.

Let $n \geq 1$ be an integer. Let M be a compact, connected, smooth manifold of dimension $2n$ with non-empty boundary ∂M. Let $W_{g,1}$ be the manifold obtained by taking a g-fold boundary connected sum of $W_{1,1} \coloneqq (S^n \times S^n) \backslash \mathrm{int}(D^{2n})$. By choosing a $(2n-1)$-disc in ∂M, we define the boundary connected sum $M \natural W_{g,1}$ along this disc. We define the space $\mathrm{BDiff}_\partial(M_\infty)$ as the homotopy colimit over the stabilization maps $\mathrm{BDiff}_\partial(M \natural W_{g,1}) \to \mathrm{BDiff}_\partial(M \natural W_{g+1,1})$, induced by boundary connected sum with $W_{1,1}$. A parameterized version of the Pontrjagin–Thom construction yields a map

$$\mathrm{BDiff}_\partial(M_\infty) \to (\Omega^\infty \mathbf{MT}\theta_M)_{\mathrm{Aut}_{\partial M}(\theta_M)} \tag{1}$$

to a purely homotopy-theoretic object, which we now describe: Consider a *Moore–Postnikov n-factorization* $\tau_M = \theta_M \circ l_M : M \to B \to \mathrm{BO}(2n)$ of a tangent classifier of M. In other words, the map l_M is an n-connected cofibration and the map θ_M is an n-coconnected fibration. The spectrum $\mathbf{MT}\theta_M$ denotes the *Thom spectrum* of the formal inverse of θ_M. The latter admits an action by the group-like monoid $\mathrm{Aut}_{\partial M}(\theta_M)$ of self weak equivalences of B over $\mathrm{BO}(2n)$ and under ∂M. The target of the map (1) is the homotopy orbits of $\Omega^\infty \mathbf{MT}\theta_M$ by this action.

Theorem ([GR17b]). *Let M be a compact, connected, smooth $2n$-manifold with non-empty boundary. For a Moore-Postnikov n-factorization $\tau_M = \theta_M \circ l_M : M \to B \to \mathrm{BO}(2n)$, the parameterized Pontrjagin-Thom map*

$$\mathrm{BDiff}_\partial(M_\infty) \to (\Omega^\infty \mathbf{MT}\theta_M)_{\mathrm{Aut}_{\partial M}(\theta_M)}$$

is acyclic onto the path component hit, that is, it induces an isomorphism on homology groups with any local coefficient system pulled back from the target.

This result is another instance of describing a geometric object, namely, the ring of *characteristic classes* of certain manifold bundles, by a purely homotopy theoretic object. Observe also that M is a submanifold of M_g for all $g \geq 1$. Moreover, one can show that the handle dimension of the complement $M_g - M^\circ \cong W_{g,1}$ of this submanifold is n. Hence, if $n \geq 3$, the embedding space $\mathrm{Emb}_{\partial_1}(W_{g,1}, M_g)$ can be completely described via embedding calculus. In the next subsubsection, we will see some consequences of applying the parameterized isotopy extension theorem for the submanifold $M \subset M_g$, by combining the work of Galatius and Randal-Williams with Goodwillie-Weiss embedding calculus.

We finish this subsubsection by explaining some direct corollaries of the theorem above. As mentioned before, the target of the parameterized Pontrjagin-Thom map above lies completely within the realm of homotopy theory. The following facts simplify the computation of this homotopy type in certain cases:

- If the pair $(M, \partial M)$ is c-connected for some $c \leq n-1$, then the monoid $\mathrm{Aut}_{\partial M}(\theta_M)$ is $(n-c-1)$-coconnected. In particular, if $(M, \partial M)$ is $(n-1)$-connected, then $\mathrm{Aut}_{\partial M}(\theta_M)$ is contractible. See [GR19, Lemma 12.4.15]

This fact effectively bounds the complexity of the homotopy type of $\mathrm{Aut}_{\partial M}(\theta_M)$. In turn, as any mapping space of spaces, its homotopy type is amenable to homotopy theoretic techniques.

- The reduced rational cohomology of $\Omega^\infty \mathbf{MT}\theta_M$ is a free graded commutative algebra on the reduced rational spectrum cohomology $\mathbf{MT}\theta_M$, which in turn, by the Thom isomorphim, is isomorphic to the cohomology $\mathrm{H}^*(B; \mathbf{Q})$, after shifting degrees by $-2n$.

From these two facts, we obtain the following corollary.

Corollary ([GR14])**.** *There is an isomorphism of graded commutative algebras*

$$\mathbf{Q}[\kappa_b : b \in \mathcal{B}, |b| > 2n] \xrightarrow{\cong} \lim_{g} \mathrm{H}^*(\mathrm{BDiff}_\partial(W_{g,1}); \mathbf{Q})$$

where $\mathcal{B}$ is the set of monomials on the Pontrjagin classes p_i for $i = \lceil \frac{n+1}{4} \rceil, \cdots, n-1$, and the degree $|\kappa_b| = |b| - 2n$.

This isomorphism recovers the Madsen and Weiss' resolution of the Mumford conjecture [MW07], as mentioned above, by taking $n = 1$. Moreover, the classes κ_b are often called the *Miller–Morita–Mumford classes*, as they generalize the characteristic classes for surface bundles with the same name. These classes can be defined geometrically via a fiberwise integration map.

1.4.3. Applications. In this subsubsection, we expand on some successful applications of the approach explained in this section. As a first example, let us come back to the *Smale conjecture* and diffeomorphism groups of spheres. As mentioned before, the action of $\mathrm{O}(d+1)$ on the sphere S^d induces a map $\mathrm{BO}(d+1) \to \mathrm{BDiff}(S^d)$. One can show that the homotopy fiber of this map is equivalent to the loop space of the classifying space $\mathrm{BDiff}_\partial(D^d)$, and hence, the moduli space of d-dimensional discs encodes the *failure* of the Smale conjecture in dimension d.

Using the classical approach, Farrell and Hsiang [FH78] proved that there is an isomorphism

$$\pi_*(\mathrm{BDiff}_\partial(D^d)) \otimes \mathbf{Q} \xrightarrow{\cong} \begin{cases} 0 & \text{if } d \text{ is even} \\ \mathrm{K}_{*+1}(\mathbf{Z}) \otimes \mathbf{Q} & \text{if } d \text{ is odd} \end{cases}$$

in the concordance stable range $* \leq \phi(D^d)$, where $\mathrm{K}_*(\mathbf{Z})$ are the algebraic K-theory groups of $\mathbf{Z}$, which are completely computed by the work of Borel. In recent years, there has been significant progress in extending this computation exploiting the approach discussed in this section, such as [Ran16, Kra22, KR24, KR21]. We will now explain some of these advances, starting with the even-dimensional case. Kupers and Randal-Williams [KR24] combined the works mentioned above to vastly extend this computation for d even. The authors exploited a delooping of the parameterized isotopy extension theorem

$$\mathrm{BDiff}_\partial(D^{2n}) \longrightarrow \mathrm{BDiff}_\partial(W_{g,1}) \longrightarrow \mathrm{BEmb}^{\cong}_{\partial_1}(W_{g,1}, W_{g,1})$$

induced by the inclusion $D^{2n} \subset W_{g,1}$, by combining extensive computations of the embedding calculus tower for the embedding space $\mathrm{Emb}^{\cong}_{\partial_1}(W_{g,1}, W_{g,1})$, together with a computation of the homotopy groups of the total space, using the work of Galatius and Randal-Williams. The main result of loc.cit. is the following.

Theorem ([KR24]). *Assume $2n \geq 6$, then the rational homotopy group*

$$\pi_k(\mathrm{BDiff}_\partial(D^{2n})) \otimes \mathbf{Q}$$

(i) vanishes if $k \leq 2n - 2$,
(ii) vanishes if k does not belong to

$$\mathcal{G} := \bigcup_{r \geq 2} [2r(n-2) - 1, 2r(n-1) + 1]$$

and is not congruent to $2n - 1 \bmod 4$,
(iii) is isomorphic to $\mathbf{Q}$ if $k \geq 2n - 1$, $k \equiv 2n - 1 \bmod 4$, and does not belong to $\mathcal{G}$.

This result not only completely computes these groups in a range, but shows that, except from the copies of $\mathbf{Q}$ from (iii), these homotopy groups are concentrated in *bands*, that is, vanish away from $\mathcal{G}$. A similar phenomenon is conjectured to hold for odd dimensions, see [Ran23].

Odd-dimensional discs. This thesis will focus on odd-dimensional manifolds and its moduli spaces. In the case of a disc, Krannich [Kra22] and later Krannich and Randal-Williams [KR21] were able to provide a comparably extensive computation of the rational homotopy groups of $\mathrm{BDiff}_\partial(D^{2n+1})$, using the work of Botvinnik and Perlmutter [BP17], which is the main motivation for this thesis, and to which we will return in the next subsection.

To explain their approach, we define the group of concordance diffeomorphisms $\mathrm{C}(M)$ for a manifold M to be the group of diffeomorphisms of $M \times [0, 1]$ which are the identity on $M \times \{0\} \cup \partial M \times [0, 1]$. There is a fiber sequence

$$\mathrm{BDiff}_\partial(M \times [0, 1]) \to \mathrm{BC}(M) \to \mathrm{BDiff}_\partial(M)_r$$

induced by restricting a concordance diffeomorphism to $M\times\{1\}$, where $\mathrm{BDiff}_\partial(M)_r$ is the classifying space of the collection of components of $\mathrm{Diff}_\partial(M)$ hit by the restriction map $\mathrm{C}(M) \to \mathrm{Diff}_\partial(M)$. In [Kra22, KR21], the authors compute the homotopy groups of $\mathrm{BC}(D^{2n})$ and provide a new computation of $\mathrm{BDiff}_\partial(D^{2n})$ via different methods. This allows them to compute the rational homotopy groups of $\mathrm{BDiff}_\partial(D^{2n+1})$, using the fiber sequence above. The classifying space $\mathrm{BC}(D^{2n})$ fits in a parameterized isotopy extension fiber sequence

$$\mathrm{BC}(D^{2n}) \to \mathrm{BDiff}_D(V_g) \to \mathrm{BEmb}^{\cong}_{\partial_1}((V_g, W_{g,1}), (V_g, W_{g,1}))$$

where the total space denotes the classifying space of the diffeomorphism group of $V_g = (S^n \times D^{n+1})^{\natural g}$ fixed pointwise on a codimension 0 disc D in its boundary, and the base space is a delooping of the space of self-embeddings of the pair $(V_g, W_{g,1})$. The aforementioned work of Botvinnik and Perlmutter [BP17] provides a description of the stable homology of the total space, see Section 1.5. The (delooping of the) space of embeddings on the right can be approach with similar methods as in the even-dimensional case, even though, in [KR21], the authors use more direct methods. The main result of loc.cit. is the following.

Theorem ([KR21])**.** *In degrees $k \leq 3n-8$, there is an isomorphism*

$$\pi_k(\mathrm{BDiff}_\partial(D^{2n+1})) \otimes \mathbf{Q} \cong \mathrm{K}_{k+1}(\mathbf{Z}) \otimes \mathbf{Q} \oplus \begin{cases} \mathbf{Q} & k \equiv 2n-2 \bmod 4,\ k \geq 2n-2 \\ 0 & \textit{otherwise.} \end{cases}$$

The additional copy of $\mathbf{Q}$ in this computation comes from the same source as the one in (iii) from the theorem above, namely it is detected by the *Pontrjagin-Weiss classes*, coming from the homotopy groups of $\mathrm{Top} \coloneqq \mathrm{colim}_d \mathrm{Homeo}(\mathbf{R}^d)$, using the topological invariance of Pontrjagin classes, established by Novikov [Nov65]. More precisely, these classes witness the failure of certain expected relations between topological Pontrjagin classes, which was discovered by Weiss [Wei22], where the author put forward the first instance of the *modern approach*, which has been explained at length in the past subsubsections. These classes are now known to be non trivial in all degrees for dimension ≥ 6, as a result of [KK25]. On another note, it is worth mentioning that there have been substantial results in the detecting certain non-trivial characteristic classes of smooth D^d-bundles, especially put forward by Watanabe, which exploit completely different methods which lie outside the scope of this thesis, see e.g. [Wat08, Wat09, Wat18, BW23, BW26].

Why discs are important. In the beginning of this subsubsection, we motivated the study of $\mathrm{BDiff}_\partial(D^d)$ by explaining that it measures the failure of the Smale conjecture in high dimensions. However, there is an arguably better argument to study the homotopy type of this space. To explain this, we will explain two instances of a phenomenon in this field, as brought forward by Krannich and Kupers [KK24b]: 2*-tangential invariance in moduli spaces of manifolds.*

We say that an embedding of codimension 0 compact d-dimensional smooth manifolds $e : M \hookrightarrow N$ is an *equivalence on tangential* 2*-types*, if there is a map $\theta : B \to \mathrm{BO}(d)$ such that the tangent classifier $\tau_N : N \to \mathrm{BO}(d)$ admits a 2-connected lift up to homotopy $\ell_N : N \to B$ along θ such that $e^*\ell_N : M \to \mathrm{BO}(d)$ is also 2-connected. The first instance of 2*-tangential invariance* is the following. Consider the fiber sequence induced by the submanifold $e(M) \subset N$

$$\mathrm{Emb}_{\partial_1}(C, N)_\iota \to \mathrm{BDiff}_\partial(M) \to \mathrm{BDiff}_\partial(N)_{\iota_0}$$

where C is $N - e(M)^\circ$. If, additionally, the embedding $(\partial N \cap C) \times [0,1] \hookrightarrow C$ is an equivalence on tangential 2-types, then the embedding calculus tower for the left-most space converges–this is [KK24a, Thm. B] which generalizes the convergence result of Goodwillie–Klein–Weiss mentioned above. Thus, roughly speaking, under some 2-tangential equivalence assumptions, the difference between the moduli spaces $\mathrm{BDiff}_\partial(M)$ and $\mathrm{BDiff}_\partial(N)$ is (almost) completely homotopy-theoretic.

Returning back to the disc, given a simply connected spin d-manifold M, then any embedding $D^d \hookrightarrow M$ is an equivalence of tangential 2-types, and hence, by the discussion above, the homotopy theoretic difference between $\mathrm{BDiff}_\partial(D^d)$–heavily understood by the work above mentioned–and $\mathrm{BDiff}_\partial(M)$ is faithfully described by the embedding calculus tower.

The second instance of 2*-tangential invariance* is the following. In the previous section, we mentioned that the range in which the *surgery-pseudoisotopy program* faithfully describes $\mathrm{BDiff}_\partial(M)$ is given by the concordance stable range $\phi(M)$. This range is defined as follows. There is a concordance stabilization map

$$s_M : \mathrm{C}(M) \to \mathrm{C}(M \times [0,1]),$$

see e.g. [Igu88, Chapter II , §1], and $\phi(M)$ is defined as the minimum integer k such that $s_{M\times[0,1]^m}$ is k-connected for all $m \geq 0$. Recently, Goodwillie–Krannich–Kupers [GKK23] showed that, if $e : M \hookrightarrow N$ is an equivalence on tangential 2-types, then the induced map $\mathrm{hofib}(s_M) \to \mathrm{hofib}(s_N)$ is roughly $\frac{3}{2}d$-connected. Roughly speaking, from this result one concludes that, below the range around $\frac{3}{2}d$, the concordance stable range $\phi(M)$ only depends on the tangential 2-type.

In the case of discs, as a consequence of the rational computations of [KR24, KR21], one is able to show that for any simply connected spin manifold M of dimension at least 10, the rational concordance stable range $\phi^{\mathbb{Q}}(M)$–given by the same definition but replacing k-connected by rationally k-connected–is exactly $d-4$, see [GKK23]. From that, one sees that the rational homotopy type of

$\mathrm{BDiff}_\partial(M)$ can be described via the classical approach in a larger range than initially expected.

To conclude this subsection, we have seen that the modern approach, coming out of the vision of Weiss [Wei22], has been successful in understanding the homotopy types of moduli spaces of manifolds, especially for simply connected spin manifolds[5]. This relied heavily on the study of embedding spaces and descriptions of stable moduli spaces of manifolds, in full generality for even dimensional manifolds by Galatius and Randal-Williams [GR17b], and in the specific case of odd-dimensional discs by Botvinnik and Perlmutter [BP17]. The main goal of this thesis is to extend the latter work to a larger class of odd-dimensional manifolds, thus (in principle) enabling the pursuit of this program for all odd-dimensional manifolds. We expand on this goal and make the result of this thesis precise in the next subsection.

1.5. The thesis problem: odd dimensions.

As mentioned before, this thesis concerns moduli spaces of odd-dimensional manifolds. More concretely, the goal of this thesis is to answer the following question.

Question. Is there an odd-dimensional analogue of Galatius and Randal-Williams' work on stable moduli spaces of even-dimensional manifolds [GR14, GR17b]?

As alluded to above, Botvinnik and Perlmutter [BP17] established a description of the homology of a certain stable moduli space in the context of odd dimensions, which we now explain. Let $n \geq 1$ and $g \geq 0$ be integers, V_g denote the g-fold boundary connected sum of $S^n \times D^{n+1}$, and fix a codimension 0 disc $D \subset \partial V_g$. As in the even-dimensional case, there are maps

$$\mathrm{BDiff}_D(V_g) \to \mathrm{BDiff}_D(V_{g+1})$$

induced by the inclusion of $V_g \subset V_{g+1}$. Denote the homotopy colimit of this system of maps by $\mathrm{BDiff}_D(V_\infty)$. Let $\tau_{>n}\,\mathrm{BO}(2n+1)$ be n-th connected cover of $\mathrm{BO}(2n+1)$ and $\Sigma^\infty_+\tau_{>n}\,\mathrm{BO}(2n+1)$ be its suspension spectrum. The main theorem of [BP17] is the following description of the homology of this space.

Theorem ([BP17])**.** *Assume that $n \geq 4$, then there exists a map*

$$\mathrm{BDiff}_D(V_\infty) \to \Omega^\infty\Sigma^\infty_+\tau_{>n}\,\mathrm{BO}(2n+1)$$

which is acyclic onto the path component hit.

[5]We refer also to [Ran23] for a great exposition of these results.

This result is, once again, a complete description of the purely geometric object on the left in terms of a purely homotopy theoretic object on the right. As mentioned before, this result was crucial to Krannich and Randal-Williams' computation of rational homotopy groups of $\mathrm{BDiff}_\partial(D^{2n+1})$ away from the concordance stable range [Kra22, KR21]. In addition, this result was earlier also exploited by Kupers [Kup19] to deduce finiteness properties of moduli spaces of 2-connected odd-dimensional manifolds.

We can see this result as an analogue of Galatius and Randal-Williams' work (see result above) for the specific case $M = D^{2n}$, namely a computation of the homology of the stable moduli space $\mathrm{hocolim}_g \mathrm{BDiff}_\partial(W_{g,1})$. In light of this analogy, we are motivated to ask the following revised question.

Thesis question. Is there a description of the homology of moduli spaces of certain odd-dimensional manifolds after stabilization by V_g, which is an analogue of Galatius and Randal-Williams' work [GR14, GR17b] in its fullest generality, and recovers the work of Botvinnik and Perlmutter [BP17] for the case of a disc?

The answer to this question is yes. We now move to state the main output of our thesis, which justifies this claim. We explain our setting in analogy with the even-dimensional setting of Galatius and Randal-Williams from Section 1.4.2. Our main objects of study are compact smooth manifold triads $(N, \partial^h N, \partial^v N)$ of dimension $d = 2n+1$, that is, compact smooth d-dimensional manifolds N with boundary, along with a decomposition $\partial N = \partial^h N \cup \partial^v N$ of their boundary into the union of two codimension 0 compact submanifolds $\partial^h N$ and $\partial^v N$ such that $\partial(\partial^h N) = \partial^h N \cap \partial^v N = \partial(\partial^v N)$. We assume that $\partial^{hv} N := \partial^h N \cap \partial^v N$ is non-empty and that N is connected. The role of the space $\mathrm{BDiff}_\partial(M)$ above is now played by the classifying space $\mathrm{BDiff}_{\partial^v}(N)$ of the topological group of diffeomorphisms of N which are the identity on $\partial^v N$ but are allowed to move on $\partial^h N$.

In this context, the role of $W_{g,1}$ above is now played by the triad $(V_g, W_{g,1}, D^{2n})$ where V_g is the manifold obtained by g-fold boundary connected sum of $S^n \times D^{n+1}$ and D^{2n} is a codimension 0 disc in ∂V_g: The boundary of V_g is the g-fold connected sum of $S^n \times S^n$, so we are using the fact that $W_{g,1}$ is diffeomorphic to $\sharp_g(S^n \times S^n)$ after removing an open $2n$-disc. Similarly, by picking a $2n$-dimensional half-disc in $\partial^v N$ whose equatorial $(2n-1)$-disc lies in $\partial^h N$, boundary connected sum yields the triad $(N \natural V_g, \partial^h N \natural W_{g,1}, \partial^v N)$. We define the space $\mathrm{BDiff}_{\partial^v}(N_\infty)$ to be the homotopy colimit over the maps $\mathrm{BDiff}_{\partial^v}(N \natural V_g) \to \mathrm{BDiff}_{\partial^v}(N \natural V_{g+1})$ induced by boundary connected sum with $(V_1, W_{1,1}, D^{2n})$. This is the analog of the left hand side in (1). We proceed by defining the analog of right hand side in our context.

We say that a factorization $(X, X') \to (Y, Y') \to (Z, Z')$ of a map of pairs $(X, X') \to (Z, Z')$ is a *Moore–Postnikov k-factorization of pairs* if $X' \to Y' \to Z'$ is a Moore–Postnikov k-factorization and $X \cup_{X'} Y' \to Y \to Z$ a Moore–Postnikov

$(k+1)$-factorization, both in the absolute sense. Such a factorization always exists (see Lemma 7.1.2 below), so we may fix a Moore–Postnikov n-factorization of pairs $\Theta_N \circ l_N : (N, \partial^h N) \to (B, B^\partial) \to (\mathrm{BO}(2n+1), \mathrm{BO}(2n))$ of a compatible pair of tangent classifiers $\tau_N : (N, \partial^h N) \to (\mathrm{BO}(2n+1), \mathrm{BO}(2n))$. We denote by $\mathbf{MT}\Theta_N$ the cofiber of the canonical map of Thom spectra $\Sigma^{-1}\mathbf{MT}\theta_{\partial^h N} \to \mathbf{MT}\theta_N$ induced by the map of pairs $\Theta_N = (\theta_N, \theta_{\partial^h N})$. The group-like monoid $\mathrm{Aut}_{\partial^v N}(\Theta_N)$ of self equivalences of the pair (B, B^∂) under $(\partial^v N, \partial^{hv} N)$ and over $(\mathrm{BO}(2n+1), \mathrm{BO}(2n))$ acts on this spectrum. A parameterized version of the Pontrjagin–Thom construction for pairs yields a map

$$\mathrm{BDiff}_{\partial^v}(N_\infty) \longrightarrow (\Omega^\infty \mathbf{MT}\Theta_N)_{\mathrm{Aut}_{\partial^v N}(\Theta_N)}\,. \tag{2}$$

We now state our main result, which can be seen as an analog for odd-dimensional triads to the one of Galatius and Randal-Williams described above.

Theorem A. *Let $(N, \partial^h N, \partial^v N)$ be a compact smooth $(2n+1)$-dimensional manifold triad where N is connected, $(N, \partial^h N)$ is 1-connected, $\partial^{hv} N \neq \emptyset$, and $n \geq 3$. For a Moore–Postnikov n-factorization $\tau_N = \Theta_N \circ l_N$, the map*

$$\mathrm{BDiff}_{\partial^v}(N_\infty) \longrightarrow (\Omega^\infty \mathbf{MT}\Theta_N)_{\mathrm{Aut}_{\partial^v N}(\Theta_N)}$$

is acyclic onto the path component it hits.

Remark. We highlight the following points:

- (*i*) Specialized to $n \geq 4$ and to the triad $(D^{2n+1}, D^{2n}_+, D^{2n}_-)$, Theorem A recovers the result by Botvinnik and Perlmutter [BP17, Thm. A] above mentioned, where $D^{2n}_\pm$ are the upper and lower hemispheres of $\partial D^{2n+1} = S^{2n}$ (see Section 7.3). Our work can be seen as a generalization of this result to a more general class of triads (and incidentally to $n \geq 3$–although the extension of [BP17, Thm. A] to dimension 7 was also previously obtained by Krannich and Kupers). However, our methods are much closer in spirit to the proof of the analogous result by Galatius and Randal-Williams. In addition, we also consider a more general notion of tangential structures for triads (see (*iii*) below) than the one considered in [BP17], and thus also recover the more general result Theorem A* of loc.cit.
- (*ii*) The right-hand side of (2) simplifies considerably in many cases. For example, if N is an h-cobordism, that is, if the inclusions $\partial^h N \hookrightarrow N \hookleftarrow \partial^v N$ are equivalences, then the right-hand side is equivalent to $\Omega^\infty\Sigma^\infty_+ B$ for B fitting in a Moore–Postnikov n-factorization $N \to B \to \mathrm{BO}(2n+1)$ (see Section 7.3). In particular, the rational cohomology ring of any of its path components is isomorphic to the free graded commutative $\mathbb{Q}$-algebra $\mathbb{Q}[\tilde{\mathrm{H}}^*(B;\mathbb{Q})]$, so the same holds for $\mathrm{BDiff}_{\partial^v}(N_\infty)$.

(*iii*) Galatius and Randal-Williams' result above is a corollary of a more general result on the moduli space $\mathrm{BDiff}^{\theta}_{\partial}(M)$ of θ-structures on M: For a map $\theta : B \to \mathrm{BO}(2n)$, a θ-structure on M is a fiberwise isomorphism $TM \to \theta^*\gamma_{2n}$, where γ_{2n} is the universal $2n$-vector bundle. In this spirit, we also prove a more general result, stated as Theorem A* in Section 7, that identifies the homology of stabilization of the moduli space $\mathrm{BDiff}^{\Theta}_{\partial^v}(N)$ of Θ-structures on N : For a map of pairs $\Theta = (\theta, \theta^{\partial}) : (B, B^{\partial}) \to (\mathrm{BO}(2n+1), \mathrm{BO}(2n))$, a Θ-structure on $(N, \partial^h N, \partial^v N)$ is a pair of compatible θ- and θ^{∂}-structures on N and $\partial^h N$, respectively (which includes compatible orientations, spin structures and framings, for example). Theorem A follows by applying this result to $\Theta = \mathrm{id}$. See Section 7.3 for other examples of Θ-structures and their corresponding moduli spaces.

(*iv*) Our result is not only analogous to the one of Galatius–Randal-Williams, but also compatible with it in the sense that there exists a commutative square

$$\begin{array}{ccc} \mathrm{BDiff}_{\partial^v}(N_\infty) & \xrightarrow{(2)} & \left(\Omega^\infty \mathbf{MT}\Theta_N\right)_{\mathrm{Aut}_{\partial^v N}(\Theta_N)} \\ \downarrow & & \downarrow \\ \mathrm{BDiff}_{\partial}(\partial^h N_\infty) & \xrightarrow{(1)} & \left(\Omega^\infty \mathbf{MT}\theta_{\partial^h N}\right)_{\mathrm{Aut}_{\partial^{hv} N}(\theta_{\partial^h N})} \end{array}$$

where the left vertical map is given by restricting diffeomorphisms and the right vertical map is induced by the canonical map $\mathbf{MT}\Theta_N \to \mathbf{MT}\theta_{\partial^h N}$ and the restriction of self-equivalences of (B, B^{∂}) to B^{∂}.

1.5.1. Stable moduli spaces of even-dimensional nullbordisms. Our main result Theorem A will be deduced from a stronger, although more technical, group completion-type result on moduli spaces of nullbordisms, which we now move to explain. A comparable deduction happens in the work of Galatius and Randal-Williams [GR14, GR17b], so we explain this case first. One of the most useful objects exploited in loc.cit. is the *cobordism category* Cob_θ *of θ-manifolds* for a map of spaces $\theta : B \to \mathrm{BO}(d)$. This is a category internal to topological spaces: Roughly speaking, the objects of Cob_θ are $(d-1)$-dimensional closed smooth manifolds P together with a vector bundle map $l_P : TP \oplus \varepsilon^1 \to \theta^*\gamma_d$ which is a fiberwise isomorphism, where γ_d is the universal d-dimensional vector bundle over $\mathrm{BO}(d)$. A morphism $W : P \rightsquigarrow Q$ in Cob_θ is a d-dimensional compact smooth manifold W such that $\partial W = P \sqcup Q$, *i.e.* a *cobordism*, together with a vector bundle map $l_W : TW \to \theta^*\gamma_d$, extending l_P and l_Q on its boundary (see [GR14, Defn. 2.6] for more details). We highlight the following facts about Cob_θ :

① Given a compact smooth d-dimensional manifold M as above, the space of interest $\mathrm{BDiff}_{\partial}(M)$ is a path component of the morphism space $\mathsf{Cob}(\emptyset, \partial M)$, where $\mathsf{Cob} \coloneqq \mathsf{Cob}_{\mathrm{id}}$ for $\mathrm{id} : \mathrm{BO}(d) \to \mathrm{BO}(d)$ the identity map. More generally, for any map $\theta : B \to \mathrm{BO}(d)$, the space $\mathrm{BDiff}^{\theta}_{\partial}(M)$ as in *(iii)* above is a union of path components of $\mathsf{Cob}_{\theta}(\emptyset, \partial M)$.

② For any map $\theta : B \to \mathrm{BO}(d)$, Galatius, Madsen, Tillmann, and Weiss [Gal+09] proved that the classifying space $\mathrm{B}\mathsf{Cob}_{\theta}$ is equivalent to $\Omega^{\infty}\,\mathbf{MT}\theta$, where $\mathbf{MT}\theta$ is the Thom spectrum of the formal inverse $-\theta$.

③ When $d = 2n$, there exists a *distinguished* morphism $H_P : P \rightsquigarrow P$ for every non-empty connected object $P \in \mathsf{Cob}_{\theta}$ such that the post-composition map $H_P \circ (-) : \mathsf{Cob}_{\theta}(\emptyset, P) \to \mathsf{Cob}_{\theta}(\emptyset, P)$ of H_P extends the stabilization map $\mathrm{BDiff}_{\partial}(M \natural W_{g,1}) \to \mathrm{BDiff}_{\partial}(M \natural W_{g+1,1})$ defined above along the inclusion from ①, in case $P = \partial M$ and $\theta = \mathrm{id}$. The underlying cobordism of H_P is the connected sum $(P \times [0, 1]) \sharp W_1$, where $W_1 \coloneqq S^n \times S^n = \partial V_1$. Moreover, the space $\mathrm{BDiff}_{\partial}(M_{\infty})$ is a path component of the space

$$\mathsf{Cob}(\emptyset, P)[(H_P)^{-1}] \coloneqq \operatorname{hocolim}\left(\mathsf{Cob}(\emptyset, P) \overset{H_P\circ(-)}{\to} \mathsf{Cob}(\emptyset, P) \to \cdots\right)$$

given by "inverting" the action of this morphism, when $P = \partial M$.

In [GR17b], the authors reduce the proof that the map (1) is acyclic, using standard homotopy-theoretic methods together with ①–③, to the following statement about the spaces $\mathsf{Cob}_{\theta}(\emptyset, P)[(H_P)^{-1}]$ for a general map $\theta : B \to \mathrm{BO}(2n)$: The canonical map

$$\mathsf{Cob}_{\theta,n}(\emptyset, P)[(H_P)^{-1}] \to \Omega_{[\emptyset,P]}\mathrm{B}\mathsf{Cob}_{\theta} \tag{3}$$

taking a morphism to the path represented by it, is acyclic. Here, the source is the subspace of $\mathsf{Cob}_{\theta}(\emptyset, P)$ of those morphisms $W : \emptyset \rightsquigarrow P$ such that $l_W : W \to B$ is n-connected. Most of the work in [GR17b] resides in the proof of the latter statement, for which the authors use a mixture of parameterized surgery (as developed in [GR14]), a variant of the classical group completion theorem, and geometric/surgery-theoretic arguments.

Interlude: Stable diffeomorphism classification. In contrast to the acyclicity of the map (1), the statement that (3) is acyclic already has content on H_0: It recovers a classical result of Kreck [Kre99] on the classification of even-dimensional manifolds up to stable diffeomorphism, which we briefly recall. Let P be a closed $(2n-1)$-dimensional manifold, we say that two compact connected $2n$-dimensional manifolds M_0 and M_1 with boundary, together with identifications of ∂M_i with P for $i = 0, 1$, are *stable diffeomorphic relative to* P if there exists an integer $g \geq 0$ such that $M_0 \sharp W_g$ and $M_1 \sharp W_g$ are diffeomorphic relative to P. In [Kre99, Thm. 2], Kreck established a necessary and sufficient condition for two such manifolds to be stably diffeomorphic, which is purely bordism-theoretic. It turns out that the acyclicity of (3) on H_0 implies this result (see Remark 6.4.2 for more details).

Therefore, one can view the latter as a *family version* of Kreck's result. Below we will explain that a similar perspective applies in our setting.

1.5.2. Stable moduli spaces of odd-dimensional triad nullbordisms. We move now to explaining the aforementioned stronger but more technical, group completion type result in our context. We also consider a category internal to topological spaces, namely the *cobordism category* $\mathsf{Cob}_\Theta^\partial$ *of* Θ*-manifolds with boundary* for a map of pairs of spaces $\Theta = (\theta, \theta^\partial) : (B, B^\partial) \to (\mathrm{BO}(d), \mathrm{BO}(d-1))$: The objects of this category are $(d-1)$-dimensional compact smooth manifolds with boundary $(P, \partial P)$ together with a map of pairs of vector bundles $l_P : (TP \oplus \varepsilon^1, T\partial P \oplus \varepsilon^1) \to (\theta^* \gamma_d, \theta^\partial \gamma_{d-1})$ (see *(iii)*). A morphism $W : P \rightsquigarrow Q$ is a *triad cobordism* between P and Q, that is, a d-dimensional compact manifold triad $(W, \partial^h W, \partial^v W)$ as above where $\partial^v W = P \sqcup Q$, along with a map of pairs of vector bundles $l_W : (TW, T\partial^h W) \to (\theta^* \gamma_d, (\theta^\partial)^* \gamma_{d-1})$ compatible with l_P and l_Q (see Definition 3.1.9 below for more details). We highlight now facts about this category, which are analogous to (1)–(3) from above:

- (1^∂) Given a compact smooth d-dimensional manifold triad $(N, \partial^h N, \partial^v N)$ as above, then $\mathrm{BDiff}_{\partial^v}(N)$ is a path component of the morphism space $\mathsf{Cob}^\partial(\emptyset, \partial^v N)$. Again, the moduli space $\mathrm{BDiff}^\Theta_{\partial^v}(N)$ defined in *(iii)* is a union of path components of $\mathsf{Cob}_\Theta^\partial(\emptyset, \partial^v N)$ for any map of pairs $\Theta : (B, B^\partial) \to (\mathrm{BO}(d), \mathrm{BO}(d-1))$.
- (2^∂) Genauer [Gen11] showed that $\mathrm{B}\mathsf{Cob}_\Theta^\partial$ is equivalent to $\Omega^\infty \mathbf{MT}\Theta$, where $\mathbf{MT}\Theta$ is the cofiber of the canonical map of Thom spectra $\Sigma^{-1}\mathbf{MT}\theta^\partial \to \mathbf{MT}\theta$ induced by $\Theta = (\theta, \theta^\partial)$.
- (3^∂) When $d = 2n+1$, there exists a *distinguished* morphism $H_P : P \rightsquigarrow P$ for every object $P \in \mathsf{Cob}_\Theta^\partial$ with non-empty connected boundary such that the left action map $H_P \circ (-) : \mathsf{Cob}_\Theta^\partial(\emptyset, P) \to \mathsf{Cob}_\Theta^\partial(\emptyset, P)$ extends the stabilization map $\mathrm{BDiff}_{\partial^v}(N \natural V_g) \to \mathrm{BDiff}_{\partial^v}(N \natural V_{g+1})$ defined above along the inclusion in (1^∂), in case $P = \partial^v N$ and $\Theta = \mathrm{id}$. The underlying cobordism of H_P is the boundary connected sum $(P \times [0,1]) \natural V_1$, performed away from $P \times \{0,1\}$. Moreover, the space $\mathrm{BDiff}_{\partial^v}(N_\infty)$ is a path component of

$$\mathsf{Cob}^\partial(\emptyset, P)[(H_P)^{-1}] \coloneqq \operatorname{hocolim}\left(\mathsf{Cob}^\partial(\emptyset, P) \overset{H_P \circ (-)}{\to} \mathsf{Cob}^\partial(\emptyset, P) \to \cdots\right)$$

  given by "inverting" the action of this morphism, when $P = \partial^v N$.

In much of the same spirit as in the even-dimensional case, we reduce the proof of our main result (Theorem A) to the following theorem. For $k \geq -2$ an integer, we say that a map of pairs of spaces $f : (X, X') \to (Y, Y')$ is *strongly k-connected* if $f|_{X'} : X' \to Y'$ is k-connected and the induced map $X \cup_{X'} Y' \to Y$ is $(k+1)$-connected, where the source denotes the homotopy pushout.

Theorem B (Theorem 6.0.2). *Fix $n \geq 3$ and a map of pairs $\Theta : (B, B^\partial) \longrightarrow (\mathrm{BO}(2n+1), \mathrm{BO}(2n))$. If the pair (B, B^∂) is 1-connected and B^∂ is path-connected, then for any $P \in \mathsf{Cob}^\partial_\Theta$ such that $\partial P \neq \emptyset$ and $\mathsf{Cob}^\partial_{\Theta,n}(\emptyset, P) \neq \emptyset$, the canonical map*

$$\mathsf{Cob}^\partial_{\Theta,n}(\emptyset, P)[(H_P)^{-1}] \longrightarrow \Omega_{[\emptyset,P]}\mathrm{B}\mathsf{Cob}^\partial_\Theta$$

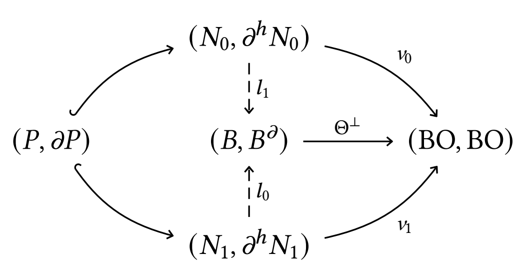
is acyclic. Here, $\mathsf{Cob}^\partial_{\Theta,n}(\emptyset, P)$ denotes the subspace of $\mathsf{Cob}^\partial_\Theta(\emptyset, P)$ of those morphisms $W : \emptyset \rightsquigarrow P$ such that $l_W : (W, \partial^h W) \longrightarrow (B, B^\partial)$ is strongly n-connected.

This result is the core of the present work. Its proof follows the overall strategy in the even-dimensional case of Galatius and Randal-Williams [GR14, GR17b], but requires a number of generalizations (of varying difficulty) of geometric/surgery-theoretic techniques for even-dimensional manifolds with boundary (some classical and some developed in loc.cit.) to the context of odd-dimensional manifold triads.

Stable diffeomorphism classification of triads. As in the even-dimensional case, Theorem B already has content on H_0: It yields an analog to Kreck's result [Kre99] for odd-dimensional triads. Since this result has not been (to our knowledge) previously established and might be of independent interest, we explain it now in full detail: Let $n \geq 3$ be an integer, P be a compact $2n$-manifold with boundary, and $(N_i, \partial^h N_i, \partial^v N_i)$ be $(2n+1)$-dimensional manifold triads for $i = 0, 1$, together with an identification of $\partial^v N_i$ with P. Assume also that $\partial^h N_i$ is connected for $i = 0, 1$. We say that:

(*a*) The triads N_0 and N_1 are *stably diffeomorphic* if there exists an integer $g \geq 0$ such that the triads $(N_0 \natural V_g, \partial^h N_0 \sharp W_g, \partial^v N_0)$ and $(N_1 \natural V_g, \partial^h N_1 \sharp W_g, \partial^v N_1)$ are diffeomorphic as triads relative to $(P, \partial P)$ (where the boundary connected sum is performed away from $\partial^v N_i$).

(*b*) The triads N_0 and N_1 have *the same stable normal n-type* if there exist a map of pairs $\Theta^\perp : (B, B^\partial) \longrightarrow (\mathrm{BO}, \mathrm{BO})$ and strongly n-connected maps of pairs $l_i : (N_i, \partial^h N_i) \longrightarrow (B, B^\partial)$ for $i = 0, 1$ making the following diagram commute up to homotopy



where ν_i is the classifying map for the stable normal bundle $(\nu_{N_i}, \nu_{\partial^h N_i})$ of the pair $(N_i, \partial^h N_i)$.

(*c*) Given a map of pairs $\Theta^\perp : (B, B^\partial) \to (\mathrm{BO}, \mathrm{BO})$, the triads N_0 and N_1 *admit bordant* $\Theta^\perp$*-smoothings* if there exist choices of strongly n-connected maps $l_i : (N_i, \partial^h N_i) \to (B, B^\partial)$ making the diagram above commute, such that the bordism class of manifolds with boundary

$$[N_0 \cup_P (-N_1)] \in \Omega_{2n+1}^{\Theta^\perp}$$

vanishes, where $\Omega_k^{\Theta^\perp}$ is the relative bordism group of k-dimensional compact manifolds with boundary $(M, \partial M)$ together with a lift $l_M : (M, \partial M) \to (B, B^\partial)$ of its stable normal bundle along $\Theta^\perp$. The bordism relation is triad bordism (in the sense of Definition 2.3.1 but $\partial_0 = \emptyset$) together with lifts of the stable normal bundle of the bordism compatible with the given lifts in its vertical boundary. Here, $N_0 \cup_P (-N_1)$ denotes the manifold obtained by the gluing of N_0 and N_1 along the identifications with P, together with the map $l_0 \cup_P (-l_1)$ to (B, B^∂), where $-l_1$ is the restriction of the map $l_1 \oplus \varepsilon^1 : (N_1 \times [0,1], \partial^h N_1 \times [0,1]) \to (B, B^\partial)$ to $N_1 \times \{1\}$, using an inwards-pointing trivialization of the normal bundle of $N_1 \times \{1\}$ in $N_1 \times [0,1]$ (see [Sto15, p. 17]).

The classical Pontrjagin-Thom construction for relative bordisms yields an isomorphism $\Omega_k^{\Theta^\perp} \cong \pi_k(\mathbf{M}\Theta^\perp)$, where $\mathbf{M}\Theta^\perp$ is the cofiber of the map of Thom spectra $\mathbf{M}(\theta^\perp)^\partial \to \mathbf{M}\theta^\perp$ induced by $\Theta^\perp = (\theta^\perp, (\theta^\perp)^\partial)$ (see e.g. [Sto15, p. 25] or [Lau00, Thm. 3.1.5]). The following result is deduced from Theorem B below in Section 6.4.

Corollary C (Stable diffeomorphism classification)**.** *Let $n \geq 3$ be an integer, P be a compact $2n$-manifold with boundary, and $(N_i, \partial^h N_i, \partial^v N_i)$ be $(2n+1)$-dimensional manifold triads for $i = 0, 1$, together with an identification of $\partial^v N_i$ with P. Assume:*

(i) The manifold $\partial^h N_i$ is non-empty and connected for $i = 0, 1$.
(ii) The pair $(N_i, \partial^h N_i)$ is 1-connected for $i = 0, 1$.

Then the triads N_0 and N_1 are stably diffeomorphic relative to P if and only if

1. *their relative Euler characteristics agree, i.e. $\chi(N_0, \partial^h N_0) = \chi(N_1, \partial^h N_1)$, and*
2. *they have the same stable normal n-type $\Theta^\perp$ and admit bordant $\Theta^\perp$-smoothings.*

On the assumptions. We conclude this introduction by commenting on the assumptions of our results:

(*i*) It would be interesting to see whether Theorem B, and consequently Theorem A and Corollary C, also holds in the excluded dimensions 3 and 5. One starting point could be to see whether Corollary C holds in these dimensions, possibly via more direct methods.

(*ii*) We believe that the 1-connectivity conditions in our results are necessary. For example, it seems plausible that one can find an explicit counterexample of Corollary C when the 1-connectivity condition is dropped.

Structure of this thesis. Section 2 introduces various categories of pairs that are used throughout, while introducing the notion of *strong connectivity* of maps of pairs of spaces. We recall the work of Borodzik, Némethi, and Ranicki on handle decompositions for triads [BNR16] and prove a *geometrical connectivity* statement in Proposition 2.3.17 relating strong connectivity with existence of certain handle decompositions. In Section 3, we recall the definition of the *cobordism category of manifolds with boundary* from [Gen11, BP17, Ste21] and variants thereof. The main result of this section is Theorem 3.1.10, where we prove a generalization of results of [BP17] for this larger class of tangential structures, which is an analog of [GR14, Thm. 3.1]. In Section 4 and Section 5 resides the core of the proof of Theorem B and Theorem A^*, where we prove a "stable stability" statement, which is analogous to [GR17b, Thm. 2.15]. The main geometrical input for this proof is a modification result for embeddings of discs into manifolds with boundary via surgery. We deduce Theorem B and a generalization thereof from the stable stability phenomena, in Section 6. In Section 7, we deduce Theorem A^* from Theorem B, by exploiting the notion of Moore–Postnikov factorizations of pairs. We finish by presenting further simplifications of Theorem A^* in the context of certain examples in Section 7.3.

2. Pairs of spaces and manifolds.

This section introduces and studies properties of pairs of spaces, manifolds, and vector bundles. We start by introducing the notion of *strong connectivity* of maps of pairs of spaces and prove closure properties of this notion. We introduce manifold pairs and an appropriate notion of *handle decompositions* on them and prove that this notion behaves well with the notion of strong connectivity.

2.1. Strong connectivity.

Throughout this paper, by *space* we mean a compactly generated space in the sense of [Str09, Defn. 1.1]. We consider the category of spaces **Top** as a model category with the Quillen model structure (see [Hir09, Defn. 7.10.6]), which turns it into a self-enriched cartesian closed model category (see [Hir09, Notation 7.10.2] and [Str09, Prop. 2.12][6]). In this model category, we denote the homotopy pushout of a diagram $X \leftarrow Y \rightarrow Z$ as $X \cup_Y Z$. When the map $Y \rightarrow X$ is a Hurewicz cofibration, the homotopy pushout is equivalent to the strict pushout (this follows from [DI04, Thm. A.7], as homotopy colimits in the Quillen and the Strom model structure agree). Thus, we can model $X \cup_Y Z$ using the double mapping cylinder construction (see [MV15, Defn. 3.6.3]). When $Y \rightarrow X$ is a Hurewicz cofibration, we implicitly denote the strict pushout by $X \cup_Y Z$. In particular, we do so in the case where $Y \rightarrow X$ is the inclusion of a compact smooth submanifold of a smooth manifold. We denote by $X \times_Y Z$ the homotopy pullback of a diagram $X \rightarrow Y \leftarrow Z$. Similarly, we often model this space using the path space construction (see [MV15, Defn. 3.2.4]).

By a *pair of spaces*, we mean a map of spaces $A' \rightarrow A$. If the map is implicit, we denote it by (A, A'). A map of pairs $f = (\alpha, \alpha') : (A, A') \rightarrow (B, B')$ is the data of two maps of spaces $\alpha : A \rightarrow B$ and $\alpha' : A' \rightarrow B'$ such that the square

$$\begin{array}{ccc} A' & \xrightarrow{\alpha'} & B' \\ \downarrow & & \downarrow \\ A & \xrightarrow{\alpha} & B \end{array}$$

commutes. Given a map of pairs $f = (\alpha, \alpha') : (A, A') \rightarrow (B, B')$, we often denote the individual maps α and α' by $f|_A$ and by $f|_{A'}$, respectively. Given two pairs (A, A') and (B, B'), we denote the mapping space of pairs by $\mathrm{Map}((A, A'), (B, B'))$ seen as subspace of $\mathrm{Map}(A', B') \times \mathrm{Map}(A, B)$. Given a map $f' : A' \rightarrow B'$, we denote the (strict) fiber of the projection $\mathrm{Map}((A, A'), (B, B')) \rightarrow \mathrm{Map}(A', B')$ at f' by $\mathrm{Map}_{f'}(A, B)$ or $\mathrm{Map}_{A'}(A, B)$ if f' is implicit.

[6]We point out a typo in this statement. The conclusion should read "the category of compactly generated spaces is cartesian closed". In fact, the statement about compactly generated weak Hausdorff spaces is given in [Str09, Prop. 2.24].

2.1.1. Connectivity for pairs. We introduce a convenient notion of connectivity in the category of pairs of spaces, which will be heavily used throughout the entire paper. Recall that for $k \geq -1$, a map of spaces is k-connected if all its homotopy fibers are $(k-1)$-connected. In particular, any map is (-1)-connected.

Definition 2.1.1. Let $k \geq -1$. A map of pairs of spaces $f : (A, A') \to (B, B')$ is *strongly k-connected* if $f|_{A'}$ is k-connected and the map $A \cup_{A'} B' \to B$ is $(k+1)$-connected.

Remark 2.1.2. We point out that this notion implies that the individual maps $f|_{A'}$ and $f|_A$ are k-connected, the first by definition and the second by the following argument. The second condition in Definition 2.1.1 is, by definition, the condition that the square

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

is homotopy $(k+1)$-cocartesian, in the sense of [MV15, Defn. 3.7.1]. We conclude that $f|_A$ is k-connected by applying [MV15, Prop. 3.7.13.2.(a)] to this square, using the fact that $f|_{A'}$ is k-connected.

Lemma 2.1.3 (2 out of 3)**.** *Let $k \geq -1$. Let $f : (A, A') \to (B, B')$ and $g : (B, B') \to (C, C')$ be maps of pairs, then the following holds:*

1. *If f and g are strongly k-connected, then $g \circ f$ is strongly k-connected.*
2. *If f is strongly $(k-1)$-connected and $g \circ f$ are strongly k-connected, then g is strongly k-connected.*
3. *If g is strongly k-connected and $g \circ f$ are strongly $(k-1)$-connected, then f is strongly $(k-1)$-connected, provided $k \geq 2$.*

Proof. Properties 1 and 2 follow from classical properties of connectivity of maps of spaces (see [MV15, Prop. 2.6.15]) and [MV15, Prop. 3.7.26]. For property 3, the same properties of connectivity of maps of space imply that the map $f|_{A'}$ is $(k-1)$-connected. It remains to prove that $A \cup_{A'} B' \to B$ is k-connected. Consider the following homotopy pushout square of spaces

$$\begin{array}{ccc} A \cup_{A'} B' & \longrightarrow & B \\ \downarrow & & \downarrow \\ A \cup_{A'} C' & \longrightarrow & B \cup_{B'} C' \end{array}$$

given by identifying the bottom left corner with $A \cup_{A'} B' \cup_{B'} C'$. By hypothesis, the map $A \cup_{A'} C' \to C$ is k-connected. Since the map $B \cup_{B'} C' \to C$ is $(k+1)$-connected, it follows that $A \cup_{A'} C' \to B \cup_{B'} C'$ is k-connected. On the other hand,

$A \cup_{A'} B' \to A \cup_{A'} C'$ is k-connected, since $g|_{B'} : B' \to C'$ is k-connected. Since $k \geq 2$, the former induces an isomorphism on fundamental groupoids and thus, the map $A \cup_{A'} B' \to B$ is k-connected by [Wal71, Prop. A]. □

Given a commutative square of maps of pairs, we call it a *homotopy pushout square* if both the square consisting of the maps between the targets of the pairs and the square of the maps between the sources of the pairs are homotopy pushout squares of spaces. Analogously, a *homotopy pullback square* is a commutative square that restricts to a homotopy pullback square of spaces both in the targets and the sources.

Lemma 2.1.4 (Pushouts)**.** *Let $k \geq -1$. Assume that the square*

$$\begin{array}{ccc} (A, A') & \xrightarrow{\alpha} & (C, C') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ (B, B') & \longrightarrow & (D, D'). \end{array}$$

is a homotopy pushout square of pairs. If the map f is strongly k-connected, then so is g. Conversely, if g is strongly k-connected, then so is f, provided both $\alpha|_A$ and $\alpha|_{A'}$ induce isomorphisms on fundamental groupoids.

Proof. We start by observing that, if $f|_{A'}$ is k-connected, then so is $g|_{C'}$, as k-connected maps are closed under taking homotopy pushouts. Conversely, if $g|_{C'}$ is k-connected and $\alpha|_{A'}$ induces an isomprhism of fundamental groupoids, then $f|_{A'}$ is k-connected by [Wal71, Prop. A]. We proceed by showing that if f is strongly k-connected, than $C \cup_{C'} D' \to D$ is $(k+1)$-connected, and thus g is strongly k-connected, and its converse under the additional assumptions stated in the result. To do so, we prove now the subclaim that the following left square in

$$\begin{array}{ccc} A \cup_{A'} B' & \longrightarrow & C \cup_{C'} D' \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array} \qquad \begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ A \cup_{A'} B' & \longrightarrow & C \cup_{C'} D' \end{array}$$

is a homotopy pushout square. By the pasting lemma for homotopy pushout squares [MV15, Prop. 3.7.26], it suffices to prove that the right square above is a homotopy pushout square. Consider the following diagrams

$$\begin{array}{ccccc} A' & \longrightarrow & A & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ B' & \longrightarrow & A \cup_{A'} B' & \longrightarrow & C \cup_{C'} D' \end{array} \qquad \begin{array}{ccccc} A' & \longrightarrow & C' & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ B' & \longrightarrow & D' & \longrightarrow & C \cup_{C'} D' \end{array} .$$

Since the left square in the left diagram is a homotopy pushout by definition, it suffices to prove that the pasted square is a homotopy pushout. For that, we observe that it can be factored as in the right diagram above as the pasting of two homotopy pushout squares. This finishes the proof of the subclaim. We conclude that, if $A \cup_{A'} B' \to B$ is $(k+1)$-connected, then so is $C \cup_{C'} D' \to D$, thus establishing that g is strongly k-connected.

For the converse claim (under the additional assumption that $\alpha|_A$ and $\alpha|'_A$ induce fundamental groupoid isomorphisms), we have that $A \cup_{A'} B' \to C \cup_{C'} D'$ is a fundamental groupoid isomorphism, since $\alpha|_A$ also is. By [Wal71, Prop A], the map $A \cup_{A'} B' \to B$ is $(k+1)$-connected, as $C \cup_{C'} D' \to D$ also is. Hence, f is strongly k-connected. ◻

For $k \geq -1$, we say that a pair of spaces (A, A') is k-connected if the map of spaces $A' \to A$ is k-connected.

Lemma 2.1.5. *Let $f : (A, A') \to (B, B')$ be a map of pairs such that $f|_{A'}$ is k-connected.*

(i) If (A, A') is 0-connected and f is strongly k-connected, then the map $f_* : \pi_i(A, A', x) \to \pi_i(B, B', f(x))$ is an isomorphism for $i < k$ and a surjection for $i = k$ for all basepoints $x \in A'$.
(ii) If (A, A') is 1-connected, then f is strongly k-connected if and only if the map $f_* : \pi_i(A, A', x) \to \pi_i(B, B', f(x))$ is an isomorphism for $i < k+1$ and a surjection for $i = k+1$ for all basepoints $x \in A'$.

Proof. We start by proving the first claim. Note that the condition on f_* is equivalent to the square

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

being homotopy $(k-1)$-cartesian (in the sense of [MV15, Defn. 3.3.1]), while the assumption is equivalent to it being homotopy $(k+1)$-cocartesian. We know that $A' \to B'$ is k-connected and $A' \to A$ is 0-connected. Thus, the Blakers-Massey theorem implies that homotopy $(k+1)$-cocartesianness of the square above implies homotopy $(k-1)$-cartesianness (see [MV15, Thm. 4.2.3]). This finishes the proof of the claim. The proof of the second claim follows similarly by observing that, since (A, A') is 1-connected, homotopy $(k+1)$-cocartesianness of the square above is equivalent to homotopy k-cartesianness by the Blakers-Massey theorem and its dual (see [MV15, Thm. 4.2.3/4]). ◻

2.1.2. Triad homotopy groups.

It will be convenient to have an analog of the relative homotopy groups in the context of maps of pairs of spaces, which we start by recalling. When studying maps of spaces $f : X \to Y$, it is convenient to define the *relative homotopy groups* $\pi_k(Y, X, x)$ as π_0 of the mapping space of maps $(D^k, \partial D^k) \to (Y, X)$ in the category of pairs of pointed spaces $\mathbf{Top}_*^{[1]}$, where X is pointed at x and Y at $f(x)$. These agree with the homotopy groups at x of the homotopy fiber of f at $f(x)$ in one degree lower. We now consider an analogous concept for maps of pairs and record some facts about them. Let $D^k_+ \subset \mathbf{R}^k$ be the subset of those tuples $x = (x_1, \cdots, x_k)$ such that $||x|| \leq 1$ and $x_1 \geq 0$. Let $\partial_0 D^k_+$ be the subset of D^k_+ of those points where $x_1 = 0$ and $\partial_1 D^k_+$ of those points with norm 1. Let $\partial_{01} D^k$ be the intersection $\partial_0 D^k_+ \cap \partial_1 D^k_+$.

Definition 2.1.6 (Triad homotopy groups)**.** Let $k \geq 2$ and $f : (A, A') \to (B, B')$ a map of pairs and $a \in A'$. The k-th *triad homotopy group* $\pi_k(B, B', A, a)$ is the set of path components of the space of squares

$$\begin{array}{ccc} (\partial_0 D^k_+, \partial_{01} D^k_+) & \xrightarrow{\alpha} & (A, A') \\ \downarrow & & \downarrow f \\ (D^k_+, \partial_1 D^k_+) & \xrightarrow{\beta} & (B, B') \end{array}$$

seen as the mapping space in the category of functors from $[1] \times [1]$ to $\mathbf{Top}_*$ from the left vertical map of pairs to the rightmost vertical map of pairs.

Warning 2.1.7. We note that, although A' is not in the notation, these groups depend heavily on A'. We believe that adding A' in the notation would make it too cumbersome, so we choose to drop it.

See [BM49] for this definition when $A, B' \subset B$ and $A' = A \cap B$. As in this special case, when $k = 2$ this is just a set with no extra structure. When $k \geq 3$, one can define a group structure given by stacking and for $k \geq 4$, this group structure is abelian. We can map this group (or set) to the $(k-2)$-nd homotopy group of the total homotopy fiber $\mathrm{tofib}_a(f) \coloneqq \mathrm{hofib}_a(A' \to B' \times_B A)$, modelled using the path-space construction, in the following way. By unwrapping the definition, we see that $\mathrm{tofib}_a(f)$ is the space of tuples $(x, \gamma_0, \gamma_1, \Gamma) \in A' \times (B')^I \times A^I \times B^{I \times I}$ such that γ_0 is a path in B' from $f(x)$ to $f(a)$, γ_1 is a path in A from $\iota_A(x)$ to $\iota_A(a)$, where $\iota_A : A' \to A$, and Γ is a homotopy from $\iota_{B'} \circ \gamma_0$ to $f \circ \gamma_1$, where $\iota_B : B' \to B$ (see also [MV15, Prop. 5.5.8]). We consider this space as based in a, that is, at the tuple $(a, \mathrm{const}_{f(a)}, \mathrm{const}_{\iota_A(a)}, \mathrm{const}_{f(\iota(a))})$. We can define a map $\pi_k(B, B', A, a) \to \pi_{k-2}(\mathrm{tofib}_a(f), a)$ taking the class $[\beta, \alpha]$ of a diagram as above to the tuple $(\alpha|_{\partial_{01} D^k_+}, \beta|_{\partial_1 D^k_+}, \alpha, \beta)$ by looking at α as a map from $\frac{\partial_{01} D^k_+ \times I}{\partial_{01} D^k_+ \times \{1\}}$ and all the other maps analogously. It is not difficult to see that this map is a bijection for $k = 2$ and a group isomorphism for $k \geq 3$.

These triad homotopy groups are related to relative homotopy groups in the following way. We have a sequence of maps

$$\cdots \to \pi_k(A, A', a) \xrightarrow{f_*} \pi_k(B, B', f(a)) \xrightarrow{p} \pi_k(B, B', A, a) \xrightarrow{\delta} \pi_{k-1}(A, A', a) \to \cdots$$

where p takes the class of a map $(D^k, \partial D^k, *) \to (B, B', f(a))$ to the diagram where α is constant, and δ sends $[\beta, \alpha]$ to $[\alpha]$. This sequence is compatible with the identification of the relative and triad homotopy groups with the homotopy groups of the fibers of ι_A and ι_B and of the total homotopy fibers. We conclude that this sequence is exact. We finish with the following lemma, which will be useful in constructing maps of pairs.

Lemma 2.1.8. *Let $k \geq 2$ and $i \leq k$ be integers. Assume (A, A') is 0-connected and $f : (A, A') \to (B, B')$ is a strongly k-connected map of pairs. Then every commutative square*

$$\begin{array}{ccc} (\partial_0 D^i_+, \partial_{01} D^i_+) & \longrightarrow & (A, A') \\ \downarrow & \overset{g}{\nearrow} & \downarrow f \\ (D^i_+, \partial_1 D^i_+) & \longrightarrow & (B, B') \end{array}$$

admits a filler g making the upper triangle commute strictly and the bottom one up to homotopy of pairs. If (A, A') is 1-connected, then the same conclusion holds for $i \leq k + 1$.

Proof. This follows by combining Lemma 2.1.5 with the long exact sequence from above to deduce that $\pi_i(B, B', A)$ vanishes for every basepoint. From a nullhomotopy of the homotopy class of this square, we can construct such a filler g. We leave this check to the reader. □

2.2. Pairs of vector bundles.

We consider now an appropriate notion of a vector bundle over a pair of spaces. The following is essentially a recollection of [Ste21, Section 4]. For the entirety of this paper, a *bundle map* between vector bundles will always be assumed to be fiberwise injective.

Definition 2.2.1. A *pair of vector bundles* (ξ, ξ') is a map of pairs $(\pi, \pi') : (\xi, \xi') \to (X, X')$ such that both π and π' are vector bundles and the map $\xi' \to \xi$ is a bundle map. Given two vector bundle pairs $\pi_X : (\xi, \xi') \to (X, X')$ and $\pi_Y : (\eta, \eta') \to (Y, Y')$, a *map of pairs of vector bundles* $f : (\xi, \xi') \to (\eta, \eta')$ is the data of a commutative square of pairs of spaces

$$\begin{array}{ccc} (\xi, \xi') & \xrightarrow{(f, f')} & (\eta, \eta') \\ \downarrow \pi_X & & \downarrow \pi_Y \\ (X, X') & \longrightarrow & (Y, Y') \end{array}$$

such that f and f' are bundle maps.

Definition 2.2.2. Let (ξ, ξ') be a pair of vector bundles. A *collar* on (ξ, ξ') is a bundle map $c : \xi' \oplus \varepsilon^1 \to \xi$ extending $\xi' \to \xi$ which is a fiberwise isomorphism, where ε^1 denotes the trivial vector bundle of rank 1. We call the pair $((\xi, \xi'), c)$ a *collared pair of vector bundles.* A map $f : (\xi, \xi') \to (\eta, \eta')$ between collared pairs of vector bundles is *collared* if the resulting square commutes:

$$\begin{array}{ccc} \xi' \oplus \varepsilon^1 & \xrightarrow{f' \oplus \mathrm{id}} & \eta' \oplus \varepsilon^1 \\ \downarrow & & \downarrow \\ \xi & \xrightarrow{\quad f \quad} & \eta \end{array} .$$

We denote the space of collared bundle maps by $\mathrm{Bun}^{\mathrm{col}}(\xi, \eta)$, topologized with the compact-open topology. Given a collared pair of vector bundles (ξ, ξ') over (X, X') and $\iota : (A, A') \to (X, X')$, we have a collared pair of vector bundles $((\iota|_A)^*\xi, (\iota|_{A'})^*\xi')$ over (A, A') defined by pulling back the individual bundles and the collar. Given a collared map $f|_A : ((\iota|_A)^*\xi, (\iota|_{A'})^*\xi') \to (\eta, \eta')$ of collared vector bundle pairs, let $\mathrm{Bun}^{\mathrm{col}}_A(\xi, \eta)$ be the subspace of collared bundle maps extending $f|_A$.

Remark 2.2.3. Unravelling the definition, one sees that we have a strict pullback square of spaces

$$\begin{array}{ccc} \mathrm{Bun}^{\mathrm{col}}_A(\xi, \eta) & \longrightarrow & \mathrm{Bun}_A(\xi, \eta) \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{A'}(\xi', \eta') & \longrightarrow & \mathrm{Bun}_{A'}(\xi' \oplus \varepsilon^1, \eta) \end{array}$$

where $\mathrm{Bun}_A(\xi, \eta)$ denotes the space of bundle maps of vector bundles $\xi \to \eta$ extending a map $f|_A : \xi|_A \to \eta$. The left vertical map takes a collared bundle map (f, f') and maps it to f'. The right vertical map takes a bundle map $f : \xi \to \eta$ to the composite $f|_{X'} = f \circ c : \eta' \oplus \varepsilon^1 \to \eta|_{X'} \to \eta$.

Throughout the entirety of this paper, we model the classifying space $\mathrm{BO}(d)$ of the d-dimensional orthogonal group as the colimit of the Grassmanian manifolds $\mathrm{Gr}_d(d + k)$ of d-planes along the standard inclusions $\mathrm{Gr}_d(d + k) \to \mathrm{Gr}_d(d + k + 1)$. Let γ_d be the universal vector bundle of dimension d over $\mathrm{BO}(d)$ given by the colimit over k of the space of pairs (V, v) of $V \in \mathrm{Gr}_d(d + k)$ a d-plane in $\mathbf{R}^{d+k}$ and $v \in V$. There is a pair of vector bundles $\gamma_d : (\gamma_d, \gamma_{d-1}) \to (\mathrm{BO}(d), \mathrm{BO}(d - 1))$, whose underlying map of vector bundles is the usual stabilization map. This vector bundle pair admits a preferred collar given by the canonical map $\gamma_{d-1} \oplus \varepsilon^1 \to \gamma_d$. Given a map $\Theta = (\theta, \theta') : (B, B') \to (\mathrm{BO}(d), \mathrm{BO}(d - 1))$ of pairs, then the vector bundle $\Theta^*\gamma_d : (\theta^*\gamma_d, \theta'^*\gamma_{d-1}) \to (B, B')$ admits a collar given by pulling back the collar along Θ. Given a collared vector bundle pair (ξ, ξ') of dimension $l < d$, we have a preferred collar for $(\xi, \xi') \oplus \varepsilon^{d-l} \coloneqq (\xi \oplus \varepsilon^{d-l}, \xi' \oplus \varepsilon^{d-l})$ induced by the collar $\xi' \oplus \varepsilon^1 \to \xi$.

Remark 2.2.4 (Universality of γ)**.** Let $((\xi, \xi'), c)$ be a collared d-dimensional vector bundle pair over (X, X'), $(A, A') \to (X, X')$ be a cofibration and $f|_A : (\xi|_A, \xi'|_{A'}) \to (\gamma_d, \gamma_{d-1})$ a collared bundle map, then the space $\mathrm{Bun}^{\mathrm{col}}_A(\xi, \gamma_d)$ is weakly contractible. This follows by first observing that the square in Remark 2.2.3 is a homotopy pullback square, since the right-hand vertical map is a fibration of spaces (see the proof of Proposition A.5). If we take $(\eta, \eta') = (\gamma_d, \gamma_{d-1})$, then all of the spaces except the one in the initial vertex are weakly contractible, by universality of γ_d and γ_{d-1}. We conclude that the space in the initial vertex, that is $\mathrm{Bun}^{\mathrm{col}}_A(\xi, \gamma_d)$, is weakly contractible.

2.3. Pairs of manifolds and triads.

Throughout this paper, we follow the convention that *manifolds* are smooth and allowed to have boundary, without further mention. A d-dimensional *manifold pair* is a pair (M, K) where M is a d-dimensional compact manifold and K is a codimension 0 compact submanifold of ∂M. The map of pairs $(TM, TK) \to (M, K)$ given by the differential of the inclusion is a pair of vector bundles. An inwards-pointing vector field on K induces a collar on this vector bundle pair. We implicity carry such an inwards-pointing vector field and induced collar whenever we have a manifold pair. A map of manifold pairs $f : (M, K) \to (M', K')$ is a map of pairs of spaces where both maps are smooth, $f^{-1}(K') = K$, and $Df : (TM, TK) \to (TM', TK')$ is a collared bundle map. A related notion that we will use is that of a triad. A d-dimensional *manifold triad* is a triple $(W, \partial^h W, \partial^v W)$ where $(W, \partial^h W)$ and $(W, \partial^v W)$ are manifold pairs such that $\partial^h W \cup \partial^v W = \partial W$ and $\partial(\partial^h W) = \partial^h W \cap \partial^v W = \partial(\partial^v W)$. We denote $\partial^h W \cap \partial^v W$ by $\partial^{hv} W$. When dealing with these objects, we will often only remember the manifold pairs $(W, \partial^h W)$ and $(\partial^v W, \partial^{hv} W)$, and the inclusion map between them. The following spaces of smooth maps will appear often throughout:

(a) Given two manifold pairs (M, K) and (M', K'), the space $\mathrm{Emb}((M, K), (M', K'))$ (or $\mathrm{Emb}(M, M')$ when the subspaces are implicit) denotes the space of smooth embeddings of pairs, that is, smooth maps of pairs which are topological embeddings and whose differentials are fiberwise injective. For a subpair $(N, L) \subseteq (M, K)$ and an embedding $e : (N, L) \to (M', K')$, we denote by $\mathrm{Emb}_N((M, K), (M', K'))$ the space of smooth embeddings which extend e on (N, L).

(b) Given a manifold triad $(W, \partial^h W, \partial^v W)$, we denote by $\mathrm{Diff}_{\partial^v}(W)$ the space of diffeomorphisms of W fixing $\partial^v W$ pointwise, whose differential is a collared bundle map. This space is a topological group where multiplication is given by composition of diffeomorphism and inverses are given by taking the

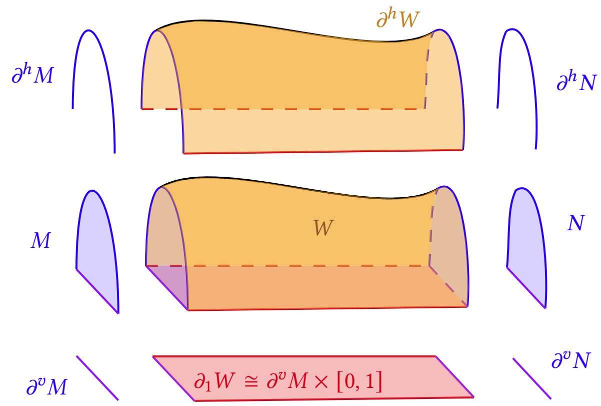


Figure 1: This is a triad cobordism $(W, \partial^h W, \partial_1 W, \partial^v W)$ from $(M, \partial^h M, \partial^v M)$ to $(N, \partial^h N, \partial^v N)$.

inverse diffeomorphism.

Both of these spaces (and any subspace of the space of smooth maps) are equipped with the Whitney C^∞-topology. By definition, the diffeomorphism group $\mathrm{Diff}_{\partial^v}(W)$ acts on $\mathrm{Bun}^{\mathrm{col}}_{\partial^v}(TW, \eta)$ for any vector bundle pair.

2.3.1. Handle decompositions for pairs of manifolds. In this section, we introduce handle decompositions for cobordisms between triads (as defined below, see also Figure 2). To do so, we are inspired by the work of Borodzik, Némethi, and Ranicki in [BNR16]. Recall that a *4-ad* is a compact d-dimensional manifold W together with a decomposition $\partial W = \partial_0 W \cup \partial_1 W \cup \partial_2 W$ into codimension 0 compact submanifolds such that $\partial(\partial_i W) = \bigcup_{i \neq j} \partial_i W \cap \partial_j W$ and that $\partial(\partial_i W \cap \partial_j W) = \bigcap_k \partial_k W$. We use the notation $\partial_{ij} W := \partial_i W \cap \partial_j W$ and $\partial_{012} W := \partial_0 W \cap \partial_1 W \cap \partial_2 W$.

Definition 2.3.1. Let $(M, \partial^h M, \partial^v M)$ and $(N, \partial^h N, \partial^v N)$ be two $(d-1)$-dimensional manifold triads. A d-dimensional manifold 4-ad $(W, \partial_0 W, \partial_1 W, \partial_2 W)$ is called a *triad cobordism from M to N* if there is an equality of triads $(\partial_2 W, \partial_{02} W, \partial_{12} W) = (M, \partial^h M, \partial^v M) \sqcup (N, \partial^h N, \partial^v N)$ and there is a diffeomorphism of pairs $(\partial_1 W, \partial_{01} W) \cong (\partial^v M \times [0,1], \partial^{hv} M \times [0,1])$ relative to $(\partial^v M, \partial^{hv} M)$, seen as a subspace of the second factor given by the inclusion at $\{0\}$. In this case, we denote $\partial^h W := \partial_0 W$ and $\partial^v W = \partial_2 W$. We denote a triad cobordism W from M to N by $W : M \rightsquigarrow N$. Observe that the definition implies that $\partial^h W$ is a cobordism with trivial boundary (or a $r\partial$-cobordism in the sense of [Wal71]) from $\partial^h M$ to $\partial^h N$ (see Figure 1).

We say that two triad cobordisms $W : M \rightsquigarrow N$ and $W' : M \rightsquigarrow N$ are *diffeomorphic* if they are diffeomorphic as 4-ads where the diffeomorphism sends M in W to M in W' (and thus N to N), not necessarily via the identity. We call the subspaces $\partial^h W$ and $\partial^v W$ of a triad W by *horizontal* and *vertical boundary*, respectively.

Remark 2.3.2. This notion can be seen as a generalization of a cobordism between manifolds with boundary with trivialized boundary (or $r\partial$-cobordisms in the sense of [Wal71]) by taking $\partial^h M = \partial^h N = \emptyset$. Moreover, given a triad cobordism W, then $\partial^h W$ is a $r\partial$-cobordism.

For $k \geq 0$, recall the definition of D^k_+ from Definition 2.1.6. We see this as a triad $(D^k_+, \partial_0 D^k_+, \partial_1 D^k_+)$ given by $\partial_0 D^k$ to be the subset of ∂D^k_+ where the first coordinate is 0. Denote by $\partial_{01} D^k_+ = \partial_0 D^k_+ \cap \partial_1 D^k_+$. For $k = 0$, this is the triad $(\{0\}, \emptyset, \emptyset)$.

Definition 2.3.3 (Handles)**.** Let $d \geq k$ be non-negative integers and let $W : M \rightsquigarrow N$ be a d-dimensional triad cobordism.

(*i*) Let $f : S^{k-1} \times D^{d-k} \to N$ be an embedding disjoint from ∂N. Define the 4-ad $(W', \partial^h W', \partial_1 W', \partial^v W')$ given by

$$W' := W \cup_f D^k \times D^{d-k}, \qquad \partial^h W' := \partial^h W,$$
$$\partial^v W' := \partial^v W \backslash f(S^{k-1} \times D^{d-k})^\circ \cup D^k \times S^{d-k-1}, \qquad \partial_1 W' := \partial_1 W.$$

There is a unique triad structure on $N' := N \backslash f(S^{k-1} \times D^{d-k})^\circ \cup D^k \times S^{d-k-1}$ making W' into a triad cobordism from M to N'. We say W' is obtained by *attaching an interior k-handle* to W along f. The subspace $(D^k \times \{0\}, \emptyset) \subset (W', \partial^h W')$ is called the *core* and $(\{0\} \times D^{d-k}, \emptyset)$ the *cocore* of this handle.

(*ii*) Let $f : (S^{k-1} \times D^{d-k}_+, S^{k-1} \times \partial_0 D^{d-k}) \to (N, \partial^h N)$ be an embedding disjoint from $\partial^v N$. Define the 4-ad $(W', \partial^h W', \partial_1 W', \partial^v W')$ given by

$$W' := W \cup_f D^k \times D^{d-k}_+, \qquad \partial^h W' := \partial^h W \cup_f D^k \times \partial_0 D^{d-k}_+,$$
$$\partial^v W' := \partial^v W \backslash f(S^{k-1} \times D^{d-k})^\circ \cup_f D^k \times \partial_1 D^{d-k}_+, \qquad \partial_1 W' := \partial_1 W.$$

There is a unique triad structure on $N' := N \backslash f(S^{k-1} \times D^{d-k}_+)^\circ \cup_f D^k \times \partial_1 D^{d-k}_+$ making W' into a triad cobordism from M to N'. We say W' is obtained by *attaching a right k-handle* to W along f. The subspace $D^k \times \{0\} \subset \partial^h W'$ is called the *core* and $(\{0\} \times D^{d-k}_+, \{0\} \times \partial_0 D^{d-k}_+) \subset (W', \partial^h W')$ the *cocore* of this handle.

(*iii*) Let $f : (\partial_1 D^k_+ \times D^{d-k}, \partial_{01} D^k_+ \times D^{d-k}) \to (N, \partial^h N)$ be an embedding disjoint from $\partial^v N$. Define the 4-ad $(W', \partial^h W', \partial_1 W', \partial^v W')$ given by

$$W' := W \cup_f D^k_+ \times D^{d-k}, \qquad \partial^h W' := \partial^h W \cup_f \partial_0 D^k_+ \times D^{d-k},$$
$$\partial^v W' := \partial^v W \backslash f(\partial_1 D^k_+ \times \partial D^{d-k})^\circ \cup D^k_+ \times \partial D^{d-k}, \qquad \partial_1 W' := \partial_1 W.$$

There is a unique triad structure on $N' := N \backslash f(\partial_1 D^k_+ \times D^{d-k})^\circ \cup D^k_+ \times \partial D^{d-k}$ making W' into a triad cobordism from M to N'. We say W' is obtained by

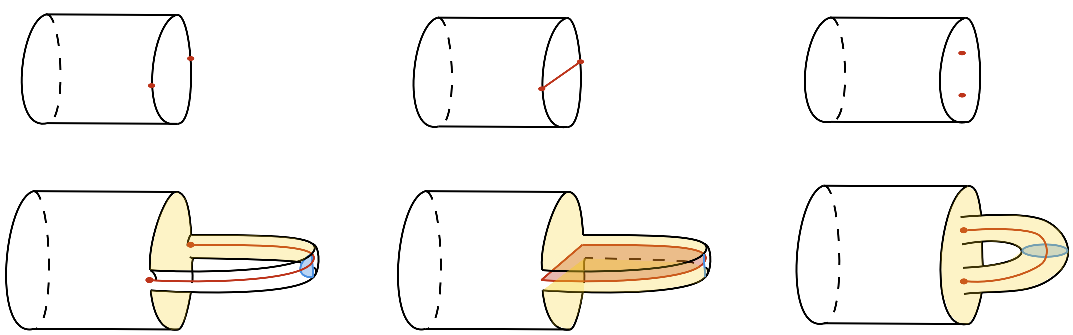


Figure 2: Here are depictions of the three types of handle attachments of Definition 2.3.3 to a trivial triad cobordism for $d = 3$. The red portions in the top pictures represent the attaching maps. The red and blue portions in the bottom pictures represent the core and cocores, respectively. The manifold N' (that is, the outgoing boundary of the cobordism W' is depicted in light yellow.

attaching a left k-handle to W along f. The subspace $(D^k_+ \times \{0\}, \partial_0 D^k_+ \times \{0\}) \subset (W', \partial^h W')$ is called the *core* and $\{0\} \times D^{d-k} \subset \partial^h W'$ the *cocore* of this handle.

A *triad handle decomposition* of a triad cobordism $W : M \rightsquigarrow N$ relative to M is a filtration of triples $\cdots \subset (W_i, \partial^h W_i, \partial_1 W_i) \subset (W_{i+1}, \partial^h W_{i+1}, \partial_1 W_{i+1}) \subset \cdots$ for $i = 0, \cdots, k$ for some k where $(W_i, \partial^h W_i, \partial_1 W_i, \partial^v W_i)$ is a triad cobordism from M to some triad M_i satisfying the following properties:

1. $(W_0, \partial^h W_0, \partial_1 W_0, \partial^v W_0) = (M \times [0,1], \partial^h M \times [0,1], \partial^v M \times [0,1], M \times \{0,1\})$
2. $(W_k, \partial^h W_k, \partial_1 W_k, \partial^v W_k) = (W, \partial^h W, \partial_1 W, \partial^v W)$
3. for $i = 0, \cdots, k-1$, the cobordism W_{i+1} is diffeomorphic to a cobordism obtained from W_i by attaching an interior, a right or a left handle to M_i. This diffeomorphism is assumed to be relative to M_i.

Observe that a triad handle decomposition of $W : M \rightsquigarrow N$ induces a handle decomposition (in the sense of [Wal71]) of the cobordism $\partial^h W : \partial^h M \rightsquigarrow \partial^h N$ given by $\partial^h W_i$ for $i = 0, \cdots, k$. More precisely, an interior k-handle induces a trivial cobordism on the horizontal boundary, a right k-handle induces a k-handle and a left k-handle induces a $(k-1)$-handle.

We move now to establishing some properties of these objects. These properties were observed in [BNR16] for the special case that $\partial_1 W = \emptyset$ and proved using Morse theory. We will deduce our results from theirs by adapting some of the arguments to the more general case when $\partial_1 W \neq \emptyset$. We recall now the relevant notions in loc.cit for the reader's convinience. We assume however the reader's familiarity with this reference for the proofs, as we will only explain the necessary Morse-theoretic modifications.

Remark 2.3.4 (Morse theory for manifolds with boundary)**.** In [BNR16], the authors study *cobordisms of manifolds with boundary* which are simply triad cobordisms W in the sense of Definition 2.3.1 where $\partial_1 W = \emptyset$. To avoid confusion, we will use the notation (Ω, Y) for a cobordism between manifolds with boundary Σ_1 and Σ_2 from [BNR16, Definition 1.1], and $(W, \partial^h W, \partial_1, \partial^v W)$ for triad cobordisms. Here are the necessary definitions and facts from [BNR16]:

1. A *Morse function* of a cobordism of manifolds with boundary (Ω, Y) is a smooth function $f : \Omega \longrightarrow [0,1]$ such that $f(\Sigma_0) = 0$ and $f(\Sigma_1) = 1$, whose critical points have non degenerate Hessian and are not in $\Sigma_0 \cup \Sigma_1$, and ∇f is everywhere tangent to Y. See [BNR16, Defn. 1.4].
2. If (Ω, Y) admits a Morse function with a single critical point c, then it is diffeomorphic to a handle attachment in the sense of Definition 2.3.3 relative to Σ_0 of:
   (a) Interior type if c is in the interior of Ω,
   (b) Left type if $c \in Y$ is *boundary stable* in the sense of [BNR16, Defn 2.4],
   (c) Right type if $c \in Y$ is *boundary unstable* in the sense of loc.cit.

   This is [BNR16, Theorem 2.27], by observing that the definitions of handles in [BNR16, Defs. 2.11 and 2.12] are special cases of Definition 2.3.3 for $\partial_1 W = \emptyset$.

To import results from [BNR16] to our setting, we see a triad cobordism $(W, \partial^h W, \partial_1 W, \partial^v W)$ as the cobordism $(\Omega, Y) = (W, \partial^h W \cup \partial_1 W)$ between the manifolds with boundary M and N.

Proposition 2.3.5 (Existence)**.** *Let $W : M \rightsquigarrow N$ be a triad cobordism, then there exists a triad handle decomposition of W relative to M. In fact, such decomposition can be found where either right or left handles are not present.*

Proof. By combining [BNR16, Lemma 2.10] with [BNR16, Thm. 2.27], we see that a Morse function on a cobordism of manifolds with boundary (Ω, Y) from Σ_0 to Σ_1 (in the sense of [BNR16]) gives rise to a filtration of the pair (Ω, Y) where at each step is obtained from the previous by attaching a handle of one of types above (where we see Ω as a 4-ad by taking $\partial_1 \Omega = \emptyset$). Moreover, these handles are attached along subsets of the stable and unstable manifolds of each critical point. We now specify to $(\Omega, Y) = (W, \partial^h W \cup \partial_1 W)$. Since $\partial_1 W$ is diffeomorphic to a product $\partial^v M \times [0,1]$, we see that there exists a Morse function on W without critical points in $\partial_1 W$, as one can find a classical Morse function on $Y = \partial^h W \cup \partial_1 W$ without critical points in $\partial_1 W$ and then extend it to a Morse function on W by [BNR16, Lemma 2.1]. Moreover, by definition, the gradient flow (for some metric

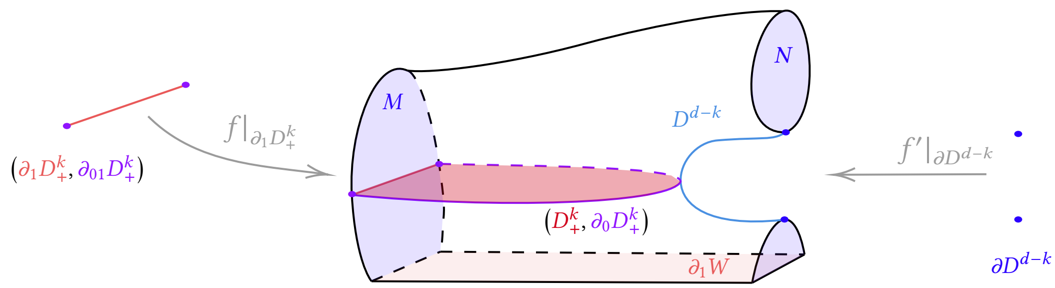


Figure 3: This is a triad cobordism $W : M \rightsquigarrow N$ obtained from the trivial triad cobordism $(M \times [0,1], \partial^h M \times [0,1], \partial^v M \times [0,1], M \times \{0,1\})$ by attaching a left k-handle to $M \times \{1\}$. On the other hand, it is obtained from the trivial cobordism $N \times [0,1]$ by attaching a right $(d-k)$-handle to $N \times \{0\}$ (compare with Lemma 2.3.8). We also depict the attaching map f, belt sphere f' (see Definition 2.3.7), the core $(D^k_+, \partial_0 D^k_+)$ and cocore D^{d-k}.

on W) of this Morse function is everywhere tangent to $\partial_1 W$. We see that there exists a collar of $\partial_1 W$ which is disjoint from the stable and unstable manifolds of all critical points. Thus, the induced filtration of pairs $(W, \partial^h W)$ induces the trivial filtration on $\partial_1 W$ and thus corresponds to a triad handle decomposition. We conclude that triad handle decompositions exist. The extra conclusion follows from the proof of [BNR16, Lemma 2.1], where a Morse function is constructed where the boundary critical points are solely of one type. □

Remark 2.3.6 (Triad handle decompositions from classical)**.** Let $(M, \partial^h M)$ be a manifold pair and $W' : \partial^h M \rightsquigarrow \partial^h N$ be a cobordism with trivial boundary (also called $r\partial$-cobordism in [Wal71]) and let $V : \partial^{hv} M \rightsquigarrow \partial^{hv} N$ be the trivial cobordism between the boundaries. Observe that the 4-ad $W \coloneqq ((W' \cup_{\partial^h M} M) \times [0,1], W' \times \{0\} \cup_{\partial^h N} \partial^h N \times [0,1], (\partial^v M \cup V) \times [0,1], M \times \{0\} \sqcup (W' \cup_{\partial M} M) \times \{1\})$ is a triad cobordism. Additionally, given any handle decomposition on W' relative to $\partial^h M$ (in the sense of loc.cit.), one can define a triad handle decomposition on W relative to M by attaching right handles just as prescribed by the handle decomposition on W'. In other words, a filtration on W' relative to ∂M induces one on $(W' \cup_{\partial^h M} M) \times [0,1]$ relative to $M \times [0,1]$. This filtration is obtained by attaching right handles at each stage along (collars of) the attaching maps of W'. Conversely, given any triad cobordism $W : M \rightsquigarrow N$ that admits a triad handle decomposition with only right handles is diffeomorphic to $(\partial^h W \cup_{\partial^h M} M) \times [0,1]$. One can see this by using the gradient flow of a Morse function, in the sense of [BNR16], with only boundary critical points to retract W into a collar of $\partial^h W \cup_{\partial^h M} M$.

Definition 2.3.7. Let $W : M \rightsquigarrow N$ be a d-dimensional triad cobordism, we call W *elementary relative to* M (resp. N) if it admits a triad handle decomposition relative to $(M, \partial^h M)$ (resp. N) given by a single handle attachment. Such cobordism is called of *interior, left* or *right type* depending on the type of the handle attachment and of index k if the handle attached is a k-handle (see Figure 3). When dealing with elementary triad cobordisms, we often fix such a triad handle decomposition. In this case, we denote the core and cocore of the handle of W by core_W and cocore_W respectively. We denote the intersection core_W with M by attach_W. Similarly, we denote the intersection of cocore_W with N by belt_W, and call it the *belt sphere*.

Given the cobordism W, we can define $W^* : N \rightsquigarrow M$ given by the same triad seen as a cobordism from N to M. The following lemma follows by unwrapping definitions.

Lemma 2.3.8. *Let $k \geq 0$. Let $W : M \rightsquigarrow N$ be a d-dimensional elementary triad cobordism of left (resp. interior) type of index k relative to M. Then W^* is elementary of a right (resp. interior) type and index $(d-k)$ relative to N. Moreover, such triad handle decomposition can be arranged so that $\mathsf{core}_{W^*} = \mathsf{cocore}_W$ and $\mathsf{cocore}_{W^*} = \mathsf{core}_W$.*

Remark 2.3.9. In the context above, note that there exists a diffeomorphism $\phi : M \backslash \nu(\mathsf{attach}_W) \cong N \backslash \nu(\mathsf{belt}_W)$, where $\nu(-)$ denotes an open tubular neighborhood. This follows by the definition of handle attachments. Thus, given a submanifold of M which is disjoint from attach_W we can consider it as a submanifold of N by choosing a small enough tubular neighborhood of attach_W and taking the image under ϕ.

Given two triad cobordisms $W : P \rightsquigarrow Q$ and $W' : Q \rightsquigarrow R$, then its *composition* $W' \circ W : P \rightsquigarrow R$ is the union $W \cup_Q W'$. In this case, we call W and W' composable. We recall that, we say that two triad cobordisms $W : M \rightsquigarrow N$ and $W' : M \rightsquigarrow N$ are diffeomorphic if they are diffeomorphic as triads where the diffeomorphism sends M in W to M in W' (and thus N to N). Given a triad cobordism $W : P \rightsquigarrow Q$, we call P the *ingoing boundary* and Q the *outgoing boundary.*

Proposition 2.3.10 (Handle rearrangement)**.** *Let $P \overset{W}{\rightsquigarrow} Q \overset{W'}{\rightsquigarrow} R$ be two elementary triad cobordisms relative to their ingoing boundaries. If $\mathsf{belt}_W \cap \mathsf{attach}_{W'} = \emptyset$ for some triad handle decomposition of W and W', then there exist two elementary triad cobordisms $P \overset{M}{\rightsquigarrow} Q' \overset{M'}{\rightsquigarrow} R$ such that M (resp. M') is of the same type and index as W' (resp. W) relative to their ingoing boundaries and the composition is diffeomorphic to $W' \circ W$ relative to P.*

Proof. Once again, we deduce this statement by observing that the analogous statement in [BNR16] (in this case Thm. 4.1) can be improved to our setting. By [BNR16, Prop. 2.35], for any elementary triad cobordism W there exists a Morse function with exactly one critical point. We want to argue that that critical point can be assumed to lie outside $\partial_1 W$. If it lies in the interior of $\partial_1 W$, then the restriction of the Morse function to $\partial_1 W$ is a Morse function. However, since $\partial_1 W$ is a trivial cobordism, it cannot be an elementary one, thus we reach a contradiction. If the critical point lies in the boundary, then using a collar on $\partial^h W$, we can push the critical point outside of $\partial_1 W$. Thus, $W' \circ W$ has a Morse function with two critical points disjoint from $\partial_1 W$. In [BNR16, Thm. 4.1], a new Morse function is created where the critical points are swapped. Here the hypothesis are satisfied since $\mathsf{belt}_W \cap \mathsf{attach}_{W'} = \emptyset$ implies that the stable and unstable manifolds of the two critical points are disjoint. Thus, we see that this new Morse function does not have any critical points on $\partial_1(W' \circ W)$ which implies that it gives a triad handle decomposition of $W' \circ W$ given by first attaching a handle of index and type of W' and after one of index and type of W. Thus, we obtain the desired result. □

Whenever the conclusion of Proposition 2.3.10 holds, we say W and W' *can be rearranged*. The following is presented in [BNR16, Table 1] and follows from transversality and dimension counting.

Corollary 2.3.11. *Let $P \overset{W}{\rightsquigarrow} Q \overset{W'}{\rightsquigarrow} R$ be two elementary triad cobordisms of index k and k' of any type relative to their ingoing boundaries. Then W and W' can be rearranged if one of the following conditions holds:*

1. *W' is of interior or left type and W is of any type provided $k \geq k'$. When W' is of interior type and W is of left type, no condition on k and k' is necessary.*
2. *Both cobordisms are of right type and $k \geq k'$.*
3. *W' is of right type and W is of left type and $k > k'$.*
4. *W' is of right type and W is of interior type.*

Proof. By transversality of embeddings of pairs (see [Gen11, Lemma 9.2][7]), we can always isotope the attaching map of the handle of W', resulting in a diffeomorphic cobordism relative to Q, so belt_W and $\mathsf{attach}_{W'}$ intersect transversely. Thus by dimension counting, the hypothesis of Proposition 2.3.10 is satisfied. The cases where no dimension assumption is needed follow from the fact that either belt_W is contained in the interior of Q and $\mathsf{attach}_{W'}$ in ∂Q, or vice-versa. □

The following propositions constitute the main results of [BNR16] and [BM24]. Given a cobordism $W : P \rightsquigarrow R$, a factorization of W is a sequence of composable cobordisms W_i such that the composition is diffeomorphic to W relative to P. The following three results follow from [BNR16, Thm. 3.1], [BM24, Thm. 1.2] and

[7]Genauer proves that the subspace of smooth maps which are transversal to a fixed smooth map is open and dense in the space of smooth maps of pairs. To conclude that the space of transversal embeddings of pairs is open in the space of smooth maps, it suffices to prove that embeddings of pairs are open in the space of smooth maps. This follows from analogous arguments to [Hir76, Prop. 2.1.4].

[BNR16, Thm. 5.1], respectively, by observing that the constructions made in those proofs can be assumed to leave ∂_1 untouched (as in Proposition 2.3.10) and thus generalize to our context.

Proposition 2.3.12 (Splitting)**.** *Let $W : P \rightsquigarrow R$ be an elementary triad cobordism of interior type of index $0 \leq k \leq d-1$ attached to a component of P connected to $\partial^h P$. Then there exists a factorization $P \overset{M}{\rightsquigarrow} Q \overset{M'}{\rightsquigarrow} R$ of W where M is elementary of left type and M' is elementary of right type and both have index k. Moreover, we have belt_W and $\mathsf{attach}_{W'}$ intersect transversely at one point.*

Proposition 2.3.13 (Merging)**.** *Let $P \overset{W}{\rightsquigarrow} Q \overset{W'}{\rightsquigarrow} R$ be two elementary triad cobordisms of index k, where W is of left and and W' is of right type relative to their ingoing boundaries. If belt_W and $\mathsf{attach}_{W'}$ intersect transversely at one point, then $W' \circ W$ is an elementary triad cobordism of interior type and index k.*

Proposition 2.3.14 (Cancellation)**.** *Let $P \overset{W}{\rightsquigarrow} Q \overset{W'}{\rightsquigarrow} R$ be two elementary triad cobordisms of the same type and index k and $k+1$ respectively. If belt_W and $\mathsf{attach}_{W'}$ intersect transversely at one point, then $W' \circ W$ is diffeomorphic to the $(P \times [0,1], \partial^h P \times [0,1])$ relative to P.*

Remark 2.3.15. It is important to remark that if W and W' are either of right or of left type, then either belt_W or $\mathsf{attach}_{W'}$ is completely contained in ∂Q. This implies that the condition in Proposition 2.3.14 can be checked in the boundary cobordism $\partial^h W' \circ \partial^h W$. In practice, when we wish to cancel a right or left handle, we produce a canceling (classical) handle on the boundary cobordism and then extend it to a triad handle of the same type.

For a triad $(P, \partial^h P, \partial^v P)$, we call the cobordism $(P \times [0,1], \partial^h P \times [0,1], \partial^v P \times [0,1], P \times \{0\} \sqcup P \times \{1\})$ the *product cobordism*. We say that a cobordism $W : P \rightsquigarrow Q$ is *trivial* if it is diffeomorphic to the product cobodism $P \times [0,1]$ relative to P, where P is seen in the product cobordism as the inclusion of $P \times \{0\}$.

Proposition 2.3.16 (Standard presentation)**.** *Let $W : P \rightsquigarrow Q$ be a triad cobordism, then W admits a decomposition of the form*

$$W \cong W_0 \cup W_0^R \cup W_1^L \cup W_1 \cup W_1^R \cup \cdots \cup W_{d-1}^R \cup W_d^L \cup W_d$$

where W_i is a composition of elementary interior cobordisms of index i and $W_i^{L/R}$ is a composition of elementary left/right cobordisms of index i, relative to their ingoing boundaries. Moreover, for any of the following conditions, there exists such a decomposition satisfying it:

(i) The cobordisms W_i^L are trivial for all $i \geq 0$.
(ii) The cobordisms W_i^R are trivial for all $i \geq 0$.

(iii) The cobordisms W_i are trivial for all $0 < j < d$.

Proof. The first result follows from Proposition 2.3.5 and Corollary 2.3.11. The second result follows from the second claim of Proposition 2.3.5 and from Proposition 2.3.12. ◻

2.3.2. Strong geometrical connectivity. Handle decompositions of cobordisms between closed manifolds are made to be akin to cell structures of CW pairs. Classical algebraic topology tells us that if a map $X \to Y$ of spaces if k-connected, then there exists a CW pair (Z, X) made from cells of dimension at least $k+1$ along with a weak equivalence $Z \to Y$ under X. Wall [Wal71] proved a geometric analog of this statement, where a map is replaced by the inclusion of one of the boundary components of a cobordism and cells are replaced by handles. We wish to prove a similar statement for triad cobordisms, triad handle decompositions and strong connectivity. We start by noticing how different handle attachments behave homotopically. Let $k \geq 0$ be an integer, then the following holds:

(I) If W' is a triad cobordism obtained from W by attaching a right k-handle, then note that the pair $(W', \partial^h W')$ is equivalent to $(W \cup_{S^{k-1}} D^k, \partial^h W' \cup_{S^{k-1}} D^k)$. Clearly, the map $\partial^h W \to \partial^h W'$ is $(k-1)$-connected. Moreover, the map $\partial^h W' \cup_{\partial^h W} W \to W'$ is an equivalence. This uses the fact that the strict pushout $\partial^h W' \cup_{\partial^h W} W$ is the homotopy pushout (since $\partial^h W \to W$ is a Hurewicz cofibration, see the discussion in the beginning of Section 2.1). In particular, the map $(W, \partial^h W) \to (W', \partial^h W')$ is strongly $(k-1)$-connected.

(II) Let $k \geq 1$. If W' is a triad cobordism obtained from W by attaching a left k-handle, then note that the pair $(W', \partial^h W')$ is equivalent to $(W \cup_{\partial_1 D^k_+} D^k_+, \partial^h W' \cup_{\partial_{01} D^k_+} \partial_0 D^k_+)$. In particular, the map $W \cup_{\partial^h W} \partial^h W' \to W'$ is equivalent to the attachment of a k-cell and thus $(k-1)$-connected. Similarly, the map $\partial^h W \to \partial^h W'$ is equivalent to the attachment of a $(k-1)$-cell and thus $(k-2)$-connected. Thus the map $(W, \partial^h W) \to (W', \partial^h W')$ is strongly $(k-2)$-connected. Additionally, the map $W \to W'$ is an equivalence.

(III) If W' is a triad cobordism obtained from W by attaching an interior k-handle, then note that the pair $(W', \partial^h W')$ is equivalent to $(W \cup_{S^{k-1}} D^k, \partial^h W')$. Since the map on horizontal boundaries is an equivalence, it follows that $(W, \partial^h W) \to (W', \partial^h W')$ is strongly $(k-2)$-connected.

We are ready now for the analog of [Wal71, Thm. 3] for triad cobordisms.

Proposition 2.3.17 (Geometrical connectivity)**.** *Let $W : R \rightsquigarrow S$ be a triad cobordism of dimension d and $-1 \leq k \leq d-5$ be an integer. If the inclusion $(R, \partial^h R) \hookrightarrow (W, \partial^h W)$ is strongly k-connected, then there exists a triad handle decomposition relative to $(R, \partial^h R)$ with only right handles of index at least $k+1$, only interior handles of index at least $k+2$, and no left handles.*

Proof. By Proposition 2.3.5, there exists a triad handle decomposition of W relative to R consisting of only right and interior handles. By Corollary 2.3.11 and Proposition 2.3.16, we can assume that any triad cobordism $W : R \rightsquigarrow S$ has a triad handle decomposition of the form

$$W \cong W_0^R \cup W_1^R \cup \cdots \cup W_{d-1}^R \cup W_0 \cup \cdots \cup W_d$$

where once again W_i^R is a composite of elementary right cobordisms of index i and W_i a composite of elementary interior cobordisms of index i. Denote by $W^R : R \rightsquigarrow R'$ and $W^I : R' \rightsquigarrow S$ the unions of W_i^R and W_i, respectively. Observe that the cobordism $\partial^h W : \partial^h R \rightsquigarrow \partial^h S$ is diffeomorphic to $\partial^h W^R : \partial^h R \rightsquigarrow \partial^h R'$ relative to $\partial^h R$, since $\partial^h W^I$ is a trivial cobordism since W^I is obtained by attaching only interior handles. Moreover, the cobordism $\partial^h W \cong \partial^h W^R$ satisfies the hypothesis of [Wal71, Thm. 3] since it is a classical cobordism between manifolds with trivial boundary (or a $r\partial$-cobordism) and $k \leq (d-1)-4 = d-5$. Thus, there exists a handle decomposition of $\partial^h W$ relative to $\partial^h R$ whose handles have index at least $k+1$. Given this handle decomposition, we can extend it to a triad handle decomposition of W^R containing only right handles of index at least $k+1$, by Remark 2.3.6. We proceed now to show that W^I admits a triad handle decomposition given by interior handles of index at least $k+2$. The cobordism W^I can be viewed as a $r\partial$-cobordism in the sense of [Wal71] (whose boundary cobordism is $\partial^h W^R \cup \partial_1 W^R$). Hence, a handle decomposition in the sense of loc. cit. induces a triad handle decomposition solely given by interior handles of the same indices as the ones in the handle decomposition. Thus, if we show that the inclusion $R' \to W^I$ is $(k+1)$-connected, applying Wall's result again finishes the claim. By assumption, the map $R \cup_{\partial^h R} \partial^h W \to W$ is $(k+1)$-connected. Since W^R is obtained from R by attaching right handles, we conclude from (I) above that the maps $R \cup_{\partial^h R} \partial^h W^R \to W^R$ and $R' \to W^R$ are equivalences. Thus it follows that $R' \to W^R \to W$ is a composite of an equivalence with a $(k+1)$-connected map and thus it is $(k+1)$-connected. This finishes the proof. □

3. Cobordism categories with boundary.

As mentioned in the introduction, the proof of the main result involves a version of the classical cobordism category (see e.g. [Gal+09]) in the context of manifolds with boundary and triad cobordisms between them. In this section, we define the cobordism category $\mathsf{Cob}_\Theta^\partial$ of Θ-manifolds with boundary and variations thereof for a map of pairs $\Theta : (B, B^\partial) \to (\mathrm{BO}(d), \mathrm{BO}(d-1))$. The main result of this section (stated below in Theorem 3.1.10) states that the classifying space of $\mathsf{Cob}_\Theta^\partial$ is equivalent to that of a certain subcategory of highly connected cobordisms. This result is analogous to [GR14, Thm. 3.1] for cobordism categories of closed manifolds and similar to [BP17, Thm A.3] in the context of manifolds with boundary.

Our proof adopts a similar strategy. We assume the reader has prior knowledge of these references, since we will at times explain our proofs by comparison and only indicate the modifications needed in our context.

3.1. Cobordism categories of Θ-triads.

In this subsection, we introduce the cobordism category $\mathsf{Cob}_\Theta^\partial$ and state the main result of this section. To do so, we start by defining certain spaces of manifolds with boundary, which will be the object and morphism spaces in our category. For an integer $N \geq 0$, let $\mathbf{R}_+^N := [0, +\infty) \times \mathbf{R}^{N-1}$ and $\partial \mathbf{R}_+^N = \{0\} \times \mathbf{R}^{N-1}$. We denote the union along the inclusions $\mathbf{R}_+^N \cong \mathbf{R}_+^N \times \{0\} \hookrightarrow \mathbf{R}_+^{N+1}$ by $\mathbf{R}_+^\infty$, and $\partial \mathbf{R}_+^\infty$ for the analogous union of $\partial \mathbf{R}_+^N$. Let $n, k \geq 0$ and $U \subset \mathbf{R}^k \times \mathbf{R}_+^{n-k}$ be an open subset and observe that $\partial U = U \cap \partial(\mathbf{R}^k \times \mathbf{R}_+^{n-k})$. Let $d \geq 0$ and $\Theta = (\theta, \theta^\partial) : (B, B^\partial) \to (\mathrm{BO}(d), \mathrm{BO}(d-1))$ be a map of pairs. We will need to import the following definitions from [GR14] and [BP17]:

(a) The space $\Psi_\theta^\partial(U)$ is the space of pairs (M, ℓ) of a d-dimensional submanifold with boundary M of U satisfying certain conditions and a bundle map $\ell : TM \to \theta^* \gamma_d$ from [BP17, Defn. 2.1]. From the same reference, for $l \leq d$ the space $\Psi_{\theta_l}^\partial(U)$ is the space of pairs (M, ℓ) of a l-dimensional submanifold with boundary M of U satisfying certain conditions and a bundle map $\ell : TM \oplus \varepsilon^{d-l} \to \theta^* \gamma_d$.

(b) The space $\Psi_{\theta^\partial}(\partial U)$ is the space of pairs (M, ℓ) of a $(d-1)$-dimensional submanifold M of ∂U satisfying certain conditions and a bundle map $\ell : TM \to (\theta^\partial)^* \gamma_{d-1}$ from [GR14, Defn. 2.2]. From the same reference, for $l \leq d-1$ the space $\Psi_{\theta_l}(\partial U)$ is the space of pairs of a l-dimensional submanifold M of ∂U and a bundle map $\ell : TM \oplus \varepsilon^{d-1-l} \to \theta^* \gamma_{d-1}$.

By definition, there exists a map $\hat{\partial} : \Psi_{\theta_l}^\partial(U) \to \Psi_{\theta_{l-1}}(\partial U)$ taking a pair (M, ℓ) to $(\partial M, \ell|_{\partial M})$ (see also [BP17, Lemma B.1]). We also have a map $\Psi_{\theta_l^\partial}(\partial U) \to \Psi_{\theta_l}(\partial U)$ by taking (M, ℓ) to $(M, \iota \circ \ell)$ where $\iota \circ \ell$ is the composite $TM \oplus \varepsilon^{d-1-l} \oplus \varepsilon^1 \to (\theta^\partial)^* \gamma_{d-1} \oplus \varepsilon^1 \to \theta^* \gamma_d$, where ι is induced by Θ.

Definition 3.1.1 (Spaces of manifolds). We define $\Psi_\Theta^\partial(U)$ to be the strict pullback $\Psi_\theta^\partial(U) \times_{\Psi_{\theta_{d-1}}(\partial U)} \Psi_{\theta^\partial}(\partial U)$. As a set, $\Psi_\Theta^\partial(U)$ is the set of pairs (M, ℓ) where M is a d-dimensional neat submanifold with boundary of U and $\ell : (TM, T\partial M) \to (\theta^* \gamma_d, (\theta^\partial)^* \gamma_{d-1})$ is a map of pairs of vector bundles on M in the sense of Definition 2.2.1 satisfying the following properties:

(*i*) M is closed as a subspace of U;

(*ii*) there exists $\epsilon > 0$ such that $M \cap (\mathbf{R}^k \times [0, \epsilon) \times \partial \mathbf{R}_+^{n-k}) = [0, \epsilon) \times \partial M$ and the restriction of ℓ to $[0, \epsilon) \times \partial M \subset M$ is given by

$$T([0, \epsilon) \times \partial M) \to T(\partial M) \oplus \varepsilon^1 \overset{\ell_{\partial M} \oplus \varepsilon^1}{\to} (\theta^\partial)^* \gamma_{d-1} \oplus \varepsilon^1 \to \theta^* \gamma_d$$

where the leftmost map is the bundle map covering the projection $[0,\epsilon)\times\partial M$ given by the splitting of the source given by the vector e_1 in the coordinate $[0,\epsilon)$.

We call such a pair (M,ℓ) a Θ-manifold and ℓ a Θ-structure on M. For $l \leq d$, let $\Psi^{\partial}_{\Theta_l}(U)$ be the strict pullback $\Psi^{\partial}_{\theta_l}(U) \times_{\Psi_{\theta_{l-1}}(\partial U)} \Psi_{\theta^{\partial}_{l-1}}(\partial U)$.

Remark 3.1.2. Let $(M,\ell) \in \Psi^{\partial}_{\Theta}(U)$, we observe that the first condition of (ii) induces a preferred collar of $(TM, T\partial M)$ (in the sense of Definition 2.2.2) by taking the inwards pointing vector field given by the basis vector on the $[0,\epsilon)$-coordinate. One can check that, with this collar, the map ℓ of collared pairs of vector bundles is collared, by the second part of (ii).

Remark 3.1.3. By [BP17, Lemma B.1], the map $\hat{\partial} : \Psi^{\partial}_{\theta_l}(U) \to \Psi_{\theta_{l-1}}(\partial U)$ is a Serre fibration, so the square

$$\begin{array}{ccc} \Psi^{\partial}_{\Theta_l}(U) & \longrightarrow & \Psi^{\partial}_{\theta_l}(U) \\ \downarrow & & \downarrow \\ \Psi_{\theta^{\partial}_{l-1}}(\partial U) & \longrightarrow & \Psi_{\theta_{l-1}}(\partial U). \end{array}$$

is a homotopy pullback square and the left vertical map is a Serre fibration.

We define now convenient subspaces of the spaces of manifolds defined above.

Definition 3.1.4. Define $\psi^{\partial}_{\Theta}(n,k) \subset \Psi^{\partial}_{\Theta}(\mathbf{R}^k \times \mathbf{R}^{n-k}_{+})$ consisting of those pairs (M,ℓ) such that M is contained in $\mathbf{R}^k \times [0,1) \times (-1,1)^{n-k-1}$. Analogously, we define the spaces $\psi^{\partial}_{\Theta_l}(n,k)$ for $l \leq k$. For $i \in \mathbf{Z}$, denote by $\psi^{\partial}_{\Theta}(\infty+i,k)$ the colimit of these $\psi^{\partial}_{\Theta}(n+i,k)$ under the stabilizations over n induced by the inclusion $\mathbf{R}^k\times[0,1)\times(-1,1)^{n-k-1} \cong \mathbf{R}^k\times[0,1)\times(-1,1)^{n-k-1}\times\{0\} \subset \mathbf{R}^k\times[0,1)\times(-1,1)^{n-k}$.

Notation 3.1.5. Let $(W,\ell) \in \psi^{\partial}_{\Theta}(\infty+1,1)$. Denote by $x_0 : \mathbf{R}\times\mathbf{R}^{\infty}_{+} \to \mathbf{R}$ the projection onto the first factor. For $K \subset \mathbf{R}$, denote $W|_K \coloneqq W \cap x_0^{-1}(K)$ and $\partial^h W|_K \coloneqq \partial W \cap W|_K$ if both subsets are manifolds (possibly with corners). If $K = \{a\}$ is a singleton, we denote $\partial W|_a \coloneqq \partial^h W|_a$. When $K = [a,b]$ and $W|_{[a,b]}$ is a manifold, then $\partial W|_{[a,b]} = \partial^h W|_{[a,b]} \cup W|_a \cup W|_b$ so we denote $W|_a \cup W|_b$ by $\partial^v W|_{[a,b]}$. We see this triad also as a triad cobordism $W|_{[a,b]} : W|_a \rightsquigarrow W|_b$ by setting $\partial_1 = \emptyset$ in the sense of Definition 2.3.1.

Definition 3.1.6 (Cobordism category)**.** Define a non-unital topological category (in the sense of [GR14, Section 2.3]) $\mathsf{Cob}^{\partial}_{\Theta}$ having $\psi^{\partial}_{\Theta_{d-1}}(\infty,0)$ as space of objects. The morphism space is the subspace of $\mathbf{R}\times\psi^{\partial}_{\Theta}(\infty+1,1)$ consisting of those pairs $(t,(W,\ell))$ where there exists an $\epsilon > 0$ such that

$$W|_{(-\infty,\epsilon)} = (\mathbf{R}\times W|_0)|_{(-\infty,\epsilon)} \in \Psi^{\partial}_{\Theta}((-\infty,\epsilon)\times\mathbf{R}^{\infty}_{+})$$

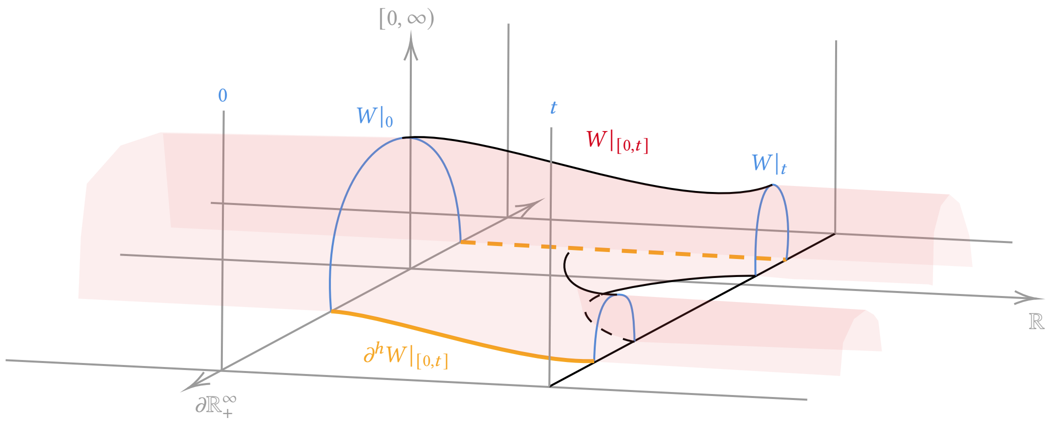


Figure 4: This is an example of the underlying submanifold of a morphism $(t,(W,\ell))$ in $\mathsf{Cob}^\partial_\Theta$.

and

$$W|_{(t-\epsilon,\infty)} = (\mathbf{R}\times W|_t)|_{(t-\epsilon,\infty)} \in \Psi^\partial_\Theta((t-\epsilon,\infty)\times\mathbf{R}^\infty_+).$$

Here, $\mathbf{R}\times W|_a$ denotes the manifold $\mathbf{R}\times W|_a \subset \mathbf{R}\times\mathbf{R}^\infty_+$ along with the map $(T(\mathbf{R}\times W|_a), T(\mathbf{R}\times\partial W|_a)) \longrightarrow \epsilon^1\oplus(TW|_a, T\partial W|_a) \longrightarrow (\theta^*\gamma_d, (\theta^\partial)^*\gamma_{d-1})$. The source and target of $(t,(W,\ell))$ are $(W|_0,\ell|_0)$ and $(W|_t,\ell|_t)$, respectively. Composition is given by "stacking" as defined in [GR14, Defn. 2.6] (see Figure 4).

It is convenient to define a variant of the cobordism category above where the objects and morphisms contain a fixed submanifold. Let L be a compact $(d-1)$-dimensional submanifold of $[0,1)\times(-\frac{1}{2},0]\times(-1,1)^{\infty-2}$ (possibly with corners) such that $\partial L = L\cap\partial([0,1)\times(-\frac{1}{2},0]\times(-1,1)^{\infty-2})$. Let $\partial^h L \coloneqq L\cap\{0\}\times(-\frac{1}{2},0]\times(-1,1)^{\infty-2}$ and $\partial^v L \coloneqq L\cap[0,1)\times\{0\}\times(-1,1)^{\infty-2}$. We require that near $\{0\}\times(-\frac{1}{2},0]\times\mathbf{R}^{\infty-2}$ it agrees with $[0,1)\times\partial^h L$ and that near $[0,1)\times\{0\}\times\mathbf{R}^{\infty-2}$ it agrees with $(-1,0]\times\partial^v L$. Additionally, let $(\ell_L,\ell_{\partial^h L}) : (TL, T\partial^h L)\oplus\epsilon^1 \longrightarrow (\theta^*\gamma_d, (\theta^\partial)^*\gamma_{d-1})$ be a map of pairs of vector bundles such that the restriction of ℓ_L to $T[0,\epsilon)\times\partial^h L$ agrees with the composition

$$T[0,\epsilon)\times\partial^h L \longrightarrow TL\oplus\epsilon^1 \overset{\ell_{\partial^h L}}{\longrightarrow} (\theta^\partial)^*\gamma_{d-1}\oplus\epsilon^1 \longrightarrow \theta^*\gamma_d,$$

and the restriction to $T(-\epsilon,0]\times\partial^v L$ is a product structure, that is, the family $t\mapsto \ell_L|_{\{t\}\times\partial^v L}$ is a constant homotopy of maps of pairs $(TL, T\partial L) \longrightarrow (\theta^*\gamma_d, (\theta^\partial)^*\gamma_{d-1})$ for $t<\epsilon$ for some ϵ.

Definition 3.1.7. Define the non-unital topological category $\mathsf{Cob}^\partial_{\Theta,L}$ whose space of objects is the space $\psi^\partial_{\Theta_{d-1},L}(\infty,0)\subset\psi^\partial_{\Theta_{d-1}}(\infty,0)$ consisting of those (M,ℓ) such that $M\cap[0,1)\times(-\infty,0]\times\mathbf{R}^{\infty-2} = L$ as Θ-manifolds, that is, ℓ agrees with ℓ_L in this subspace. Its space of morphisms is the subspace $\psi^\partial_{\Theta_{d-1},L}(\infty+1,1)\subset\psi^\partial_{\Theta_{d-1}}(\infty+1,1)$

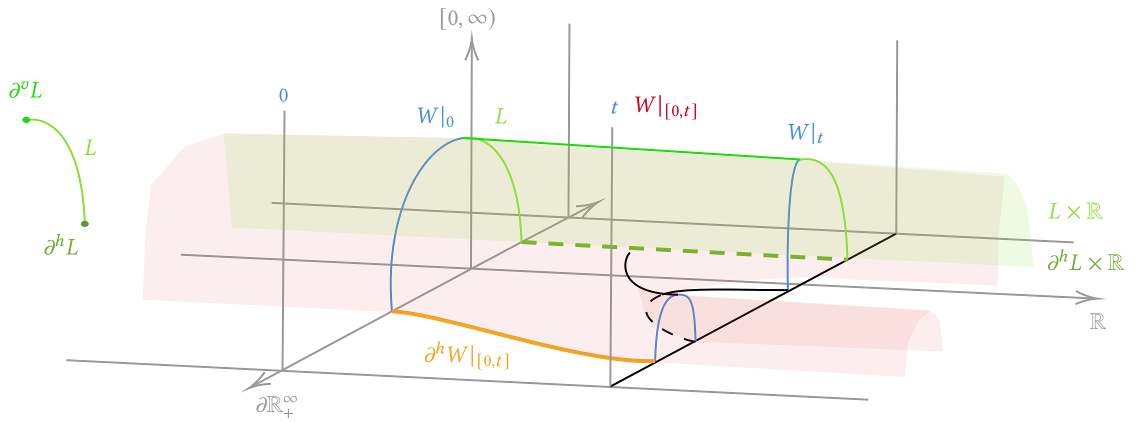


Figure 5: This is an example of the underlying submanifold of a morphism (t, W) in $\mathsf{Cob}^{\partial}_{\Theta,L}$.

of those $(t, (W, \ell))$ such that $W \cap (\mathbf{R} \times [0, 1) \times (-\infty, 0] \times \mathbf{R}^{\infty-2}) = \mathbf{R} \times L$ as Θ-manifolds (see Figure 5).

Recall from [GR14, Section 2.3] that for a non-unital topological category C, we define its classifying space $B\mathsf{C}$ as the geometric realization $||N_\bullet \mathsf{C}||$ of its nerve $N_\bullet \mathsf{C}$, which is a semi-simplicial space. The following is proved exactly as [GR14, Cor. 2.17].

Proposition 3.1.8. *For any L as above, the map induced by inclusion* $B\mathsf{Cob}^{\partial}_{\Theta,L} \to B\mathsf{Cob}^{\partial}_{\Theta}$ *is a weak equivalence.*

We will be interested in certain subcategories of these categories. A *subcategory* C of $\mathsf{Cob}^{\partial}_{\Theta,L}$ is a pair of a collection of objects $\mathrm{ob}(\mathsf{C}) \subset \mathrm{ob}(\mathsf{Cob}^{\partial}_{\Theta,L})$ and a collection of morphisms $\mathrm{mor}(\mathsf{C}) \subset \mathrm{mor}(\mathsf{Cob}^{\partial}_{\Theta,L})$ such that:

- (a) $\mathrm{ob}(\mathsf{C})$ is a union of path components of $\mathrm{ob}(\mathsf{Cob}^{\partial}_{\Theta,L})$;
- (b) the images of source and target maps $s_0, s_1 : \mathrm{mor}(\mathsf{C}) \subset \mathrm{mor}(\mathsf{Cob}^{\partial}_{\Theta,L}) \to \mathrm{ob}(\mathsf{Cob}^{\partial}_{\Theta,L})$ lie in the $\mathrm{ob}(\mathsf{C})$;
- (c) for every $a, b \in \mathsf{C}$, the morphism space $\mathsf{C}(a, b)$ defined by the strict pullback $\mathrm{mor}(\mathsf{C}) \times_{\mathrm{ob}(\mathsf{C})^{\times 2}} \{(a, b)\}$, given by the source and target maps, is a union of path components of $\mathsf{Cob}^{\partial}_{\Theta,L}(a, b)$;
- (d) for every $(P, \ell_P) \in \mathsf{C}$ and $t > 0$, the morphism $(t, [0, t] \times P)$ with the *cylindrical Θ-structure* $\ell_P \oplus \varepsilon^1$ is in C.

Unwrapping the definition, C is a non-unital topological category and the inclusion $\mathsf{C} \to \mathsf{Cob}^{\partial}_{\Theta}$ is a functor of non-unital topological categories. The following definition is analogous to [GR14, Def. 2.9].

Definition 3.1.9. Let $k \geq -1$ be an integer. Let $\mathsf{Cob}^{\partial,(k)}_{\Theta,L}$ be the subcategory of $\mathsf{Cob}^{\partial}_{\Theta,L}$ consisting of the same objects and those morphisms $(t,(W,\ell))$ where the inclusion of $\partial W|_t \to \partial^h W|_{[0,t]}$ is k-connected. When $k = \lfloor \frac{d-3}{2} \rfloor$, we denote $\mathsf{Cob}^{\partial,(k)}_{\Theta,L}$ by $\mathsf{Cob}^{\partial,b}_{\Theta,L}$. Let now $k \geq -2$. Let $\mathsf{Cob}^{\partial,k}_{\Theta,L}$ be the subcategory of $\mathsf{Cob}^{\partial,b}_{\Theta,L}$ with the same objects and those morphisms $(t,(W,\ell))$ where the inclusion $(W|_t, \partial W|_t) \to (W|_{[0,t]}, \partial^h W|_{[0,t]})$ is strongly k-connected in the sense of Definition 2.1.1.

The main result of this section (and the only one used in later sections) is the following result, whose proof ocupies the remainder of this section. Recall the definition of a triad handle decomposition from Definition 2.3.3. Once and for all, we see $(L, \partial^h L, \emptyset, \partial^v L)$ as a triad cobordism from $(\partial^v L, \partial^{hv} L, \emptyset)$ to $\emptyset$.

Theorem 3.1.10. *Let $d \geq 5$ and $k \geq -2$ be integers such that $2k \leq d-3$. Assume $(L, \partial^h L)$ admits a triad handle decomposition only using handles of index at most $d-k-3$ and of any type relative to $\emptyset$. Then the map*

$$\mathsf{BCob}^{\partial,k}_{\Theta,L} \to \mathsf{BCob}^{\partial}_{\Theta}$$

is a weak equivalence.

3.1.1. Flexible models.

To prove Theorem 3.1.10, it is convenient to model the homotopy types of the classifying spaces of these subcategories of $\mathsf{Cob}^{\partial}_{\Theta,L}$ in a more tractable way, similar to [GR14, Defs. 2.13 and 2.18]. We start by introducing two models for these classifying spaces.

Definition 3.1.11. Let $\mathsf{D}^{\partial}_{\Theta} \subset \mathbf{R} \times \mathbf{R}_{>0} \times \psi^{\partial}_{\Theta}(\infty+1,1)$ be the subspace of tuples $(t,\epsilon,(W,\ell))$ such that $[t-\epsilon, t+\epsilon]$ consists of *regular values* for $x_0 : W \to \mathbf{R}$, that is, points $s \in \mathbf{R}$ such that the maps $x_0 : W \to \mathbf{R}$ and $x_0|_{\partial W}$ have s has a regular value. We define a partial order on $\mathsf{D}^{\partial}_{\Theta}$ by $(t,\epsilon,(W,\ell)) < (t',\epsilon',(W',\ell'))$ if and only if $(W,\ell) = (W',\ell')$ and $t+\epsilon < t'-\epsilon'$. Denote by $\mathsf{D}^{\partial}_{\Theta,L}$ the full subposet of $\mathsf{D}^{\partial}_{\Theta}$ consisting of those $(t,\epsilon,(W,\ell))$ such that $W \cap (\mathbf{R} \times [0,+\infty) \times (-\infty,0] \times \mathbf{R}^{\infty-2}) = \mathbf{R} \times L$ as Θ-manifolds. If $\mathsf{C} \subset \mathsf{Cob}^{\partial}_{\Theta,L}$ is a subcategory, we denote by $\mathsf{D}^{\partial,\mathsf{C}}_{\Theta,L} \subset \mathsf{D}^{\partial}_{\Theta,L}$ the smallest subposet containing the objects $(t,\epsilon,(W,\ell))$ where $W|_t$ is an object of C and of morphisms $(t,\epsilon,(W,\ell)) < (t',\epsilon',(W',\ell'))$ where $W'|_{[t,t']} = W|_{[t,t']} : W|_t \rightsquigarrow W|_{t'}$ is a morphism of C.

Proposition 3.1.12. *Let $\mathsf{C} \subset \mathsf{Cob}^{\partial}_{\Theta,L}$ be a subcategory of $\mathsf{Cob}^{\partial}_{\Theta,L}$, then there is a weak equivalence $\mathsf{BC} \simeq \mathsf{BD}^{\partial,\mathsf{C}}_{\Theta,L}$.*

Proof. The proof follows analogously as the proof of [GR14, Prop. 2.14] by replacing the appropriate terms and using [Gen11, Lemma 10.1] instead of Ehresmann's fibration lemma. □

We introduce now a "more flexible model" in the spirit of [GR14, Defn. 2.18]. Roughly speaking, these models have better "mapping in"-properties than the previous model.

Definition 3.1.13. Let $p \geq 0$ be an integer and define $\mathsf{X}^{\partial}_{\Theta,L}$ to be the semisimplicial space where the space of p-simplices consists of tuples $a \in \mathbf{R}^{p+1}$, $\epsilon \in (\mathbf{R}_{>0})^{p+1}$ and $(W, \ell) \in \Psi^{\partial}_{\Theta}((a_0 - \epsilon_0, a_p + \epsilon_p) \times \mathbf{R}^{\infty}_{+})$ satisfying:

(i) For all $i = 0, \cdots, p-1$, we have $a_i + \epsilon_i < a_{i+1} - \epsilon_{i+1}$;
(ii) W lies in $(a_0 - \epsilon_0, a_p + \epsilon_p) \times [0, 1) \times (-1, 1)^{\infty-1}$;
(iii) W agrees with $(a_0 - \epsilon_0, a_p + \epsilon_p) \times L$ as Θ-manifold pairs on the subspace $U \coloneqq (a_0 - \epsilon_0, a_p + \epsilon_p) \times [0, +\infty) \times (-\infty, 0] \times \mathbf{R}^{\infty-2}$, that is, they agree on $\Psi^{\partial}_{\Theta}(U)$.

Given $\mathsf{C} \subset \mathsf{Cob}^{\partial}_{\Theta,L}$ a subcategory, let $\mathsf{X}^{\partial,\mathsf{C}}_{\Theta,L}$ be the subspace of $\mathsf{X}^{\partial}_{\Theta,L}$ consisting of those tuples where for each regular value t of x_0, $W|_t \in \mathsf{C}$ and for each pair of regular values $t_0 < t_1$ contained in the union over i of $(a_i - \epsilon_i, a_i + \epsilon_i)$, the cobordism $W|_{[t_0,t_1]}$ is a morphism in C.

We can map the nerve of the topological poset $\mathsf{D}^{\partial,\mathsf{C}}_{\Theta,L}$ to $\mathsf{X}^{\partial,\mathsf{C}}_{\Theta,L}$ taking a p-simplex $(a_0, \cdots, a_p; \epsilon_0, \cdots \epsilon_p, (W, \ell))$ to the same tuple where we consider the restriction of (W, ℓ) to $(a_0 - \epsilon_0, a_p + \epsilon_p) \times \mathbf{R}^{\infty}_{+}$. The following result is proved using exactly the same method as in [GR14, Prop. 2.20].

Proposition 3.1.14. *Let $\mathsf{C} \subset \mathsf{Cob}^{\partial}_{\Theta,L}$ be a subcategory of $\mathsf{Cob}^{\partial}_{\Theta,L}$, then the map $N_{\bullet}\mathsf{D}^{\partial,\mathsf{C}}_{\Theta,L} \to \mathsf{X}^{\partial,\mathsf{C}}_{\Theta,L}$ is a weak equivalence after geometric realization.*

3.2. Surgery on morphisms.

In this section, we prove Theorem 3.1.10. To do so, we consider the following filtrations from Definition 3.1.9

$$\mathsf{Cob}^{\partial,k}_{\Theta,L} \subset \mathsf{Cob}^{\partial,k-1}_{\Theta,L} \subset \cdots \subset \mathsf{Cob}^{\partial,0}_{\Theta,L} \subset \mathsf{Cob}^{\partial,-1}_{\Theta,L} = \mathsf{Cob}^{\partial,b}_{\Theta,L}$$

and

$$\mathsf{Cob}^{\partial,(k)}_{\Theta,L} \subset \mathsf{Cob}^{\partial,(k-1)}_{\Theta,L} \subset \cdots \subset \mathsf{Cob}^{\partial,(-1)}_{\Theta,L} \subset \mathsf{Cob}^{\partial,(-2)}_{\Theta,L} = \mathsf{Cob}^{\partial}_{\Theta,L}.$$

The main inputs to the proof of Theorem 3.1.10 are the two results stated below, which give conditions to the inclusions of the subcategories in the filtrations above to induce an equivalence on classifying spaces.

Proposition 3.2.1 (Surgery on the interior)**.** *Assume $d \geq 5$. Suppose that the following conditions are satisfied:*

(i) $2k \leq d - 3$;

(ii) $(L, \partial^h L)$ *admits a triad handle decomposition where interior and left handles have index at most* $d-k-3$ *relative to* $\emptyset$.

Then the map induced by inclusion $\mathsf{BCob}^{\partial,k}_{\Theta,L} \to \mathsf{BCob}^{\partial,k-1}_{\Theta,L}$ *is a weak equivalence.*

Proposition 3.2.2 (Surgery on the boundary)**.** *Assume* $d \geq 3$. *Suppose that the following conditions are satisfied:*

(i) $2k \leq d-3$;

(ii) $(L, \partial^h L)$ *admits a triad handle decomposition where right handles have index at most* $d-k-3$ *and left handles have index at most* $d-k-2$ *relative to* $\emptyset$.

Then the map induced by inclusion $\mathsf{BCob}^{\partial,(k)}_{\Theta,L} \to \mathsf{BCob}^{\partial,(k-1)}_{\Theta,L}$ *is a weak equivalence.*

These results are analogous to [GR14, Thm. 3.1] and [BP17, Thms. A.2 and B.4]. Assuming these results, we can now prove our main statement of this section.

Proof of Theorem 3.1.10. This follows by applying Proposition 3.2.2 to conclude that the inclusions in the second filtration induce equivalences on classifying spaces for $k \leq \lfloor \frac{d-3}{2} \rfloor$ and thus the inclusion $\mathsf{Cob}^{\partial,b}_{\Theta,L} \to \mathsf{Cob}^{\partial}_{\Theta,L}$ induces an equivalence of classifying spaces. Applying Proposition 3.2.1, we deduce that the inclusions in the first filtration are equivalences after taking classifying spaces if $2k \leq d-3$. Thus, we obtain that the maps $\mathsf{BCob}^{\partial,k}_{\Theta,L} \to \mathsf{BCob}^{\partial,b}_{\Theta,L} \to \mathsf{BCob}^{\partial}_{\Theta,L}$ are equivalences. The result now follows by post-composing this with the equivalence of Proposition 3.1.8. ◻

3.2.1. The approach due to Galatius and Randal-Williams. The remainder of this section is devoted to proving Propositions 3.2.1 and 3.2.2. We start by explaining the overall strategy of the proofs. We employ a strategy originally due to Galatius and Randal-Williams in [GR14]. Let us consider a general context. Let $\mathsf{A} \subset \mathsf{B}$ be subcategories of $\mathsf{Cob}^{\partial}_{\Theta,L}$. We wish to prove that the inclusion $\mathsf{A} \hookrightarrow \mathsf{B}$ induces an equivalence on classifying spaces. To do so, we will construct an augmented bi-semisimplicial space[8] $(\mathsf{D}^{\partial,\mathsf{A}}_{\Theta,L})_{\bullet,\bullet}$ where $(\mathsf{D}^{\partial,\mathsf{A}}_{\Theta,L})_{\bullet,-1} = N_\bullet \mathsf{D}^{\partial,\mathsf{B}}_{\Theta,L}$ along with maps $\iota : N_\bullet \mathsf{D}^{\partial,\mathsf{A}}_{\Theta,L} \to (\mathsf{D}^{\partial,\mathsf{A}}_{\Theta,L})_{\bullet,0}$ of semi-simplicial spaces fitting in a commutative square of solid arrows

$$\begin{array}{ccccc}
\mathsf{BD}^{\partial,\mathsf{A}}_{\Theta,L} & & \xrightarrow[\simeq]{3.1.14} & & \left|\mathsf{X}^{\partial,\mathsf{A}}_{\Theta,L}\right| \\
\big\downarrow \iota & & \overset{\mathsf{K}^1}{\nearrow} & & \big\downarrow \mathrm{inc} \\
\left|\left(\mathsf{D}^{\partial,\mathsf{A}}_{\Theta,L}\right)_{\bullet,\bullet}\right| & \xrightarrow{p} & \mathsf{BD}^{\partial,\mathsf{B}}_{\Theta,L} & \xrightarrow[\simeq]{3.1.14} & \left|\mathsf{X}^{\partial,\mathsf{B}}_{\Theta,L}\right|
\end{array} \tag{4}$$

[8]Here, we mean that this bi-semisimplicial space is augmented in the second coordinate. In other words, it is a semi-simplicial object in the category of augmented semi-simplicial spaces.

where the map p is induced by the augmentation, satisfying the following properties:

1. (*Contractability of surgery data*) The map p is a weak equivalence.
2. (*Parameterized surgery move*) There exists a dashed lift K^1 as above making the upper triangle strictly commute and a homotopy K^t making the lower triangle commute up to homotopy.

By 1 and commutativity of the lower triangle, the map K^1 induces an injection on all homotopy groups. By the commutativity of the upper triangle, the map K^1 induces also a surjection on all homotopy groups. Thus K^1 is a weak equivalence and thus so are the vertical maps. By Proposition 3.1.12 and Proposition 3.1.14, the right vertical map is equivalent to the map $\mathsf{BA} \to \mathsf{BB}$ and hence the latter is a weak equivalence. The remaining sections are devoted to constructing this bisimplical space and this diagram for A and B the steps of the filtrations above.

3.2.2. Proof of Proposition 3.2.1. Recall the definition of the submanifolds $V \coloneqq (-2, 0) \times \mathbf{R}^k \times \mathbf{R}^{d-k}$ and $\bar{V} \coloneqq [-2, 0] \times \mathbf{R}^k \times \mathbf{R}^{d-k}$ of $\mathbf{R}^{k+1} \times \mathbf{R}^{d-k}$ from [BP17, Section A.1]. We start with some notation.

Notation 3.2.3. We fix once and for all an infinite set Ω. Let (Λ, δ, e) be a triple where $\Lambda \subset \Omega$ is a finite and $\delta : \Lambda \to [p]^\vee$ is a function and $e : \Lambda \times \bar{V} \hookrightarrow \mathbf{R}\times[0, 1)\times(0, 1)\times(-1, 1)^{\infty-2}$. Here $[p]^\vee$ means the set of monotone non-decreasing maps from $[p] \coloneqq \{0, \cdots , p\}$ to $[1]$ (see [BP17, p. 149]). We say that such triple is a *(interior) surgery datum* for a p-simplex $(a, \epsilon, (W, \ell))$ in the nerve of $\mathsf{D}^{\partial,\mathsf{C}}_{\Theta,L}$ if the triple satisfies conditions (i) to (v) in [BP17, Def. A.3.]. We call the subspaces $D_i = e(\delta^{-1}(i)\times\partial_- D^{k+2}\times\{0\})$ the *cores* of (the trace of) the surgery datum (Λ, δ, e) for $i = 0, \cdots , p$, where $\partial_- D^{k+2} \coloneqq \{(x_1, \cdots , x_{k+2}) \in \mathbf{R}^{k+2}|\ \sum_{i=1}^{k+2} x_i^2 = 1,\ x_1 \geq 0\}$. Notice that since the third coordinate is restricted to $(0, 1)$, the embedding e is automatically disjoint from L.

We define now the bi-semisimplicial spaces promised above in the context of the first filtration. The following definition is essentially [BP17, Def. A.3].

Definition 3.2.4. Given a p-simplex $x = (a, \epsilon, (W, \ell))$ of $\mathsf{D}^{\partial,k-1}_{\Theta,L}$. Let $\mathsf{Z}_0(x)$ to be set of surgery data (Λ, δ, e) for x such that the map

$$\left(W|_{a_i} \cup D_i|_{[a_{i-1},a_i]}, \partial W|_{a_i}\right) \to \left(W_{[a_{i-1},a_i]}, \partial^h W|_{[a_{i-1},a_i]}\right)$$

is strongly k-connected for each $i = 1, \cdots , p$. Define $\mathsf{Z}_p(x)$ in the same way as [BP17, Def.A.5]. Similarly to [BP17] define the bi-semisimplicial space given by $(\mathsf{D}^{\partial,k}_{\Theta,L})_{p,q}$ to be the space of pairs (x, y) where x is a p-simplex of $(\mathsf{D}^{\partial,k-1}_{\Theta,L})_\bullet \coloneqq N_\bullet \mathsf{D}^{\partial,k-1}_{\Theta,L}$ and $y \in \mathsf{Z}_q(x)$ topologized as a subspace of

$$\left(\mathsf{D}^{\partial,k-1}_{\Theta,L}\right)_p \times \left(\coprod_{\Lambda\subset\Omega} C^\infty(\Lambda \times \bar{V}, \mathbf{R} \times \mathbf{R}^\infty_+)\right)^{(p+2)(q+1)}.$$

We can define the vertical left map ι for the square (4) in this setting as the inclusion of the empty surgery data, that is, $\Lambda = \emptyset$. We prove now the first property of the square (4) in this context. The following result should be compared to [BP17, Thm. A.6] and [GR14, Thm. 3.4].

Proposition 3.2.5. *Assume the hypothesis of Proposition 3.2.1, then the augmentation map*

$$\left(\mathsf{D}^{\partial,k}_{\Theta,L}\right)_{\bullet,\bullet} \to \left(\mathsf{D}^{\partial,k-1}_{\Theta,L}\right)_{\bullet}$$

is a weak equivalence after geometric realization.

Proof. This proof is very similar to the proof of [GR14, Theorem 3.4]. However, there are steps where extra care is needed. We assume the reader's familiarity with this reference. We do not give a complete and detailed proof, but instead we sketch the strategy, which follows by analogy to loc.cit., and focus on the points where it differs. For every $p \geq 0$, the semi-simplicial space $(\mathsf{D}^{\partial,k}_{\Theta,L})_{p,\bullet}$ is augmented over $(\mathsf{D}^{\partial,k}_{\Theta,L})_p$. We will show that the augmentation induces a weak equivalence on geometric realizations for every p. This implies the claim by taking geometric realization in the p-coordinate. To prove that the augmentation map induces a weak equivalence, we use the criterion of [GR14, Thm. 6.2]. For that, one must check that this augmented semi-simplicial space is an *augmented topological flag complex* in the sense of [GR14, Def. 6.1]. We sketch the argument here for completeness. Just as in the case of this reference (see the remark in p.334 of [GR17a]), our augmented semi-simplicial space does not satisfy this criteria of [GR14, Thm. 6.2]. However, one can find an equivalent model which weakens some of the conditions in the definition of this space and satisfies this criteria, analogous to $\tilde{D}^{\kappa}_{\theta,L}(\mathbf{R}^N)$ in [GR14, Defn. 6.8], by only assuming that the embeddings in a surgery datum are immersions which restrict to embeddings on the cores. We proceed implicitly with this weaker model. To apply [GR14, Thm. 6.2], we must check that, for every p, this augmented topological flag complex satisfies conditions $(i)-(iii)$ of loc.cit. To verify condition (i), one proceeds exactly as [GR14, Prop. 6.10], using that the embeddings in our surgery data are contained in the interior.

Condition (ii) is equivalent to the claim that $\mathsf{Z}_0(x)$ is non-empty for every $x \in \mathsf{D}^{\partial,k-1}_{\Theta,L}$. We will proceed as in [GR14, Prop. 6.13] to check this condition. We explain the part of the argument which is different in our case in detail now. Let $x = (a, \epsilon, (W, \ell))$ be in $\mathsf{D}^{\partial,k-1}_{\Theta,L}$. It follows from the assumptions, the pair $(W|_{[a_{i-1},a_i]}, W|_{a_i})$ is $(k-1)$-connected for every i and the pair $(\partial^h W|_{[a_{i-1},a_i]}, \partial W|_{a_{i+1}})$ is $b \coloneqq \lfloor \frac{d-3}{2} \rfloor$-connected. Thus, we see that $(W|_{[a_{i-1},a_i]}, W|_{a_i})$ is already k-connected, since $k \leq b$ and $(W|_{a_i}, \partial W|_{a_i}) \longrightarrow (W|_{[a_{i-1},a_i]}, \partial^h W|_{[a_{i-1},a_i]})$ is strongly $(k-1)$-connected. By Whitehead's theorem, $W|_{[a_{i-1},a_i]}$ is equivalent to a CW complex X obtained from $W|_{a_i}$ by attaching cells of dimension at least

$k+1$. These cells can be assumed to be embedded in $W|_{[a_{i-1},a_i]}$ since $2(k+1) < d$, so we get an embedding

$$\hat{e}_{i,0} : \Lambda_{i,0} \times (D^{k+1}, \partial D^{k+1}) \to (W|_{[a_{i-1}+\epsilon_{i-1}, a_i+\epsilon_i]}, W|_{a_i+\epsilon_i})$$

for finite set $\Lambda_{i,0} \subset \Omega$ such that $W|_{a_i+\epsilon_i} \cup \mathrm{im}(\hat{e}_{i,0}) \to W|_{[a_{i-1}+\epsilon_{i-1},a_i+\epsilon_i]}$is $(k+1)$-connected for every i. This implies that the map

$$(W|_{a_i+\epsilon_i} \cup \mathrm{im}(\hat{e}_{i,0}), \partial W|_{a_i+\epsilon_i}) \to (W|_{[a_{i-1}+\epsilon_{i-1},a_i+\epsilon_i]}, \partial^h W|_{[a_{i-1}+\epsilon_{i-1},a_i+\epsilon_i]})$$

is strongly $(k+1)$-connected. These embeddings can be assumed to be disjoint from $\partial^h W|_{[a_{i-1}+\epsilon_{i-1},a_i+\epsilon_i]}$, since they can be defined in the interior of $W|_{a_i+\epsilon_i}$. These embeddings can be isotoped to be disjoint from $L \times \mathbf{R}$ by the following argument. By taking a triad handle decomposition of L (where L is seen as a triad cobordism from $(\partial^v L, \partial^{hv} L, \emptyset)$ to $\emptyset$ by taking $\partial_0 L = \partial^h L$ and $\partial_1 L = \emptyset$) as in the hypothesis, it suffices to make these embeddings disjoint from the products of the cores by $\mathbf{R}$. The cores of the right handles are automatically disjoint since they are collars of submanifolds of $\partial^h W|_{[a_{i-1}+\epsilon_{i-1},a_i+\epsilon_i]}$. The left and interior cores have dimension at most $d-k-2$ so $\hat{e}_{i,0}$ is generically disjoint since $(d-k-2)+(k+1) < d$, by transversality. Once again by the fact that $2(k+1) < d$, $\hat{e}_{i,0}$ and $\hat{e}_{j,0}$ can be assumed to be disjoint for all $i \neq j$. One proceeds now as in [GR14, Prop. 6.13] to extend these embeddings to a surgery datum. Thus, we conclude that $\mathsf{Z}_0(x)$ is non-empty and so condition (ii) is verified.

The verification of (iii) goes exactly as [GR14, Prop. 6.12] since any tuple $(k+1)$-dimensional submanifolds of a d-dimensional manifold can be made pairwise disjoint by transversality. This finishes the proof by using [GR17a, Theorem 6.2]. □

We move now to establish the second property of (4) in this context. Before that, we record an application of the Blakers-Massey theorem needed for this proof.

Lemma 3.2.6. *Let (M,K) be a 1-connected compact manifold pair of dimension $d \geq 3$. Let $\phi : (P, \partial P) \hookrightarrow (M,K)$ be an embedding of pairs, where P is a simply connected manifold with boundary of dimension $k \geq 0$ with no closed components and path-connected boundary. Let (M', K') be the manifold pair obtained from (M,K) by removing an open tubular neighborhood of ϕ. If $d \geq k+3$, then the induced map $\pi_i(M',K') \to \pi_i(M,K)$ is an isomorphism for $i \leq d-k-1$ and a surjection for $i = d-k$ for every basepoint.*

Proof. Start by observing that the claim is equivalent to proving that the square

$$\begin{array}{ccc} K' & \longrightarrow & M' \\ \downarrow & & \downarrow \\ K & \longrightarrow & M \end{array}$$

is homotopy $(d-k-1)$-cartesian in the sense of [MV15, Defn. 3.3.1]. We establish this by proving that this square is homotopy $(d-k)$-cocartesian and applying the Blakers-Massey theorem. In other words, we prove now that the $M' \cup_{K'} K \to M$ is $(d-k)$-connected. This map is a pushout of the map $S(\nu_{P,M}) \cup_{S(\nu_{\partial P,K})} D(\nu_{\partial P,K}) \hookrightarrow D(\nu_{P,M})$ where $\nu_{P,M}$ and $\nu_{\partial P,K}$ are the normal bundles of P in M and ∂P in K respectively (which are oriented since P is simply-connected), and $D(-)$ and $S(-)$ are the disc and sphere bundles of a vector bundle (with a metric). The right-hand side is a d-dimensional manifold with boundary homotopy equivalent to P and the left-hand side is its boundary. By Poincaré duality, we have that $\mathrm{H}_i(D(\nu_{P,M}), \partial D(\nu_{P,M})) \cong \mathrm{H}^{d-i}(P) \cong \mathrm{H}_{k-d+i}(P, \partial P)$ which vanishes for $k-d+i \leq 0$, since P has no closed components and hence $\partial P \hookrightarrow P$ is 0-connected. Hence $\mathrm{H}_i(D(\nu_{P,M}), \partial D(\nu_{P,M}))$ vanishes for $i \leq d-k$. Since $d \geq k+3$ and P is simply connected, we deduce that $\partial D(\nu_{P,M})$ is simply connected. Thus, the map $\partial D(\nu_{P,M}) \hookrightarrow D(\nu_{P,M})$ is $(d-k)$-connected. Hence, the square above is homotopy $(d-k)$-cocartesian in the sense of [MV15, Defn. 3.7.1]. Additionally, the map $K' \to K$ is $(d-1-k)$-connected by transversality, since ∂P is $(k-1)$-dimensional. We prove now that $K' \to M'$ is 1-connected. Since $d \geq k+2$, we have $\pi_0(K') = \pi_0(K)$ and $\pi_0(M') \to \pi_0(M)$ so $\pi_0(M') = \pi_0(K')$. Moreover, since $d \geq k+3$, the same applies for fundamental groups and thus $\pi_1(K')$ surjects onto $\pi_1(M')$. We conclude by applying the Blakers-Massey theorem, as stated in [MV15, Thm. 4.2.3], to deduce that the square above is homotopy $(d-k-1)$-cartesian. □

The following result is similar to [BP17, Lemma A.9] and uses [GR14, Prop. 3.6] (see Remark 3.2.8 below for more details on this similarity).

Proposition 3.2.7. *Assume the hypothesis of Proposition 3.2.1, then there exists a homotopy*

$$\mathsf{K} : I \times \left(\mathsf{D}^{\partial,k}_{\Theta,L}\right)_{p,0} \to \left(\mathsf{X}^{\partial,k-1}_{\Theta,L}\right)_p$$

such that the image of $(t,(W,\ell))$ lies in $\left(\mathsf{X}^{\partial,k}_{\Theta,L}\right)_p$ if $t=1$ or if $(W,\ell) \in \left(\mathsf{D}^{\partial,k}_{\Theta,L}\right)_p$.

Proof. We use the map K constructed in [BP17, (A.1)], that uses the standard family of [GR14, Prop. 3.6], and verify the claim in the statement above. Once again, we assume familiarity of the reader with both constructions. Denote the image of $(t,(a,\epsilon,(W,\ell),e))$ under K by $(a,\epsilon,(W_t,\ell))$ (since this homotopy is constant on the parameters a and ϵ). The manifold W_t is obtained from W_0 by removing the interior of e and attaching the manifold $P_t \in \Psi_\theta(V)$ from [GR14, Prop 3.6]. Since, by [BP17, Rmk. A.8] the family is constant on its horizontal boundary, we immediately have a θ^∂-structure in $\partial^h W_t$ from $\partial^h W$ by taking the constant family of θ^∂ structures $\ell|_{\partial^h W_t} := \ell|_{\partial^h W}$. Thus, we can see $W_t \in \Psi^\partial_{\Theta,L}(\mathbf{R}^\infty)$. It remains to check that the image of K lies in $(\mathsf{X}^{\partial,k}_{\Theta,L})_p$ if $t=1$ or $(W,\ell) \in (\mathsf{D}^{\partial,k}_{\Theta,L})_p$. Let us start by proving the case $(a, \epsilon, (W, \ell)) \in (\mathsf{D}^{\partial,k}_{\Theta,L})_p$. Let $t \in [0, 1]$, we have to show that, for two regular values $a < b$ of x_0, the map

$$(W_t|_b, \partial W_t|_b) \longrightarrow (W_t|_{[a,b]}, \partial^h W_t|_{[a,b]})$$

is strongly k-connected, provided $(W|_b, \partial W|_b) \longrightarrow (W|_{[a,b]}, \partial^h W|_{[a,b]})$ is strongly k-connected. Since the family $\partial^h W_t$ is constant in t, it suffices to prove that

$$W_t|_b \cup_{\partial W_t|_b} \partial^h W_t|_{[a,b]} \longrightarrow W_t|_{[a,b]}$$

is $(k + 1)$-connected. Denote the manifold $W|_{[a,b]} \backslash \mathrm{im}(e)^\circ$ by X. Observe that the pair $(W_t|_{[a,b]}, \partial^h W_t|_{[a,b]})$ is the union of $(X, \partial^h W_t|_{[a,b]})$ with $P_t|_{[a_\lambda,b_\lambda]}$ for all $\lambda \in \Lambda_{i,0}$ for some i. We start by proving that

$$Y \coloneqq X|_b \cup_{\partial X|_b} \partial^h W_t|_{[a,b]} \longrightarrow X$$

is $(k+1)$-connected. The pair (X, Y) is obtained from $(W|_{[a,b]}, W|_b \cup_{\partial W|_b} \partial^h W|_{[a,b]})$ by cutting out a neighborhood of embedded copies of $(D^{k+1}, \partial D^{k+1})$. By applying Lemma 3.2.6 for each copy iteratively and $2k \leq d - 3$, we conclude that $\pi_i(X, Y)$ vanishes for $i \leq k + 1 \leq d - (k + 1) - 1$, since it is isomorphic to $\pi_i(W|_{[a,b]}, W|_b \cup_{\partial W|_b} \partial^h W|_{[a,b]})$, which vanishes by assumption. Since connectivity of maps is preserved under pushouts, the map $W_t|_b \cup_{\partial W_t|_b} \partial^h W_t|_{[a,b]} = W_t|_b \cup_{X|_b} Y \longrightarrow W_t|_b \cup_{X|_b} X$ is also $(k + 1)$-connected, since $\partial X|_b = \partial^h W_t|_b$. By construction, the map $X \cup_Y W_t|_b \longrightarrow W_t|_{[a,b]}$ is a pushout of a $(k + 1)$-connected map, by Property (iv) of [BP17, Lemma A.7] (see square (A.4) in [BP17]). We conclude that, the map $W_t|_b \cup \partial^h W_t|_{[a,b]} \longrightarrow W_t|_{[a,b]}$ is $(k + 1)$-connected. This establishes the case $(a, \epsilon, (W, \ell)) \in N_p \mathsf{D}^{\partial,k}_{\Theta,L}$.

We now concern ourselves with the case $t = 1$ and $(a, \epsilon, (W, \ell)) \in (\mathsf{D}^{\partial,k-1}_{\Theta,L})_p$. In [GR14, Lemma 3.7], the proof is divided in three steps. We focus on Step 2 of loc.cit., since it is the non-trivial step. More precisely, we have to show that, for two regular values $a \in (a_{i-1} - \epsilon_{i-1}, a_{i-1} + \epsilon_{i-1})$ and $b \in (a_i - \epsilon_i, a_i + \epsilon_i)$ of x_0, the map

$$W_1|_b \cup_{\partial W_1|_b} \partial^h W_1|_{[a,b]} \longrightarrow W_1|_{[a,b]}$$

is $(k + 1)$-connected. It suffices to prove the analogous statement for $\widetilde{W}_1$ obtained from W by only doing the surgeries for $\Lambda_{i,0}$. Isotope the submanifold D_i to a submanifold $\widetilde{D}_i$ so it is disjoint from the image of e_i, using the fact that it has trivial normal bundle. Once again by Lemma 3.2.6, since $k + 1 \leq d - (k + 1) - 1$, we deduce that the map

$$X|_b \cup \widetilde{D}_i|_{[a,b]} \cup_{\partial W|_b} \partial^h W|_{[a,b]} \longrightarrow X,$$

is $(k + 1)$-connected, since it is obtained from $W|_b \cup \widetilde{D}_i|_{[a,b]} \cup_{\partial W|_b} \partial^h W|_{[a,b]} \longrightarrow W|_{[a,b]}$, which is $(k + 1)$-connected by assumption, by removing a neighborhood of an $(k+1)$-disc. Once again by Property (iv) of [BP17, Lemma A.7], we deduce that

$$\widetilde{W}_1|_b \cup_{\partial \widetilde{W}_1|_b} \partial^h \widetilde{W}_1|_{[a,b]} \cup \widetilde{D}_i|_{[a,b]} \to \widetilde{W}_1|_{[a,b]}$$

is $(k+1)$-connected. However, by Property (v) of [BP17, Lemma A.7], $\widetilde{D}_i$ retracts to $\widetilde{W}_1|_b$ so the map $\widetilde{W}_1|_b \hookrightarrow \widetilde{W}_1|_b \cup \widetilde{D}_i|_{[a,b]}$ is an equivalence. This implies

$$\widetilde{W}_1|_b \cup_{\partial \widetilde{W}_1|_b} \partial^h \widetilde{W}_1|_{[a,b]} \to \widetilde{W}_1|_{[a,b]}$$

is $(k+1)$-connected. This finishes the proof. □

Remark 3.2.8. The difference between this statement and [BP17, Lemma A.9] is two-fold. Firstly, we work with a more general notion of tangential structure and therefore must produce a family of manifold pairs with such structure. Secondly, our definition of $\mathsf{D}^{\partial,k}_{\Theta,L}$ and $\mathsf{X}^{\partial,k}_{\Theta,L}$ differs from the analogous one in that paper in two ways. First, we do not assume that $\partial W|_t$ is $(n-2)$-connected for every regular value t. Second, their connectivity condition of the morphisms is dual to the notion of strong connectivity in the sense that the square in Remark 2.1.2 is assumed to be homotopy k-cartesian instead of homotopy $(k+1)$-cocartesian (see also Lemma 2.1.5 for a relation between the two).

Proof of Proposition 3.2.1. Consider the square (4) for $\mathsf{A} = \mathsf{Cob}^{\partial,k}_{\Theta,L}$ and $\mathsf{B} = \mathsf{Cob}^{\partial,k-1}_{\Theta,L}$, where the bottom left space is defined in Definition 3.2.4 and the left vertical map as the inclusion of the empty surgery datum. By Proposition 3.2.5, the map p is a weak equivalence and by Proposition 3.2.7 and an analogous extension as in [GR14, p. 301] to a homotopy between geometric realizations gives a lift K. By the discussion below (4), we conclude that the map $\mathsf{A} \hookrightarrow B$ induces an equivalence on classifying spaces. □

3.2.3. Proof of Proposition 3.2.2.

Recall now the definition of $V := (-2,0) \times \mathbf{R}^k \times \mathbf{R}^{d-k+1}_+$ and $\bar{V} := [-2,0] \times \mathbf{R}^k \times \mathbf{R}^{d-k+1}_+$ from [BP17, Section B.2].

Notation 3.2.9. We fix again an infinite set Ω. Let (Λ, δ, e) be a triple where $\Lambda \subset \Omega$ is a finite and $\delta : \Lambda \to [p]^\vee$ (see [BP17, p. 149]) is a function and $e : \Lambda \times \bar{V} \hookrightarrow \mathbf{R} \times [0,1) \times (0,1) \times (-1,1)^{\infty-1}$. We say that such triple is a *boundary surgery datum* for a p-simplex $(a, \epsilon, (W, \ell))$ in the nerve of $\mathsf{D}^{\partial,\mathsf{C}}_{\Theta,L}$ if the triple satisfies conditions (i) to (vi) in [BP17, Def. B.5]. We call the subspace $D_i = e(\delta^{-1}(i) \times \partial_- D^{k+1} \times \{0\})$ the *cores* of the trace of the surgery datum (Λ, δ, e) for $i = 0, \cdots, p$. The space of such data is topologized similarly to [BP17, Def B.5]. Notice that since the third coordinate is restricted to $(0,1)$, the embedding e is automatically disjoint from L.

Definition 3.2.10. Given a p-simplex $x = (a, \epsilon, (W, \ell))$ of $\mathsf{D}^{\partial,(k-1)}_{\Theta,L}$. Let $\mathsf{Z}_0(x)$ to be space of surgery data (Λ, δ, e) for x such that the map

$$\partial W|_{a_i} \cup D_i \to \partial^h W|_{[a_{i-1}, a_i]}$$

is k-connected for each $i = 1, \cdots, p$. Define $\mathsf{Z}_p(x)$ in the same way as [BP17, Def.B.5]. Similarly to [BP17] define the bi-semisimplicial space given by $(\mathsf{D}^{\partial,(k)}_{\Theta,L})_{p,q}$ to be the space of pairs (x, y) where x is a p-simplex of $\mathsf{D}^{\partial,(k-1)}_{\Theta,L}$ and $y \in \mathsf{Z}_q(x)$.

The following result should be compared to [BP17, Lemma B.6] and [GR14, Thm. 3.4] and is proved in the same way as Proposition 3.2.5.

Proposition 3.2.11. *Assume the hypothesis of Proposition 3.2.2, then the augmentation map*

$$\left(\mathsf{D}^{\partial,(k)}_{\Theta,L}\right)_{\bullet,\bullet} \to \left(\mathsf{D}^{\partial,(k-1)}_{\Theta,L}\right)_{\bullet}$$

is a weak equivalence after geometric realization.

The following result is essentially [BP17, Prop. B7].

Proposition 3.2.12. *Assume the hypothesis of Proposition 3.2.2, there exists a homotopy*

$$\mathsf{K} : I \times \left(\mathsf{D}^{\partial,(k)}_{\Theta,L}\right)_{p,0} \to \left(\mathsf{X}^{\partial,(k-1)}_{\Theta,L}\right)_{p}$$

such that the image of $(t, (W, \ell))$ lies in $\left(\mathsf{X}^{\partial,(k)}_{\Theta,L}\right)_p$ if $t = 1$ or $(W, \ell) \in \left(\mathsf{D}^{\partial,(k)}_{\Theta,L}\right)_p$.

Proof. Such family was constructed in [BP17, Prop. B.7] as a θ-manifold pair. We can follow exactly the same proof by replacing the map $\Psi^{\partial}_{\theta}(V) \to \Psi^{\partial}_{\theta_{d-1}}(V)$ by $\Psi^{\partial}_{\Theta}(V) \to \Psi^{\partial}_{\theta^{\partial}}(V)$. The former is a Serre fibration by [BP17, Lemma B.1]. By definition of $\Psi^{\partial}_{\Theta}(V)$ as a pullback, so is the latter. This property is the only necessary ingredient for the proof of [BP17, Prop. B.7]. The rest of the argument follows from [GR14, Lemma 3.7] applied to $\partial^h W$. □

Proof of Proposition 3.2.2. This follows analogously as the proof of Proposition 3.2.1 by replacing Proposition 3.2.5 by Proposition 3.2.11 and Proposition 3.2.7 by Proposition 3.2.12 and considering (4) for $\mathsf{A} = \mathsf{Cob}^{\partial,(k)}_{\Theta,L}$ and $\mathsf{B} = \mathsf{Cob}^{\partial,(k-1)}_{\Theta,L}$. □

4. Stable stability.

One of the key steps in the proof of Theorem A and a generalization of it allowing tangential structures is establishing some form of "stable stability" in the context of cobordism categories of manifolds with boundary. The first goal of this section is to make this wish into a precise claim (see Theorem 4.1.11 below). The second goal of this section is to reduce this statement to two closure properties of the subcategory of "stably stable cobordisms" (see Theorem 4.3.16 and Theorem 4.4.4 below). The proof of these statements is deferred to the next section. From now on, we specialize our study to odd-dimensional manifolds and cobordism categories whose morphisms are odd-dimensional cobordisms. Our definitions and proofs are close in spirit to [GR17b], in particular the proof of Theorem 2.15 in loc.cit. Nevertheless, we follow a slightly different strategy, but we refer to the analogous concepts and proofs in [GR17b] as they appear.

4.1. The statement.

The goal of this subsection is to state stable stability in the form of Theorem 4.1.11 below. The first important definition towards this goal is that of a Θ-end for a map of pairs $\Theta : (B, B^\partial) \to (\mathrm{BO}(d), \mathrm{BO}(d-1))$. To define this, consider the following definitions. Let (M, K) be a d-dimensional manifold pair (see Section 2.3), recall that (TM, TK) is a collared vector bundle pair, where the collar is given by an inwards-pointing vector field along K. A Θ-*structure* on (M, K) is a collared bundle map $\ell : (TM, TK) \to (\theta^*\gamma_d, (\theta^\partial)^*\gamma_{d-1})$. The *space of* Θ-*structures* on (M, K) is $\mathrm{Bun}^{\mathrm{col}}(TM, \Theta^*\gamma_d)$ as in Definition 2.2.2. A *framing* on a pair (M, K) is a fr_d-structure for $\mathrm{fr}_d : (\mathrm{EO}(d), \mathrm{EO}(d-1)) \to (\mathrm{BO}(d), \mathrm{BO}(d-1))$. The pair $(\mathbf{R}^d_+, \partial\mathbf{R}^d_+)$ admits the canonical framing coming from the basis $\{e_1, \cdots, e_d\}$.

Assumptions 4.1.1. For the entirety of this subsection, we fix the following choices:

- I An odd integer $d = 2n + 1 \geq 7$.
- II A map of pairs $\Theta : (B, B^\partial) \to (\mathrm{BO}(2n+1), \mathrm{BO}(2n))$ such that B^∂ is path-connected.
- III A collared bundle map $\tau : \mathrm{fr}_d^*(\gamma_{2n+1}, \gamma_{2n}) \to \Theta^*(\gamma_{2n+1}, \gamma_{2n})$. This induces a preferred Θ-structure for every framed $(2n+1)$-manifold pair. In particular, we have a preferred choice of Θ-structure on $(\mathbf{R}^{2n+1}_+, \partial\mathbf{R}^{2n+1}_+)$, called the *basepoint* Θ-*structure.* Moreover, up to homotopy τ is determined by the choice of basepoint Θ-structure on $(\mathbf{R}^{2n+1}_+, \partial\mathbf{R}^{2n+1}_+)$.
- IV A compact $(d-1)$-dimensional submanifold L of $[0,1) \times (-\frac{1}{2}, 0] \times (-1,1)^{\infty-2}$ and Θ-structure ℓ_L on L satisfying the assumptions of Definition 3.1.7. We assume that $\partial^{hv} L \neq \emptyset$.

We now introduce the notion of a *standard framing*. The following definitions are necessary for the definition of Θ-ends and are analogous to [GR17b, Defn.2.9]. For every integer $k \geq 0$, we consider the embedding

$$\rho : (D^k, \partial D^k) \hookrightarrow (\mathbf{R}^{k+1}_+, \partial\mathbf{R}^{k+1}_+)$$

taking $x = (x_1, \cdots, x_k) \in D^k \subset \mathbf{R}^k$ to $(\sqrt{1 - |x|^2}, x_1, \cdots, x_k)$. Define (V_1, W_1) to be the submanifold pair $(S^n \times \rho(D^{n+1}), S^n \times \rho(\partial D^{n+1})) \subset (\mathbf{R}^{n+1} \times \mathbf{R}^{n+2}_+, \mathbf{R}^{n+1} \times \partial\mathbf{R}^{n+2}_+)$ induced by ρ and the inclusion $S^n \subset \mathbf{R}^{n+1}$. Recall the triad $(D^k_+, \partial_0 D^k_+, \partial_1 D^k_+)$ from Definition 2.1.6.

Definition 4.1.2. The *right standard framing* ξ^r on $(S^n \times D^{n+1}_+, S^n \times \partial_0 D^{n+1}_+)$ is the framing induced by the codimension 0 embedding of pairs

$$(S^n \times D^{n+1}_+, S^n \times \partial_0 D^{n+1}_+) \hookrightarrow (\mathbf{R}^{n+1} \times \mathbf{R}^n_+, \mathbf{R}^{n+1} \times \partial \mathbf{R}^n_+) \cong (\mathbf{R}^{2n+1}_+, \partial \mathbf{R}^{2n+1}_+)$$
$$(x; y_0, \cdots, y_n) \mapsto (2e^{-\frac{y_1}{2}} x; y_0, y_2, \cdots, y_n)$$

We say that a Θ-structure on $S^n \times D^{n+1}_+$ is *standard* if it is homotopic to $\ell_r \coloneqq \tau \circ \xi^r$, that is, in the same path component in $\mathrm{Bun}^{\mathrm{col}}(T(S^n \times D^{n+1}_+), \Theta^* \gamma)$.

Definition 4.1.3. The *left standard framing* ξ^l on $(D^{n+1} \times D^n, \partial D^{n+1} \times D^n)$ is the framing induced by the codimension 0 embedding of pairs

$$(D^{n+1} \times D^n, \partial D^{n+1} \times D^n) \hookrightarrow (\mathbf{R}^{n+2}_+ \times \mathbf{R}^{n-1}, \partial \mathbf{R}^{n+2}_+ \times \mathbf{R}^{n-1}) = (\mathbf{R}^{2n+1}_+, \partial \mathbf{R}^{2n+1}_+)$$
$$(x; y_1, \cdots, y_n) \mapsto (2e^{-\frac{y_1}{2}} \rho(x); y_2, y_3, \cdots, y_n)$$

We say that a Θ-structure on $D^{n+1} \times D^n$ is *standard* if it is homotopic to $\ell_l \coloneqq \tau \circ \xi^l$, that is, in the same path component in $\mathrm{Bun}^{\mathrm{col}}(T(D^{n+1} \times D^n), \Theta^* \gamma)$.

Definition 4.1.4 (Standard Θ-structure)**.** Fix a closed $2n$-disc D inside the interior of the product of the lower hemispheres $D^n_- \times D^n_- \subset S^n \times S^n$ and denote the complement $W_1 \backslash \mathrm{int}(D)$ by $W_{1,1}$. A Θ-structure ℓ on the pair $(V_1, W_{1,1})$ is *standard* if both structures $\bar{e}^* \ell$ and $\bar{f}^* \ell$ are standard, where $\bar{e}$ and $\bar{f}$ are the embeddings defined by

$$\bar{e} : (S^n \times D^{n+1}_+, S^n \times \partial_0 D^{n+1}_+) \hookrightarrow (S^n \times \rho(D^{n+1}), S^n \times \rho(\partial D^{n+1})) \subset \mathbf{R}^{n+1} \times \mathbf{R}^{n+2}_+$$

induced by the inclusion $D^{k+1}_+ \subset D^{k+1}$ and ρ, and

$$\bar{f} : (D^{n+1} \times D^n, S^n \times D^n) \hookrightarrow (S^n \times \rho(D^{n+1}), S^n \times \rho(\partial D^{n+1})) \subset \mathbf{R}^{n+1} \times \mathbf{R}^{n+2}_+$$
$$(x; y) \mapsto \left(y, \sqrt{1 - |y|^2}; \rho(x)\right)$$

Lemma 4.1.5. *The space of standard Θ-structures on $(V_1, W_{1,1})$ extending the base-point Θ-structure (see III) on a disc in $\partial W_{1,1}$ is non-empty and connected.*

Proof. This is an adaptation of [GR17a, Lemma 7.6/7]. Denote the images of $\bar{e}$ and $\bar{f}$ by E and F, respectively. Observe that the intersection $E \cap F$ is a contractible pair, *i.e.* equivalent to $(*, *)$ as a pair. Two framings on a contractible manifold pair are homotopic if and only if they induce the same orientation. Since both $\bar{e}$ and $\bar{f}$ are orientation preserving embeddings, it follows that the restrictions of a right standard framing on E and a left standard framing on F to $E \cap F$ are homotopic. Thus, we can homotope them to agree on this subspace pair. This defines a standard framing on $E \cup F$, which is isotopy equivalent to V_1 as pairs. This proves the first claim.

Let us denote the space of standard Θ-structures on $(V_1, W_{1,1})$ by X. Denote by X_0 (resp. X_1) the path component of $\mathrm{Bun}^{\mathrm{col}}(T(S^n \times D^{n+1}_+), \Theta^*\gamma)$ (resp. $\mathrm{Bun}^{\mathrm{col}}(T(D^{n+1} \times D^n), \Theta^*\gamma)$) of standard Θ-structures. Denote also by Z the path component of $\mathrm{Bun}^{\mathrm{col}}(T(E \cap F), \Theta^*\gamma)$ given by the basepoint Θ-structure (here we are identifying $E \cap F$ with $(\mathbf{R}^{2n+1}_+, \partial\mathbf{R}^{2n+1}_+)$). One can show that the restriction maps $X_i \to Z$ are Serre fibrations (see Proposition A.5), and thus X is equivalent to the homotopy pullback $X_0 \times_Z X_1$. Denote by X^* the space of standard Θ-structures on $(V_1, W_{1,1})$ extending the basepoint Θ-structure on a disc D in $\partial W_{1,1}$. We want to show that X^* is path-connected. Denote also by X_0^* and X_1^* the analogous spaces of Θ-structures extending the basepoint Θ-structure on D. Observe that the previous pullback description of X implies that X^* is equivalent to the product $X_0^* \times X_1^*$, since Z is equivalent to the space of Θ-structures on D. Thus, to prove that X^* is path-connected, it suffices to show that X_i^* is path-connected for $i = 0, 1$. For $i = 0$, this now follows by the fact that X_0 is path-connected and the fact that we have a fiber sequence $X_0^* \to X_0 \to Z$ that admits a section $s : Z \to X_0$ by pre-composing a Θ-structure on $E \cap F$ with the embedding from Definition 4.1.2. For $i = 1$, this follows verbatim. □

As in [GR17b, Defn. 2.5], it will be convenient to consider a version of the cobordism category $\mathsf{Cob}^{\partial}_{\Theta,L}$ defined in Definition 3.1.7 where the submanifold L from 6.3.1 is removed in the following way.

Definition 4.1.6. Let L be as in Assumptions 4.1.1. Define the non-unital topological category $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ to have objects $(M^\circ, \ell|_{M^\circ})$ for $(M, \ell) \in \mathsf{Cob}^{\partial}_{\Theta,L}$, where $(M^\circ, \partial^h M^\circ) \coloneqq (M \backslash \operatorname{int}(L), \partial M \backslash \operatorname{int}(\partial^h L))$. Here $\operatorname{int}(L) \coloneqq L \backslash \partial^v L$ and $\operatorname{int}(\partial^h L) = \partial^h L \backslash \partial^{hv} L$. The morphisms are triples $(t, (W^\circ, \ell|_{W^\circ}))$ for $(t, (W, \ell))$ a morphism in $\mathsf{Cob}^{\partial}_{\Theta,L}$, where $(W^\circ, \partial^h W^\circ) \coloneqq (W \backslash \operatorname{int}(L) \times [0,t], \partial^h W \backslash \operatorname{int}(\partial^h L) \times [0,t])$. The topology on the space of objects and morphisms is the unique one such that the assignment $(M \mapsto M^\circ)$ from $\mathsf{Cob}^{\partial}_{\Theta,L}$ to $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ is an isomorphism of topological categories. Similarly, define $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ to be the subcategory with the same objects and those morphisms $W : M \rightsquigarrow M'$ such that $(M', \partial^h M') \to (W, \partial^h W)$ is strongly $(n-1)$-connected.

Remark 4.1.7. Observe that given $W : M \rightsquigarrow N$ in $\mathsf{Cob}^{\partial}_{\Theta,L}$, then the 4-ad $(W^\circ, \partial^h W^\circ, \partial^v L \times [0,1], M^\circ \sqcup N^\circ)$ is a triad cobordism in the sense of Definition 2.3.1.

Remark 4.1.8. Recall that $\mathsf{Cob}^{\partial,n-1}_{\Theta,L}$ is a subcategory of $\mathsf{Cob}^{\partial}_{\Theta,L}$ of those morphisms $W : M \rightsquigarrow N$ such that $(N, \partial N) \to (W, \partial^h W)$ is strongly $(n-1)$-connected. It turns out that $\mathsf{Cob}^{\partial,n-1}_{\Theta,L}$ is not, in general, isomorphic to $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$. It will be clear later that $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ is more convenient for our purposes. However, under some conditions on L (which will be satisfied in our case), we will see that $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ is isomorphic to $\mathsf{Cob}^{\partial,n-1}_{\Theta,L}$ (see Lemma 6.2.6 below).

We are now ready for the definition of a Θ-end. From now on, we directly work with the underlying manifolds and Θ-structures where L is cut out. Therefore, we drop the notation of M° unless necessary, that is, when we talk about an object $(M, \ell) \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$, then $(M, \partial^h M) = (N^\circ, \partial^h N^\circ)$ for some $N \in \mathsf{Cob}^{\partial}_{\Theta,L}$.

Definition 4.1.9. Let L be as in Assumptions 4.1.1, a Θ-*end* K in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ is a sequence of composable morphisms

$$\{K|_{[i,i+1]} : K|_i \rightsquigarrow K|_{i+1}\}_{i\geq 0}$$

in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ such that:

(i) the inclusions $(K|_i, \partial^h K|_i) \hookrightarrow (K|_{[i,i+1]}, \partial^h K|_{[i,i+1]}) \hookleftarrow (K|_{i+1}, \partial^h K|_{i+1})$ are strongly $(n-1)$-connected, for every $i \geq 0$;
(ii) for every $i \geq 0$, there exists an embedding $\omega : (V_1, W_{1,1}) \hookrightarrow (K|_{[i,i+1]}, \partial^h K|_{[i,i+1]})$ in a path component which intersects $\partial^v L$. We require that $\mathrm{im}(\omega)$ is disjoint from $\partial^v L$ and that $\omega^* \ell_{K_{[i,i+1]}}$ is a standard Θ-structure.

For any Θ-end K in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$, define

$$\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P, K|_i) \subset \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}(P, K|_i)$$

to be the subspace of those Θ-cobordisms (s, W) such that $\ell_W : (W, \partial^h W) \to (B, B^\partial)$ is strongly n-connected. The following is similar to [GR17b, Lemma 2.14].

Lemma 4.1.10. *For every $i \geq 0$, the subspace $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P, K|_i) \subset \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}(P, K|_i)$ determines a subfunctor*

$$\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(-, K|_i) : (\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L})^{op} \to \mathbf{Top}$$

of $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}(-, K|_i)$. Moreover, post-composition with $K|_{[i,i+1]}$ defines a natural transformation

$$\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(-, K|_i) \Rightarrow \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(-, K|_{i+1}).$$

Proof. The first claim is equivalent to the following: given $M : Q \rightsquigarrow P$ in $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ and $W : P \rightsquigarrow K|_i$ such that ℓ_W is strongly n-connected, then $\ell_{W\circ M}$ is strongly n-connected. To prove this, it suffices to observe that $(W, \partial^h W) \hookrightarrow (M \cup_P W, \partial^h M \cup_{\partial^h P} \partial^h W)$ is strongly $(n-1)$-connected, by Lemma 2.1.3. This follows since $(P, \partial^h P) \hookrightarrow (M, \partial^h M)$ is strongly $(n-1)$-connected, using Lemma 2.1.4. The second claim follows exactly by the same argument using that $(K|_i, \partial^h K|_i) \hookrightarrow (K|_{[i,i+1]}, \partial^h K|_{[i,i+1]})$ is strongly $(n-1)$-connected. □

Denote by $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P, K|_\infty)$ the homotopy colimit of $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P, K|_i)$ under the stabilization maps $K|_{[i,i+1]} \circ (-)$. By Lemma 4.1.10, this defines a presheaf of spaces on $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$, by taking a functorial model of the homotopy colimit, e.g. the telescope. We recall from the introduction that a map of spaces $f : X \to Y$

is called an *abelian homology equivalence* if $f_*: \mathrm{H}_k(X; f^*\mathcal{L}) \to \mathrm{H}_k(Y;\mathcal{L})$ is an isomorphism for all *abelian* local systems $\mathcal{L}: \Pi_1(Y) \to \mathbf{Ab}$, that is, those $\mathcal{L}$ such that for every $x \in X$ the action of the commutator subgroup of $\pi_1(X,x)$ on $\mathcal{L}(x)$ is trivial. We are now ready to state the main result of this section. Recall the Assumptions 4.1.1.

Theorem 4.1.11 (Stable stability)**.** *For any* Θ*-end* K*, the functor* $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(-,K|_\infty)$ *takes morphisms to abelian homology equivalences, provided* (B, B^∂) *is* 1*-connected.*

The path to the proof of this statement is inspired by the proof of [GR17b, Thm. 2.15] and will roughly go as follows. First, we will reduce this statement to a claim about the behaviour of the functor $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(-,K|_\infty)$ under certain simpler types of morphisms. Second, for each of these types, we further reduce this statement to a certain *closure property*, which is then proved in the next section. Theorem 4.1.11 is the only result we use in later sections. The next subsection establishes the first step in the proof of this statement.

4.2. Elementary simplifications of Theorem 4.1.11.

From now on we fix a Θ-end K. The goal of this subsection is to reduce the proof of Theorem 4.1.11 for K to the verification that certain "elementary" morphisms in $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ are sent to abelian homology equivalences by $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(-,K|_\infty)$. To do so, consider the following definition. Let $\mathcal{W}$ be the collection of morphisms $M: P \rightsquigarrow Q$ of $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ such that $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(Q,K|_\infty) \to \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P,K|_\infty)$ is an abelian homology equivalence. Notice that Theorem 4.1.11 is equivalent to $\mathcal{W} = \mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$. We start by showing that $\mathcal{W}$ is a subcategory which contains all isomorphisms in the sense of the definition below. Analogously to the definition below Proposition 3.1.8, a subcategory of $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ is a collection of morphisms closed under composition, which is the union of entire path components and contains $[0,t] \times P$ with the cylindrical Θ-structure $\ell_P \oplus \varepsilon^1$.

Definition 4.2.1. Let C be a subcategory of $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ and $W: M \rightsquigarrow N$ be a morphism in C. We say that W is an *isomorphism* if there exists another morphism $W': N \rightsquigarrow M$ in C such that $W \circ W'$ and $W' \circ W$ are in the same path components of $[0,s] \times N$ and $[0,s] \times M$ (both with the cylindrical Θ-structure) in $\mathsf{C}(N,N)$ and in $\mathsf{C}(M,M)$, respectively, for some $s > 0$.

Example 4.2.2. Let $t > 0$ and $P \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$. For any Θ-structure ℓ on $[0,t]\times P$, the morphism $W := (t, [0,t]\times P, \ell)$ is an isomorphism: let W^* be the dual Θ-cobordism given by precomposing ℓ with map on tangent bundles induced by the reflection around the axis $\{\frac{t}{2}\} \times \mathbf{R}^\infty_+$; observe that W and W^* are composable morphisms, whose compositions have Θ-structure homotopic to $[0,2t] \times \ell_P$ relative to both ends. Hence, W and W^* are mutual inverses and hence are isomorphisms. In general, any morphism (t, W) whose underlying manifold is diffeomorphic to $[0, s] \times P$ for some $s > 0$ is an isomorphism.

Lemma 4.2.3. *The collection $\mathcal{W}$ is a subcategory of $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ which contains all isomorphisms.*

Proof. Start by observing that $\mathcal{W}$ is closed under composition since abelian homology equivalences are closed under composition. To show that $\mathcal{W}$ contains entire path components of the morphism spaces $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}(P, Q)$, observe that a path in $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}(P, Q)$ from $(t, (W, \ell))$ to $(t', (W', \ell'))$ induces a homotopy between the induced pre-composition maps by these morphisms, since the composition law is continuous. Thus, $\mathcal{W}$ contains entire path components since abelian homology equivalences are homotopy invariant. To show that $[0, s] \times P$ with the cylindrical Θ-structure is in $\mathcal{W}$, it suffices to observe that the induced pre-composition map is isotopic to the identity by scaling the s-parameter and thus a homotopy equivalence.

We are left to show that all isomorphisms are in $\mathcal{W}$. Let $W : M \rightsquigarrow N$ be an isomorphism in $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ and $W' : N \rightsquigarrow M$ be another morphism in $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ such that $W \circ W'$ and $W' \circ W$ are in the same path component as the corresponding cylinders $[0, s] \times N$ or $[0, s] \times M$. This implies that the induced maps of W and W' are homotopy inverses to each other. Thus, $(-) \circ W$ is a homotopy equivalence and hence an abelian homology equivalence. ◻

Lemma 4.2.4. *The subcategory $\mathcal{W} \subset \mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ has the 2-out-of-3 property.*

Proof. This follows from the 2-out-of-3 property for abelian homology equivalences, which we prove now. Let $g \circ f : X \to Y \to Z$ be two maps of spaces, where at least two out of the three maps f, g or $g \circ f$ are abelian homology equivalences. Thus, in particular, all maps are homology equivalences since homology equivalences satisfy the 2-out-of-3 property. In particular $\mathrm{H}_1(X) \cong \mathrm{H}_1(Y) \cong \mathrm{H}_1(Z)$. Suppose g and $g \circ f$ are abelian homology equivalences and let $\mathcal{L}$ be an abelian system on Y. Then, since $\mathrm{H}_1(Y) \cong \mathrm{H}_1(Z)$, there exists a local system $\mathcal{L}'$ on Z such that $\mathcal{L} = g^*\mathcal{L}'$. This implies that for f induces an isomorphism for all abelian systems on Y. The remaining cases are simpler since one always considers abelian local systems on Z and systems which are pulled back from abelian systems are themselves abelian. ◻

Recall the definition of elementary triad cobordisms, their type and index from Definition 2.3.7. Often we will use results in Section 2.3 to deduce consequences for morphisms in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$. We will hinge on the following remark.

Remark 4.2.5 (Factorizations = Triad handle decompositions)**.** Let $W : M \rightsquigarrow N$ be a morphism in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$. Given a triad handle decomposition of the underlying triad cobordism of W in the sense of Definition 2.3.3, there exists a path in

$\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}(M,N)$ from W to a composition

$$M \overset{W_1}{\rightsquigarrow} M_1 \overset{W_2}{\rightsquigarrow} M_2 \rightsquigarrow \cdots \rightsquigarrow M_k = N$$

whose underlying triad cobordisms W_i are elementary of the types and indices present in the triad handle decomposition. This is seen by the following argument: The triad handle decomposition induces a decomposition $W = W_1 \cup_{M_1} W_2 \cup_{M_2} \cup \cdots \cup_{M_{k-1}} W_k$ where W_i is a triad elementary cobordism from M_i to M_{i+1}. We can embed the underlying triad W such that M_i lies in the subspace $\{i\} \times \mathbf{R}^{\infty}_{+}$, the cobordisms W_i lie in $[i, i+1] \times \mathbf{R}^{\infty}_{+}$ and satisfy the appropriate collaring conditions to be seen as morphisms in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}(M_i, M_{i+1})$ for all $i \geq 0$. By the Whitney's embedding theorem, one can see that this embedding of W is isotopic to the original inclusion of W into $\mathbf{R} \times \mathbf{R}^{\infty}_{+}$. Moreover, such an isotopy can be promoted to a path in the morphism space $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}(M,N)$, hence justifying the original claim.

We move now to the first simplification of Theorem 4.1.11.

Lemma 4.2.6. *If $\mathcal{W}$ contains every morphism $M : P \rightsquigarrow Q$ in $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ whose underlying triad cobordism is elementary relative to Q of the following type:*

(i) left type and index $n+1 \leq k < 2n+1$;
(ii) right type and index $n \leq k < 2n$;
(iii) interior type and index $2n$,

then $\mathcal{W} = \mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$.

Proof. By combining Propositions 2.3.12 and 2.3.17 and Remark 4.2.5, every morphism M in $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ admits a factorization into elementary triad cobordisms of index at least n for right type, $n+1$ for left type and $2n+1$ for interior type relative to Q, up to a path in the morphism space $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}(P,Q)$. By Lemma 4.2.4, M is in $\mathcal{W}$ if all elementary pieces are. By hypothesis, all the elementary pieces are in $\mathcal{W}$ except the ones of left or interior type of index $2n+1$ and right type of index $2n$. We prove now that our assumptions imply that these cases lie in $\mathcal{W}$ too.

Let $M : P \rightsquigarrow Q$ be a morphism of $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ whose underlying cobordism is elementary of interior type and index $2n+1$ relative to Q. Let $\phi : S^{2n} \hookrightarrow Q$ be the attaching map of the interior $(2n+1)$-handle of such a triad handle decomposition. By invariance of domain, this embedding is a diffeomorphism onto one component of Q. In particular, this component is disjoint from the boundary and therefore disjoint from $\partial^v L$. Choose an embedding $\varphi : S^0 \times D^{2n} \hookrightarrow Q$ sending one disc to the image of ϕ and the other to the interior of any other component (which exists since $\partial^{hv} L \neq \emptyset$ and by the discussion above). Let U be the result of attaching an interior handle to Q along this attaching map. This cobordism admits a Θ-structure extending ℓ_Q since B is path-connected, and thus we can see it as a morphism $U : Q \rightsquigarrow R$ in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$. Moreover, the inclusion $(R, \partial^h R) \to (U, \partial^h U)$

is strongly $(2n-2)$-connected and thus it lies in $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$. The composition $U \circ M$ consists of a $2n+1$ and a $2n$-handle of interior type relative to R such that the belt (recall Definition 2.3.7) of the latter intersects the attaching map of the former exactly at one point, by construction. Thus, by Proposition 2.3.14, $U \circ M$ is diffeomorphic to $R \times [0,1]$ as a cobordism relative to R, which is in $\mathscr{W}$ by Lemma 4.2.3 and Example 4.2.2. By hypothesis, U is in $\mathscr{W}$ and thus so is M, by Lemma 4.2.4.

Let $M : P \rightsquigarrow Q$ be a morphism in $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ whose underlying cobordism is elementary of right type of index $2n$ relative to Q. By invariance of domain, the attaching map $\phi : S^{2n-1} \hookrightarrow \partial^h Q$ is a diffeomorphism onto one path component of Q disjoint from $\partial^v L$. Choose similarly an embedding $\varphi : S^0 \times D^{2n-1} \hookrightarrow \partial^h Q$ sending one disc to the image of ϕ and the other to any other component of $\partial^h Q$ and let $U : Q \rightsquigarrow R$ be the result of adding a right handle along this embedding. Since B^∂ is path-connected, U admits a Θ-structure and is strongly $(n-1)$-connected relative to its target, so it lies in $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$. The composition $U \circ M$ consists of a right $2n$-handle and a left $2n$-handle such that the belt of the latter intersects the attaching sphere of the former in exactly one point. By Proposition 2.3.13, this is isomorphic to an elementary interior cobordism of index $2n$ relative to Q. This is in $\mathscr{W}$ by hypothesis and since U is by assumption, so is M.

Let $M : P \rightsquigarrow Q$ be a morphism in $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ whose underlying cobordism is elementary of left type of index $2n+1$ relative to Q. Consider the attaching map $\phi : (D^{2n}, S^{2n-1}) \hookrightarrow (Q, \partial^h Q)$. Once again by invariance of domain, this embedding is a diffeomorphism onto one component of Q. In particular, this component does not contain the boundary and therefore is disjoint from $\partial^v L$. The map $\phi|_{S^{2n-1}}$ is closed and open and thus a diffeomorphism onto a component of $\partial^h Q$. Let $\varphi : S^0 \times D^{2n-1} \hookrightarrow \partial^h Q$ be an embedding sending one disc to the image of ϕ and the other one to another path component (once again exists by the discussion above, since $\partial^{hv} L$ is non-empty). Let $U : Q \rightsquigarrow R$ be the result of a 1-right handle attachment at φ. By the same reason as before, U admits a Θ-structure and is strongly $(n-1)$-connected relative to its target, so it lies in $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$. By Proposition 2.3.14, the composite $U \circ M$ is a cancelling pair of a left $2n$ and left $(2n+1)$-handle and thus is in $\mathscr{W}$. Since $U \in \mathscr{W}$ by hypothesis, we see that $M \in \mathscr{W}$. This finishes the proof. □

We introduce the following definition and notation, which will be convenient for the proof of Theorem 4.1.11.

Notation 4.2.7 (Basepoint component)**.** Notice that $\partial^{hv} L$ has finitely many connected components (by compactness of L) and is non-empty. By definition, each $K|_{[i,i+1]}$ contains an embedded copy of V_1 in some path component intersecting $\partial^v L$. There exists a path component $\partial_0 \partial^v L$ of $\partial^{hv} L$ such that for infinitely many i, $\partial^h K|_{[i,i+1]}$ contains an embedded copy of $\partial^h V_1 = W_{1,1}$ in a path component of $\partial^h K|_{[i,i+1]}$ intersecting $\partial_0 \partial^v L$. Choose a path component $\partial^v_0 L$ of $\partial^v L$

such that $\partial(\partial_0^v L)$ contains $\partial_0\partial^v L$ and call it the *basepoint component*. Observe that $(\partial_0^v L, \partial_0\partial^v L)$ is a pair in the sense of Section 2.3. It follows that for infinitely many i, $K|_{[i,i+1]}$ contains an embedded copy of V_1 in a path component intersecting $\partial_0^v L$.

Assumptions 4.2.8. Fix a basepoint component $(\partial_0^v L, \partial_0\partial^v L)$ as above. For the purposes of Theorem 4.1.11, we may assume that for every $i \geq 0$, the cobordism $K|_{[i,i+1]}$ contains a V_1 in a path component intersecting $\partial_0^v L$ by the following argument: By composing the appropriate number of $K|_{[i,i+1]}$ and rescaling the first coordinate, we produce a Θ-end K' with the same underlying manifold (that is, the union of all $K'_{[i,i+1]}$) and isotopy equivalent submanifold of $\mathbf{R} \times \mathbf{R}^\infty_+$, that has this property. This assumption will not affect the conclusion of Theorem 4.1.11, since it takes the colimit over i. Such colimit for K agrees with the one for K'.

Given a morphism $M : P \rightsquigarrow Q$ such that the underlying cobordism is elementary of any type and of index $k \geq n$ relative to Q, we say that it is *attached to the basepoint component* if the attaching map of the k-handle lies in a component of Q intersecting the basepoint component $\partial_0^v L$ and the horizontal boundary of the attaching map of k-handle lies in a component of $\partial^h Q$ intersecting the component $\partial_0\partial^v L \subset \partial(\partial_0^v L)$ for some handle decomposition of M relative to Q.

Lemma 4.2.9. *The conclusion of Lemma 4.2.6 still holds if additionally M is assumed to be attached to the basepoint component.*

Proof. It suffices to prove that an elementary triad cobordism $M : P \rightsquigarrow Q$ of type and index as in the statement but not attached in the basepoint component lies in $\mathcal{W}$. We proceed by checking this for each type. Let $M : P \rightsquigarrow Q$ be elementary of right type and index $n \leq k < 2n$ relative to Q and let $\phi : \partial D^k \times D^{2n-k} \hookrightarrow \partial^h Q$ be its attaching map. Choose an embedding $\varphi : S^0 \times D^{2n-1} \hookrightarrow \partial^h Q$ sending one disc to the component of the image of ϕ but disjoint to it (possible because $k < 2n$) and the other disc to the basepoint component $\partial_0\partial^v L$. Define $T : Q \rightsquigarrow S$ to be an elementary triad cobordism relative to Q given by attaching a right handle along φ with any Θ-structure extending ℓ_Q (this is again possible since B^∂ is path connected, as in Lemma 4.2.6). Since $\partial^h T$ is the trace of the surgery of $\partial^h Q$ at φ, the belt of this handle lies in the basepoint component in S. Thus T is an elementary left cobordism of index $2n$ relative to S attached to the basepoint component in S and thus in $\mathcal{W}$. By Proposition 2.3.10 and Remark 4.2.5, there exists a factorization $M' \circ T' : P \rightsquigarrow R \rightsquigarrow S$ of $T \circ M$, where T' is right elementary with index 1 relative to P and M' is right elementary of index k. The attaching map of T' in $\partial^h P$ is one disc in the basepoint component and another disc, thus be the same argument above, T' is left elementary of index $2n$ attached to the basepoint component at R. Thus, $T' \in \mathcal{W}$. However, the component of the image of the attaching map of M' is the basepoint component, since T' performed the 1-surgery between the component of ϕ and the basepoint component. Thus, $M \in \mathcal{W}$.

When $M : P \rightsquigarrow Q$ is elementary of left type and index $n+1 \leq k < 2n+1$ relative to Q, we proceed similarly by defining $\varphi : S^0 \times D^{2n-1} \hookrightarrow \partial^h Q$ to send one disc to the basepoint component and another to the component of the image of the attaching map of the $(k-1)$-handle in $\partial^h M$. Define $T : Q \rightsquigarrow S$ to be the elementary triad cobordism given by attaching a right handle along φ along with any Θ-structure extending ℓ_Q (this is again possible since B^∂ is path connected, as above). We obtain a factorization $T \circ M \cong M' \circ T' : P \rightsquigarrow S$ where $T' : P \rightsquigarrow R$ is left elementary of index $2n$ relative to R and M' left elementary of index k relative to S. Proceeding exactly as in the last paragraph, we see that the morphisms T and T' are in $\mathcal{W}$ since they are both attached to the basepoint component. By the same argument as above, M' is attached to the basepoint component and thus lies in $\mathcal{W}$. This implies that $M \in \mathcal{W}$.

When $M : P \rightsquigarrow Q$ is of interior type and index $2n$, we proceed similarly by defining $\varphi : S^0 \times D^{2n} \hookrightarrow Q$ to send one disc to (interior of) the basepoint component and another to the component of (but disjoint from) the image of the attaching map of the $2n$-handle in $\partial^h M$. We define $T : Q \rightsquigarrow S$ to be the elementary interior cobordism of index 1 relative to Q along with any Θ-structure extending ℓ_Q (this is now possible since B is path connected, as in Lemma 4.2.6). Since T is interior elementary of index $2n$ relative to S and attached to the basepoint component, it lies in $\mathcal{W}$. Proceeding similarly to before, we deduce that $M \in \mathcal{W}$. This finishes the proof. □

4.3. The closure property for middle handles.

We move now to the second goal of this section: reducing the proof of Theorem 4.1.11 to two *closure properties* of the subcategory $\mathcal{W}$, as mentioned in the introduction. This will be done in the next two subsections, where we will state these properties and prove that they imply Theorem 4.1.11. By Lemma 4.2.9, it suffices to prove that certain elementary triad cobordisms are in $\mathcal{W}$ to show that $\mathcal{W} = \mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$. Roughly speaking, we will check this condition by induction on the index of the elementary triad cobordisms. In this subsection, we concern ourselves with the base case of this induction, which is proved using the first closure property (see Theorem 4.3.16 below).

Roughly speaking, the main strategy can be seen as a generalization of the strategy of Lemma 4.2.9 without assuming T and T' are in $\mathcal{W}$ but having $W' \in \mathcal{W}$ to deduce that $W \in \mathcal{W}$. The next two subsubsections introduce models for "rearrangements" of morphisms and "traces of surgeries". These notions are also used in Section 4.4 so we phrase them in higher generality.

4.3.1. Models for rearrangements. We start by modelling rearrangements (see Proposition 2.3.10 for the terminology and also Remark 4.3.4 below for the precise relation) of morphisms in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$. This is inspired by [GR17b, Section 4.1].

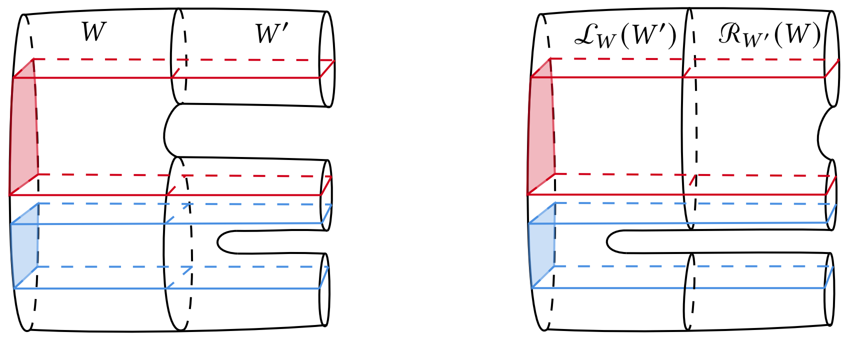


Figure 6: In this picture, the red and blue shaded areas represent the supports of W and W' respectively. Notice that the support of $\mathcal{L}_W(W')$ is the blue shaded area and the support of $\mathcal{R}_{W'}(W)$ is the red shaded area.

Definition 4.3.1. Let $(t, W) : M \rightsquigarrow N$ be a morphism in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$. The support $\mathrm{supp}(W)$ of W is the smallest closed subset A of $\mathbf{R}^\infty_+$ such that

$$W \cap ([0,t] \times \mathbf{R}^\infty_+ \backslash A) = [0,t] \times M \backslash A$$

as Θ-manifolds (recall Definition 3.1.1 for the terminology).

Definition 4.3.2. Let $(t, W) : M \rightsquigarrow N$ and $(t', W') : N \rightsquigarrow R$ be morphisms in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ such that $\mathrm{supp}(W) \cap \mathrm{supp}(W') = \emptyset$. We define the *rearrangement* (or *interchange of support*) to be the morphisms

$$\mathcal{R}_{W'}(W) = (W \backslash [0,t] \times \mathrm{supp}(W')) \cup ([0,t] \times R \backslash \mathrm{supp}(W))$$

and

$$\mathcal{L}_W(W') = (W' \backslash [0,t'] \times \mathrm{supp}(W)) \cup ([0,t'] \times M \backslash \mathrm{supp}(W'))$$

See Figure 6 for an example.

Remark 4.3.3. Notice that $\mathcal{R}_{W'}(W)$ is a morphism in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ from $N' \coloneqq (M \backslash \mathrm{supp}(W')) \cup (R \backslash \mathrm{supp}(W))$ to $(N \backslash \mathrm{supp}(W')) \cup (R \backslash \mathrm{supp}(W)) = R$, since $N \backslash \mathrm{supp}(W') = R \backslash \mathrm{supp}(W')$. Similarly, $\mathcal{L}_W(W')$ is a morphism from M to N'. Moreover, there exists a path $\tau(W, W') : [0,1] \to \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}(M, R)$ from $W' \circ W$ to $\mathcal{R}_{W'}(W) \circ \mathcal{L}_W(W')$. See [GR17b, p. 148] for more details. In particular, both compositions are diffeomorphic relative to their ends.

Remark 4.3.4. The term interchange of support is used in [GR17b]. We will also use the term *rearrangement* since when W and W' are elementary, then $\mathcal{R}_{W'}(W)$ and $\mathcal{L}_W(W')$ are a model for the rearrangement of W and W' (in the sense of Proposition 2.3.10). That is, $\mathcal{R}_{W'}(W)$ and $\mathcal{L}_W(W')$ are elementary of type and index of W and W', respectively, and $W' \circ W \cong \mathcal{R}_{W'}(W) \circ \mathcal{L}_W(W')$. More precisely, the underlying cobordism of $\mathcal{R}_{W'}(W)$ is obtained from N' by attaching a handle at the attaching map of W in $M \backslash \operatorname{supp}(W') \subset N'$ and the underlying cobordism of $\mathcal{L}_W(W')$ is obtained from M by attaching a handle at the attaching map of W' in $N \backslash \operatorname{supp}(W) \subset M$.

4.3.2. Models for traces of surgeries. In Lemma 4.2.9, we saw that it suffices to prove that certain elementary triad cobordisms are in $\mathcal{W}$. We now provide models for elementary triad cobordisms (recall Definition 2.3.7), which come with choices of a core, cocore, and attaching and belt spheres. Moreover, the support of these models is easy to describe. We prove below in Lemma 4.3.10 that any elementary triad cobordism can be realized as such model up to composition with an isomorphism in the sense of Definition 4.2.1.

Construction 4.3.5 (Interior traces)**.** Let $k \geq 0$ be an integer. Let $A^I \coloneqq \{0\} \times \partial D^k \times \mathbf{R}^{2n+1-k} \subset [0,1] \times \mathbf{R}^k \times \mathbf{R}^{2n+1-k}$ and let T^I be a submanifold of $[0,1] \times \mathbf{R}^k \times \mathbf{R}^{2n+1-k}$ seen as a triad with $\partial^h T^I = \emptyset$ satisfying the following properties:

(i) T^I agrees with $[0,1] \times A^I$ outside $[0,1] \times D^k \times D^{2n+1-k}$,
(ii) $T^I \cap ([0,\epsilon) \times \mathbf{R}^{2n+1}) = [0,\epsilon) \times A^I$, for some $0 < \epsilon < 1$,
(iii) $T^I \cap ((1-\epsilon, 1] \times \mathbf{R}^{2n+1}_+) = (1-\epsilon, 1] \times P$ for a manifold P diffeomorphic to $\mathbf{R}^k \times \partial D^{2n+1-k}_+$, for some $0 < \epsilon < 1$.

such that the triad T^I is obtained from $A^I = T^I \cap \{0\} \times \mathbf{R}^k \times \mathbf{R}^{2n+1-k}$ by attaching an interior k-handle along the inclusion

$$\partial D^k \times D^{2n+1-k} \hookrightarrow \partial D^k \times \mathbf{R}^{2n+1-k}.$$

Such submanifold exists: take the graph of the function $f : D^k \times D^{2n+1-k} \longrightarrow [-1,1]$ given by $f(x,y) = \sum_{i=1}^k x_i^2 - \sum_{j=1}^{2n+1-k} y_j^2$, scale and translate it to have image $[0,1]$. In order to satisfy the properties above, one must add a ϵ-collar to it. The core and cocore of T^I are the core and cocore of the interior k-handle in the sense of Definition 2.3.3.

Construction 4.3.6 (Right traces)**.** Let $k \geq 0$ be an integer. Let $(A^R, \partial A^R) \coloneqq \{0\} \times \partial D^k \times (\mathbf{R}^{2n+1-k}_+, \partial \mathbf{R}^{2n+1-k}_+)$ and let $(T^R, \partial^h T^R)$ be a submanifold pair of $[0,1] \times \mathbf{R}^k \times (\mathbf{R}^{2n+1-k}_+, \partial \mathbf{R}^{2n+1-k}_+)$ satisfying the following properties:

(i) $(T^R, \partial^h T^R)$ agrees with $[0,1] \times A^R$ outside $[0,1] \times D^k \times (D^{2n+1-k}_+, \partial_0 D^{2n+1-k}_+)$,
(ii) $(T^R, \partial^h T^R) \cap ([0,\epsilon) \times \mathbf{R}^k \times (\mathbf{R}^{2n+1-k}_+, \partial \mathbf{R}^{2n+1-k}_+)) = [0,\epsilon) \times (A^R, \partial A^R)$, for some $0 < \epsilon < 1$,
(iii) $(T^R, \partial^h T^R) \cap ((1-\epsilon, 1] \times \mathbf{R}^k \times (\mathbf{R}^{2n+1-k}_+, \partial \mathbf{R}^{2n+1-k}_+)) = (1-\epsilon, 1] \times (P, \partial^h P)$ for a manifold pair $(P, \partial^h P)$ diffeomorphic to $(\mathbf{R}^k \times \partial_1 D^{2n+1-k}_+, \mathbf{R}^k \times \partial_{01} D^{2n+1-k}_+)$, for some $0 < \epsilon < 1$.

such that $(T^R, \partial^h T^R)$ is obtained from $(A^R, \partial A^R)$ by attaching a right k-handle along the inclusion

$$(\partial D^k \times D_+^{2n+1-k}, \partial D^k \times \partial_0 D_+^{2n+1-k}) \hookrightarrow (\partial D^k \times \mathbf{R}_+^{2n+1-k}, \partial D^k \times \partial \mathbf{R}_+^{2n+1-k}).$$

Such submanifold triad exists: we can use the already constructed T^I and restrict it to the subspace $[0,1] \times \mathbf{R}^k \times \mathbf{R}_+^{2n+1-k} \subset [0,1] \times \mathbf{R}^k \times \mathbf{R}^{2n+1-k}$. The core and cocore of T^R are the core and cocore of the right k-handle in the sense of Definition 2.3.3.

Construction 4.3.7 (Left traces)**.** Let $k \geq 0$. Let $(A^L, \partial A^L) \coloneqq \{0\} \times (\partial_1 D_+^k, \partial_{01} D_+^k) \times \mathbf{R}^{2n+1-k}$ and let $(T^L \partial^h T^L)$ be a submanifold pair of $[0,1] \times (\mathbf{R}_+^k, \partial \mathbf{R}_+^k) \times \mathbf{R}^{2n+1-k}$ satisfying the following properties:

(i) $(T^L, \partial^h T^L)$ agrees with $[0,1] \times (A^L, \partial A^L)$ outside $[0,1] \times (D_+^k, \partial_0 D_+^k) \times D^{2n+1-k}$,
(ii) $(T^L, \partial^h T^L) \cap ([0,\epsilon) \times (\mathbf{R}_+^k, \partial \mathbf{R}_+^k) \times \mathbf{R}^{2n+1-k}) = [0,\epsilon) \times (A^L, \partial A^L)$, for some $0 < \epsilon < 1$,
(iii) $(T^L, \partial^h T^L) \cap ((1-\epsilon, 1] \times (\mathbf{R}_+^k, \partial \mathbf{R}_+^k) \times \mathbf{R}^{2n+1-k}) = (1-\epsilon, 1] \times (P', \partial P')$ for a manifold pair P' diffeomorphic to $(\mathbf{R}_+^k, \partial \mathbf{R}_+^k) \times \partial D^{2n+1-k}$, for some $0 < \epsilon < 1$.

such that the $(T^L, \partial^h T^L)$ is obtained from $(A^L, \partial A^L)$ by attaching a left k-handle along the inclusion

$$(\partial_1 D_+^k \times D^{2n+1-k}, \partial_{01} D_+^k \times D^{2n+1-k}) \hookrightarrow (\partial_1 D_+^k \times \mathbf{R}^{2n+1-k}, \partial_{01} D_+^k \times \mathbf{R}^{2n+1-k}).$$

Such submanifold triad exists: we can use the already constructed T^I and restrict it to the subspace $[0,1] \times \mathbf{R}_+^k \times \mathbf{R}^{2n+1-k} \subset [0,1] \times \mathbf{R}^k \times \mathbf{R}^{2n+1-k}$. The core and cocore of T^L are the core and cocore of the left k-handle in the sense of Definition 2.3.3.

Definition 4.3.8 (Trace of a surgery)**.** Let $N \in \mathsf{Cob}^{\partial}_{\Theta, \partial^v L}$ and $\sigma : \mathbf{R}^k \times (\mathbf{R}_+^{2n+1-k}, \partial \mathbf{R}_+^{2n+1-k}) \hookrightarrow (\mathbf{R}_+^{\infty}, \partial \mathbf{R}_+^{\infty})$ be an embedding of pairs such that $\sigma^{-1}(N) = A^R$ as manifold pairs. The *trace* $\mathsf{tr}(\sigma) : N \rightsquigarrow N_\sigma$ of σ is the elementary triad cobordism of right type and index k relative to N given by

$$([0,1] \times N \backslash \sigma(A^R)) \cup ((\mathrm{id}_{[0,1]} \times \sigma)(T^R)).$$

For an embedding $\sigma : \mathbf{R}^k \times \mathbf{R}^{2n+1-k} \hookrightarrow \mathbf{R}_+^{\infty} \backslash \partial \mathbf{R}_+^{\infty}$ such that $\sigma^{-1}(N \backslash \partial^h N) = A^I$, denote by $\mathsf{tr}(\sigma) : N \rightsquigarrow N_\sigma$ the elementary interior cobordism of index k relative to N contructed analogously using T^I. For an embedding of pairs $\sigma : (\mathbf{R}_+^k, \partial \mathbf{R}_+^k) \times \mathbf{R}^{2n+1-k} \hookrightarrow (\mathbf{R}_+^{\infty}, \partial \mathbf{R}_+^{\infty})$ such that $\sigma^{-1}(N) = A^L$, denote by $\mathsf{tr}(\sigma) : N \rightsquigarrow N_\sigma$ the elementary left cobordism of index k relative to N constructed analogously using T^L. Let $r : [0,1] \times \mathbf{R}_+^{\infty} \to [0,1] \times \mathbf{R}_+^{\infty}$ given by $r(t,x) = (1-t, x)$. We define the *reverse trace* $\overline{\mathsf{tr}}(\sigma) : N_\sigma \rightsquigarrow N$ of σ to be the image of $\mathsf{tr}(\sigma)$ under the diffeomorphism r. In other words, $\overline{\mathsf{tr}}(\sigma)$ is given by

$$([0,1] \times N \backslash \sigma(A^R)) \cup ((\mathrm{id}_{[0,1]} \times \sigma)(r(T^R))).$$

Denote by core_σ and cocore_σ to be the image of core and cocores of T^R, T^I or T^L under σ. We call σ an *attaching map* of interior, right or left type on N.

Remark 4.3.9 (How to construct Θ-structures on $\mathsf{tr}(\sigma)$)**.** In general, the cobordism $\mathsf{tr}(\sigma)$ admits no preferred Θ-structure making it a morphism of $\mathsf{Cob}^{\partial}_{\Theta,L}$. Let us provide a general recipe to endow this cobordism with such a structure. We will focus on the case when σ is of interior type. Let T^I_0 be the submanifold of T^I given by $A^I \cup [0,1] \times \partial D^k \times \mathbf{R}^{2n+1-k} \backslash D^{2n+1-k}$. By pulling back the structure ℓ_N along $\sigma|_{\sigma^{-1}(N)}$, we can endow T^I_0 with a Θ-structure. Thus, to endow $\mathsf{tr}(\sigma)$ with a Θ-structure it suffices to extend the latter structure from T^I_0 to T^I. In particular, given such an extension ℓ, the morphism $(\mathsf{tr}(\sigma), \ell)$ has support at most $\sigma(D^k \times D^{2n+1-k}_+)$. One proceeds analogously for the right and left types.

When considering Θ-structures on $\mathsf{tr}(\sigma)$, we will always implicitely assume that they are constructed as above in Remark 4.3.9. In particular, a morphism with underlying cobordism $\mathsf{tr}(\sigma)$ will always have support at most $\sigma(D^k \times D^{2n+1-k}_+)$, $\sigma(D^k \times D^{2n+1-k})$ or $\sigma(D^k_+ \times D^{2n+1-k})$ depending if σ is of right, interior or left type.

Lemma 4.3.10. *Let $W : M \rightsquigarrow N$ be a morphism in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ whose underlying cobordism is elementary (in the sense of Definition 2.3.7). Then there exists an embedding σ of the same type (see Definition 4.3.8) of W relative to M and an extension ℓ of the Θ-structure ℓ_M to*

$$\mathsf{tr}(\sigma) : M \rightsquigarrow M_\sigma \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$$

such that there exists an isomorphism $U : M_\sigma \rightsquigarrow N$ in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ such that $U \circ \mathsf{tr}(\sigma)$ and W are in the same component of $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}(M,N)$. Dually, there exists σ' and an extension of ℓ_N such that $\overline{\mathsf{tr}}(\sigma') : N_{\sigma'} \rightsquigarrow N$ after pre-composed with an isomorphism is in the same component of W.

Proof. The proof is completely analogous to [GR17b, Lemma 4.5]. ◻

4.3.3. The closure property. We move now to stating the first closure property, which will be essential for the base case of the induction alluded to in the beginning of this subsection.

Definition 4.3.11. Let $W : M \rightsquigarrow N$ be a morphism in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ and let σ be an attaching map on N in the sense of Definition 4.3.8 disjoint from the support of W and ℓ be a Θ-structure on $\mathsf{tr}(\sigma)$. Define the *surgery on W along σ*, denoted by

$$\chi(W, \sigma, \ell) : \chi(M, \sigma) \rightsquigarrow \chi(N, \sigma),$$

to be the morphism $\mathcal{R}_{\mathsf{tr}(\sigma)}(W)$, which is well defined since $\mathrm{supp}(\mathsf{tr}(\sigma))$ is disjoint from the support of W.

Remark 4.3.12. Let $W : M \rightsquigarrow N$ be a morphism in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ whose underlying triad cobordism is elementary relative to N, and let σ be an attaching map in N disjoint from the attaching map of the handle of W. Then the underlying cobordism of $\chi(W,\sigma)$ is an elementary triad cobordism relative to $\chi(N,\sigma)$ between the results of the surgery $\chi(M,\sigma)$ and $\chi(N,\sigma)$ on M and N along σ. This cobordism is built by attaching a handle of the type and index of W to the attaching map of W. This is well defined because σ is disjoint from the attaching map of the handle of W. In particular, the underlying cobordism $\chi(W,\sigma,\ell)$ does not depend on the choice of ℓ (see also Remark 4.3.4).

Notation 4.3.13 (Translation of σ)**.** Given an right attaching map $\sigma : (\partial\mathbf{R}^k \times \mathbf{R}^{2n+1-k}_+, \partial\mathbf{R}^k \times \partial\mathbf{R}^{2n+1-k}_+) \hookrightarrow (\mathbf{R}^\infty_+, \partial\mathbf{R}^\infty_+)$ in N, we denote by $\sigma_p = \sigma(-, 3p\cdot e_1 + (-))$ for $p \geq 1$, where e_1 is the first basis vector of $\partial\mathbf{R}^{2n+1-k} \cong \mathbf{R}^{2n-k}$. Observe that this is an attaching map on N. Similarly, for an interior or left attaching map σ, one defines σ_p by restricting in the second factor to the disc translated by the first basis vector of $\mathbf{R}^{2n+1-k}$.

One sees that σ_1 is disjoint from the support of $\mathsf{tr}(\sigma)$. Therefore, it is disjoint from the support of $\chi(W,\sigma,\ell)$. To define a Θ-structure on $\mathsf{tr}(\sigma_p)$ we proceed similarly to Remark 4.3.9. Assume σ is of interior type, let $T^I_{0,t} \coloneqq A^I \cup [0,1] \times \partial D^k \times \mathbf{R}^{2n+1-k} \backslash (D^{2n+1-k} + t\cdot e_1)$ and pull back the structure ℓ_N to obtain a Θ-structure ℓ_t on it. For every $t > 0$, translation along the e_1-coordinate, we have a diffeomorphism $T^I_0 \cong T^I_{0,t}$, which induces a structure ℓ_t on T^I_0. Assume we are given a $(0,\infty)$-parameter family[9] $\hat{\ell}_t$ of Θ-structures on T such that $\hat{\ell}_t$ extends ℓ_t in T^I_0. This induces a structure on each $\mathsf{tr}(\sigma_p)$ by taking $\hat{\ell}_{3p}$ making it into a morphism in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$. Proceed analogously for right and left types.

Definition 4.3.14. Let $W : M \rightsquigarrow N$ be a morphism in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ and let σ an attaching map disjoint from the support of W and ℓ_t be a $(0,\infty)$-family of Θ-structures as above. Define the *p-th (translated) iterated surgery on W along σ* to be the morphism given by the inductive formula

$$\chi^p(W,\sigma,\ell) \coloneqq \chi(\chi^{p-1}(W,\sigma,\ell), \sigma_p, \ell_{3p})$$

and the initial value $\chi^0(W,\sigma,\ell) = W$.

Remark 4.3.15. If W lies in $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$, then so does $\chi(W,\sigma,\ell)$. This follows by using Proposition 2.3.17 and Remark 4.3.12: for any handle decomposition of W, there exists a handle decomposition of $\chi(W,\sigma,\ell)$ with same the number of handles in each index as one of W.

[9]Here for a submanifold $T \in \Psi^\partial(U)$ (see Definition 3.1.1), a $(0,\infty)$-*parameter family of* Θ-*structures* is simply a path $(0,\infty) \to \Psi^\partial_\Theta(U)$ lifting the constant path at T along the projection $\Psi^\partial_\Theta(U) \to \Psi^\partial(U)$ which forgets the Θ-structure.

Recall that we fix the choices of Assumptions 4.1.1.

Theorem 4.3.16. *Assume (B, B^∂) is 1-connected. Let $W : M \rightsquigarrow N$ be a morphism in $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ and σ an attaching map of index n for right type or $n+1$ for left type disjoint from the support of W and ℓ_t be a $(0,\infty)$-family of Θ-structures as above. If $\chi^p(W,\sigma,\ell) \in \mathscr{W}$ for every $p \geq 1$, then so is W.*

As mentioned before, we will defer the proof of this statement to the next section.

4.3.4. Deducing stable stability from Theorem 4.3.16.

In this subsubsection, we deduce the following result.

Proposition 4.3.17. *Let $W : M \rightsquigarrow N$ be a morphism whose underlying cobordism is elementary relative to N of right type and index n or elementary of left type and index $n+1$ attached to the basepoint component. Then $W \in \mathscr{W}$.*

This will be the base case in an induction proving Theorem 4.1.11. We proceed in a similar way to [GR17b, Sections 3 and 4]. Roughly speaking, we first start by proving this claim directly in the case where the handles are attached *trivially*. Secondly, we use Theorem 4.3.16 to reduce to this case, by constructing an attaching map σ such that the handles in $\chi^p(W,\sigma,\ell)$ are trivially attached.

Let us give a few more words about this construction. We will construct σ to be an attaching map in N of the same type and index W where $\sigma|_{\sigma^{-1}(N)}$ is "parallel" to an attaching map of the handle of W. By Remark 4.3.12, $\chi(W,\sigma,\ell)$ is elementary of the type of W with unique handle attached to $\chi(N,\sigma)$. However, since surgery along its attaching map is performed, this handle is attached trivially. The same holds for $\chi^p(W,\sigma,\ell)$. Based on this idea, we shall use Theorem 4.3.16 to deduce stable stability for W. (See Figure 7).

§4.3.4.1. *Stability for V_1.* We start by proving that $\mathscr{W}$ contains a certain restricted class of morphisms. Let us first define the following notion.

Definition 4.3.18 (Triad connected sum)**.** Let $(W, \partial^h W, \partial^v W)$ and $(W', \partial^h W', \partial^v W')$ be two d-dimensional manifold triads such that $\partial^v W$ and $\partial^v W'$ are non-empty. Let $e : (D^{d-1}_+, \partial_0 D^{d-1}_+) \hookrightarrow (\partial^v W, \partial^{hv} W)$ and $e' : (D^{d-1}_+, \partial_0 D^{d-1}_+) \hookrightarrow (\partial^v W', \partial^{hv} W')$ be two embeddings. Define the *triad connected sum* $W \natural W'$ as the triad obtained by attaching a right 1-handle to $W \sqcup W'$ along $e \sqcup e'$. The result on horizontal and vertical boundaries are boundary connected sums.

Remark 4.3.19. If $\partial^{hv} W$ and $\partial^{hv} W$ are connected, then the diffeomorphism type of $W \natural W'$ only depends on the local orientations induced by the embeddings e and e'. If either W or W' admit a orientation-reversing diffeomorphism, then the diffeomorphism type of the triad $W \natural W'$ is independent of e and e'. For example,

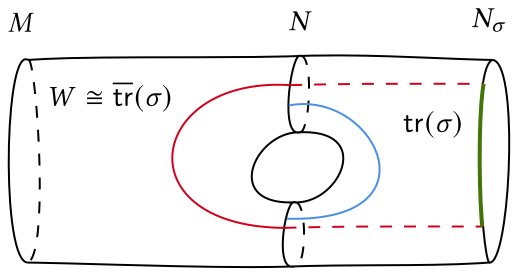


Figure 7: In this picture, we see a morphism $W : M \rightsquigarrow N$ whose underlying cobordism is elementary. We represent its core relative to N as the red arc. Observe that if perform surgery on a translation of the attaching map of W, the attaching map becomes trivial. In the picture, we see that the attaching map of W seen in N_σ bounds a disc, here depicted in green. However by Definition 4.3.11, $\chi(W, \sigma, \ell)$ can be obtained by taking N_σ and attaching a handle precisely along this attaching map. Thus, it is a trivial handle attachment.

this is the case for the triad V_1 (see the definition in the introduction or above Definition 4.1.2). Unless otherwise specified, if W is a cobordism from P to Q then we assume that the connected sum is done in ∂Q and the result is a cobordism from P to $Q \natural \partial^v W$.

Let $P \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$, let

$$H_P : P' \rightsquigarrow P$$

be any morphism in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ whose underlying triad cobordism is given by taking the triad connected sum (see Definition 4.3.18) of $P \times [0,1]$ with V_1 on the basepoint component (recall Notation 4.2.7) in $P \times \{0\}$ away from $\partial^v L$ and whose Θ-structure is such that the restriction to V_1 is standard in the sense of Definition 4.1.4. More explicitly, the underlying triad cobordism of H_P is $(P \times [0,1] \natural V_1, \partial^h P \times [0,1] \natural W_{1,1}, \partial^v L \times [0,1], P \times \{0\} \natural \partial^v V_1 \sqcup P \times \{1\})$, so in particular $P' \cong P$. This Θ-structure exists since we are in the basepoint component and by Lemma 4.1.5. This is in $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ since the pair $(V_1, \partial^h V_1)$ is n-connected. Similarly, let ${}_P H : P \rightsquigarrow P'$ be any morphism whose underlying triad cobordism is given by the triad connected sum of $P \times [0,1]$ with V_1 along $P \times \{1\}$ in the basepoint component with standard Θ-structure.

Proposition 4.3.20. *For any $P \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$, then H_P and ${}_P H$ are in $\mathcal{W}$.*

Proof. This is proven analogously to [GR17b, Thm. 3.1] (see also [GR14, Lemma 7.15] for the same argument) by using Lemma 4.1.5 in place of [GR17b, Lemma 2.12]. □

§4.3.4.2. *Reflections of cobordisms.* We will now see that the morphism ${}_PH$ can be expressed as a composition $W \circ r(W)$ of an morphism W whose underlying triad cobordism is elementary and its "reflection" $r(W)$, where the unique handle of W is attached trivially. We shall apply Theorem 4.3.16 to a general morphism M whose underlying cobordism is elementary and its reflection to reduce stable stability to the trivially attached case, i.e. Proposition 4.3.20. We start by making the notion of the reflection of a Θ-cobordism precise.

Construction 4.3.21 (Reflection)**.** Let $W : M \rightsquigarrow N$ be $\overline{\mathrm{tr}}(\sigma)$ for some $\sigma : \mathbf{R}^n \times \mathbf{R}^{n+1}_+ \hookrightarrow \mathbf{R}^\infty_+$ (which we call of *right type*) or $\sigma : \mathbf{R}^{n+1}_+ \times \mathbf{R}^n \hookrightarrow \mathbf{R}^\infty_+$ (which we call of *left type*) as in Definition 4.3.8. Let $r : [0,1] \times \mathbf{R}^\infty_+ \to [0,1] \times \mathbf{R}^\infty_+$ given by $r(t,x) = (1-t,x)$. Let $r(W) : N \rightsquigarrow M$ whose underlying submanifold of $[0,1] \times \mathbf{R}^\infty_+$ is $r(W)$. The manifold $r(W)$ is left (resp. right) elementary of index $n+1$ (resp. n) relative to M if σ is of right (resp. left) type. More specifically: if σ is of right type, the manifold W contains the single handle

$$\mathsf{cocore}_\sigma = (\mathrm{id}_{[0,1]} \circ \sigma)(\mathsf{cocore}_{T^R}) : (D^{n+1}_+ \times \mathbf{R}^n, \partial_0 D^{n+1}_+ \times \mathbf{R}^n) \to W$$

relative to M. If σ is of left type, the manifold M contains the single handle

$$\mathsf{cocore}_\sigma = (\mathrm{id}_{[0,1]} \circ \sigma)(\mathsf{cocore}_{T^L}) : (D^n \times \mathbf{R}^{n+1}_+, D^n \times \partial \mathbf{R}^{n+1}_+) \to W$$

relative to M. Thus, it suffices to determine a Θ-structure on $r(\mathsf{cocore}_\sigma)$ which extends ℓ_M in $r(\mathsf{belt}_\sigma) \subset M$, where $\mathsf{belt}_\sigma = \mathsf{cocore}_\sigma \cap N$. In the right case, we define a Θ-structure on $r(\mathsf{cocore}_\sigma)$ by insisting that the union

$$(D^{n+1} \times D^n, \partial D^{n+1} \times D^n) \cong r(\mathsf{cocore}_\sigma)(D^{n+1}_+ \times D^n) \cup_M \mathsf{cocore}_\sigma(D^{n+1}_+ \times D^n) \subset r(W) \cup_M W$$

has a standard Θ-structure (see below the proof of existence of such structure). In the left case, we define a Θ-structure on $r(\mathsf{cocore}_\sigma)$ by insisting that the union

$$(S^n \times D^{n+1}_+, S^n \times \partial_0 D^{n+1}_+) \cong r(\mathsf{cocore}_\sigma)(D^n \times D^{n+1}_+) \cup_M \mathsf{cocore}_\sigma(D^n \times D^{n+1}_+) \subset r(W) \cup_M W$$

has a standard Θ-structure (see below the proof of existence of such structure).

Proof of existence of such structure. We prove that there exists a Θ-structure on $r(\mathsf{cocore}_\sigma)$ such that the union above has standard Θ-structure. We will focus on the first case, as the second is completely analogous. Pick a standard Θ-structure ℓ^{std} on $(D_R, \partial^h D_R) \coloneqq (D^{n+1} \times D^n, \partial D^{n+1} \times D^n)$ and let $\ell \coloneqq (\mathsf{cocore}_\sigma|_{D_R})^* \ell_M$. We start by observing that ℓ^{std} and ℓ are homotopic as bundle maps. We will show that ℓ is homotopic to the basepoint Θ-structure fixed in Assumptions 4.1.1, as ℓ^{std} is by definition. This will use that σ is in the basepoint component. We prove in the case that $(\theta^* \gamma_{2n+1}, (\theta^\partial)^* \gamma_{2n})$ is orientable, in which case all Θ-manifolds are orientable. If this assumption is not satisfied, there exists only one Θ-structure on D_R as B^∂ is path-connected so the claim follows trivially. By assumption, there exists a path in $\partial^h W$ from a point in the image of cocore_σ and a point in the basepoint component $\partial_0^v L$. Thus, for any embedding e of D_R inducing the same orientation as cocore_σ into the basepoint component, we have that $e^*\ell_{\partial^v L}$ is homotopic to ℓ. Moreover, by the definition of the basepoint component (see Notation 4.2.7), we see that there exists a path from it to an embedded copy of V_1 with standard Θ-structure. One key feature of the standard Θ-structure is that when restricted to an orientation preserving embedding of the disc it is the basepoint Θ-structure. We assume that the embedding σ induces the same orientation as the one induced by the embedding of this disc after composing with the embedding of the copy of V_1. (If this is not the case, we can change σ by an orientation reversing diffeomorphism, since it induces the same underlying manifold after taking $\overline{\mathsf{tr}}$.) We conclude that $e^*\ell_{\partial^v L}$ is homotopic to the basepoint Θ-structure, which implies that ℓ is homotopic to the basepoint Θ-structure. By the (bundle) homotopy extension property of the pair $(S^n \times D^n, D_R)$, we can extend the homotopy from ℓ to the basepoint Θ-structure from D_R to a homotopy from a Θ-structure extending ℓ to $S^n \times D^n$ to a standard Θ-structure. □

Remark 4.3.22. As in Remark 4.3.9, the support of $\mathsf{tr}(\sigma)$ is $\sigma(D^n \times D_+^{n+1})$ or $\sigma(D_+^{n+1} \times D^n)$ if σ is of right or left type. It is clear to see $\mathrm{supp}(M) = \mathrm{supp}(r(M))$.

§4.3.4.3. *Identifying $\chi^p(W, \sigma, \ell)$.* Let $W : M \rightsquigarrow N$ be $\overline{\mathsf{tr}}(\sigma)$ as in Construction 4.3.21, where σ lies in the basepoint component. It is easy to see that $\mathrm{supp}(W)$ is disjoint from $\mathrm{supp}(\mathsf{tr}(\sigma))$. Moreover, it is also clear that $\mathrm{supp}(W)$ is disjoint from $\mathrm{supp}(\mathsf{tr}(\sigma_p))$ (recall Definition 4.3.14) for all $p \geq 1$. The trace $\mathsf{tr}(\sigma)$ is elementary of the type of σ relative to N. We fix a $(0, \infty)$-family of Θ-structures ℓ_t as in 4.3.14 by insisting, as before, that the union of the core of W relative to N with the core of $\mathsf{tr}(\sigma)$ relative to N is standard.

Lemma 4.3.23. *Let $W = \overline{\mathsf{tr}}(\sigma)$ be as above. Then $\chi^p(W \circ r(W), \sigma, \ell)$ is the composition of $H_{\chi^p(N,\sigma,\ell)}$ with an isomorphism, for every $p \geq 1$. In particular, it is in $\mathscr{W}$.*

Proof. Let $X \coloneqq \chi^p(W \circ r(W), \sigma, \ell)$, we show that there exists an embedding of pairs $h : (V_1, W_{1,1}) \hookrightarrow (X, \partial^h X)$ with standard induced Θ-structure and a path γ from a point in $h(W_{1,1})$ to a point in $\partial^h \chi^p(N, \sigma, \ell)$ in $\partial^h X$ disjoint from h in its interior such that the complement of h and a tubular neighborhood of γ (i.e. an embedding $\widetilde{\gamma} : ([0,1] \times D_+^{2n}, [0,1] \times \partial_0 D_+^{2n}) \to (X, \partial^h X)$ such that $\widetilde{\gamma}|_{[0,1]\times\{0\}} = \gamma$) is diffeomorphic to an interval cobordism (and thus, an isomorphism by Definition 4.2.1). We also note that the complement of h and a tubular neighborhood of γ is diffeomorphic to the union of the complement of h with a disc $(D_+^{2n+1}, \partial_1 D_+^{2n+1})$ along $h(\partial^v V_1, \partial^{hv} V_1)$.

Assume σ is of right type. The union of cocore_σ with $r(\mathsf{cocore}_\sigma)$ induces an embedding

$$f' : (D^{n+1} \times D^n, \partial D^{n+1} \times D^n) \hookrightarrow r(W) \cup_P W$$

with standard Θ-structure, by definition. Since the image of f' is inside the support of W, it is disjoint from the support of $\mathrm{tr}(\sigma_p)$. The rearrangement (recall Definition 4.3.2) defines an embedding f into X. (Moreover this embedding is isotopic to f' in $X \circ \mathcal{L}_{W\circ r(W)}(\mathrm{tr}(\sigma_p)) = \mathrm{tr}(\sigma_p) \circ W \circ r(W)$, by Remark 4.3.3). The core of $W = \overline{\mathrm{tr}}(\sigma)$ is disjoint from the support of $\mathrm{tr}(\sigma_p)$ and intersects the cocore of W transversely at one point (by definition of handle attachment). In other words, it intersects the image of f' transversely at one point and thus, its image after the rearrangement will have the same intersection behavior with f. However, the embedding of the attaching map of W can be seen also in $\chi^p(N, \sigma, \ell)$ giving an embedding

$$g : (S^{n-1} \times D^{n+1}_+, S^{n-1} \times \partial_0 D^{n+1}_+) \hookrightarrow (\chi^p(N, \sigma, \ell), \partial^h \chi^p(N, \sigma, \ell))$$

since it is disjoint from the support of $\mathrm{tr}(\sigma_p)$. However, this embedding extends to an embedding

$$e' : (D^n \times D^{n+1}_+, D^n \times \partial_0 D^{n+1}_+) \hookrightarrow (\chi^p(N, \sigma, \ell), \partial^h \chi^p(N, \sigma, \ell))$$

since the target is the result of right surgery at a translate of g. Moreover, the induced Θ-structure on e after glued with the one of the core of W (after rearrangement) is standard, by definition. This produces an embedding

$$e : (S^n \times D^{n+1}_+, S^n \times \partial_0 D^{n+1}_+) \hookrightarrow (X, \partial^h X)$$

with standard Θ-structure. By the discussion above, the cores of e and f intersect transversely at one point in the horizontal boundary. Therefore, the union of these embeddings is diffeomorphic to $(V_1, W_{1,1})$ with standard Θ-structure. Thus, we define h as the gluing of these embeddings. We can choose a path γ in $\partial^h X$ connecting $\chi^p(N, \sigma, \ell)$ and this embedding and disjoint from such image, since $n \geq 2$.

We finish the proof by showing that the union of the complement of h with a disc $(D^{2n+1}_+, \partial_1 D^{2n+1}_+)$ along $h(\partial^v V_1, \partial^{hv} V_1)$ is diffeomorphic to an interval cobordism[10]. We claim that this complement is diffeomorphic to

$$(Y, \partial^h Y) := (X \backslash f'(D^{n+1} \times \mathrm{int}(D^n)) \cup D^{n+2}_+ \times \partial D^n, \partial^h X \backslash f'(\partial D^{n+1} \times D^n) \cup \partial_1 D^{n+2}_+ \times \partial D^n)$$

which is the result of doing surgery along f' of left type and index $n+1$, that is, the outgoing boundary of an elementary triad cobordism of left type and index $n+2$ relative to $(X, \partial^h X)$. This can be seen from the fact that $(V_1, W_{1,1})$ is obtained as the

[10] We thank Oscar Randal-Williams for explaining the argument underlying the analagous claim in the proof of [GR17b, Lemma 4.8], which inspired the present proof.

outgoing boundary of an elementary triad cobordism relative to $(D^{2n+1}_+, \partial_1 D^{2n+1}_+)$ of left type and index $n+2$ along the embedding $\bar{f}$ from Definition 4.1.4. To show now that $(Y, \partial^h Y)$ is the trivial triad cobordism, we show that the inverse surgery move applied to the trivial cobordism recovers $(X, \partial^h X)$. We use the fact that $(X, \partial^h X)$ is the result of composing an elementary triad cobordism W of right type and index n relative to $\chi^p(N, \sigma, \ell)$ with its reflection $r(W)$, which can be seen by unravelling definitions, and that f' is the inclusion of the union of the cocore of W with its reflection in $r(W)$, together with the following claim. Let $d, k \geq 0$ be integers.

Claim 1. *Let $W : R \rightsquigarrow S$ be an elementary triad cobordism of dimension d of right type and index k relative to S. Let $\phi : (\partial D^k \times D^{d-k}_+, \partial D^k \times \partial_1 D^k_+) \hookrightarrow (S, \partial^h S)$ be an attaching map of the unique handle of W. Then $W \circ r(W)$ is the result of doing surgery of right type and index $k-1$ to $S \times [-2,2]$ along the embedding $\tilde{\phi}$*

$$\begin{array}{ccc} (\partial D^k \times D^{d-k+1}_+, \partial D^k \times \partial_1 D^{d-k+1}_+) & \xrightarrow{\cong} & (\partial D^k \times D^{d-k}_+ \times [-1,1], \partial D^k \times \partial_1 D^{d-k}_+ \times [-1,1]) \\ & & \downarrow{\scriptstyle \phi \times [-1,1]} \\ & & S \times [-1,1] \subseteq S \times [-2,2] \end{array}$$

Proof of Claim 1. By definition, the result of doing surgery of right type and index $k-1$ to $S \times [-2,2]$ along the embedding $\tilde{\phi}$ is the outgoing boundary of the elementary triad cobordism relative to $S \times [-2,2]$ of right type and index k given by attaching a handle along $\tilde{\phi}$. Unravelling definitions (see Definition 2.3.3), we see that this result is the pair

$$(S\times[-2,2]\backslash\tilde{\phi}(\partial D^k\times D^{d-k+1}_+)\cup D^k\times\partial_0 D^{d-k+1}_+, \partial^h S\times[-2,2]\backslash\tilde{\phi}(\partial D^k\times D^{d-k+1}_+)\cup D^k\times\partial_{01} D^{d-k+1}_+).$$

We will see this pair as a union of two pairs M_0 and M_1, where $M_0 \cong W$, $M_1 \cong r(W)$ and $M_0 \cap M_1 \cong R$, where the last identification is compatible with the previous two. We take M_0 as

$$(S\times[-2,0]\backslash\tilde{\phi}(\partial D^k\times D^{d-k+1}_{+-})\cup D^k\times\partial_0 D^{d-k+1}_{+-}, \partial^h S\times[-2,0]\backslash\tilde{\phi}(\partial D^k\times D^{d-k+1}_{+-})\cup D^k\times\partial_{01} D^{d-k+1}_{+-})$$

where D^{d-k+1}_{+-} is the subspace of $D^{d-k+1}_+ \cong D^{d-k}_+ \times [-1,1]$ of the points where the last coordinate is non-positive. One can see that this is diffeomorphic to $(S\times[-2,0]\cup_\phi D^k\times\partial_0 D^{d-k+1}_{+-}, \partial^h S\times[-2,0]\cup_\phi D^k\times\partial_{01} D^{d-k+1}_{+-})$, hence diffeomorphic to W. We take M_1 as

$$(S\times[0,2]\backslash\tilde{\phi}(\partial D^k\times D^{d-k+1}_{++})\cup D^k\times\partial_0 D^{d-k+1}_{++}, \partial^h S\times[0,2]\backslash\tilde{\phi}(\partial D^k\times D^{d-k+1}_{++})\cup D^k\times\partial_{01} D^{d-k+1}_{++})$$

where D^{d-k+1}_{++} is the subspace of $D^{d-k+1}_+ \cong D^{d-k}_+ \times [-1,1]$ of the points where the last coordinate is non-negative. The same argument as before identifies M_1 with $r(W)$. The union $M_0 \cup M_1$ is thus diffeomorphic to $W \circ r(W)$, which completes the proof. □

This finishes the case when σ is of right type. The case when σ is of left type follows analogously by dualizing the arguments above. □

We are now ready to prove Proposition 4.3.17.

Proof of Proposition 4.3.17. By Lemma 4.3.10, it suffices to assume that W is $\overline{\mathrm{tr}}(\sigma)$ for some σ, where σ lies in the basepoint component. Let ℓ be the constant path on the chosen Θ-structure for $\mathrm{tr}(\sigma)$. By combining Theorem 4.3.16 and Lemma 4.3.23, we see that $W \circ r(W) \in \mathscr{W}$. Since the reflection of an elementary of right type and index n is an elementary of left type and index $n+1$ and vice-versa, we have that $r(W) \circ r(r(W)) \in \mathscr{W}$. Consider now the equality $W \circ (r(W) \circ r(r(W))) = (W \circ r(W)) \circ r(r(W))$. One concludes that all morphisms induce isomorphisms on $\mathrm{H}_1(-, \mathbf{Z})$. On the other hand, pre-composition with $r(W)$ induces surjections on homology with abelian coefficients from its target, since $W \circ r(W) \in \mathscr{W}$, and injections on homology with abelian coefficients on the target of $r(r(W))$, since $r(W) \circ r(r(W)) \in \mathscr{W}$. Since the abelian coefficients in all three spaces are the same (as the maps above induce isomorphisms on $\mathrm{H}_1(-, \mathbf{Z})$), it follows that $r(W) \in \mathscr{W}$. Thus, by Lemma 4.2.4, we see $W \in \mathscr{W}$. □

4.4. The closure property for higher handles.

We move now to stating the main closure property Theorem 4.4.4 for higher dimensional handles, which will be the main input for the induction step to prove Theorem 4.1.11. This strategy is very similar to the previous subsection but instead of parallel surgery, we apply "meridian" surgery. To make this precise, we consider the following definition. Let σ be an attaching map to N of one of the following types:

(*a*) $\sigma : \mathbf{R}^k \times \mathbf{R}_+^{2n+1-k} \hookrightarrow \mathbf{R}_+^\infty$ for $n < k < 2n$; (Right)
(*b*) $\sigma : \mathbf{R}_+^k \times \mathbf{R}^{2n+1-k} \hookrightarrow \mathbf{R}_+^\infty$ for $n+1 < k < 2n+1$; (Left)
(*c*) $\sigma : \mathbf{R}^k \times \mathbf{R}^{2n+1-k} \hookrightarrow \mathbf{R}_+^\infty \backslash \partial \mathbf{R}_+^\infty$ for $n+1 < k < 2n+1$. (Interior)

Let $\iota : D^{k-1} \longrightarrow \partial D^k$ for the embedding onto the lower hemisphere which is inverse to the stereographic projection $(y_1, \cdots, y_k) \mapsto \frac{1}{1-y_k}(y_1, \cdots, y_{k-1})$. Consider D^k as a subspace of $D^{k-1} \times D^1$ with coordinates $(y, z) = (y_1, \cdots, y_{k-1}, z)$. Consider the family of embedding

$$\mu_t : \partial D^{2n+1-k} \times D^k \hookrightarrow \partial D^k \times \mathbf{R}^{2n+1-k}$$

given by $\mu_t(x; y, z) = (\iota(y), t(1 + \frac{1}{3}z)x)$ for $t \in (2, \infty)$. From this family, we will only need the following properties:

(*i*) The image $\mu_t(\partial D^{2n+1-k} \times \{0\})$ is $\{-e_k\} \times t \cdot \partial D^{2n+1-k}$, the *meridian sphere* of σ to $e_k \in \partial D^k$ of radius t, and the image of $\partial D^{2n+1-k} \times D^k$ lies in $\partial D^k \times \mathbf{R}^{2n+1-k} \backslash D^{2n+1-k}$.

(ii) If $t' > 2t$, then the images of μ_t and $\mu_{t'}$ are disjoint.
(iii) For all t, we have $\mu_t^{-1}(\partial_1 D_+^k \times \mathbf{R}^{2n+1-k}) = \partial D^{2n+1-k} \times D_+^k$ and $\mu_t^{-1}(\partial D^k \times \mathbf{R}_+^{2n+1-k}) = \partial_1 D_+^{2n+1-k} \times D^k$.

Definition 4.4.1 (The family of meridian attaching maps)**.** For the right type, let $\chi : \partial D^k \times \mathbf{R}_+^{2n+1-k} \to \mathbf{R}^k \times \mathbf{R}_+^{2n+1-k}$ be the inclusion induced by $\partial D^k \hookrightarrow \mathbf{R}^k$. For the left type, let $\chi : \partial_1 D_+^k \times \mathbf{R}^{2n+1-k} \to \mathbf{R}_+^k \times \mathbf{R}^{2n+1-k}$ be the inclusion induced by $\partial_1 D_+^k \hookrightarrow \mathbf{R}_+^k$. For interior type, let $\chi : \partial D^k \times \mathbf{R}^{2n+1-k} \to \mathbf{R}^k \times \mathbf{R}^{2n+1-k}$ be the inclusion induced by $\partial D^k \hookrightarrow \mathbf{R}^k$. For each of the types above, denote $\phi_t^\sigma = \sigma \circ \chi \circ \mu_t$ using property (iii) of μ above. We can extend ϕ_t^σ to an attaching map disjoint from support of $\overline{\mathsf{tr}}(\sigma)$ of index $2n+1-k$ of left, right and interior type for σ of right, left and interior type and index k, respectively such that the images of the extensions of ϕ_t^σ and $\phi_{t'}^\sigma$ are disjoint if $t' > 2t$ by (ii). For simplicity, we denote such family of extensions also by ϕ_t^σ.

Remark 4.4.2 (Θ-structures for meridian surgery)**.** To define Θ-structures on $\mathsf{tr}(\phi_t^\sigma)$, we can consider the extension $\hat{\mu}_t : D^{2n+1-k} \times D^k \hookrightarrow [0,2] \times \partial D^k \times \mathbf{R}^{2n+1-k}$ of μ_t given by $\hat{\mu}_t(x,y,z) = (\frac{3}{2}(1-|x|^2)(1+\frac{1}{3}z), \iota(y), t(1+\frac{1}{3}z)x)$. Let ℓ be a Θ-structure on $[0,2] \times \partial D^k \times \mathbf{R}^{2n+1-k}$ if σ is of interior type, on $[0,2] \times \partial D^k \times \mathbf{R}_+^{2n+1-k}$ if σ is of right type and on $[0,2] \times \partial_1 D_+^k \times \mathbf{R}^{2n+1-k}$ if σ is of left type. We can extend $\mu_t^* \ell$ to get a Θ-structure on $\mathsf{tr}(\phi_t^\sigma)$. Moreover, the space of such extensions is contractible.

Definition 4.4.3. Let $W : M \rightsquigarrow N$ be a morphism in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ and let σ an attaching map whose support contains the support of W and ϕ_t^σ be as in Definition 4.4.1. Let ℓ be a Θ-structure as in 4.4.2. Define the *p-th iterated (meridian) surgery on W along ϕ* $\chi^p(W, \phi^\sigma, \ell)$ to be given by the inductive formula

$$\chi^p(W, \phi^\sigma, \ell) \coloneqq \chi(\chi^{p-1}(W, \phi^\sigma, \ell), \phi_{3p}^\sigma, \mu_{3p}^* \ell)$$

and the initial value $\chi^0(W, \phi^\sigma, \ell) = W$. This is well defined since ϕ_{3p}^σ is disjoint from the support of σ.

We state now the second closure property we use. This will play the same role as Theorem 4.3.16 but now for the induction step in the proof of Theorem 4.1.11. We will finish by assembling all the pieces and giving the proof of the later. Recall that we conisder the choices fixed in Assumptions 4.1.1.

Theorem 4.4.4. *Assume (B, B^∂) is 0-connected. Let $W : M \rightsquigarrow N$ be a morphism in $\mathsf{Cob}^{\partial, n-1}_{\Theta,\partial^v L}$ and σ an attaching map of index k satisfying the bounds of (a)–(c) above for each type whose support contains the support of W and ℓ be a Θ-structure as in 4.4.2. If $\chi^p(W, \phi^\sigma, \ell) \in \mathscr{W}$ for every $p \geq 1$, then so is W.*

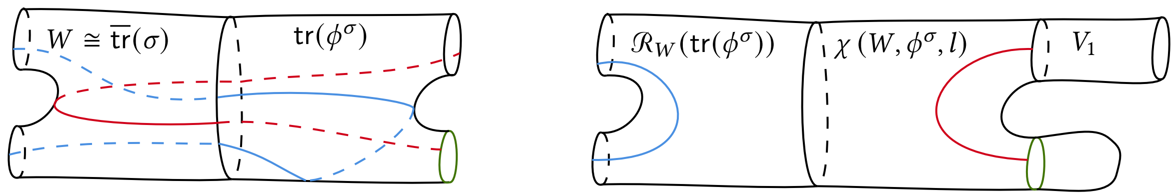


Figure 8: In this picture, we see an morphism $W : M \rightsquigarrow N$ whose underlying triad cobordism is elementary of index k relative to N composed with the trace of a surgery along the "meridian sphere". Observe that W does not admit a left inverse, that is, there does not exist any bordism $W' : N \rightsquigarrow M$ where the composite $W' \circ W$ is an isomorphism. However, as the picture illustrates, $\chi(W, \phi^\sigma, \ell)$ does. This right inverse V_1 is obtained by attaching a handle $2n - k + 2$ relative to $\chi(N, \phi^\sigma)$ along the green circle. This morphism is elementary of index $k - 1$ relative to its target and thus, by induction, is in $\mathcal{W}$.

4.4.1. Deducing stable stability from Theorem 4.4.4. Let W be $\overline{\mathrm{tr}}(\sigma)$ for an attaching map σ as in Theorem 4.4.4. We have produced maps ϕ_p^σ disjoint from the support of W, whose attaching map are isotopic to the meridian of σ at x. Geometrically, doing "meridian surgery" using ϕ^σ produces a left-invertible morphism, whose inverse is already in $\mathcal{W}$ by induction. (See Figure 8 for a depiction of this fact in the classical case without boundary[11].) The next proposition establishes exactly this.

Lemma 4.4.5. *There exist a Θ-structure ℓ on $\mathrm{tr}(\phi^\sigma)$ as in Remark 4.4.2 and a morphism $V_p' : \chi^p(N, \phi^\sigma, \ell) \rightsquigarrow S_p$ in $\mathrm{Cob}_{\Theta, \partial^v L}^{\partial, n-1}$ for every $p \geq 1$, such that the underlying cobordism is elementary of the same type of σ and strictly lower index relative to S_p, and that $V_p' \circ \chi^p(W, \phi^\sigma, \ell)$ is an isomorphism.*

Proof. We focus on the case where σ is of right type as the remaining cases are analogous. We start by choosing the Θ-structure ℓ on $\mathrm{tr}(\phi^\sigma)$. To do so, start by observing that there exists an embedding of underlying pairs $e : \mathrm{tr}(\phi^\sigma) \hookrightarrow N \times [0, 1]$ relative to N by the following argument: It suffices to find an embedding of the core of the unique handle of $\mathrm{tr}(\phi^\sigma)$ extending its attaching map. More precisely, we must find an embedding $(D_+^{2n+1-k} \times D^k, \partial_0 D_+^{2n+1-k} \times D^k) \hookrightarrow (N \times [0, 1], \partial^h N \times [0, 1])$ extending the map $\phi^\sigma : (\partial_1 D_+^{2n+1-k} \times D^k, \partial_{01} D_+^{2n+1-k} \times D^k) \hookrightarrow (N \times \{0\}, \partial^h N \times \{0\})$. This is possible since the attaching map $\phi^\sigma|_{(\partial_1 D_+^{2n+1-k}, \partial_{01} D^{2n+1-k})}$ is trivial, that is, it extends to $(D_+^{2n+1-k}, \partial_0 D_+^{2n+1-k})$ using σ. Observe that the pullback $e^*(\ell_N \oplus \varepsilon^1)$ of the cylindrical structure may not be cylindrical away from the support of $\mathrm{tr}(\phi^\sigma)$, however we can modify the cylindrical structure on

[11]The picture is slightly misleading as the red arc on the right seems to be homotopic to the cocore of $\mathrm{tr}(\phi^\sigma)$ in the left picture, which is false. Unfortunately, this is a disadvantage of low dimensions where not all rearrangements can be realized.

$N\times[0,1]$ to another structure ℓ' such that $e^*\ell'$ is cylindrical away from the support of $\mathsf{tr}(\phi^\sigma)$. Let ℓ be $e^*\ell'$ for such a choice of Θ-structure on $N \times [0,1]$.

We will now construct the morphism V_p'. We will restrict ourselves to the case $p=1$ as the other cases are given by iterating the construction for this case. We define the underlying cobordism of V_1' to be complement of the embedding e and we choose the Θ-structure ℓ_V on V_1' given by pulling back ℓ' along the inclusion $V_1' \hookrightarrow N \times [0,1]$. As the embedding e can be chosen to be disjoint from $\partial^v L$, we can see (V_1', ℓ_V) as a morphism of $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ with source $\chi^1(N,\phi^\sigma,\ell)$. We denote its target by S_1. Since the underlying cobordism V_1' is the complement of $\mathsf{tr}(\phi^\sigma)$ in $N \times [0,1]$, we make the following observations:

1. The cobordism V_1' is elementary of left type and index $2n-k+2$ relative to $\chi^1(N,\phi^\sigma,\ell)$. Thus, it is elementary of right type and index $k-1$ relative to S_1 and thus lies in the subcategory $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ and satisfies the first condition of the claim.
2. There exists a handle decomposition of V_1' relative to $\chi^1(N,\phi^\sigma,\ell)$ whose attaching map intersects the belt sphere of $\mathsf{tr}(\phi^\sigma)$ exactly at one point, as V_1' cancels $\mathsf{tr}(\phi^\sigma)$ in the sense of Proposition 2.3.14.

However, the belt sphere of ϕ^σ is, by definition, isotopic to the attaching map of the unique handle (see Remark 4.3.12) of $\chi^1(W,\phi^\sigma,\ell)$. Thus, the handle decomposition of V_1' can be arranged so that its attaching map relative to $\chi^1(N,\phi^\sigma,\ell)$ intersects the attaching map of $\chi^1(W,\phi^\sigma,\ell)$ relative to $\chi^1(N,\phi^\sigma,\ell)$ exactly at one point. Thus $V_1' \circ \chi^p(W,\phi^\sigma,\ell)$ is diffeomorphic to a product cobordism by Proposition 2.3.14 and hence an isomorphism by Example 4.2.2. We conclude that V_1' satisfies our desired properties, hence we have finished the proof. □

We can now state and prove "stable stability" for higher elementary triad cobordisms.

Proposition 4.4.6. *Let $W : M \rightsquigarrow N$ be a morphism whose underlying cobordism is elementary of index $n < k < 2n$ for right type, $n+1 < k < 2n+1$ for interior type and $n+1 < k < 2n+1$ for left type attached to the basepoint component, then $W \in \mathcal{W}$.*

Proof. Once again, we can assume without loss of generality that W is $\overline{\mathsf{tr}}(\sigma)$ as above. We proceed by induction in the following way for each type. We start with right type. Assume by induction that the result is true for $k-1$. The base case is Proposition 4.3.17. Define ϕ_p^σ as above and note that V_p in Lemma 4.4.5 is in $\mathcal{W}$ by induction for every $p \geq 1$. Thus, we have $\chi^p(W,\phi^\sigma,\ell) \in \mathcal{W}$. Apply now Theorem 4.3.16 to deduce the result. Thus, we conclude that the result holds for elementary triad cobordism of right type. The same argument applies for the left type, where the base case is index $n+1$ and is established by Proposition 4.3.17. For interior type, the base case (index $n+1$) is established by using Proposition 2.3.12

to factor this morphism as the composite of a pair of morphisms whose underlying triad cobordisms are elementary of left and right type of index $n+1$. Since we have proved this claim for right and left elementary triad cobordisms of index $n+1$, we conclude that both factors are in $\mathcal{W}$. Thus, we establish the base case for interior handles. The induction step goes exactly as the previous cases. This finishes the proof. □

We finish by putting everything together and prove Theorem 4.1.11.

Proof of Theorem 4.1.11. This follows by combining Proposition 4.3.17 and Proposition 4.4.6 with Lemma 4.2.9 to deduce that $\mathcal{W} = \mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$. □

5. Proof of the closure properties.

The goal of this section is to prove Theorems 4.3.16 and 4.4.4. We start by sketching the overall strategy, which is inspired by the proof of [GR17b, Thm. 2.15]. Let $W : M \rightsquigarrow N$, σ and ℓ_t be as in the hypothesis of either claim. We shall construct *augmented semi-simplicial spaces* $X(N)_\bullet \to \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(N, K|_\infty)$ and $X(M)_\bullet \to \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(M, K|_\infty)$ satisfying the following properties:

(*i*) The map $(-)\circ W : \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(N, K|_\infty) \to \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(M, K|_\infty)$ lifts to a map $(-)\circ W : X(N)_\bullet \to X(M)_\bullet$ of semi-simplicial spaces (see Lemmas 5.1.7 and 5.2.5).

(*ii*) The induced map $||X(N)_\bullet|| \to \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(N, K|_\infty)$ is an equivalence and similarly for the version with M instead of N (see Theorems 5.1.10 and 5.1.19 and Proposition 5.2.8).

(*iii*) The map $X(N)_{p-1} \to X(M)_{p-1}$ is an abelian homology equivalence, for any $p \geq 1$ (see Propositions 5.1.27 and 5.2.10).

Since abelian homology equivalences are preserved under homotopy colimits, then (*iii*) implies that $||X(N)_\bullet|| \to ||X(M)_\bullet||$ is an abelian homology equivalence. By (*ii*), we conclude that $W \in \mathcal{W}$. The construction of such augmented semi-simplicial spaces takes the form of Definitions 5.1.1 and 5.1.3 for the middle-dimensional case, and Definitions 5.2.1 to 5.2.3 for the higher-dimensional case. We fix again the data present in the Assumptions 4.1.1 and a Θ-end K.

5.1. The middle-dimensional case.

The goal of this section is to prove Theorem 4.3.16. Although the hypotheses of Theorem 4.3.16 require (B, B^∂) to be 1-connected, we do not start by imposing such a condition. The only step that requires this extra assumption is Lemma 5.1.14. We start by defining the augmented semi-simplicial space mentioned in the sketch above.

Definition 5.1.1 (Right resolution)**.** Let $n \geq 1$, $P, Q \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ and $(s, W) \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P, Q)$. Let

$$\phi : (\partial D^n \times (0,\infty) \times \mathbf{R}^n_+, \partial D^n \times (0,\infty) \times \partial\mathbf{R}^n_+) \hookrightarrow (P, \partial^h P)$$

be an embedding and $\ell : (0,\infty) \to \mathrm{Bun}^{\mathrm{col}}(T(D^n \times D^{n+1}_+), \Theta^*\gamma_{2n+1})$ a map, such that $\ell_t|_{\partial D^n \times D^{n+1}_+} = (\phi^*\ell_P)|_{\partial D^n \times (t\cdot e_1 + D^{n+1}_+)}$, where $\mathrm{Bun}^{\mathrm{col}}(T(D^n \times D^{n+1}_+), \Theta^*\gamma_{2n+1})$ denotes the space of collared bundle maps $(T(D^n \times D^{n+1}_+), T(D^n \times \partial_0 D^{n+1}_+)) \to (\theta^*\gamma_{2n+1}, (\theta^\partial)^*\gamma_{2n})$. Define $\mathcal{K}(W, \phi, \ell)_0$ to be the space of triples $(t, c, \mathcal{L})$ where $t \in (0,\infty)$, $c : (D^n \times D^{n+1}_+, D^n \times \partial_0 D^{n+1}_+) \hookrightarrow (W, \partial^h W)$ an embedding of pairs (recall Section 2.3) and $\mathcal{L} : [0,1] \to \mathrm{Bun}^{\mathrm{col}}_{\partial^v}(T(D^n \times D^{n+1}_+), \Theta^*\gamma_{2n+1})$ satisfying the following properties:

(*i*) *(Collared translation of ϕ by t near P)* For some $\delta > 0$, we have $c(x, v) = \phi(\frac{x}{|x|}, v + t\cdot e_1) + (1 - |x|)\cdot e_0$, where e_0 is the first coordinate vector of $\mathbf{R} \times \mathbf{R}^\infty_+$ for all x such that $1 - |x| < \delta$.
(*ii*) The image $(C, \partial^h C) = c(D^n \times D^{n+1}_+, D^n \times \partial_0 D^{n+1}_+)$ is disjoint from $([0,s] \times \partial^v L) \cup (\{s\} \times Q)$ and $c^{-1}(P) = \partial D^n \times D^{n+1}_+$.
(*iii*) $\mathcal{L}(0) = c^*\ell_W$ and $\mathcal{L}(1) = \ell_t$.
(*iv*) The map $\ell_W|_{W\backslash C} : (W\backslash C, \partial^h W \backslash \partial^h C) \to (B, B^\partial)$ is strongly n-connected.

This space is topologized as a subspace of

$$\mathbf{R} \times \mathrm{Emb}(D^n \times D^{n+1}_+, [0,\infty) \times \mathbf{R}^\infty_+) \times \mathrm{Bun}^{\mathrm{col}}_{\partial^v}(D^n \times D^{n+1}_+, \Theta^*\gamma)^{[0,1]}$$

where the middle space denotes the space of embeddings of pairs $e : (D^n \times D^{n+1}_+, D^n \times \partial_0 D^{n+1}_+) \hookrightarrow ([0,\infty) \times \mathbf{R}^\infty_+, [0,\infty) \times \partial\mathbf{R}^\infty_+)$. Define $\mathcal{K}(W, \phi, \ell)_p$ to be the subspace of $\mathcal{K}(W, \phi, \ell)_0^{\times(p+1)}$ consisting of those tuples $(t_i; c_i; \mathcal{L}_i)_{i=0,\cdots,p}$ such that:

(*i*) $t_0 < t_1 < \cdots < t_p$.
(*ii*) The embeddings c_i are pairwise disjoint.
(*iii*) The map $\ell_W|_{W\backslash C} : (W\backslash C, \partial^h W\backslash \partial^h C) \to (B, B^\partial)$ is strongly n-connected, for C the union of the images of all c_i.

Lemma 5.1.2. *The subspaces $\mathcal{K}(W, \phi, \ell)_p \subset \mathcal{K}(W, \phi, \ell)_0^{\times(p+1)}$ are preserved under forgetting factors. Therefore, it induces the structure of a semi-simplicial space to $\mathcal{K}(W, \phi, \ell)_\bullet$.*

Proof. We want to show that if we restrict the domain of the projection map $\mathrm{proj}_i : \mathcal{K}(W, \phi, \ell)_0^{\times(p+1)} \to \mathcal{K}(W, \phi, \ell)_0^{\times p}$ to the $\mathcal{K}(W, \phi, \ell)_p$, the image lies in $\mathcal{K}(W, \phi, \ell)_{p-1}$. It is obvious to see that the conditions (*i*) and (*ii*) of Definition 5.1.1 are closed under forgetting some factors. It remains to prove that, if $M = W\backslash \cup_j C_j \to B$ is strongly n-connected, then so is $M' = W\backslash \cup_{j\neq i} C_j \to B$. We prove that the inclusion $M \to M'$ is strongly $(n-1)$-connected and apply Lemma 2.1.3.

The cobordism M is obtained from M' by removing the core of a right n-handle. Therefore, M' is obtained from M by attaching its dual handle, which is a left $(n+1)$-handle. Thus by (ii) in Section 2.3.2, the map $M \to M'$ is $(n-1)$-connected. □

Definition 5.1.3 (Left resolution)**.** Let $n \geq 1$, $P, Q \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ and $(s, W) \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P, Q)$. Let

$$\phi : (\partial_1 D^{n+1}_+ \times (0,\infty) \times \mathbf{R}^{n-1}, \partial_{01} D^{n+1}_+ \times (0,\infty) \times \mathbf{R}^{n-1}) \hookrightarrow (P, \partial^h P)$$

be an embedding and $\ell : (0,\infty) \to \mathrm{Bun}^{\mathrm{col}}(T(D^{n+1}_+ \times D^n), \Theta^*\gamma_{2n+1})$ a map, such that $\ell_t|_{\partial_1 D^{n+1}_+ \times D^n} = \phi^* \ell_P|_{\partial_1 D^{n+1}_+ \times (t\cdot e_1 + D^n)}$, where $\mathrm{Bun}^{\mathrm{col}}(T(D^{n+1}_+ \times D^n), \Theta^*\gamma_{2n+1})$ denotes the space of collared bundle maps $(T(D^{n+1} \times D^n), T(D^{n+1}_+ \times \partial_0 D^n)) \to (\theta^*\gamma_{2n+1}, (\theta^{\partial})^*\gamma_{2n})$. Define $\mathcal{K}(W, \phi, \ell)_0$ to be the space of triples $(t, c, \mathcal{L})$ where $t \in (0,\infty)$, $c : (D^{n+1}_+ \times D^n, \partial_0 D^{n+1}_+ \times D^n) \hookrightarrow (W, \partial^h W)$ an embedding of pairs and $\mathcal{L} : [0,1] \to \mathrm{Bun}^{\mathrm{col}}_{\partial^v}(T(D^{n+1}_+ \times D^n), \Theta^*\gamma_{2n+1})$ satisfying the following properties:

(i) *(Collared translation of ϕ by t near P)* For some $\delta > 0$, we have $c(x, v) = \phi(\frac{x}{|x|}, v + t \cdot e_1) + (1 - |x|) \cdot e_0$ where e_0 is the first coordinate vector of $\mathbf{R} \times \mathbf{R}^{\infty}_+$ for all x such that $1 - |x| < \delta$.

(ii) The image $(C, \partial^h C) = c(D^{n+1}_+ \times D^n, \partial_0 D^{n+1}_+ \times D^n)$ is disjoint from $([0,s] \times \partial^v L) \cup (\{s\} \times Q)$ and $c^{-1}(P) = \partial_1 D^{n+1}_+ \times D^n$.

(iii) $\mathcal{L}(0) = c^*\ell_W$ and $\mathcal{L}(1) = \ell_t$.

(iv) The map $\ell_W|_{W\backslash C} : (W\backslash C, \partial^h W \backslash \partial^h C) \to (B, B^{\partial})$ is strongly n-connected.

This space is topologized in the same way as Definition 5.1.1. Define $\mathcal{K}(W, \phi, \ell)_p$ to be the subspace of $\mathcal{K}(W, \phi, \ell)_0^{\times(p+1)}$ consisting of those tuples $(t_i; c_i; \mathcal{L}_i)_{i=0,\cdots,p}$ such that:

(i) $t_0 < t_1 < \cdots < t_p$;

(ii) the embeddings c_i are pairwise disjoint;

(iii) the map $\ell_W|_{W\backslash C} : (W\backslash C, \partial^h W \backslash \partial^h C) \to (B, B^{\partial})$ is strongly n-connected, for C the union of the images of all c_i.

Lemma 5.1.4. *The subspaces $\mathcal{K}(W, \phi, \ell)_p \subset \mathcal{K}(W, \phi, \ell)_0^{\times(p+1)}$ are preserved under forgetting factors. Therefore, it induces the structure of a semi-simplicial space to $\mathcal{K}(W, \phi, \ell)_\bullet$.*

Proof. As before, it suffices to prove that, if $M = W \backslash \cup_j C_j \to B$ is strongly n-connected, then so is $M' = W \backslash \cup_{j \neq i} C_j \to B$ also is. We prove that the inclusion $M \to M'$ is strongly $(n-1)$-connected. The cobordism M is obtained from M' by removing the core of a left $(n+1)$-handle. Therefore, M' is obtained by attaching its dual handle, which is a right n-handle. Thus by (i) in Section 2.3.2, the map $M \to M'$ is strongly $(n-1)$-connected. The claim follows now by Lemma 2.1.3. □

We now introduce the augmented semi-simplicial spaces as promissed in the introduction to this section.

Definition 5.1.5. Given $P, Q \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ and ϕ and ℓ as in one of the above definitions. Define $\mathcal{K}_{\phi,\ell}(P,Q)_p$ to be the space of pairs (W, x) where $W \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P,Q)$ and $x \in \mathcal{K}(W,\phi,\ell)_p$. For every $p \geq 0$, this space maps to $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P,Q)$ by forgetting x.

Lemma 5.1.6. *The homotopy fiber of the map* $||\mathcal{K}_{\phi,\ell}(P,Q)_\bullet|| \to \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P,Q)$ *over* $W \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P,Q)$ *is equivalent to* $||\mathcal{K}(W,\phi,\ell)_\bullet||$.

Proof. By [ER19, Lemma 2.14][12], it suffices to show that $\mathcal{K}_{\phi,\ell}(P,Q)_p \to \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P,Q)$ is a quasi-fibration for each $p \geq 0$. This follows analogously to [GR17a, Lemma 7.17]. □

5.1.1. Functoriality of the resolutions.

We now establish "functoriality" properties of these resolutions with respect to pre- and post-composition. In this section, we take $\mathcal{K}_{\phi,\ell}(P,Q)$ to be of any of the types above. Start by observing that post-composition with a cobordism $W' : Q \rightsquigarrow S$ and a triple $(t, e, \mathcal{L}) \in \mathcal{K}(W,\phi,\ell)$ induces a triple $(t, \iota(c), \mathcal{L})$ in $\mathcal{K}(W \cup_Q W', \phi, \ell)$ since c is disjoint from Q, where $\iota : W \hookrightarrow W \cup_Q W'$ is the inclusion.

Lemma 5.1.7. *Let* $W' : Q \rightsquigarrow S$ *be a morphism in* $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ *which is strongly* $(n-1)$*-connected relative to* Q*, then post-composition defines a map of augmented semi-simplicial spaces*

$$W' \circ (-) : \mathcal{K}_{\phi,\ell}(P,Q)_\bullet \to \mathcal{K}_{\phi,\ell}(P,S)_\bullet$$

by taking a tuple $(W, (t, c, \mathcal{L}))$ *to the tuple* $(W \cup_Q W', (t, \iota(c), \mathcal{L}))$.

Proof. It suffices to prove that, given $(W, (t, c, \mathcal{L})) \in \mathcal{K}_{\phi,\ell}(P,Q)_p$ then the restriction of the Θ-structure to $(W \cup W')\backslash C$ is strongly n-connected. This manifold is equal to $(W\backslash C) \cup W'$, since C is disjoint from $Q = W \cap W'$, so it suffices to prove that $W\backslash C \to (W\backslash C) \cup W'$ is strongly $(n-1)$-connected, by Lemma 2.1.3. This follows by Lemma 2.1.4 since $(Q, \partial^h Q) \hookrightarrow (W', \partial^h W')$ is strongly $(n-1)$-connected. □

As a consequence of this lemma, the maps $K_{[i,i+1]} \circ (-) : \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(N, K|_i) \to \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(N, K|_{i+1})$ lift to maps of augmented semi-simplicial spaces, and also for M. Thus, we can define the augmented semi-simplicial spaces $X(M)$ and $X(N)$ from the introduction by levelwise taking the homotopy colimit along these lifts. We denote these spaces by $\mathcal{K}_{\phi,\ell}(M, K|_\infty)$ amd $\mathcal{K}_{\phi,\ell}(N, K|_\infty)$, respectively. We will return to this closer to the end of the proof of Theorem 4.3.16. We move now to constructing a similar lift of the pre-composition map.

[12]To use this result, one is forced to prove that all spaces forming the semi-simplicial space are compactly generated. However, the statement is invariant under changing X_p by its Kellyfication kX_p since the map $X \to kX$ is a weak equivalence (and the identity on underlying sets) for every space X.

Construction 5.1.8. Fix a diffeomorphism $\epsilon_s : (D^n \times D^{n+1}_+, D^n \times \partial_0 D^{n+1}_+) \cong ((s \cdot e_1 + D^n) \times D^{n+1}_+, (s \cdot e_1 + D^n) \times \partial_0 D^{n+1}_+) \cup [0,s] \times (\partial D^n \times D^{n+1}_+, \partial D^n \times \partial_0 D^{n+1}_+) =: B_s \cup A_s = D_s$. Given $(t, c, \mathcal{L}) \in \mathcal{K}(W, \phi, \ell)$ for ϕ of right type and $(s, W') : P' \rightsquigarrow P$ such that $\mathrm{supp}(W) \cap \phi = \emptyset$, define $c_s : D_s \hookrightarrow W' \cup W$ by taking B_s to c and A_s to $[0,s] \times c|_{\partial D^n \times D^{n+1}_+}$. The triple $(t, \epsilon_s^{-1}(c_s), \epsilon_s^{-1}(\mathcal{L}_s))$, where $\mathcal{L}_s$ is given by extending the path of Θ-structure $\mathcal{L}$ along the collar A_t, satisfies properties $(i) - (iii)$ of Definitions 5.1.1 and 5.1.3. We prove in the next lemma that such definition satisfies (iv). When ϕ is of left type, one proceeds in a similar way by adding now a collar $[0,s] \times (\partial_1 D^{n+1}_+ \times D^n, \partial_{01} D^{n+1}_+ \times D^n)$ to $(D^{n+1}_+ \times D^n, \partial_0 D^{n+1}_+ \times D^n)$. (For more details, see [GR17b, Defn. 4.13].)

Lemma 5.1.9. *Let $(s, W') : P' \rightsquigarrow P$ be a morphism in $\mathsf{Cob}^{\partial, n-1}_{\Theta, \partial^v L}$ such that $\mathrm{supp}(W) \cap \phi = \emptyset$, then pre-composition as defined above*

$$(-) \circ W' : \mathcal{K}_{\phi,\ell}(P, Q)_\bullet \to \mathcal{K}_{\phi,\ell}(P', Q)_\bullet$$

is a map of augmented semi-simplicial spaces. Moreover, for $W'' : Q \rightsquigarrow S$, then the pre and post-composition maps commute strictly in the natural way.

Proof. As mentioned before, it suffices to prove that the construction above satisfies (iv). Let $(W, (t, c, \mathcal{L}))$ be a p-simplex of $\mathcal{K}_{\phi,\ell}(P, Q)$. We wish to prove that the Θ-structure of $(W' \cup W) \backslash ([-s, 0] \times \partial^v C \cup C)$ is strongly n-connected and apply Lemma 2.1.3. It suffices to prove that the inclusion

$$W \backslash C \to (W' \cup W) \backslash ([-s, 0] \times \partial^v C \cup C)$$

is strongly $(n-1)$-connected. By Lemma 2.1.4, this follows if $P \backslash \partial^h C \to W' \backslash C$ is strongly $(n-1)$-connected. However, this follows by Lemma 2.1.4, since $P \to W'$ is strongly $(n-1)$-connected and $(P \backslash \partial^v C, \partial^h P \backslash \partial^{hv} C) \to (P, \partial^h P)$ induces an isomorphism of fundamental groupoids, since $n \geq 3$. ◻

This establishes *(i)* from the introduction, namely that, we have a map of semi-simplicial spaces $(-) \circ W : \mathcal{K}_{\phi,\ell}(M, K|_\infty)_\bullet \to \mathcal{K}_{\phi,\ell}(N, K|_\infty)_\bullet$ lifting the map $(-) \circ W : \mathsf{Cob}^{\partial}_{\Theta, \partial^v L, n}(N, K|_\infty) \to \mathsf{Cob}^{\partial}_{\Theta, \partial^v L, n}(M, K|_\infty)$.

5.1.2. Contractability of left core complexes of middle dimension.

The goal of this subsubsection is to establish *(ii)* from the introduction for the left type. More precisely, we prove the following result.

Theorem 5.1.10. *Let $P \in \mathsf{Cob}^{\partial}_{\Theta, \partial^v L}$ and K a Θ-end. Let ϕ and $\ell : (0, \infty) \to \mathrm{Bun}^{\mathrm{col}}(T(D^{n+1}_+ \times D^n), \Theta^* \gamma_{2n+1})$ be as in Definition 5.1.3. Then the map induced by augmentation $||\mathcal{K}_{\phi,\ell}(P, K|_\infty)_\bullet|| \to \mathsf{Cob}^{\partial}_{\Theta, \partial^v L, n}(P, K|_\infty)$ is an equivalence, provided (B, B^∂) is 1-connected.*

By Lemma 5.1.6, to prove this statement it suffices to prove that the homotopy colimit of the realizations of $\mathcal{K}(W \cup K_{[i,i+j]}, \phi, \ell)_\bullet$ is contractible, for any $W \in \mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P, K|_i)$. To do so, we define the following variations of the semi-simplicial spaces $\mathcal{K}(-)$. Fix W, ϕ and ℓ as in Definition 5.1.3.

Definition 5.1.11. Define $\overline{\mathcal{K}}(W, \phi, \ell)_0$ to be the space of triples $(t, c, \mathcal{L})$ where $t \in (0, \infty)$,

$$c : (D^{n+1}_+ \times D^n, \partial_0 D^{n+1}_+ \times D^n) \to (W, \partial^h W)$$

an immersion of pairs and $\mathcal{L} : [0,1] \to \mathrm{Bun}^{\mathrm{col}}_{\partial^v}(D^{n+1}_+ \times D^n)$ satisfying the following properties:

(i) c is an embedding when restricted to $(D^{n+1}_+ \times \{0\}, \partial_0 D^{n+1}_+ \times \{0\})$ and for some $\delta > 0$, we have $c(x, v) = \phi(\frac{x}{|x|}, v + t \cdot e_1) + (1 - |x|) \cdot e_0$ where e_0 is the first coordinate vector of $\mathbf{R} \times \mathbf{R}^\infty_+$ for all x such that $1 - |x| < \delta$.
(ii) The image $C = c(D^{n+1}_+ \times \{0\}, \partial_0 D^{n+1}_+ \times \{0\})$ is disjoint from $([0, s] \times \partial^v L) \cup (\{s\} \times Q)$ and $c^{-1}(P) = \partial_1 D^{n+1}_+ \times D^n$.
(iii) $\mathcal{L}(0) = c^* \ell_W$ and $\mathcal{L}(1) = \ell_t$.
(iv) The map $\ell_W|_{W \backslash C} : (W \backslash C, \partial^h W \backslash \partial^h C) \to (B, B^\partial)$ is strongly n-connected.

Define $\overline{\mathcal{K}}(W, \phi, \ell)_p$ to be the subspace of $\overline{\mathcal{K}}(W, \phi, \ell)_0^{\times(p+1)}$ constisting of those tuples $(t_i; c_i; \mathcal{L}_i)_{i=0,\cdots,p}$ such that:

(i) $t_0 < t_1 < \cdots < t_p$.
(ii) The embeddings $c_i|_{D^{n+1} \times \{0\}}$ are pairwise disjoint.
(iii) The map $\ell_W|_{W \backslash C} : (W \backslash C, \partial^h W \backslash \partial^h C) \to (B, B^\partial)$ is strongly n-connected, for C the union of the images of all $c_i|_{D^{n+1} \times \{0\}}$.

This defines a semi-simplicial space $\overline{\mathcal{K}}(W, \phi, \ell)_\bullet$. Denote by $\overline{\mathcal{K}}^\delta(W, \phi, \ell)_\bullet$ the semi-simplicial space where $\overline{\mathcal{K}}^\delta(W, \phi, \ell)_p$ is the set $\overline{\mathcal{K}}(W, \phi, \ell)_p$ with the discrete topology. The identity induces a map of semi-simplicial spaces

$$\overline{\mathcal{K}}^\delta(W, \phi, \ell)_\bullet \to \overline{\mathcal{K}}(W, \phi, \ell)_\bullet.$$

Definition 5.1.12. Define $\widetilde{\mathcal{K}}^\delta(W, \phi, \ell)_0$ to be the set of triples $(t, c, \mathcal{L})$ where $t \in (0, \infty)$,

$$c : (D^{n+1}_+ \times D^n, \partial_0 D^{n+1}_+ \times D^n) \to (W, \partial^h W)$$

an immersion of pairs and $\mathcal{L} : [0,1] \to \mathrm{Bun}^{\mathrm{col}}_{\partial^v}(D^{n+1}_+ \times D^n)$ satisfying properties $(i) - (iii)$ of Definition 5.1.11. Define $\widetilde{\mathcal{K}}^\delta(W, \phi, \ell)_p$ to be the subset of $\widetilde{\mathcal{K}}^\delta(W, \phi, \ell)_0^{\times(p+1)}$ constisting of those tuples $(t_i; c_i; \mathcal{L}_i)_{i=0,\cdots,p}$ such that:

(i) $t_0 < t_1 < \cdots < t_p$.
(ii) the embeddings $c_i|_{D^{n+1} \times \{0\}}$ are pairwise disjoint.

This defines a semi-simplicial space $\widetilde{\mathcal{K}}^\delta(W, \phi, \ell)_\bullet$, where all spaces have the discrete topology. The identity induces a map

$$\overline{\mathcal{K}}^\delta(W, \phi, \ell)_\bullet \to \widetilde{\mathcal{K}}^\delta(W, \phi, \ell)_\bullet.$$

Definition 5.1.13. Define $\widehat{\mathcal{K}}^{\delta}(W,\phi,\ell)_0$ to be the set of triples $(t,c,\mathcal{L})$ where $t\in(0,\infty)$,

$$c:(D^{n+1}_+\times D^n,\partial_0 D^{n+1}_+\times D^n)\to(W,\partial^h W)$$

an immersion of pairs and $\mathcal{L}:[0,1]\to\mathrm{Bun}^{\mathrm{col}}_{\partial v}(D^{n+1}_+\times D^n)$ satisfying:

- (i') c is a self-transverse immersion with no triple points when restricted to $(D^{n+1}_+\times\{0\},\partial_0 D^{n+1}_+\times\{0\})$ and for some $\delta>0$, we have $c(x,v)=\phi(\frac{x}{|x|},v+t\cdot e_1)+(1-|x|)\cdot e_0$where e_0 is the first coordinate vector of $\mathbf{R}\times\mathbf{R}^{\infty}_+$ for all x such that $1-|x|<\delta$.

and $(ii)-(iii)$ of Definition 5.1.11. Define $\widehat{\mathcal{K}}^{\delta}(W,\phi,\ell)_p$ to be the subset of $\widehat{\mathcal{K}}^{\delta}(W,\phi,\ell)_0^{\times(p+1)}$ constisting of those tuples $(t_i;c_i;\mathcal{L}_i)_{i=0,\cdots,p}$ such that:

- (i) $t_0<t_1<\cdots<t_p$.
- (ii) the embeddings $c_i|_{D^{n+1}_+\times\{0\}}$ are pairwise transverse and without triple intersections.

This defines a semi-simplicial space $\widehat{\mathcal{K}}^{\delta}(W,\phi,\ell)_\bullet$, where all spaces have the discrete topology. The identity induces a map

$$\widetilde{\mathcal{K}}^{\delta}(W,\phi,\ell)_\bullet\to\widehat{\mathcal{K}}^{\delta}(W,\phi,\ell)_\bullet.$$

Putting all these variations together, we have the following string of maps

$$\mathcal{K}(W,\phi,\ell)_\bullet\to\overline{\mathcal{K}}(W,\phi,\ell)_\bullet\leftarrow\overline{\mathcal{K}}^{\delta}(W,\phi,\ell)_\bullet\to\widetilde{\mathcal{K}}^{\delta}(W,\phi,\ell)_\bullet\to\widehat{\mathcal{K}}^{\delta}(W,\phi,\ell)_\bullet.$$

The strategy to prove that the leftmost semi-simplicial space has contractible realization after taking the homotopy colimit of the maps $K_{[i,i+1]}\circ(-)$ will go as follows. First, we prove that the rightmost semi-simplicial set has contractible realization. Secondly, we prove that all arrows above induce a weak equivalence after realization and taking homotopy colimits. The first step does not require taking this homotopy colimit, however this is necessary for the second step. We start with the first step. This is the first point in our proof where the assumption that (B,B^{∂}) is 1-connected is necessary.

Lemma 5.1.14. *The space* $||\widehat{\mathcal{K}}^{\delta}(W,\phi,\ell)_\bullet||$ *is contractible, provided* (B,B^{∂}) *is* 1-*connected.*

Proof. We start by showing that $\widehat{\mathcal{K}}^{\delta}(W,\phi,\ell)_0$ is non-empty. Since (B,B^{∂}) and thus also $(W,\partial^h W)$ (by $n\geq 2$) are 1-connected, Lemma 2.1.8 assures the existence of a lift

$$\begin{array}{ccccc}
(\partial_1 D^{n+1}_+\times D^n,\partial_{01}D^{n+1}_+\times D^n) & \xrightarrow{\phi} & (P,\partial^h P) & \hookrightarrow & (W,\partial^h W)\\
\big\downarrow & & {\scriptstyle\hat{c}}\nearrow & & \big\downarrow{\scriptstyle \ell_W}\\
(D^{n+1}_+\times D^n,\partial_0 D^{n+1}_+\times D^n) & & \xrightarrow{\quad\ell_t\quad} & & (B,B^{\partial})
\end{array}$$

for any $t \in (0, \infty)$, which can be assumed to be covered by a bundle map between the tangent bundles, making the upper triangle commute and the lower one commute up to homotopy of bundle maps, since ℓ_t is covered by a bundle map. Thus by Smale-Hirsch, we can homotope it to be an immersion of pairs. We can now define a triple $(t, \hat{c}, \mathcal{L})$ where $\mathcal{L}$ is a bundle homotopy witnessing that the bottom triangle is homotopy commutative through bundle maps, such that it defines a point in $\widehat{\mathcal{K}}^{\delta}(W, \phi, \ell)_0$. We proceed by proving contractability assuming non-emptyness. For $k \geq 0$, let $f : \partial I^k \to ||\widehat{\mathcal{K}}^{\delta}(W, \phi, \ell)_\bullet||$ be a map, which we can assume to be simplicial with respect to a triangulation of ∂I^k by the simplicial approximation theorem. Thus, for each vertex $v_i \in \partial I^k$ of this triangulation, we have an element $f(v_i) = (t_i, c_i, \mathcal{L}_i) \in \widehat{\mathcal{K}}^{\delta}(W, \phi, \ell)_0$. Fix such a v_i. Let $(t, c', \mathcal{L})$ be a triple such that $t \neq t_j$ for any j, c' is the result of an isotopy of c_i which extends the translation of ϕ_{t_i} to ϕ_t and $\mathcal{L}$ be the path extending the path ℓ on P. By transversality, we can find a regular homotopy of pairs from $c'|_{D_+^{n+1} \times \{0\}}$ relative to its vertical boundary (and hence, constant on ϕ_t) such the result c is transverse to all c_j and without triple intersections. Therefore, f is in the star of $f(v_j)$ and so f extends to the cone of ∂I^k. □

We now move to comparing the different variations of the resolutions defined above. Let $W : P \rightsquigarrow K|_i$ for some i and denote by $\mathcal{K}(W_j, \phi, \ell)_p := \mathcal{K}(W_i \cup K|_{[i,i+j]}, \phi, \ell)_p$ for $j \geq 0$. Denote by $\mathcal{K}(W_\infty, \phi, \ell)_p$ the (strict) colimit of $\mathcal{K}(W_j, \phi, \ell)_p$ along post composition map $K|_{[i,i+j]} \circ (-)$ (see Lemma 5.1.7). One can check that this is equivalent to the homotopy colimit, since the maps are inclusions. Since the geometric realization commutes with homotopy colimits, we have an equivalence

$$\text{hocolim}_j\, ||\mathcal{K}(W_j, \phi, \ell)_\bullet|| \to ||\mathcal{K}(W_\infty, \phi, \ell)_\bullet||.$$

We use the analogous notation for the variations of $\mathcal{K}(W, \phi, \ell)$ defined previously.

Proposition 5.1.15. *The map*

$$\widetilde{\mathcal{K}}^{\delta}(W_\infty, \phi, \ell)_\bullet \to \widehat{\mathcal{K}}^{\delta}(W_\infty, \phi, \ell)_\bullet$$

induces an equivalence on geometric realizations.

Proof. Denote the source of this map by $X_\bullet$ and the target by $Y_\bullet$. Let $k \geq 0$ be an integer and $F : (I^k, \partial I^k) \to (||Y_\bullet||, ||X_\bullet||)$ be a map, which we can assume to be simplicial with respect to some triangulation of I^k. We want to show that F is homotopic to a map whose image is contained in $||X_\bullet||$. We argue now that it suffices to find a map of pairs F' homotopic to F where $F'(\sigma) \in X_p$ for every simplex σ of dimension p for $p = \{0, 1\}$: This follows by two properties. First, the map $Y_p \to Y_0^{\times(p+1)}$ given by the differentials is injective. Second, for $p \geq 1$, $v = (v_0, v_1, \cdots, v_p) \in Y_p$ belongs to X_p if and only if $(v_i, v_j) \in X_1$ for every $0 \leq i, j \leq p$. We start by finding a homotopic map where the images of all vertices

Figure 9: In this picture, we represent the region $U_0 \times D^n$ in blue and $U_1 \times D^n$ in red in a chart x.

v of I^k are in X_0. In other words, given $F(v) = (t, c, \mathcal{L})$, it suffices to find another vertex $w_v = (t + \epsilon, c', \mathcal{L}') \in Y_0$ for arbitrarily small enough $\epsilon > 0$ where c' is embedded, once restricted to its core, and transverse to c (and to all immersions given by the images of the other vertices, which c was already transverse) without triple intersections: by applying this technique to all interior vertices v and rechoosing F on those vertices to be w_v, we produce a new map, since $(F(v), w_{v'}) \in Y_1$ whenever $(F(v), F(v')) \in Y_1$. This map is homotopic to F since for each v, $(F(v), w_v) \in Y_1$.

The embedding c' will be constructed in a few steps via local modifications around the self-intersection points. Firstly, we fix a "chart" around the self-intersection points and all modifications will essentially be supported in this chart. Secondly, we define a local modification to c eliminating intersection points, provided c satisfies a certain hypothesis (D). Thirdly, we define a local modification that takes any embedding into one satisfying (D). This last step will the only place where the infinite supply of embedded V_1's is necessary. This strategy is inspired by the constructions in the proof of [GR17b, Prop. 5.5].

Parameterizing self-intersection loci. Let v be an interior vertex of I^k and $(t, c, \mathcal{L})$ its image under F. Let $j \geq 0$ be an integer such that the image of c lies in W_j. Denote the self-intersection locus of $c|_{D^{n+1}_+ \times \{0\}}$ by $(K, \partial K)$, that is the 1-dimensional submanifold pair of $(W_j, \partial^h W_j)$ of those points where the pre-image of c is more than one point. This is a disjoint union of manifolds diffeomorphic to $([0, 1], \{0, 1\})$ and $(S^1, \emptyset)$. Fix a path component K'. We present a method to rechoose c on a neighborhood of the pre-image of K' so its self-intersection locus is $K \backslash K'$. By hypothesis, $c^{-1}(K')$ has two connected components diffeomorphic to K', which we denote by K_0 and K_1. Observe that the normal bundle of K_i in $D \coloneqq D^{n+1}_+ \times \{0\}$ is trivial: The tangent bundle of K_i is trivial and its sum with the normal bundle is the pullback of $(TD^{n+1}_+, T\partial_0 D^{n+1}_+)$, which is trivial. Thus, the normal bundle is an n-dimensional collared vector bundle pair which is stably trivial, thus it is trivial. Moreover, the normal bundle of K' in W_j splits as the sum of the normal bundles of K_0 and K_1 in D. Thus, we can find a (framed) tubular neighborhood $x : (K' \times D^n \times D^n, \partial K' \times D^n \times D^n) \hookrightarrow (W_j, \partial^h W_j)$ disjoint from $\partial^v W_j$ such that, up to scaling the $\{0\} \times D^n$-direction, there exists an immersion c_0 which agrees with c outside of a neighborhood of $\mathrm{im}(x)$ such that:

(i) there exists a neighborhood U_0 of K_0 in $D^{n+1}_+ \times \{0\}$ such that $\mathrm{im}(x) \cap c_0(U_0) = x(K' \times D^n \times \{0\})$. Additionally, $\mathrm{im}(x) \cap c_0(U_0 \times D^n) = x(K \times D^n \times \frac{1}{2}D^n)$.

(ii) there exists a neighborhood U_1 of K_1 in $D^{n+1}_+ \times \{0\}$ such that $\mathrm{im}(x) \cap c_0(U_1) = x(K' \times \{0\} \times D^n)$. Additionally, $\mathrm{im}(x) \cap c_0(U_1 \times D^n) = x(K \times \frac{1}{2}D^n \times D^n)$.

By replacing c by c_0, we can assume without loss of generality that $c = c_0$, since this change is supported away from $\partial_1 D^{n+1}_+ \times D^n$. (See Figure 9 for the restriction of the immersion to the chart x for the case K' is diffeomorphic to $[0,1]$.)

The hypothesis (D). Let us introduce the following hypothesis on the immersion c: there exists an embedding $f : (K' \times S^n \times D^n, \partial K' \times S^n \times D^n) \hookrightarrow (W, \partial^h W)$ whose core (that is, $\mathrm{im}(f|_{K' \times S^n \times \{0\}})$) intersects c exactly in $f(K' \times \{x\} \times \{0\})$ for some $x \in S^n$, and an embedding $\gamma : (K' \times [0,1], \partial K' \times [0,1]) \hookrightarrow (D^{n+1}_+, \partial_0 D^{n+1}_+)$ disjoint from $c^{-1}(K') \backslash K_0$ such that $\mathrm{im}(\gamma(-,0)) = K_0$ and $c(\gamma(-,1)) = f(K' \times \{x\} \times \{0\})$. Assume additionally that the framings of the normal bundle of $c(\gamma(-,0))$ and $c(\gamma(-,1))$ induced by the identifications above are in the same path component. We say c satisfies (D) if this hypothesis holds. We shall describe a construction which can be applied to c satisfying (D) and later modify c to satisfy (D). Assume c satisfies (D). Choose a chart x_f as before for the intersection of c and f satisfying the analogous properties where K_0 lies in the source of c and K_1 on the source of f. We can find a framed tubular neighborhood

$$\hat{\gamma} : (K' \times [0,1] \times D^n \times D^{n-1}, \partial K' \times [0,1] \times D^n \times D^{n-1}) \hookrightarrow (W_j, \partial^h W_j)$$

of $c(\mathrm{im}(\gamma))$ satisfying the following properties:

(i) when restricted to $K' \times \{0\} \times D^n \times D^{n-1}$, $\hat{\gamma}$ agrees with x restricted to $K' \times D^n \times D^{n-1}_+ \subset K' \times D^n \times \partial D^n$, where D^{n-1}_+ is the upper hemisphere of ∂D^n, under the clear identification $K' \times D^n \times D^{n-1} \cong K' \times D^n \times D^{n-1}_+$.

(ii) when restricted to $K' \times \{1\} \times D^n \times D^{n-1}$, $\hat{\gamma}$ agrees with x_f restricted to $K' \times D^n \times D^{n-1}_- \subset K' \times D^n \times \partial D^n$, where D^{n-1}_- is the lower hemisphere of ∂D^n.

Denote by $\hat{\gamma}_0$ the restriction of $\hat{\gamma}$ to $K' \times [0,1] \times \partial(\frac{1}{2}D^n) \times D^{n-1} \cong [0,1] \times \partial(\frac{1}{2}D^n) \times D^n$. Let $c' : (D^{n+1}_+ \times D^n, \partial_0 D^{n+1}_+ \times D^n) \hookrightarrow (W_j, \partial^h W_j)$ be the following immersion: Outside the $U_1 \times D^n$, it agrees with c. Inside $\frac{1}{2}U_1 \times D^n$ it is given by gluing $\hat{\gamma}_0$ with the image of f except the pre-image of x. This construction requires isotopies of $x(K' \times \frac{1}{2}D^n \times \partial D^n)$ to agree with $\hat{\gamma}(K' \times \{0\} \times \partial(\frac{1}{2}D^n) \times D^{n-1})$ and similarly for x_f. This produces an immersion c' which agrees with c outside $U_1 \times D^n$, whose core has intersection locus $K \backslash K'$. We remark that in this step no additional copies of V_1 were used (see Figure 10 for a picture of this move).

Modifying c to satisfy (D). We shall modify c in the neighborhood U_0 of K_0. In this

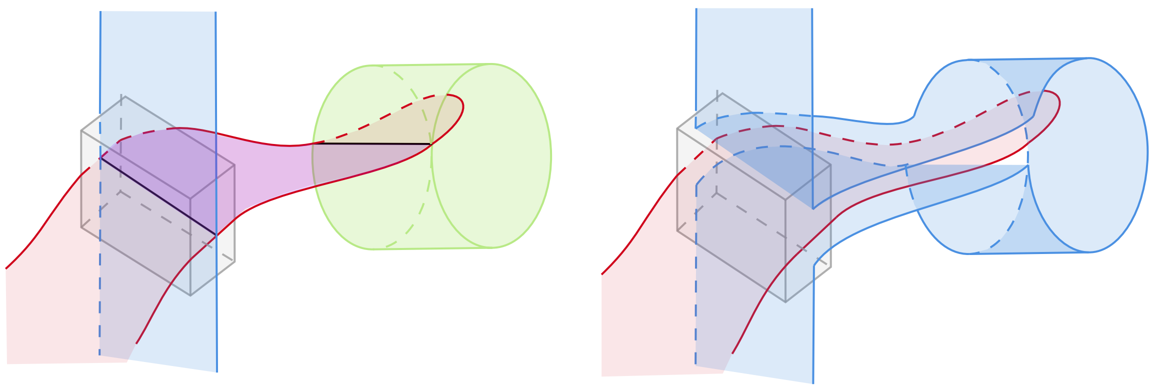


Figure 10: In this picture, we represent the move explained assuming the hypothesis (D) for the case $K' \cong [0,1]$. On the left, we see the two strands of the immersion c around the chart x (colored in grey) around the intersection submanifold $c(K')$ (colored in black) colored in blue and red as in Figure 9. In this case, U_0 (resp. U_1) is the intersection of the red (resp. blue) strand with the grey box. To prevent visual cluttering, we represent only the core of c. In green, we see the "geometric dual" embedding f and its intersection with c (colored in black). In light purple, we see the embedded path γ of embeddings of K' between the black strips. On the right, we see the effect of the move. We see that the restriction to $\partial^h W$ is simply an embedded (double) connected sum with the spheres $f(\{0\} \times S^n \times \{0\})$ and $f(\{1\} \times S^n \times \{0\})$. This move can be thought of a K'-parameterized connected sum between c and f along the intersection $c(K')$.

step, we use the available copies of V_1 and the hypothesis $n \geq 3$. Take the tubular neighborhood U_0 of K_0 framed as $K_0 \times D^n$, using the chart x, and let $\hat{U}_0$ be the subspace $K_0 \times (\frac{1}{4}D^n + \frac{1}{2} \cdot e_1)$ of U_0. Recall the embedding $\bar{f} : (D^n_+ \times D^{n+1}, D^n_+ \times \partial D^{n+1}) \hookrightarrow (V_1, W_{1,1})$ from Definition 4.1.4. Let $f : (S^n \times K' \times D^n, S^n \times \partial K' \times D^n) \hookrightarrow (V_1, W_{1,1})$ be the embedding induced by the inclusion $K' \times D^n \subset D^{n+1}$ given by $[0,1] \times D^n \hookrightarrow D^{n+1}$ (given by the first coordinate) or $S^1 \times D^1 \times D^{n-1} \hookrightarrow D^2 \times D^{n-1} \cong D^{n+1}$ (given by an embedding $S^1 \times D^1 \hookrightarrow D^2$, for example as in Definition 4.1.2) for K' diffeomorphic to $[0,1]$ or S^1 respectively. The intersection of $\bar{f}(\{0\} \times D^{n+1})$ with $f(S^n \times K' \times \{0\})$ is exactly $f(\{N\} \times K' \times \{0\})$, where $N = (0, \cdots, 0, 1)$. By assumption, the cobordism W_{j+1} contains an embedded copy of V_1 disjoint from the immersion c. We find an embedded $\gamma' : (K' \times [0,1], \partial K' \times [0,1]) \hookrightarrow (W_{j+1}, \partial^h W_{j+1})$ where $\gamma'(K' \times 0)$ agrees with $c(K_0 \times \frac{1}{2}e_1)$ and $\gamma'(K' \times 1)$ with $f(\{N\} \times K' \times \{0\})$ which is disjoint from c outside $K \times \{0\}$ and from all other embeddings c_i given by images of the other vertices under F: This is possible by transversality since the dimension of the image of γ', which is 2, is larger than the codimension of the immersion c, which is n. We can modify the immersion c around $\hat{U}_0 \times D^n$ by removing $K_0 \times (\frac{1}{4}\mathrm{int}(D^n) + \frac{1}{2} \cdot e_1) \times D^n$ and replacing it by the gluing of a framed neighborhood of γ' restricted to its sphere bundle and a neighborhood of $f(N \times K' \times \{0\})$ in $\bar{f}$. We see that the embedding f intersects the modified embedding precisely where $\bar{f}$ does. Thus, this creates an embedding f satisfying the hypothesis of (D). The embedding γ is induced by the embedding γ' which now connects the two intersection loci inside the image of the modifieed embedding. This concludes this step.

Rechoosing F on vertices up to homotopy. We shall define a map $F_v : (I^k, \partial I^k) \to (||Y_\bullet||, ||X_\bullet||)$ homotopic to F. Choose a compactly supported vector field on W_j extending the translation in $\phi(\partial_1 D^{n+1}_+ \times (0,\infty) \times \mathbf{R}^{n-1}) \subset P$ by the unit vector of $(0,\infty)$. By taking its flow, we produce a family $\psi_\epsilon : W_j \to W_j$ of diffeomorphism with the property that $\psi_\epsilon(\phi(x,t,v)) = \phi(x, t+\epsilon, v)$ for $\epsilon > 0$. Let c_ϵ be the immersion $\psi_\epsilon(c)$ and notice that $(t+\epsilon, c_\epsilon, \psi_\epsilon^* \mathcal{L}) \in Y_0$. Consider the immersion c' obtained from c_ϵ by first modifying it so it satisfies (D) for every path component of the self-intersection loci and further modifying it to remove all of those components, by applying the construction above iteratively. In other words, c' is an embedding to $W_{j'}$ for some $j' \geq j$ which agrees with c_ϵ outside a neighborhood of K. Moreover, the modifications above can be chosen to be disjoint from c and thus c' is disjoint from c. To produce a path $\mathcal{L}'$ from $(c')^* \ell_{W_{j'}}$ to $\ell_{t+\epsilon}$, we shall construct a path from $c_\epsilon^* \ell_{W_{j'}}$ to $(c')^* \ell_{W_{j'}}$. When modifying c_ϵ to satisfy (D), we replaced a part of this immersion by a part of the embedding $\bar{f}$. However, $\bar{f}^* \ell_{W_{j'}}$ is standard in the sense of Definition 4.1.3, so extends to $(\mathbf{R}^{n-1} \times \mathbf{R}^{n+2}_+, \mathbf{R}^{n-1} \times \partial \mathbf{R}^{n+2}_+)$. Thus the homotopy class of the Θ-structure does not change after this modification. Similarly, the second modification is done by gluing c with the embedding f.

However, in this particular case, $f^*\ell_{W_j}$ extends over a contractible space since it is a subspace of the embedding $\bar{e}$ in $(V_1, W_{1,1})$ which again has standard structure. We conclude that $c_\epsilon^*\ell_{W_{j'}}$ and $(c')^*\ell_{W_{j'}}$ are homotopic relative to $\partial_1 D_+^{n+1} \times D^n$. We let $\mathcal{L}'$ be the composite of the path $\mathcal{L}_\epsilon$ with such a homotopy. Let $F_v(w) = F(w)$ for all vertices $w \neq v$ and $F_v(v) = (t+\epsilon, c', \mathcal{L}') \in X_0$. This assignment extends to 1-simplices uniquely since c' stays transverse to any other immersions, to which c was transverse. Moreover, it extends to higher simplices uniquely as they are determined by the 0− and 1-simplices. Moreover, F_v is homotopic to F since $(F_v(v), F(v)) \in Y_1$. We let F' be the result of iterating this modification to all vertices of the triangulation of I^k. This produces a homotopic map whose vertices map to $X_\bullet$.

Rechoosing F' to lie in $X_\bullet$. By construction, the images of all the vertices along F' lie in X_0. In other words, all immersions c given by $F'(v) = (t, c, \mathcal{L})$ are actually embeddings when restricted to its cores. To produce a map F'' where all the 1-simplices lie in X_1, it suffices to modify these embeddings c to be pairwise disjoint. To do so, we simply apply the same technique above for removing self-intersections of the immersion

$$\coprod_{(t,c,\mathcal{L})\in F'((I^k)_0)} c : \coprod (D_+^{n+1} \times D^n, \partial_0 D_+^{n+1} \times D^n) \to (W_\infty, \partial^h W_\infty).$$

The self-intersection locus of this immersion is precisely all the pairwise intersections of the embeddings c. Thus, modifying such immersion to an embedding is making all 1-simplices lie in X_1. Proceeding as above, the result will be a homotopic map F'' sending 0- and 1-simplices to $X_\bullet$ and thus, sending any simplex of any dimension to $X_\bullet$. This can be applied to any such map F and thus, concludes the proof. ◻

Proposition 5.1.16. *The map*

$$\overline{\mathcal{K}}^\delta(W_\infty, \phi, \ell)_\bullet \to \widetilde{\mathcal{K}}^\delta(W_\infty, \phi, \ell)_\bullet$$

induces an equivalence on geometric realizations.

Lemma 5.1.17. *Let $x = \{(t_i, c_i, \mathcal{L}_i)\}_{i=0,\cdots,p}$ be a p-simplex of $\widetilde{\mathcal{K}}^\delta(W, \phi, \ell)$. Then $x \in \overline{\mathcal{K}}^\delta(W, \phi, \ell)_p$ if and only if the map $\partial^h W \backslash \partial^h C \to B^\partial$ is n-connected.*

Proof. The "only if" direction is immediate from the definition. Given a p-simplex $x = \{(t_i, c_i, \mathcal{L}_i)\}_{i=0,\cdots,p}$ in $\widetilde{\mathcal{K}}^\delta(W, \phi, \ell)_\bullet$, the cobordism W is obtained from $W\backslash C$ by attaching a right handle of index n at the belt of the each left handle c_i. For the "if" direction, it suffices to show $W\backslash C \cup_{\partial^h W\backslash \partial^h C} B^\partial \to B$ is $(n+1)$-connected. We can replace the source by the equivalent $W\backslash C \cup_{\partial^h W\backslash\partial^h C} \partial^h W \cup_{\partial^h W} B^\partial$. Since $W\backslash C \to W$ is equivalent to the attachment of a right handle, we have $W\backslash C \cup_{\partial^h W\backslash\partial^h C} \partial^h W \simeq W$.

Thus, the map above is equivalent to the map $W \cup_{\partial^h W} B^\partial \to B$ which is $(n+1)$-connected by hypothesis. $\square$

Proof of Proposition 5.1.16. Denote the source and target of the map in the statement by $Z_\bullet$ and $X_\bullet$ respectively. Let $F : (I^k, \partial I^k) \to (||X_\bullet||, ||Z_\bullet||)$ be a map, which we can assume to be simplicial with respect to some triangulation of I^k, and σ be a simplex of I^k. We shall modify F up to homotopy so that the cores of $F(\sigma)$ satisfy the condition of Lemma 5.1.17. Let C_σ be the union of the images of c_i for $F(\sigma) = \{(t_i, c_i, \mathcal{L}_i)\}$. Let $K_\sigma(F)$ be the kernel of the map $\pi_{n-1}(\partial^h W \backslash \partial^h C_\sigma) \to \pi_{n-1}(B^\partial)$. Let $I_\sigma(F)$ be the image of $\pi_n(\partial^h W \backslash \partial^h C_\sigma) \to \pi_n(B^\partial)$. Since the map $\partial^h W \backslash \partial^h C_\sigma \to \partial^h W$ is $(n-1)$-connected, it suffices to replace F so that $K_\sigma(F)$ vanishes and $I_\sigma(F) = \pi_n(B^\partial)$. If $n \geq 3$, then $\pi_1(\partial^h W \backslash \partial^h C_\sigma) \cong \pi_1(\partial^h W) \cong \pi_1(B^\partial)$. In this case, we apply the Hurewicz theorem to deduce that $\pi_n(\partial^h W, \partial^h W \backslash \partial^h C_\sigma)$ is generated as a $\pi_1(B^\partial)$-module by the meridian spheres $c_i|_{\{0\}\times \partial D^n}$. This module surjects onto $K_\sigma(F)$, which proves that the latter is a finitely generated $\mathbf{Z}[\pi_1(B^\partial)]$-module. If $n = 2$, then the Seifert-Van Kampen theorem shows that $K_\sigma(F)$ is normally generated by the meridian circles of the removed handles (as pointed out in [GR17b, p. 166]). Moreover since $n \geq 2$, if $K_\sigma(F)$ vanishes, then $\pi_1(\partial^h W \backslash \partial^h C_\sigma) \cong \pi_1(B^\partial)$ and the following exact sequence of $\pi_1(B^\partial)$-modules

$$\cdots \to \pi_n(\partial^h W \backslash \partial^h C_\sigma) \to \pi_n(B^\partial) \to \pi_n(B^\partial, \partial^h W \backslash \partial^h C_\sigma) \to 0$$

shows that $\pi_n(B^\partial)$ is generated by $I_\sigma(F)$ and finitely many elements lifted from the rightmost module, since the latter is finitely generated.

Killing $K_\sigma(F)$. We describe a modification on the embeddings c_i and describe its main properties. This is analogous to [GR17b, p. 166] and similar to the modification of c to satisfy (D) in the proof of Proposition 5.1.15. We assume $n \geq 2$. Let v be a vertex of σ and $F(v) = (t, c, \mathcal{L})$ and let $j \geq 0$ be such that $\mathrm{im}(c) \subset W_j$. We start by considering the perturbation c_ϵ for some $\epsilon > 0$ as defined in Proposition 5.1.15. Recall the embedding $\bar{f} : (D^{n+1} \times D^n, \partial D^{n+1} \times D^n) \hookrightarrow (V_1, W_{1,1})$. Denote also by $\bar{f}$ the composition of this embedding with the inclusion of a fixed copy of V_1 in $K|_{[j,j+1]}$ seen now in W_{j+1}. Choose an embedded path γ in $\partial^h W_{j+1}$ from $c_\epsilon(x,0)$ for some point $x \in \partial_0 D^{n+1}$ and $\bar{f}(x,0)$ disjoint from $\partial^{hv} W_j$ and C_σ, and from $\mathrm{im}(c_\epsilon)$ and $\mathrm{im}(\bar{f})$ except at times $t = \{0,1\}$. Using a thickening of this path, we can construct an embedded connected sum of the cores of c_ϵ and $\bar{f}$ and further thicken it to an embedding $c' : (D^{n+1}_+ \times D^n, \partial_0 D^{n+1} \times D^n) \hookrightarrow (W_{j+1}, \partial^h W_{j+1})$ which agrees with c_ϵ outside of a neighborhood of x. As in Proposition 5.1.15, since the Θ-structure $\bar{f}^* \ell_{W_{j+1}}$ is standard, the structures $(c')^* \ell_{W_{j+1}}$ and $c_\epsilon^* \ell_{W_{j+1}}$ are homotopic. By choosing such homotopy and composing it with the path $\mathcal{L}_\epsilon$, we construct a path $\mathcal{L}'$ from $(c')^* \ell_{W_{j+1}}$ to $\ell_{t+\epsilon}$. Denote $F_v : (I^k, \partial I^k) \to (||X_\bullet||, ||Z_\bullet||)$ the map definedd on vertices by $F_v(w) = F(w)$ for $w \neq v$ and $F_v(v) = (t+\epsilon, c', \mathcal{L}')$. For ϵ small enough, we see that this map is well-defined and homotopic to F relative

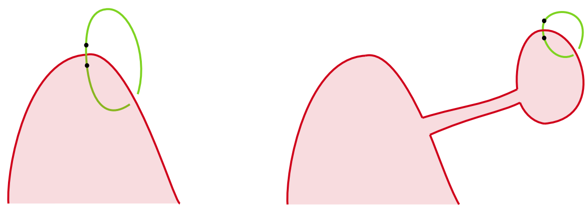


Figure 11: In this picture, we see the effect of an embedded D_+^{n+1} in W, represented in red, having a "geometrically dual sphere", represented in green, on the meridian class. The meridian class, depicted in black in this picture, lives in $\pi_{n-1}(\partial^h W \backslash \partial_0 D_+^{n+1})$. Notice that when a geometrically dual sphere exists, in particular it follows that the meridian class vanishes. Observe that taking an embedded connected sum with a disc D^{n+1} admitting a dual sphere creates a dual sphere for the result. The move described is a special case of this fact, where the source of this additional disc comes from the existing V_1. Additionally, this dual sphere allows us to modify the initial disc to be disjoint from a fixed embedded n-sphere in $\partial^h W$. This is a 0-dimensional version of hypothesis (D) in Proposition 5.1.15.

to ∂I^k. (See Figure 11 for a picture portraying this move.) The following claim establishes properties of this construction.

Claim 2. *Given k, F and v as above, the map F_v is homotopic to F relative to ∂I^k satisfies the following properties:*

(i) for every simplex $\sigma \in I^k$, there exists a surjection $K_\sigma(F) \to K_\sigma(F_v)$, sending the class of the meridian of each simplex of F to the meridian of each simplex of F_v except v and whose kernel contains the class of the meridian of v.

(ii) for every simplex $\sigma \in I^k$, we have $I_\sigma(F) \subseteq I_\sigma(F_v)$.

Proof. Apply [GR14, Claim 5.6] to the horizontal boundary of c. The only difference on our construction that we take a connected sum with the sphere $\bar{f}$ instead of $\bar{e}$, but the proof follows verbatim. □

Apply the construction above and Claim 2 for every interior vertex $v \in I^k$. This produces a homotopic map F' where the meridian of all vertices are trivial in $K_\sigma(F')$. Since this kernel is generated by such meridians, we have $K_\sigma(F') = 0$. By induction on the dimension of the simplices, we can obtain F' such that $K_\sigma(F') = 0$ for every simplex σ of I^k.

Adding elements to $I_\sigma(F')$. Since $K_\sigma(F) = 0$, it follows that $\pi_n(B^\partial)$ is generated by $I_\sigma(F')$ along with finitely many other elements. Choose finitely many elements

$$\{g_\alpha : S^n \to \partial^h W_\infty\}_\alpha$$

such that $[l_{\partial^h W_\infty} \circ g_\alpha] \in \pi_n(B^\partial)$ generate $\pi_n(B^\partial)$ along with $I_\sigma(F')$. If g_α is disjoint from C_σ, then this class lifts to $\pi_n(\partial^h W_\infty \backslash C_\sigma)$ and thus, lies in $I_\sigma(F')$. We shall modify F' and g_α so this is the case. This modification is heavily inspired by the proof of [GR17b, Prop. 5.5] and in fact it restricts to it when restricted to the horizontal boundary. We may suppose that all the g_α are smooth and transverse to the embeddings given by the images of the vertices of σ along F'. Let v be a vertex of σ and $F'(v) = (t, c, \mathcal{L})$ and let $(t+\epsilon, c_\epsilon, \mathcal{L}_\epsilon)$ be an ϵ-perturbation as before. By compactness, the intersection of $c_\epsilon(D^{n+1}_+ \times \{0\})$ with g_α is a finite set of points. Let p be such an intersection point, then we can find a chart $x : (D^{n+1}_+ \times D^n, \partial_0 D^{n+1}_+ \times D^n) \hookrightarrow (W_\infty, \partial^h W_\infty)$ disjoint from $\partial^v W_\infty$ where $x(0,0) = p$ such that:

(i) $x(D^{n+1}_+ \times D^n) \cap c_\epsilon(D^{n+1}_+ \times \{0\}) = x(D^{n+1}_+ \times \{0\})$ and $x(D^{n+1}_+ \times D^n) \cap g_\alpha = x(\{0\} \times D^n)$;
(ii) additionally, $x(D^{n+1}_+ \times D^n) \cap c_\epsilon(D^{n+1}_+ \times D^n) = x(D^{n+1}_+ \times \frac{1}{2}D^n)$.

We start by finding an embedded path in $\partial^h W_\infty$ from a point in $x(\partial(\partial_0 D^{n+1}_+ \times D^n))$ to an embedded copy of $(V_1, W_{1,1})$ disjoint from $\partial^v W_\infty$ and the remaining of images under F'. This is possible because $n \geq 2$. We can thicken this path to define an embedded boundary connected sum

$$x' : ((D^{n+1}_+ \times D^n)\natural V_1, (\partial_0 D^{n+1}_+ \times D^n)\natural W_{1,1}) \hookrightarrow (W_\infty, \partial^h W_\infty).$$

There are two prefered disjoint embeddings

$$j_0 : (D^{n+1}_+ \times \frac{1}{2}D^n, \partial_0 D^{n+1}_+ \times \frac{1}{2}D^n) \hookrightarrow ((D^{n+1}_+ \times D^n)\natural V_1, (\partial_0 D^{n+1}_+ \times D^n)\natural W_{1,1})$$
$$j_1 : \{0\} \times D^n \hookrightarrow (\partial_0 D^{n+1}_+ \times D^n)\natural W_{1,1}$$

which agree with the canonical inclusion near the (vertical) boundary: identify $((D^{n+1}_+ \times D^n)\natural V_1, (\partial_0 D^{n+1}_+ \times D^n)\natural W_{1,1}) \cong (S^n \times D^{n+1}, S^n \times S^n \backslash (D^n_-, D^n_-)$ and take the submanifolds $(\frac{1}{2}D^n_+ \times D^{n+1}_+, \frac{1}{2}D^n_+ \times \partial_1 D^{n+1}_+)$ and $D^n_+ \times \{0\}$ respectively. We define the embedding c' to agree with c_ϵ away from x and to be the embedding j_0 on the remaining domain. Similarly, we define g'_α to agree with g_α away from x and the map j_1 inside the domain of x. It is clear to see that c' and g'_α intersect transversely where c and g_α do except the point p. (See Figure 12 for a depiction of this move.) Observe that the c' is the result of the construction before Claim 2, thus we can define a triple $(t+\epsilon, c', \mathcal{L}')$ in X_0. Let F'_v be the simplicial map that agrees with F' for all vertices different from v and that $F'_v(v) = (t+\epsilon, c', \mathcal{L}')$. This map is homotopic to F' by the same reason as before. By Claim 2, we still have that $K_\sigma(F'_v) = 0$ and that $I_\sigma(F') \subseteq I_\sigma(F'_v)$. After repeating this process for every vertex of σ, we obtain a map F''_α and a map g''_α. However, the map g''_α is obtained by taking multiple embedded connected sums with sphere with standard (and in particular nullhomotopic) Θ-structure. Thus the class $[l_{\partial^h W_\infty} \circ g''_\alpha]$ agrees with $[l_{\partial^h W_\infty} \circ g_\alpha]$

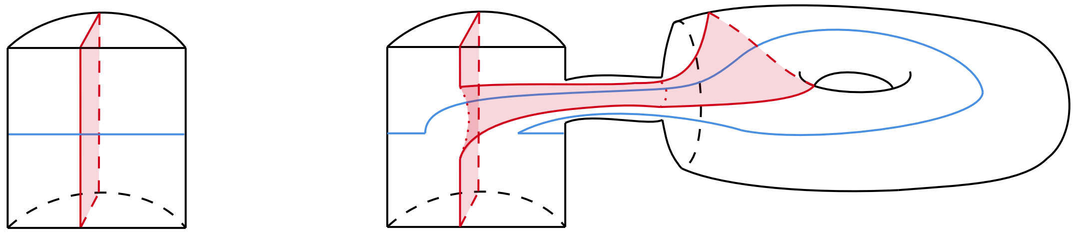


Figure 12: On the left, we see the chart x where the red portion represents the intersection of the core of c with the domain of x and the blue arc the intersection of g_α. On the right, we see the result of the explained modification. The red portion in the right picture represents the embedding j_0 and the blue arc the embedding j_1. Notice that this move simulaneously modifies the embedding e as in Figure 11 and takes an embedded connected sum of the blue arc with the "geometric dual" of the red portion in V_1. This is a 0-dimensional version of the move depicted in Figure 10.

in $\pi_n(B^\partial)$. But, since g''_α is disjoint from the cores of $F''(\sigma)$, it lies in $I_\sigma(F'')$. By finite generation and applying this process to every simplex, we produce a map F'' homotopic to F relative to ∂I^k where $K_\sigma(F'') = 0$ and $I_\sigma(F'') = \pi_n(B^\partial)$ for every simplex σ. This finishes the proof of this statement. □

Proposition 5.1.18. *The maps* $\mathcal{K}(W_\infty, \phi, \ell)_\bullet \to \overline{\mathcal{K}}(W_\infty, \phi, \ell)_\bullet \leftarrow \overline{\mathcal{K}}^\delta(W_\infty, \phi, \ell)_\bullet$ *induce an equivalence on geometric realizations.*

Proof. One proceeds exactly as [GR17b, Lemma 5.3] for the left equivalence and [GR17b, Lemma 5.7] for the right equivalence. □

Proof of Theorem 5.1.10. Combine Lemma 5.1.6, Proposition 5.1.18, Proposition 5.1.16, Proposition 5.1.15 and Lemma 5.1.14. □

5.1.3. Contractability of right core complexes of middle dimension.

In this section, we prove the analogous result for the right type, establishing *(ii)* from the introduction in this case.

Theorem 5.1.19. *Let* $P \in \mathsf{Cob}^\partial_{\Theta,\partial^v L}$ *and* K *a* Θ*-end. Let* ϕ *and* $\ell : (0,\infty) \to$ $\mathrm{Bun}^{\mathrm{col}}(T(D^n \times D^{n+1}_+), \Theta^*\gamma_{2n+1})$ *be as in Definition 5.1.1. Then the map* $||\mathcal{K}_{\phi,\ell}(P, K|_\infty)_\bullet|| \to$ $\mathsf{Cob}^\partial_{\Theta,\partial^v L,n}(P, K|_\infty)$ *is an equivalence, provided* (B, B^∂) *is* 1*-connected.*

We proceed similarly as before by defining variants of $\mathcal{K}(W, \phi, l)$, deduce their contractability from [GR17b, Section 5] and compare them to $\mathcal{K}(W, \phi, \ell)$. Let (s, W), ϕ and ℓ be as in Definition 5.1.1.

Definition 5.1.20. Define $\overline{\mathcal{K}}(W,\phi,\ell)_0$ to be the space of triples $(t, c, \mathcal{L})$ where $t \in (0,\infty)$,

$$c : D^n \times D^n \hookrightarrow \partial^h W$$

an immersion and $\mathcal{L} : [0,1] \to \mathrm{Bun}^{\mathrm{col}}_{\partial^v}(D^n \times D^n, (\theta^\partial)^*\gamma_{2n})$ satisfying the following properties:

(i) c is an embedding when restricted to $D^n \times \{0\}$ and for some $\delta > 0$, we have $c(x,v) = \phi(\frac{x}{|x|}, v + t \cdot e_1) + (1 - |x|) \cdot e_0$ where e_0 is the first coordinate vector of $\mathbf{R} \times \mathbf{R}^\infty$.

(ii) The image $C = c(D^n \times \{0\})$ is disjoint from $([0,s] \times \partial^v L) \cup (\{s\} \times Q)$ and $c^{-1}(\partial^h P) = \partial D^n \times D^n$.

(iii) $\mathcal{L}(0) = c^*\ell_{\partial^h W}$ and $\mathcal{L}(1) = \ell_t|_{D^n \times \partial_0 D^{n+1}_+}$.

(iv) The map $\ell_W|_{W \backslash C} : W \backslash C \to B$ is strongly n-connected.

Define $\overline{\mathcal{K}}(W,\phi,\ell)_p$ to be the subspace of $\overline{\mathcal{K}}(W,\phi,\ell)_0^{\times(p+1)}$ constisting of those tuples $(t_i; c_i; \mathcal{L}_i)_{i=0,\cdots,p}$ such that:

(i) $t_0 < t_1 < \cdots < t_p$.

(ii) The embeddings $c_i|_{D^n \times \{0\}}$ are pairwise disjoint.

(iii) The map $\ell_W|_{W \backslash C} : W \backslash C \to B$ is strongly n-connected, for C the union of the images of all $c_i|_{D^n \times \{0\}}$.

This defines a semi-simplicial space $\overline{\mathcal{K}}(W,\phi,\ell)_\bullet$. Denote by $\overline{\mathcal{K}}^\delta(W,\phi,\ell)_\bullet$ the semi-simplicial space where $\overline{\mathcal{K}}^\delta(W,\phi,\ell)_p$ is the set $\overline{\mathcal{K}}(W,\phi,\ell)_\bullet$ with the discrete topology. The identity induces a map of semi-simplicial spaces

$$\overline{\mathcal{K}}^\delta(W,\phi,\ell)_\bullet \to \overline{\mathcal{K}}(W,\phi,\ell)_\bullet.$$

Definition 5.1.21. Define $\widetilde{\mathcal{K}}^\delta(W,\phi,\ell)_0$ to be the space of triples $(t, c, \mathcal{L})$ as in Definition 5.1.20 satisfying $(i) - (iii)$ along with

(iv') The map $\ell_{\partial^h W}|_{\partial^h W \backslash \partial^h C} : \partial^h W \backslash C \to B^\partial$ is n-connected.

Define $\widetilde{\mathcal{K}}^\delta(W,\phi,\ell)_p$ to be the subspace of $\widetilde{\mathcal{K}}^\delta(W,\phi,\ell)_0^{\times(p+1)}$ consisting of those tuples $(t_i; c_i; \mathcal{L}_i)_{i=0,\cdots,p}$ satisfying $(i) - (ii)$ and

(iii') The map $\ell_{\partial^h W}|_{\partial^h W \backslash \partial^h C} : \partial^h W \backslash B^\partial$ is n-connected, where C is the union of all images of c_i.

This defines a semi-simplicial space $\widetilde{\mathcal{K}}^\delta(W,\phi,\ell)_\bullet$ with a map to $\overline{\mathcal{K}}^\delta(W,\phi,\ell)_\bullet$.

In this case, we are able to extract contractability directly from the work of [GR17b], since our embedding is completely contained in the boundary. Similarly to last subsubsection, we denote by $\mathcal{K}(W_\infty,\phi,\ell)$ the strict colimit of the maps $K|_{[i,i+1]} \circ (-)$, and similarly for all variations of $\mathcal{K}$.

Lemma 5.1.22. *The space* $||\widetilde{\mathcal{K}}^\delta(W_\infty,\phi,\ell)_\bullet||$ *is contractible.*

Proof. Recall the semi-simplicial space $\overline{Y}^{\delta}(\partial^h W)_\bullet$ of [GR17b, Defn. 5.1] and notice that this is precisely $\widetilde{\mathcal{K}}^{\delta}(W, \phi, \ell)_\bullet$ for any W. The claim follows by [GR17b, Prop. 5.4/5]. □

To compare the geometric realizations of the discrete semi-simplicial spaces defined above, we will employ a similar (but dual) strategy to Proposition 5.1.16 once again using the infinite supply of V_1's at hand.

Proposition 5.1.23. *The map*

$$\overline{\mathcal{K}}^{\delta}(W_\infty, \phi, \ell)_\bullet \to \widetilde{\mathcal{K}}^{\delta}(W_\infty, \phi, \ell)_\bullet$$

induces an equivalence on geometric realizations, provided (B, B^∂) is 1-connected.

Proof. Denote the source and target of this map by $X_\bullet$ and $Y_\bullet$. Let $k \geq 0$ and $F : (I^k, \partial I^k) \to (||Y_\bullet||, ||X_\bullet||)$ be a map, which we can assume to be simplicial with respect to some triangulation of I^k, and σ a simplex of I^k for which $\ell_{W\backslash C_\sigma}$ is not (necessarily) strongly n-connected, where C_σ is the union of all cores of $F(\sigma)$. We shall replace F up to homotopy so that $F(\sigma) \in X_p$. The inclusion $(W\backslash C_\sigma, \partial^h W\backslash\partial^h C_\sigma) \hookrightarrow (W, \partial^h W)$ is strongly $(n-1)$-connected. Since $n \geq 2$ and $\partial^h W\backslash\partial^h C_\sigma \to B^\partial$ is n-connected, we see $\pi_1(\partial^h W\backslash\partial^h C_\sigma) \cong \pi_1(B^\partial)$. Since the map of spaces $W\backslash C_\sigma \to W$ is an equivalence, it follows $\pi_1(W\backslash C_\sigma) \cong \pi_1(B)$ and thus $(W\backslash C_\sigma, \partial^h W\backslash\partial^h C_\sigma)$ is 1-connected. By Lemma 2.1.5, we can detect strong n-connectivity on maps of relative homotopy groups. Set $K_\sigma(F)$ to be the kernel of the map

$$\pi_n(W\backslash C_\sigma, \partial^h W\backslash\partial^h C_\sigma) \to \pi_n(B, B^\partial)$$

and $I_\sigma(F)$ to be the image of the same map in degree $n+1$. We see that $F(\sigma)$ lies in X_p if and only if $K_\sigma(F)$ and $\pi_{n+1}(B, B^\partial)/I_\sigma(F)$ vanish, since we know $\ell_{\partial^h W\backslash\partial^h C}$ is n-connected. For $n \geq 2$, the maps above are of $\mathbf{Z}[\pi_1(B^\partial)]$-modules. We start by proving that $K_\sigma(F)$ is finitely generated as such. Note that the triad homotopy group (see [BP17, Section 3.1]) $\pi_{n+1}(B, B^\partial, W\backslash C_\sigma)$ surjects into $K_\sigma(F)$. Moreover, by comparing the long exact sequences for the maps of pairs $W\backslash C_\sigma \to W \to B$ and the four-lemma, we have a surjection $\pi_{n+1}(W, \partial^h W, W\backslash C_\sigma) \to \pi_{n+1}(B, B^\partial, W\backslash C_\sigma)$. Since $W\backslash C_\sigma \to W$ is an equivalence (as it is equivalent to the attachment of a left handle), we have an isomorphism $\pi_{n+1}(W, \partial^h W, W\backslash C_\sigma) \cong \pi_n(\partial^h W, \partial^h W\backslash\partial^h C_\sigma)$. The latter group is a finitely generated $\mathbf{Z}[\pi_1(B^\partial)]$-module generated by the horizontal boundaries of belt (or meridian) spheres of C_σ by the Hurewicz theorem, if $n \geq 3$, and normally finitely generated, if $n = 2$ (see [GR17b, p. 166] or the proof of Proposition 5.1.16). In other words, the $\mathbf{Z}[\pi_1(B^\partial)]$-module $K_\sigma(F)$ is generated by the belt spheres $\{0\} \times \partial_1 D^{n+1}_+ \subset D^n \times D^{n+1}_+$ composed with the embeddings of $F(\sigma)$. Similar to before, if $K_\sigma(F) = 0$, then we have a long exact sequence

$$\cdots \to \pi_{n+1}(W\backslash C_\sigma, \partial^h W\backslash\partial^h C_\sigma) \to \pi_{n+1}(B, B^\partial) \to \pi_{n+1}(B, B^\partial, W\backslash C_\sigma) \to 0.$$

As mentioned above, the $\mathbf{Z}[\pi_1(B^\partial)]$-module $\pi_{n+1}(B, B^\partial, W\backslash C_\sigma)$ is finitely generated and thus, $\pi_{n+1}(B, B^\partial)$ is generated by $I_\sigma(F)$ along with finitely many elements coming from lifts of the leftmost map.

Killing $K_\sigma(F)$. We describe a modification on the embeddings c_i and describe its main properties. This is analogous to [GR17b, p. 166] and similar to the modification of c to satisfy (D) in the proof of Proposition 5.1.15. Let v be a vertex of σ and $F(v) = (t, c, \mathcal{L})$ and let $j \geq 0$ be such that $\mathrm{im}(c) \subset W_j$. Recall the embedding $\bar{e} : S^n \times \partial_0 D^{n+1}_+ \hookrightarrow W_{1,1}$. Denote also by $\bar{e}$ the composition of this embedding with the inclusion of a fixed copy of V_1 in $K|_{[j,j+1]}$ seen now in W_{j+1}. Choose an embedded path γ in $\partial^h W_{j+1}$ from $c_\epsilon(x, 0)$ for some point $x \in D^n_+$ and $\bar{e}(x, 0)$ disjoint from $\partial^{hv} W_j$ and C_σ, and from $\mathrm{im}(c_\epsilon)$ and $\mathrm{im}(\bar{e})$ except at times $t = \{0, 1\}$. Using a thickening of this path, we can construct an embedded connected sum of the cores of c_ϵ and $\bar{e}$ and further thicken it to an embedding $c' : D^n \times D^n \hookrightarrow \partial^h W_{j+1}$ which agrees with c_ϵ outside of a neighborhood of x. As in Proposition 5.1.15 and Proposition 5.1.16, since the Θ-structure $\bar{e}^* \ell_{\partial^h W_{j+1}}$ is standard, the structures $(c')^* \ell_{\partial^h W_{j+1}}$ and $c_\epsilon^* \ell_{\partial^h W_{j+1}}$ are homotopic. By choosing such homotopy and composing it with the path $\mathcal{L}_\epsilon$, we construct a path $\mathcal{L}'$ from $(c')^* \ell_{\partial^h W_{j+1}}$ to $\ell_{t+\epsilon}|_{D^n \times \partial_0 D^{n+1}_+}$. We remark that, once restricted to the horizontal boundary, this construction is exactly the one explained in [GR17b, p. 166]. We conclude by [GR17b, Claim 5.6] that it does not affect the fact that $\partial^h W\backslash C_\sigma \to B^\partial$ is already n-connected, so we stay in $Y_\bullet$. Denote $F_v : (I^k, \partial I^k) \to (||X_\bullet||, ||Y_\bullet||)$ the map definedd on vertices by $F_v(w) = F(w)$ for $w \neq v$ and $F_v(v) = (t + \epsilon, c', \mathcal{L}')$. For ϵ small enough, we see that this map is well-defined and homotopic to F relative to ∂I^k. The following claim establishes properties of this construction.

Claim 3. *The map F_v is homotopic to F relative to ∂I^k and satisfies the following properties:*

(i) for every simplex $\sigma = (v, \omega_1, \cdots, \omega_p) \in I^k$, there exists a surjection $K_\sigma(F) \to K_\sigma(F_v)$, sending the class of the meridian of each simplex of g to the meridian of each simplex of F_v except v and whose kernel contains the class of the meridian of v.

(ii) for every simplex $\sigma \in I^k$, we have $I_\sigma(F) \subseteq I_\sigma(F_v)$.

Proof of Claim 3. This is essentially analogous to [GR17b, Claim 5.6]. We start with (i). Let X to be the union of $\mathrm{im}(c) = C$, the thickened path γ and the V_1 used for the move above, in such a way that the image of modified embedding c' lies in X and $\partial^h X = \partial^h W \cap X$. Let also C_ω be the image of the cores of ω_i. Consider the following maps

$$K_\sigma(F') \leftarrow \ker(\pi_n(W\backslash(X \cup C_\omega), \partial^h W\backslash(\partial^h X \cup \partial^h C_\omega)) \to \pi_n(B, B^\partial)) \to K_\sigma(F).$$

We start by proving that the right map is an isomorphism. It suffices to prove that the map $\pi_n(W\backslash(X\cup C_\omega), \partial^h W\backslash(\partial^h X\cup\partial^h C_\omega)) \to \pi_n(W\backslash(C\cup C_\omega), \partial^h W\backslash(\partial^h C\cup \partial^h C_\omega))$ is an isomorphism. Observe that the following square is a pushout square of pairs

$$\begin{array}{ccc} (D^{2n}_+, \partial_0 D^{2n}_+) & \longrightarrow & (W\backslash(X\cup C_\omega), \partial^h W\backslash(\partial^h X\cup\partial^h C_\omega)) \\ \downarrow & & \downarrow \\ (V_1, W_{1,1}) & \longrightarrow & (W\backslash(C\cup C_\omega), \partial^h W\backslash(\partial^h C\cup\partial^h C_\omega)) \end{array}$$

since the right map is obtained by taking a connected sum of pairs with $(V_1, W_{1,1})$. Given a pair of path connected spaces (Y, Y'), one finds an isomorphism

$$\pi_i(Y\vee V_1, Y'\vee W_{1,1}) \cong \pi_i(Y, Y') \oplus (\pi_i(V_1, W_{1,1}) \otimes \operatorname{coker}(\mathbf{Z}[\pi_1 Y'] \to \mathbf{Z}[\pi_1 Y])),$$

for $i \leq n$ (for example, using the relative Hurewicz theorem or as a corollary of the Hilton-Milnor theorem). By taking (Y, Y') to be the pair in the upper right corner in the square above, one sees that the right vertical map induces an isomorphism in relative homotopy groups in degrees at most n. We proceed to prove that the left horizontal map above is surjective and its kernel contains the class of the meridian of v, which finishes (i). Recall that f is isotopic to the inclusion of $S^n\times\{*\}\subset S^n\times D^{n+1}$ for some $*\in\partial D^{n+1}$. Thus, V_1 is obtained from a neighborhood of f by attaching a left $(n+1)$-handle at the meridian of f. We conclude that we have a (homotopy) pushout square of pairs

$$\begin{array}{ccc} (\partial_1 D^{n+1}_+, \partial_{01} D^{n+1}_+) & \longrightarrow & (W\backslash(X\cup C_\omega), \partial^h W\backslash(\partial^h X\cup\partial^h C_\omega)) \\ \downarrow & & \downarrow \\ (D^{n+1}_+, \partial_0 D^{n+1}_+) & \longrightarrow & (W\backslash(C'\cup C_\omega), \partial^h W\backslash(\partial^h C'\cup\partial^h C_\omega)) \end{array}$$

where C' denotes the image of c' and the top map is the inclusion of the meridian sphere of f. By (II) in Section 2.3.2, the right map is strongly $(n-1)$-connected and moreover, the class of the meridian sphere of f is in the kernel of the map on relative homotopy groups. We conclude that the map

$$\pi_n(W\backslash(X\cup C_\omega), \partial^h W\backslash(\partial^h X\cup\partial^h C_\omega)) \to \pi_n(W\backslash(C'\cup C_\omega), \partial^h W\backslash(\partial^h C'\cup\partial^h C_\omega))$$

is surjective and the class of the meridian of $g'(v)$ (which is simply the meridian of f) is in the kernel. This implies (i) by compositing to $\pi_n(B, B^\partial)$.

We finish by proving (ii). Let $\psi : (D^{n+1}, \partial D^{n+1}) \to (W\backslash(C\cup C_\omega), \partial^h W\backslash(\partial^h C\cup \partial^h C_\omega))$ and suppose its image intersects the image of c', otherwise, it is automatically in $I_\sigma(F')$. It must do so inside $W_{1,1}$ (in a finite set of points) as the path λ can be isotoped away from it, since $n\geq 2$, and the image of c' is a collar of an embedding in $W_{1,1}$. We can construct a new map ψ' which agrees with ψ outside this finite set of points given by the boundary connected sum with translates of the embedding $\bar{f}$ to remove intersection points exactly as in the proof of Proposition 5.1.16 and explained in Figure 12. The only difference is the fact that the chart x is already in V_1. Moreover, for this dual sphere $\bar{f}$, since the tangential structure on V_1 is assumed to be standard, we have $[l_W \circ \psi] = [l_W \circ \psi'] \in \pi_{n+1}(B, B^\partial)$. Hence, the class induced by ψ in $I_\sigma(F)$ also lies in $I_\sigma(F')$. This proves (ii). □

Apply the construction above and Claim 3 for every interior vertex $v \in I^k$. This produces a homotopic map F' where the meridian of all vertices are trivial in $K_\sigma(F')$. Since this kernel is generated by such meridians, we have $K_\sigma(F') = 0$. By induction on the dimension of the simplices, we can obtain F' such that $K_\sigma(F') = 0$ for every simplex σ of I^k.

Adding elements to $I_\sigma(F')$. We apply exactly the same strategy as the analogous step in Proposition 5.1.16 by choosing elements $\{\psi_i\}_i$ of $\pi_{n+1}(B, B^\partial)$ that generate this module along with $I_\sigma(F')$. To replace them by other representatives which are disjoint from C_σ, apply the same move as in Proposition 5.1.16 with the small difference that the embedding j_0 is restricted to $D^{n+1}_+ \times \{0\}$ and j_1 is extended to $\frac{1}{2}D^{n+1}_+ \times D^n$ and the charts are modified in the analogous way. This move will replace both C_σ and ψ_i by connected sums with the embeddings $\bar{e}$ and $\bar{f}$ respectively, which have standard structures. Proceed exactly as in Proposition 5.1.16 using Claim 3 instead of Claim 2. This finishes the proof. □

Proposition 5.1.24. *The maps $\mathcal{K}(W_\infty, \phi, \ell)_\bullet \to \overline{\mathcal{K}}(W_\infty, \phi, \ell)_\bullet \leftarrow \overline{\mathcal{K}}^\delta(W_\infty, \phi, \ell)_\bullet$ induce an equivalence on geometric realizations.*

Proof. One proceeds exactly as [GR17b, Lemma 5.7] for the right equivalence. We focus on the leftmost map for $p = 0$. Observe that, by contracting the t-coordinate, $\mathcal{K}(W, \phi, \ell)_0$ is equivalent to (a point-set model for) the homotopy fiber of the map

$$\mathrm{Emb}_{\partial^v}(D^n \times D^{n+1}_+, W) \to \mathrm{Bun}^{\mathrm{col}}_{\partial^v}(D^n \times D^{n+1}_+, \Theta^*\gamma)$$

at the point ℓ_0, where the source is the space of embeddings of pairs c extending ϕ, taking an embedding e to $e^*\ell_W$. On the other hand, $\overline{\mathcal{K}}(W, \phi, \ell)_0$ is equivalent to the analogous space where c is an embedding instead of only an immersion with embedded core (see [GR17b, Lemma 5.3]). This space is equivalent to (a point-set model for) the homotopy fiber of the analogous map

$$\mathrm{Emb}_\partial(D^n \times \partial_0 D^{n+1}_+, \partial^h W) \to \mathrm{Bun}_\partial(D^n \times \partial_0 D^{n+1}_+, (\theta^\partial)^*\gamma_{2n})$$

at ℓ_0. Moreover, the map above in $p = 0$ is simply the induced map on the homotopy fiber induced by the restriction maps between the embedding and bundle spaces. We claim that both maps are equivalences: between embedding spaces one uses the parameterized isotopy extension theorem to deduce that it is a Serre fibration and its fiber identifies with the space of embeddings of $D^n \times D^{n+1}_+$ fixed in $D^n \times \partial_0 D^{n+1}_+$, which is contractible; and between bundle spaces one uses Proposition A.5 to deduce similarly that its fiber is contractible. Hence the map on homotopy fibers is an equivalence, which finishes the proof for $p = 0$. The higher cases are completely analogous. □

Proof of Theorem 5.1.19. Combine Lemma 5.1.22, Proposition 5.1.23 and Proposition 5.1.24. □

5.1.4. Resolutions are degree-wise cuts. We establish now *(iii)* from the introduction, which involves specifying the data of ϕ and ℓ from the results above. Let $W : M \rightsquigarrow N$ be a morphism in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$, σ an attaching map and ℓ_t be a family of Θ-structure on $\mathsf{tr}(\phi)$ indexed by $t \in (0,\infty)$ satisfying the hypothesis of Theorem 4.3.16. For the remaining of this subsection, we fix $\phi \coloneqq \sigma|_{\partial D^n \times (0,\infty)\times \mathbf{R}^n_+}$ when σ is of right type and $\phi \coloneqq \sigma|_{\partial_1 D^{n+1}_+ \times (0,\infty)\times \mathbf{R}^{n-1}}$ when σ is of left type. By assumption, the image of ϕ lands away from the support of W, thus we see this map both in N and in M (as $N\backslash \operatorname{supp}(W) = M\backslash \operatorname{supp}(W)$). We define ℓ to be the restriction of the family ℓ_t to $D^n \times D^{n+1}_+$ for the right type and to $D^{n+1}_+ \times D^n$ for the left type. Our first goal is to identify the space of p-simplices of $\mathcal{K}(W,\phi,\ell)$ in terms of the iterated surgeries $\chi^{p+1}(W,\sigma,\ell)$ (recall the definition from Definition 4.3.14).

Recall that we have the subspaces core_σ and cocore_σ of $\mathsf{tr}(\sigma)$, whose parameterization we fix. Given the multiple surgery $\chi^p(W,\sigma,\ell)$, one denotes by $\mathsf{core}_{\sigma,i}$ to be the union of $\mathsf{core}_{\sigma_i} \subset \mathsf{tr}(\sigma_i)$ with $[0,i-1]\times \operatorname{im}(\sigma_i) \subset \cup_{j<i}\mathsf{tr}(\sigma_j) = \mathsf{tr}(\sigma^{i-1})$. This is possible since σ_i is disjoint from the traces of all $\mathsf{tr}(\sigma_j)$ for $j < i$.

Notation 5.1.25. Let $\mathcal{L}_i$ be the constant path in ℓ_{3i}. Given $(s,X) \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(\chi^p(N,\sigma,\ell),K|_i)$, we claim that the tuple

$$(s+p+1, X \circ \mathsf{tr}(\sigma^{p+1}); (3i; \mathsf{core}_{\sigma,i}, \mathcal{L}_i)_{i=1,\cdots,p+1}) \in \mathcal{K}_{\phi,\ell}(N,K|_i)_p.$$

where $\mathsf{tr}(\sigma^{p+1})$ is the union of all $\mathsf{tr}(\sigma_i)$ for $i \leq p+1$. Since $\mathsf{tr}(\sigma^p)$ is isotopy equivalent to the union of all cores, it follows that the complement of such cores is isotopy equivalent to X. Thus, by assumption, it has strongly n-connected Θ-structure. This establishes property *(iv)* in the definition above. The remaining properties hold by construction. This defines a map

$$(-)\circ \sigma^{p+1} : \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(\chi^{p+1}(N,\sigma,\ell),K|_i) \to \mathcal{K}_{\phi,\ell}(N,K|_i)_p.$$

Similarly, we have the analogous map replacing N by M and using the cores of $\mathcal{L}_W(\mathsf{tr}(\sigma^{p+1}))$ coming from the cores of $\mathsf{tr}(\sigma^{p+1})$. We denote this map also by $(-)\circ\sigma^{p+1}$.

Lemma 5.1.26. *The following square*

$$\begin{array}{ccc} \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(\chi^{p+1}(N,\sigma,\ell),K|_i) & \xrightarrow{(-)\circ\sigma^{p+1}} & \mathcal{K}_{\phi,\ell}(N,K|_i)_p \\ \big\downarrow{\scriptstyle (-)\circ\chi^{p+1}(W,\sigma,\ell)} & & \big\downarrow{\scriptstyle (-)\circ W} \\ \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(\chi^{p+1}(M,\sigma,\ell),K|_i) & \xrightarrow{(-)\circ\sigma^{p+1}} & \mathcal{K}_{\phi,\ell}(M,K|_i)_p \end{array}$$

is homotopy commutative.

Proof. If we post-compose the horizontal maps with the projection maps $\mathcal{K}_{\phi,\ell}(-,Q) \to \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(-,Q)$, then we have the square

$$\begin{array}{ccc} \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(\chi^{p}(N,\sigma,\ell),K|_i) & \xrightarrow{(-)\circ\mathsf{tr}(\sigma^{p+1})} & \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(N,K|_i)_p \\ \big\downarrow{\scriptstyle (-)\circ\chi^{p+1}(W,\sigma,\ell)} & & \big\downarrow{\scriptstyle (-)\circ W} \\ \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(\chi^{p}(M,\sigma,\ell),K|_i) & \xrightarrow{(-)\circ\mathcal{L}_W(\mathsf{tr}(\sigma^{p+1}))} & \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(M,K|_i)_p \end{array}$$

which is homotopy commutative, since we have a diffeomorphism $W \circ \mathsf{tr}(\sigma^{p+1}) \cong \mathcal{L}_W(\mathsf{tr}(\sigma^{p+1})) \circ \chi^{p+1}(W,\sigma,\ell)$, by Remark 4.3.3. To verify that the original square commutes, it suffices to observe that the diffeomorphism can be chosen so it takes the cores of the bottom map to the ones defined using the top map. This is done exactly as [GR17b, p. 157]. ◻

Proposition 5.1.27. *The map*

$$(-) \circ \sigma^{p+1} : \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(\chi^{p+1}(N,\sigma,\ell),K|_i) \to \mathcal{K}_{\phi,\ell}(N,K|_i)_p$$

is an equivalence, for every $p \geq 0$. The analogous result for M instead of N holds too.

Proof. We follow the strategy of [GR17b, p. 156]. Let E be the subspace of

$$\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(N,K|_i) \times \mathrm{Emb}_N(\mathsf{tr}(\sigma^{p+1}),[0,\infty)\times\mathbf{R}^{\infty}_{+}) \times \mathrm{Bun}^{\mathrm{col}}_{\partial^v}(\mathsf{tr}(\sigma^{p+1}),\Theta^*\gamma)^{[0,1]}$$

where the second factor is the space of embeddings of pairs extending the inclusion of N, consisting of those triples $((s,W'),e,\mathcal{L})$ where the image of e lies in W', the complement has strongly n-connected Θ-structure, $\mathcal{L}(0) = e^*\ell_{W'}$ and $\mathcal{L}(1) = l_{\mathsf{tr}(\sigma^{p+1})}$. Taking such a triple to $((s,W'),(3i,e|_{\mathsf{core}_{\sigma,i}},\mathcal{L}|_{\mathsf{core}_{\sigma,i}})_{i=0,\cdots,p})$ provides a map $E \to \mathcal{K}_{\phi,\ell}(N,K|_i)_p$. This map is an equivalence since $\mathsf{tr}(\sigma^{p+1})$ is isotopy equivalent to the union of all its cores and (a collar on) N. Consider the projection

$$E \to \mathrm{Emb}_N(\mathsf{tr}(\sigma^{p+1}),[0,\infty)\times\mathbf{R}^{\infty}_{+}).$$

We claim that this projection is a fibration by the parameterized isotopy extension theorem (see [Ste21, Appendix A]). We sketch a proof here for completeness: if we restrict to the path components of E of those triples where the first coordinate is a fixed cobordism $W' : N \rightsquigarrow K|_i$. Ignoring the third coordinate, we describe E alternatively by the space of pairs $(e_0 : \mathsf{tr}(\sigma^{p+1}) \hookrightarrow W', e_1 : W' \hookrightarrow [0,\infty) \times \mathbf{R}^\infty_+)$ such that both e_0 and e_1 extend (in a collared way) the embedding of N in $\{0\} \times \mathbf{R}^\infty_+$ and additionally e_1 extends a (translated) embedding of $K|_i$ in $\{0\} \times \mathbf{R}^\infty_+$ (to guarantee that it is disjoint from $K|_i$) balanced by the action of $\mathrm{Diff}_{\partial^v}(W')$. The projection above can now be seen as composition of e_0 and e_1. This map is equivariant with respect to the action of $\mathrm{Diff}_{\{0\}\times\mathbf{R}^\infty_+}([0,\infty) \times \mathbf{R}^\infty_+)$. Moreover, by [Ste21, Appendix A], the target admits local cross sections and thus, by [Pal60, Thm. A], any equivariant map to it is a (locally trivial) Serre fibration.

The base of this fibration is contractible by [Gen11, Thm. 2.7]. The fiber over the standard inclusion of $\mathsf{tr}(\sigma^{p+1})$ is the equivalent to the subspace of $(t, W) \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(N, K|_i)$ such that $\mathsf{tr}(\sigma^{p+1}) \subset W'$, which is homeomorphic to $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(\chi^p(N,\sigma,\ell), K|_i)$. The map $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(\chi^p(N,\sigma,\ell), K|_i) \to E$ is an equivalence and therefore, we have factored the map $(-) \circ \sigma^{p+1}$ into two equivalences. This finishes the proof. □

From Lemma 5.1.26 and Proposition 5.1.27, we see that if $\chi^{p+1}(W,\sigma,\ell) \in \mathcal{W}$, then the induced map on resolutions is an abelian homology equivalence on p-simplices. This establishes *(iii)* of our strategy.

5.1.5. Proof of the closure property. This section is dedicated to the proof of Theorem 4.3.16. Let $W : M \rightsquigarrow N$ be a morphism in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$, σ an attaching map satisfying the hypothesis and ℓ_t be a family of Θ-structure on $\mathsf{tr}(\sigma)$ indexed by $t \in (0,\infty)$. We assume that $\chi^p(W,\sigma,\ell) \in \mathcal{W}$ for every $p \geq 1$. We wish to prove that $W \in \mathcal{W}$. Recall the definition of ϕ from the previous subsubsection. Consider the square

$$\begin{array}{ccc} ||\mathcal{K}_{\phi,\ell}(N, K|_\infty)_\bullet|| & \longrightarrow & ||\mathcal{K}_{\phi,\ell}(M, K|_\infty)_\bullet|| \\ \downarrow & & \downarrow \\ \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(N, K|_\infty) & \longrightarrow & \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(M, K|_\infty) \end{array}$$

obtained by Lemma 5.1.9. By Theorem 5.1.10 and Theorem 5.1.19, the vertical maps are weak equivalences, so it suffices to prove that the top map is an abelian homology equivalence. For $p \geq 0$, consider now the square

$$\begin{array}{ccc} \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(\chi^{p+1}(N,\sigma,\ell), K|_\infty) & \xrightarrow{(-)\circ\sigma^{p+1}} & \mathcal{K}_{\phi,\ell}(N, K|_\infty)_p \\ \downarrow{\scriptstyle (-)\circ\chi^{p+1}(W,\sigma,\ell)} & & \downarrow{\scriptstyle (-)\circ W} \\ \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(\chi^{p+1}(M,\sigma,\ell), K|_\infty) & \xrightarrow{(-)\circ\sigma^{p+1}} & \mathcal{K}_{\phi,\ell}(M, K|_\infty)_p \end{array}$$

which commutes by Lemma 5.1.26 and whose horizontal maps are equivalences by Proposition 5.1.27. The leftmost vertical map is an abelian homology equivalence by hypothesis, thus so is the right map. Thus the top map in the first square is a degree-wise abelian homology equivalence of semi-simplicial spaces. Thus, so is the map after taking geometric realizations (see [GR17b, p. 158] for an argument). This finishes the proof of Theorem 4.3.16.

5.2. The higher dimensional case.

The main goal of this subsection is to prove Theorem 4.4.4. We proceed similarly to the previous section. Recall the definition of μ_t from Section 4.4. We start by defining augmented semi-simplicial spaces over $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P,Q)$ for right, left and interior type.

Definition 5.2.1 (Right resolution)**.** Let $n < k < 2n$ be an integer, $P, Q \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ and $(s, W) \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P,Q)$. Let

$$\chi : (\partial D^k \times \mathbf{R}^{2n+1-k}_+ \backslash D^{2n+1-k}_+, \partial D^k \times \partial \mathbf{R}^{2n+1-k}_+ \backslash \partial_0 D^{2n+1-k}_+) \hookrightarrow (P, \partial^h P)$$

be an embedding and ℓ a Θ-structure on $[0,2]\times\partial D^k\times\mathbf{R}^{2n+1-k}_+$ such that $\ell|_{\{0\}\times\partial D^k\times\mathbf{R}^{2n+1-k}_+\backslash D^{2n+1-k}_+} = \chi^*\ell_P$. Let $\phi^\chi_t = \chi \circ \mu_t$ for every $t \in (2,\infty)$. Define $\mathcal{K}(W,\chi,\ell)_0$ to be the space of triples $(t, c, \mathcal{L})$ where $t \in (2,\infty)$, $c : (D^{2n+1-k}_+ \times D^k, \partial_0 D^{2n+1-k}_+ \times D^k) \hookrightarrow (W, \partial^h W)$ an embedding of pairs and $\mathcal{L} : [0,1] \to \mathrm{Bun}^{\mathrm{col}}_{\partial^v}(T(D^{2n+1-k}_+ \times D^k), \Theta^*\gamma_{2n+1})$ is a path of structures fixed as $\mu_t^*\ell$ on $\partial_1 D^{2n+1-k}_+ \times D^k$ satisfying the following properties:

- (i) *(Collared expansion of meridian of χ by t near P)* For some $\delta > 0$, we have $c(x,v) = \phi^\chi_t(\frac{x}{|x|}, v) + (1-|x|)\cdot e_0$ where e_0 is the first coordinate vector of $\mathbf{R}\times\mathbf{R}^\infty_+$ for all x such that $1-|x| < \delta$.
- (ii) The image $(C, \partial^h C) = c(D^{2n+1-k}_+ \times D^k, \partial_0 D^{2n+1-k}_+ \times D^k)$ is disjoint from $([0,s]\times\partial^v L) \cup (\{s\}\times Q)$ and $c^{-1}(P) = \partial_1 D^{2n+1-k}_+ \times D^k$.
- (iii) $\mathcal{L}(0) = c^*\ell_W$ and $\mathcal{L}(1) = \mu_t^*\ell$.

This space is topologized in the same way as the resolutions from the previous subsection. Define $\mathcal{K}(W,\chi,\ell)_p$ to be the subspace of $\mathcal{K}(W,\chi,\ell)_0^{\times(p+1)}$ consisting of those tuples $(t_i; c_i; \mathcal{L}_i)_{i=0,\cdots,p}$ such that:

- (i) $t_0 < t_1 < \cdots < t_p$.
- (ii) The embeddings c_i are pairwise disjoint.

This defines a semi-simplicial space $\mathcal{K}(W,\chi,\ell)_\bullet$ in the same way as in the last subsection.

Definition 5.2.2 (Left resolution)**.** Let $n+1 < k < 2n+1$ be an integer, $P, Q \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ and $(s, W) \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P,Q)$. Let

$$\chi : (\partial_1 D^k_+ \times \mathbf{R}^{2n+1-k} \backslash D^{2n+1-k}, \partial_{01} D^k_+ \times \mathbf{R}^{2n+1-k} \backslash D^{2n+1-k}) \hookrightarrow (P, \partial^h P)$$

be an embedding and ℓ a Θ-structure on $[0,2]\times\partial_1 D^k_+\times\mathbf{R}^{2n+1-k}$ such that $\ell|_{\{0\}\times\partial_1 D^k_+\times\mathbf{R}^{2n+1-k}\backslash D^{2n+1-k}} = \chi^*\ell_P$. Let $\phi^\chi_t = \chi \circ \mu_t$ for every $t \in (2,\infty)$. Define $\mathcal{K}(W,\chi,\ell)_0$ to be the space of triples

$(t, c, \mathcal{L})$ where $t \in (2, \infty)$, $c : (D^{2n+1-k} \times D^k_+, D^{2n+1-k} \times \partial_0 D^k_+) \hookrightarrow (W, \partial^h W)$ an embedding of pairs and $\mathcal{L} : [0,1] \to \mathrm{Bun}^{\mathrm{col}}_{\partial^v}(T(D^{2n+1-k} \times D^k_+), \Theta^*\gamma_{2n+1})$ is a path of structures fixed as $\mu_t^* \ell$ on $\partial D^{2n+1-k} \times D^k_+$ satisfying the following properties:

(i) *(Collared expansion of meridian of χ by t near P)* For some $\delta > 0$, we have $c(x, v) = \phi_t^\chi(\frac{x}{|x|}, v) + (1 - |x|) \cdot e_0$ where e_0 is the first coordinate vector of $\mathbf{R} \times \mathbf{R}^\infty_+$ for all x such that $1 - |x| < \delta$.

(ii) The image $(C, \partial^h C) = c(D^{2n+1-k} \times D^k_+, D^{2n+1-k} \times \partial_0 D^k_+)$ is disjoint from $([0, s] \times \partial^v L) \cup (\{s\} \times Q)$ and $c^{-1}(P) = \partial D^{2n+1-k} \times D^k$.

(iii) $\mathcal{L}(0) = c^* \ell_W$ and $\mathcal{L}(1) = \mu_t^* \ell$.

This space is topologized in the same way as the resolutions from the previous subsection. Define $\mathcal{K}(W, \chi, \ell)_p$ to be the subspace of $\mathcal{K}(W, \chi, \ell)_0^{\times(p+1)}$ consisting of those tuples $(t_i; c_i; \mathcal{L}_i)_{i=0,\cdots,p}$ such that:

(i) $t_0 < t_1 < \cdots < t_p$.

(ii) The embeddings c_i are pairwise disjoint.

This defines a semi-simplicial space $\mathcal{K}(W, \chi, \ell)_\bullet$.

Definition 5.2.3 (Interior resolution)**.** Let $n + 1 < k < 2n + 1$ be an integer, $P, Q \in \mathsf{Cob}^\partial_{\Theta, \partial^v L}$ and $(s, W) \in \mathsf{Cob}^\partial_{\Theta, \partial^v L, n}(P, Q)$. Let

$$\chi : \partial D^k \times \mathbf{R}^{2n+1-k} \backslash D^{2n+1-k} \hookrightarrow P \backslash \partial^h P$$

be an embedding and ℓ is a Θ-structure on $[0, 2] \times \partial D^k \times \mathbf{R}^{2n+1-k}$ such that $\ell|_{\{0\} \times \partial D^k \times \mathbf{R} \backslash D^{2n+1-k}} = \chi^* \ell_P$. Let $\phi_t^\chi = \chi \circ \mu_t$ for every $t \in (2, \infty)$. Define $\mathcal{K}(W, \chi, \ell)_0$ to be the space of triples $(t, c, \mathcal{L})$ where $t \in (2, \infty)$, $c : D^{2n+1-k} \times D^k \hookrightarrow W \backslash \partial^h W$ an embedding of pairs and $\mathcal{L} : [0, 1] \to \mathrm{Bun}^{\mathrm{col}}_{\partial^v}(T(D^{2n+1-k} \times D^k), \Theta^* \gamma_{2n+1})$ is a path of structures fixed as $\mu_t^* \ell$ on $\partial D^{2n+1-k} \times D^k$, where $\mathrm{Bun}_\partial(T(D^{2n+1-k} \times D^k), \theta^* \gamma_{2n+1})$ satisfying the following properties:

(i) *(Collared translation of the meridian of χ by t near P)* For some $\delta > 0$, we have $c(x, v) = \phi_t^\chi(\frac{x}{|x|}, v) + (1 - |x|) \cdot e_0$ where e_0 is the first coordinate vector of $\mathbf{R} \times \mathbf{R}^\infty_+$ for all x such that $1 - |x| < \delta$.

(ii) The image $C = c(D^{2n+1-k} \times D^k)$ is disjoint from $([0, s] \times \partial^v L) \cup (\{s\} \times Q)$ and $c^{-1}(P) = \partial D^{2n+1-k} \times D^k$.

(iii) $\mathcal{L}(0) = c^* \ell_W$ and $\mathcal{L}(1) = \ell_t$.

This space is topologized in the same way as the resolutions from the previous subsection. Define $\mathcal{K}(W, \chi, \ell)_p$ to be the subspace of $\mathcal{K}(W, \chi, \ell)_0^{\times(p+1)}$ consisting of those tuples $(t_i; c_i; \mathcal{L}_i)_{i=0,\cdots,p}$ such that:

(i) $t_0 < t_1 < \cdots < t_p$.

(ii) The embeddings c_i are pairwise disjoint.

This defines a semi-simplicial space $\mathcal{K}(W, \chi, \ell)_\bullet$.

Definition 5.2.4. Given $P, Q \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ and χ and ℓ as in one of the above definitions. Define $\mathcal{K}_{\chi,\ell}(P,Q)_p$ to be the space of pairs (W,x) where $W \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P,Q)$ and $x \in \mathcal{K}(W,\chi,\ell)_p$. This space maps to $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P,Q)$ by forgetting x for every $p \geq 0$.

5.2.1. Functoriality of the resolutions. We continue by remarking "functoriality" properties of these resolutions with respect to pre- and post-composition. This will play the same role as the analogous step for the middle dimensional case.

Lemma 5.2.5. *Let $W' : Q \rightsquigarrow S$ be a morphism in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ which is strongly $(n-1)$-connected relative to Q, then post-composition defines a map of augmented semi-simplicial spaces*

$$W' \circ (-) : \mathcal{K}_{\chi,\ell}(P,Q)_\bullet \to \mathcal{K}_{\chi,\ell}(P,S)_\bullet.$$

Proof. By the same argument as in Lemma 4.1.10, post-composition defines a map $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P,Q) \to \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P,S)$. Since the embeddings in $\mathcal{K}_{\chi,\ell}(P,Q)$ are disjoint from Q, post composition with inclusion in W' induces data in $\mathcal{K}_{\chi,\ell}(P,S)$. The verification of the conditions above follows by unravelling definitions. □

Construction 5.2.6. We follow exactly the strategy of Construction 5.1.8. Fix a diffeomorphism $\epsilon_s : (D^k \times D^{2n+1-k}_+, D^k \times \partial D^{2n+1-k}_+) \cong ((s\cdot e_1 + D^k) \times D^{2n+1-k}_+, (s\cdot e_1 + D^k) \times \partial_0 D^{2n+1-k}_+) \cup [0,s] \times (\partial D^k \times D^{2n+1-k}_+, \partial D^k \times \partial_0 D^{2n+1-k}_+) =: B_s \cup A_s = D_s$. Given $(t, c, \mathcal{L}) \in \mathcal{K}(W,\chi,\ell)$ for χ of left type and $(s, W') : P' \rightsquigarrow P$ such that $\mathrm{supp}(W) \cap \chi = \emptyset$, define $c_s : D_s \hookrightarrow W' \cup W$ by taking B_s to c and A_s to $[0,s] \times c|_{\partial D^k \times D^{2n+1-k}_+}$. The triple $(t, \epsilon_s^{-1}(c_s), \epsilon_s^{-1}(\mathcal{L}_s))$, where $\mathcal{L}_s$ is given by extending the path of Θ-structure $\mathcal{L}$ along the collar A_t, satisfies properties $(i)-(iii)$. When χ is of right type, one proceeds in a similar way by adding now a collar $[0,s] \times (\partial_1 D^k_+ \times D^{2n+1-k}, \partial_{01} D^k_+ \times D^{2n+1-k})$ to $(D^k_+ \times D^{2n+1-k}, \partial_0 D^k_+ \times D^{2n+1-k})$. For interior type, one proceed analogously. (For more detail, see [GR17b, Defn. 4.13].)

Lemma 5.2.7. *Let $(s, W') : P' \rightsquigarrow P$ be a morphism in $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ such that* $\mathrm{supp}(W) \cap \mathrm{im}(\chi) = \emptyset$*, then pre-composition defined above*

$$(-) \circ W' : \mathcal{K}_{\chi,\ell}(P,Q)_\bullet \to \mathcal{K}_{\chi,\ell}(P',Q)_\bullet$$

is a map of augmented semi-simplicial spaces. Moreover, for $W'' : Q \rightsquigarrow S$, then the pre and post-composition maps commute strictly in the natural way.

Proof. This follows exactly as Lemma 5.2.5. □

5.2.2. Contractability of core complexes for higher handles. We establish the analog of Theorem 5.1.10 and Theorem 5.1.19 for higher handles. Notice that the statement has no mention of a Θ-end, since the augmented semi-simplicial spaces are already resolutions before statibilizing with the Θ-end. The analogous phenomenon was observed in [GR17b, Theorem 6.6].

Proposition 5.2.8. *Let $P, Q \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$, χ and ℓ be as in Definition 5.2.4. Then $||\mathcal{K}_{\chi,\ell}(P,Q)_\bullet|| \to \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P,Q)$ is an equivalence, provided (B, B^∂) is 0-connected.*

Proof. By the same argument as in Lemma 5.1.6, it suffices to prove that $\mathcal{K}(W,\chi,\ell)_\bullet$ has contractible realisation for every $W \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P,Q)$. We can replace $\mathcal{K}(W,\chi,\ell)_\bullet$ by $\overline{\mathcal{K}}(W,\chi,\ell)$ of 0-simplices $(t,c,\mathcal{L})$ where c is only assumed to be an immersion whose restriction to its core is an embedding and higher simplices $(t_i, c_i, \mathcal{L}_i)_i$ if the cores of c_i are pairwise disjoint. This is analogous to Definition 5.1.11. By the same argument of Proposition 5.1.18, it follows that the map $||\mathcal{K}(W,\chi,\ell)_\bullet|| \to ||\overline{\mathcal{K}}(W,\chi,\ell)_\bullet||$ is an equivalence. One verifies that $\overline{\mathcal{K}}(W,\chi,\ell)$ produces a topological flag complex (augmented over a point) as in Proposition 3.2.5. (See the proof of [GR17b, Thm. 6.6] for a similar argument.) Thus, it suffices to verify the conditions $(i)-(iii)$ of [GR14, Thm. 6.2] for this augmented flag complex. Condition (i) is vacuous because X_{-1} is a point. Condition (ii) is equivalent to $\mathcal{K}(W,\chi,\ell)_0$ being non-empty. We separate by cases and start with the case of χ being of right type. Consider the following diagram

$$\begin{array}{ccccc}
(\partial_1 D^{2n+1-k}_+ \times D^k, \partial_{01} D^{2n+1-k}_+ \times D^k) & \xrightarrow{\phi_3^\chi} & (P, \partial^h P) & \hookrightarrow & (W, \partial^h W) \\
\big\downarrow & & \overset{\hat{c}}{\dashrightarrow} & & \big\downarrow \ell_W \\
(D^{2n+1-k}_+ \times D^k, \partial_0 D^{2n+1-k}_+ \times D^k) & & \xrightarrow{\mu_3^*\ell} & & (B, B^\partial)
\end{array}$$

which admits a lift making the top triangle commute strictly and the lower one up to homotopy, since ℓ_W is strongly n-connected and $k > n$ by Lemma 2.1.8, since (B, B^∂) and thus $(W, \partial^h W)$ are 0-connected. By Smale-Hirsch theory as in Lemma 5.1.14, we may assume that $\hat{c}$ is an immersion of pairs, as $\mu_3^*\ell$ is covered by a bundle map. Moreover, since $2(2n+1-k) < 2n+1$, this immersion can be assumed to be an embedding once restricted to $D^{2n+1-k}_+ \times \{0\}$. By scaling the tubular neighborhood, we can homotope this immersion to be an embedding c. Choose $\mathcal{L}$ to be the path of Θ-structures given by the homotopy of the bottom triangle above, and thus $(3, c, \mathcal{L}) \in \mathcal{K}(W,\chi,\ell)_0$. The right and interior types follow exactly the same strategy, where the appropriate lifting problem will have a solution by the connectivity of ℓ_W and the restriction on k. We finish with condition (iii). Given a p-simplex $v = \{(t_i, c_i, \mathcal{L}_i)\}_{i=0,\cdots,p}$ we have to find $(t, c, \mathcal{L})$ such that $\{(t,c,\ell)\} \cup v$ is a $(p+1)$-simplex. Take $t = t_0 - \epsilon$ for all i, consider c the embedding c_0 composed with an expansion on its vertical boundary (in P) so it lies in ϕ_t^χ. This is done in the same way as c_ϵ in the proof of Proposition 5.1.15. We can isotope c relative to its vertical boundary such that it is disjoint to all c_i by transversality [Gen11, Lemma 9.2], since the handle dimension of the cores is strictly smaller than half of the total dimension. Taking $\mathcal{L}$ to be $\mathcal{L}_0$ composed with the expansion in P. We see that $\{(t, c, \ell)\} \cup v$ is a $(p+1)$-simplex. □

5.2.3. Resolutions are degree-wise cuts. As in the middle dimensional case, we proceed now by specifying our choice of χ and ℓ. Let $W : M \rightsquigarrow N$ be a morphism in $\mathsf{Cob}_{\Theta,\partial^v L}^{\partial,n-1}$, σ an attaching map of index k and ℓ be a Θ-structure satisfying the hypothesis of Theorem 4.4.4. For the remaining of this subsection, we set $\chi \coloneqq \sigma|_{\partial D^k \times \mathbb{R}_+^{2n+1-k} \setminus D_+^{2n+1-k}}$ if σ is of right type, $\chi \coloneqq \sigma|_{\partial_1 D_+^k \times \mathbb{R}^{2n+1-k} \setminus D^{2n+1-k}}$ for the left type and $\chi \coloneqq \sigma|_{\partial D^k \times \mathbb{R}^{2n+1-k} \setminus D^{2n+1-k}}$ for the interior type. Once again, we see χ both in N and in M.

Recall the definition of ϕ^σ from Definition 4.4.1. We proceed exactly as in Section 5.1.4 to define a morphism $(p+1, \mathsf{tr}(\phi^{\sigma,p+1})) : P \rightsquigarrow \chi^{p+1}(P, \phi^\sigma, \ell)$ given by the union of the traces $\mathsf{tr}(\phi_{3i}^\sigma)$ for $1 \leq i \leq p+1$, whose Θ-structure is induced by ℓ. Inside this cobordism, we have embeddings $\mathsf{core}_{\phi^\sigma,i}$ for $i = 1, \cdots, p+1$. We define a map

$$(-) \circ \phi^{\sigma,p+1} : \mathsf{Cob}_{\Theta,\partial^v L,n}^{\partial}(\chi^{p+1}(N, \sigma, \ell), K|_i) \to \mathcal{K}_{\chi,\ell}(N, K|_i)_p$$

by taking (s, X) to $(s + p + 1, X \circ \mathsf{tr}(\phi^{\sigma,p+1}); (3i; \mathsf{core}_{\phi^\sigma,i}; \mathcal{L}_i)_{i=1,\cdots,p+1})$ where $\mathcal{L}_i$ is the constant path at $\mu_{3i}^* \ell$. Similarly, we have the analogous map replacing N by M and using the cores of $\mathcal{L}_W(\mathsf{tr}(\phi^{\sigma,p+1}))$ coming from the cores of $\mathsf{tr}(\phi^{\sigma,p+1})$. We denote this map also by $(-) \circ \phi^{\sigma,p+1}$.

Lemma 5.2.9. *The following square*

$$\begin{array}{ccc} \mathsf{Cob}_{\Theta,\partial^v L,n}^{\partial}(\chi^{p+1}(N, \sigma, \ell), K|_i) & \xrightarrow{(-)\circ\phi^{\sigma,p+1}} & \mathcal{K}_{\chi,\ell}(N, K|_i)_p \\ \big\downarrow{\scriptstyle (-)\circ\chi^{p+1}(W,\sigma,\ell)} & & \big\downarrow{\scriptstyle (-)\circ W} \\ \mathsf{Cob}_{\Theta,\partial^v L,n}^{\partial}(\chi^{p+1}(M, \sigma, \ell), K|_i) & \xrightarrow{(-)\circ\phi^{\sigma,p+1}} & \mathcal{K}_{\chi,\ell}(M, K|_i)_p \end{array}$$

is homotopy commutative.

Proof. This follows verbatim with the strategy of Lemma 5.1.26. □

Proposition 5.2.10. *The map*

$$(-) \circ \phi^{\sigma,p+1} : \mathsf{Cob}_{\Theta,\partial^v L,n}^{\partial}(\chi^{p+1}(N, \sigma, \ell), K|_i) \to \mathcal{K}_{\chi,\ell}(N, K|_i)_p$$

is an equivalence, for every $p \geq 0$. The analogous claim for M also holds.

Proof. This follow verbatim with the strategy of Proposition 5.1.27. □

5.2.4. Proof of the closure property. We finish by proving Theorem 4.4.4. Let $W : M \rightsquigarrow N$ be a morphism in $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$, σ an attaching map of index k and ℓ be a Θ-structure satisfying the hypothesis of Theorem 4.4.4. Assume $\chi^p(W,\phi^\sigma,\ell) \in \mathcal{W}$ for every $p \geq 1$. Recall the definition of χ from the previous subsubsection. Consider the square

$$\begin{array}{ccc} ||\mathcal{K}_{\chi,\ell}(N,K|_\infty)_\bullet|| & \longrightarrow & ||\mathcal{K}_{\chi,\ell}(M,K|_\infty)_\bullet|| \\ \downarrow & & \downarrow \\ \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(N,K|_\infty) & \longrightarrow & \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(M,K|_\infty) \end{array}$$

obtained by Lemma 5.1.9. By Proposition 5.2.8, the vertical maps are equivalences. We proceed exactly as in Section 5.1.5 to deduce that the upper map is an abelian homology equivalence by using Proposition 5.2.10 instead of Proposition 5.1.27. This finishes the proof. The proofs for the left and interior cases are analogous.

6. Stable homology and group completion.

In this section, we use the main results of the last sections, namely Theorems 3.1.10 and 4.1.11, to prove a generalization of Theorem B. This will be crucial to deduce Theorem A and more generally, a version with tangential structures in the next section. We start by stating the main result of this section. To do so, consider the following definitions. Throughout this section, we fix again the data in Assumptions 4.1.1. Although our main result of this section will not feature the L, its proof requires us to choose a specific L and thus establish preliminary results where a general L is present. We specialize to our desired L by the end of this section. In Definition 4.1.9, we introduced the notion of a Θ-end in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$, but we now consider a variation which does not use the L.

Definition 6.0.1. A Θ-*end* K in $\mathsf{Cob}^{\partial}_{\Theta}$ is a sequence of composable morphisms

$$\{K|_{[i,i+1]} : K|_i \rightsquigarrow K|_{i+1}\}_{i\geq 0}$$

in $\mathsf{Cob}^{\partial}_{\Theta}$ such that:

(i) for every $i \geq 0$, the inclusions $(K|_i, \partial K|_i) \hookrightarrow (K|_{[i,i+1]}, \partial^h K|_{[i,i+1]})$ and $(K|_{i+1}, \partial K|_{i+1}) \hookrightarrow (K|_{[i,i+1]}, \partial^h K|_{[i,i+1]})$ are strongly $(n-1)$-connected,

(ii) for every $i \geq 0$, there exists an embedding $\omega : (V_1, W_{1,1}) \hookrightarrow (K|_{[i,i+1]}, \partial^h K|_{[i,i+1]})$ such that $\omega^*\ell_{K_{[i,i+1]}}$ is a standard Θ-structure in the sense of Definition 4.1.4.

For a pair of objects $P,Q \in \mathsf{Cob}^{\partial}_{\Theta}$, consider the map $\mathsf{Cob}^{\partial}_{\Theta}(P,Q) \to \Omega_{[P,Q]}\mathsf{BCob}^{\partial}_{\Theta}$ that takes a morphism to the path in $\mathsf{BCob}^{\partial}_{\Theta}$ represented by this morphism. For an object $P \in \mathsf{Cob}^{\partial}_{\Theta}$, let $\mathcal{N}^{\partial}_{\Theta}(P)$ denote $\mathsf{Cob}^{\partial}_{\Theta}(\emptyset, P)$ and let $\mathcal{N}^{\partial}_{\Theta,n}(P)$ be the subspace of $\mathcal{N}^{\partial}_{\Theta}(P)$ consisting of those morphisms (W,ℓ_W) such that ℓ_W :

$(W, \partial^h W) \to (B, B^\partial)$ is strongly n-connected. Given a Θ-end K in $\mathsf{Cob}^\partial_\Theta$, denote by $\mathscr{N}^\partial_{\Theta,n}(K|_\infty)$ the homotopy colimit over i of the maps $K|_{[i,i+1]} \circ (-) : \mathscr{N}^\partial_{\Theta,n}(K|_i) \to \mathscr{N}^\partial_{\Theta,n}(K|_{i+1})$ (which are well defined by the first assumption in Definition 6.0.1, see Lemma 4.1.10.).

We have a map $\mathscr{N}^\partial_{\Theta,n}(K|_\infty) \to \Omega_{[\emptyset, K|_\infty]}\mathrm{B}\mathsf{Cob}^\partial_\Theta$ where the target denotes the homotopy colimit of the maps $\Omega_{[\emptyset,K|_i]}\mathrm{B}\mathsf{Cob}^\partial_\Theta \to \Omega_{[\emptyset,K|_{i+1}]}\mathrm{B}\mathsf{Cob}^\partial_\Theta$ given by concatenating loops with the path represented by the morphism $K|_{[i,i+1]}$. Recall that a map of spaces $f : X \to Y$ is *acyclic* if for all local systems $\mathscr{L}$ on Y, the induced map $\mathrm{H}_*(X; f^*\mathscr{L}) \to \mathrm{H}_*(Y, \mathscr{L})$ is an isomorphism in all degrees. We are now ready to state the main result of this section. Recall from Assumptions 4.1.1 that we assume $2n + 1 \geq 7$ and B^∂ is path-connected.

Theorem 6.0.2. *Assume (B, B^∂) is 1-connected. Let K be a Θ-end in $\mathsf{Cob}^\partial_\Theta$ such that $\partial K|_0 \neq \emptyset$ and $\mathscr{N}^\partial_{\Theta,n}(K|_0) \neq \emptyset$. Then the map*

$$\mathscr{N}^\partial_{\Theta,n}(K|_\infty) \to \Omega_{[\emptyset,K|_\infty]}\mathrm{B}\mathsf{Cob}^\partial_\Theta$$

is acyclic.

The majority of the remainder of this section is devoted to proving this statement. We will proceed similarly to [GR17b, Section 7], so we attempt to make our exposition as close to this reference as possible. To prove this statement, we will need two preliminary versions of Theorem 6.0.2, which are stated and proved in the next two subsections. Finally, the last subsection is devoted to the proof of Corollary C.

6.1. The group completion argument.

The main goal of this subsection is to prove the first preliminary version of the statement above using Theorem 4.1.11. Recall that we have fixed an L as in Assumptions 4.1.1. Given two objects $P, Q \in \mathsf{Cob}^\partial_{\Theta,\partial^v L}$, denote by $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P,Q)$ the subspace of $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}(P,Q)$ (see Definition 4.1.6) of those (W, ℓ_W) such that $\ell_W : (W, \partial^h W) \to (B, B^\partial)$ is strongly n-connected. Given a Θ-end K in $\mathsf{Cob}^\partial_{\Theta,\partial^v L}$ (see Definition 4.1.9), we denote by $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P, K|_\infty)$ the homotopy colimit of the post-composition maps by $K|_{[i,i+1]}$, similar to the definition above. Before stating the main result of this subsection, we introduce the following notion. For $k \geq 2$, we say that the pair (B, B^∂) is (E_k) if there exists a strongly k-connected map $(X, X') \to (B, B^\partial)$ from a CW complex pair (X, X') such that X' has finite k-skeleton and the pair (X, X') has finite $(k+1)$-skeleton (i.e. the $(k+1)$-skeleton of the relative CW filtration of X relative to X' is finite). We now state the first preliminary version of Theorem 6.0.2.

Proposition 6.1.1. *Assume (B, B^∂) is 1-connected and (E_n) . Let K be a Θ-end in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ such that $\ell|_{K|_0}$ is strongly $(n-1)$-connected. If $P \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$, then the map*

$$\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P, K|_\infty) \to \Omega_{[P,K|_\infty]}\mathrm{B}\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$$

is acyclic.

Remark 6.1.2. Observe that if Θ satisfies the assumption of Theorem 6.0.2, there exists a nullbordism W of $K|_0$ with a strongly n-connected map to (B, B^∂). Since W and $\partial^h W$ are compact manifolds, they admit finite CW structures. Thus (B, B^∂) satisfies the condition (E_n).

6.1.1. An easier case.

We start the proof of Proposition 6.1.1 by establishing an intermediate version, where the hypothesis on K and Θ are stronger.

Proposition 6.1.3. *Assume (B, B^∂) is 1-connected and (E_n). Let K be a Θ-end in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ such that $\ell|_{K|_0}$ is strongly $(n-1)$-connected and $\ell|_{K|_{[i,+\infty)}}$ is strongly n-connected for every i. If $P \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$, then the map*

$$\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P, K|_\infty) \to \Omega_{[P,K|_\infty]}\mathrm{B}\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$$

is acyclic.

The proof of this result will use Theorem 4.1.11 in conjuction with the generalized group completion theorem for topological categories, as stated in [GR17b, Thm. A.1]. To use these results, we need to prove the following lemmas. The overall goal of these lemmas is to relate the morphism spaces of the category $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ and the subfunctor $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(-, K|_\infty)$, which is the object of study of Theorem 4.1.11.

Lemma 6.1.4. *In the context of Proposition 6.1.3, the inclusion $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P, K|_i) \subseteq \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P, K|_i)$ is an equality, for every $i \geq 0$.*

Proof. It suffices to prove that for any morphism $W : P \rightsquigarrow K|_i$ such that $\ell|_W$ is strongly n-connected, then $(K|_i, \partial^h K|_i) \to (W, \partial^h W)$ is strongly $(n-1)$-connected. Start by observing that the map $(K|_i, \partial^h K|_i) \to (K|_{[i,+\infty)}, \partial^h K|_{[i,+\infty)})$ is strongly $(n-1)$-connected. To do this, we show that $(K|_i, \partial^h K|_i) \to (K|_{[i,i+j]}, \partial^h K|_{[i,i+j]})$ is strongly $(n-1)$-connected for all $j \geq 1$. For $j = 1$, this follows from the definition of a Θ-end. By induction, it suffices to prove that $(K|_{[i,i+j]}, \partial^h K|_{[i,i+j]}) \to (K|_{[i,i+j+1]}), \partial^h K|_{[i,i+j+1]}))$ is strongly $(n-1)$-connected. Observe the following homotopy pushout square

$$\begin{array}{ccc}
(K|_{i+j}, \partial^h K|_{i+j}) & \longrightarrow & (K|_{[i,i+j]}, \partial^h K|_{[i,i+j]}) \\
\downarrow & & \downarrow \\
(K|_{[i+j,i+j+1]}, \partial^h K|_{[i+j,i+j+1]}) & \longrightarrow & (K|_{[i,i+j+1]}, \partial^h K|_{[i,i+j+1]})
\end{array}$$

where the left vertical map is strongly $(n-1)$-connected by definition. Applying Lemma 2.1.4, we have that the right vertical map is strongly $(n-1)$-connected, as claimed. Now the following square

$$\begin{array}{ccc} (K|_i, \partial^h K|_i) & \longrightarrow & (W, \partial^h W) \\ \downarrow & & \downarrow \\ (K|_{[i,\infty)}, \partial^h K|_{[i,\infty)}) & \longrightarrow & (B, B^\partial) \end{array}$$

implies that the composite $(K|_i, \partial^h K|_i) \to (B, B^\partial)$ is strongly $(n-1)$-connected. By Lemma 2.1.3, we deduce that $(K|_i, \partial^h K|_i) \to (W, \partial^h W)$ is strongly $(n-1)$-connected, since $n \geq 3$. This finishes the proof. □

Before we prove Proposition 6.1.3, we need the following lemma about the condition (E_n). For a map of pairs $f : (X, X') \to (Y, Y')$, recall the definition of the triad homotopy group $\pi_*(Y, Y', X)$ from Definition 2.1.6.

Lemma 6.1.5. *Let $k \geq 2$. Let (B, B') be a 1-connected pair satisfying (E_k) such that B' is path-connected. Then the following claims hold:*

(i) B' satisfies Wall's finiteness condition (F_k) (in the sense of [Wal65, p. 57])

(ii) Let (W, W') be a CW pair such that both W and W' have finite k-skeleton relative to $\emptyset$. Let $f : (W, W') \to (B, B')$ be a strongly $(k-1)$-connected map such that $f|_{W'} : W' \to B'$ is k-connected, then the triad homotopy group $\pi_{k+1}(B, B', W)$ is finitely generated as a $\mathbf{Z}[\pi_1 W']$-module.

Proof. Let $\alpha : (X, X') \to (B, B')$ be strongly k-connected map from a CW pair (X, X') such that X' has finite k-skeleton and the pair (X, X') has finite $(k+1)$-skeleton. For (i), we observe that $\alpha|_{X'} : X' \to B'$ can be made into a weak equivalence by attaching cells of dimension at least $k+1$. Let Y be the CW complex given by the result of that procedure. Then Y satisfies Wall's (F_k) condition by [Wal65, Thm. A]. Since B' is weakly equivalent to Y, we conclude that B' satisfies Wall's (F_k) condition.

We focus now on (ii). From the hypothesis, one obtains that the spaces W, W', B and B' are path-connected, hence we will drop the basepoint from the notation of the relative and triad homotopy groups that appear in this proof. Additionally, we obtain that $f : W \to B$ is k-connected, since it factors as $W \to W \cup_{W'} B' \to B$, both of which are k-connected maps. We denote $\pi' \coloneqq \pi_1(W') \cong \pi_1(B')$ and $\pi \coloneqq \pi_1(W) \cong \pi_1(B)$. Consider the long exact sequence of the map of pairs $(W, B) \to (W', B')$ induced by f

$$\cdots \to \pi_{k+1}(B', W') \xrightarrow{f_*} \pi_{k+1}(B, W) \to \pi_{k+1}(B, B', W) \to \pi_k(B', W') \to \cdots$$

of $\mathbf{Z}[\pi']$-modules. Since $f|_{W'}$ is k-connected, the rightmost module vanishes, and hence $\pi_{k+1}(B, B', W)$ is isomorphic to the cokernel of f_*. Moreover, since

$f|_W$ and $f|_{W'}$ are k-connected, the Hurewicz homomorphisms $\pi_{k+1}(B', W') \to \mathrm{H}_{k+1}(B', W'; \mathbf{Z}[\pi'])$ and $\pi_{k+1}(B, W) \to \mathrm{H}_{k+1}(B, W; \mathbf{Z}[\pi])$ are isomorphisms. Under these isomorphisms, the map f_* corresponds to the composite

$$\mathrm{H}_{k+1}(B', W'; \mathbf{Z}[\pi']) \to \mathrm{H}_{k+1}(B', W'; f^*\mathbf{Z}[\pi]) \xrightarrow{f_*} \mathrm{H}_{k+1}(B, W; \mathbf{Z}[\pi]).$$

Denote the kernel of $\mathbf{Z}[\pi'] \to \mathbf{Z}[\pi]$ by K. Since $\pi' \to \pi$ is surjective, the left map above fits in the long exact sequence

$$\cdots \to \mathrm{H}_{k+1}(B', W'; \mathbf{Z}[\pi']) \to \mathrm{H}_{k+1}(B', W'; f^*\mathbf{Z}[\pi]) \to \mathrm{H}_k(B', W'; K) \to \cdots .$$

Since $f|_{W'}$ is k-connected, the $\mathbf{Z}[\pi']$-module $\mathrm{H}_k(B', W'; K)$ vanishes. We conclude that $\pi_{k+1}(B, B', W)$ is isomorphic to the cokernel of the map of $\mathbf{Z}[\pi']$-modules $f_* : \mathrm{H}_{k+1}(B', W'; f^*\mathbf{Z}[\pi]) \to \mathrm{H}_{k+1}(B, W; \mathbf{Z}[\pi])$. Once again by k-connectivity of $f|_{W'}$, this cokernel is isomorphic to $\mathrm{H}_{k+1}(B, B' \cup_{W'} W; \mathbf{Z}[\pi])$. Thus, it suffices to prove that the latter module is finitely generated as a $\mathbf{Z}[\pi]$-module, as this implies that it is finitely generated as a $\mathbf{Z}[\pi']$-module, as $\pi' \to \pi$ is surjective. We will do so by showing that the chain complex of $\mathbf{Z}[\pi]$-modules $C_*(B, B' \cup_{W'} W; \mathbf{Z}[\pi])$ of relative chains with coefficients in $\mathbf{Z}[\pi]$ is quasi-isomorphic to a chain complex C_* such that $C_* = 0$ for $* \leq k$ and C_{k+1} is finitely generated. We will use the following general facts about chain complexes of R-modules for a ring R. We say that a chain complex of R-modules has *finite k-skeleton* if it is quasi-isomorphic to a chain complex which is degreewise finitely generated projective in degrees at most k.

(i) *Let $i \in \mathbf{Z}$, given a cofiber sequence of chain complexes of R-modules $A \to B \to C$ such that A and B have finite $i-1$ and i-skeletons, respectively, then C has finite i-skeleton*: this follows since C is quasi-isomorphic to the cone of $A \to B$, which in degree $d \in \mathbf{Z}$ is given by $A_{d-1} \oplus B_d$.

(ii) *Let $i \geq 0$ and let C be a chain complex of projective R-modules supported in non-negative degrees. If $\mathrm{H}_*(C) = 0$ for $* \leq i$, then C is quasi-isomorphic to a chain complex D supported in degrees at least $i+1$. If, additionally, C_* is finitely generated projective for $* \leq i+l$ for a fixed $l \geq 0$, then D_* can be assumed to be finitely generated projective for $* \leq i+l$* : This is essentially an algebraic analogue of Whitehead's cell trading lemma [Whi50, Lemma 15] (see also [FP93, Lemma 6.1]). By induction on i, we can assume that $C_* = 0$ for $* \leq i-1$. Since $\mathrm{H}_i(C) = 0$, it follows that $d : C_{i+1} \to C_i$ is surjective. Choose a section $s : C_i \to C_{i+1}$ of d, which is possible since C_i is projective. Let $D(C_i)$ be the chain complex concentrated in degrees i and $i+1$ with value C_i in both entries, and whose only possible non-trivial differential is the identity. Let C' be the cone of the chain map $s : D(C_i) \to C$ given by id_{C_i} in degree i and s in degree $i+1$. In particular, $C'_* = C_i$ for $* \neq i+1, i+2$ and $C'_* = C_* \oplus C_i$ for $* = i+1, i+2$. Since $D(C_i)$ is acyclic, it follows that

$C \to C'$ is a quasi-isomorphism. Moreover, we have a degreewise split injective map $(0, \mathrm{id}) : D(C_i) \to C'$ which is the identity in degree i and is $0 \oplus \mathrm{id}_{C_i} : C_i \to C_{i+1} \oplus C_i$ in degree $i+1$. Let D be the degreewise cokernel of $(0, \mathrm{id})$, which is quasi-isomorphic to C, as it is also the cofiber of $(0, \mathrm{id})$, as this map is degreewise injective. Moreover, we see that $D_* = 0$ for $* \leq i$, $D_{i+1} = C_{i+1}$, D_{i+2} is the direct sum of projective modules, and $D_* = C_*$ for $* \geq i+3$. Finally, one traces that the additional finite generation hypothesis on C implies the desired finite generation on D.

Start by observing that we have a cofiber sequence of $\mathbf{Z}[\pi]$-modules

$$C_*(W, W'; \mathbf{Z}[\pi]) \to C_*(B, B'; \mathbf{Z}[\pi]) \to C_*(B, B' \cup_{W'} W; \mathbf{Z}[\pi]).$$

The chain complex $C_*(W, W'; \mathbf{Z}[\pi])$ is the cofiber of $C_*(W'; \mathbf{Z}[\pi]) \to C_*(W; \mathbf{Z}[\pi])$, which is map of chain complexes with finite k-skeleta, by taking cellular chains (which are degreewise free modules) and using the assumptions on the skeleta of W' and W. From *(i)* above, we deduce that $C_*(W, W'; \mathbf{Z}[\pi])$ has finite k-skeleton. On the other hand, the chain complex $C_*(B, B'; \mathbf{Z}[\pi])$ is the cofiber of the connecting map $C_{*+1}(B, B' \cup_{X'} X; \mathbf{Z}[\pi]) \to C_*(X, X'; \mathbf{Z}[\pi])$, where (X, X') is the source of the map α defined in the first line of this proof. By taking cellular chains and the assumption on (X, X'), we obtain that $C_*(X, X'; \mathbf{Z}[\pi])$ has finite $(k+1)$-skeleton. Since $B' \cup_{X'} X \to B$ is $(k+1)$-connected, we can apply *(ii)* to obtain that $C_{*+1}(B, B' \cup_{X'} X; \mathbf{Z}[\pi])$ is quasi-isomorphic to a chain complex concentrated in degrees at least $k+1$. In particular, this former chain complex has finite k-skeleton. From *(i)*, we conclude that $C_*(B, B' \cup_{W'} W; \mathbf{Z}[\pi])$ has finite $(k+1)$-skeleton. Now applying *(ii)* to this chain complex, we obtain that $C_*(B, B' \cup_{W'} W; \mathbf{Z}[\pi])$ is quasi-isomorphic to a chain complex concentrated in degrees at least $k+1$ and whose entry in degree $k+1$ is finitely generated, as $B' \cup_{W'} W \to B$ is k-connected. This finishes this proof. □

Lemma 6.1.6. *In the context of Proposition 6.1.3, the map induced by inclusion*

$$\mathsf{Cob}^{\partial, n-1}_{\Theta, \partial^v L, n}(P, K|_\infty) \to \mathsf{Cob}^{\partial, n-1}_{\Theta, \partial^v L}(P, K|_\infty)$$

is an equivalence.

Proof. Start by noticing that $\mathsf{Cob}^{\partial, n-1}_{\Theta, \partial^v L, n}(P, K|_i)$ is a union of path components of $\mathsf{Cob}^{\partial, n-1}_{\Theta, \partial^v L}(P, K|_i)$ for every $i \geq 0$. So it suffices to prove that the induced map on π_0 is surjective after taking the colimit over i. Moreover, it suffices to prove that given a morphism $W : P \rightsquigarrow K|_i$, there exists a $j \geq i$ such that $(W \cup K|_{[i,j]}, \partial^h(W \cup K|_{[i,j]}) \to (B, B^\partial)$ is strongly n-connected.

Since $(K|_i, \partial^h K|_i) \to (W, \partial^h W)$ and $(K|_i, \partial^h K|_i) \to (B, B^\partial)$ (see Lemma 6.1.4) are strongly $(n-1)$-connected, we have that $(W, \partial^h W) \to (B, B^\partial)$ is strongly $(n-1)$-connected by Lemma 2.1.3. We show that there exists j, such that the map $(W \cup K|_{[i,j]}, \partial^h(W \cup K|_{[i,j]})) \to (B, B^\partial)$ is strongly n-connected. This is inspired by [GR17b, Lemma 7.7]. We first show that there exists j, such that the map $\partial^h(W \cup K|_{[i,j]}) \to B^\partial$ is n-connected. By Lemma 6.1.5, B^∂ is (F_n) and so $\pi_n(B^\partial, \partial^h W)$ is a finitely generated $\mathbf{Z}[\pi_1(B^\partial)]$-module, since $n \geq 3$. This implies that $\ker(\pi_{n-1}(\partial^h W) \to \pi_{n-1}(B^\partial))$ is a finitely generated $\mathbf{Z}[\pi_1(B^\partial)]$-module. Choose a finite basis $\{a_p\}_p$ for this module. It follows from the hypothesis, there exists an integer $l \geq 0$ such that the images of a_p in $\pi_{n-1}(\partial^h(W \cup K|_{[i,i+l]}))$ vanish for all p. Similarly, we need to add a finite number of elements of $\pi_n(B^\partial)$ to the image of $\pi_n(\partial^h(W \cup K|_{[i,i+l]}))$ to generate $\pi_n(B^\partial)$ as $\mathbf{Z}[\pi_1(B^\partial)]$-modules. Thus there exists an $l' \geq l$ such that, additionally, the map $\pi_n(\partial^h(W \cup K|_{[i,i+l']})) \to \pi_n(B^\partial)$ is surjective. Therefore, the map $\partial^h(W \cup K|_{[i,i+l']}) \to B^\partial$ is n-connected.

Let $W' \coloneqq W \cup K|_{[i,i+l']}$ and consider now the group $\pi_{n+1}(B, B^\partial, W')$. By Lemma 6.1.5, this group is finitely generated as a $\mathbf{Z}[\pi_1(\partial^h W')]$-module. This implies that $\ker(\pi_n(W', \partial^h W') \to \pi_n(B, B^\partial))$ is finitely generated as a $\mathbf{Z}[\pi_1(\partial^h W')]$-module. Choose a finite basis $\{b_q\}_q$ for this module. By hypothesis, there exists an integer $m \geq 0$ such that the images of b_p in $\pi_n(W' \cup K|_{[l',l'+m]}, \partial^h(W' \cup K|_{[l',l'+m]}))$ vanish for all q. Similarly, we can generate $\pi_{n+1}(B, B^\partial)$ as a module over $\pi_1(B^\partial) \cong \pi_1(\partial^h W')$ by the image of $\pi_{n+1}(W' \cup K|_{[l',l'+m]}, \partial^h(W' \cup K|_{[l',l'+m]}))$ and finitely many elements. Thus, there exists an $m' \geq m$ such that, additionally, the map $\pi_{n+1}(W' \cup K|_{[l',l'+m']}, \partial^h(W' \cup K|_{[l',l'+m']})) \to \pi_{n+1}(B, B^\partial)$ is surjective. Therefore, the map $(W \cup K|_{[i,i+l'+m']}, \partial^h(W \cup K|_{[i,i+l'+m']}) \to (B, B^\partial)$ is strongly n-connected. This finishes the proof. □

Proof of Proposition 6.1.3. We apply [GR17b, Thm. A.14] to the category $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$. Condition (i) follows as in the proof of [GR17b, Prop. 7.5]. To verify (ii), we observe that the functor $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}(-, K|_\infty)$ is equivalent to $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(-, K|_\infty)$ by Lemma 6.1.6 and Lemma 6.1.4. The latter takes all morphisms to abelian homology equivalences by Theorem 4.1.11, which is precisely (ii). Thus, we obtain that the map

$$\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}(P, K|_\infty) \to \Omega_{[P,K|_\infty]} B\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$$

is acyclic. By pre-composing with the map in Lemma 6.1.6, we finish the proof. □

6.1.2. Θ-surgery. The goal of this subsubsection is to provide a general construction which will allow us to produce a Θ-end K' satisfying the hypothesis of Proposition 6.1.3 from a Θ-end K satisfying only the hypothesis of Proposition 6.1.1. In the next subsubsection, we will relate these Θ-ends to give a proof of the latter statement. This construction takes the form of the following result.

Proposition 6.1.7. *Assume (B, B^∂) is (E_n) and let R be an object in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ such that $\ell_R : (R, \partial^h R) \to (B, B^\partial)$ is strongly $(n-1)$-connected. If $(R, \partial^h R)$ is 1-connected, then there exists a morphism $W_R : R \rightsquigarrow R'$ in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ satisfying the following properties:*

(a) *The inclusions* $(R, \partial^h R) \hookrightarrow (W_R, \partial^h W_R) \hookleftarrow (R', \partial^h R')$ *are strongly* $(n-1)$*-connected;*
(b) $\ell : (W_R, \partial^h W_R) \to (B, B^\partial)$ *is strongly* n*-connected.*

One can see this result as an analog of [Kre99, Prop. 4] for triads. In fact, the strategy is analogous to the proof of loc.cit. (see also [GR17b, Lemma 7.6]). We start with two constructions that will be essential to the proof of Proposition 6.1.7. Let $(M, \ell_M) \in \mathsf{Cob}^\partial_{\Theta, \partial^v L}$ and $k \geq 0$ be an integer. Assume we are given an embedding $e : S^k \to \partial^h M \backslash \partial^{hv} M$ and a map $\ell : D^{k+1} \to B^\partial$ extending $e^*\ell_M$ on ∂D^{k+1}. We say that e and ℓ *occur as the trace of a right* Θ*-surgery* if there exists a morphism $(W, \ell_W) : M \rightsquigarrow M'$ which admits a handle decomposition with a single right $(k+1)$-handle (recall Definition 2.3.3) relative to M, whose attaching map is a thickening of e such that there exists a homotopy commutive diagram

$$\begin{array}{ccc} (M \cup_e D^{k+1}, \partial^h M \cup_e D^{k+1}) & \xrightarrow{\iota} & (W, \partial^h W) \\ & \searrow^{\ell_M \cup \ell} & \downarrow \ell_W \\ & & (B, B^\partial) \end{array}$$

where ι is the inclusion map induced by the core of the single right $(k+1)$-handle. The following lemma provides a sufficient condition for this to be the case. This will be useful to construct morphisms in $\mathsf{Cob}^\partial_{\Theta, \partial^v L}$ with prescribed Θ-structure.

Lemma 6.1.8. *Let* $e : S^k \to \partial M$ *be an embedding and* $\ell : D^{k+1} \to B^\partial$ *be an extension of the* Θ*-structure* $e^*\ell_M$*. Then* e *and* ℓ *occur as the trace of a right* Θ*-surgery, provided* $k \leq n-1$.

Proof. This is essentially [GR14, Section 4.1] by applying it to ∂M and extending the trace and θ^∂-structure to a collar. □

The second construction is analogous to the first but for left surgery. Let again $(M, \ell_M) \in \mathsf{Cob}^\partial_{\Theta, \partial^v L}$ and $k \geq 0$ be an integer. Let $e : (\partial_1 D^{k+1}_+, \partial_{01} D^{k+1}_+) \to (M, \partial M)$ be an embedding along with a map $\ell : (D^{k+1}_+, \partial_0 D^{k+1}) \to (B, B^\partial)$ that extends $e^*\ell_M$ in $(\partial_1 D^{k+1}_+, \partial_{01} D^{k+1}_+)$. We say that e and ℓ *occur as the trace of a left* Θ*-surgery* if there exists a morphism $(W, \ell_W) : M \rightsquigarrow M'$ which admits a handle decomposition with a single left $(k+1)$-handle (recall Definition 2.3.3) whose attaching map is a thickening of e such that there exists a homotopy commutive diagram

$$\begin{array}{ccc} (M \cup_e D^{k+1}_+, \partial^h M \cup_e \partial_1 D^{k+1}_+) & \xrightarrow{\iota} & (W, \partial^h W) \\ & \searrow^{\ell_M \cup \ell} & \downarrow \ell_W \\ & & (B, B^\partial) \end{array}$$

where ι is the inclusion map induced by the core of the single left $(k+1)$-handle. The next lemma is analogous to Lemma 6.1.8.

Lemma 6.1.9. *Let* $e : (\partial_1 D^{l+1}_+, \partial_{01} D^{l+1}_+) \to (M, \partial M)$ *be an embedding and* $\ell_D :$ $(D^{l+1}_+, \partial_0 D^{l+1}) \to (B, B^\partial)$ *be an extension of the* Θ*-structure* $e^*\ell_M$*. Then* ℓ_D *occurs as the trace of a left* Θ*-surgery, provided* $l \leq 2n-1$*.*

Proof. We follow the strategy of [GR14, Section 4.1]. The embedding e induces an embedding

$$e_1 = [0,1] \times e : [0,1] \times (\partial_1 D^{l+1}_+, \partial_{01} D^{l+1}_+) \to [0,1] \times (M, \partial M).$$

Let $A := (D^{l+1}_+ \backslash \mathrm{int}(\frac{1}{2} D^{l+1}_+), \partial_0 D^{l+1}_+ \backslash \mathrm{int}(\frac{1}{2}\partial_0 D^{l+1}_+))$ and identify it with $[0,1] \times (\partial_1 D^{l+1}_+, \partial_{01} D^{l+1}_+)$. As in [GR14, Section 4.1], ℓ_M induces a collared bundle map of pairs $T(D^{l+1}_+, \partial_0 D^{l+1}_+)|_A \to (\theta^*\gamma_{2n+1}, (\theta^\partial)^*\gamma_{2n})$. We start by proving that such a map admits an extension to a collared bundle map $T(D^{l+1}_+, \partial_0 D^{l+1}_+) \to (\theta^*\gamma_{2n+1}, (\theta^\partial)^*\gamma_{2n})$ which covers the map ℓ_D. Note that it suffices to find a collared bundle map $T(D^{l+1}_+, \partial_0 D^{l+1}_+) \to \ell_D^*(\theta^*\gamma_{2n+1}, (\theta^\partial)^*\gamma_{2n})$over the pair $(D^{l+1}_+, \partial_0 D^{l+1}_+)$ extending the given map on the subspace A. Since both collared bundles are trivial, as they are bundles over a contractible pair, this problem is equivalent to finding a lift of the following lifting problem

$$\begin{array}{ccc} (\partial_1 D^{l+1}_+, \partial_{01} D^{l+1}_+) & \longrightarrow & (V_l(2n+1), V_{l-1}(2n)) \\ \downarrow & \nearrow & \\ (D^{l+1}_+, \partial_1 D^{l+1}_+) & & \end{array}$$

where $V_i(j)$ is the Stiefel manifold of i-frames on $\mathbf{R}^j$ and the top map is induced by the trivialization over the subspace A. The upper right pair is $(2n-1)$-connected, thus a lift always exists if $l \leq 2n-1$. Choose now such a collared bundle map of pairs $T(D^{l+1}_+, \partial_0 D^{l+1}_+) \to (\theta^*\gamma_{2n+1}, (\theta^\partial)^*\gamma_{2n})$, which induces a collared vector bundle pair $(V, V') \to (D^{l+1}_+, \partial_0 D^{l+1}_+)$ of dimension $2n-l$ and collar bundle map

$$T(D^{l+1}_+, \partial_0 D^{l+1}_+) \oplus (V, V') \to (\theta^*\gamma_{2n+1}, (\theta^\partial)^*\gamma_{2n}),$$

which is a fiberwise isomorphism and extends the given map. The disc bundle of (V, V') is then identified with the normal bundle of e_1 once restricted to A. Since the bundle (V, V') has contractible base, then its disc bundle is diffeomorphic to $(D^{l+1}_+ \times D^{2n-l}, \partial_0 D^{l+1}_+ \times D^{2n-l})$ Consider now the pushout of pairs

$$\begin{array}{ccc} (D(V), D(V'))|_A & \longrightarrow & (D(V), D(V')) \\ \downarrow & & \downarrow \\ ([0,1] \times M, [0,1] \times \partial M) & \longrightarrow & (W, \partial^h W) \end{array}$$

which is a model for attachment of a left $(l+1)$-handle along a thickening of e. This description comes with a preferred Θ-structure given by the bundle maps defined above. In other words, ℓ_D occurs as a left trace of a surgery. This finishes the proof. □

We remark that the discussion above did not use the fact that the dimension of the cobordisms is odd in any essential way. We are now ready to start the proof of Proposition 6.1.7. This will be done by constructing W_R step by step increasing the connectivity of its Θ-structure. We start by deducing the existence of an intermediate morphism, where *(b)* is satisfied in the boundary.

Lemma 6.1.10. *Let R be an object of $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ satisfying the conditions of Proposition 6.1.7. Then there exists a morphism $(W_0, \ell_0) : R \rightsquigarrow R_0$ in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ satisfying properties (a) and where the map $\ell_0 : \partial^h W_0 \to B^\partial$ is n-connected.*

Proof. This follows by applying the construction of the proof of [GR17b, Lemma 7.6] to $P = \partial^h R$ and extending it to a morphism in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ using right handles (see Remark 2.3.6), which is possible by Lemma 6.1.8. The property *(a)* follows from (I) from Section 2.3.2. The map ℓ_0 is n-connected by property (*iv*) in the proof of [GR17b, Lemma 7.6]. □

Lemma 6.1.11. *Let R be an object of $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ satisfying the conditions of Proposition 6.1.7. Then there exists a morphism $(W_1, \ell_1) : R \rightsquigarrow R_1$ in $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ satisfying properties (a) and where the map $\ell_1|_{\partial^h W_1} : \partial^h W_1 \to B^\partial$ is n-connected and the map $(\ell_1)_* : \pi_i(W_1, \partial^h W_1) \to \pi_i(B, B^\partial)$ is an isomorphism for $i \leq n$.*

Proof. Let $(W_0, \ell_0) : R \rightsquigarrow R_0$ be a morphism satisfying the properties of Lemma 6.1.10. Observe that $\pi_1(\partial^h W_0) \cong \pi_1(B^\partial)$, since $n \geq 2$. We start by observing that the triad homotopy group (recall from Definition 2.1.6) $\pi_{n+1}(B, B^\partial, W_0)$ is a finitely generated $\mathbf{Z}[\pi_1(B^\partial)]$-module, by Lemma 6.1.5. We construct now the surgery data in order to construct the cobordism W_1. The $\mathbf{Z}[\pi_1(B^\partial)]$-module

$$\ker\Big((\ell_0)_* : \pi_n(W_0, \partial^h W_0) \to \pi_n(B, B^\partial)\Big)$$

is finitely generated: consider the long exact sequence on homotopy groups

$$\cdots \to \pi_{n+1}(B, B^\partial) \to \pi_{n+1}(B, B^\partial, W_0) \to \pi_n(W_0, \partial^h W_0) \to \pi_n(B, B^\partial) \to \cdots.$$

This is an exact sequence of $\mathbf{Z}[\pi_1(B^\partial)]$-modules, since $\partial^h W_0 \to B^\partial$ is an isomorphism on π_1. However, the module $\pi_{n+1}(B, B^\partial, W_0)$ surjects to the kernel considered above and thus the latter is also finitely generated. Pick generators $\beta_1, \cdots, \beta_q$ of this kernel as a $\mathbf{Z}[\pi_1(B^\partial)]$-module. Since the map $\pi_n(R_0, \partial^h R_0) \to \pi_n(W_0, \partial^h W_0)$ is surjective, we can represent these generators by maps

$$\beta_i : (D^n, \partial D^n) \to (R_0, \partial^h R_0).$$

To represent these elements by embeddings of pairs to the interior of $(R_0, \partial^h R_0)$, we use a result by Hudson [Hud72, Thm. 1], best stated for our purposes in [BP17, Thm. 5.13], which requires $n \geq 3$. In the notation of [BP17], we use this result for $(P, \partial_0 P, \partial_1 P) \coloneqq (D^n, \partial D^n, \emptyset)$ and $(W, \partial_0 W, \partial_1 W) \coloneqq (R_0, \partial^h R_0, \partial^v R_0)$, since the inclusion $\partial^h R \longrightarrow R$ is 1-connected. Once such embeddings are found, we can make them disjoint by [BP17, Thm. C.1], as $n \geq 3$. Once again, use Lemma 6.1.9 along with nullhomotopies of ℓ restricted to β_i to define the morphism $W_1 : R \rightsquigarrow R_1$ given by the simultaneous attachment of left $(n+1)$-handles to W_0 along thickenings of the embeddings β_i. We finish by verifying the properties claimed in the statement. Once again by (II) from Section 2.3.2, $(W_1, \partial^h W_1)$ is strongly $(n-1)$-connected relative to $(W_0, \partial^h W_0)$ and thus relative to $(R, \partial^h R)$. By Lemma 2.3.8 and (I) from Section 2.3.2, the inclusion of $(R_1, \partial^h R_1)$ to $(W_1, \partial^h W_1)$ is $(n-1)$-connected. The cobordism $\partial^h W_1$ is obtained from $\partial^h W_0$ by attaching trivial n-cells along $\beta_i|_{\partial D^n}$ (by Lemma 6.1.10), which implies that $\partial^h W_0 \longrightarrow \partial^h W_1$ admits a section collapsing these cells. Thus, the map $\pi_i(\partial^h W_0) \longrightarrow \pi_i(\partial^h W_1)$ is an isomorphism for $i \leq n-1$ and injective for $i = n$. Thus, the map $\partial^h W_1 \longrightarrow B^\partial$ is n-connected, as $\partial^h W_0 \longrightarrow B^\partial$ so is. Finally, the group $\pi_i(W_1, \partial^h W_1)$ is isomorphic to $\pi_i(W_0, \partial^h W_0) \cong \pi_i(R, \partial^h R)$ for $i < n$, and thus also to $\pi_i(B, B^\partial)$. For $i = n$, the group is a quotient of $\pi_n(W_0, \partial^h W_0)$ by the submodule generated by the classes β_i, and isomorphic to $\pi_n(B, B^\partial)$. This finishes the proof. □

To finish the proof of Proposition 6.1.7, recall the definition of triad connected sum from Definition 4.3.18.

Remark 6.1.12. Given two Θ-triads W and W' such that ℓ_W and $\ell_{W'}$ hit the same path component of B. Then $W \natural W'$ admits a Θ-structure extending ℓ_W and $\ell_{W'}$ using the following argument: Pick a path from $\ell_W(e(0))$ and $\ell_{W'}(e(0))$ in B^∂ and define Θ-structure on $(W \natural W', \partial^h(W \natural W')) \simeq (W \cup_{e(0)} [0,1] \cup_{e'(0)} W', \partial^h W \cup_{e(0)} [0,1] \cup_{e'(0)} \partial^h W')$ given by this path on $[0,1]$. In most cases, we can take embeddings such that $\ell_W(e(0))$ and $\ell_{W'}(e(0))$ coincide. In this case, we take the constant path.

Proof of Proposition 6.1.7. Let $W_1 : R \rightsquigarrow R_1$ be a morphism satisfying the properties of Lemma 6.1.11. We can pick a finite collection of $\zeta_0, \zeta_1, \cdots, \zeta_m \in \pi_{n+1}(B, B^\partial)$ that generate $\pi_{n+1}(B, B^\partial)$ as a $\mathbf{Z}[\pi_1(B^\partial)]$-module along with the image of $\pi_{n+1}(W_2, \partial^h W_2)$. This is possible since $\pi_{n+1}(B, B^\partial, W_2)$ is a finitely generated $\mathbf{Z}[\pi_1(B^\partial)]$-module, by the same argument as in loc.cit. Let $(F_i, \partial^h F_i)$ be the n-disc bundle given by a lift of the form

$$\begin{array}{ccccc} & & & & (\mathrm{BO}(n), \mathrm{BO}(n)) \\ & & & \nearrow & \downarrow \\ (D^{n+1}, \partial D^{n+1}) & \longrightarrow & (B, B^\partial) & \longrightarrow & (\mathrm{BO}(2n+1), \mathrm{BO}(2n)). \end{array}$$

which exists and is unique by obstruction theory since $\pi_i(\mathrm{BO}(2n+1), \mathrm{BO}(2n), \mathrm{BO}(n)) \cong \pi_i(\mathrm{BO}(2n+1), \mathrm{BO}(2n))$ vanishes for $i < 2n+1$. Once again, we endow such pair with the Θ-structure given by the composite $(F_i, \partial^h F_i) \to (D^{n+1}, \partial D^{n+1}) \to (B, B^\partial)$. Let W_P be the cobordism given by a triad connected sum of W_2 with the triad determined by $(F_i, \partial^h F_i)$. Similar to before, we can obtain F_i from D^{2n+1}_+ by attaching a left $(n+1)$-handle so *(a)* is automatic by (I) and (II) from Section 2.3.2, and Lemma 2.3.8. By construction, the map $\pi_{n+1}(W_R, \partial^h W_R) \to \pi_{n+1}(B, B^\partial)$ is surjective. By Lemma 2.1.5, we obtain *(b)*. By the same argument as in Lemma 6.1.11, we do not lose the established properties of W_1. This completes the proof since $(W_R, \partial^h W_R)$ is 1-connected, as $n \geq 2$. □

6.1.3. Proof of Proposition 6.1.1. In this subsubsection, we use the general construction from the previous subsubsection and Proposition 6.1.3 to prove the main result of this subsection. We start with the following lemma which uses this construction.

Lemma 6.1.13. *Assume (B, B^∂) is 1-connected and (E_n). If $P \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ such that $\ell|_P$ is strongly $(n-1)$-connected, then there exists a Θ-end K such that $K|_0 = P$ and $\ell|_{K|_{[i,+\infty)}}$ is strongly n-connected for every $i \geq 0$.*

Proof. Given any object $P \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ such that $\ell|_P$ is strongly $(n-1)$-connected, it suffices to construct a morphism $W_P : P \rightsquigarrow P'$ in $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ such that:

- (i) W_P is strongly $(n-1)$-connected relative to both ends;
- (ii) W_P contains an embedded copy of V_1 with standard Θ-structure (see Definition 4.1.4)
- (iii) $\ell|_{W_P}$ is strongly n-connected.

Given such construction, $\ell|_{P'}$ is strongly $(n-1)$-connected, so we can iterate this contruction to produce the Θ-end $W_P \cup W_{P'} \cup W_{(P')'} \cup \cdots$. This is constructed by applying Proposition 6.1.7 (producing a morphism satisfying (i) and (iii) of above) and doing a triad connected sum (recall Definition 4.3.18) with V_1 with a standard Θ-structure (to satisfy (ii)). This does not affect (i) or (iii) since $W \to W \natural V_1$ is strongly n-connected. (This is possible since $(P, \partial^h P)$ is 1-connected, given the assumption on (B, B^∂) and $n \geq 3$.) □

We are now ready to prove the first preliminary version of our main result of this section.

Proof of Proposition 6.1.1. We start by fixing a Θ-end $\{P|_{[i,i+1]} : P|_i \rightsquigarrow P|_{i+1}\}$ with $P|_0 = P$. For example, take $P|_{[i,i+1]} = {}_{P|_i}H$ (as defined before Proposition 4.3.20). For any two Θ-ends K and K' such that $K|_0 = K'|_0$, we construct a zig-zag of abelian homology equivalences

$$\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P, K|_\infty) \to \cdots \leftarrow \mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P, K'|_\infty)$$

which commutes with an analogous zig-zag of equivalences

$$\Omega_{[P,K|_\infty]}\mathrm{BCob}^{\partial,n-1}_{\Theta,\partial^v L} \to \cdots \leftarrow \Omega_{[P,K'|_\infty]}\mathrm{BCob}^{\partial,n-1}_{\Theta,\partial^v L}.$$

The proof follows by letting K' be the Θ-end constructed in Lemma 6.1.13 and applying Proposition 6.1.3 and Remark 6.1.2. This implies that the map in the claim is an abelian homology equivalence. However, the target of this map has abelian fundamental group, and thus the map is actually acyclic, since all local systems are abelian (see [GR17b, Appendix A]).

We construct the aforementioned zig-zags in the following way

$$\begin{array}{ccc}
\mathrm{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P,K|_\infty) & \longrightarrow & \operatorname{hocolim}_j \operatorname{hocolim}_i \mathrm{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P|_j,K|_i) \\
 & & \uparrow \\
 & & \mathrm{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P|_\infty,K|_0) = \mathrm{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P|_\infty,K'|_0) \\
 & & \downarrow \\
\mathrm{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P,K'|_\infty) & \longrightarrow & \operatorname{hocolim}_j \operatorname{hocolim}_i \mathrm{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P|_j,K'|_i)
\end{array}$$

where the horizontal maps are induced by pre-composition by $P|_*$ and the vertical maps by post-composition by K. The horizontal maps are abelian homology equivalences by Theorem 4.1.11, since they are the sequential colimit of abelian homology equivalences of the form

$$\mathrm{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P|_i,K|_\infty) \to \mathrm{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P|_{i-1},K|_\infty)$$

and such equivalences are closed under sequential colimits. Since $\ell_{K|_i}$ is strongly $(n-1)$-connected (since $\ell_{K|_0}$ is and $K|_{[i,i+1]}$ is strongly $(n-1)$-connected relative to both ends), we see that $\mathrm{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P|_j,K|_i) = \mathrm{Cob}^{\partial}_{\Theta,\partial^v L,n}(P|_j,K|_i)$ since $n \geq 2$. We can thus apply Theorem 4.1.11. The vertical maps are abelian homology equivalences by the analogous statement of Theorem 4.1.11 where stabilization is done on the left. More precisely, the top vertical map is a colimit of maps of the form

$$\mathrm{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P|_\infty,K|_i) \to \mathrm{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P|_\infty,K|_{i+1}).$$

By the same argument as above, both spaces can be replaced by the analogous ones for $\mathrm{Cob}^{\partial}_{\Theta,\partial^v L,n}(-,-)$. Applying Theorem 4.1.11 to the Θ-end $\overline{P}|_{-*}$ given by the reflection (see [GR17b, p. 140]) and composing it with the reflection equivalence

$$\mathrm{Cob}^{\partial}_{\Theta,\partial^v L,n}(P,Q) \simeq \mathrm{Cob}^{\partial}_{\Theta,\partial^v \overline{L},n}(\overline{Q},\overline{P})$$

where the Θ-structure on $\overline{Q}$ is the pre-composition of ℓ_Q with the equivalence of vector bundle pairs $(-1) \oplus \mathrm{id} : \varepsilon^1 \oplus TQ \simeq \varepsilon^1 \oplus TQ$ (see [GR14, Defn. 5.1] for more details on the reflection automorphism). It is clear that the diagram above is compatible with the analogous diagram for the different (colimits of) loop spaces of $\mathrm{BCob}^{\partial}_{\Theta,\partial^v L}$. All the maps in that diagram are induced by concatenation with paths, and hence are weak equivalences. This finishes the proof. □

6.2. Weak maps of Θ-ends.

The goal of this subsection is to prove a second preliminary version of Theorem 6.0.2, which builds on Proposition 6.1.1. More precisely, this version takes the following form. Recall the definition of triad handle decomposition from Definition 2.3.3.

Proposition 6.2.1. *Assume (B, B^∂) is 1-connected and $(L, \partial^h L)$ admits a triad handle decomposition with handles of any type and of index at most $n-1$ relative to $\emptyset$, and the maps $\partial^v L \to L$ and $\partial^{hv} L \to \partial^h L$ are fundamental groupoid isomorphisms. Let K be a Θ-end and $P \in \mathsf{Cob}^{\partial, n-1}_{\Theta, \partial^v L}$ such that $\mathsf{Cob}^{\partial}_{\Theta, \partial^v L, n}(P, K|_0)$ is non-empty. Then*

$$\mathsf{Cob}^{\partial}_{\Theta, \partial^v L, n}(P, K|_\infty) \to \Omega_{[P, K|_\infty]} \mathsf{B}\mathsf{Cob}^{\partial}_{\Theta, \partial^v L}$$

is acyclic.

Remark 6.2.2. If $n \geq 3$, then the condition that $\partial^v L \to L$ and $\partial^{hv} L \to \partial^h L$ are fundamental groupoid isomorphisms, follows from the condition that $(L, \partial^h L)$ admits a triad handle decomposition with handles of any type and of index at most $n-1$ relative to $\emptyset$, by (I) to (III) from Section 2.3.2.

Once again, we proceed by constructing a Θ-end satisfying the conditions of Proposition 6.1.1 from one which only satisfies the hypothesis from above. In this case, these Θ-ends are related in a different way to the one from the last subsection. To make this precise, it is convenient to introduce the following definition.

Definition 6.2.3 (Weak maps of Θ-ends.)**.** Let K' and K be Θ-ends in $\mathsf{Cob}^{\partial}_{\Theta, \partial^v L}$. A *weak map of* Θ*-ends* $V : K' \rightsquigarrow K$ is a collection of morphisms $V_i : K'|_i \rightsquigarrow K|_i$ in $\mathsf{Cob}^{\partial}_{\Theta, \partial^v L}$ for every $i \geq 0$ such that the composition $K'|_i \rightsquigarrow K'|_{i+1} \rightsquigarrow K|_{i+1}$ is diffeomorphic to the composition $K'|_i \rightsquigarrow K|_i \rightsquigarrow K|_{i+1}$ for every $i \geq 0$ and for every $i \geq 0$, the inclusion $(K'|_i, \partial^h K'|_i) \to (V_i, \partial^h V_i)$ is strongly $(n-1)$-connected.

By fixing diffeomorphisms issuing the condition above, one produces a map

$$V_* : \mathsf{Cob}^{\partial}_{\Theta, \partial^v L, n}(P, K'|_\infty) \to \mathsf{Cob}^{\partial}_{\Theta, \partial^v L, n}(P, K|_\infty)$$

by taking homotopy colimits of the homotopy coherent diagram given by the maps $V_i \circ (-) : \mathsf{Cob}^{\partial}_{\Theta, \partial^v L, n}(P, K'|_i) \to \mathsf{Cob}^{\partial}_{\Theta, \partial^v L, n}(P, K|_i)$. The reason why, in the definition above, we do not fix diffeomorphisms is that the property that this map is an abelian homology equivalence does not depend on the choice of diffeomorphisms. More precisely, if such a map, given a certain choice, is an abelian homology equivalence, then any other map induced by another choice also is one. This follows from [GR17b, Lemma A.10] and was pointed out in [GR17b, p. 183]. Therefore, the property that V_* is an abelian homology equivalence is a well-defined property of the weak map of Θ-ends V.

Lemma 6.2.4. *Assume the hypothesis of Proposition 6.2.1. Let $V : K' \rightsquigarrow K$ be a weak map of Θ-ends, then the map*

$$V_* : \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P, K'|_\infty) \to \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P, K|_\infty)$$

is an abelian homology equivalence.

Proof. As in Proposition 6.1.1, we start by fixing a Θ-end $\{P|_{[i,i+1]} : P|_i \rightsquigarrow P|_{i+1}\}$ with $P|_0 = P$. Consider the following square

$$\begin{array}{ccc} \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P, K'|_\infty) & \xrightarrow{V_*} & \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P, K|_\infty) \\ \downarrow & & \downarrow \\ \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P|_\infty, K'|_\infty) & \xrightarrow{V_*} & \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P|_\infty, K|_\infty) \end{array}$$

where the vertical maps are the maps induced by V_* after taking homotopy colimits with respect to stabilization on the left by the Θ-end. The vertical left map is the homotopy colimit of the abelian homology equivalences of the form $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P|_i, K'|_\infty) \to \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P|_{i-1}, K'|_\infty)$, by Theorem 4.1.11. Hence, the map to the homotopy colimit is also an abelian homology equivalence. The same applies for the right vertical map. On the other hand, the lower horizontal map is the homotopy colimit over i of maps $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P|_\infty, K'|_i) \to \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P|_\infty, K|_i)$ which are abelian homology equivalences by the dual of Theorem 4.1.11 (see the proof of Proposition 6.1.1). This finishes the proof by 2-out-of-3 property for abelian homology equivalences (see Lemma 4.2.4). □

Following the result above, our goal is to find a weak map of Θ-ends $K' \rightsquigarrow K$ such that $\ell|_{K'|_i}$ is strongly $(n-1)$-connected, for any Θ-end K satisfying the hypothesis of Proposition 6.2.1. The following lemma achieves precisely that.

Lemma 6.2.5. *Let K be a Θ-end and $P \in \mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ such that $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P, K|_0) \neq \emptyset$, then there exists a weak map of Θ-ends $K' \rightsquigarrow K$ such that $\ell|_{K'|_i}$ is strongly $(n-1)$-connected, for every $i \geq 0$.*

Proof. Let $W \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P, K|_0)$ and fix a triad handle decomposition of $(W, \partial^h W)$ relative to $(P, \partial^h P)$ (see Definition 2.3.3). Define $V_0 : K'|_0 \rightsquigarrow K|_0$ to be the union of all right handles of index at least n, and interior and left handles of index at least $n+1$. Thus, by (i) to (iii) from Section 2.3.2, the inclusion $(K'|_0, \partial^h K'|_0) \hookrightarrow (V_0, \partial^h V_0)$ is strongly $(n-1)$-connected. The inclusion of $K'|_0 \hookrightarrow W$ is strongly $(n-1)$-connected, since it is obtained by attaching right handles of index n, interior handles of index at least $n+1$ and left handles of index at least $n+2$. Thus, the map $(K'|_0, \partial^h K'|_0) \to (B, B^\partial)$ is strongly $(n-1)$-connected. Consider the composition $K|_{[0,1]} \circ V_0 : K'|_0 \rightsquigarrow K|_1$. By assumption, $K|_{[0,1]} \cong M \circ H_{K|_0}$ for some morphism M. The handle $H_{K|_0}$ is attached trivially to $K|_0$ by definition, so it can be made disjoint from the belts of the handles in V since $n \geq 1$. Hence, by Proposition 2.3.10 and Remark 4.2.5, the composition $K|_{[0,1]} \circ V_0 : K'|_0 \rightsquigarrow K|_1$ factors as $V_1 \circ H_{K'|_0}$ for some $V_1 : K'|_1 \rightsquigarrow K|_1$. Clearly, the inclusions $(K'|_0, \partial^h K'|_0) \hookrightarrow (H_{K'|_0}, \partial^h H_{K'|_0}) \hookleftarrow (K'|_1, \partial^h K'|_1)$ are strongly $(n-1)$-connected. Thus, so is the map $(K'|_1, \partial^h K'|_1) \to (B, B^\partial)$. The inclusion $(K'|_1, \partial^h K'|_1) \hookrightarrow (V_1, \partial^h V_1)$ is strongly $(n-1)$-connected by the following argument: Consider the following (homotopy) pushout square of pairs

$$\begin{array}{ccc} (K'|_1, \partial^h K'|_1) & \longrightarrow & (H_{K'|_0}, \partial^h H_{K'|_0}) \\ \downarrow & & \downarrow \\ (V_1, \partial^h V_1) & \longrightarrow & (V_1 \circ H_{K'|_0}, \partial^h (V_1 \circ H_{K'|_0})) \end{array}$$

where both horizontal maps are strongly $(n-1)$-connected and the right vertical map is strongly $(n-1)$-connected since the composition $(K'|_0, \partial^h K'|_0) \to (H_{K'|_0}, \partial^h H_{K'|_0}) \to (V_1 \circ H_{K'|_0}, \partial^h (V_1 \circ H_{K'|_0})) \cong (K|_{[0,1]} \circ V_0, \partial^h (K|_{[0,1]} \circ V_0))$ is strongly $(n-1)$-connected. Thus, by Lemma 2.1.4 and $n \geq 2$, the leftmost vertical map is strongly $(n-1)$-connected. We can iterate this construction to obtain a weak map between $K' = \{H_{K'|_i} : K'|_i \rightsquigarrow K'|_{i+1}\}_{i \geq 0}$ and K with the desired property. □

Before we prove Proposition 6.2.1, we return briefly to the difference between the categories $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ and $\mathsf{Cob}^{\partial,n-1}_{\Theta,L}$. Recall that the functor $(M \mapsto M^\circ) : \mathsf{Cob}^{\partial}_{\Theta,L} \to \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ is an isomorphism of topological categories, but under this map $\mathsf{Cob}^{\partial,n-1}_{\Theta,L}$ does not map in general into $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$. However, the next lemma gives a condition for that to happen. This will be necessary to use both Theorem 3.1.10 and Theorem 4.1.11, which concern these two different categories.

Lemma 6.2.6. *The image of functor $(M \mapsto M^\circ) : \mathsf{Cob}^{\partial,n-1}_{\Theta,L} \to \mathsf{Cob}^{\partial}_{\Theta,\partial^v L}$ is $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$, provided the maps in $\partial^v L \to L$ and $\partial^{hv} L \to \partial^h L$ induce an isomorphism on fundamental groupoids. In this case, the functor $(M \mapsto M^\circ) : \mathsf{Cob}^{\partial,n-1}_{\Theta,L} \to \mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L}$ is an isomorphism of categories.*

Proof. Let $W : N \rightsquigarrow M$ be a morphism in $\mathsf{Cob}^{\partial,n-1}_{\Theta,L}$. Start by noticing that the assumption implies that the map $(M^\circ, \partial^h M^\circ) \to (M, \partial M)$ induces an isomorphism on fundamental groupoids: This follows from the left homotopy pushout square of pairs in the following diagram

$$\begin{array}{ccc} (\partial^v L, \partial^{hv} L) & \longrightarrow & (M^\circ, \partial^h M^\circ) \\ \downarrow & & \downarrow \\ (L, \partial^h L) & \longrightarrow & (M, \partial M) \end{array} \qquad \begin{array}{ccc} (M^\circ, \partial^h M^\circ) & \longrightarrow & (M, \partial M) \\ \downarrow & & \downarrow \\ (W^\circ, \partial^h W^\circ) & \longrightarrow & (W, \partial^h W) \end{array}$$

and by Seifert-Van Kampen. Consider now the right homotopy pushout square of pairs above. Since the top map induces an isomorphism on fundamental groupoids, then by Lemma 2.1.4 the right vertical map is strongly $(n-1)$-connected if and only if the leftmost is. ◻

We are now ready to finish the proof of the second preliminary version of Theorem 6.0.2.

Proof of Proposition 6.2.1. Let $K' \rightsquigarrow K$ be a weak map such that $\ell|_{K'|_i}$ is strongly $(n-1)$-connected, which exists by Lemma 6.2.5. Given $W \in \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P, K'|_i)$, we have that $K'|_i \to W$ is strongly $(n-1)$-connected since $\ell|_{K'|_i}$ and $\ell|_W$ are strongly $(n-1)$ and n-connected respectively and $n \geq 2$. Thus, $\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P, K'|_i) = \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P, K'|_i)$ for every $i \geq 0$. Consider the following commutative diagram

$$\begin{array}{ccccc}
\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P, K'|_\infty) & & \longrightarrow & & \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(P, K|_\infty) \\
\| & & & & \downarrow \\
\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L,n}(P, K'|_\infty) & & & & \\
\downarrow & & & & \\
\Omega_{[P,K'|_\infty]} B\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L} & \longrightarrow & \Omega_{[P,K|_\infty]} B\mathsf{Cob}^{\partial,n-1}_{\Theta,\partial^v L} & \longrightarrow & \Omega_{[P,K|_\infty]} B\mathsf{Cob}^{\partial}_{\Theta,\partial^v L}.
\end{array}$$

By Proposition 6.1.1, the left vertical map is an acyclic and by Lemma 6.2.4 the top map is an abelian homology equivalence. The bottom right map is an equivalence by Theorem 3.1.10 and Lemma 6.2.6, since $(L, \partial^h L)$ is obtained by handles of index at most $n-1$ from $\emptyset$ and the maps $\partial^v L \to L$ and $\partial^{hv} L \to \partial^h L$ are fundamental groupoid isomorphisms. Since the bottom map is an equivalence, we see that the right vertical map is an abelian homology equivalence and thus, acyclic since the fundamental group of the target is abelian. ◻

6.3. Proof of Theorem 6.0.2.

In this subsection, we prove the main result of this section using the preliminary versions established above. Until now, we have proved all our statements for general classes of submanifolds L. We will now make use of this additional generality to choose a specific L, which depends on the Θ-end K in the hypothesis of Theorem 6.0.2. We start by constructing this L.

Construction 6.3.1. Let K be a Θ-end in $\mathsf{Cob}^{\partial}_{\Theta}$ such that $\partial K|_0 \neq \emptyset$ and pick a triad handle decomposition of $(K|_0, \partial K|_0, \emptyset)$ relative to $\emptyset$ with only left and interior handles, which is possible by Proposition 2.3.5. Let L be the union of all handles of index at most $n-1$. This is a Θ-triad by restricting. By isotoping the embedding $(K|_0, \partial K|_0) \hookrightarrow (\mathbf{R}^\infty_+, \partial \mathbf{R}^\infty_+)$, we can assume that $L = K|_0 \cap ([0, +\infty) \times (-\infty, 0] \times \mathbf{R}^{\infty-2})$. Therefore, we can assume $K|_0 \in \mathsf{Cob}^{\partial}_{\Theta,L}$. Note that, $(L, \partial^h L)$ is built from

$(\partial^v L, \partial^{hv} L)$ by attaching right and interior handles of index at least $n+1$. Since $n \geq 2$, we see from (I) and (III) from Section 2.3.2 that $\partial^{hv} L \longrightarrow \partial^h L$ and $\partial^v L \longrightarrow L$ are fundamental groupoid isomorphisms. Hence, $(L, \partial^h L)$ satisfies the hypothesis of Proposition 6.2.1, since $\partial^{hv} L \neq \emptyset$ as $\partial K|_0 \neq \emptyset$.

The following lemma relates the notion of a Θ-end in $\mathsf{Cob}^{\partial}_{\Theta}$ to a Θ-end in $\mathsf{Cob}^{\partial}_{\Theta, \partial^v L}$. This will allow us to use Proposition 6.2.1 for our specific choice of L to deduce our main result.

Lemma 6.3.2. *The embedding $K \hookrightarrow [0,\infty) \times [0,1) \times (-1,1)^{\infty-2}$ may be isotoped, along with a bundle homotopy of its Θ-structure, such that $K \cap ([0,\infty) \times [0,1) \times (-\infty, 0] \times \mathbf{R}^{\infty-2}) = [0,\infty) \times L$ relative to $K|_0$ as a Θ-manifold. Moreover, K° is a Θ-end in $\mathsf{Cob}^{\partial, n-1}_{\Theta, \partial^v L}$.*

Proof. We start by showing that any embedding $\{i\} \times L \hookrightarrow K|_i$ can be extended to an embedding $[i, i+1] \times L \hookrightarrow K|_{[i,i+1]}$. By Proposition 2.3.17, we can find a triad handle decomposition of $K|_{[i,i+1]}$ relative to $K|_{i+1}$ with right handles of index at least n and interior handles of index at least $n+1$. This induces a triad handle decomposition of $K|_{[i,i+1]}$ relative to $K|_i$ with left handles of index at most $n+1$ and interior handles of index at most n. One can extend the embedding of L if its image can be made disjoint from the attaching maps of such handles. It suffices to make the cores of L disjoint from such attaching maps (since L is isotopy equivalent to the union of its cores). The attaching maps of the left and interior handles are submanifolds of dimension at most n and the cores of all handles in $\partial^h L$ have dimension at most $n-1$. Since $n + (n-1) < 2n$, we can isotope the attaching maps of the left and interior handles in $K|_i$ to be disjoint from all cores in $\partial^h L$ by transversality. By iterating this procedure, we have an embedding of $e : [0,\infty) \times L \hookrightarrow K$ relative to $K|_0$. It suffices to prove that we can isotope K in $[0,\infty) \times [0,1) \times (-1,1)^{\infty-2}$ to have the claimed property. This follows from the isotopy extension theorem: We can isotope (as maps of pairs) the image of $[0,\infty) \times L \hookrightarrow K \hookrightarrow [0,\infty) \times [0,1) \times (-1,1)^{\infty-2}$ to be the inclusion (recall that $[0,\infty) \times L$ is a submanifold of the target). By the isotopy extension theorem, we can extend such an isotopy to an isotopy of K. We can improve such an isotopy to be one of Θ-triads by choosing a bundle homotopy of the Θ-structure $e^* \ell|_K$ to $\ell|_{[0,\infty) \times L}$ (which exists since both are just extensions of $\ell|_L$). This is possible, since the embedding of L is a Hurewicz cofibration. We finish by proving that $K^\circ = K \backslash [0,\infty) \times L$ is a Θ-end in $\mathsf{Cob}^{\partial, n-1}_{\Theta, \partial^v L}$. The first condition is verified by considering the following pushout square

$$\begin{array}{ccc} (M^\circ, \partial^h M^\circ) & \longrightarrow & (M, \partial M) \\ \downarrow & & \downarrow \\ (W^\circ, \partial^h W^\circ) & \longrightarrow & (W, \partial^h W) \end{array}$$

for any morphism $W : M \rightsquigarrow M'$ in $\mathsf{Cob}^{\partial,n-1}_{\Theta,L}$. The top map is a pushout of the map $(\partial^v L, \partial^{hv} L) \to (L, \partial^h L)$ (see Lemma 6.2.6), which is an isomorphism on fundamental groupoids by assumption, and thus an isomorphism on fundamental groupoids, by the Seifert–Van Kampen theorem. Thus, by Lemma 2.1.4, if the right vertical map is strongly $(n-1)$-connected than so is the left one. Once again by transversality, the embedded copy of V_1 in $K|_{[i,i+1]}$ (which can be seen as trivially attached pair of left $(n+1)$ and right n-handle) can be made disjoint from $[i, i+1] \times L$ for each $i \geq 0$. This verifies the second condition and finishes the proof. □

We are now equipped for the proof of the main result of this section.

Proof of Theorem 6.0.2. Pick L as in Construction 6.3.1. An analogous contruction to [GR17b, Lemma 2.17] produces an equivalence $\mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(\overline{L}, K|^{\circ}_{\infty}) \to \mathscr{N}^{\partial}_{\Theta,n}(K|_{\infty})$ (and thus the former is non-empty) and a commutative diagram

$$\begin{array}{ccc} \mathsf{Cob}^{\partial}_{\Theta,\partial^v L,n}(\overline{L}, K|^{\circ}_{\infty}) & \longrightarrow & \Omega_{[\overline{L},K|^{\circ}_{\infty}]}\mathsf{BCob}^{\partial}_{\Theta,\partial^v L} \\ \downarrow & & \downarrow \\ \mathscr{N}^{\partial}_{\Theta,n}(K|_{\infty}) & \longrightarrow & \Omega_{[\emptyset,K|_{\infty}]}\mathsf{BCob}^{\partial}_{\Theta} \end{array}.$$

The proof of this statement follows analogously to loc.cit. Here $\overline{L}$ denotes the reflection of L along the hyperplane $[0,1) \times \{0\} \times (-\infty,\infty)^{\infty-2}$ (recall the definition of L above Definition 3.1.7). The vertical compositions are equivalences by once again the analog of [GR17b, Lemma 2.17] and Proposition 3.1.8. The top map is acyclic by Proposition 6.2.1, thus so is the bottom map. □

6.4. Stable diffeomorphism classification.

In this subsection, we prove Corollary C on the classification of manifold triads up to stable diffeomorphism using Theorem 6.0.2. This result is akin to Kreck's seminal work [Kre99] on stable diffeomorphism classification of even-dimensional manifolds with boundary. We now recall the setting of Corollary C: Let $n \geq 3$ be an integer, P be a compact $2n$-manifold with boundary, and $(N_i, \partial^h N_i, \partial^v N_i)$ be $(2n+1)$-dimensional manifold triads (see Section 2.3) for $i = 0, 1$, together with an identification of $\partial^v N_i$ with P. Assume also that $\partial^h N_i$ is connected for $i = 0, 1$.

Proof of Corollary C. We start with the "only if"-direction, namely, we assume that N_0 and N_1 are stably diffeomorphic relative to P and fix $g \geq 0$ and a diffeomorphism $\hat{f}$ between the triads $N_i \natural V_g$ for $i = 0, 1$ as in *(a)* from the introduction. The statement about relative Euler characteristics follows from the equality $\chi(N_i\natural V_g, \partial^h N_i\sharp W_g) = \chi(N_i, \partial^h N_i) + (-1)^{n+1}g$. We prove now the second statement. To define a stable normal n-type, take a Moore-Postnikov n-factorization of maps of pairs $\nu_0 = \Theta^\perp \circ \ell_0 : (N_0, \partial^h N_0) \to (B, B^\partial) \to (\mathrm{BO}, \mathrm{BO})$ in the sense of the introduction (see also Definition 7.1.1 below). By definition, the map ℓ_0 is strongly n-connected, so it suffices to produce a strongly n-connected map $\ell_1 : (N_1, \partial^h N_1) \to (B, B^\partial)$ to conclude that N_0 and N_1 have the same stable normal n-type. Take a Moore-Postnikov factorization of pairs $\nu_{N_0\natural V_g} = \Theta_g^\perp \circ l^g : (N_0\natural V_g, \partial^h N_0\sharp W_g) \to (B_g, B_g^\partial) \to (\mathrm{BO}, \mathrm{BO})$. One can check that there is an induced homotopy commutative diagram

$$\begin{array}{ccccc} (N_0, \partial^h N_0) & \xrightarrow{\ell_0} & (B, B^\partial) & \xrightarrow{\Theta^\perp} & (\mathrm{BO}, \mathrm{BO}) \\ \downarrow & & \downarrow b & & \downarrow \mathrm{id} \\ (N_0\natural V_g, \partial^h N_0\sharp W_g) & \xrightarrow{\ell^g} & (B_g, B_g^\partial) & \xrightarrow{\Theta_g^\perp} & (\mathrm{BO}, \mathrm{BO}) \end{array} .$$

We claim that b is a weak equivalence, which can be seen from the fact that the map $(*, *) \to (V_g, W_{g,1})$ is strongly $(n-1)$-connected, and that the pair $(V_g, W_{g,1})$ is stably parallelizable and thus the map of pairs to $(\mathrm{BO}, \mathrm{BO})$ classifying its stable normal bundle is nullhomotopic. Thus, without loss of generality, we can assume the map ℓ_0 factors through $(N_0\natural V_g, \partial^h N_0\sharp W_g)$. We define now ℓ_1 to be the composite of the inclusion $(N_1, \partial^h N_1) \hookrightarrow (N_1\natural V_g, \partial^h N_1\sharp W_g)$, the diffeomorphism $\hat{f}^{-1} : (N_1\natural V_g, \partial^h N_1\sharp W_g) \cong (N_0\natural V_g, \partial^h N_0\sharp W_g)$ and the map ℓ^g. It remains to check that ℓ_1 is strongly n-connected. This can be seen again by naturality of Moore-Postnikov factorizations of pairs and the fact that the map $(*, *) \to (V_g, W_{g,1})$ is strongly $(n-1)$-connected and that $(V_g, W_{g,1})$ is stably parallelizable. We leave this check to the reader. We conclude that N_0 and N_1 have the same stable normal n-type $\Theta^\perp$. Finally, to see that N_0 and N_1 admit bordant $\Theta^\perp$-smoothings, we will construct a $\Theta^\perp$-nullbordism of $(N_0 \cup_P N_1, \ell_0 \cup (-\ell_1))$. This is inspired by [CS11, Lemma 2.2]. Consider the $(2n+2)$-dimensional triad cobordism (recall Definition 2.3.1) M_i obtained from $N_i \times [0, 1]$ by attaching g-many trivial left $(n+1)$-handles (recall Figure 3), then its vertical boundary is diffeomorphic to the disjoint union of N_i and $N_i\natural V_g$: This can be seen by observing that $(V_1, W_{1,1})$ can be obtained from $(D_+^{2n+1}, \partial_0 D_+^{2n+1})$ by removing a neighborhood of a $(\partial_1 D_+^{n+1}, \partial_{01} D_+^{n+1})$ and gluing $(D_+^{n+1} \times S^n, \partial_0 D_+^{n+1})$. Let $\hat{M}$ be the triad bordism given by gluing M_0 and M_1 along $\hat{f} : N_0\natural V_g \to N_1\natural V_g$. This is a triad cobordism from N_0 to N_1, which we can see as triad whose vertical boundary is $N_0 \cup \partial_1 \hat{M} \cup N_1$, which is diffeomorphic to $N_0 \cup_P N_1$. Thus, we can see $\hat{M}$ as a nullbordism of $N_0 \cup_P N_1$. We proceed now to extend ℓ_0 to a $\Theta^\perp$-structure $\hat{\ell}$ on $\hat{M}$ such that $\hat{\ell}|_{N_1}$ is strongly n-connected. As the left handles of M_0 relative to N_0 are attached trivially, we can find a $\Theta^\perp$-structure ℓ_{M_0} on M_0 by extending ℓ_0 trivially along the core of the handle, which is possible as the core is parallelizable. We must now extend this structure from M_0 to $\hat{M}$. In other words, we must solve the following lifting problem of maps of pairs

$$\begin{array}{ccc} (M_0, \partial^h M_0) & \xrightarrow{\ell_{M_0}} & (B, B^\partial) \\ \downarrow & \overset{\hat{\ell}}{\nearrow} & \downarrow \Theta^\perp \\ (\hat{M}, \partial^h \hat{M}) & \xrightarrow{\nu_{\hat{M}}} & (\mathrm{BO}, \mathrm{BO}) \end{array} .$$

To do so, observe that $(\hat{M}, \partial^h \hat{M})$ is obtained from $(M_0, \partial^h M_0)$ by attaching g-many right $(n+1)$-handles, since M_1 is obtained from N_1 by attaching g-many left $(n+1)$-handles, by Lemma 2.3.8. In particular, we have $\hat{M}$ is equivalent to the union $M_0 \cup_{\partial^h M_0} \partial^h \hat{M}$ (see (I) from Section 2.3.2), and so it suffices to solve th lifting problem of maps of spaces given by restriction of the problem above of pairs to the sources of the pairs involved. By definition, the map $(\theta^\perp)^\partial : B^\partial \to \mathrm{BO}$ is n-coconnected. Since $\partial^h M_0 \to \partial^h \hat{M}$ is an n-connected map of CW complexes, we see that such a lift exists by obstruction theory. Moreover by (I) and (II) from Section 2.3.2, the map $(N_i, \partial^h N_i) \hookrightarrow (\hat{M}, \partial^h \hat{M})$ is strongly $(n-1)$-connected. We show now that $\hat{\ell}|_{N_1}$ is strongly n-connected. Since ℓ_0 is strongly n-connected, we deduce that $\hat{\ell}$ is strongly n-connected by Lemma 2.1.3. We conclude that $\hat{\ell}|_{M_1}$ is also strongly n-connected, since the map $(M_1, \partial^h M_1) \to (\hat{M}, \partial^h \hat{M})$ is strongly n-connected. On the other hand, we have $M_1 \simeq N_1$ and $\partial^h M_1 \simeq N_1 \vee (S^n)^{\vee g}$ by (II) in Section 2.3.2. Since the map $B^\partial \to \mathrm{BO}$ is n-coconnected and $\nu_1|_{(S^n)^{\vee g}}$ is nullhomotopic, we deduce $\hat{\ell}|_{(S^n)^{\vee g}}$ is nullhomotopic. Thus, we conclude that $\partial N_1 \to B^\partial$ is n-connected. It remains to show that $N_1 \cup_{\partial N_1} B^\partial \to B$ is $(n+1)$-connected. Since $\hat{\ell}|_{M_1}$ is also strongly n-connected, we see that $N_1 \cup_{\partial N_1 \vee (S^n)^{\vee g}} B^\partial \to B$ is $(n+1)$-connected. However, the target of the latter map is equivalent to $(N_1 \cup_{\partial N_1} B^\partial) \vee (S^{n+1})^{\vee g}$ and the restriction of the map to $(S^{n+1})^{\vee g}$ is nullhomotopic since it is once post-composed with $\theta^\perp$, which is an n-coconnected map. We conclude that $N_1 \cup_{\partial N_1} B^\partial \to B$ is $(n+1)$-connected, hence finishing the proof that N_0 and N_1 admit bordant $\Theta^\perp$-smoothings. This establishes the "only if"-direction.

We move now to the "if"-direction. We start by proving the case $\partial^{hv} N_i \neq \emptyset$ and later deduce the general case from the special case. Start by choosing maps of pairs $\Theta^\perp : (B, B^\partial) \to (\mathrm{BO}, \mathrm{BO})$ as in *(b)* from the introduction, a strongly n-connected $\ell_i : (N_i, \partial^h N_i) \to (B, B^\partial)$ lift of ν_i as in *(c)* from the introduction such that $N_0 \cup_P (-N_1)$ is $\Theta^\perp$-nullbordant. Fix an embedding of $(P, \partial P)$ into $(\mathbf{R}^\infty_+, \partial \mathbf{R}^\infty_+)$. Let $\Theta : (B', (B')^\partial) \to (\mathrm{BO}(2n+1), \mathrm{BO}(2n))$ be the map given by pulling back $\iota \circ \Theta^\perp$ along the stabilization map $(\mathrm{BO}(2n+1), \mathrm{BO}(2n)) \to (\mathrm{BO}, \mathrm{BO})$, where $\iota : \mathrm{BO} \to \mathrm{BO}$ is the map induced by taking a matrix to its inverse. Observe that $\iota \circ \nu_i$ is the classifying map for the stable tangent bundle of $(N_i, \partial^h N_i)$. Thus, ℓ_i lifts uniquely (up to homotopy) along $(B', (B')^\partial) \to (B, B^\partial)$ to a map $\ell_i' : (N_i, \partial^h N_i) \to (B', (B')^\partial)$, as the stable tangent classifier lifts to the unstable tangent classifier. Observe that ℓ_i' is also strongly n-connected since the map $(\mathrm{BO}(2n+1), \mathrm{BO}(2n)) \to (\mathrm{BO}, \mathrm{BO})$ is strongly $(2n-1)$-connected and ι is an equivalence. Therefore, by choosing embeddings of N_0 and N_1 into $(-\infty,0]\times\mathbf{R}^\infty_+$ extending the embedding of $(P,\partial P)$ (using the fixed identifications), we can consider $(N_i,\ell_i')\in\mathscr{N}^\partial_{\Theta,n}(P)$ as in Theorem 6.0.2. Consider a Θ-end K in the sense of Definition 6.0.1 given by taking connect sums with V_1 starting in $P\times[0,1]$. In particular, $K|_0=P$. Then, one can check that N_0 and N_1 are stably diffeomorphic if the classes $[N_0,\ell_0']$ and $[N_1,\ell_1']$ in $\pi_0(\mathscr{N}^\partial_{\Theta,n}(K|_\infty))$ agree. We proceed now to check the latter condition. By Theorem 6.0.2, this condition is equivalent to these classes agreeing in $\pi_0(\Omega_{[\emptyset,\partial^v N_0]}\mathrm{BCob}^\partial_\Theta)$ after applying the map present in loc.cit. Observe that we can consider $(N_1,-\ell_1')\in\mathsf{Cob}^\partial_\Theta(\partial^v N_0,\emptyset)$ for the fixed map ℓ_1' as in (c) from the introduction, and concatenation by the path induced by $(N_1,-\ell_1')$ in $\mathrm{BCob}^\partial_\Theta$ induces an isomorphism $\pi_0(\Omega_{[\emptyset,P]}\mathrm{BCob}^\partial_\Theta)\cong\pi_0(\Omega_{[\emptyset,\emptyset]}\mathrm{BCob}^\partial_\Theta)$. By applying [Gen11, Main Cor. 4.6] to π_1, there is an isomorphism $\pi_0(\Omega_{[\emptyset,\emptyset]}\mathrm{BCob}^\partial_\Theta)\cong\pi_0(\mathbf{MT}\Theta)$ where $\mathbf{MT}\Theta$ is the cofiber of the canonical map of Thom spectra $\Sigma^{-1}\mathbf{MT}\theta^\partial\to\mathbf{MT}\theta$ (recall the definition from the introduction). On the other hand, there is a stabilization map $s:\mathbf{MT}\Theta\to\Sigma^{-2n-1}\mathbf{M}\Theta^\perp$ (see proof of Lemma 6.4.1 below for more details). One can check that the composite

$$\pi_0(\mathscr{N}^\partial_{\Theta,n}(K|_\infty))\xrightarrow{\cong}\pi_0(\Omega_{[\emptyset,\partial^v N_0]}\mathrm{BCob}^\partial_\Theta)\xrightarrow{\cong}\pi_0(\Omega_{[\emptyset,\emptyset]}\mathrm{BCob}^\partial_\Theta)\xrightarrow{\cong}\pi_0(\mathbf{MT}\Theta)\xrightarrow{\pi_0(s)}\pi_{2n+1}(\mathbf{M}\Theta^\perp)\cong\Omega^{\Theta^\perp}_{2n+1}$$

takes the class $[N_i,\ell_i']$ to the relative bordism class of $[N_i\cup_P N_1,\ell_i'\cup(-\ell_1')]$. By hypothesis, the image of N_0 under this composite vanishes. On the other hand, the image of N_1 is $N_1\cup_P(-N_1)$, which vanishes by definition (by taking $N_1\times[0,1]$). Hence, the images of N_0 and N_1 under this composite agree. We conclude that the classes $[N_0,\ell_0']$ and $[N_1,\ell_1']$ agree on $\pi_0(\mathscr{N}^\partial_{\Theta,n}(K|_\infty))$ by the following fact (which we prove separately below): The map $\pi_0(s)\times\chi^{\mathrm{rel}}:\pi_0(\mathbf{MT}\Theta)\to\Omega^{\Theta^\perp}_{2n+1}\times\mathbf{Z}$ is injective, where χ^{rel} takes a Θ-bordism class of manifolds with boundary to its relative Euler characteristic (one can check that the relative Euler characteristic is bordism invariant). This finishes the proof of the "if"-direction for the case $\partial^{hv}N_i\neq\emptyset$.

We finish by deducing the general case from this special case. Assume $\partial^{hv}N_i=\emptyset$ and let N_i° be the triad $(N_i,\partial N_i\backslash D_i^\circ,D_i)$ where D_i is a codimension 0 disc in ∂N_i. Observe that the hypothesis implies that $\chi(N_0,\partial^h N_0^\circ)=\chi(N_1,\partial^h N_0^\circ)$, since $\chi(N_i,\partial^h N_i^\circ)=\chi(N_i,\partial N_i)+1$. Similarly, the pairs $(N_i,\partial N_i)$ and $(N_i,\partial^h N_i^\circ)$ have the same stable normal n-type $\Theta^\perp$, as the inclusion $\partial^h N_i^\circ\hookrightarrow\partial N_i$ is $(2n-1)$-connected. Moreover, we know that N_0 and N_1 admit bordant Θ-smoothings, so there exists ℓ_i and W as in (c) from the introduction (which now has empty vertical boundary) such that N_0 and N_1 agree in $\Omega^{\Theta^\perp}_{2n+1}$. Let W be a $\Theta^\perp$-bordism between N_0 and N_1. We can assume that there is a path in $\partial^h W$ between a point in D_0 and a point in D_1 by the following argument: since ℓ_i is strongly n-connected and $\partial^h N_i$ is connected, we know that $\ell_i|_{\partial^h N_i}$ hit the same path component of B^∂. Take the triad W and attach a right 1-handle along the horizontal boundary in the path components hit by D_0 and D_1. Denote the result of this procedure by W'. Since

$\partial^h N_0$ and $\partial^h N_1$ hit the same component of B^∂, we can extend the $\Theta^\perp$-structure from W to W', making the latter into a $\Theta^\perp$-bordism between N_0 and N_1, which satisfies this extra assumption. One can consider a thickening of that path to define a $\Theta^\perp$-nullbordism of $N_0^\circ \cup_D N_1^\circ$. We leave this step to the reader. Thus, we see that $N_0^\circ \cup_D N_1^\circ$ vanishes in this bordism group. Hence, N_0° and N_1° admit bordant $\Theta^\perp$-smoothings. Thus, by the special case, we see that N_0° and N_1° are stably diffeomorphic. Since this diffeomorphism extends the chosen identification of their vertical boundaries, we see that it extends to a stable diffeomorphism between N_0 and N_1. Thus, we have established the general case and hence finished the proof. ◻

Lemma 6.4.1. *The map $\pi_0(s) \times \chi^{rel} : \pi_0(\mathbf{MT}\Theta) \to \Omega_{2n+1}^{\Theta^\perp} \times \mathbf{Z}$ is injective.*

Proof. This is inspired by [Kra19, Lemma 1.4]. Let us start by defining the map s in detail. Recall that we have the map $\Theta = (\theta, \theta^\partial) : (B', (B')^\partial) \to (\mathrm{BO}(2n+1), \mathrm{BO}(2n))$ which is obtained by pulling back $\iota \circ \Theta^\perp$ along the stabilization map $(\mathrm{BO}(2n+1), \mathrm{BO}(2n)) \to (\mathrm{BO}, \mathrm{BO})$. Thus, the map $B^\partial \to \mathrm{BO}(2n+1) \to \mathrm{BO}$ is the classifying map of the stable vector bundle $-(\theta^\partial)^*\gamma$ of rank $-2n$. Applying the Thom spectrum functor to this classifying map produces a map $\mathbf{MT}\theta^\partial \to \Sigma^{-2n}\mathbf{M}(\theta^\perp)^\partial$, where $(\theta^\perp)^\partial \coloneqq \Theta^\perp|_{B^\partial}$. Similarly, we have a map $\mathbf{MT}\theta \to \Sigma^{-2n-1}\mathbf{M}\theta^\perp$ given by the Thomified classifying map of the stable vector bundle $-\theta^*\gamma$. The map $s : \mathbf{MT}\Theta \to \Sigma^{-2n-1}\mathbf{M}\Theta^\perp$ is defined as the induced map on cofibers between the maps $\Sigma^{-1}\mathbf{MT}\theta^\partial \to \mathbf{MT}\theta$ and $\Sigma^{-2n-1}\mathbf{M}(\theta^\perp)^\partial \to \Sigma^{-2n-1}\mathbf{M}\theta^\perp$. Let $\Theta_{2n+2} = (\theta_{2n+2}, \theta^\partial_{2n+2}) : (B'', (B'')^\partial) \to (\mathrm{BO}(2n+2), \mathrm{BO}(2n+1))$ be the map obtained by pulling back $\iota \circ \Theta^\perp$ along the stabilization map $(\mathrm{BO}(2n+2), \mathrm{BO}(2n+1)) \to (\mathrm{BO}, \mathrm{BO})$. Then, clearly the map s factors as the composite $\mathbf{MT}\Theta \to \Sigma\,\mathbf{MT}\Theta_{2n+2} \to \Sigma^{-2n-1}\mathbf{M}\Theta^\perp$. The map $\Sigma\,\mathbf{MT}\Theta_{2n+2} \to \Sigma^{-2n-1}\mathbf{M}\Theta^\perp$ can be seen to be 1-connected and thus an isomorphism on π_0 (this follows since the maps $\mathbf{MT}\theta^\partial_{2n+1} \to \Sigma^{-2n-1}\mathbf{M}(\theta^\perp)^\partial$ and $\mathbf{MT}\theta_{2n+2} \to \Sigma^{-2n-2}\mathbf{M}(\theta^\perp)^\partial$ are 0-connected, see [Kra19, p. 5]). Thus, it suffices to prove that the map $\pi_0(s) \times \chi^{\mathrm{rel}} : \pi_0(\mathbf{MT}\Theta) \to \pi_{-1}(\mathbf{MT}\Theta_{2n+2}) \times \mathbf{Z}$ is injective. We shall now study the fiber $\mathbf{MT}\Xi \coloneqq \mathrm{fib}(\mathbf{MT}\Theta \to \Sigma\,\mathbf{MT}\Theta_{2n+2})$. The fibers of the maps $\mathbf{MT}\theta \to \Sigma\,\mathbf{MT}\theta_{2n+2}$ and $\mathbf{MT}\theta^\partial \to \Sigma\mathbf{MT}\theta^\partial_{2n+1}$ are equivalent to $\Sigma^\infty_+ B''$ and $\Sigma^\infty_+(B'')^\partial$, respectively (see [Gol16, Prop. 3.0.7]). Thus, we have a cofiber sequence of spectra $\Sigma^\infty_+ B'' \to \mathbf{MT}\Xi \to \Sigma^\infty_+(B'')^\partial$. We conclude that there is an isomorphism $\pi_0(\mathbf{MT}\Xi) \cong \pi_0(\Sigma^\infty_+(B'')^\partial) \oplus \mathrm{im}(\pi_0(\Sigma^\infty_+ B'') \to \pi_0(\mathbf{MT}\Xi))$. One checks that B'' and $(B'')^\partial$ are path-connected and thus $\pi_0(\Sigma^\infty_+ B'') \cong \mathbf{Z} \cong \pi_0(\Sigma^\infty_+(B'')^\partial)$. We proceed by showing the following claims:

(*a*) *The composite $\pi_0(\Sigma^\infty_+ B'') \to \pi_0(\mathbf{MT}\Xi) \to \pi_0(\mathbf{MT}\Theta)$ is zero*: We start by producing a splitting of the cofiber sequence $\Sigma^\infty_+ B'' \to \mathbf{MT}\Xi \to \Sigma^\infty_+(B'')^\partial$ at the level of π_0. To do so, we start by observing that the group $\pi_0(\mathbf{MT}\Xi)$ has the following geometric description: It is isomorphic to the bordism group of triples $[M, E, \phi]$ where:

1. $M = (M, \partial^h M, \partial^v M)$ is a $(2n+2)$-dimensional triad (in the sense of Section 2.3);
2. $E = (E, E^\partial)$ is a $(2n+2)$-dimensional collared vector bundle pair over $(M, \partial^h M)$ together with a collared bundle map $(E, E^\partial) \longrightarrow ((\theta_{2n+2})^*\gamma_{2n+2}, (\theta^\partial_{2n+2})^*\gamma_{2n+1})$ (in the sense of Definition 2.2.2):
3. ϕ is a stable isomorphism of stable vector bundle pairs $(TM, T\partial^h M) \cong (E, E^\partial)$.

Such a triple is *nullbordant* if there exists a $(2n+3)$-dimensional 4-ad $(W, \partial_0 W, \partial_1 W, \partial_2 W)$ (recall Definition 2.3.1) such that $(\partial_2 W, \partial_{02} W, \partial_{12} W) = (M, \partial^h M, \partial^v M)$, a $(2n+2)$-dimensional collared vector bundle pair (F, F^∂) over $(W, \partial_0 W)$ which restricts to (E, E^∂) on $(M, \partial^h M)$, and a stable isomorphism $(TW, T\partial_0 W) \cong (F, F^\partial) \oplus \varepsilon^1$ compatible with ϕ. One sees this by combining [Lau00, Thm. 3.1.5] with the observation that $\mathbf{MT}\Xi$ is equivalent to the total cofiber of the following commutative square of spectra

$$\begin{array}{ccc} \Sigma^{-2}\mathbf{MT}\theta^\partial & \longrightarrow & \Sigma^{-1}\mathbf{MT}\theta^\partial_{2n+2} \\ \downarrow & & \downarrow \\ \Sigma^{-1}\,\mathbf{MT}\theta & \longrightarrow & \mathbf{MT}\theta_{2n+2} \end{array}$$

On the other hand, as $\Sigma^\infty_+ B''$ is equivalent to the cofiber of $\Sigma^{-1}\,\mathbf{MT}\theta \to \mathbf{MT}\theta_{2n+2}$, we see that $\pi_0(\Sigma^\infty_+ B'')$ is isomorphic to the bordism group of triples $[M', E', \phi']$ where M' is a $(2n+2)$-dimensional compact manifold with boundary, E' is a $(2n+2)$-dimensional vector bundle over M together with a bundle map $E' \longrightarrow (\theta_{2n+2})^*\gamma_{2n+2}$, and $\phi' : TM \cong E'$ is a stable isomorphism of bundles. Under these geometric desciptions, the map $\pi_0(\Sigma^\infty_+ B'') \longrightarrow \pi_0(\mathbf{MT}\Xi)$ takes $[M', E', \phi']$ to the triple $[(M, \emptyset, \partial M), (E', \emptyset), \phi']$. This map admits a splitting, namely the map which takes a triple $[M, E, \phi] \in \pi_0(\mathbf{MT}\Xi)$ to the triple $[M, E|_M, \phi|_M]$, where one forgets the triad structure on M. We conclude that we have an isomorphism $\pi_0(\mathbf{MT}\Xi) \cong \pi_0(\Sigma^\infty_+ B'') \oplus \pi_0(\Sigma^\infty_+ (B'')^\partial)$. We now return to proving the initial claim. For that, we show that a generator $a \in \pi_0(\Sigma^\infty_+ B'') \subset \pi_0(\mathbf{MT}\Xi)$ is hit by the map $\pi_0(\mathbf{MT}\Theta_{2n+2}) \longrightarrow \pi_0(\mathbf{MT}\Xi)$, and hence maps to zero in $\pi_0(\mathbf{MT}\Theta)$ by exactness. Let $x \in \pi_0(\mathbf{MT}\Theta_{2n+2})$ be a class representing the triple $[D^{2n+2}, TD^{2n+2}, \phi]$ where $(TD^{2n+2}, T\partial D^{2n+2})$ is seen with some Θ_{2n+2}-structure. First, we show that x maps to $a + b \in \pi_0(\mathbf{MT}\Xi)$ where $a \in \pi_0(\Sigma^\infty_+ B'')$ is a generator and $b \in \pi_0(\Sigma^\infty_+ (B'')^\partial)$: Consider the map $\chi^{\mathrm{rel}} : \pi_0(\Sigma^\infty_+ B'') \longrightarrow \mathbf{Z}$ which takes $[M', E', \phi']$ to $\chi(M', \partial M')$ and observe that it is surjective and thus an isomorphism, as B'' is path-connected. However, the image a of x in $\pi_0(\Sigma^\infty_+ B'')$ hits 1 along χ^{rel}. Hence, a is a generator of $\pi_0(\Sigma^\infty_+ B'')$. Second, we prove that the image $b \in$

$\pi_0(\Sigma^\infty_+(B'')^\partial)$ vanishes: Observe that the map $\pi_0(\Sigma^\infty_+(B'')^\partial) \to \pi_0(\mathbf{MT}\theta^\partial)$ is injective by the proof of [Kra19, Lemma 1.4] (see also below in the proof of (b)). However, the class b maps to 0 along the latter map as it is in the image of $\pi_0(\mathbf{MT}\theta^\partial_{2n+1})$, by definition. Thus, b vanishes. We conclude that $x \in \pi_0(\mathbf{MT}\Theta_{2n+2})$ hits a generator of the summand $\pi_0(\Sigma^\infty_+ B'')$ in $\pi_0(\mathbf{MT}\Xi)$ and hence the map $\pi_0(\Sigma^\infty_+ B'') \to \pi_0(\mathbf{MT}\Xi) \to \pi_0(\mathbf{MT}\Theta)$ is zero.

(b) *The composite* $\mathbf{Z} \cong \pi_0(\Sigma^\infty_+(B'')^\partial) \hookrightarrow \pi_0(\mathbf{MT}\Xi) \to \pi_0(\mathbf{MT}\Theta) \xrightarrow{\chi^{\text{rel}}} \mathbf{Z}$ *is an isomorphism*: Consider the following diagram

$$\begin{array}{ccccc} \pi_0(\mathbf{MT}\Xi) & \longrightarrow & \pi_0(\mathbf{MT}\Theta) & \xrightarrow{\chi^{\text{rel}}} & \mathbf{Z} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle 2\cdot} \\ \mathbf{Z} \cong \pi_0(\Sigma^\infty_+(B'')^\partial) & \longrightarrow & \pi_0(\mathbf{MT}\theta^\partial) & \xrightarrow{\chi} & \mathbf{Z} \end{array}$$

which commutes, since given a $(2n+1)$-manifold with boundary M, then $\chi(\partial M) = 2\chi(M, \partial M)$. By the proof of [Kra19, Lemma 1.4], we can observe that the bottom composition is the map $2 \cdot (-) : \mathbf{Z} \to \mathbf{Z}$. Thus, we conclude that the generator of inclusion of the summand $\pi_0(\Sigma^\infty_+(B'')^\partial)$ hits $1 \in \mathbf{Z}$ along the top composite, since its image along the right vertical map is 2. This finishes the claim.

Using (a) and (b), we finish the proof of this claim. Recall that it suffices to prove that $\pi_0(s) \times \chi^{\text{rel}} : \pi_0(\mathbf{MT}\Theta) \to \pi_{-1}(\mathbf{MT}\Theta_{2n+2}) \times \mathbf{Z}$ is injective. By exactness, the kernel of $\pi_0(s)$ is the image of the map $\pi_0(\mathbf{MT}\Xi) \to \pi_0(\mathbf{MT}\Theta)$. By (a), this image equals the image of the summand $\pi_0(\Sigma^\infty_+(B'')^\partial)$. By (b), this image is detected by χ^{rel}. Hence, we conclude that $\pi_0(s) \times \chi^{\text{rel}}$ is injective. □

Remark 6.4.2. Corollary C should be seen as an analog of [Kre99, Thm. C] for odd-dimensional triads. Although in loc.cit. only the "if"-direction is stated, the converse direction holds as well (see [CS11, Lemma 2.2]). A similar proof as above recovers [Kre99, Thm. C] and its converse from the work of Galatius and Randal-Williams [GR17b, Thm. 1.5] (see [Kra19, Lemma 1.4] for the analog of Lemma 6.4.1).

7. General tangential structures.

In this section, we state and prove our main theorem. We start by defining the main object of study. Let $n \geq 3$ and $(N, \partial^h N, \partial^v N)$ be a $(2n+1)$-dimensional manifold triad (see Section 2.3) and $\Theta = (\theta, \theta^\partial) : (B, B^\partial) \to (\mathrm{BO}(2n+1), \mathrm{BO}(2n))$ be a map of pairs. Recall that we fix an inwards-pointing vector field on $\partial^h N$ inducing a collar on the vector bundle pair $(TN, T\partial^h N)$. On the other hand, the canonical collar on $(\gamma_{2n+1}, \gamma_{2n})$ induces a collar on $(\theta^*\gamma_{2n+1}, (\theta^\partial)^*\gamma_{2n})$ (see discussion below Remark 2.2.3). Given a collared bundle map $\ell^v : (TN|_{\partial^v N}, T\partial^h N|_{\partial^{hv}N}) \to (\theta^*\gamma_{2n+1}, (\theta^\partial)^*\gamma_{2n})$ and a smooth embedding of pairs $e^v : (\partial^v N, \partial^{hv} N) \hookrightarrow (\{0\} \times \mathbf{R}^\infty_+, \{0\} \times \partial\mathbf{R}^\infty_+)$ (and a collared vector bundle map $(T\partial^v N \oplus \varepsilon^1, T\partial^h N \oplus \varepsilon^1) \to (TN|_{\partial^v N}, T\partial^h N|_{\partial^{hv}N})$ over the identity), we can see $(\partial^v N, \ell^v)$ as an object in $\mathsf{Cob}^\partial_\Theta$.

Definition 7.0.1 (Moduli space of Θ-structures)**.** Given the definitions above, we denote by $\mathscr{M}^\partial_\Theta(N; \ell^v)$ the path components of $\mathscr{N}^\partial_\Theta(\partial^v N) := \mathsf{Cob}^\partial_\Theta(\emptyset, \partial^v N)$ of all morphisms whose underlying submanifold pair is abstractly diffeomorphic to $(N, \partial^h N)$ relative to $\partial^v N$.

Remark 7.0.2. One can check that the space $\mathscr{M}^\partial_\Theta(N; \ell^v)$ is weakly equivalent to the balanced product

$$\mathrm{Emb}_{\partial^v}(N, (-\infty, 0] \times \mathbf{R}^\infty_+) \times_{\mathrm{Diff}_{\partial^v}(N)} \mathrm{Bun}^{\mathrm{col}}_{\partial^v}(TN, \Theta^*\gamma_{2n+1})$$

of the actions of $\mathrm{Diff}_{\partial^v}(N)$ on the space of embeddings of pairs $e : (N, \partial^h N) \hookrightarrow ((-\infty, 0] \times \mathbf{R}^\infty_+, (-\infty, 0] \times \partial\mathbf{R}^\infty_+)$ extending e^v (see Section 2.3) and on the space $\mathrm{Bun}^{\mathrm{col}}_{\partial^v}(TN, \Theta^*\gamma_{2n+1})$ of those collared bundle maps $(TN, T\partial^h N) \to (\theta^*\gamma_{2n+1}, (\theta^\partial)^*\gamma_{2n})$ extending ℓ^v (see Definition 2.2.2). The diffeomorphism group $\mathrm{Diff}_{\partial^v}(N)$ acts on the latter space by pre-composition with the differential of a diffeomorphism (which is a collared bundle map of pairs, see Section 2.3). We remark that the action of $\mathrm{Diff}_{\partial^v}(N)$ on $\mathrm{Emb}_{\partial^v}(N, (-\infty, 0] \times \mathbf{R}^\infty_+)$ is free and admits local cross-sections (by an analogous argument to [Pal60, Thm. B], see also [Ste21, Theorem A.1]). Moreover, it is contractible (see e.g. [Gen11, Thm 2.7] for a similar argument) and thus $\mathscr{M}^\partial_\Theta(N; \ell^v)$ is a model for the homotopy orbits $(\mathrm{Bun}^{\mathrm{col}}_{\partial^v}(TN, \Theta^*\gamma_{2n+1}))_{\mathrm{Diff}_{\partial^v}(N)}$. In particular, for $\Theta = \mathrm{id}_{\mathrm{BO}(2n+1)}$, then this is a model for the classifying space $\mathrm{BDiff}_{\partial^v}(N)$, by Remark 2.2.4.

By picking path components in $\partial^{hv} N$ (and assuming $\partial^{hv} N \neq \emptyset$), we have a morphism $({}_{\partial^v N}H, \ell^{\mathrm{std}}) : \partial^v N \rightsquigarrow (\partial^v N)'$ (see the definition above Proposition 4.3.20) in $\mathsf{Cob}^\partial_\Theta$ given by taking a triad connected sum of $\partial^v N \times [0, 1]$ with V_1 at $\partial^h N \times \{1\}$, where ℓ^{std} is chosen by requiring that the restriction to V_1 is standard (see Definition 4.1.4 for the definition). To make sense of the concept of standard Θ-structure, we need a basepoint Θ-structure (see Assumptions 4.1.1). In this context, we fix such a basepoint Θ-structure by requiring that the restriction of ℓ^v to a disc in the chosen component of $\partial^{hv} N$ is homotopic to it. With this choice, a Θ-structure ℓ^{std} with the above requirement exists by Lemma 4.1.5. By post-composing with this morphism, we have a map

$$\mathscr{M}^\partial_\Theta(N; \ell^v) \to \mathscr{M}^\partial_\Theta(N \cup_{\partial^v N} ({}_{\partial^v N}H); \ell^v_1),$$

where $\ell^v_1 := \ell^{\mathrm{std}}|_{(\partial^v N)'}$. By letting $N_0 := N$ and N_g be defined by the gluing $N_{g-1} \cup_{\partial^v N_{g-1}} ({}_{\partial^v N_{g-1}}H)$ and the corresponding Θ-structure on $\partial^v N_g$ be ℓ^v_g, we can define the following space

$$\mathscr{M}^\partial_\Theta(N_\infty; \ell^v) := \mathrm{hocolim}\Big(\cdots \to \mathscr{M}^\partial_\Theta(N_g; \ell^v_g) \to \mathscr{M}^\partial_\Theta(N_{g+1}; \ell^v_{g+1}) \to \cdots\Big).$$

Our main result describes the homology of the space above, assuming the pair $(N, \partial^h N)$ is 1-connected. Our first goal is to state our main result. To do so, we start by introducing the necessary concepts. After that, we proceed by assembling the results of the previous sections to prove this result. We finish this section by giving some examples of computations.

7.1. The main theorem.

We will need the following definitions before we can state our main result. It will be convenient to fix a model structure on the category of pairs of spaces. Consider the *projective model structure*[13] on the category of pairs of compactly generated spaces generated by the Quillen model structure on the category of spaces. Weak equivalences and fibrations in the category of pairs are taken objectwise and cofibrations are those maps $(A, A') \to (B, B')$ such that $A' \to B'$ and $A \cup_{A'} B' \to B$ are cofibrations, where here $A \cup_{A'} B'$ means the strict pushout.

Definition 7.1.1. Let $g : (Y, Y') \to (Z, Z')$ be a map of pairs and $k \geq 0$ an integer, we say g is *strongly k-coconnected* if $g|_{Y'}$ is k-coconnected (*i.e.* all its homotopy fibers are $(k-1)$-truncated) and if all its total homotopy fibers $\mathrm{tohofib}(f) \coloneqq \mathrm{hofib}(Y' \to Y \times_Z Z')$ are $(k-1)$-truncated. A factorization $(X, X') \to (Y, Y') \to (Z, Z')$ of $g \circ f$ is called a *Moore-Postnikov k-factorization* if f is a strongly k-connected cofibration and g is a strongly k-coconnected fibration.

Lemma 7.1.2. *Let $f : (X, X') \to (Z, Z')$ be a map of pairs and $k \geq 0$ an integer, then there exists a Moore-Postnikov k-factorization $u \circ l : (X, X') \to (Y, Y') \to (Z, Z')$.*

Proof. Consider a Moore-Postnikov k-factorization $u' \circ \ell' : X' \to Y' \to Z'$ of $f|_{X'}$. We can assume ℓ' is a cofibration and u' a fibration. We now define Y to fit in a Moore-Postnikov $(k+1)$-factorization of $X \cup_{X'} Y' \to Y \to Z$. Once again, we assume that the leftmost map is a cofibration and the right map a fibration. This produces a factorization $u \circ l : (X, X') \to (Y, Y') \to (Z, Z')$. By construction, ℓ is both a cofibration of pairs and strongly k-connected. One checks that u is strongly k-coconnected since $Y' \to Z'$ and $Y \to Z$ are k and $(k+1)$-coconnected, respectively. On the other hand, they are also fibrations. This finishes the construction. □

One can check that these factorizations are unique up to homotopy (either directly or as consequence of Proposition A.3). Given a cofibration $\iota : (X, X') \to (Y, Y')$ and a fibration $u : (Y, Y') \to (Z, Z')$, we denote by $\mathrm{Aut}_X(u)$ the space of

[13] Here, we mean the Reedy model structure on the category of functors from the poset $\mathcal{C} \coloneqq \{0 < 1\}$ to the category of spaces, where we take 0 to have degree 0 and 1 to have degree 1 and $\overrightarrow{\mathcal{C}} = \mathcal{C}$ and $\overleftarrow{\mathcal{C}}$ to be the identity morphisms. See [Hir09, Thm. 15.3.4].

self weak equivalences of (Y, Y') under ι and over u. Composition makes this space into a group-like topological monoid[14].

Let $n \geq 3$ and $(N, \partial^h N, \partial^v N)$ be a $(2n+1)$-dimensional manifold triad and $\Theta : (B, B^\partial) \to (\mathrm{BO}(2n+1), \mathrm{BO}(2n))$ be a map of pairs. Given a collared bundle map $\ell : (TN, T\partial^h N) \to (\theta^* \gamma_{2n+1}, (\theta^\partial)^* \gamma_{2n})$ and letting $\ell = u \circ \ell' : (N, \partial^h N) \to (B', B'^\partial) \to (B, B^\partial)$ be a Moore-Postnikov n-factorization of the underlying map of spaces, then the monoid $\mathrm{Aut}_{\partial^v N}(u)$ acts on the collared bundle $(\theta'^* \gamma_{2n+1}, (\theta'^\partial)^* \gamma_{2n})$ via collared bundle maps, where $\Theta' = (\theta', \theta'^\partial)$ is the composite $\Theta \circ u$.

Recall from the introduction that $\mathbf{MT}\Theta'$ denotes the cofiber of the canonical map of Thom spectra $\Sigma^{-1} \mathbf{MT}\, \theta'^\partial \simeq \mathbf{MT}(\theta \oplus \varepsilon^1) \to \mathbf{MT}\, \theta'$ induced by the collar. Then, by the discussion above, the monoid $\mathrm{Aut}_{\partial^v N}(u)$ acts on the spectrum $\mathbf{MT}\Theta'$. Finally, denote by $\mathscr{M}^\partial_\Theta(N; \ell^v)_\ell$ the path component of ℓ in $\mathscr{M}^\partial_\Theta(N; \ell^v)$, where $\ell^v := \ell|_{\partial^v N}$. If $\partial^{hv} N \neq \emptyset$ and after picking a path component of $\partial^{hv} N$, denote by $\mathscr{M}^\partial_\Theta(N_\infty; \ell^v)_\ell$ the homotopy colimit of the maps $\mathscr{M}^\partial_\Theta(N_g; \ell^v_g)_{\ell_g} \to \mathscr{M}^\partial_\Theta(N_{g+1}; \ell^v_{g+1})_{\ell_{g+1}}$, where ℓ_g is the gluing of the map ℓ with the maps $\ell^{\mathrm{std}}_{g'}$ on ${}_{\partial^v N_{g'}}H$ for all $g' \leq g$. We are now ready to state our main theorem.

Theorem A*. *Let $n \geq 3$. Let $\Theta : (B, B^\partial) \to (\mathrm{BO}(2n+1), \mathrm{BO}(2n))$ be a map of pairs, $(N, \partial^h N, \partial^v N)$ be a manifold triad such that N is connected, $(N, \partial^h N)$ is 1-connected and $\partial^{hv} N$ is non-empty, and $\ell : (TN, T\partial^h N) \to (\theta^* \gamma_{2n+1}, (\theta^\partial)^* \gamma_{2n})$ be a Θ-structure. Let $\ell = u \circ \ell' : (N, \partial^h N) \to (B', (B')^\partial) \to (B, B^\partial)$ be a Moore-Postnikov n-factorization, then there exists a map*

$$\mathscr{M}^\partial_\Theta(N_\infty; \ell^v)_\ell \to (\Omega^\infty \mathbf{MT}\Theta')_{\mathrm{Aut}_{\partial^v N}(u)}$$

which is acyclic onto the path component it hits, where $\Theta' = \Theta \circ u$.

By specializing this result to $\Theta = \mathrm{id}_{\mathrm{BO}(2n+1)}$ and by Remark 7.0.2, we conclude Theorem A. The rest of this section is dedicated to the proof of this statement and presenting some examples of potential interest.

Remark 7.1.3. In the statement above, the map $\mathscr{M}^\partial_\Theta(N_\infty; \ell^v)_\ell \to (\Omega^\infty \mathbf{MT}\Theta')_{\mathrm{Aut}_{\partial^v N}(u)}$ is more precisely a morphism in the category $\mathbf{Top}[w.e.^{-1}]$, that is, the localization of the category of spaces at weak equivalences. The property of a map being acyclic is invariant under weak equivalence, so it is a well-defined property for morphisms in $\mathbf{Top}[w.e.^{-1}]$. More concretely, we construct a span of spaces $\mathscr{M}^\partial_\Theta(N_\infty; \ell^v)_\ell \leftarrow X \to (\Omega^\infty \mathbf{MT}\Theta')_{\mathrm{Aut}_{\partial^v N}(u)}$ where the leftmost map is a weak equivalence and the rightmost map is acyclic.

[14] This monoid is group-like since any such weak equivalence is a homotopy equivalence. This follows from the general fact that weak equivalences between fibrant-cofibrant objects in a model category are homotopy equivalences (see [Hir09, Thm. 7.5.10]), and the fact that (Y, Y') is fibrant-cofibrant in the category of pairs under (X, X') and over (Z, Z') (see [Hir09, Thm. 7.6.5.(3)])

7.2. Proof of Theorem A*.

In this subsection, we assemble all the results from previous section into the proof of Theorem A*. This result will be a consequence of Theorem 6.0.2 by specializing to a chosen Θ'-end and by removing the condition of strong n-connectivity for the Θ-structure. Before we proceed, we fix the standing assumptions hidden behind Theorem A*.

Assumptions 7.2.1. As mentioned before, the data above Theorem A* allows us to see $(\partial^v N, \ell^v)$ as an object of $\mathsf{Cob}_\Theta^\partial$. Moreover, the collection of morphisms $\{(({}_{\partial^v N}H)_g, \ell_g^{\text{std}})\}_g$ is a Θ-end in the sense of Definition 6.0.1. We fix now further choices necessary to define the map in Theorem A*:

(I) A *basepoint* Θ'-structure on $(\mathbf{R}_+^{2n+1}, \partial\mathbf{R}_+^{2n+1})$ such that after composing with u it is homotopic to the basepoint Θ-structure; this is possible by taking the pullback of ℓ' along inclusion of the fixed disc in the chosen component of $\partial^{hv}N$;

(II) Choices of Θ'-structures $\ell_g'^{\text{ std}}$ on $({}_{\partial^v N}H)_g$ such that once restricted to V_1, they are standard (with respect to the basepoint Θ'-structure); this implies that $u \circ \ell_g'^{\text{ std}}$ is a standard Θ-structure once restricted to V_1 and thus by Lemma 4.1.5, it is homotopic to ℓ_g^{std} on V_1 and thus on $({}_{\partial^v N}H)_g$.

As before, the data above allows us to see $(\partial^v N; \ell'^v)$ as an object of $\mathsf{Cob}_{\Theta'}^\partial$. The collection $\{(({}_{\partial^v N}H)_g, \ell_g'^{\text{ std}})\}_g$ is then a Θ'-end.

7.2.1. Bundle maps.

In Section 6, we restricted ourselves to the study of the moduli spaces of Θ-manifolds where the underlying map of the Θ-structure is strongly n-connected. We start by describing the space where this condition is not assumed and relate it to a version of the former. Consider the fixed Moore-Postnikov factorization of maps of pairs

$$\Theta : (B', B'^\partial) \xrightarrow{(u,u^\partial)} (B, B^\partial) \xrightarrow{\Theta'} (\mathrm{BO}(2n+1), \mathrm{BO}(2n))$$

from Theorem A*. Recall that u is a fibration, by Definition 7.1.1. We have a collared map of vector bundle pairs $u : (\theta'^*\gamma_{2n+1}, (\theta'^\partial)^*\gamma_{2n}) \to (\theta^*\gamma_{2n+1}, \theta^*\gamma_{2n})$. Post-composition with u induces a map

$$\mathrm{Bun}^{\mathrm{col}}_{\partial^v,n}(TN, \Theta'^*\gamma) \to \mathrm{Bun}^{\mathrm{col}}_{\partial^v}(TN, \Theta^*\gamma),$$

where the source is the subspace of $\mathrm{Bun}^{\mathrm{col}}(TN, \Theta'^*\gamma)$ of those maps extending $\ell'^v := \ell'|_{\partial^v N}$ whose underlying map of pairs is strongly n-connected. Moreover, this map factors through the homotopy orbits of the source by the action of $\mathrm{Aut}_{\partial^v}(u)$. See [GR17b, p. 190] for the analogous definition in the closed case. The following claims that this map is a weak equivalence, under some hypothesis on u.

Lemma 7.2.2. *The induced map* $(\mathrm{Bun}^{\mathrm{col}}_{\partial^v,n}(TN,\Theta'^*\gamma))_{\mathrm{Aut}_{\partial^v}(u)} \to \mathrm{Bun}^{\mathrm{col}}_{\partial^v}(TN,\Theta^*\gamma)$ *is a weak equivalence onto the path components which it hits. The same claim holds after replacing* N *by* N_g *with the* Θ'*-structure from* (II) *for any* $g \geq 0$.

Proof. We start by noticing that this claim is equivalent to proving that the homotopy fibers of this map are either empty or contractible. We start by studying the the map $\mathrm{Bun}^{\mathrm{col}}_{\partial^v,n}(TN,\Theta'^*\gamma) \to \mathrm{Bun}^{\mathrm{col}}_{\partial^v}(TN,\Theta^*\gamma)$ and its homotopy fibers. By Proposition A.7, this map is a Serre fibration, since u is a fibration and $(\partial^v N, \partial^{hv} N) \to (N, \partial^h N)$ is a cofibration. Moreover, the (homotopy) fiber over a point $g \in \mathrm{Bun}^{\mathrm{col}}_{\partial^v}(TN,\Theta^*\gamma)$ identifies with the union of those path components of $\mathrm{Map}^{(B,B^\partial)}_{(\partial^v N,\partial^{hv}N)}((N,\partial^h N),(B',B'^\partial))$ of strongly n-connected maps, by the same result. If this space is non-empty, we pick a point and call it $f : (N,\partial^h N) \to (B',B'^\partial)$. Consider the map

$$\mathrm{Map}^{(B,B^\partial)}_{(\partial^v N,\partial^{hv}N)}((B',B'^\partial),(B',B'^\partial)) \to \mathrm{Map}^{(B,B^\partial)}_{(\partial^v N,\partial^{hv}N)}((N,\partial^h N),(B',B'^\partial)) \quad (5)$$

given by pre-composition with f. We prove that this map is a weak equivalence onto the components hit. Restricting maps of pairs to the sources of the pairs induces a map

$$\mathrm{Map}^{B^\partial}_{\partial^{hv}N}(B'^\partial,B'^\partial) \to \mathrm{Map}^{B^\partial}_{\partial^{hv}N}(\partial^h N,B'^\partial),$$

which is a weak equivalence by the proof of [GR17b, Lemma 9.2]. Here, we use that $\partial^{hv}N \to \partial^h N$ and $\partial^{hv}N \to B'^\partial$ are cofibrations, $u^\partial : B'^\partial \to B^\partial$ is an n-coconnected fibration. By Proposition A.1, these restriction maps to the sources of the pairs are Serre fibrations. Here, we use that $(\partial^v N,\partial^{hv}N) \to (N,\partial^h N)$ and $(\partial^v N,\partial^{hv}N) \to (B',B'^\partial)$ are cofibrations and u is a fibration. Now, to show that (5) is a weak equivalence, it suffices to show that the induced map on homotopy fibers of the restriction maps is a weak equivalence. For a map $f' : B'^\partial \to B'^\partial$, the induced map on fibers agrees with the pre-composition map

$$\mathrm{Map}^{B}_{\partial^v N\cup_{\partial^{hv}N} B'^\partial}(B',B') \to \mathrm{Map}^{B}_{\partial^v N\cup_{\partial^{hv}N} B'^\partial}(N\cup_{\partial^h N} B'^\partial,B')$$

induced by $f : N \to B'$ (see Remark A.0.2). This map is a weak equivalence, once again by the proof of [GR17b, Lemma 9.2] (and observing that it does not depend on the fact that the pair $(W,\partial W)$ is a manifold and its boundary). To apply this result, we use that the maps $\partial^v N \cup_{\partial^{hv}N} B'^\partial \to B'$ and $\partial^v N \cup_{\partial^{hv}N} B'^\partial \to N \cup_{\partial^h N} B'^\partial$ are cofibrations (which is a consequence of $\partial^v N \cup_{\partial^{hv}N} \partial^h N \to N$ being a cofibration), that $f : N \cup_{\partial^h N} B'^\partial \to B'$ is $(n+1)$-connected and $B' \to B$ is an n-coconnected fibration (which follows from the Definition 7.1.1). We conclude that the map (5) is a weak equivalence. Observe now that, since f is strongly n-connected, the subspace of the source of (5) consisting of weak equivalences hits the path components of strongly n-connected maps. On the other hand, by the uniqueness of Moore-Postnikov factorizations (see the discussion below Lemma 7.1.2), we see that every component of the subspace of those strongly n-connected maps is hit by the subspace consisting of weak equivalences. By definition, the latter subspace is $\mathrm{Aut}_{\partial^v N}(u)$. Thus, we see that (5) restricts to a weak equivalence between $\mathrm{Aut}_{\partial^v N}(u)$ and the fiber over g of the map $\mathrm{Bun}^{\mathrm{col}}_{\partial^v,n}(TN, \Theta'^*\gamma) \to \mathrm{Bun}^{\mathrm{col}}_{\partial^v}(TN, \Theta^*\gamma)$. By the same reasoning as in [GR17b, Lemma 9.2], we conclude that after taking homotopy orbits on the source of the latter map, its homotopy fibers are either empty or contractible. This finishes the proof. The case of N_g follows verbatim by observing that the proof above only used that $(\partial^v N, \partial^{hv} N) \to (N, \partial^h N)$ is a cofibration. ◻

We now apply this result above to the moduli spaces $\mathscr{M}^\partial_\Theta(-)$. Note that $\mathrm{Aut}_{\partial^v N}(u)$ acts on the category $\mathsf{Cob}^\partial_{\Theta'}$ via continuous endofunctors by changing the Θ'-structures by post-composition. Since the elements of $\mathrm{Aut}_{\partial^v N}(u)$ are weak equivalences $(B', B'^\partial) \to (B', B'^\partial)$ under the map $\ell'^v : (\partial^v N, \partial^{hv} N) \to (B', B'^\partial)$, the object $(\partial^v N, \ell'^v)$ is fixed under this action. This induces an action on $\mathscr{N}^\partial_{\Theta'}(\partial^v N)$ and thus on $\mathscr{M}^\partial_{\Theta',n}(N; \ell'^v)$, where $\mathscr{M}^\partial_{\Theta',n}(N; \ell'^v)$ denotes the path components of $\mathscr{M}^\partial_\Theta(N; \ell'^v)$ of those (W, ℓ_W) where ℓ_W is strongly n-connected. Similarly, we have a map $(\mathscr{M}^\partial_{\Theta',n}(N; \ell'^v))_{\mathrm{Aut}_{\partial^v}(u)} \to \mathscr{M}^\partial_\Theta(N; \ell^v)$ given by post-composition with u. We deduce from Lemma 7.2.2 that this map is a weak equivalence onto the path components hit.

Corollary 7.2.3. *Then the induced map $(\mathscr{M}^\partial_{\Theta',n}(N; \ell'^v))_{\mathrm{Aut}_{\partial^v}(u)} \to \mathscr{M}^\partial_\Theta(N; \ell^v)$ is a weak equivalence onto the path components hit. The same claim holds after replacing N by N_g for any $g \geq 0$.*

Proof. This follows from Lemma 7.2.2, the equivalence $\mathscr{M}^\partial_\Theta(N; l) \simeq (\mathrm{Bun}^{\mathrm{col}}_{\partial^v}(TN, \Theta^*\gamma))_{\mathrm{Diff}_{\partial^v}(N)}$ from Remark 7.0.2, and the fact that the actions of $\mathrm{Diff}_{\partial^v}(N)$ and $\mathrm{Aut}_{\partial^v}(u)$ commute and thus, one can form the homotopy orbits in any order. ◻

Note that the equivalence of Corollary 7.2.3 is natural with respect to the stabilization map $\mathscr{M}^\partial_{\Theta',n}(N_g; \ell'^v_g) \to \mathscr{M}^\partial_{\Theta',n}(N_{g+1}; \ell'^v_{g+1})$ on the target and the analogous maps after taking homotopy orbits by $\mathrm{Aut}_{\partial^v N_g}(u)$ and $\mathrm{Aut}_{\partial^v N_{g+1}}(u)$, respectively. These two monoids are equivalent for every $g \geq 0$, since $\partial^v N_g \cong \partial^v N_{g+1}$ and under such identification $\ell'^v_g|_{\partial^v N_g}$ and $\ell'^v_{g+1}|_{\partial^v N_{g+1}}$ of (II) are homotopic, since ℓ'^{std}_g and $\ell'^{\mathrm{std}}_{g+1}$ are standard (and thus extend to a contractible space). Thus, we have an equivalence after taking homotopy colimits

$$\left(\mathscr{M}^\partial_{\Theta',n}(N_\infty; \ell'^v)\right)_{\mathrm{Aut}_{\partial^v}(u)} \to \mathscr{M}^\partial_\Theta(N_\infty; \ell^v)$$

onto the path components which are hit. In the context of the $\mathrm{Aut}_{\partial^v N}(u)$-action defined above Corollary 7.2.3, the maps

$$\mathscr{M}^\partial_{\Theta',n}(N; \ell'^v) \to \mathscr{N}^\partial_{\Theta'}(\partial^v N) \to \Omega_{[\emptyset, \partial^v N]} \mathsf{BCob}^\partial_{\Theta'}$$

are equivariant. We are now ready to prove our main result.

*Proof of Theorem A**. Consider the notation and choices fixed in Assumptions 4.1.1 and 7.2.1. We can apply Theorem 6.0.2 to the Θ'-end given by $K|_0 \coloneqq \partial^v N$ and $K|_{[i,i+1]} \coloneqq ({}_{\partial^v N}H)_i$. Note that by assumption $\partial K|_0 = \partial^{hv}N \neq \emptyset$, (B', B'^{∂}) is 1-connected and B'^{∂} is path-connected, since ℓ' is strongly n-connected and $n \geq 3$. By restricing to the component of N, we obtain that the map $\mathscr{M}^{\partial}_{\Theta',n}(N_\infty;\ell'^v) \to \Omega_{[\emptyset,\partial^v N|_\infty]}\mathsf{BCob}^{\partial}_{\Theta'}$ is acyclic. Consider now the span

$$\mathscr{M}^{\partial}_{\Theta}(N_\infty;\ell^v) \leftarrow \left(\mathscr{M}^{\partial}_{\Theta',n}(N_\infty;\ell'^v)\right)_{\mathrm{Aut}_{\partial^v}(u)} \to \left(\Omega_{[\emptyset,(\partial^v N,\ell'^v)]}\mathsf{BCob}^{\partial}_{\Theta'}\right)_{\mathrm{Aut}_{\partial^v}(u)},$$

where the leftmost map is an equivalence onto the hit path components by the discussion below Corollary 7.2.3. Since sequential homotopy colimits and homotopy orbits commute and acyclicity is preserved after taking homotopy orbits, the right map is acyclic onto the path components hit. By [Gen11, Main Thm. 4.5], there exists an equivalence $\mathrm{B}\mathsf{Cob}^{\partial}_{\Theta'} \simeq \Omega^{\infty-1}\,\mathbf{MT}\Theta'$, which is a composition of equivariant maps with respect to the actions of $\mathrm{Aut}_{\partial^v}(u)$ (see the zig-zag in p.538 of loc.cit). This finishes the proof, by taking the component of ℓ in the leftmost space and picking a Θ'-nullbordism of $(\partial^v N, \ell'^v)$ (e.g. $(N, \partial^h N)$ itself) to identify $\Omega_{[\emptyset,(\partial^v N,\ell'^v)]}\Omega^{\infty-1}\,\mathbf{MT}\Theta'$ with $\Omega^{\infty}\,\mathbf{MT}\Theta'$. □

7.3. Some examples.

In this subsection, we expand on a few examples and prove the claims made in the introduction. Throughout the entire subsection, we let $(N, \partial^h N, \partial^v N)$ be a connected $(2n+1)$-manifold triad such that $(N, \partial^h N)$ is 1-connected for $n \geq 3$ and $\partial^{hv}N \neq \emptyset$. We start by describing the Thom spectra and automorphism monoid present in Theorem A* in specific cases.

Induced tangential structures. Let $\theta : B \to \mathrm{BO}(2n+1)$ be a map of spaces. We can define the map of pairs $\iota_*\theta : (B, B\times_{\mathrm{BO}(2n+1)}\mathrm{BO}(2n)) \to (\mathrm{BO}(2n+1), \mathrm{BO}(2n))$. One has that $\mathbf{MT}\,\iota_*\theta \simeq \Sigma^{\infty}_{+}B$. (See [Gal+09, Prop. 3.1] for a proof when $\theta = \mathrm{id}_{\mathrm{BO}(2n+1)}$ and [Gol16, Prop. 3.0.7] in general.) Given a map of pairs $\Theta : (B, B^{\partial}) \to (\mathrm{BO}(2n+1), \mathrm{BO}(2n))$, we denote by $\iota^*\Theta$ the underlying map $B \to \mathrm{BO}(2n+1)$. We are interested in knowing when the structure $\Theta_N : (B, B^{\partial}) \to (\mathrm{BO}(2n+1), \mathrm{BO}(2n))$ given by the strong Moore-Postnikov n-factorization (see Definition 7.1.1) of $\tau_N : (N, \partial^h N) \to (\mathrm{BO}(2n+1), \mathrm{BO}(2n))$ is equivalent to $\iota_*\iota^*\Theta_N$.

Lemma 7.3.1. *In the context of the above, the map Θ_N is equivalent to $\iota_*\iota^*\Theta_N$ if and only if $(N, \partial^h N)$ is n-connected. More generally, let $\theta : B \to \mathrm{BO}(2n+1)$ be a map and l a $\iota_*\theta$-structure on $(N, \partial^h N)$. Let $l = u \circ l' : (N, \partial^h N) \to (B', (B')^{\partial}) \to (B, B\times_{\mathrm{BO}(2n+1)}\mathrm{BO}(2n))$ be a Moore Postnikov n-factorization of pairs, then $\Theta_l \coloneqq \Theta\circ u$ is equivalent to $\iota_*\iota^*\Theta_l$ if and only if $(W, \partial^h W)$ is n-connected.*

Proof. We prove the second statement, as the first one follows by applying the second to $\theta = \mathrm{id}_{\mathrm{BO}(2n+1)}$. We start with the "only if"-direction. Suppose $\Theta_l : (B', (B')^\partial) \to (\mathrm{BO}(2n+1), \mathrm{BO}(2n))$ is equivalent to $\iota_*\iota^*\Theta_l$, thus it follows that $(B', (B')^\partial)$ is $2n$-connected, as it is a pullback of a $2n$-connected map $\mathrm{BO}(2n) \to \mathrm{BO}(2n+1)$. Thus, we see that $\pi_i(W, \partial^h W) \cong \pi_i(B', (B')^\partial)$ for $i \leq n$ by Lemma 2.1.5, since $(W, \partial^h W) \to (B', (B')^\partial)$ is strongly n-connected and $(W, \partial^h W)$ is 1-connected. For the "if"-direction, consider the factorization $(B')^\partial \to B' \times_{\mathrm{BO}(2n+1)} \mathrm{BO}(2n) \to B'$. The first map is n-coconnected by definition, and the second map is $2n$-connected, since it is the pullback of an $2n$-connected map. On the other hand, the pair $(B', (B')^\partial)$ is n-connected, since $(W, \partial^h W)$ is. Since $n \geq 1$, we have that $(B')^\partial \to B' \times_{\mathrm{BO}(2n+1)} \mathrm{BO}(2n)$ is also n-connected, again by Lemma 2.1.5. Thus, it is an equivalence. This finishes the proof. □

Contractible automorphisms. In certain cases, the grouplike monoid $\mathrm{Aut}_{\partial^v N}(u)$ is contractible. This eliminates substantial complexity in computing the (co)homology of the stable moduli space $\mathscr{M}^\partial_\Theta(N_\infty)$ via Theorem A*. The following is an application of Proposition A.3.

Lemma 7.3.2. *Let $\Theta : (B, B^\partial) \to (\mathrm{BO}(2n+1), \mathrm{BO}(2n))$ and l_N a Θ-structure on $(N, \partial^h N)$. Denote a Moore-Postnikov n-factorization of pairs $u \circ l'_N : (N, \partial^h N) \to (B', (B')^\partial) \to (B, B^\partial)$ of l_N. Assume that $(N, \partial^v N)$ and $(\partial^h N, \partial^{hv} N)$ are $(n-1)$-connected, then $\mathrm{Aut}_{\partial^v N}(u)$ is contractible. Moreover, if $(B')^\partial \to B' \times_B B^\partial$ is an equivalence, the condition on $(\partial^h N, \partial^{hv} N)$ can be removed.*

Moduli spaces of h-cobordisms stabilized. We call a triad $(N, \partial^h N, \partial^v N)$ an *h-cobordism* if the inclusions $\partial^h N \to N$ and $\partial^v N \to N$ are equivalences. Let $\theta : B \to \mathrm{BO}(2n+1)$ be a map and l_N be a $\iota_*\theta$-structure on N.

Corollary 7.3.3. *For an h-cobordism $(N, \partial^h N, \partial^v N)$, there exists a map*

$$\mathscr{M}^\partial_{\iota_*\theta}(N_\infty; l^v_N)_{l_N} \to \Omega^\infty \Sigma^\infty_+ B'$$

which is acyclic onto the path component it hits, where $N \to B' \to B$ is the classical Moore-Postnikov n-factorization of the map $l_N : N \to B$.

Proof. This follows by combining Theorem A*, Lemma 7.3.1, Lemma 7.3.2 and the fact that the underlying map of the strong Moore-Postnikov factorization of this map of pairs in this case is a classical Moore-Postnikov factorization of $\tau_N : N \to B$. We leave this check to the reader. □

Remark 7.3.4. By taking B n-connected and W to be the triad $(D^{n+1}_+, \partial_0 D^{n+1}_+, \partial_1 D^{n+1}_+)$ (as defined in Definition 2.1.6), this result recovers [BP17, Thm. A*] and extends it to dimension 7.

A rational computation. Let A be $2n$-dimensional manifold with non-empty boundary. Consider the triad $(A \times [0,1], A \times \{0\}, \partial A \times [0,1] \cup A \times \{1\})$, and observe that it satisfies the conditions of Theorem A (or Theorem A* for $\Theta = \mathrm{id}_{\mathrm{BO}(2n+1)}$). The space $\mathrm{BDiff}_{\partial^v}(A \times [0,1] \natural V_\infty)$ can be viewed as a space of V_g-stabilized concordances of the manifold A. We compute its rational cohomology as follows.

Corollary 7.3.5. *Assume A is aspherical and parallelizable. There exists an isomorphism*

$$\mathrm{H}^*(\mathrm{BDiff}_{\partial^v}(A \times [0,1] \natural V_\infty); \mathbb{Q}) \cong \mathbb{Q}[\mathrm{H}^*(A;\mathbb{Q}) \otimes \mathbb{Q}[p_{[(n+1)/4]}, \cdots, p_n]]$$

as graded commutative $\mathbb{Q}$-algebras.

Proof. This follows by applying Corollary 7.3.3 to the trivial h-cobordism $A \times [0,1]$ along with the following classical facts: the rational cohomology $\mathbb{Q}$-algebra $\mathrm{H}^*(\Omega_0^\infty \Sigma_+^\infty X)$ is isomorphic to $\mathbb{Q}[\widetilde{\mathrm{H}}^*(X;\mathbb{Q})]$ of a space X with finite dimensional $\mathbb{Q}$-homology in each degree; since A is parallelizable and aspherical, the composition $A \to A \times \tau_{>n} \mathrm{BO}(2n+1) \to \mathrm{BO}(2n+1)$ is a Moore-Postnikov n-factorization, where $\tau_{>n} \mathrm{BO}(2n+1)$ is the n-connected cover of $\mathrm{BO}(2n+1)$; finally the $\mathbb{Q}$-algebra $\mathrm{H}^*(\tau_{>n} \mathrm{BO}(2n+1); \mathbb{Q})$ is the free graded commutative $\mathbb{Q}$-algebra on the Pontrjagin classes $p_{[(n+1)/4]}, \cdots, p_n$, where p_i lives in degree $4i$. □

The initial structure. We finish by considering the *initial* tangential structure on N. We consider its tangent classifies $\tau_N : (N, \partial^h N) \to (\mathrm{BO}(2n+1), \mathrm{BO}(2n))$. The identity map of N induces a τ_N-structure on N. In this case, taking the Moore-Postnikov factorization of $\mathrm{id} : (N, \partial^h N) \to (N, \partial^h N)$ gives the identity of N. Thus, $\mathbf{MT}\Theta$ is the cofiber of the map $\Sigma^{-1} \mathrm{Th}(-T\partial^h N) \to \mathrm{Th}(-TN)$, which is equivalent to the Spanier-Whitehead dual $\mathrm{D}(N/\partial^v N)$ by Atiyah duality. By Proposition A.3 for $k = l = 0$, we see that $\mathrm{Aut}_{\partial^v N}(\mathrm{id})$ is contractible. Thus, we obtain the following result.

Corollary 7.3.6. *There exists a map*

$$\mathscr{M}^{\partial}_{\tau_N}(N_\infty; \mathrm{id}|_{\partial^v N})_{\mathrm{id}} \to \Omega^\infty \mathrm{D}(N/\partial^v N)$$

which is acyclic onto the path component it hits.

A. Appendix: Mapping spaces of pairs.

Recall from the discussion above Definition 7.1.1 that we consider the category of pairs of spaces $\mathbf{Top}^{[1]}$ as a model category with the projective model structure induced by the Quillen model structure on $\mathbf{Top}$. Given pairs (A, A') and (B, B'), one can define the category of pairs of spaces under (A, A') and over (B, B').

We consider this category with the under-over model structure (i.e. the model structure where maps are fibrations, cofibrations, or weak equivalences if and only if the underlying map in $\mathbf{Top}^{[1]}$ is, see [Hir09, Thm 7.6.5]). This category is enriched in spaces by taking the space of maps

$$\mathrm{Map}_{(A,A')}^{(B,B')}((X,X'),(Y,Y'))$$

with the compact-open topology. We can also see this category as enriched in simplicial sets by taking $\mathrm{Sing}_\bullet$.

Proposition A.1. *Let $(A,A') \to (X,X')$ be a cofibration of pairs, $(Y,Y') \to (B,B')$ a fibration of pairs, and $f : (A,A') \to (Y,Y')$ and $g : (X,X') \to (Y,Y')$ be maps of pairs. Then the restriction map*

$$\mathrm{Map}_{(A,A')}^{(B,B')}((X,X'),(Y,Y')) \to \mathrm{Map}_{A'}^{B'}(X',Y')$$

is a fibration. Moreover, the homotopy fiber over a map $\alpha : X' \to Y'$ is equivalent to $\mathrm{Map}^{B}_{A\cup_{A'}X'}(X,Y)$, where Y is seen as under $A \cup_{A'} X'$ using $g \circ f$ and α.

Proof. Let $I \to I'$ be a trivial cofibration in **Top** and consider a diagram of the following form

$$\begin{array}{ccc} I' & \longrightarrow & \mathrm{Map}_{(A,A')}^{(B,B')}((X,X'),(Y,Y')) \\ \downarrow & \nearrow & \downarrow \\ I & \longrightarrow & \mathrm{Map}_{A'}^{B'}(X',Y'). \end{array}$$

The data of such a lift is equivalent to the data of a map $I \times (X,X') \to (Y,Y')$ over (B,B') such that its restriction to $I \times X' \to Y'$ is the bottom horizontal map, the restriction to $I' \times (X,X') \to (Y,Y')$ is the top horizontal map and the restriction to $I \times (A,A') \to (Y,Y')$ is constant at f. In other words, we have to find a lift of the following diagram

$$\begin{array}{ccc} (I' \times X) \cup_{I' \times (A \cup_{A'} X')} I \times (A \cup_{A'} X') & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow \\ I \times X & \longrightarrow & B \end{array}$$

which is possible if the left vertical map is a trivial cofibration. This map is the pushout product of $I' \to I$ and $A \cup_{A'} X' \to X$ (see [Hov07, Defn.4.2.1]). Since the category of spaces is a monoidal model category, by [Hov07, Defn. 4.2.6, Prop. 4.2.11][15], the pushout product of two cofibrations is a cofibration and trivial if one of them is. By assumption, $I' \to I$ is a trivial cofibration and $A \cup_{A'} X' \to X$ is a cofibration, thus the left vertical map above is a trivial cofibration. This finishes the proof. □

[15]Note that in this reference, our definition of spaces agrees with the definition of k-spaces, while compactly generated spaces are assumed to weakly Hausdorff.

Remark A.0.2. By Proposition A.1, we have a fiber sequence of the form

$$\mathrm{Map}^{B}_{A\cup_{A'}X'}(X,Y) \to \mathrm{Map}^{(B,B')}_{(A,A')}((X,X'),(Y,Y')) \to \mathrm{Map}^{B'}_{A'}(X',Y').$$

It is convenient to have a variant of this fiber sequence where the middle space is the subspace of those maps $(X,X') \to (Y,Y')$ which are strongly k-connected (recall from Definition 2.1.1). To do so, observe that the map $\mathrm{Map}^{B}_{A\cup_{A'}X'}(X,Y) \to \mathrm{Map}^{B}_{A\cup_{A'}Y'}(X\cup_{X'}Y',Y)$ is a homeomorphism. Thus, we have fiber sequence analogous to the one above, where the base is the subspace of those maps $X' \to Y'$ which are k-connected, the total space is the subspace of those maps $(X,X') \to (Y,Y')$ which are strongly k-connected and the fiber is the subspace of $\mathrm{Map}^{B}_{A\cup_{A'}Y'}(X\cup_{X'}Y',Y)$ of those maps which are $(k+1)$-connected.

We can describe the homotopy type of these mapping spaces given connectivity and coconnectivity of the pairs involved. Recall the definition of strong connectivity and coconnectivity from Definitions 2.1.1 and 7.1.1.

Proposition A.3. *Let $k,l \geq 0$ be integers. Let $(A,A') \to (X,X')$ be a cofibration of pairs and $(Y,Y') \to (B,B')$ be a strongly l-coconnected fibration of pairs. If both $A \to X$ and $A' \to X'$ are k-connected, then the space*

$$\mathrm{Map}^{(B,B')}_{(A,A')}((X,X'),(Y,Y'))$$

is a $(l-k-2)$-type. In particular, it is contractible if $k \geq l-1$, provided it is non-empty. Furthermore, if $Y' \to Y\times_B B'$ is an equivalence, then we can remove the k-connectivity assumption of $A' \to X'$.

Proof. We start by noticing that, under our assumptions, the space $\mathrm{Map}^{(B,B')}_{(A,A')}((X,X'),(Y,Y'))$ is a model for the derived mapping space in the category of pairs under (A,A') and over (B,B'), since (X,X') is cofibrant and (Y,Y') is fibrant in this category. However, the derived mapping space of a model category only depends on the class of weak equivalences (see [DK80, Corollary 4.7] together with the fact that this category is a closed simplicial model category in the sense of [Qui67, Defn. 2.2.2]). We define the following model structure with the same class of weak equivalences on $\mathbf{Top}^{[1]}$. The *injective model structure* on $\mathbf{Top}^{[1]}$ is model structure where the weak equivalences and cofibrations are objectwise and the fibrations are those maps $(X,X') \to (Y,Y')$ such that $X \to Y$ and $X' \to X\times_Y Y'$ are fibrations of spaces. Similarly to before, this induces a model structure on the over-under category. Since the injective model structure has the same weak equivalences, our space of interest is equivalent to the derived mapping space in this category. The latter is equivalent to the same mapping space but replacing $(Y,Y') \to (B,B')$ by an injective fibration, as this gives a fibrant replacement of the target. For simplicity of notation, we denote this replacement also by (Y,Y'). Let Z be the mapping space $\mathrm{Map}^{(B,B')}_{(A,A')}((X,X'),(Y,Y'))$. Let $i \geq 0$ and given a map $\alpha : S^i \to Z$ be a map, then α can be extended to D^{i+1} if and only if the following lifting problem can be solved

$$\begin{array}{ccc} (D^{i+1}\times A\cup S^i\times X, D^{i+1}\times A'\cup S^i\times X') & \longrightarrow & (Y,Y') \\ \downarrow & \nearrow & \downarrow \\ (D^{i+1}\times X, D^{i+1}\times X') & \longrightarrow & (B,B'). \end{array}$$

From the assumptions, we conclude the following facts:

(i) the maps $D^{i+1}\times A\cup S^i\times X \to D^{i+1}\times X$ and $D^{i+1}\times A'\cup S^i\times X' \to D^{i+1}\times X'$ are $(k+i+1)$-connected cofibrations. This can be seen by induction on cells on the k-connected pairs (X,A) and (X',A').

(ii) the maps $Y \to B$ and $Y' \to Y\times_B B'$ are l-coconnected fibrations.

By obstruction theory, we have no obstructions to solving the leftmost lifting problem

$$\begin{array}{ccc} D^{i+1}\times A\cup S^i\times X & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow \\ D^{i+1}\times X & \longrightarrow & B \end{array} \qquad \begin{array}{ccc} D^{i+1}\times A'\cup S^i\times X' & \longrightarrow & Y' \\ \downarrow & \nearrow & \downarrow \\ D^{i+1}\times X' & \longrightarrow & Y\times_B B' \end{array}$$

if $k+i+1 \geq l$, that is, when $i \geq l-k-1$. Provided $i \geq l-k-1$ and fixing a lift, we can solve the initial lifting problem if we can solve the right problem above where the bottom horizontal map is the unique map to the pullback induced by the map $D^{i+1}\times X' \to B'$ and the chosen lift $D^{i+1}\times X' \to D^{i+1}\times X \to Y'$. The same obstruction theoretic argument implies that such a problem can be solved provided $i \geq l-k-1$. This implies that Z is a $(l-k-2)$-type. If the map $Y' \to Y\times_B B'$ is an equivalence, then no conditions are needed on the left vertical map of the third lifting problem to solve it. Thus, no condition on the cofibration $A' \to X'$ is necessary to deduce that Z is an $(l-k-2)$-type in this case. ◻

Proposition A.4. *Let $(A,A') \to (X,X')$ be a cofibration of pairs, $(Y,Y') \to (Z,Z') \to (B,B')$ two composable fibrations of pairs, and $(A,A') \to (Y,Y')$ and $(X,X') \to (Y,Y')$ be maps of pairs. Then the post-composition map*

$$\mathrm{Map}^{(B,B')}_{(A,A')}((X,X'),(Y,Y')) \to \mathrm{Map}^{(B,B')}_{(A,A')}((X,X'),(Z,Z'))$$

is a fibration.

Proof. The claim follows if we show that the simplicially enriched category of pairs of spaces under (A,A') and over (B,B') is a simplicial model category (in the sense of [Hir09, Defn. 9.1.6]) when taken with the projective model structure, as this claim is a special case of axiom M7. This follows by combining the general fact that the category of diagrams indexed in a Reedy category of a simplicial model category is a simplicial model category, when taken with the Reedy model structure (see [Hir09, Thm. 15.3.4.(3)]), and the fact that over-under categories of a simplicial model categories are simplicial model categories. ◻

We establish now similar properties for spaces of collared bundle maps.

Proposition A.5. *Let* $\iota : (A, A') \to (X, X')$ *be a cofibration of pairs,* (ξ, ξ') *a collared vector bundle over* (X, X')*,* (η, η') *a collared vector bundle over* (Y, Y') *and* $f : (\iota^*\xi, \iota^*\xi') \to (\eta, \eta')$ *be a collared bundle map. The restriction map*

$$\mathrm{Bun}^{\mathrm{col}}_A(\xi, \eta) \to \mathrm{Bun}_{A'}(\xi', \eta')$$

is a fibration. Moreover, the homotopy fiber over $\alpha : \xi' \to \eta'$ *is equivalent to* $\mathrm{Bun}_{A \cup_{A'} X'}(\xi, \eta)$*.*

Proof. This statement will follow from the following claim, whose proof we leave to the reader: the restriction map $\mathrm{Bun}(\xi, \eta) \to \mathrm{Bun}(\xi|_{X'}, \eta)$ is a fibration, if $X' \to X$ is a cofibration. We show now our desired statement using this claim. From the pullback decomposition of Remark 2.2.3, we observe that it suffices to show that $\mathrm{Bun}_A(\xi, \eta) \to \mathrm{Bun}_{A'}(\xi|_{X'}, \eta)$ is a fibration, as $\xi|_{X'} \cong \xi' \oplus \varepsilon^1$. For that, consider the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Bun}_A(\xi, \eta) & \longrightarrow & \mathrm{Bun}(\xi, \eta) & \longrightarrow & \mathrm{Bun}(\xi|_A, \eta) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Bun}_{A'}(\xi|_{X'}, \eta) & \longrightarrow & \mathrm{Bun}(\xi|_{X'}, \eta) & \longrightarrow & \mathrm{Bun}(\xi|_{A'}, \eta) \end{array}$$

where the rows are fibration sequences, by the claim above as $A \to X$ and $A' \to X'$ are cofibrations. One can see that the left vertical map is a fibration if the map

$$\mathrm{Bun}(\xi, \eta) \to \mathrm{Bun}(\xi|_{X'}, \eta) \times_{\mathrm{Bun}(\xi|_{A'}, \eta)} \mathrm{Bun}(\xi|_A, \eta)$$

induced by the commutativity of the left square, where $\times$ denotes the strict pullback, is a fibration. However, this map is homeomorphic to the restriction map $\mathrm{Bun}(\xi, \eta) \to \mathrm{Bun}(\xi|_{X' \cup_{A'} A}, \eta)$, which is fibration by the claim above using that $X' \cup_{A'} A \to X$ is a cofibration. This finishes the proof of the first claim. The claim about the homotopy fiber follows by observing that $\mathrm{Bun}_{A \cup_{A'} X'}(\xi, \eta)$ is the strict fiber of the restriction map $\mathrm{Bun}_A(\xi, \eta) \to \mathrm{Bun}_{A'}(\xi|_{X'}, \eta)$. ◻

Remark A.0.6. Assume that the dimensions of ξ and η agree. Similarly to A.0.2, the (homotopy) fiber of the map above $\mathrm{Bun}_{A \cup_{A'} X'}(\xi, \eta)$ is homeomorphic to $\mathrm{Bun}_{A \cup_{A'} Y'}(\xi \cup \eta' \oplus_{\xi' \oplus \varepsilon^1} \varepsilon^1, \eta)$. On the other hand, by forgetting the bundle map, one obtains a map of fiber sequences between the one describes in Proposition A.5 and the one in Proposition A.1.

Proposition A.7. *Let $\iota : (A, A') \to (X, X')$ be a cofibration of pairs, (ξ, ξ') a collared vector bundle over (X, X'), (η_Y, η'_Y) a collared vector bundle over (Y, Y'), (η_Z, η'_Z) a collared vector bundle over (Z, Z') and $f : (\iota^*\xi, \iota^*\xi') \to (\eta_Y, \eta'_Y)$ be a collared bundle map. Given a collared bundle map $u : (\eta_Y, \eta'_Y) \to (\eta_Z, \eta'_Z)$ whose underlying map is a fibration, the map induced by post-composition with u*

$$\mathrm{Bun}^{\mathrm{col}}_A(\xi, \eta_Y) \to \mathrm{Bun}^{\mathrm{col}}_A(\xi, \eta_Z)$$

is a fibration. Moreover, the homotopy fiber over a map g is equivalent to $\mathrm{Map}^{(Z,Z')}_{(A,A')}((X, X'), (Y, Y'))$, where (X, X') is seen over (Z, Z') using g.

Proof. We start by showing that the map above is a Serre fibration. Let $I' \to I$ be a trivial cofibration in **Top** and consider a diagram of the form

$$\begin{array}{ccc} I' & \longrightarrow & \mathrm{Bun}^{\mathrm{col}}_A(\xi, \eta_Y) \\ \big\downarrow & \nearrow & \big\downarrow \\ I & \longrightarrow & \mathrm{Bun}^{\mathrm{col}}_A(\xi, \eta_Z) \end{array} .$$

Observe that the analogous lifting problem where collared bundle mapping spaces are replaced by mapping spaces of the underlying pairs can be solved by Proposition A.4. In other words, we can find a map $l : I \times (X, X') \to (Y, Y')$ extending the underlying map of the top arrow above, such that after composing with the map $(Y, Y') \to (Z, Z')$, it agrees with the underlying map of the bottom map of the square above. The original lifting problem can be solved if we can find a collared bundle map $I \times (\xi, \xi') \to (l^*\eta_Y, l^*\eta'_Y)$ over the identity of (X, X') compatibly with the existing bundle maps given by the square above. To do that, observe that a collared bundle map $I \times (\xi, \xi') \to (l^*\eta_Y, l^*\eta'_Y)$ over $I \times X$ is equivalent to a section of the map of pairs $(\mathrm{Hom}(\xi, l^*\eta_Y), \mathrm{Hom}(\xi', l^*\eta'_Y)) \to I \times (X, X')$ which is induced by the collars of ξ and η. Each individual map in this map of pairs is a fiber bundle and thus a fibration. We conclude that the solution of the lifting problem above is reduced to the following lifting problem

$$\begin{array}{ccc} I \times (A, A') \cup_{I' \times (A,A')} I' \times (X, X') & \longrightarrow & (\mathrm{Hom}(\xi, l^*\eta_Y), \mathrm{Hom}(\xi', l^*\eta'_Y)) \\ \big\downarrow & \nearrow & \big\downarrow \\ I \times (X, X') & \longrightarrow & (\mathrm{Hom}(\xi, (u \circ l)^*\eta_Z), \mathrm{Hom}(\xi', (u \circ l)^*\eta'_Z)) \end{array}$$

where the bottom map is induced by the bottom map of the original lifting problem. Now, this can be solved uniquely since the right vertical map is a homeomorphism. This proves the first claim.

The proof of the identification of the homotopy fibers follows the same idea of the proof of the first claim. One can identify the fiber of this map with the fiber of the map between the underlying mapping spaces. This fiber identifies precisely with $\mathrm{Map}^{(Z,Z')}_{(A,A')}((X, X'), (Y, Y'))$. We leave this verification to the reader. □

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