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Felix Neussel and Oliver Stein

Institute for Operations Research (IOR), Karlsruhe Institute of Technology (KIT), Karlsruhe, Germany

ABSTRACT

We study the supportedness of nondominated points of multiobjective optimization problems, that is, whether they can be obtained via weighted sum scalarization. One key question is how supported points behave under an efficiency-preserving transformation of the original problem. Under a differentiability assumption, we characterize the transformations that preserve both efficiency and supportedness as the component-wise transformations with strictly increasing and convex components. In addition, we consider transformations that can render originally unsupported points supported in the transformed problem. This enables algorithms to find nondominated points by applying the weighted sum scalarization to a transformed problem.

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1. Introduction

This paper investigates effects of image space coordinate transformations for multiobjective optimization problems of the form

$$\min f(x) \text{ s.t. } \quad x \in X, \tag{MOP}$$

with a nonempty set $X \subseteq \mathbb{R}^n$ of feasible points and a continuous vector-valued objective function $f : X \to \mathbb{R}^m$. We do not impose any convexity assumptions on the component functions of f or on the set X. The image set of X under f is denoted by $Y := f(X)$. Throughout this paper, we consider solution concepts based on the standard ordering cone $\mathbb{R}^m_{\geqq}$, the set of vectors in $\mathbb{R}^m$ with non-negative components.

One concept for the algorithmic solution of problems like *MOP* is the weighted sum scalarization, whose solutions are referred to as supported points of *MOP*. Supported points are always weakly efficient for *MOP* (cf. [1]), while the converse does generally not hold. It is well known that if *MOP* is convexlike, that is, the upper image set $Y + \mathbb{R}^m_{\geqq}$ is convex, the weakly efficient points are characterized as the supported points (cf. [1]).

CONTACT Felix Neussel felix.neussel@kit.edu Institute for Operations Research (IOR), Karlsruhe Institute of Technology (KIT), Kaiserstr. 12, Karlsruhe 76131, Germany

In [2], image space transformations $\Phi : \mathbb{R}^m \to \mathbb{R}^m$ are considered, along with the transformed problem

$$\min\ \Phi(f(x))\ \text{s.t.}\quad x \in X, \qquad (MOP_\Phi)$$

such that MOP and MOP_Φ share the same set of efficient points. Such transformations can generate desirable properties like smoothness or convexity. However, if the weighted sum scalarization is the intended solution concept, it is crucial that all supported points of MOP stay supported for MOP_Φ. This paper focuses on identifying and characterizing supportedness respecting transformations (SRTs), that is, image space transformations that preserve the supported points.

A prominent example studied in the literature is the p-th power transformation, defined by $\Phi^p(f(x)) = (f_1(x)^p, \dots, f_m(x)^p)$, under the assumption that $Y \subseteq \mathbb{R}^m_>$. As noted in [2], applying the weighted sum scalarization to MOP_{Φ^p} is equivalent to the compromise programming approach with ℓ_p-norms, with the origin as utopia point of the original problem. Combined with Lemma 5.5 in [2], this establishes that the p-th power transformation is an SRT.

There exist simple examples in which this transformation even renders previously unsupported points supported. In [3], it is shown that when X is finite, the p-th power transformation with a sufficiently large p ensures that all nondominated points become supported. A lower bound for p is also provided. However, the task of computing this bound is of the same complexity as the exact computation of the nondominated set with the filter method of Jahn, Graef and Younes [4]): if $Y = \{y^1, \dots, y^J\} \subset \mathbb{R}^m_>$, computation of the bound requires $mJ(J-1)$ evaluations of terms of the form $\log(a/b)$ in the worst case, while the filter method requires the same number of scalar comparisons. For the special case that all points in Y have integer components and are contained in a box, an easy-to-compute lower bound for p is provided in [5].

In the continuous case, [6] studies the p-th power transformation under the assumption that the set of nondominated points (Definition 2.3) is the graph of a twice continuously differentiable function. Under this condition, it is shown that for some sufficiently large $p \in [0, \infty)$, every nondominated point of MOP_{Φ^p} is supported. Unfortunately, the required assumption is restrictive: even in the linear case, the set of nondominated points typically contains kinks and is thus not the graph of a differentiable function. This limitation is acknowledged in [6], where it is further shown that if the graph of the function is twice continuously differentiable in a neighbourhood of a nondominated point, then this point becomes locally supported under the p-th power transformation: it is attained as a locally minimal point of a weighted sum. Yet, solving a weighted sum problem locally does not guarantee the global efficiency of the solution. Similar comments apply to the exponential transformation from [7].

This paper is organized as follows. Section 2 introduces basic definitions and preliminary results from the literature. In Section 3, we define image space transformations that preserve both efficiency and supportedness and, under mild assumptions, characterize them as component-wise monotone transformation with convex components. Section 4 discusses the capability of such transformations to generate new supported points. Finally, Section 5 offers concluding remarks.

2. Preliminaries

2.1. Solution concepts

In multiobjective optimization there exist three main concepts to generalize the notion of a (global) minimal point from the single-objective case. For their introduction we use the following notation for relations between vectors (cf. [1]). It replaces the usual inequality sign $\leq$ by the sign $\leqq$ and the re-defines the sign $\leq$.

Definition 2.1: For $y^1, y^2 \in \mathbb{R}^m$ with $m \in \mathbb{N}$ we define

(a) $y^1 \leqq y^2 \ :\Leftrightarrow\ y^1_j \leqq y^2_j,\ j \in [m] := \{1, \ldots, m\}$,
(b) $y^1 \leq y^2 \ :\Leftrightarrow\ y^1 \leqq y^2$ and $y^1 \neq y^2$,
(c) $y^1 < y^2 \ :\Leftrightarrow\ y^1_j < y^2_j,\ j \in [m]$.

In the case $y^1 \leq y^2$ one says that y^1 dominates y^2, and for $y^1 < y^2$ that y^1 strictly dominates y^2. The inequalities $y^1 \geqq y^2$, $y^1 \geq y^2$ and $y^1 > y^2$ are defined analogously.

Note that for scalars the inequality $y^1 \leq y^2$ is equivalent to $y^1 < y^2$, so that for scalars we shall only use the relations $y^1 \leqq y^2$ and $y^1 < y^2$. The relation $y^1 \leq y^2$ is a relevant concept only for $m \geqq 2$.

With the cones

$$\begin{aligned} \mathbb{R}^m_{\geqq} &:= \{y \in \mathbb{R}^m \mid y \geqq 0\}, \\ \mathbb{R}^m_{\geq} &:= \{y \in \mathbb{R}^m \mid y \geq 0\} = \mathbb{R}^m_{\geqq} \setminus \{0\}, \text{ and} \\ \mathbb{R}^m_{>} &:= \{y \in \mathbb{R}^m \mid y > 0\} = \text{int } \mathbb{R}^m_{\geqq}, \end{aligned}$$

one may write a relation like $y^1 \leq y^2$ equivalently as $y^2 - y^1 \in \mathbb{R}^m_{\geq}$, etc. One may also define ordering structures on $\mathbb{R}^m$ by replacing the standard ordering cone (or Pareto cone) $\mathbb{R}^m_{\geqq}$ by other convex cones, but in the present paper we focus on the standard case of component-wise inequalities.

Definition 2.2:

(a) For $Y \subseteq \mathbb{R}^m$ a point $\bar{y} \in Y$ is called weakly nondominated, if no $y \in Y$ with $y < \bar{y}$ exists.
(b) For *MOP* a point $\bar{x} \in X$ is called weakly efficient, if $f(\bar{x})$ is a weakly nondominated point of $f(X)$.
(c) We denote the sets of weakly nondominated points of Y and of weakly efficient points of *MOP* by Y_{wnd} and X_{we}, respectively.

Definition 2.3:

(a) For $Y \subseteq \mathbb{R}^m$ a point $\bar{y} \in Y$ is called nondominated, if no $y \in Y$ with $y \leq \bar{y}$ exists.
(b) For *MOP* a point $\bar{x} \in X$ is called efficient, if $f(\bar{x})$ is a nondominated point of $f(X)$.

(c) We denote the sets of nondominated points of Y and of efficient points of MOP by Y_{nd} and X_e, respectively.

Each nondominated point of a set Y is also weakly nondominated, but not vice versa. The analogous statement is true for efficient and weakly efficient points of MOP. The third solution concept is stronger than nondominatedness and efficiency, respectively. We use the notions of proper nondominatedness and proper efficiency in the sense of Geoffrion [8], since it is tailored to the component-wise structure of the standard ordering cone $\mathbb{R}^m_{\geqq}$. Other properness concepts are due to, for example, Benson [9], Borwein [10] and Henig [11].

Definition 2.4:

(a) For $Y \subseteq \mathbb{R}^m$ the point $\bar{y} \in Y$ is called properly nondominated, if some real number $K > 0$ exists such that for all $y \in Y$ and all $i \in [m]$ with $y_i < \bar{y}_i$ some $j \in [m]$ with $y_j > \bar{y}_j$ and

$$\frac{\bar{y}_i - y_i}{y_j - \bar{y}_j} \leqq K$$

exists.

(b) For MOP the point $\bar{x}$ is called properly efficient, if $f(\bar{x})$ is a properly nondominated point of $f(X)$.

(c) We denote the sets of properly nondominated points of Y and of properly efficient points of MOP by Y_{pnd} and X_{pe}, respectively.

The following decision space solution concepts generalize the notion of strict (local) minimal points from single-objective optimization.

Definition 2.5: A point $\bar{x} \in X$ is called strictly efficient if there is no $x \in X$, $x \neq \bar{x}$, such that $f(x) \leqq f(\bar{x})$.

Definition 2.6 ([12]): A point $\bar{x} \in X$ is called strictly local efficient of order 2 if there exist some $a > 0$ and a neighbourhood U of $\bar{x}$ such that for all $x \in X \cap U \setminus \{\bar{x}\}$, it holds that

$$(f(x) + \mathbb{R}^m_{\geqq}) \cap B(f(\bar{x}), a\|x - \bar{x}\|^2) = \emptyset,$$

where $B(y, \delta)$ denotes an open ball centred at y with radius δ.

For motivations and illustrations of these solution concepts we refer to, e.g. [1, 13].

Finally, for a nonempty set $Y \subseteq \mathbb{R}^m$ the vector α with extended real-valued entries

$$\alpha_j = \inf_{y \in Y} y_j, \; j \in [m] \tag{1}$$

is called ideal point point of Y. In this paper, we adopt the usual convention $\inf_{y \in Y} y_j = -\infty$ if y_j is not bounded from below on Y. Thus, the vector α exclusively has real-valued entries if and only if Y is bounded from below. By the Weierstrass theorem, all appearing infima are attained as minimal values, if Y is compact. We refer to a vector $\widehat{\alpha} \in \mathbb{R}^m$ with $\widehat{\alpha} < \alpha$ as utopia point of Y.

2.2. Scalarization techniques

A well-established approach for solving multiobjective problems is scalarization, where the original problem is replaced by a parametric single-objective problem. In this work, we focus on two scalarization methods: the weighted sum and the ε-constraint scalarization. The weighted sum scalarization depends on a parameter $w \in \mathbb{R}^m_{\geq}$ (or $w \in \mathbb{R}^m_{>}$) and is given by

$$\min\ w^\top f(x) \text{ s.t. } \quad x \in X \tag{$\widetilde{WS}(w)$}$$

in the decision space. Since our analysis mostly takes place in the image space, we also consider the image space version:

$$\min\ w^\top y \text{ s.t. } \quad y \in Y. \tag{$WS(w)$}$$

The ε-constraint scalarization depends on an index parameter $i \in [m]$ and parameter vector $b \in \mathbb{R}^m$ (historically referred to as ε) and is given by

$$\min f_i(x) \text{ s.t. } \quad f_j(x) \leqq b_j,\ j \in [m] \setminus \{i\},\ x \in X. \tag{$R(i,b)$}$$

An image space formulation of this scalarization also exists, but is not used in this paper.

2.3. Supportedness and convexity

Various definitions of supported points can be found in the literature. A comprehensive overview is given in [14]. We adopt the following definition, which, according to [14] belongs to the most general class of supportedness concepts.

Definition 2.7: A point $y \in Y$ is called supported for Y if there exists some $w \in \mathbb{R}^m_{\geq}$ such that y is a minimal point of $WS(w)$. The corresponding preimages $x \in X$ with $f(x) = y$ are called supported for *MOP*.

A useful characterization of supportedness is given in the following result, which holds under external stability: if for every point $\bar{y} \in Y \setminus Y_{nd}$ there exists some point $y \in Y_{nd}$ with $y \leq \bar{y}$, we say that Y_{nd} is externally stable (cf. [15]). If Y_{nd} is externally stable, the underlying problem *MOP* is said to possess the domination property (cf. [16]).

Theorem 2.8 (Lemma 8.4 in [14]): *Let Y_{nd} be externally stable. Then a point $y \in Y$ is a supported point of Y if and only if there is no convex combination $\sum \lambda_i y^i$ of points $y^1, \dots, y^r \in Y_{nd} \setminus \{y\}$ such that $\sum \lambda_i y^i < y$.*

The next two results establish the connection between supportedness and weak nondominance.

Theorem 2.9 ([1], Theorem 3.4): *Every supported point of Y is weakly nondominated.*

If the upper image set $Y + \mathbb{R}^m_{\geqq}$ is convex, we refer to *MOP* as convexlike. A sufficient condition for *MOP* to be convexlike is the convexity of X and of the objectives $f_j, j \in [m]$.

Theorem 2.10 ([1], Theorem 3.5): *Let MOP be convexlike. Then the set of supported points of Y equals the set of weakly nondominated points of Y.*

The following classical result states a helpful characterization of convexity for univariate functions.

Theorem 2.11 ([17], Theorem 1.1.8): *Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be continuous. Then f is convex if and only if it is midpoint convex, i.e. for all $x^1, x^2 \in I$ it holds that*

$$f\left(\frac{x^1 + x^2}{2}\right) \leqq \frac{f(x^1) + f(x^2)}{2}.$$

2.4. Component-wise monotone transformations

We now define the type of image space transformations studied in this paper. We call a set $\mathcal{Y}$ a box if it can be written as $\mathcal{Y} = \mathcal{Y}_1 \times \ldots \times \mathcal{Y}_m$ with not necessarily closed or bounded intervals $\mathcal{Y}_j \subseteq \mathbb{R}^1, j \in [m]$. In this sense, the whole space $\mathbb{R}^m$ itself is a box.

Definition 2.12: For boxes $\mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^m$ we call a bijective mapping $\Phi : \mathcal{Y} \to \mathcal{Z}$ a component-wise monotone transformation (CMT) if it is of the form

$$\Phi(y) = P \begin{pmatrix} \varphi_1(y_1) \\ \vdots \\ \varphi_m(y_m) \end{pmatrix}$$

with a permutation matrix P and strictly increasing functions $\varphi_j : \mathcal{Y}_j \to \mathcal{Z}_j$, $j \in [m]$. If additionally $\mathcal{Y}$ and $\mathcal{Z}$ are open and Φ is a $\mathcal{C}^1$-diffeomorphism, Φ is called a component-wise monotone $\mathcal{C}^1$-transformation, or $\mathcal{C}^1$-CMT for short.

More generally, [2] considers transformations that leave the set of efficient points invariant. Under differentiability, these transformations are characterized as the $\mathcal{C}^1$-CMTs. Moreover, [2] proves that CMTs preserve weak efficiency, and under additional assumptions, proper efficiency. Since permutation matrices P only change the order of the objectives, we restrict ourselves to the identity permutation matrix $P = I$ throughout this paper.

We point out that CMTs are tailored to multiobjective optimization problems with the standard ordering cone $\mathbb{R}^m_{\geqq}$. For general ordering cones, it cannot be expected that image space transformations that leave the set of efficient points (with respect to this general ordering cone) invariant have this component-wise structure.

2.5. Optimality conditions

We recall a first order necessary condition for proper efficiency and a second order sufficient condition for strict local efficiency of order 2. For more details on the following concepts from nonlinear programming we refer to [18]. In the remainder of this section, we assume that there exist sufficiently smooth functions $g : \mathbb{R}^n \to \mathbb{R}^q$ and $h : \mathbb{R}^n \to \mathbb{R}^r$ such that the feasible set of *MOP* is given by $X = \{x \in \mathbb{R}^n \mid g(x) \leqq 0, h(x) = 0\}$, and that f is sufficiently smooth as well. We define the indices of active constraints (active index set) as

$$I_0(\bar{x}) = \{i \in [q] \mid g_i(\bar{x}) = 0\}.$$

For first order approximations of the feasible set at a point $\bar{x}$, both the tangent cone

$$T(\bar{x}, X) := \{d \in \mathbb{R}^n \mid \exists (d^k, t^k)_{k\in\mathbb{N}}, d^k \to d, t^k \searrow 0 \text{ with } \bar{x} + t^k d^k \in X \,\forall\, k \in \mathbb{N}\}$$

and the linearization cone

$$L_{\leqq}(\bar{x}, X) = \{d \in \mathbb{R}^n \mid \nabla g_i(\bar{x})^\top d \leqq 0,\ i \in I_0(\bar{x}),\ \nabla h_j(\bar{x})^\top d = 0,\ j \in [r]\}$$

are commonly used. If $T(\bar{x}, X) = L_{\leqq}(\bar{x}, X)$ holds, the Abadie constraint qualification (ACQ) is said to hold.

Theorem 2.13 (Proposition 4.3 in [19]): *Let $\bar{x}$ be a properly efficient point of MOP at which the ACQ holds. Then there exist $\bar{\kappa} \in \mathbb{R}^m_>$, $\bar{\lambda} \in \mathbb{R}^q_{\geqq}$, $\bar{\mu} \in \mathbb{R}^r$ such that*

$$\sum_{i=1}^m \bar{\kappa}_i \nabla f_i(\bar{x}) + \sum_{j=1}^q \bar{\lambda}_j \nabla g_j(\bar{x}) + \sum_{k=1}^r \bar{\mu}_k \nabla h_k(\bar{x}) = 0, \tag{2}$$

$$\bar{\lambda}_j g_j(\bar{x}) = 0, \quad j \in [q]. \tag{3}$$

Finally, we state a second order sufficient condition for a strict local efficient point of order 2 from [12], for which we introduce

$$L_{\leqq}(\bar{x}, f) := \{d \in \mathbb{R}^n \mid \nabla f_i(\bar{x})^\top d \leqq 0,\ i \in [m]\}.$$

Theorem 2.14 (Corollary 5.3 in [12]): *Let $\bar{x} \in X$ and suppose that there exist $\bar{\kappa} \in \mathbb{R}^m_{\geqq}$, $\bar{\lambda} \in \mathbb{R}^q_{\geqq}$, $\bar{\mu} \in \mathbb{R}^r$ such that (2) and (3) hold, and*

$$d^\top \left(\sum_{i=1}^m \bar{\kappa}_i D^2 f_i(\bar{x}) + \sum_{j=1}^q \bar{\lambda}_j D^2 g_j(\bar{x}) + \sum_{k=1}^r \mu_k D^2 h_k(\bar{x}) \right) d > 0 \tag{4}$$

is fulfilled for all $d \in L_{\leqq}(\bar{x}, X) \cap L_{\leqq}(\bar{x}, f) \setminus \{0\}$. Then $\bar{x}$ is a strict local efficient point of order 2.

3. Supportedness respecting transformations

In this section we introduce a class of CMTs under which no supported points are lost in the transformed problem.

Definition 3.1: Let $\mathcal{Y}, \mathcal{Z}$ be boxes and $\Phi : \mathcal{Y} \to \mathcal{Z}$ a CMT. Then Φ is called a supportedness respecting transformation (SRT) if for all $Y \subseteq \mathcal{Y}$ and all supported points y of Y, the transformed point $\Phi(y)$ is a supported point of $\Phi(Y)$. If in addition $\mathcal{Y}$ and $\mathcal{Z}$ are open and Φ is a $\mathcal{C}^1$-diffeomorphism, we call Φ a supportedness respecting $\mathcal{C}^1$-transformation, or $\mathcal{C}^1$-SRT for short.

In the remainder of this section, we will characterize $\mathcal{C}^1$-SRTs as $\mathcal{C}^1$-CMTs with convex components. This requires a lemma on the convex hull of a set.

Definition 3.2: Let $Y \subseteq \mathbb{R}^m$. The intersection of all convex supersets of Y is called the convex hull conv Y of Y.

Lemma 3.3: *Let $Y \subseteq \mathbb{R}^m$ and $\bar{y} \in Y$. Then $\bar{y}$ is a supported point of Y if and only if it is a supported point of the convex hull* conv Y *of Y.*

Proof: "$\Rightarrow$": Let $\bar{y}$ be a supported point of Y. Then there exists some $w \geq 0$ such that $w^\top y \geqq w^\top \bar{y}$ for all $y \in Y$ or equivalently, Y is a subset of the halfspace $H_{\geqq} := \{y \in \mathbb{R}^m \mid w^\top y \geqq w^\top \bar{y}\}$. This means that $H_{\geqq}$ is a convex superset of Y and thus, $Y \subseteq \text{conv } Y \subseteq H_{\geqq}$ holds. We thus have $w^\top y \geqq w^\top \bar{y}$ for all $y \in \text{conv } Y$, which in combination with $\bar{y} \in Y \Rightarrow \bar{y} \in \text{conv } Y$ means that $\bar{y}$ is a supported point of conv Y.

"$\Leftarrow$": Let $\bar{y}$ be a supported point of conv Y, then there exists some $w \geq 0$ such that $w^\top \bar{y} \leqq w^\top y$ holds for all $y \in \text{conv } Y$. From $Y \subseteq \text{conv } Y$ and $\bar{y} \in Y$ it follows that $\bar{y}$ is a supported point of Y. ∎

The next result shows that a CMT is an SRT if its components are convex functions.

Theorem 3.4: *Let $\mathcal{Y}, \mathcal{Z}$ be boxes and $\Phi : \mathcal{Y} \to \mathcal{Z}$ be a CMT where in addition, the components φ_i, $i \in [m]$, are convex. Then Φ is an SRT.*

Proof: Let $Y \subseteq \mathcal{Y}$ and let y be a supported point of Y. From Lemma 3.3 we know that y is also a supported point of conv Y. It follows from Theorem 2.9 that y is a weakly nondominated point of conv Y. Since Φ is a CMT, $\Phi(y)$ is a weakly nondominated point of $\Phi(\text{conv } Y)$. We now artificially write the components of Φ as $\Phi_i : \mathcal{Y} \to \mathcal{Z}_i$, $\Phi_i(y) = \varphi_i(y_i)$. It is easy to verify that all Φ_i, $i \in [m]$, are convex and thus the auxiliary problem

$$\min \Phi(y) \text{ s.t. } \quad y \in \text{conv } Y$$

is convexlike. From Theorem 2.10 it thus follows that $\Phi(y)$ is a supported point of $\Phi(\text{conv } Y)$. Since $\Phi(Y) \subseteq \Phi(\text{conv } Y)$ and $\Phi(y) \in \Phi(Y)$, it follows that $\Phi(y)$ is a supported point of $\Phi(Y)$. ∎

In the following theorem we will also establish the converse of Theorem 3.4 in the sense that the components of a $\mathcal{C}^1$-SRT must be convex. The proof proceeds by contraposition:

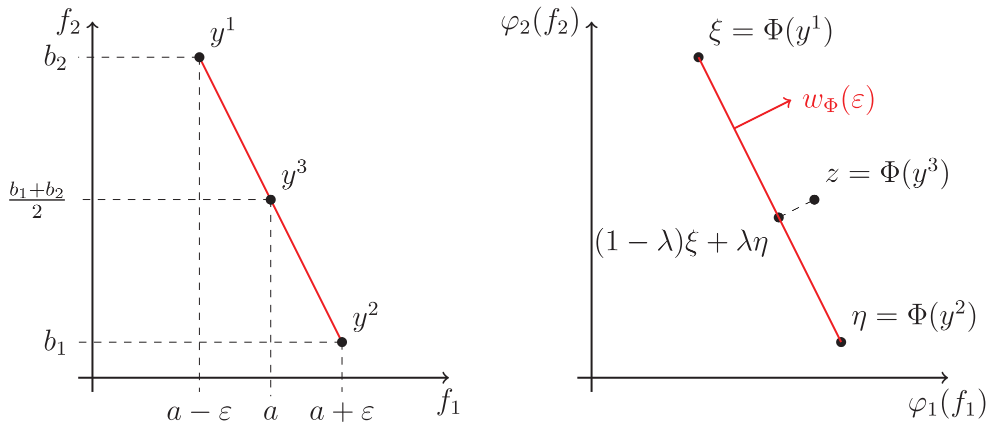

Figure 1. Sketch of the proof of Theorem 3.5.

we assume a $\mathcal{C}^1$-CMT $\Phi : \mathcal{Y} \to \mathcal{Z}$ where at least one component, say φ_2, is nonconvex and then construct a set $Y \subseteq \mathcal{Y}$, such that all points in Y are supported, but an unsupported point exists in $\Phi(Y)$.

The main construction of the proof is illustrated for $m = 2$ in Figure 1. Since by Theorem 2.11 midpoint convexity is also violated, we can choose b_1, b_2 such that $\varphi_2((b_1 + b_2)/2) > (\varphi_2(b_1) + \varphi_2(b_2))/2$. We then define $Y := \{y^1, y^2, y^3\}$ as shown on the left-hand side of Figure 1. All three points lie on a line and are thus supported. The scalar $\varepsilon > 0$ is chosen sufficiently small, such that the primary variation in Y appears along the coordinate associated with the nonconvex component φ_2. Using Taylor's theorem, we show that for such an ε there exists some weight vector $w_\Phi(\varepsilon)$ such that in the transformed set $\Phi(Y)$ it holds that

$$w_\Phi^\top(\varepsilon)\Phi(y^1) = w_\Phi^\top(\varepsilon)\Phi(y^2) < w_\Phi^\top(\varepsilon)\Phi(y^3).$$

This situation is illustrated on the right-hand side of Figure 1, where it becomes evident that $\Phi(y^3)$ is not supported. To formalize this, we invoke Theorem 2.8 and show that there exists some convex combination of $\Phi(y^1)$ and $\Phi(y^2)$ that strictly dominates $\Phi(y^3)$. This dominating point is obtained by projecting $\Phi(y^3)$ onto the line connecting $\Phi(y^1)$ and $\Phi(y^2)$. The generalization to $m > 2$ is straightforward and involves extending the three points to three $(m-2)$-dimensional boxes, which only play a minor role in the argument.

Theorem 3.5: *Let $\mathcal{Y}, \mathcal{Z}$ be open boxes and $\Phi : \mathcal{Y} \to \mathcal{Z}$ a $\mathcal{C}^1$-CMT. Then Φ is an SRT if and only if all φ_i, $i \in [m]$, are convex.*

Proof: In view of Theorem 3.4, we only need to show "$\Leftarrow$", which we will do by contraposition. Assume there exists some $j \in [m]$ such that φ_j is nonconvex. W.l.o.g., let $j = 2$. By Theorem 2.11, there exist $b_1, b_2 \in \mathcal{Y}_2$, $b_1 < b_2$ such that

$$\varphi_2(\frac{1}{2}(b_1 + b_2)) > \frac{1}{2}(\varphi_2(b_1) + \varphi_2(b_2)). \tag{5}$$

Choose a point $a \in \mathcal{Y}_1$ and $\varepsilon > 0$ such that $[a-\varepsilon, a+\varepsilon] \subset \mathcal{Y}_1$ and closed intervals $Y_i = [\underline{y}_i, \overline{y}_i] \subset \mathcal{Y}_i$ with $\underline{y}_i < \overline{y}_i$ for $i = 3, \dots, m$. Set

$$Y^1(\varepsilon) := \{a-\varepsilon\} \times \{b_2\} \times Y_3 \times \cdots \times Y_m,$$
$$Y^2(\varepsilon) := \{a+\varepsilon\} \times \{b_1\} \times Y_3 \times \cdots \times Y_m,$$
$$Y^3(\varepsilon) := \{a\} \times \left\{\frac{1}{2}(b_1+b_2)\right\} \times Y_3 \times \cdots \times Y_m,$$

and

$$Y(\varepsilon) := Y^1(\varepsilon) \cup Y^2(\varepsilon) \cup Y^3(\varepsilon).$$

All points in $Y(\varepsilon)$ are supported. This can be seen by choosing the weight vector $w(\varepsilon) := (b_2 - b_1, 2\varepsilon, 0, \dots, 0)^\top \in \mathbb{R}^m_\geq$ and computing that for all points $y \in Y(\varepsilon)$, it holds that $w^\top y = a(b_2 - b_1) + \varepsilon(b_1 + b_2)$. In particular, all points in $Y(\varepsilon)$ are weakly nondominated. It is not hard to see that the points

$$y^1_{nd} := (\alpha - \varepsilon, b_2, \underline{y}_3, \dots, \underline{y}_m)^\top,$$
$$y^2_{nd} := (\alpha + \varepsilon, b_1, \underline{y}_3, \dots, \underline{y}_m)^\top,$$
$$y^3_{nd} := \left(\alpha, \frac{b_1+b_2}{2}, \underline{y}_3, \dots, \underline{y}_m\right)^\top$$

are even nondominated. Let $Z_i := [\underline{z}_i, \overline{z}_i] := [\varphi_i(\underline{y}_i), \varphi_i(\overline{y}_i)]$, $i = 3, \dots, m$, and define

$$Z^1(\varepsilon) := \{\varphi_1(a-\varepsilon)\} \times \{\varphi_2(b_2)\} \times Z_3 \times \cdots \times Z_m,$$
$$Z^2(\varepsilon) := \{\varphi_1(a+\varepsilon)\} \times \{\varphi_2(b_1)\} \times Z_3 \times \cdots \times Z_m,$$
$$Z^3(\varepsilon) := \{\varphi_1(a)\} \times \left\{\varphi_2\left(\frac{1}{2}(b_1+b_2)\right)\right\} \times Z_3 \times \cdots \times Z_m,$$

as well as

$$Z(\varepsilon) := \Phi(Y(\varepsilon)) = Z^1(\varepsilon) \cup Z^2(\varepsilon) \cup Z^3(\varepsilon).$$

Choose $w_\Phi(\varepsilon) := (\varphi_2(b_2) - \varphi_2(b_1), \varphi_1(a+\varepsilon) - \varphi_1(a-\varepsilon), 0, \dots, 0)^\top \in \mathbb{R}^m_\geq$ and let $z^{12} \in Z^1(\varepsilon) \cup Z^2(\varepsilon)$. We then have

$$w_\Phi^\top(\varepsilon) z^{12} = \varphi_1(a+\varepsilon)\varphi_2(b_2) - \varphi_1(a-\varepsilon)\varphi_2(b_1).$$

For $z^3 \in Z^3(\varepsilon)$, we have

$$\begin{aligned} w_\Phi^\top(\varepsilon)(z^3 - z^{12}) = {} & (\varphi_2(b_2) - \varphi_2(b_1))\varphi_1(a) + (\varphi_1(a+\varepsilon) - \varphi_1(a-\varepsilon))\varphi_2\left(\frac{b_1+b_2}{2}\right) \\ & - \varphi_1(a+\varepsilon)\varphi_2(b_2) + \varphi_1(a-\varepsilon)\varphi_2(b_1), \end{aligned}$$

and using a first order Taylor expansion of both $\varphi_1(a+\varepsilon)$ and $\varphi_1(a-\varepsilon)$ at the point a, we obtain

$$\begin{aligned} w_\Phi^\top(\varepsilon)(z^3 - z^{12}) &= (\varphi_2(b_2) - \varphi_2(b_1))\varphi_1(a) + (2\varepsilon(\varphi_1'(a) + \omega(\varepsilon))\varphi_2\left(\frac{b_1+b_2}{2}\right) \\ &\quad - [\varphi_1(a) + \varepsilon(\varphi_1'(a) + \omega(\varepsilon))]\varphi_2(b_2) \\ &\quad + [\varphi_1(a) - \varepsilon(\varphi_1'(a) + \omega(\varepsilon))]\varphi_2(b_1) \\ &= 2\varepsilon(\varphi_1'(a) + \omega(\varepsilon))\left[\varphi_2\left(\frac{1}{2}(b_1+b_2)\right) - \frac{1}{2}(\varphi_2(b_1) + \varphi_2(b_2))\right], \end{aligned}$$

where ω denotes (possibly different) terms with $\omega(\varepsilon) \xrightarrow{\varepsilon \to 0} 0$. Because of (5) and $\varphi_1'(a) > 0$, we thus have $w_\Phi^\top(\varepsilon)(z^3 - z^{12}) > 0$ when ε is sufficiently small. Note that the term $w_\Phi^\top(\varepsilon)(z^3 - z^{12})$ does not depend on the concrete choices of z^{12} and z^3, and thus, neither does ε. We now choose $\varepsilon > 0$ such that for all $z^1 \in Z^1(\varepsilon), z^2 \in Z^2(\varepsilon)$ and $z^3 \in Z^3(\varepsilon)$, we have

$$w_\Phi^\top(\varepsilon)z^1 = w_\Phi^\top(\varepsilon)z^2 < w_\Phi^\top(\varepsilon)z^3. \tag{6}$$

We will use Theorem 2.8 to show that $Z^3(\varepsilon)$ contains an unsupported point. To this end we define

$$\begin{aligned} \xi &:= \Phi(y_{nd}^1) = (\varphi_1(a-\varepsilon), \varphi_2(b_2), \underline{z}_3, \dots, \underline{z}_m)^\top \in Z^1(\varepsilon), \\ \eta &:= \Phi(y_{nd}^2) = (\varphi_1(a+\varepsilon), \varphi_2(b_1), \underline{z}_3, \dots, \underline{z}_m)^\top \in Z^2(\varepsilon), \\ z &:= (\varphi_1(a), \varphi_2((b_1+b_2)/2), \bar{z}_3, \dots, \bar{z}_m)^\top \in Z^3(\varepsilon), \end{aligned}$$

and $w := w_\Phi(\varepsilon)$. In particular, it follows from (6) that we have

$$w^\top \xi = w^\top \eta < w^\top z. \tag{7}$$

Since Φ is a CMT, the points ξ and η are nondominated points of $\Phi(Y(\varepsilon))$. We will show that there exists some $\lambda \in (0,1)$ with

$$(1-\lambda)\xi + \lambda\eta < z. \tag{8}$$

We find such a λ by projecting the point z onto the line through ξ and η, i.e.we impose

$$(z - ((1-\lambda)\xi + \lambda\eta))^\top(\xi - \eta) = 0,$$

resulting in

$$\lambda = \frac{(\eta_1 - \xi_1)(z_1 - \xi_1) + (\xi_2 - \eta_2)(\xi_2 - z_2)}{(\eta_1 - \xi_1)^2 + (\xi_2 - \eta_2)^2}.$$

By inserting the definitions, one sees that

$$\begin{aligned} &0 < (\eta_1 - \xi_1)(z_1 - \xi_1) < (\eta_1 - \xi_1)^2 \text{ and} \\ &0 < (\xi_2 - \eta_2)(\xi_2 - z_2) < (\xi_2 - \eta_2)^2 \end{aligned}$$

hold, which means that in fact, $\lambda \in (0,1)$. We now show that every component of $z - ((1-\lambda)\xi + \lambda\eta)$ is positive. For $i = 3, \dots, m$, we simply have

$$z_i - ((1-\lambda)\xi_i + \lambda\eta_i) = \bar{z}_i - ((1-\lambda)\underline{z}_i + \lambda\underline{z}_i) = \bar{z}_i - \underline{z}_i > 0.$$

For the first two components, note that $w = (\xi_2 - \eta_2, \eta_1 - \xi_1, 0, \dots, 0)^\top$ holds, and we can thus write

$$\lambda = \frac{1}{w^\top w} w^\top \begin{pmatrix} \xi_2 - z_2 \\ z_1 - \xi_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We thus have

$$w^\top w(z - ((1-\lambda)\xi + \lambda\eta)) = w^\top w(z - \xi) - w^\top \begin{pmatrix} \xi_2 - z_2 \\ z_1 - \xi_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} (\eta - \xi).$$

For the first component, this results in

$$\begin{aligned} w^\top w(z_1 - ((1-\lambda)\xi_1 + \lambda\eta_1)) &= w^\top \left(w(z_1 - \xi_1) - \underbrace{(\eta_1 - \xi_1)}_{=w_2} \begin{pmatrix} \xi_2 - z_2 \\ z_1 - \xi_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) \\ &= w^\top \left(\begin{pmatrix} w_1(z_1 - \xi_1) \\ w_2(z_1 - \xi_1) \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} w_2(\xi_2 - z_2) \\ w_2(z_1 - \xi_1) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) \\ &= w_1(w^\top (z - \xi)) \\ &> 0, \end{aligned}$$

where the inequality holds due to (7) and $w_1 > 0$. Dividing by $w^\top w > 0$ shows that we have $z_1 - ((1-\lambda)\xi_1 + \lambda\eta_1) > 0$, and a similar computation yields $z_2 - ((1-\lambda)\xi_2 + \lambda\eta_2) > 0$. In conclusion, we showed that (8) holds. Additionally, it is not hard to see that $Z(\varepsilon)_{nd}$ is externally stable and thus, by Theorem 2.8, z is not a supported point of $Z(\varepsilon) = \Phi(Y(\varepsilon))$. Since all points in $Y(\varepsilon)$ are supported, Φ is not an SRT. ■

Theorem 3.5 provides the desired characterization of C^1-diffeomorphisms that preserve both efficiency and supportedness: the C^1-diffeomorphisms that preserve efficiency are exactly the C^1-CMTs (cf. [2]), and Theorem 3.5 characterizes the C^1-CMTs that also preserve supportedness, namely the C^1-SRTs, as the C^1-CMTs with convex components.

4. Generation of supported points

In this section we turn our attention to SRTs that not only preserve supportedness, but can also render unsupported points supported. In Section 4.1, we establish an equivalence between the supportedness of a nondominated point and strong Lagrangian duality for a special case of the ε-constraint scalarization $R(i, b)$. Section 4.2 presents results from the literature that provide conditions under which a certain p-th power transformation guarantees strong duality for single-objective problems. In Section 4.3, these results are applied to $R(i, b)$ to prove that a multiobjective p-th power transformation can render a nondominated point supported if the feasible set of *MOP* is polyhedral. Section 4.4 briefly outlines an extension to the fully nonconvex case, and in Section 4.5, we state a new result on the compromise programming approach with ℓ_p-norms that is equivalent to our findings on the p-th power transformation.

4.1. Supportedness and duality

In this section, we relate the supportedness of an efficient point to the Lagrangian dual problem of a particular ε-constraint scalarization. A connection between Lagrangian duality and scalarizations is mentioned in [20], Chapter 5.3.3. In Theorem 4.2, we formalize this observation and provide a proof based on similar ideas as in the proof of Theorem 4.6 in [1].

Consider a general single-objective inequality-constrained problem

$$\min_x \widetilde{f}(x) \text{ s.t. } \quad g(x) \leqq b, \; x \in X, \tag{P}$$

where $X \subseteq \mathbb{R}^n, \widetilde{f} : X \to \mathbb{R}$, $g : X \to \mathbb{R}^q$ and $b \in \mathbb{R}^q$. The associated Lagrangian function is given by

$$L(x, \lambda) := \widetilde{f}(x) + \lambda^\top (g(x) - b).$$

The Lagrangian dual of P consists of maximizing the dual objective

$$\psi(\lambda) := \inf_{x \in X} L(x, \lambda)$$

over all $\lambda \in \mathbb{R}^q_{\geqq}$, i.e.

$$\max \; \psi(\lambda) \text{ s.t. } \quad \lambda \geqq 0. \tag{D}$$

By the weak duality theorem, for a primal-dual feasible pair (x, λ), the inequality $\psi(\lambda) \leqq \widetilde{f}(x)$ always holds (cf. [20]). If this inequality is fulfilled with equality at such a pair of points, strong duality is said to hold at (x, λ), and the points are optimal for their respective problems.

Lemma 4.1: *Let x be feasible for P and $\lambda \in \mathbb{R}^q_{\geqq}$ such that $\widetilde{f}(x) = \psi(\lambda)$. Then x is a minimal point of P and λ a maximal point of D.*

Whenever such a pair (x, λ) exists, we say that strong duality holds for P. In the next theorem, we apply this duality framework to a scalarization $R(i, b)$, with $b = f(\bar{x})$, which is denoted $R(i, f(\bar{x}))$.

Theorem 4.2: *Let $\bar{x} \in X$ be efficient. Then $\bar{x}$ is a supported point of MOP if and only if strong duality holds for $R(i, f(\bar{x}))$ for some $i \in [m]$.*

Proof: The dual problem of $R(i, f(\bar{x}))$ is given by

$$\max\ \psi(\lambda) := \left(\inf_{x \in X} f_i(x) + \sum_{j \in [m] \setminus \{i\}} \lambda_j (f_j(x) - f_j(\bar{x})) \right) \text{ s.t. } \quad \lambda \geqq 0. \tag{9}$$

"$\Leftarrow$": By strong duality, there exist some $\widetilde{x} \in X$ with $f_j(\widetilde{x}) \leqq f_j(\bar{x}), j \in [m] \setminus \{i\}$, and some $\bar{\lambda} \geqq 0$ such that $\psi(\bar{\lambda}) = f_i(\widetilde{x})$ holds. By Lemma 4.1, $\widetilde{x}$ is thus a minimal point of $R(i, f(\bar{x}))$. Since $\bar{x}$ is efficient, it is also a minimal point of $R(i, f(\bar{x}))$ (proof of Theorem 4.5 in [1]) and we have $\psi(\bar{\lambda}) = f_i(\bar{x})$. If we choose $\bar{w} := (\bar{\lambda}_1, \ldots, \bar{\lambda}_{i-1}, 1, \bar{\lambda}_{i+1}, \ldots \bar{\lambda}_m)^\top$, we obtain

$$f_i(\bar{x}) = \psi(\bar{\lambda}) = \left(\inf_{x \in X} \bar{w}^\top f(x) \right) - \sum_{j \in [m] \setminus \{i\}} \bar{\lambda} f_j(\bar{x}),$$

and, by rearranging,

$$\inf_{x \in X} \bar{w}^\top f(x) = \sum_{j \in [m] \setminus \{i\}} \bar{\lambda} f_j(\bar{x}) + f_i(\bar{x}) = \bar{w}^\top f(\bar{x}).$$

The point $\bar{x}$ is thus a minimal point of $\widetilde{WS}(w)$ with $w = \bar{w}$, i.e. $\bar{x}$ is a supported point of *MOP*.

"$\Rightarrow$": Since $\bar{x}$ is supported, there exists some $\bar{w} \in \mathbb{R}^m_{\geq}$ such that $\bar{x}$ is a minimal point of $\widetilde{WS}(w)$ with $w = \bar{w}$. We thus have

$$\inf_{x \in X} \bar{w}^\top f(x) = \bar{w}^\top f(\bar{x}).$$

There exists some $i \in [m]$ such that $\bar{w}_i > 0$, and since $\bar{w}$ can be scaled arbitrarily, we can assume w.l.o.g. that $\bar{w}_i = 1$ holds. Now, if we define the vector $\bar{\lambda} := (\bar{w}_1, \ldots, \bar{w}_{i-1}, \bar{w}_{i+1}, \ldots, \bar{w}_m)$, rearranging yields $f_i(\bar{x}) = \psi(\bar{\lambda})$. Since $\bar{\lambda} \geqq 0$, $\bar{x} \in X$ and $f_j(\bar{x}) \leqq f_j(\bar{x}), j \in [m] \setminus \{i\}$, strong duality holds for $R(i, f(\bar{x}))$. ■

4.2. Strong duality via power transformation

Beginning in the 1990s, several authors have studied the so-called p-th power transformation for single-objective problems with inequality constraints and an abstract set constraint. In the original formulation by [21], the objective function, constraints and right-hand sides are raised to a power of p. This problem is equivalent to the original one if the objective, constraints, and right-hand sides are positive, which can be enforced by applying an exponential function first or by adding a sufficiently large constant, if they are bounded from below.

In [21], it was shown that for sufficiently large p, the p-th power transformation satisfies the assumption of the local duality theorem (cf. [22]), making it solvable by certain primal-dual algorithms. Subsequent works like [23] and [24] extended this result to a broader setting and introduced additional variants of the approach.

Later, in [25] and [26], it was shown that under stronger assumptions, even strong Lagrangian duality holds for the p-th power transformation if p is sufficiently large. The same result was later shown in [27] under slightly weaker conditions.

From this point forward, we assume that the problem data of P fulfil $\widetilde{f} : X \to \mathbb{R}_>$, $g : X \to \mathbb{R}^q_>$, and $b \in \mathbb{R}^q_>$. The positivity assumption is not restrictive, as it can be ensured by applying the exponential function to all components. Note that the scalarization $R(i, b)$ is a special case of P and can thus be treated by the p-th power transformation. For $p \in \mathbb{R}$ and a vector $a \in \mathbb{R}^m$, we define

$$a^p := (a_1^p, \dots, a_m^p)^\top.$$

Let $p \geqq 1$. The p-th power transformation of P is given by

$$\min_x \widetilde{f}(x)^p \text{ s.t. } \quad g(x)^p \leqq b^p, \; x \in X. \tag{$Q(p)$}$$

The positivity of $\widetilde{f}$, g and b ensures that the local and global minimal points of P and $Q(p)$ coincide.

We now summarize the main result from [25], which states that under certain conditions, strong duality holds for $Q(p)$ if p is sufficiently large.

Let $\nabla_x L(\bar{x}, \bar{\lambda})$ and $D_x^2 L(\bar{x}, \bar{\lambda})$ denote the gradient and Hessian matrix of the Lagrangian of P with respect to x at a point $(\bar{x}, \bar{\lambda})$. We define the sets

$$C(\bar{x}, \bar{\lambda}) := \{d \in \mathbb{R}^n \mid \nabla \widetilde{f}(\bar{x})^\top d = 0, \; \nabla g_j(\bar{x})^\top d = 0 \text{ for all } j \text{ with } \bar{\lambda}_j > 0\},$$
$$F(\bar{x}, \bar{\lambda}) := \{x \in X \mid \widetilde{f}(x) \leqq \widetilde{f}(\bar{x}), \; g_j(x) \leqq b_j \text{ for all } j \text{ with } \bar{\lambda}_j > 0\}.$$

Assumption 4.3: Assume that X is compact and P uniquely solvable with minimal point $\bar{x}$ and that there exists some $\bar{\lambda} \in \mathbb{R}^m_{\geqq}$ such that

(a) we have

$$\nabla_x L(\bar{x}, \bar{\lambda})^\top d \geqq 0 \quad \forall d \in T(\bar{x}, X), \tag{10a}$$

$$\sum_{i=1}^m \bar{\lambda}_i (g_i(\bar{x}) - b_i) = 0 \tag{10b}$$

(b) for all $d \in C(\bar{x}, \bar{\lambda}) \cap T(\bar{x}, X) \setminus \{0\}$, it holds that $d^\top D_x^2 L(\bar{x}, \bar{\lambda}) d > 0$,
(c) there exists some neighbourhood U of $\bar{x}$ such that $X \cap U$ is convex,
(d) it holds that

$$F(\bar{x}, \bar{\lambda}) = \{\bar{x}\}. \tag{11}$$

It was shown in [23] that if a local minimal point of $Q(p)$ satisfies parts (a), (b) and (c) of Assumption 4.3, the Lagrangian L_p of $Q(p)$ possesses a local saddle point $(\bar{x}, \bar{\lambda}_p)$, if p is sufficiently large. The component $\bar{\lambda}_p$ of this local saddle point is defined via the multiplier $\bar{\lambda}$, whose existence is part of Assumption 4.3. This result was strengthened in [25], where it was shown that under Assumption 4.3, L_p has a global saddle point if p is sufficiently large.

According to Section 5.4.2 in [20], strong duality holds for a problem P if and only if its Lagrangian has a saddle point. This proves the following theorem.

Theorem 4.4: *Let Assumption 4.3 be fulfilled. Then there exists some $\bar{p} \geqq 1$ such that for all $p \geqq \bar{p}$, strong duality holds for $Q(p)$.*

In Section 4.3, we will discuss how restrictive Assumption 4.3 is and provide a sufficient condition that is tailored to multiobjective optimization.

4.3. Supportedness via power transformation

From now on, we assume that X is compact, the objective function f of MOP is twice continuously differentiable on X, and that the image set of MOP satisfies $Y \subseteq \mathbb{R}^m_>$. The latter assumption is not restrictive. By the compactness of X and continuity of f, Y is compact. Thus, one can choose a utopia point $\widehat{\alpha}$ and consider the transformed problem MOP_Φ with $\Phi(y) = y - \widehat{\alpha}$.

Under the assumption $Y \subseteq \mathbb{R}^m_>$ we will show that for some sufficiently large $p \geqq 1$, the SRT

$$\Phi^p : \mathbb{R}^m_> \to \mathbb{R}^m_>, \quad \Phi^p(y) = (y_1^p, \ldots, y_m^p)^\top$$

can render an efficient point $\bar{x}$ supported for MOP_Φ with $\Phi = \Phi^p$. We employ the ε-constraint scalarization

$$\min f_m(x) \text{ s.t. } \; f_i(x) \leqq f_i(\bar{x}),\ i \in [m-1],\ x \in X, \qquad (R(m,f(\bar{x})))$$

where we chose the index $i = m$ for convenience, although the following arguments are valid for any index. For $p \geqq 1$, the problem MOP_{Φ^p} denotes MOP_Φ with $\Phi = \Phi^p$. The ε-constraint scalarization of MOP_{Φ^p} with $i = m$ and $b = \Phi^p(f(\bar{x}))$ is given by

$$\min f_m(x)^p \text{ s.t. } \; f_i(x)^p \leqq f_i(\bar{x})^p, \quad i \in [m-1],\ x \in X. \qquad (R^p(m,f(\bar{x})^p))$$

Lemma 4.5: *Let X be compact, $Y \subseteq \mathbb{R}^m_>$, and $\bar{x}$ be an efficient point of MOP. If Assumption 4.3 holds for $R(m,f(\bar{x}))$ at $\bar{x}$, then there exists some $\bar{p} \geqq 1$ such that for all $p \geqq \bar{p}$, the point $\bar{x}$ is a supported point of MOP_{Φ^p}.*

Proof: The problem $R(m,f(\bar{x}))$ is a special case of P, and $R^p(m,f(\bar{x})^p)$ is its p-th power transformation $Q(p)$. Since X is compact and Assumption 4.3 holds for $R(m,f(\bar{x}))$ at $\bar{x}$, Theorem 4.4 implies the existence of some $\bar{p} \geqq 1$ such that for all $p \geqq \bar{p}$, strong duality holds for $R^p(m,f(\bar{x})^p)$. It thus follows from Theorem 4.2 that $\bar{x}$ is a supported point of MOP_{Φ^p}. ■

In the remainder of this section, we discuss how restrictive Assumption 4.3 is, and in the subsequent Lemma 4.7, we provide a sufficient condition for said assumption, which is more tailored to the context of multiobjective optimization.

We first briefly discuss the assumption for a general single-objective problem, ignoring the special structure of $R(m,f(\bar{x}))$. Assumption 4.3(a) generalizes the KKT conditions in the presence of the abstract set constraint $x \in X$. According to Corollary 6.15 in [28], this condition necessarily holds at a local minimal point under a certain constraint qualification. At an interior point $\bar{x}$ of X, part (b) is implied if $\bar{x}$ is a nondegenerate local minimal point (cf. [29]). At a boundary point however, the definition of a nondegenerate local minimal point involves a Lagrangian function that aggregates all active constraints, including the ones describing the set X, while the Lagrangian employed in Assumption 4.3 only takes the explicit constraints into account. This can be overcome if the explicit constraints of P describe a compact set and X is described by inequalities as well, since X can then be chosen such that the feasible set of P lies in its interior. In that case, part (b) can be considered mild, since it is a topologically generic assumption (cf. [30, 31]). If this generic assumption holds, the uniqueness of the minimal point $\bar{x}$ is only violated in degenerate situations, and part (c) of the assumption holds as well, since $\bar{x}$ is an interior point of X. In conclusion, all parts of the assumption mentioned so far can be considered mild. Unfortunately, part (d) is a more restrictive condition. The following example illustrates that it can be violated even in a nondegenerate case.

Example 4.6: The problem

$$\min 2 - x^2 \quad \text{s.t.} \quad g_1(x) = -x \leqq 1,\ g_2(x) = 1 + x \leqq 1, x \in X = [-100, 100]$$

has the unique globally minimal point $\bar{x} = -1$ with multipliers $\bar{\lambda}_1 = 2, \bar{\lambda}_2 = 0$, but we have

$$F(\bar{x}, \bar{\lambda}) = \{x \in \mathbb{R} \mid 2 - x^2 \leqq 1, -x \leqq 1\} = \{-1\} \cup [1, 100].$$

Furthermore, this violation persists under small perturbations of the problem data.

Assumption 4.3(d) also appears in [26] and [27]. In [25], it was shown that it is satisfied for convex programming problems. But in such cases, under a mild constraint qualification, strong duality already holds without requiring power transformations. Unfortunately, the authors also argued that this assumption is indeed indispensable.

Luckily, part (d) turns out to be less restrictive when applied to $R(m, f(\bar{x}))$. In this case the set X is the feasible set of the original problem MOP, and the only explicit constraints,

$$f_i(x) \leqq f_i(\bar{x}),\ i \in [m-1],$$

are active at $\bar{x}$ by construction. We will show that under the additional assumption that $\bar{x}$ is properly efficient for MOP, the multipliers corresponding to these explicit constraints are positive. Thus, Assumption 4.3(d) becomes equivalent to the condition that $R(m, f(\bar{x}))$ is uniquely solvable.

There is however a drawback in having X represent the feasible set of MOP. It is now possible that a feasible point lies on the boundary of X and thus, part (c) is not guaranteed to hold, and part (b) is no longer the generic sufficient condition for strict local minimality. As a remedy, we will from now on assume that X is a polyhedron described by linear functions $g_j, j \in [q]$ and $h_k, k \in [r]$, i.e.

$$X = \{x \in \mathbb{R}^n \mid g_j(x) \leqq 0,\ j \in [q],\ h_k(x) = 0,\ k \in [r]\}.$$

Thus, part (c) is again guaranteed to hold since X is convex, and part (b) again takes the form the generic second order sufficient condition for a local minimal point, since the second derivatives of the linear functions describing X would vanish if one computed the Hessian of a Lagrangian that also takes into account the constraints describing X.

Lemma 4.7: *Let X be a polyhedron and $\bar{x}$ a strictly and properly efficient point of MOP. Furthermore, suppose that $\bar{x}$ satisfies the sufficient condition for a strict local efficient point of order 2 from Theorem 2.14 together with its multipliers $\widetilde{\kappa} \in \mathbb{R}^m_>$, $\widetilde{\lambda} \in \mathbb{R}^q_{\geqq}$, $\widetilde{\mu} \in \mathbb{R}^r$, which exist according to Theorem 2.13. Then Assumption 4.3 holds for $R(m, f(\bar{x}))$ at $\bar{x}$.*

Proof: Since $\bar{x}$ is strictly efficient, it follows from the definition that for all $x \neq \bar{x}$ that are feasible for $R(m, f(\bar{x}))$, it holds that $f_m(x) > f_m(\bar{x})$. Thus, $\bar{x}$ is the unique minimal point of $R(m, f(\bar{x}))$.

The condition (10b) of Assumption 4.3(a), i.e.

$$\sum_{i=1}^{m-1} \kappa_i (f_i(\bar{x}) - f_i(\bar{x})) = 0$$

is obviously fulfilled at $\bar{x}$ for arbitrary κ. The Lagrangian of $R(m, f(\bar{x}))$ is given by

$$L(x, \kappa) := f_m(x) + \sum_{i=1}^{m-1} \kappa_i (f_i(x) - f_i(\bar{x})),$$

and to prove that (10a) holds, we must show that there exists some $\bar{\kappa} \in \mathbb{R}^{m-1}_{\geqq}$ such that $(\nabla f_m(\bar{x}) + \sum_{i=1}^{m-1} \bar{\kappa}_i \nabla f_i(\bar{x}))^\top d \geqq 0$ for all $d \in T(\bar{x}, X)$. Since $\bar{x}$ is assumed to be properly efficient, and since the ACQ holds at $\bar{x}$ by the polyhedrality of X, we may apply Theorem 2.13. Hence there exist $\widetilde{\kappa} \in \mathbb{R}^m_>$, $\widetilde{\lambda} \in \mathbb{R}^q_{\geqq}$, $\widetilde{\mu} \in \mathbb{R}^r$ such that $\widetilde{\lambda}_j g_j(\bar{x}) = 0$ for $j \in [q]$, and

$$\sum_{i=1}^{m} \widetilde{\kappa}_i \nabla f_i(\bar{x}) + \sum_{j=1}^{q} \widetilde{\lambda}_j \nabla g_j(\bar{x}) + \sum_{k=1}^{r} \widetilde{\mu}_k \nabla h_k(\bar{x}) = 0.$$

If we divide both sides by $\widetilde{\kappa}_m > 0$ and define $(\bar{\kappa}, \bar{\lambda}, \bar{\mu}) := (\widetilde{\kappa}_1, \ldots, \widetilde{\kappa}_{m-1}, \widetilde{\lambda}, \widetilde{\mu})/\widetilde{\kappa}_m$, we obtain

$$\nabla f_m(\bar{x}) + \sum_{i=1}^{m-1} \bar{\kappa}_i \nabla f_i(\bar{x}) + \sum_{j=1}^{q} \bar{\lambda}_j \nabla g_j(\bar{x}) + \sum_{k=1}^{r} \bar{\mu}_k \nabla h_k(\bar{x}) = 0,$$

where $\bar{\kappa} \in \mathbb{R}^{m-1}_>$. Let $d \in T(\bar{x}, X)$. By the linearity of the constraints, the ACQ is fulfilled everywhere in X and we have $d \in L_{\leqq}(\bar{x}, X)$. It thus holds that $\nabla g_j(\bar{x})^\top d \leqq 0$ for all $j \in I_0(\bar{x})$ and $\nabla h_k(\bar{x})^\top d = 0$ for all $k \in [r]$. Furthermore, $\bar{\lambda}_j = 0$ for all $j \notin I_0(\bar{x})$. We obtain

$$\begin{aligned} &\left(\nabla f_m(\bar{x}) + \sum_{i=1}^{m-1} \bar{\kappa}_i \nabla f_i(\bar{x}) \right)^\top d \\ &\geqq \left(\nabla f_m(\bar{x}) + \sum_{i=1}^{m-1} \bar{\kappa}_i \nabla f_i(\bar{x}) + \sum_{j=1}^{q} \bar{\lambda}_j \nabla g_j(\bar{x}) + \sum_{k=1}^{r} \bar{\mu}_k \nabla h_k(\bar{x}) \right)^\top d = 0. \end{aligned}$$

For part (b), let $d \neq 0$ lie in the set $C(\bar{x},\bar{\kappa}) \cap T(\bar{x},X) = C(\bar{x},\bar{\kappa}) \cap L_{\leqq}(X,\bar{x})$. For $R(m,f(\bar{x}))$, we have

$$C(\bar{x},\bar{\kappa}) = \{d \in \mathbb{R}^n \mid \nabla f_i(\bar{x})^\top d = 0,\ i \in [m]\},$$

since $\bar{\kappa} > 0$. Note that $C(\bar{x},\bar{\kappa}) \subseteq L_{\leqq}(f,\bar{x})$ and thus, by assumption, we have that (4) holds at $(\bar{x},\widetilde{\kappa},\widetilde{\lambda},\widetilde{\mu})$. Dividing by $\widetilde{\kappa}_m$, and the fact that all entries of g and h are linear shows that we have

$$d^\top D_x^2 L(\bar{x},\bar{\kappa})d > 0.$$

Assumption 4.3(c) is fulfilled since X is a polyhedron.

For Assumption 4.3(d), note that for the problem $R(m,f(\bar{x}))$, we have

$$F(\bar{x},\bar{\kappa}) = \{x \in X \mid f_m(x) \leqq f_m(\bar{x}), f_j(x) \leqq f_j(\bar{x}), j \in [m-1]\},$$

since $\bar{\kappa}_j > 0$ for all $j \in [m-1]$. Thus, $F(\bar{x},\bar{\kappa})$ is the set of minimal points of $R(m,f(\bar{x}))$, and since $\bar{x}$ is the unique minimal point, we obtain $F(\bar{x},\bar{\kappa}) = \{\bar{x}\}$. ∎

The combination of Lemmas 4.5 and 4.7 yields the following result.

Theorem 4.8: *Let X be a bounded polyhedron, $Y \subseteq \mathbb{R}^m_>$, and $\bar{x}$ a strictly and properly efficient point of MOP. Furthermore, suppose that $\bar{x}$ satisfies the sufficient condition for a strict local efficient point of order 2 from Theorem 2.14 together with its multipliers $\widetilde{\kappa} \in \mathbb{R}^m_>$, $\widetilde{\lambda} \in \mathbb{R}^q_{\geqq}$, $\widetilde{\mu} \in \mathbb{R}^r$, which exist according to Theorem 2.13. Then there exists some $\bar{p} \geqq 1$ such that for all $p \geqq \bar{p}$, the point $\bar{x}$ is a supported point of MOP_{Φ^p}.*

4.4. General feasible sets

Theorem 4.8 requires the feasible set X to be a polyhedron. However, the capability of the p-th power transformation to generate supported points depends only on the geometry of the image set Y, but not on its specific description. Thus, if the nonlinearity of X can be "hidden" in the objective function without altering the image set, Theorem 4.8 remains applicable.

Specifically, assume that there exist some bounded polyhedron $\widetilde{X}$ and a surjective map $\Gamma \in \mathcal{C}^2(\widetilde{X},X)$. Then the problem

$$\min f(\Gamma(x)) \text{ s.t. } \quad x \in \widetilde{X} \qquad (MOP_{lin})$$

has the same image set as MOP, since $\Gamma(\widetilde{X}) = X$. To ensure that the condition that $R(m,f(\bar{x}))$ is uniquely solvable from Assumption 4.3 is preserved in the respective scalarization of MOP_{lin}, we assume that the map Γ is bijective. Note that the inverse of Γ is not required to be twice continuously differentiable and thus, it is not necessary that Γ is a $\mathcal{C}^2$-diffeomorphism.

Example 4.9: The set $X = \{x \in \mathbb{R}^2 \mid x_2 \geqq 0, \|x\|_2 = 1\}$ is not polyhedral, and also Assumption 4.3(c) is violated at every point in X. On the other hand, the set $\widetilde{X} := [0,\pi] \subseteq \mathbb{R}$ a polyhedron, and with the bijective $\mathcal{C}^2$-map Γ defined as $\Gamma(t) := (\cos t, \sin t)^\top$, it holds that $X = \Gamma(\widetilde{X})$.

Corollary 4.10: *Assume that there exist a bounded polyhedron $\widetilde{X}$ and a bijective map $\Gamma \in C^2(\widetilde{X}, X)$. Let $\bar{x} \in X$ and $\bar{t} \in \widetilde{X}$ such that $\bar{x} = \Gamma(\bar{t})$ and the assumptions of Theorem 4.8 are fulfilled for MOP_{lin} at $\bar{t}$. Then, there exists some $\bar{p} \geqq 1$ such that for all $p \geqq \bar{p}$, the point $\bar{x}$ is a supported point of MOP_{Φ^p}.*

Proof: By Theorem 4.8, there exists some $\bar{p} \geqq 1$ such that for all $p \geqq \bar{p}$, the point $\bar{t}$ is a supported point of

$$\min \Phi^p(f(\Gamma(t))) \text{ s.t. } \quad t \in \widetilde{X}.$$

Thus, $\Phi^p(f(\Gamma(\bar{t}))) = \Phi^p(f(\bar{x}))$ is a supported point of $\Phi^p(Y)$ and $\bar{x}$ a supported point of MOP_{Φ^p}, when $p \geqq \bar{p}$. ■

4.5. Compromise programming

In this final section we show that if $\bar{x}$ fulfils the assumptions of Corollary 4.10, it can also be obtained by the compromise programming approach with ℓ_p-norms (cf. [32, 33]). This scalarization is based on weighted ℓ_p-norms $\|y\|_{w,p} := (\sum_{i=1}^m |w_i y_i^p|)^{1/p}$, with $w \in \mathbb{R}^m_>$ and $p \in [1, \infty)$. In the compromise programming approach, it is even possible to have $w \in \mathbb{R}^m_\geq$, although $\|y\|_{p,w}$ is not a norm on $\mathbb{R}^m$ when w has zero-entries. The compromise programming approach is a scalarization

$$\min \|f(x) - \widehat{\alpha}\|_{w,p} \text{ s.t. } \quad x \in X \qquad (\widetilde{C}_p(w))$$

that minimizes the distance of image points to a utopia point $\widehat{\alpha}$ in a weighted ℓ_p-norm for some $p \in [1, \infty)$ and $w \in \mathbb{R}^m_\geq$. A p-th power is applied to the objective function, which does not alter the set of minimal points of the scalarization and renders a smooth objective function. This results in the image space formulation

$$\min \sum_{i=1}^m w_i(y_i - \widehat{\alpha}_i)^p \text{ s.t. } \quad y \in Y \qquad (C_p(w))$$

of the compromise programming approach. It is well known that when $w > 0$, every minimal point of $C_p(w)$ is properly nondominated. On the other hand, it was shown in [34] that if Y is closed, every nondominated point of Y can be approximated by compromise solutions in the following sense. For every nondominated point $\bar{y} \in Y$ and $\varepsilon > 0$, there exists some $\bar{p} \geqq 1$ such that for all $p \geqq \bar{p}$ there exists some $w \in \mathbb{R}^m_>$ such that for every minimal point $\widetilde{y}$ of $C_p(w)$, it holds that $\|\bar{y} - \widetilde{y}\|_\infty < \varepsilon$.

Using our new results on the p-th power transformation, we can show that certain properly efficient points cannot only be approximated by the compromise programming approach, but are even attained as minimal points of $C_p(w)$ for sufficiently large p. If MOP and $\bar{x} \in X$ fulfil the assumptions of Corollary 4.10, then $Y \subseteq \mathbb{R}^m_>$ and thus, 0 is a utopia point of Y. Thus, the compromise programming approach of *MOP* with $\widehat{\alpha} = 0$ is given by

$$\min \sum_{i=1}^{m} w_i y_i^p \text{ s.t. } \quad y \in Y, \qquad (C_p^0(w))$$

which is precisely the weighted sum scalarization of MOP_{Φ^p}. We have shown in Corollary 4.10 that there exists some $\bar{p} \geqq 1$ such that for all $p \geqq \bar{p}$, the point $\bar{x}$ is a supported point of MOP_{Φ^p}, which by definition means that there exists some $w \in \mathbb{R}^m_{\geq}$ such that $f(\bar{x})$ is a minimal point of $C_p^0(w)$. This proves the following result.

Corollary 4.11: *Assume that the assumptions of Corollary 4.10 are fulfilled at $\bar{x} \in X$. Then there exist some $\bar{p} \geqq 1$ such that for all $p \geqq \bar{p}$, there exists some $w \in \mathbb{R}^m_{\geq}$ such that $f(\bar{x})$ is a minimal point of $C_p^0(w)$.*

5. Conclusion

In this work, we have shown that component-wise image space transformations with strictly increasing and convex components preserve the set of efficient points and ensure that no supported points are lost because of the transformation, while new supported points might be generated. From a practitioner's point of view, this means that such transformations can be safely applied when the intended solution approach is based on finding supported points, like the weighted sum scalarization. Conversely, among all diffeomorphisms, transformations of this special component-wise structure are the only ones that, regardless of the problem that they are applied to, preserve efficiency and supportedness.

Furthermore, we described a connection between the concept of supportedness in multiobjective optimization and Lagrangian duality in single-objective optimization. This connection allowed us to show that for a multiobjective problem with polyhedral feasible set, but possibly nonconvex objectives, certain properly efficient points can be rendered supported by the p-th power transformation. This result had also been shown in [6] without the requirement of a polyhedral feasible set, but under the assumption that the nondominated set is the graph of a smooth function. We argue that this latter condition is restrictive and that the assumptions required in this work are more realistically achievable. In addition, Section 4.4 outlines a possible extension to the fully nonconvex case.

Two important questions from the application perspective remain open: does there exist a sufficiently large p such that all properly efficient points become supported? And is it possible to compute a lower bound on such a p? Only then, solution algorithms that identify supported points could be applied to a p-th power transformed nonconvex problem.

In the technical note [35], a lower bound for p under the assumptions of [6] was derived. However, this bound is of theoretical interest only since its computation requires an optimization over the set of efficient points. In [27], it is argued that it would be difficult to compute a lower bound for the value of $\bar{p}$ from Theorem 4.4. Even if such a bound could be computed, it would only apply to one specific properly efficient point. Unfortunately, the set of properly efficient points is generally not closed. It is thus not guaranteed that a p exists, such that all properly efficient points become supported.

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ORCID

Oliver Stein http://orcid.org/0000-0001-9514-6317

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