Flip Distance of Non-Crossing Spanning Trees: NP-Hardness and Improved Bounds

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Abstract

We consider the problem of reconfiguring non-crossing spanning trees on point sets. For a set P of n points in general position in the plane, the flip graph $\mathcal{F}(P)$ has a vertex for each non-crossing spanning tree on P and an edge between any two spanning trees that can be transformed into each other by the exchange of a single edge (coined a flip). This flip graph has been intensively studied, lately with an emphasis on determining its diameter $\mathrm{diam}(\mathcal{F}(P))$ for sets P of n points in convex position. For this case, the current best bounds are ${}^{14}\!/_{9} \cdot n - O(1) \leq \mathrm{diam}(\mathcal{F}(P)) < {}^{15}\!/_{9} \cdot n - 3$, obtained in a recent breakthrough work [Bjerkevik, Kleist, Ueckerdt, and Vogtenhuber; SODA 2025]. The crucial tool for both the upper and lower bound are so-called *conflict graphs*, which the authors stated might be the key ingredient for determining the diameter (up to lower-order terms).

In this paper, we pick up the concept of conflict graphs from the above-mentioned work and show that this tool is even more versatile than previously hoped. As our first main result, we use conflict graphs to show that computing the flip distance between two non-crossing spanning trees is NP-hard, even for point sets in convex position. Interestingly, the result still holds for more constrained flip operations, concretely, compatible flips (where the removed and the added edge do not cross) and rotations (where the removed and the added edge share an endpoint).

Additionally, we present new insights on the diameter of the flip graph, by this directly extending the line of research from [BKUV SODA25]. Their lower bound is based on a constant-size pair of trees, one of which is of a type we refer to as *stacked*. We show that if one of the trees is stacked, then the lower bound is indeed optimal up to a constant term, that is, there exists a flip sequence of length at most ${}^{14}\!/_{9} \cdot (n-1)$ to any other tree.

Lastly, we improve the lower bound on the diameter of the flip graph $\mathcal{F}(P)$ for n points in convex position to ${}^{11}\!/_{7} \cdot n - o(n)$.

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Keywords and phrases Non-crossing, spanning tree, plane graph, flip graph, reconfiguration, diameter, complexity, NP-hard, edge exchange, compatible flip, rotation, happy edge property

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1 Introduction

Non-crossing configurations such as triangulations, spanning trees, matchings, or polygonizations on planar point sets are fundamental structures in computational geometry, with wide applications. In dynamic environments or interactive settings, it is often necessary to transform one configuration into another through a sequence of local modifications, while ensuring that the non-crossing property is preserved at every step and that the intermediate configurations are of the same type. This process, known as (discrete) reconfiguration, raises natural algorithmic questions: Given two non-crossing configurations, is it possible to convert one into the other using only basic operations? If so, it is of interest to bound the length of a shortest reconfiguration sequence between any two configurations, to compute the length of an optimal sequence between two given configurations, and to efficiently determine a short(est) sequence.

These questions can be nicely rephrased in terms of a so-called *flip graph*. The flip graph of a discrete reconfiguration problem has a vertex for each configuration and an edge between any two configurations that can be transformed into one another by a single basic discrete operation, called a *flip*. In terms of the flip graph, the above-mentioned three questions then read as follows: Is the flip graph connected? What is the diameter of the flip graph? Can a shortest path (flip sequence) between two vertices of the flip graph be computed efficiently?

For many types of geometric graphs, the most classical flip operation is the exchange of one edge with another edge, see also Figure 1. For instance, it is known that the flip graph of triangulations on a set of n points in the plane in convex position [31] and has diameter exactly $2n - 10$ [35, 36]. For decades, it has been a tantalizing and important open problem in theoretical computer science whether a shortest flip sequence between two triangulations of a convex point set can be computed in polynomial time. This problem is equivalent to the complexity of computing the rotation distance between two rooted binary trees.

In this work, we consider flips between non-crossing geometric spanning trees. Let P denote a set of n points in the plane in general position, that is, no three points of P are collinear. A *non-crossing spanning tree on* P is a spanning tree that has P as its vertex set and whose edges are pairwise non-crossing straight-line segments. A *flip* in a non-crossing spanning tree T on P is the exchange of one edge of T by a different one such that the resulting graph is again a non-crossing spanning tree T' on P; see Figure 1 for an example. The according flip graph $\mathcal{F}(P)$ has a vertex for each non-crossing spanning tree on P and an edge between any two trees whenever they can be transformed into each other by a single edge flip. For two trees T and T' on P, we denote their flip distance, the length of a shortest path between T and T' in $\mathcal{F}(P)$, by $\text{dist}(T, T')$.

In 1996, Avis and Fukuda [10] showed in their famous reverse search paper that $\mathcal{F}(P)$ is connected for any n-point set P in general position, and that its radius is bounded from above by $n-2$, implying $\text{diam}(\mathcal{F}(P)) \leq 2n-4$. In 1999, Hernando, Hurtado, Márquez, Mora,

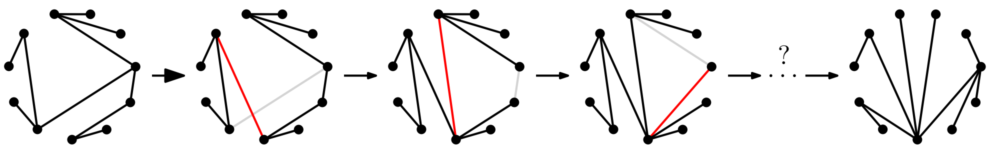


Figure 1 Reconfiguration of non-crossing spanning trees on convex point sets by edge exchange. The appearing edge is highlighted in red and the disappearing edge in gray. Can you complete the sequence to obtain the tree on the right? How many further flips are needed?

and Noy [25] showed that for convex n-point sets P, it holds that $\mathrm{diam}(\mathcal{F}(P)) \geq \lfloor 3/2 \cdot n \rfloor - 5$. Since then, the flip graph $\mathcal{F}(P)$ of non-crossing geometric spanning trees on point sets P in convex position has been subject of intensive study; see the discussion in Section 1.1 below for some details. For a long time, it was conjectured that the true value for its diameter should be $3/2 \cdot n + c$ for some constant c. Very recently, Bjerkevik, Kleist, Ueckerdt, and Vogtenhuber [13] refuted this conjecture by obtaining the currently best known bounds of $14/9 \cdot n + O(1) \leq \mathrm{diam}(\mathcal{F}(P)) \leq 15/9 \cdot n - O(1)$ for the diameter of the flip graph on non-crossing spanning trees on point sets in convex position. The crucial tool for both the upper and lower bound are so-called *conflict graphs*, which the authors introduced and stated might be the key ingredient for determining the diameter of the flip graph of spanning trees for point sets in convex position (up to lower-order terms). Their lower bound is based on a constant-size example containing a tree belonging to a family of trees that we call *stacked*.

In this work, we pick up the concept of conflict graphs from [13] and show that this tool is even more versatile than previously hoped. We prove that the flip distance between two trees can be as high as $11/7 \cdot n - o(n)$, by this directly extending the line of research from [13].

▶ **Theorem 1.** *For any set P of $n \geq 3$ points in convex position, the flip graph $\mathcal{F}(P)$ of non-crossing spanning trees on P has diameter at least $(11/7 - o(1))n$.*

We further show that if one of the trees is stacked, then the lower bound from [13] is indeed the best possible. In other words, whenever one of the trees is stacked, then there exists a flip sequence of length at most $14/9 \cdot n - O(1)$ to any other tree.

▶ **Theorem 2.** *Let T, T' be non-crossing trees on $n \geq 3$ points in convex position. If T is a stacked tree, then dist$(T, T') \leq \frac{14}{9}(n-1)$.*

Moreover, we use conflict graphs beyond the expectations of [13], showing that they can also be used to prove that computing the flip distance between two non-crossing spanning trees is NP-hard, even for point sets in convex position.

▶ **Theorem 3.** *Given two non-crossing spanning trees T, T' on a point set and an integer δ, the decision problem of whether there exists a flip sequence of length at most δ between T and T' is* NP*-complete, even for point sets in convex position.*

Interestingly, this result still holds for more constrained flip operations, concretely, *compatible* flips (where the removed and the added edge do not cross) and *rotations* (where the removed and the added edge share an endpoint). In Figure 1, while the first flip is just an edge exchange, the second flip a compatible flip, and the third flip is a rotation.

▶ **Theorem 4.** *Given two non-crossing spanning trees T, T' on a point set and an integer δ, the decision problem of whether there exists a compatible flip sequence / a rotation sequence of length at most δ between T and T' is* NP*-complete, even for point sets in convex position.*

We point out that the latter two statements constitute the first hardness results for flip graphs on point sets in convex position. Moreover, Theorem 4 refutes the long-held belief that the flip distance can be computed in polynomial time when the so-called happy edge property holds, because that property holds for compatible flips of non-crossing spanning trees on convex point sets. See Section 1.1 for more background and Table 1 for an overview of complexity status for various flip graphs of non-crossing structures on point sets.

Outline. We discuss further related work in Section 1.1 and introduce the central concepts from [13] that are required for our work in Section 2. In Section 3, we prove NP-completeness for deciding the length of a shortest (general, compatible, or rotational) flip sequence between

Table 1 Overview of complexity results. Results obtained in this work are marked by [*].

setting	flip type	convex position	general position
triangulations	edge exchange	NP-hard? [20][1]	NP-hard [32, 34]
non-crossing spanning trees	edge exchange	NP-hard [*]	NP-hard [*]
	compatible	NP-hard [*]	NP-hard [*]
	rotation	NP-hard [*]	NP-hard [*]
	slide	open	open
non-crossing spanning paths	edge exchange	in P [4]	open
non-crossing perfect matchings	edge exchange	in P [26]	NP-hard [11]
non-crossing odd matchings	edge exchange	open	NP-hard [3]

two spanning trees on a convex point set. In Section 4, we show that the flip distance between two trees on a convex n-point set is at most $^{14}/_{9} \cdot (n-1)$ if one of them is stacked. In Section 5, we improve the lower bound on the diameter of the tree flip graph for n points in convex position to $(^{11}/_{7} - o(1))n$. We conclude with a summary and open problems in Section 6. Full details and proofs of all statements marked by ($\star$) can be found in the arXiv version of our paper [12].

1.1 Related work

We review the related work in two aspects, concerning the connectedness and diameter bounds on the one hand, and the computational complexity of computing a shortest flip sequence on the other hand.

Connectedness and Diameter. For triangulations, connectedness of the flip graph for point sets was shown by Lawson [31] and for simple polygons by Hurtado, Noy, and Urrutia [28]. In both cases, the diameter of the flip graph is in $O(n^2)$. Hurtado, Noy, and Urrutia [28] provided a matching lower bound of $\Omega(n^2)$ for both settings. For triangulations of convex point sets, Sleator, Tarjan, and Thurston [36] showed that the diameter of the flip graph is at most $2n - 10$, which is tight for $n > 13$ as proven by Pournin [35]. There are many more results on flips in triangulations; see for example [21–24, 27, 37–39].

For non-crossing spanning trees of point sets in general position, Avis and Fukuda [10] showed that the flip graph is connected for flips, compatible flips, and rotations, and has a diameter at most $2n - 4$. Nichols, Pilz, Tóth, and Zehmakan [33] gave an upper bound on the diameter of $O(n \log(n))$ for empty triangle rotations. Aicholzer, Aurenhammer, and Hurtado [1] proved that the flip graph is connected for edge slides (where the source and the target edge together with some other edge of the tree form an empty triangle). They also provided an exponential upper bound on the diameter, which was later improved to $O(n^2)$ by Aichholzer and Reinhart [8]. For point sets in convex position, determining the exact diameter of the flip graph has gained considerable attention in the last couple of years. Starting from the benchmark upper bound of $2n - 4$ [10] and the conjectured to be essentially tight lower bound of $\lfloor ^{3n}/_{2} \rfloor - 5$ [25], the upper bound was subsequently improved

[1] In a very recent arXiv preprint, Dorfer [20] announced that computing the flip distance between triangulations of convex point sets is NP-complete. This (potential since not yet reviewed) result was inspired by and initiated after the submission of the work at hand, and uses similar techniques: Introducing a notion of conflict graphs, showing that the flip distance is closely tied to the size of a largest acyclic subset, and proving NP-hardness of computing the largest acyclic subset.

to $2n - \log(n)$ [2] and $2n - \sqrt{n}$ [16]. The leading constant factor of 2 was broken by Bousquet, de Meyer, Pierron, and Wesolek [15], who improved the upper bound to $1.96n$. The latest advancement before the work at hand is due to the recent work [13], which improved the upper bound to $5/3 \cdot n - 3$ and the lower bound to $14/9 \cdot n - O(1)$. We note that the lower bound of $14/9 \cdot n - O(1)$ is not only the best bound for points in general position, but also holds for restricted flip types. When restricting the flips to compatible flips or rotations in the convex setting, the best standing upper bounds on the diameter of the flip graphs of spanning trees are $5/3 \cdot n - 2$ and $7/4 \cdot (n - 1)$, respectively, due to Aichholzer, Dorfer, and Vogtenhuber [5].

For plane spanning paths, connectedness of the flip graph is an interesting open problem. So far, the flip graph has been shown to be connected for convex point sets [9] (with diameter $2n - 6$ for $n > 4$ [18]), wheel sets and double circles [6], as well as point sets with two convex layers [30]. For matchings, the connectedness problem becomes a parity issue: While the flip graph of odd matchings is known to be connected [3], the connectedness of the flip graph remains an open question for perfect matchings. Partial progress has been made for convex point sets by Hernando, Hurtado, Noy [26] and for the case when an unbounded number of edges can be exchanged by Houle, Hurtado, Noy, and Rivera-Campo [27].

Computational complexity of shortest flip sequences and the happy edge property. In several settings, there is a close connection between the existence of NP-hardness proofs or polynomial-time algorithms and the happy edge property. A reconfiguration problem has the *happy edge property* if edges that are in the initial as well as the target configuration (and thus are *happy*) can remain in the configuration throughout the entire flip sequence.

The happy edge property holds for triangulations of point sets in convex position [36], but not for triangulations of general point sets or polygons [14]. The latter fact was used to show that the flip distance problem for triangulations of point sets and polygons is NP-complete [7,32] and also APX-hard on point sets [34]. In contrast, on point sets without empty convex pentagons, the flip distance can be computed in polynomial time [21]. If we consider point sets in convex position, the flip graph is the 1-skeleton of a famous polytope, the *associahedron*. For the more general case of graph associahedra [19], the shortest path problem has been recently shown to be NP-hard [29]. Finding shortest paths in the associahedron or equivalently shortest flip sequences between triangulations of convex point sets might also be hard by a very recent arXiv preprint [20].

In the case of perfect matchings on general point sets, NP-hardness of computing a shortest flip sequence was shown via counterexamples to the happy edge property [11]. For convex point sets, the happy edge property holds and the flip distance between two matchings can be determined in linear time [26]. Similarly, for the reconfiguration of odd matchings on general point sets, the happy edge property also does not hold and the flip distance is NP-hard to compute [3].

All these results support the long standing belief that the happy edge property is tied to the complexity of graph reconfiguration problems: If the happy edge property holds, then the problem is supposedly easy to solve. If it doesn't hold, the problem should be hard to solve.

Most recently, this pattern has been broken, when Aichholzer and Dorfer [4] provided a linear time algorithm to determine the flip distance between non-crossing spanning paths of convex point sets, despite the existence of counterexamples to the happy edge property. What remained open (until now) is the other direction: Is there a graph reconfiguration problem for which it is NP-hard to compute the flip distance, even though the problem has the happy edge property? We give an affirmative answer to this question in this work: The happy edge property holds for compatible flips on non-crossing spanning trees on convex point sets [5], but the flip distance is NP-hard to compute as shown in Theorem 4.

2 Fundamental concepts: Edge pairs, conflict graphs, and blowups

Our results build on insights from Bjerkevik, Kleist, Ueckerdt, and Vogtenhuber [13]. In the following, we introduce some of their concepts.

Linear Representation. A convex point set is usually visualized by equally spaced points on a circle. A *linear representation* of a non-crossing tree T on a convex point set is constructed by cutting open the circle and unfolding the points into a horizontal line segment, called the spine. Edges in T can be illustrated nicely by semi-circles connecting their end points above the spine. Given two trees (for example the initial tree and the target tree of a flip sequence) in the same linear representation, it is convenient to represent the edges of one tree above the spine and the edges of the other tree below the spine. For an example, see Figure 2. If an edge between two consecutive vertices on the spine is contained in both trees we will in later parts of the paper depict this by drawing it as a straight line between the two vertices instead of two semicircles.

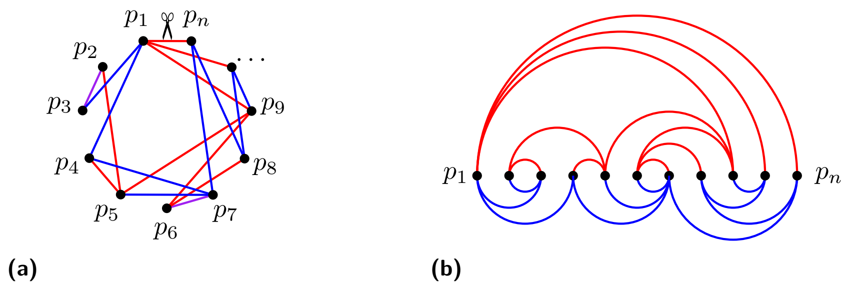


Figure 2 Figure reproduced from [13]. a Two non-crossing trees T, T' on a circularly labeled point set in convex position and b its linear representation with T above and T' below the spine.

By labeling the vertices along the linear representation by $p_1, ..., p_n$, we get a natural notion of the *length* of an edge. Concretely, the length of the edge p_ip_j is defined as $|i - j|$. We say a point p_k is *covered* by an edge p_ip_j with $i < j$ if $i \leq k \leq j$. An edge *covers* another edge if it covers both of its endpoints.

Gaps and Edge Pairing. A *gap* in the convex hull is the open line segment between two consecutive points p_i and p_{i+1} on the spine. For every tree T, there is a natural *gap-edge bijection* between gaps in the linear representation and edges of T: Each gap is assigned to the shortest edge that covers the gap. This assignment is indeed a bijection by Lemma 3.1 in [13]. For an example, consider Figure 3a. Based on the assigned gap, edges of T are partitioned into three groups. In particular, an edge is *short* if it shares both vertices with its gap, *near* if it shares one vertex with its gap, and *long* if shares no vertex with its gap.

For two trees T and T', we obtain an edge-edge bijection by pairing two edges that are assigned to the same gap of the linear representation. The authors of [13] explain how to handle nine types of edge pairs. For our results it is sufficient to concentrate on near-near pairs where the two edges are distinct. We call a gap g_i with edges e_i from T and e'_i from T'

above if e_i and e'_i are adjacent and e_i is longer than e'_i, see Figure 3b.

below if e_i and e'_i are adjacent and e_i is shorter than e'_i, see Figure 3c.

crossing if e_i and e'_i are not adjacent, equivalently, the two edges cross if drawn in the same convex point set, see Figure 3d.

The set of all above, below, and crossing gaps is denoted by A, B, and C, respectively.

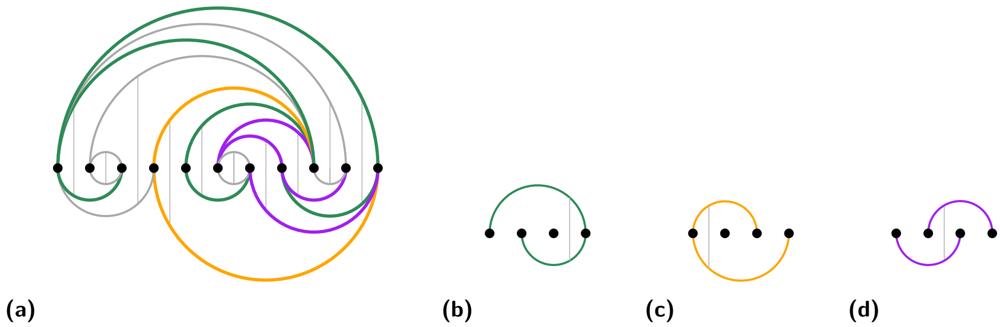


Figure 3 a Illustration of the gap-edge and edge-edge bijection. Examples of an above b, a below c, and a crossing pair d. Edges belonging to near-near pairs in A, B, and C are colored green, orange, and purple, respectively (also in several other figures).

Conflict Graphs. *Conflict graphs* were introduced as a tool to find the largest set of near-near pairs where the initial edge can be flipped directly to the target edge. The conflict graph H has a vertex for each (gap associated to a) near-near pair and a directed edge from a pair (e_i, e'_i) to (e_j, e'_j) if the edge e_i needs to be removed before the flip that adds e'_j and removes e_j can be performed. We denote the set of vertices in H by $V(H)$. There exist three types of conflicts. Specifically, there is a directed edge $\overrightarrow{g_i g_j}$ in H, from g_i to g_j if

type 1 e_i crosses e'_j, see Figure 4a.

type 2 e'_j covers e_i and e_i covers g_j, see Figure 4b.

type 3 e_i covers e'_j and e'_j covers g_i, see Figure 4c.

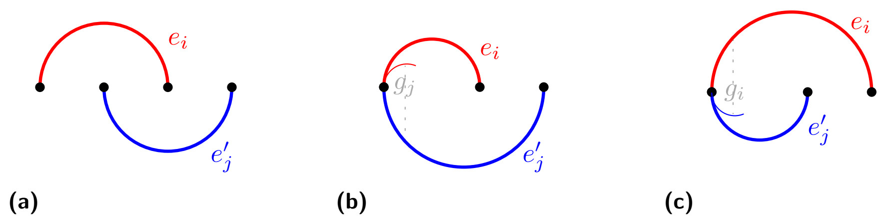


Figure 4 Figure reproduced from [13]. Examples of conflicts: a type 1, b type 2, c type 3.

We slightly abuse terminology and view the vertices of H interchangeably as gaps and as near-near pairs. Let $\mathrm{ac}(H)$ denote the largest number of vertices in H which induce an acyclic subgraph. The authors of [13] derive the following relations between acyclic sets in the conflict graph and flip distances.

▶ **Theorem 5** ([13, Theorem 2.1]). *Let T, T' be two non-crossing trees on linearly ordered points $p_1, \dots, p_n$ with corresponding conflict graph $H = H(T, T')$.*

(i) *If $V(H)$ is non-empty, then* $\mathrm{dist}(T, T') \leq \max\{\frac{3}{2}, 2 - \frac{\mathrm{ac}(H)}{|V(H)|}\}(n-1)\}$. *If $V(H)$ is empty, then* $\mathrm{dist}(T, T') \leq \frac{3}{2}(n-1)$.

(ii) *If $V(H)$ is non-empty, then there is a constant c depending only on T and T' such that for all $\bar{n} \geq 1$, we have* $\mathrm{diam}(\mathcal{F}_{\bar{n}}) \geq \left(2 - \frac{\mathrm{ac}(H)}{|V(H)|}\right) \bar{n} - c$.

Moreover, we will use the following lemma on the sets of above, below, and crossing gaps.

▶ **Lemma 6** ([13, Lemma 3.2]). *Each of A, B, and C is an acyclic set of H.*

Blowups. By default, a flip sequence based on the described edge-edge bijection is not necessarily a shortest one and can in fact be very far from optimal. The authors of [13] overcome this by introducing *blowups* for near-near pairs. For trees T and T', the β-blowup is a pair of trees $\beta \cdot T$ and $\beta \cdot T'$ obtained by the following construction. For every gap g that corresponds to a near-near pair (e, e'), insert a set $V(e) = V(e')$ of β additional points in the gap g. In $\beta \cdot T$ add an edge from each $v \in V(e)$ to the endpoint of e that is not adjacent to g. Proceed similarly in $\beta \cdot T'$ for e'. Part (ii) of Theorem 5 is obtained by considering blowups.

▶ **Lemma 7** ([13, Lemma 23, arXiv-Version])**.** *The flip distance from $\beta \cdot T$ to $\beta \cdot T'$ is at least $(\beta - 2n)(2|V(H)| - \mathrm{ac}(H))$, where $H = H(T, T')$.*

3 NP-Completeness results

In this section we sketch proofs for our hardness results. Theorem 3 states that the decision problem of whether there exists a flip sequence of length at most δ between two non-crossing spanning trees T, T' is NP-complete. In fact, this holds even for point sets in convex position. Our proof of Theorem 3 goes via an intermediate problem. Specifically, we show that it is NP-complete to decide whether a directed graph H has an acyclic subset of size at least k, even if H is the conflict graph of two non-crossing trees with a given linear representation.

▶ **Proposition 8.** *Given two non-crossing trees T, T' on a convex point set, with a linear representation and corresponding non-empty conflict graph $H = H(T, T')$, as well as an integer k, it is* NP*-complete to decide whether* $\mathrm{ac}(H) \geq k$.

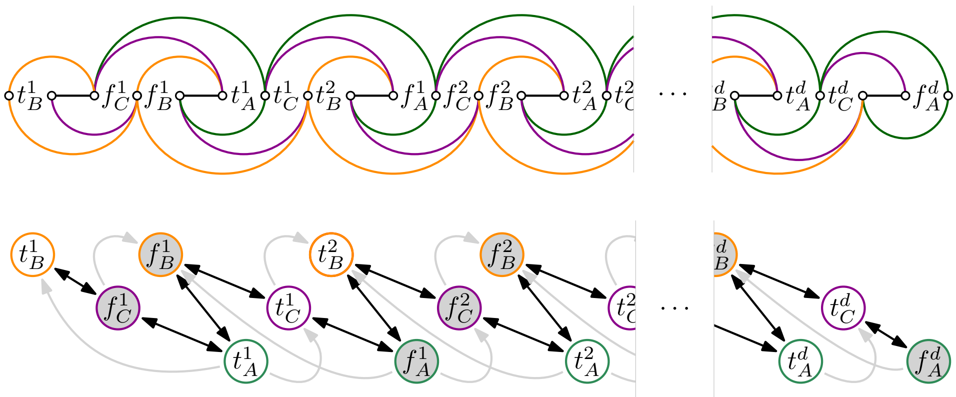
Our proof of Proposition 8 is a reduction from the NP-complete problem Planar Max-2SAT. We consider a restricted version [17], in which we are given a 2SAT-formula ϕ with variable-set X and clause-set $\mathcal{C}$ whose vertex-clause incidence graph G_ϕ admits an *aligned drawing*, namely, a drawing with all variables on a horizontal line L and no edge crossing L or any other edge. In particular, every clause lies either in the upper or lower halfplane defined by L. Additionally given an integer t, Planar Max-2SAT asks whether X admits a truth assignment such that at least t clauses in $\mathcal{C}$ are satisfied.

For fixed ϕ and t, we construct two non-crossing trees T, T' with a linear representation and an integer k such that the following are equivalent:

- The conflict graph $H = H(T, T')$ admits an acyclic subset of size at least k.
- The 2SAT-formula ϕ admits a truth assignment that satisfies at least t clauses.

To construct T and T', we model the variables and clauses in ϕ with small gadgets. In doing so, we also discuss the conflicts that occur. Recall that each gap has a type: above, below, or crossing. In fact, it will be enough for us to focus on the *mixed conflicts*, i.e., those between gaps of different types. We also remark that our construction only has conflicts $\overrightarrow{g_i g_j}$ of type 1, i.e., where e_i properly crosses e'_j as illustrated in Figure 4a.

Let $x \in X$ be a variable and $d = \deg(x)$ be its degree in G_ϕ, i.e., x occurs (negated or non-negated) in exactly d clauses. The variable-gadget for x consists of $8d + 1$ pairs of edges, of which $6d$ are near-near pairs and hence their corresponding $6d$ gaps are vertices in the conflict graph $H = H(T, T')$. The gadget is illustrated in Figure 5a and the corresponding mixed conflicts in Figure 5b. The $6d$ gaps form a path P of double conflicts in the conflict graph H, i.e., every acyclic subset is an independent set in P. We partition P into two independent sets $t(x)$ and $f(x)$ by alternating the gaps in $t(x)$ and $f(x)$ along P. Later we ensure that every acyclic subset S of H with $|S| \geq k$ either contains all of $t(x)$ or all of $f(x)$, which encodes the two possible truth assignments True and False for variable x.



(a)

(b)

Figure 5 Illustration of a variable-gadget and the graph of its mixed conflicts.

We place each variable-gadget at the position of the corresponding variable in the aligned drawing of G_ϕ in such a way that the rightmost point of any gadget coincides with the leftmost point of the next gadget (immediately to the right). Secondly, for each clause C with incident variables x and y we introduce a new gap g_x in the variable-gadget of x and a new gap g_y in the variable-gadget of y. For each $z \in \{x, y\}$, the edge pair for g_z has an edge of length 3 in the variable-gadget of z and an edge connecting the gadgets of x and y. See Figures 6a and 7a for an illustration. To ensure that different clause-gadgets do not cross, long edges are put in the halfplane that contains the clause C in the aligned drawing of G_ϕ. Further, the position of g_z on the path P of length $6\deg(z)$ in the variable-gadget of z is chosen with respect to the order of incident edges at z in the drawing of G_ϕ.

Moreover, g_z is chosen such that it has a double conflict with a gap in $t(z)$ if z appears negated in C, and a double conflict with a gap in $f(z)$ if z appears non-negated in C. This ensures that an acyclic set S contains the gap g_z only if the truth assignment of variable z satisfies the clause C. A double conflict between the two gaps g_x, g_y of the clause-gadget ensures that at most one of g_x, g_y is in any acyclic set S. There are six further mixed conflicts between g_x, g_y and the variable-gaps, as illustrated in Figures 6b and 7b. Given a truth assignment, we choose all of $t(x)$ when x is TRUE, $f(x)$ when x is FALSE, and one gap g_z in each satisfied clause. That this gives an acyclic subset S of $H(T, T')$, relies on the following observation.

▶ **Observation 9.** *For every mixed conflict $\overrightarrow{g_1 g_2}$ one of the following holds:*

- *g_1 is above*
- *g_2 is below*
- *g_1 and g_2 have a double conflict*
- *there is a g_1-g_2-path of three double conflicts with three gaps of some variable-gadget*

In fact, each of the sets A, B, C of all above, all below, and all crossing gaps, respectively, is acyclic by Lemma 6. Thus with Observation 9 it follows that every set $S \subseteq V(H)$ with no double conflict and none of the exceptional conflicts in Observation 9 is acyclic by taking first $A \cap S$, then $C \cap S$, and finally $B \cap S$ for a topological ordering.

This way, every truth assignment satisfying t of ϕ corresponds to an acyclic set $S \subseteq V(H)$ containing exactly half of all variable-gaps and exactly t of all clause-gaps. Finally, for the other direction (getting a truth assignment from an acyclic subset S), we need to ensure that for every variable x we have $t(x) \subseteq S$ or $f(x) \subseteq S$, as long as S is large enough. To this end, let $m = |\mathcal{C}|$ and for each variable-gadget we apply an m-blowup to the first and last

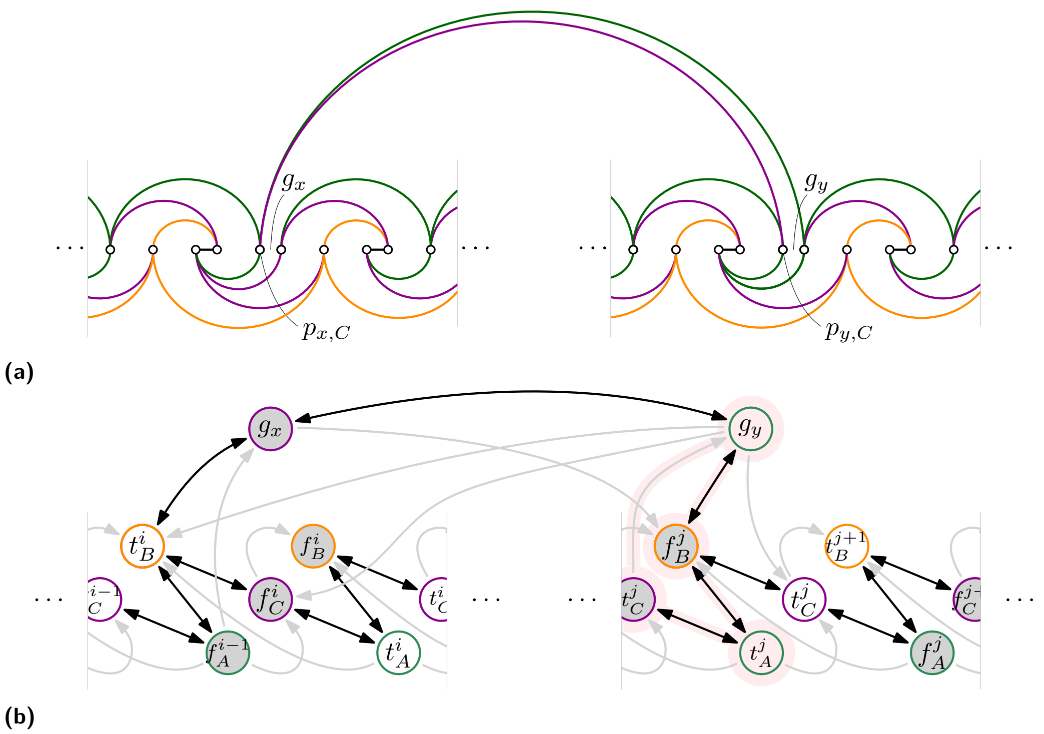


Figure 6 Illustration of a clause-gadget for a clause above the horizontal line and its conflict graph. The special conflict corresponding to the last option in Observation 9 is highlighted.

gap, a $2m$-blowup to all other variable-gaps, while applying no blowup to the clause-gaps. See Figure 8 for an illustration. Setting $k = m(12m + 6 - |X|) + t$ concludes our reduction from Planar Max-2SAT and thus the proof of Proposition 8.

In order to finally prove Theorem 3, we now reduce the problem of finding large acyclic subsets in conflict graphs to the problem of finding short flip sequences in β-blowups.

▶ **Theorem 10.** *Let T and T' be two non-crossing spanning trees on n vertices with a linear representation and non-empty conflict graph $H(T, T')$, and $k > 0$ be an integer. Let $\beta = 4n^2 + 2n$. Then the following are equivalent.*

- *The conflict graph $H = H(T, T')$ has an acyclic subset of size at least k.*
- *The flip distance between $\beta \cdot T$ and $\beta \cdot T'$ is at most $(\beta + 1)(2|V(H)| - k) + 2(n - |V(H)|)$.*

Proof. For convenience, let us denote $|H| = |V(H)|$. Assume H does not contain an acyclic set of size k, i.e., $\mathrm{ac}(H) < k$. Then by Lemma 7,

$$\begin{aligned}\mathrm{dist}(\beta \cdot T, \beta \cdot T') &\geq (\beta - 2n)(2|H| - \mathrm{ac}(H)) \geq (\beta + 1 - (2n + 1))(2|H| - k + 1)\\ &= (\beta + 1)(2|H| - k) + (\beta + 1) - (2n + 1)(2|H| - k + 1)\\ &> (\beta + 1)(2|H| - k) + 2(n - |H|).\end{aligned}$$

For the last inequality observe that $|H| < n$ and hence $(2n+1)(2|H|-k+1) \leq (2n+1)(2n-1) \leq 4n^2$, which together with $\beta = 4n^2 + 2n > 4n^2 + 2(n - |H|)$ gives the desired lower bound.

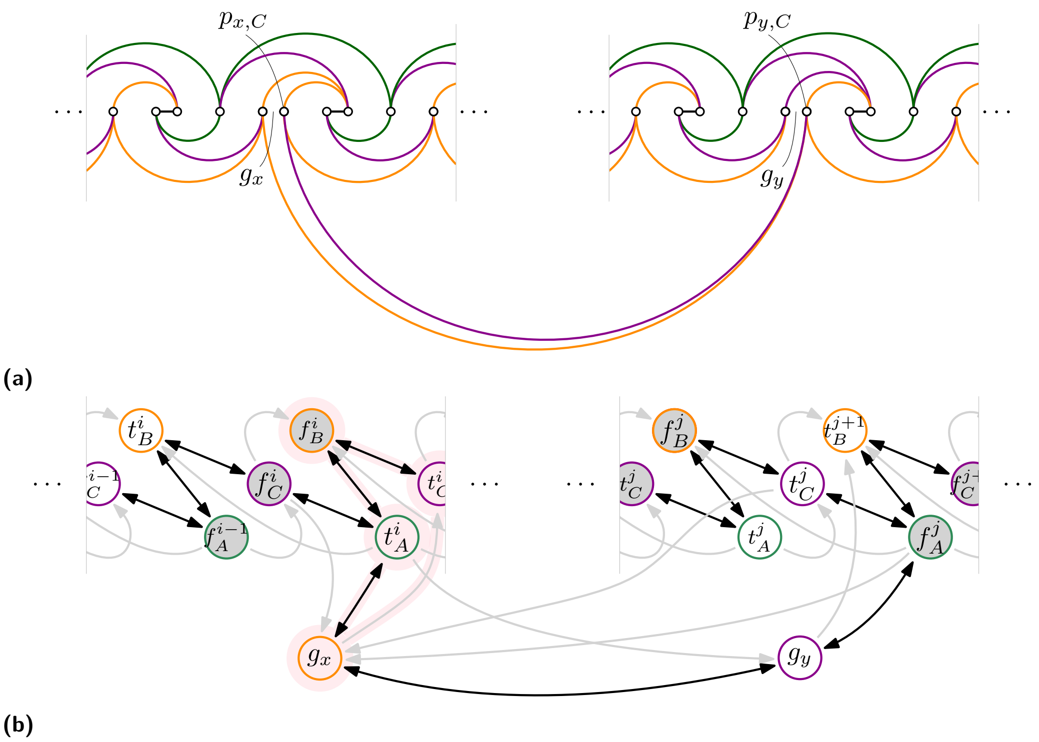


Figure 7 Illustration of a clause-gadget for a clause below the horizontal line and its conflict graph. The special conflict corresponding to the last option in Observation 9 is highlighted.

Conversely, assume $H = H(T, T')$ has an acyclic subset of size k, i.e., $\mathrm{ac}(H) \geq k$. Then, the conflict graph $H^\beta = H(\beta \cdot T, \beta \cdot T')$ of the blowup has an acyclic set of size $k \cdot (\beta + 1)$, i.e., $\mathrm{ac}(H^\beta) \geq k(\beta + 1)$. For the original trees T, T' we have n vertices and $n - 1$ gaps, of which $|H|$ are affected by the blowup. Thus, each of $\beta \cdot T$ and $\beta \cdot T'$ has $n + \beta|H|$ vertices and $n - 1 + |\beta|$ edges. Moreover, $|V(H^\beta)| = (\beta + 1)|H|$.

Using one flip for each edge pair in the largest acyclic subset and two flips for each other edge pair gives the desired bound of

$$\mathrm{dist}(\beta \cdot T, \beta \cdot T') \leq (\beta + 1)(2|H| - \mathrm{ac}(H)) + 2(n - 1 - |H|) < (\beta + 1)(2|H| - k) + 2(n - |H|).$$

and hence completes the proof. ◀

We remark that the proof of Theorem 10 produces a flip sequence that contains non-compatible flips. Next, we modify the proof to the case of flip sequences that contain only compatible flips or only rotations. In fact, a direct flip for an above pair or a below pair is already a rotation. Further, flipping any near edge to a short edge covering its associated gap is also a rotation. We do, however, need to deal with large collections of crossing edge pairs. To this end, we use the following fact.

▶ **Lemma 11** ($\star$). *Let T, T' be two trees that differ in a single crossing pair. Then the rotation flip distance between $\beta \cdot T$ and $\beta \cdot T'$ is at most $\beta + 2$.*

With Lemma 11 in place, we prove that finding short compatible flip sequences and short rotation sequences is NP-hard as well. As before, we use the shorthand notation $|H| = |V(H)|$.

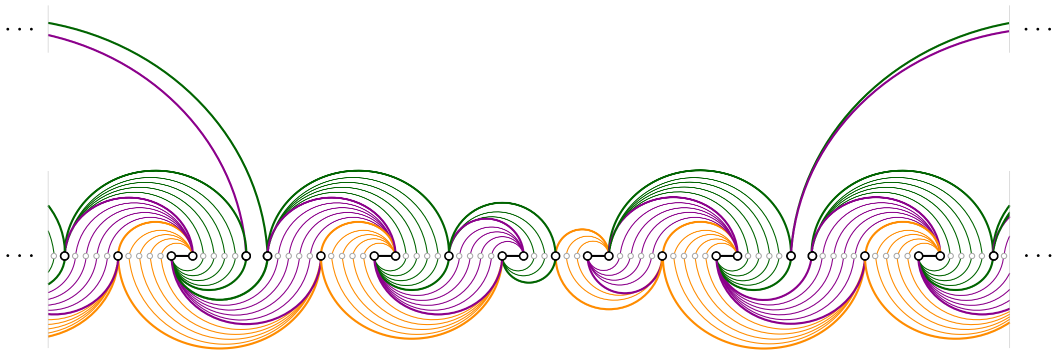


Figure 8 Applying an m-blowup (here $m = 2$) to the first and last gap of each variable-gadget and $2m$-blowups to all other gaps in variable-gadgets.

▶ **Theorem 12** ($\star$). *Let T and T' be two non-crossing spanning trees on n vertices with a linear representation, and $k > 0$ be an integer. Let $\beta = 4n^2 + 4n$. The following are equivalent.*

- *The conflict graph $H = H(T, T')$ has an acyclic subset of size at least k.*
- *The compatible flip distance* $\mathrm{dist}_{\mathrm{comp}}(\beta \cdot T, \beta \cdot T') \leq (\beta + 1)(2|H| - k) + |H| + 4(n - |H|)$.
- *The rotation flip distance* $\mathrm{dist}_{\mathrm{rot}}(\beta \cdot T, \beta \cdot T') \leq (\beta + 1)(2|H| - k) + |H| + 4(n - |H|)$.

4 Stacked Trees

In this section, we improve the upper bound for the special case where in one of the two trees, the (relevant) edges can be partitioned into independent "stacks" of nested edges. This can be seen as a first natural generalization of an upper bound where one tree is a separated caterpillar, that is, a tree with two stacks [13, Section 6, arXiv-Version]. Further, the top tree in the best previously known lower bound example in [13], depicted in Figure 9, has this property. Throughout this section, we refer to a non-crossing spanning tree on a convex point set in linear representation simply as a tree.

We call a tree T *stacked* if its edge set can be partitioned into sets $S_1, \ldots, S_k$ such that

(i) no edge in S_i covers an edge in S_j for $i \neq j$, and

(ii) each S_i is totally ordered by the covering relation.

Let T' be another tree on the same vertex set. We say that T is *stacked with respect to* T' if the subset N of edges in T that are in near-near pairs of (T, T') admits a partition into sets $S_1, \ldots, S_k$ satisfying properties (i) and (ii). Note that if T is stacked then it is stacked with respect to any T'. We refer to the edge sets S_i as *stacks (with respect to T')*.

4.1 The Special Case: Three stacks

Fix trees T and T'. We first assume that T has exactly three stacks S_1, S_2 and S_3 with respect to T'. Let $\mathcal{P}_N$ be the set of near-near pairs of (T, T'). We partition $\mathcal{P}_N$ into 9 different sets. A pair $(e, e') \in \mathcal{P}_N$ will be put into sets by the following rules.

- A_i if e belongs to stack S_i and (e, e') is an above pair.
- $B_{i,j}$, $i \neq j$ if e belongs to stack S_i, (e, e') is a below pair or a crossing pair and e' crosses an edge in a different stack S_j.
- If e belongs to stack S_i, (e, e') is a below pair or a crossing pair and e' does not cross an edge in any other stack S_j, we assign (e, e') arbitrarily to one of the two sets $B_{i,j}$.

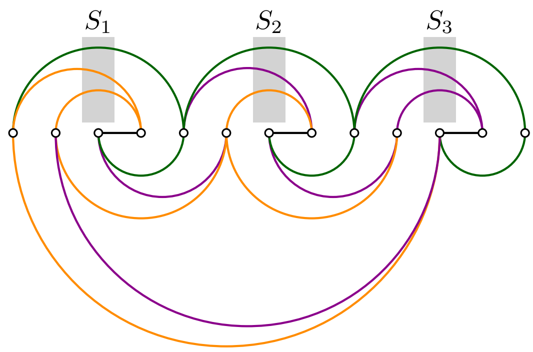


Figure 9 The best previously known lower bound example from [13]. The top tree is a stacked tree with three stacks. The bottom tree is neither stacked, nor stacked with respect to the top tree.

For this to be well-defined, we need to verify that (e, e') cannot belong to both $B_{i,j}$ and $B_{i,j'}$ for $j \neq j'$. The edge e' covers two gaps adjacent to its endpoints. One of these is covered by e. If e' intersects edges $f \in S_j$ and $g \in S_{j'}$, then the other gap is covered by both f and g, which is a contradiction, since f does not cover g or vice versa by the definition of stacks.

▶ **Lemma 13** **(⋆).** *For all choices of x, y, z such that $\{x, y, z\} = \{1, 2, 3\}$, both $H_x := A_y \cup A_z \cup B_{x,y} \cup B_{x,z}$ and $H_{x,y} := A_z \cup B_{y,z} \cup B_{x,y} \cup B_{x,z}$ are acyclic.*

We are now ready to obtain the following upper bound on the flip distance if one of the two trees is a stacked tree with three stacks.

▶ **Theorem 14.** *Let T, T' be non-crossing trees on $n \geq 3$ points in convex position. Let T be a stacked tree with three stacks. Then $\text{dist}(T, T') \leq {}^{14}/_{9}(n-1)$.*

Proof. Each element of $V(H)$ is contained in four of the nine acyclic sets $H_1, H_2, H_3, H_{1,2}, H_{2,1}, H_{1,3}, H_{3,1}, H_{2,3}, H_{3,2}$ from Lemma 13. Thus, the average size of these sets is ${}^{1}/_{9}\Sigma_{i=1}^{9}|H_i| = {}^{4}/_{9}|V(H)|$ and, consequently, the largest acyclic set is of size at least ${}^{4}/_{9}|V(H)|$. By Theorem 5(i), there exists a flip sequence from T to T' of length at most ${}^{14}/_{9}(n-1)$. ◀

4.2 Trees with arbitrarily many stacks

We now consider a pair of trees T and T' where T is a stacked tree and let $S_1, \ldots, S_k$ denote the stacks of T with respect to T'. Let G be the graph with $S_1, \ldots, S_k$ as its vertices and an edge between S_i and S_j with $i \neq j$ if there is a near edge e in S_i paired with a near edge e' that crosses an edge in S_j. Then G is a subset of a triangulation of a convex polygon and therefore 3-colorable. We let the colors be denoted by 1, 2 and 3.

We partition the set of near-near pairs into 9 different sets, based on the 3-coloring. A pair $(e, e') \in \mathcal{P}_N$ will be put into a set by the following rules.

- A_i if e belongs to a stack with color i and (e, e') is an above pair.
- $B_{i,j}$, $i \neq j$ if e belongs to a stack with color i, (e, e') is a below pair or a crossing pair and e' crosses an edge in a stack with a color $j \neq i$.
- If e belongs to a stack with color i, (e, e') is a below or a crossing pair and e' does not cross an edge in any other stack, we assign (e, e') arbitrarily to one of the two sets $B_{i,j}$.

Lemma 15, which is a slight extension of Lemma 13, can be used to prove Theorem 16 in a similar way as Theorem 14 is derived from Lemma 13.

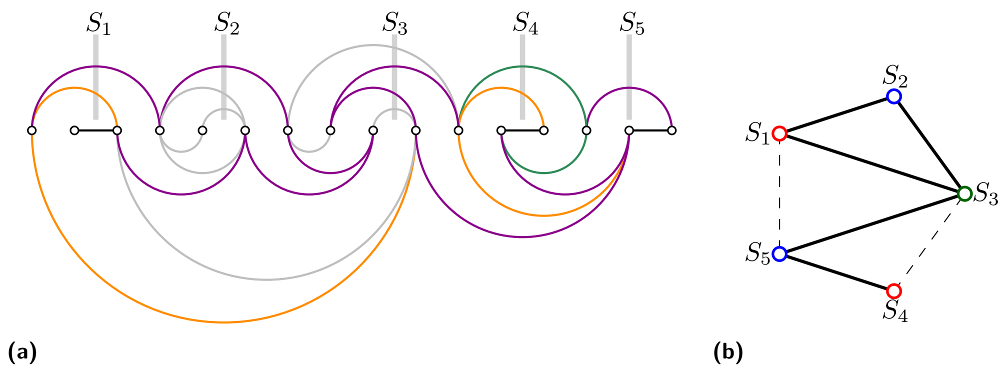


Figure 10 (a) A pair of trees where one tree is stacked and (b) the resulting graph G with a proper 3-coloring.

▶ **Lemma 15** (⋆). *Let $\{x, y, z\} = \{1, 2, 3\}$. Then $H_x = A_y \cup A_z \cup B_{x,y} \cup B_{x,z}$ and $H_{x,y} = A_z \cup B_{y,z} \cup B_{x,y} \cup B_{x,z}$ are acyclic.*

▶ **Theorem 16.** *Let T, T' be non-crossing trees on $n \geq 3$ points in convex position. Let T be stacked with respect to T'. Then* $\text{dist}(T, T') \leq {}^{14}/_{9}(n-1)$.

We remind the reader that if T is stacked, then it is stacked with respect to any T'. Thus, the same statement without "with respect to T'" is true a fortiori.

5 A new lower bound

We construct trees T^i and $\tilde{T}^i$ iteratively, and show that their conflict graph has $7i$ vertices and a largest acyclic subset of size at most $3i + 1$. The latter divided by the former tends to ${}^3/_7$, which allows us to use Theorem 5 (ii) to prove $\text{diam}(\mathcal{F}_n) \geq {}^{11}/_7 \cdot n - o(n)$.

We start with the trees T^1 and $\tilde{T}^1$ illustrated in Figure 11. These are trees on vertices $v_1, \ldots, v_8, x, y, v_9$ labeled from left to right along the spine and have short edges v_2v_3, v_6v_7 and xy as well as edges e_i and $\tilde{e}_i$ for $i = 1, \ldots, 7$, respectively.

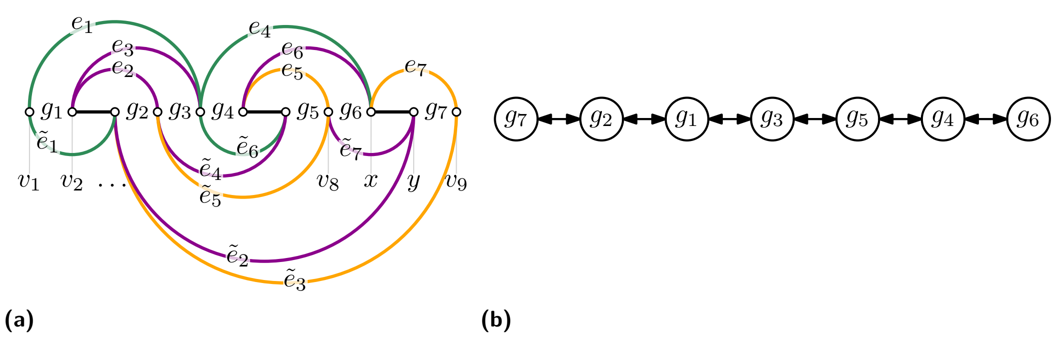


Figure 11 Illustration of T^1 and $\tilde{T}^1$ and the relevant part of the conflict graph.

Next, we construct T^ℓ and $\tilde{T}^\ell$ inductively for $\ell \geq 2$. Assume that $T^{\ell-1}$ and $\tilde{T}^{\ell-1}$ are given. Add vertices $v_1^\ell, \ldots, v_5^\ell$ from left to right to the left of $v_1^{\ell-1}$ and let $v_6^\ell = v_2^{\ell-1}$. Further, add a vertex v_7^ℓ between $v_3^{\ell-1}$ and $v_4^{\ell-1}$, a vertex v_8^ℓ between $v_8^{\ell-1}$ and x, and a vertex v_9^ℓ to the right of $v_9^{\ell-1}$. For an illustration consider Figure 12.

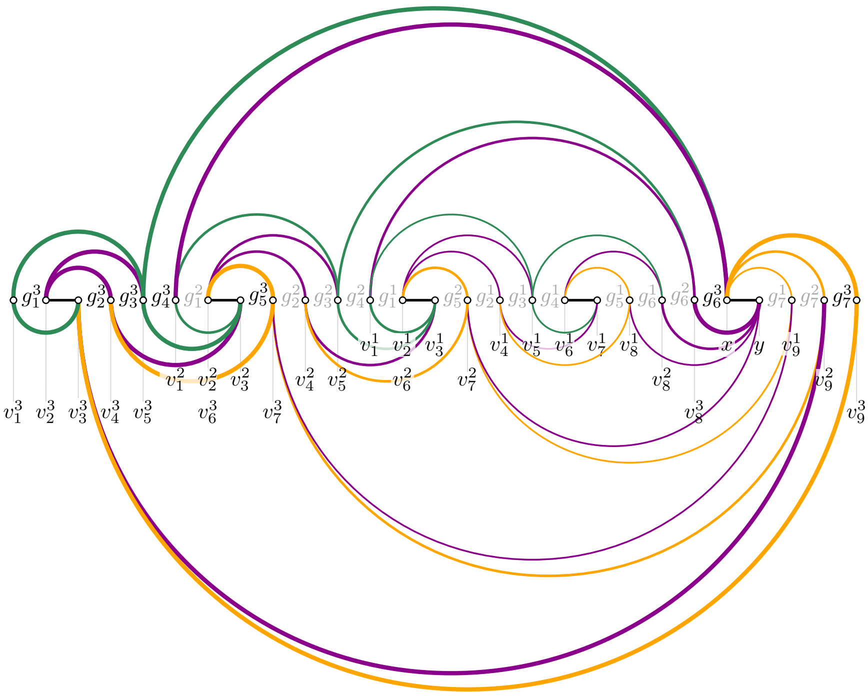


Figure 12 Illustration for the iterative construction of T^ℓ and $\tilde{T}^\ell$ from $T^{\ell-1}$ and $\tilde{T}^{\ell-1}$ for $\ell = 3$.

The edges in $T^{\ell-1}$ and $\tilde{T}^{\ell-1}$ are also present in T^ℓ and $\tilde{T}^\ell$, respectively, except that the right endpoint of $e_4^{\ell-1}$ and $e_6^{\ell-1}$ (which was x) is changed to v_8^ℓ, and the left endpoint of $\tilde{e}_2^{\ell-1}$ and $\tilde{e}_7^{\ell-1}$ (which was $v_3^{\ell-1}$) is changed to v_7^ℓ. Correspondingly, we change $g_2^{\ell-1}$ to $v_7^\ell v_4^{\ell-1}$ and $g_6^{\ell-1}$ to $v_8^{\ell-1} v_8^\ell$. We put a short edge $v_2^\ell v_3^\ell$ in both T^ℓ and $\tilde{T}^\ell$. Lastly, we add the following edges to T^ℓ and $\tilde{T}^\ell$

$$
\begin{aligned}
&e_1^\ell = v_1^\ell v_5^\ell, \quad e_2^\ell = v_2^\ell v_4^\ell, \quad e_3^\ell = v_2^\ell v_5^\ell, \quad e_4^\ell = v_5^\ell x, \quad e_5^\ell = v_6^\ell v_7^\ell, \quad e_6^\ell = v_1^{\ell-1} x, \quad e_7^\ell = x v_9^\ell,\\
&\tilde{e}_1^\ell = v_1^\ell v_3^\ell, \quad \tilde{e}_2^\ell = v_3^\ell v_9^{\ell-1}, \quad \tilde{e}_3^\ell = v_4^\ell v_3^{\ell-1}, \quad \tilde{e}_4^\ell = v_5^\ell v_3^{\ell-1}, \quad \tilde{e}_5^\ell = v_4^\ell v_7^\ell, \quad \tilde{e}_6^\ell = v_8^\ell y, \quad \tilde{e}_7^\ell = v_3^\ell v_9^\ell,
\end{aligned}
$$

with corresponding gaps

$$
g_1^\ell = v_1^\ell v_2^\ell, \quad g_2^\ell = v_3^\ell v_4^\ell, \quad g_3^\ell = v_4^\ell v_5^\ell, \quad g_4^\ell = v_5^\ell v_1^{\ell-1}, \quad g_5^\ell = v_3^{\ell-1} v_7^\ell, \quad g_6^\ell = v_8^\ell x, \quad g_7^\ell = v_9^{\ell-1} v_9^\ell.
$$

We next show properties of the trees T^ℓ and $\tilde{T}^\ell$ resulting form the iterative construction.

▶ **Lemma 17** ($\star$). *The following are true for T^ℓ and $\tilde{T}^\ell$:*

(i) *T^ℓ and $\tilde{T}^\ell$ are both non-crossing spanning trees,*

(ii) *their near-near pairs are $(e_i^j, \tilde{e}_i^j)$, with associated gaps g_i^j, for $1 \le i \le 7$ and $1 \le j \le \ell$,*

(iii) *for $1 \le j \le \ell$, the conflict graph has bidirected edges $g_7^j \leftrightarrow g_2^j \leftrightarrow g_1^j \leftrightarrow g_3^j \leftrightarrow g_5^j \leftrightarrow g_4^j \leftrightarrow g_6^j$,*

(iv) *for $1 \le j < j' \le \ell$, the conflict graph has a bidirected edge $g_6^j \leftrightarrow g_7^{j'}$.*

▶ **Lemma 18.** *The conflict graph $H(T^\ell, \tilde{T}^\ell)$ has 7ℓ vertices and its largest acyclic sets have size at most $3\ell + 1$.*

Proof. That the conflict graph has 7ℓ vertices follows immediately from Lemma 17 (ii). Observe that from every bidirected path of length seven given in Lemma 17 (iii), we can pick at most four gaps for the acyclic set. Furthermore, the only way to pick four is to pick

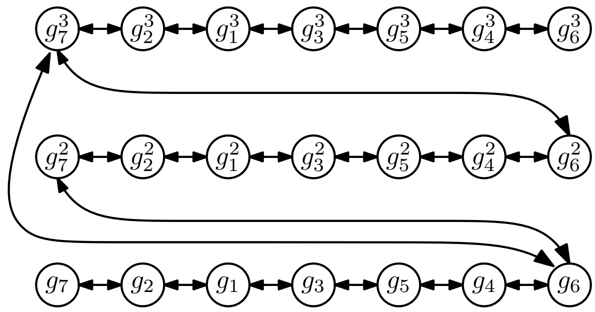


Figure 13 The relevant double conflicts in the conflict graph $H(T^\ell, \tilde{T}^\ell)$ for $\ell = 3$.

the gaps g_7^j, g_1^j, g_5^j and g_6^j, see also Figure 13. Now assume the cardinality of an acyclic set exceeds $3\ell + 1$. Then, by the pigeonhole principle, there exist at least two paths of length seven from which there are four pairs in the acyclic set. In particular, the set contains g_6^j and $g_7^{j'}$ for some $j < j'$ which are in double conflict by Lemma 17 (iv), contradicting acyclicity. ◀

Together, Lemma 18 and Theorem 5 imply the following.

▶ **Theorem 19.** *As a function of n,* $\operatorname{diam}(\mathcal{F}_n) \geq {}^{11}/_{7} \cdot n - o(n)$.

Proof. By Lemma 18 and Theorem 5 (ii), for any fixed ℓ, we have

$$\operatorname{diam}(\mathcal{F}_n) \geq \left(2 - \frac{3\ell+1}{7\ell}\right) n - c = \left(\frac{11}{7} - \frac{1}{7\ell}\right) n - c$$

for some c depending on ℓ. As a consequence, for any $\epsilon > 0$, $\operatorname{diam}(\mathcal{F}_n) \geq \left(\frac{11}{7} - \epsilon\right) n$ for sufficiently large n: Choose ℓ such that $\frac{1}{7\ell} < \epsilon$ and n large enough that $(\epsilon - \frac{1}{7\ell})n \geq c$, and calculate

$$\operatorname{diam}(\mathcal{F}_n) \geq \left(\frac{11}{7} - \frac{1}{7\ell}\right) n - c = \left(\frac{11}{7} - \epsilon\right) n + \left(\epsilon - \frac{1}{7\ell}\right) n - c \geq \left(\frac{11}{7} - \epsilon\right) n.$$

It follows that $\operatorname{diam}(\mathcal{F}_n) \geq \frac{11}{7} n - o(n)$. ◀

6 Conclusion

For spanning trees on convex point sets we proved that finding shortest flip sequences is NP-hard and we improved the current best lower bound on the diameter of the flip graph.

Interesting future directions include determining the exact diameters for flip graphs on point sets in convex as well as general position. We are also hopeful that our techniques for NP-hardness are fruitful in other settings. Very exciting is a recently announced breakthrough by Dorfer [20] concerning the complexity of computing the flip distance of triangulations on convex point sets, which is inspired by our methods.

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