Product Structure and Treewidth of Hyperbolic Uniform Disk Graphs

Thomas Bläsius
Karlsruhe Institute of Technology, Germany

Emil Dohse
Karlsruhe Institute of Technology, Germany

Deborah Haun
Karlsruhe Institute of Technology, Germany

Laura Merker
Karlsruhe Institute of Technology, Germany

Abstract

Hyperbolic uniform disk graphs (HUDGs) are intersection graphs of disks with some radius r in the hyperbolic plane, where r may be constant or depend on the number of vertices in a family of HUDGs. We show that HUDGs with constant clique number do not admit *product structure*, i.e., that there is no constant c such that every such graph is a subgraph of $H \boxtimes P$ for some graph H of treewidth at most c. This justifies that HUDGs are described as not having a grid-like structure in the literature, and is in contrast to unit disk graphs in the Euclidean plane, whose grid-like structure is evident from the fact that they are subgraphs of the strong product of two paths and a clique of constant size [Dvořák et al., '21, MATRIX Annals]. By allowing H to be any graph of constant treewidth instead of a path-like graph, we reject the possibility of a grid-like structure not merely by the maximum degree (which is unbounded for HUDGs) but due to their global structure. We complement this by showing that for every (sub-)constant r, HUDGs admit product structure, whereas the typical hyperbolic behavior is observed if r grows with the number of vertices.

Our proof involves a family of n-vertex HUDGs with radius $\log n$ that has bounded clique number but unbounded treewidth, and one for which the ratio of treewidth and clique number is $\log n / \log\log n$. Up to a $\log\log n$ factor, this negatively answers a question raised by Bläsius et al. [SoCG '25] asking whether balanced separators of HUDGs with radius $\log n$ can be covered by less than $\log n$ cliques. Our results also imply that the local and layered tree-independence number of HUDGs are both unbounded, answering an open question of Dallard et al. [arXiv '25].

2012 ACM Subject Classification Mathematics of computing → Graph theory

Keywords and phrases hyperbolic uniform disk graphs, product structure, treewidth

Digital Object Identifier 10.4230/LIPIcs.SoCG.2026.18

Related Version *Full Version*: https://arxiv.org/abs/2603.18997 [2]

Funding Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Projektnummer 524989715.

Acknowledgements We thank Torsten Ueckerdt and Marcus Wilhelm for fruitful discussions.

1 Introduction

In this paper, we investigate the structure of *hyperbolic uniform disk graphs (HUDGs)*, i.e., graphs G for which there is an r such that G is the intersection graph of disks with radius r in the hyperbolic plane. The structure of these graphs heavily depends on r, where r may be constant or a function depending on the number n of vertices for a family of HUDGs. In the literature [3, 5], HUDGs with very small r are described as almost Euclidean or grid-like, and as firmly hyperbolic or more hierarchical for larger r; see Figure 1. However, the state of the

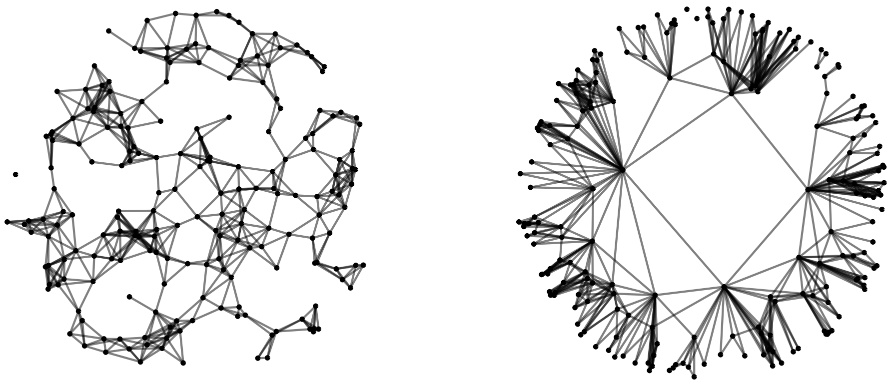

Figure 1 HUDGs with random vertex positions. Left: The radius r is so small (think of $1/\sqrt{n}$) that it is indistinguishable from the Euclidean setting. Right: The disk radius r is logarithmic in n.

art cannot fully explain in what sense the grid-like structure depends on r. We contribute to the understanding of the structure of HUDGs by formalizing when a graph has a grid-like structure and showing that this is indeed not satisfied by HUDGs, unless the radius is small. We do so by showing that HUDGs do not admit a so-called *product structure*, and thereby also deepen the understanding of this notion. Before stating our results precisely, we review what is already known on the structure of HUDGs and how it depends on the radius r.

We start with small radii and work our way up from there. Locally, the hyperbolic plane behaves like the Euclidean plane, which sets the expectation that very small r yield graphs similar to *Euclidean unit disk graphs (EUDGs)*, which are defined as intersection graphs of unit disks in the Euclidean plane. While there are always HUDGs that are not EUDGs independent of how small we choose r, every EUDG is in fact a HUDG if we choose r sufficiently small [3]. Sufficiently small here means that r shrinks with the number of vertices n. EUDGs [8] are a well studied graph class that allows for balanced separators that can be covered with $O(\sqrt{n})$ cliques [14]. We remark that balanced separators are arguably one of the main reasons why HUDGs, EUDGs, and more generally disk graphs, are interesting from an algorithmic perspective. For HUDGs, there are balanced separators that can be covered with $O((1 + 1/r) \log n)$ cliques [5]. Interestingly, this matches the $\sqrt{n}$ bound of the Euclidean case for $r \in \Theta(1/\sqrt{n})$ up to a $\log n$ factor and improves for larger r. In fact, if $r \in \Omega(1)$, this gives separators coverable with $\log n$ cliques. This gives a strong argument that the structure of HUDGs changes when going from sub-constant to at least constant r. This change in structure is also reflected in the fact that several problems can be solved more efficiently for constant r than in the Euclidean setting [5, 30]. Summarizing, there is strong evidence that the structure of HUDGs change significantly between sub-constant r and constant r.

This raises the question whether there is a similar structural difference between constant r and super-constant r. While the above separators do not continue to get smaller, there are indications that super-constant r makes a difference, and that in particular $r \in \Theta(\log n)$ is an interesting case. For HUDGs with $r \in \Theta(\log n)$, the independent set problem can be solved in polynomial time [5]. Moreover, HUDGs with $r \in \Theta(\log n)$ and random vertex positions are so-called hyperbolic random graphs [31]; a random graph model that is popular due to its resemblance to real-world networks in regards to properties like clustering, diameter, and degree distribution [22, 23, 26]. While this indicates a significant difference between constant and growing r, the state of the art does not make the structural differences explicit. The only structural insight we are aware of is that large stars, or more generally high-degree vertices with sparse neighborhoods, require r to grow logarithmically with the degree [3].

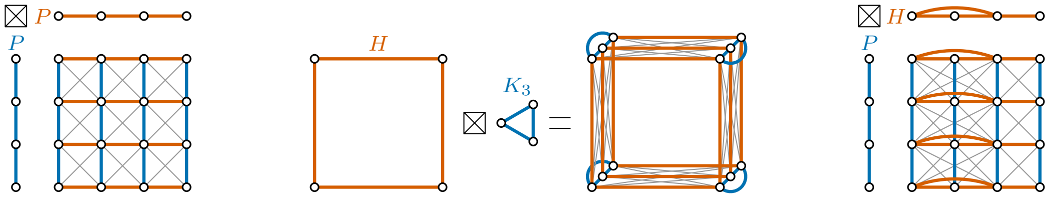

Figure 2 The strong product of two paths (left), of a graph H and a clique (middle), and a graph H with a path (right).

The main result of this paper is to show that every class of HUDGs with constant radius r has product structure, while there are HUDGs with growing radius that do not. This closes two gaps in the literature: First, it justifies why HUDGs with large radii should be considered *not* grid-like. And second, the contrast to HUDGs with constant radius shows a stark structural difference between constant radius compared to larger radii, say $\log n$.

Concerning the notion of grid-like structure, observe that $\sqrt{n} \times \sqrt{n}$-grids are EUDGs and thus also HUDGs for small radius. However, the grid-like structure of EUDGs goes far beyond merely containing grids. In fact, EUDGs are in general rather grid-like in the sense that there may be large cliques locally, but the global structure of every EUDG behaves like (the subgraph of) a grid. This notion of being "grid-like" can be formalized using the *strong product* of graphs, which we introduce formally in Section 2. For now, it suffices to know that the strong product $P_a \boxtimes P_b$ of two paths on a and b vertices, respectively, is the $a \times b$-grid with diagonals. Moreover, the strong product $H \boxtimes K_k$ of a graph H with the complete graph K_k expands each vertex of H to a k-clique as shown in Figure 2. Dvořák, Huynh, Joret, Liu, and Wood [20] show that every EUDG G is a subgraph of $P \boxtimes P \boxtimes K_k$, where P is a path and k is linear in the clique number ω of G. Equivalently, one can partition the vertices of a EUDG into sets of size $O(\omega)$ so that contracting each set into a single vertex yields a subgraph H of a grid (with diagonals), yielding a product of the form $H \boxtimes K_k$.

More generally, the concept of product structure allows us to view the graphs of a graph class as subgraphs of the product of simpler graphs; usually a graph with small treewidth and a path. We say a graph class $\mathcal{G}$ admits *product structure* if for every $G \in \mathcal{G}$, we have $G \subseteq H'' \boxtimes P$ for some graph H'' of constant treewidth and some path P. In our proofs, we show a stronger statement, namely that the graphs we consider are subgraphs of $H \boxtimes K_k$ for some hyperbolic tiling H. Since planar graphs, which include hyperbolic tilings, admit product structure, this implies $H \subseteq H' \boxtimes P \boxtimes K_3$ for some graph H of constant treewidth [16,17], so with $H'' = H' \boxtimes K_3 \boxtimes K_k$ we obtain a product of the form $H'' \boxtimes P$. Note that the above mentioned result on EUDGs can be viewed in this framework by choosing $H = P \boxtimes P$. We remark that replacing one of the paths in the product $P \boxtimes P \boxtimes K_k$ by a graph H', enables graphs that go beyond Euclidean square-grids. In particular, it allows for high-degree vertices. That is, rejecting a grid-like structure via product structure is significantly stronger than showing that a graph class does not allow for a decomposition of the form $P \boxtimes P \boxtimes K_k$. To finish the discussion on grid-like structure, let us mention that "having product structure" does not need to be seen as a binary decision. For this, the *row-treewidth* of a graph G is the smallest t such that $G \subseteq H'' \boxtimes P$ for some graph H'' of treewidth t and some path P. The row-treewidth of a graph class is the maximum row-treewidth among all contained graphs, possibly as a function depending on the number of vertices if it is unbounded. Now, among graph classes admitting product structure, we consider a class to have a stronger grid-like structure if the row-treewidth is small.

Our results that some families of HUDGs admit product structure while others do not, make HUDGs also interesting from the product structure perspective. Product structure has proved to be a useful tool for many graph-theoretic (see [28] for a survey) and algorithmic [1,6] applications. While naturally, the literature concentrates on graph classes that admit product structure [7, 15, 16, 19, 27, 28], it is also interesting to find graph classes that do not admit product structure, for an other reason than the presence of large cliques. Interestingly, many known constructions are geometric intersection graphs, namely different kinds of string graphs with constant clique number [29, 32]. By showing that HUDGs do not admit product structure, we contribute to an improved understanding of the limitations of this notion.

1.1 Main Results and Discussion

The following theorem states our main result that HUDGs do not admit product structure, even when restricted to constant clique size, rejecting the possibility of a global grid-structure.

▶ **Theorem 1.** *Hyperbolic uniform disk graphs with constant clique number do not admit product structure.*

Our proof of Theorem 1 requires the radius to grow with the graph size. This is indeed necessary as HUDGs with constant or sub-constant disk radius do admit product structure.

▶ **Theorem 2.** *Every family of hyperbolic uniform disk graphs with clique number and disk radius in $O(1)$ admits product structure. In contrast, for every super-constant r, there are families of hyperbolic uniform disk graphs with constant clique number and disk radius in $\Theta(r)$ not admitting product structure.*

This confirms the observations of previous results [3, 5] that HUDGs become similar to the Euclidean case for small radii, whereas they differ significantly for large radii. As having product structure in some sense restricts the complexity of the neighborhood of an individual vertex, this in particular indicates that HUDGs become more complex (and for some applications more interesting) when allowing the radius to grow with the graph size. We show the first part of Theorem 2 in Section 4 and the second in Section 3.

Bounds on the Row-Treewidth. We actually prove more fine-grained results that go beyond the binary question of having product structure. Recall that a graph class admits product structure if and only if it has bounded row-treewidth, i.e., a super-constant lower bound and a constant upper bound on the row-treewidth disprove and prove product structure, respectively. Our lower and upper bounds on the row-treewidth depending on clique number ω and disk radius r are shown in Table 1. The first two rows show our lower bounds that grow with r (as long as $r \in O(\log n)$), where we have the two cases $\omega \in O(1)$ and $\omega \in O(\log \log n)$. The first case is most relevant for product structure as growing clique size immediately rejects product structure, whereas the second case gives the stronger lower bound. Then, the third row gives our upper bound, whereas the last row is from the literature and is discussed at the end of this section. Before doing so, we discuss each result in Table 1 in more detail.

The first lower bound in Table 1 assumes constant clique number $\omega \in O(1)$. Note that there are two regimes for the radius r. For radii smaller than $\log n$, the lower bound grows with increasing radius until it reaches a row-treewidth of $\Omega(\log \log n)$ for radius $r \in \Theta(\log n)$. For larger radii, the lower bound stops to grow with r. We note that any super-constant

Table 1 Overview of lower and upper bounds for the row-treewidth. The lower bounds mean that for every radius r in the given regime, there exists a family of HUDGs with radius $\Theta(r)$ and the given bounds on the clique number ω and the row-treewidth. The upper bounds hold for all HUDGs with radius r and clique number ω (both of which may depend on the graph size n).

ω	row-treewidth		comment
	$r \in O(\log n)$	$r \in \Omega(\log n)$	
$O(1)$	$\Omega(\log r)$	$\Omega(\log \log n)$	Cor. 15 and 16; implies Thm. 1 and 2
$O(\log \log n)$	$\Omega(r)$	$\Omega(\log n)$	Cor. 15 and 16
variable	$O(\omega \cdot 3^{8r})$		Thm. 17; implies Thm. 2 for $r, \omega \in O(1)$
variable	$O(\omega(1 + \frac{1}{r}) \log n)$		follows from treewidth bound [5]

radius already implies super-constant row-treewidth and thus refutes product structure, yielding Theorem 1 and the second part of Theorem 2. That is, the specific lower bounds are significantly stronger than Theorem 1 as they refute product structure for every growing radius and not only for the class of all HUDGs. For the lower bound given in the next row of Table 1, we allow slightly growing clique number $\omega \in O(\log \log n)$, which yields exponentially larger lower bounds for the row-treewidth compared to the first row. We remark that all our lower bounds are linear in the size of the largest clique-minor, known as the Hadwiger number.

Turning to our upper bound (bottom part of Table 1), observe that it only depends on ω and r. Assuming both to be constant directly implies the first part of Theorem 2. Moreover, for growing clique number ω (but constant r), this is the best upper bound one can hope for, as the clique number is a lower bound on the row-treewidth. Concerning growing radius, note that the radius r contributes exponentially to the row-treewidth. While it might be possible to improve this upper bound, we note that our lower bounds for growing radius imply that there has to be a dependence on r. We remark that we strengthen our upper bound in terms of product structure. For this, recall that the upper bound on the row-treewidth means that every HUDG with radius r is a subgraph of $H'' \boxtimes P$ for some graph H'' of treewidth $O(\omega \cdot 3^{8r})$. We show the stronger statement that for every HUDG G with radius r and clique number ω, we have $G \subseteq H \boxtimes K_k \subseteq H' \boxtimes P \boxtimes K_k$, for some hyperbolic tiling H, some graph H' of treewidth 3, some path P, and $k \in O(\omega \cdot 3^{8r})$. Our proof is similar to the Euclidean case [20], but instead of the Euclidean square-grid, we construct a suitable tiling of the hyperbolic plane. Interestingly, the tiling depends on r and becomes more hyperbolic for large r and more Euclidean for small r. With our tiling, we also contribute to the rich field of constructing hyperbolic tilings with various favorable properties [9, 13, 33].

▶ **Theorem 3.** *For each $r > 0$ there is an (irregular) tiling of the hyperbolic plane such that*
- *every tile can be covered by $O(3^{8r})$ disks of radius r and*
- *each two points with distance at most $2r$ lie in the same tile or in two adjacent tiles.*

The last upper bound in Table 1 directly follows from the literature, which we discuss next.

Connections to the State of the Art. Our lower bounds are based on the fact that the row-treewidth is asymptotically lower-bounded by the treewidth in the neighborhood of a vertex [18]. Notably, the neighborhoods of HUDGs are called *strongly hyperbolic uniform disk graphs (SHUDGs)* and are an interesting graph class in their own right [3]. In fact, the much studied hyperbolic random graphs are SHUDGs with random vertex positions [31]. The core of our lower bounds for the row-treewidth of HUDGs are the following bounds on the treewidth of SHUDGs.

▶ **Theorem 4.** *There are families of n-vertex strongly hyperbolic uniform disk graphs with radius $\Theta(\log n)$ and*

- *clique number $O(\log\log n)$ and treewidth $\Omega(\log n)$*
- *clique number $O(1)$ and treewidth $\Omega(\log\log n)$.*

To connect this to the results by Bläsius, von der Heydt, Kisfaludi-Bak, Wilhelm, and Van Wordragen [5], recall that they show that n-vertex HUDGs with radius r have balanced separators that can be covered with $(1 + 1/r)\log n$ cliques. This implies that the treewidth is $O(\omega(1 + 1/r)\log n)$ [21], which is an upper bound for the row-treewidth (which is why we list it in Table 1). For r at least constant, this gives separators coverable with $O(\log n)$ cliques, i.e., treewidth $O(\omega \log n)$. The authors of [5] raise the question whether this can be improved, in particular for increasing radius r. With Theorem 4, we give a negative answer to this, up to a $\log\log n$ factor: The family with clique number $\omega \in O(\log\log n)$ has treewidth $\Omega(\log n) = \Omega(\omega \log n/ \log\log n)$.

In addition, Theorem 4 provides the first known example of a family of SHUDGs with constant clique number but unbounded treewidth. Graph classes with this property are also called *not* (tw, ω)*-bounded*. By Dallard, Milanič, and Štorgel [12], graph classes with bounded tree-independence number are (tw, ω)-bounded, so our constructed family of SHUDGs has unbounded tree-independence number. Since SHUDGs are neighborhoods of vertices in HUDGs, it follows that the local tree-independence number of HUDGs is unbounded, which in turn implies that their layered tree-independence number is unbounded [24]. This gives a negative answer to an open question of Dallard, Milanič, Munaro, and Yang [11].

▶ **Corollary 5.** *The local tree-independence number and the layered tree-independence number of hyperbolic uniform disk graphs are unbounded.*

Note that for these two answered open questions, the first comes from the HUDG side and the other is more related to product structure. This underscores that HUDGs and product structure form an interesting combination.

2 Preliminaries

We assume familiarity with some graph-theoretic terminology; also see the full version [2].

2.1 Product Structure

Separators and Treewidth. The *treewidth* $\mathrm{tw}(G)$ of a graph G is linearly tied to the *separation number*, i.e., the smallest integer s such that every subgraph of G has a balanced separator of size s [21], [10, Lemma 7.20]. If G has a clique-minor of size k, then $\mathrm{tw}(G) \geq k-1$.

Strong Product. For two graphs G and H, the *strong product* $G \boxtimes H$ is defined as follows; also see Figure 3. $G \boxtimes H$ has vertex set $V(G \boxtimes H) = V(G) \times V(H)$, i.e., every vertex $u = (u_G, u_H)$ of $G \boxtimes H$ corresponds to a vertex u_G in G and to a vertex u_H in H. Two vertices $u = (u_G, u_H)$ and $v = (v_G, v_H)$ are adjacent in $G \boxtimes H$ if one of the following is true.

- u and v are the same vertex in G and adjacent in H, i.e., $u_G = v_G$ and $u_H v_H \in E(H)$.
- u and v are the same vertex in H and adjacent in G, i.e., $u_H = v_H$ and $u_G v_G \in E(G)$.
- u and v are adjacent in G and H, i.e., $u_G v_G \in E(G)$ and $u_H v_H \in E(H)$.

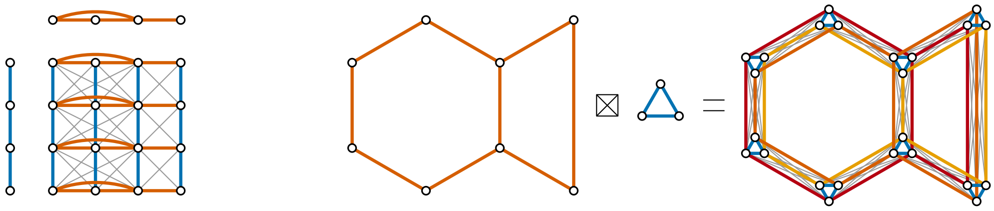

Figure 3 Left: Strong product of a graph H (red) with a path P (blue). The three different types of edges are color coded in red (edge in H, same vertex in P), blue (edge in P, same vertex in H), and gray (edge in H and P). Right: Strong product of a graph (red) with a K_3 (blue).

Row-Treewidth and Product Structure. We are usually interested in products of the form $H \boxtimes P$ where H has small treewidth and P is a path (of arbitrary length). Assume G is a subgraph of such a product $G \subseteq H \boxtimes P$ and assume that H is chosen such that $\mathrm{tw}(H)$ is minimum with this property. Then, we call $\mathrm{tw}(H)$ the *row-treewidth* of G. We say that a graph family $\mathcal{G}$ has *product structure* if the graphs in $\mathcal{G}$ have row-treewidth $O(1)$, i.e., there exists a constant that upper-bounds the row-treewidth of every graph $G \in \mathcal{G}$.

For products of the form $G = H \boxtimes K_k$, think of G as being obtained as follows. Start with H and replace each vertex with a k-clique and each edge with a complete bipartite graph between the corresponding k-cliques. The treewidth of $H \boxtimes K_k$ is $(\mathrm{tw}(H) + 1) \cdot k - 1$.

2.2 Hyperbolic Geometry and Tilings

While this paper is about graphs in the hyperbolic plane, most of it can be understood with only little knowledge of hyperbolic geometry. We thus only provide a brief introduction. Formally, hyperbolic geometry is obtained from Euclidean geometry by replacing the parallel axiom: Instead of having exactly one parallel to a line through some given point not on the line, there are infinitely many, where two lines are considered parallel if they do not intersect. Thus, many concepts familiar from Euclidean geometry still apply as long as the parallel axiom is not involved, e.g., definitions of distances, angles, disks, (regular) polygons, and congruency are unchanged and the triangle inequality holds. However, some geometric laws like trigonometric formulas work differently, which we only introduce when needed; see [25].

Basic Properties. Our illustrations sometimes use the Poincaré disk model, which maps the hyperbolic plane into a unit disk of the Euclidean plane. However, we do not need any knowledge about this model and the illustrations should be helpful regardless. The core property of the hyperbolic plane relevant for this paper is the following. The hyperbolic plane expands exponentially, while at the same time behaving like the Euclidean plane locally. To concretize, consider a disk D of radius r. If r is small, then the area and circumference of D are quadratic and linear in r, respectively, like in the Euclidean plane. If the radius r increases, the area and circumference both grow exponentially like $\Theta(e^r)$. This has two interesting effects. First, the hyperbolic plane has more space. This becomes more pronounced at bigger scales, i.e., when we look at a large region. Secondly, in Euclidean geometry one can scale objects to change their size while keeping them structurally the same, e.g., think of similar triangles. In the hyperbolic plane, however, the scale makes a big structural difference, and there is no scaling operation that changes all distances by the same factor.

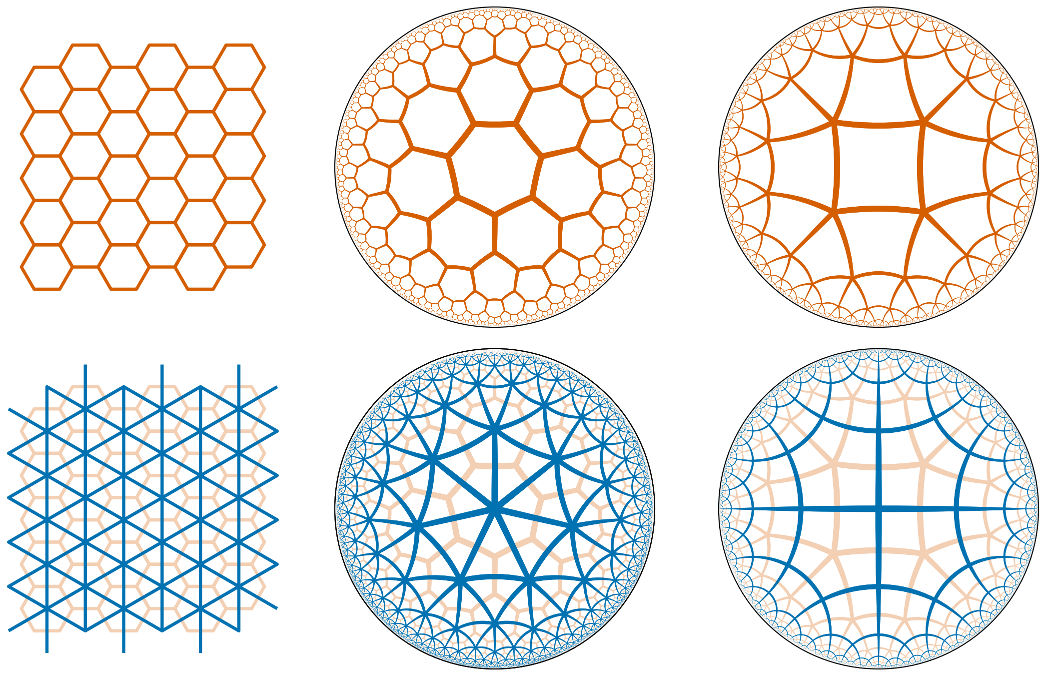

Figure 4 Regular tilings (top) together with their duals (bottom). From left to right: A Euclidean $\{6, 3\}$ tiling, a hyperbolic $\{7, 3\}$ tiling, and a hyperbolic $\{4, 5\}$-tiling. While the tiles seem to get smaller towards the boundary of the Poincaré disk, they are all congruent.

Hyperbolic Tilings. A *regular tiling* is a partition of the plane into congruent regular polygons; see Figure 4. The Euclidean plane can be tiled with triangles, squares, and hexagons. These tilings have the Schläfli symbols $\{3, 6\}$, $\{4, 4\}$, and $\{6, 3\}$, respectively, where $\{p, q\}$ indicates that each tile is a p-gon and q such p-gons meet at each corner. For any p and q with $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, we obtain a regular tiling of the hyperbolic plane. While Euclidean tilings can be scaled arbitrarily, the size of the tiles in a hyperbolic $\{p, q\}$-tiling is determined by p and q and can be derived from basic trigonometry. Observe in Figure 4 that the exponential expansion of the hyperbolic plane is reflected in the tilings, which can be seen as a discretization of the plane. We also consider (irregular) tilings, where the tiles do not need to be congruent. We usually interpret tilings as infinite[1] plane graphs. Then the corners are the vertices, the sides are the edges, and the p-gons are the *faces*. The *dual* of such a plane graph has one vertex per face and two faces are adjacent if they share an edge.

2.3 Geometric Graphs

A graph G is a *(Euclidean) unit disk graph (EUDG)* if it is the intersection graph of disks of radius 1 in the Euclidean plane. We also interpret the disk centers as the *positions* of the corresponding vertices and use terms like *distance*, i.e., two vertices are connected if and only if their distance is at most 2. We say that G is a *hyperbolic uniform disk graph (HUDG)* if there exists a *disk radius* r such that G is the intersection graph of hyperbolic disks of radius r. The radius r can depend on the graph and makes an important difference.

[1] Throughout the paper, we implicitly assume that all considered graphs are finite. Tilings are the only exception. But also for tilings, we are usually only interested in a finite subgraph.

We are usually interested in the asymptotics of r with respect to the graph size n. We say that a HUDG G *has disk radius* r if it is the intersection graph of hyperbolic disks of radius r. Moreover, we say that a family $\mathcal{G}$ of HUDGs *has disk radius* $r(n)$ if every $G \in \mathcal{G}$ with n vertices has disk radius $r(n)$. This lets us use asymptotics, i.e., $\mathcal{G}$ *has disk radius* $\Theta(r(n))$ if there exist constants c_1, c_2 such that every sufficiently large n-vertex graph $G \in \mathcal{G}$ has disk radius r_G for $c_1 r(n) \leq r_G \leq c_2 r(n)$.[2]

A graph is a *strongly hyperbolic uniform disk graph (SHUDG)* [3] if it is a HUDG with disk radius r such that all vertices have distance at most $2r$ from the origin. This means that a vertex placed at the origin would be universal, i.e., adjacent to all other vertices of the graph. Note that the neighborhood of every vertex in a HUDG induces an SHUDG. Studying SHUDGs is in some sense similar to studying families of HUDGs with large disk radius: SHUDGs with bounded average degree have disk radius $\Omega(\log n)$ [3].

3 HUDGs Do Not Admit Product Structure

In this section, we show that HUDGs with constant clique number do not admit product structure (Theorem 1). While our construction initially has disk radius $\Theta(\log n)$, we extend it to arbitrary super-constant disk radii to prove the second part of Theorem 2. This is tight as we show in Section 4 that every family of HUDGs with disk radius in $O(1)$ and constant clique number has indeed product structure. Similarly, we provide families of HUDGs for the lower bounds in the second row of Table 1 for every super-constant disk radius. These results are summarized in Corollaries 15 and 16 at the end of the section.

To reject product structure, we use that, asymptotically, the treewidth in the neighborhood of some vertex is a lower bound on the row-treewidth [16, 18]. Since SHUDGs describe the neighborhoods of vertices in a HUDG, we aim for a family of SHUDGs with constant clique number but unbounded treewidth to prove Theorem 1 (see the full version [2] for details).

▶ **Theorem 6.** *There is a family of n-vertex SHUDGs with disk radius in $\Theta(\log n)$, clique number in $O(\log \log n)$, and treewidth in $\Omega(\log n)$.*

We then identify an induced subgraph within each of the constructed SHUDGs to prove Theorem 1. In addition, Theorems 6 and 7 together prove Theorem 4.

▶ **Theorem 7.** *There is a family of n-vertex SHUDGs with disk radius in $\Theta(\log n)$, clique number in $\Theta(1)$, and treewidth in $\Omega(\log \log n)$.*

3.1 Proof of Theorem 6

For each integer $r \geq 2$, we construct an n-vertex SHUDG G_r by placing $n \in \Theta(2^r)$ vertices into the hyperbolic plane, which yields $r \in \Theta(\log n)$ for the disk radius as required by Theorems 6 and 7. First, we place a vertex at the origin which we call the *root*. Second, for $2 \leq k \leq r$, let A_k be a regular 2^k-gon with side lengths $2r$ whose center is at the origin. We align these polygons such that for each $k > 2$, every other edge of A_k has the same perpendicular bisector as some edge of A_{k-1}; see Figure 5. Now, let V_k denote the vertex set of A_k and let V_1 contain exactly the root. We say a vertex is in the *k-th level* if it is contained in V_k. The vertex set of G_r is defined as the union of all V_k, $k \in [r]$, and two vertices are adjacent if and only if their distance is at most $2r$. In particular, the sides of

[2] We note that this is the natural way to use asymptotics. We only make this explicit as asymptotics can be unintuitive in this context, e.g., "the class of all HUDGs with radius $\Theta(\log n)$" is not well-defined.

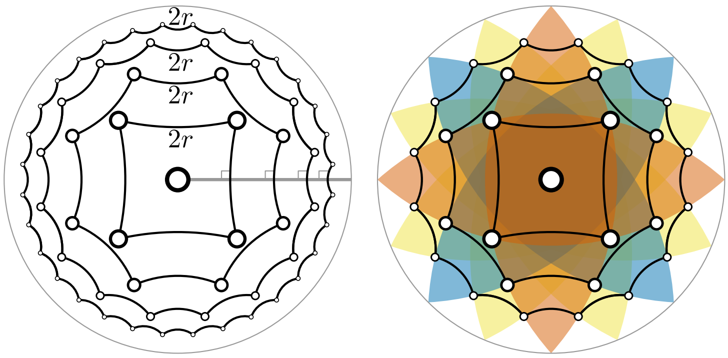

Figure 5 Left: Vertex positions of G_r constructed for Theorem 6. The k-th level, $k \geq 2$, induces a 2^k-cycle with consecutive vertices having angular distance $2\pi/2^k$. Right: Equivalent construction inspired by separators in [4], where vertices are placed at intersection points of hypercycles.

the polygons are also edges in G_r, and there are no further edges within a level as chords in a regular polygon are strictly longer than their sides. That is, each level induces a cycle, where the angular distance between any two adjacent vertices in level k is $2\pi/2^k$. Note that the number of vertices is indeed in $\Theta(2^r)$. In the full version [2, Lemma 11] we show that all vertices are adjacent to the root, i.e., G_r is a SHUDG, as required. Without going into detail, we note that the above construction is inspired by an analysis in [4]; see Figure 5.

To bound the clique number and treewidth of G_r, we count the neighbors in each level.

▶ **Lemma 8.** *For every $2 \leq i \leq j \leq r$, every vertex of G_r in level i has at least two and at most $4\sqrt{2^{j-i}}$ neighbors in level j, where the upper bound is tight up to a constant factor.*

Before giving a proof sketch, let us discuss the implications to the structure of G_r. For this, consider a vertex v in some level i and the two closest vertices in level $i+1$, one to the left and one to the right. Now let T denote the tree induced by the four edges from the root to the first level and the edges to the two closest neighbors for each vertex in levels $2, \ldots, r-1$, see Figure 6. Observe that, apart from the root having degree 4, T is a spanning binary tree. That is, the number of descendants of some vertex grows with base 2 in T, whereas the number of neighbors in G_r grows with base $\sqrt{2}$. Hence, for a vertex in level i, there are far fewer neighbors than descendants in level j, provided $j - i$ is sufficiently large. This observation is key to bounding the clique number of G_r.

Proof sketch. Observe that for a vertex v_i in level i, the two closest vertices v_j^ℓ, v_j^r in level j are adjacent and have a very small angular distance to v_i (see Figure 6). Since v_j^ℓ, v_j^r are connected to the root, one can conclude that they also have edges to v_i.

For the upper bound, we use some trigonometric formulas to bound the maximum angular distance of two adjacent vertices with fixed distances from the root. Since we know from the construction how many vertices we have in each level, this gives us the number of neighbors, see again Figure 6. We refer to [2, Lemma 13] for a full proof. ◀

By the lower bound of Lemma 8, since each two levels are connected by edges, contracting each of the r levels yields an r-clique.

▶ **Corollary 9.** *The treewidth of G_r is at least $r \in \Omega(\log n)$, where n is the number of vertices.*

Next, the upper bound enables us to conclude the following lemma for triangles in G_r.

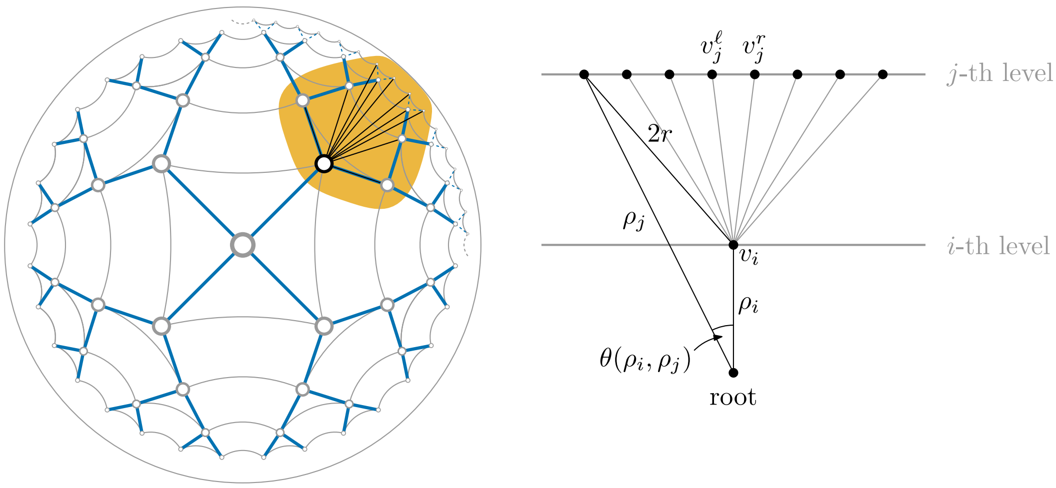

Figure 6 Left: A spanning tree (blue) guiding the structure of G_r, and the neighborhood of some vertex (black and orange). Note that the number of neighbors increases with the level, but slower than the number of descendants in the tree, whereas the angle in which a vertex has neighbors shrinks. Right: The number of neighbors a vertex v_i has in level j is determined by the angle $\theta(\rho_i, \rho_j)$ to the leftmost neighbor, where ρ_i and ρ_j are the radii of the two levels.

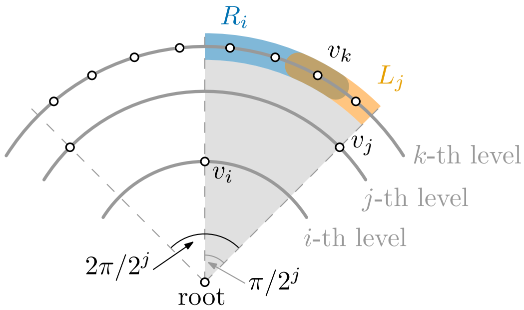

Figure 7 A triangle of G_r, where v_k lies both in the neighborhood of v_i (blue) and of v_j (orange). We conclude that the right half of the neighborhood of v_i plus the left half of the neighborhood of v_j is larger than the number of vertices in level k that lie between v_i and v_j (gray wedge).

▶ **Lemma 10.** *For every triangle in G_r with vertices v_i, v_j, and v_k on levels $2 \leq i \leq j \leq k$, respectively, we have $k - j \leq j - i + 6$.*

Proof sketch. We consider the number of neighbors v_i and v_j have in level k and compare the sum with the number of vertices in level k that lie between v_i and v_j (Figure 7). The first is $O(\sqrt{2^{k-i}})$ by Lemma 8, whereas the second is $\Omega(2^{k-j})$ by construction, yielding the desired bound, which we show in the full version [2, Lemma 15]. ◀

It follows from Lemma 10 that if a clique in G_r has vertices in levels i and $j \geq i$ with $\Delta = j - i$, then all further vertices of the clique in larger levels are in levels $j, \ldots, j + \Delta + 6$. Applying this to all pairs of vertices in a clique, we obtain that the distances $|i - j|$ between the levels i, j shrink exponentially. We formalize this in the following lemma.

▶ **Lemma 11.** *Let $r \geq 1$, $c \geq 0$ be integers, and let $S \subseteq [r]$ be a set such that for every three elements $i < j < k$ of S, we have $k - j \leq j - i + c$. Then $|S| < \log_2(r) + c + 2$.*

Proof sketch. We claim that for an optimal set S^*, we may choose the elements greedily, adding always the largest element so that $k - j \leq j - i + c$ is maintained for every $i < j < k$ in S^*. In [2, Lemma 16] we show that S^* contains roughly the c largest elements and starting from there, leaves exponentially growing gaps when smaller elements are added. ◀

Putting everything together, we conclude that the clique number of G_r is indeed $O(\log \log n)$.

▶ **Lemma 12.** *The clique number of G_r is at most $O(\log r) = O(\log \log n)$, where n is the number of vertices of G_r.*

Proof. Consider a clique C in G_r whose size we aim to bound. First recall that each level induces a cycle and thus has clique number 2. Hence, up to a factor of 2, we may assume that the vertices of C are in pairwise distinct levels. Now each three vertices of C in levels $i < j < k$ form a triangle and thus satisfy $k - j \leq j - i + 6$ by Lemma 10. Then, Lemma 11 shows that C contains only $2(\log_2(r) + 8) \in O(\log r) = O(\log \log n)$ vertices. ◀

Finally, Corollary 9 and Lemma 12 together prove Theorem 6.

3.2 Proof of Theorem 7

Our goal is to construct SHUDGs with radius $r \in \Theta(\log n)$, clique number $O(1)$, and treewidth $\Omega(\log \log n)$, which implies Theorem 1. We do so by taking a subgraph of the graph G_r for $r \geq 8$ constructed in Section 3.1 that has constant clique number but unbounded treewidth. For this, let G'_r be the subgraph of G_r that is induced by the root and the vertices in levels 2^k for $k = 3, \dots, \log_2(r)$. Recall that G_r has r levels, where the k-th level contains 2^k vertices for $k \geq 2$, and thus the r-th level contains more than half of the vertices of G_r. Thus, G'_r has $\Theta(\log r) = \Theta(\log \log n)$ levels and $\Theta(n)$ vertices, where n is the number of vertices of G_r. Further recall that in G_r, and thus also in G'_r, Lemma 8 shows that each two levels are connected by edges. Hence, we again obtain a clique-minor by contracting each level.

▶ **Lemma 13.** *The treewidth of G'_r is at least $\Omega(\log r) = \Omega(\log \log n)$, where n is the number of vertices of G'_r.*

To bound the clique number, recall that the distance between two levels i and j refers to $|i - j|$, and not, e.g., to the hyperbolic distance.

▶ **Lemma 14.** *The clique number of G'_r is at most 5.*

Proof sketch. For the main idea, recall from Lemmas 10 and 11 that in G_r, the distances between the levels used by a clique shrink exponentially from the root to the outermost level, i.e., most vertices of any clique are in the outer levels. In contrast, we choose the levels for G'_r with exponentially growing distances, i.e., mostly inner levels are chosen. Thus, for every clique in G_r, the levels chosen for G'_r hit only very few vertices of the clique. We formalize this in the full version [2, Lemma 19]. ◀

This concludes the proof that hyperbolic uniform disk graphs do not admit product structure as Lemmas 13 and 14 together show Theorem 7, which in turn proves Theorem 1.

3.3 Dependence on the Disk Radius

The graph families we construct for Theorems 6 and 7 have disk radius $\Theta(\log n)$, which we extend to arbitrary radii by adding more vertices or stopping the construction earlier, see the full version [2]. Note that for super-constant r, we obtain unbounded row-treewidth thus no product structure. This proves the second part of Theorem 2 and strengthens Theorem 4.

▶ **Corollary 15.** *For every $r \in O(\log n)$, there are families of n-vertex HUDGs with disk radius in $\Theta(r)$,*
- *clique number in $O(\log r)$, and (row-)treewidth in $\Omega(r)$, respectively*
- *clique number in $O(1)$, and (row-)treewidth in $\Omega(\log r)$.*

▶ **Corollary 16.** *For every $r \in \Omega(\log n)$, there are families of n-vertex SHUDGs with disk radius in $\Theta(r)$,*
- *clique number in $O(\log \log n)$, and (row-)treewidth in $\Omega(\log n)$, respectively*
- *clique number in $O(1)$, and (row-)treewidth in $\Omega(\log \log n)$.*

4 Product Structure for Small Disk Radius

In this section, we show that HUDGs whose disk radius and clique number are (sub-)constant admit product structure, which finishes the proof of Theorem 2. In fact, we show that every HUDG G with disk radius r and clique number ω is a subgraph of $H \boxtimes K_k$, where H is a hyperbolic tiling and $k \in O(\omega \cdot 3^{8r})$, which emphasizes their grid-structure. Since H is planar, and planar graphs are subgraphs of $H' \boxtimes P \boxtimes K_3$ with $\mathrm{tw}(H') = 3$ [16], it follows that $G \subseteq H' \boxtimes P \boxtimes K_{3k}$. This implies that $G \subseteq H'' \boxtimes P$ for the graph $H'' = H' \boxtimes K_{3k}$ of treewidth $O(\omega \cdot 3^{8r})$, i.e., G has row-treewidth $O(\omega \cdot 3^{8r})$. Thus, in the following theorem, the product $H \boxtimes K_{O(\omega \cdot 3^{8r})}$ is the strongest statement and the others follow.

▶ **Theorem 17.** *For every HUDG G with disk radius r and clique size ω it holds that $G \subseteq H \boxtimes K_{O(\omega \cdot 3^{8r})} \subseteq H' \boxtimes P \boxtimes K_{O(\omega \cdot 3^{8r})}$, where H is a (possibly irregular) hyperbolic tiling and H' is a graph of treewidth at most 3. Moreover, the row-treewidth of G is in $O(\omega \cdot 3^{8r})$.*

Note that Theorem 17 indeed implies the first part of Theorem 2 since the row-treewidth only depends on ω and r, and thus we obtain product structure for every graph class with (sub-)constant clique number and disk radius. Further note that we also obtain an upper bound on the row-treewidth if clique number or disk radius are super-constant for some family of HUDGs. However, in these cases the upper bound on the row-treewidth also grows and thus we do not obtain product structure, which is no surprise given Section 3.

Recall that [20] shows that EUDGs with clique number ω are subgraphs of $H \boxtimes K_{O(\omega)}$, where H is a square-grid with diagonals. To find the product $H \boxtimes K_k$ for HUDGs, we lift their approach to the hyperbolic setting with two major changes. Not surprisingly, we replace the Euclidean tiling by a hyperbolic tiling to adjust to the geometry. However, we also need to face different disk radii since, in contrast to [20], we cannot freely choose the size of our tiles. For large disk radii, we solve this by merging tiles, whereas we subdivide tiles for small disk radii as needed. To do so, we provide a geometric interpretation of what we aim for. Two tiles are called *adjacent* if they are adjacent in the dual, i.e., if they share an edge.

▶ **Theorem 3.** *For each $r > 0$ there is an (irregular) tiling of the hyperbolic plane such that*
- *every tile can be covered by $O(3^{8r})$ disks of radius r and*
- *each two points with distance at most $2r$ lie in the same tile or in two adjacent tiles.*

Observe that this indeed implies that every HUDG with radius r and clique number ω is a subgraph of $H \boxtimes K_k$, where H is the dual of the tiling from Theorem 3, and $k \in O(\omega \cdot 3^{8r})$. To see this, consider the partition $\mathcal{P}$ with one part for the vertices of each tile. By the first property, each part contains at most $O(\omega \cdot 3^{8r})$ vertices since each disk of radius r covers a clique and thus at most ω vertices. Thus, we have $G \subseteq Q \boxtimes K_k$ for the quotient $Q = G/\mathcal{P}$ that is obtained by contracting the vertices in each tile, and some $k \in O(\omega \cdot 3^{8r})$. Then, by

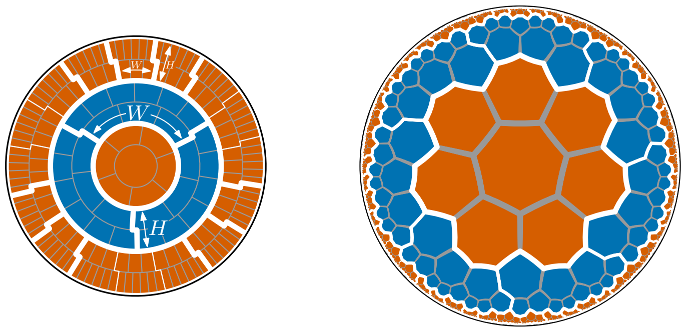

Figure 8 To ensure that non-adjacent tiles are sufficiently far from each other, we merge 7-gons as indicated by the white lines. High-level, we define rings of height $H \approx 2r$, which we cut into parts of width $W \in O(3^{4r})$ (left). Locally, however, we stick to the 7-gons from a $\{7, 3\}$-tiling (right).

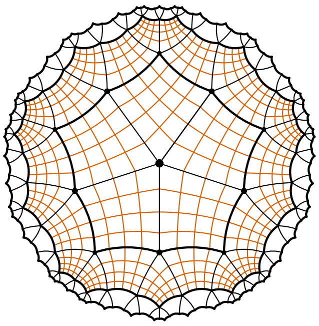

Figure 9 A $\{4, 5\}$-tiling (black) that is subdivided (red) so that each small tile can be covered by $O(1)$ disks of radius r.

the second property, the endpoints of every edge lie in the same or in adjacent tiles, so the quotient Q is a subgraph of the dual H of the tiling from Theorem 3. Together, we obtain $G \subseteq Q \boxtimes K_k \subseteq H \boxtimes K_k$, as required by Theorem 17.

The main challenge of this section is to prove Theorem 3, for which we refer to the full version [2]. For an overview, if r is at least some carefully chosen constant, we start with a regular $\{7, 3\}$-tiling whose tiles are merged until the second property of Theorem 3 is satisfied (Figure 8). Here, the tiles have a rapidly growing maximum degree to meet the requirements of the hyperbolic plane. The case for small radii then starts with a regular $\{4, 5\}$-tiling, where we subdivide the tiles until they are small enough for the first property (see Figure 9). This results in a tiling that gets more Euclidean the smaller r is, in the sense that locally, it looks like a Euclidean square grid.

5 Open Questions

In this paper, we characterize for which functions r families of HUDGs with constant clique number and disk radius r admit product structure: If $r \in O(1)$, then every family has bounded row-treewidth and thus admits product structure, otherwise there are families with unbounded row-treewidth. However, our upper bound is exponential in the disk radius, whereas our lower bound is logarithmic. It is an interesting open question whether the exponential dependency is necessary, both for Theorems 3 and 17. We conjecture that it is indeed necessary for the tiling, but believe that the base can be improved, which would also improve the bound on the row-treewidth. Note that even if there is an exponential lower bound for the tiling, it does not necessarily transfer to the row-treewidth of HUDGs.

▶ **Question 18.** *What is the row-treewidth of HUDGs with clique number ω and disk radius r?*

With our lower-bound construction, we show that there are SHUDGs with radius $\Theta(\log n)$ having treewidth $\Omega(\omega \frac{\log n}{\log \log n})$. This almost matches the upper bound of $O(\omega \cdot \log n)$ [5, 21]. We ask whether the remaining gap can be closed. Moreover, we ask whether similar lower bounds are possible for other clique numbers than $\omega \in O(\log \log n)$; specifically for constant ω.

▶ **Question 19.** *Is there a family of n-vertex HUDGs with clique number ω and treewidth $\Theta(\omega \cdot \log n)$? Is there a family with constant clique number and treewidth $\Theta(\log n)$?*

Finally, we conjecture that our lower bound for the row-treewidth stops growing at logarithmic grid radius because there are no HUDGs that require larger radius.

▶ **Question 20.** *Are there families of HUDGs that do not admit a disk representation in the hyperbolic plane such that the disk radius is $O(\log n)$?*

References

1 Shinwoo An, Seonghyuk Im, Seokbeom Kim, and Myounghwan Lee. An efficient algorithm for $\mathcal{F}$-subgraph-free edge deletion on graphs having a product structure, 2025. doi:10.48550/arXiv.2510.14674.

2 Thomas Bläsius, Emil Dohse, Deborah Haun, and Laura Merker. Product structure and treewidth of hyperbolic uniform disk graphs, 2026. doi:10.48550/arXiv.2603.18997.

3 Thomas Bläsius, Tobias Friedrich, Maximilian Katzmann, and Daniel Stephan. Strongly hyperbolic unit disk graphs. In *Symposium on Theoretical Aspects of Computer Science (STACS)*, pages 13:1–13:17, 2023. doi:10.4230/LIPIcs.STACS.2023.13.

4 Thomas Bläsius, Tobias Friedrich, and Anton Krohmer. Hyperbolic random graphs: Separators and treewidth. In *European Symposium on Algorithms (ESA)*, pages 15:1–15:16, 2016. doi:10.4230/LIPIcs.ESA.2016.15.

5 Thomas Bläsius, Jean-Pierre von der Heydt, Sándor Kisfaludi-Bak, Marcus Wilhelm, and Geert van Wordragen. Structure and independence in hyperbolic uniform disk graphs. In *Symposium on Computational Geometry (SoCG)*, pages 21:1–21:16, 2025. doi:10.4230/LIPIcs.SoCG.2025.21.

6 Prosenjit Bose, Pat Morin, and Saeed Odak. An optimal algorithm for product structure in planar graphs. In *Scandinavian Symposium and Workshops on Algorithm Theory (SWAT)*, pages 19:1–19:14, 2022. doi:10.4230/LIPIcs.SWAT.2022.19.

7 Rutger Campbell, Katie Clinch, Marc Distel, J. Pascal Gollin, Kevin Hendrey, Robert Hickingbotham, Tony Huynh, Freddie Illingworth, Youri Tamitegama, Jane Tan, and et al. Product structure of graph classes with bounded treewidth. *Combinatorics, Probability and Computing*, 33(3):351–376, 2024. doi:10.1017/S0963548323000457.

8 Brent N. Clark, Charles J. Colbourn, and David S. Johnson. Unit disk graphs. *Discrete Mathematics*, 86(1):165–177, 1990. doi:10.1016/0012-365X(90)90358-O.

9 H. S. M. Coxeter. The trigonometry of hyperbolic tessellations. *Canadian Mathematical Bulletin*, 40(2):158–168, 1997. doi:10.4153/CMB-1997-019-0.

10 Marek Cygan, Fedor V. Fomin, Łukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. *Parameterized Algorithms*. Springer International Publishing, 2015. doi:10.1007/978-3-319-21275-3.

11 Clément Dallard, Martin Milanič, Andrea Munaro, and Shizhou Yang. Layered tree-independence number and clique-based separators, 2025. doi:10.48550/arXiv.2506.12424.

12 Clément Dallard, Martin Milanič, and Kenny Štorgel. Treewidth versus clique number. II. tree-independence number. *Journal of Combinatorial Theory, Series B*, 164:404–442, 2024. doi:10.1016/j.jctb.2023.10.006.

13 Basudeb Datta and Subhojoy Gupta. Semi-regular tilings of the hyperbolic plane. *Discrete Comput Geom*, pages 531–553, 2019. doi:10.1007/s00454-019-00156-0.

14 Mark de Berg, Hans L. Bodlaender, Sándor Kisfaludi-Bak, Dániel Marx, and Tom C. van der Zanden. A framework for exponential-time-hypothesis–tight algorithms and lower bounds in geometric intersection graphs. *SIAM Journal on Computing*, 49(6):1291–1331, 2020. doi:10.1137/20M1320870.

15 Marc Distel, Robert Hickingbotham, Tony Huynh, and David R. Wood. Improved product structure for graphs on surfaces. *Discrete Mathematics & Theoretical Computer Science*, 24, 2022. doi:10.46298/dmtcs.8877.

16 Vida Dujmović, Gwenaël Joret, Piotr Micek, Pat Morin, Torsten Ueckerdt, and David R. Wood. Planar graphs have bounded queue-number. *J. ACM*, 67(4), 2020. doi:10.1145/3385731.

17 Vida Dujmović, Gwenaël Joret, Piotr Micek, Pat Morin, Torsten Ueckerdt, and David Wood. Planar graphs have bounded queue-number. In *Symposium on Foundations of Computer Science (FOCS)*, pages 862–875, 2019. doi:10.1109/FOCS.2019.00056.

18 Vida Dujmović, Pat Morin, and David R. Wood. Layered separators in minor-closed graph classes with applications. *Journal of Combinatorial Theory, Series B*, 127:111–147, 2017. doi:10.1016/j.jctb.2017.05.006.

19 Vida Dujmović, Pat Morin, and David R. Wood. Graph product structure for non-minor-closed classes. *Journal of Combinatorial Theory, Series B*, 162:34–67, 2023. doi:10.1016/j.jctb.2023.03.004.

20 Zdeněk Dvořák, Tony Huynh, Gwenael Joret, Chun-Hung Liu, and David R. Wood. *Notes on Graph Product Structure Theory*, pages 513–533. Springer International Publishing, 2021. doi:10.1007/978-3-030-62497-2_32.

21 Zdeněk Dvořák and Sergey Norin. Treewidth of graphs with balanced separations. *Journal of Combinatorial Theory, Series B*, 137:137–144, 2019. doi:10.1016/j.jctb.2018.12.007.

22 Tobias Friedrich and Anton Krohmer. On the diameter of hyperbolic random graphs. In *International Colloquium on Automata, Languages, and Programming (ICALP)*, pages 614–625. Springer-Verlag, 2015. doi:10.1007/978-3-662-47666-6_49.

23 Tobias Friedrich and Anton Krohmer. On the diameter of hyperbolic random graphs. *SIAM Journal on Discrete Mathematics*, 32(2):1314–1334, 2018. doi:10.1137/17M1123961.

24 Esther Galby, Andrea Munaro, and Shizhou Yang. Polynomial-time approximation schemes for independent packing problems on fractionally tree-independence-number-fragile graphs. In *Symposium on Computational Geometry (SoCG)*, volume 258, pages 34:1–34:15, 2023. doi:10.4230/LIPIcs.SoCG.2023.34.

25 Marvin Jay Greenberg. *Euclidean and Non-Euclidean Geometries.* W. H. Freeman and Company, 3rd edition, 1993.

26 Luca Gugelmann, Konstantinos Panagiotou, and Ueli Peter. Random hyperbolic graphs: Degree sequence and clustering. In *Automata, Languages, and Programming (ICALP)*, pages 573–585, 2012. doi:10.1007/978-3-642-31585-5_51.

27 Robert Hickingbotham, Paul Jungeblut, Laura Merker, and David R. Wood. The product structure of squaregraphs. *Journal of Graph Theory*, 105(2):179–191, 2024. doi:10.1002/jgt.23008.

28 Robert Hickingbotham and David R. Wood. Shallow minors, graph products, and beyond-planar graphs. *SIAM Journal on Discrete Mathematics*, 38(1):1057–1089, 2024. doi:10.1137/22M1540296.

29 Nikolai Karol. String graphs: Product structure and localised representations, 2025. doi:10.48550/arXiv.2511.15156.

30 Sándor Kisfaludi-Bak. Hyperbolic intersection graphs and (quasi)-polynomial time. In *Symposium on Discrete Algorithms (SODA)*, pages 1621–1638, 2020. doi:10.1137/1.9781611975994.100.

31 Dmitri Krioukov, Fragkiskos Papadopoulos, Maksim Kitsak, Amin Vahdat, and Marián Boguñá. Hyperbolic geometry of complex networks. *Physical Review E*, 82(3):036106, 2010. doi:10.1103/PhysRevE.82.036106.

32 Laura Merker, Lena Scherzer, Samuel Schneider, and Torsten Ueckerdt. Intersection graphs with and without product structure. In *International Symposium on Graph Drawing and Network Visualization (GD)*, volume 320, pages 23:1–23:19, 2024. `doi:10.4230/LIPIcs.GD.2024.23`.

33 Yanick Thurn, Manuel Schrauth, and Johanna Erdmenger. Hyperbolic tiling neighborhoods in O(1) time, 2025. `doi:10.48550/arXiv.2508.04765`.