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AGGREGATING HEAVY-TAILED RANDOM VECTORS: FROM FINITE SUMS TO LÉVY PROCESSES

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Abstract

The tail behavior of aggregates of heavy-tailed random vectors is known to be determined by the so-called principle of 'one large jump', be it for finite sums, random sums, or Lévy processes. We establish that, in fact, a more general principle is at play. Assuming that the random vectors are multivariate regularly varying on various subcones of the positive orthant $[0, \infty)^d$, first we show that their aggregates are also multivariate regularly varying on these subcones. This allows us to approximate certain tail probabilities rendered asymptotically negligible under classical regular variation. Second, we discover that depending on the structure of a particular tail event, the tail behavior of the aggregates may be characterized by more than a single large jump. Finally, we illustrate a similar phenomenon for regularly varying multivariate Lévy processes, establishing as well a relationship between regular variation of a multivariate Lévy process and multivariate regular variation of its Lévy measure on different subcones. The applicability of these results in financial and insurance risk management is discussed.

Keywords: Convolution; compound Poisson process; Lévy process; multivariate regular variation; one large jump

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1. Introduction

In this paper, we study the asymptotic behavior of tail events pertaining to independent sums of heavy-tailed random vectors and multivariate Lévy processes under the paradigm of multivariate regular variation (MRV) [10, 55]. Assessment of such tail probabilities are of interest in risk management for many finance, insurance, queueing, and environmental applications; see [5, 24, 49] in the classical univariate case and [14, Chapter 5] and [48, Chapter 7] for some expositions and results in the multivariate case. For instance, in financial and insurance systems, where return values, operational risks and insurance losses may be heavy-tailed, a key goal is to assess the risk contagion in the system. Such assessments are done using various conditional risk measures such as conditional value at risk (CoVaR), marginal expected shortfall

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(MES), marginal mean excess (MME), and systemic risk (SRISK) (cf. [1, 2, 15]), which can be applied to many financial and insurance network models, e.g., as in [17, 39, 40]. The results in this paper aim to help in the precise computation of such measures while aggregating these returns, losses, or risk values.

A well-known class of heavy-tailed distributions is that of subexponential distributions. If $Z, Z^{(1)}, \ldots, Z^{(n)}$ are independent and identically distributed (i.i.d.) random variables defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, then for fixed $x > 0$ we know that

$$\lim_{n\to\infty} \frac{\mathbb{P}(Z^{(1)} + \cdots + Z^{(n)} > tx)}{\mathbb{P}(Z > tx)} = n \tag{1}$$

if and only if Z follows a *subexponential* distribution, i.e., $\mathbb{P}(Z^{(1)} + Z^{(2)} > t) \sim 2\mathbb{P}(Z > t)$ as $t \to \infty$; Chistyakov [13] proved this for non-negative random variables, later extended to $\mathbb{R}$ by Willikens [60]. This phenomenon of 'one large jump' exhibited in (1) is so-called because a high threshold-crossing of the sum of a set of random variables occurs with the same asymptotic probability as any one of them crossing the same threshold. An important subclass of subexponential distributions are distributions with regularly varying tails (cf. [59]), and hence (1) holds for these distributions as well. A random variable Z has a *regularly varying right tail* if for $x > 0$, we have $\lim_{t\to\infty} \mathbb{P}(Z > tx)/\mathbb{P}(Z > t) = x^{-\alpha}$ for some $-\alpha < 0$, which is called the index of regular variation or tail index. We write $Z \in \mathcal{RV}_{-\alpha}$.

The aim of this work is to extend (1) to a multivariate setup by investigating aggregates of i.i.d. multivariate regularly varying (MRV) random vectors, and provide new insights into multivariate subexponentiality. A key finding of this paper is that tail events may occur in ways other than (1), which extends our understanding beyond what is expected and obtained using classical theory. In a multivariate context, an $\mathbb{R}_+^d$-valued random vector $\mathbf{Z}$ is multivariate regularly varying on $\mathbb{R}_+^d \setminus \{\mathbf{0}\}$, if there exists a measurable function $b(t) \to \infty$ as $t \to \infty$, and a non-null measure μ in the set of all Borel measures in $\mathbb{R}_+^d \setminus \{\mathbf{0}\}$ which are finite on sets bounded away from $\mathbf{0}$ such that

$$t\mathbb{P}(\mathbf{Z}/b(t) \in A) \to \mu(A) \quad \text{as } t \to \infty$$

for Borel sets A bounded away from $\mathbf{0}$ with $\mu(\partial A) = 0$; see Section 2 for details. In particular, we have $\mu(tA) = t^{-\alpha}\mu(A)$ for some $\alpha > 0$ and write $\mathbf{Z} \in \mathcal{MRV}(\alpha, b, \mu, \mathbb{R}_+^d \setminus \{\mathbf{0}\})$.

For this paper, we restrict to non-negative random vectors, and the tail sets of interest relate to positive high values, which is suited for many finance, insurance, queueing, and environmental applications as, e.g., for insurance risks and operational risks, risk managers are often only concerned with losses which are non-negative and the claim amounts are the aggregates of such losses. Now, for i.i.d. non-negative random vectors $\mathbf{Z}, \mathbf{Z}^{(1)}, \ldots, \mathbf{Z}^{(n)} \in \mathcal{MRV}(\alpha, b, \mu, \mathbb{R}_+^d \setminus \{\mathbf{0}\})$, we can deduce that

$$\mathbf{Z}^{(1)} + \cdots + \mathbf{Z}^{(n)} \in \mathcal{MRV}(\alpha, b, n\mu, \mathbb{R}_+^d \setminus \{\mathbf{0}\}) \tag{2}$$

(cf. [54, Proposition 4.1], [55, Section 7.3], and [35, Lemma 3.11]). Hence, for Borel sets A in $\mathbb{R}_+^d$ which are bounded away from $\mathbf{0}$ satisfying

$$\mu(\partial A) = 0 \quad \text{and} \quad \mu(A) > 0, \tag{3}$$

we have for fixed $n \geq 1$,

$$\mathbb{P}(\mathbf{Z}^{(1)} + \cdots + \mathbf{Z}^{(n)} \in tA) \sim \frac{n}{b^{\leftarrow}(t)}\mu(A) \sim n\mathbb{P}(\mathbf{Z} \in tA) \quad \text{as } t \to \infty.$$

Thus, for such sets we may obtain an extension of (1) to higher dimensions of the form

$$\lim_{n\to\infty}\frac{\mathbb{P}(\mathbf{Z}^{(1)}+\cdots+\mathbf{Z}^{(n)}\in tA)}{\mathbb{P}(\mathbf{Z}\in tA)}=n, \tag{4}$$

which is again a manifestation of the principle of 'one large jump'. In particular, if the margins of $\mathbf{Z}$ are tail-equivalent, $A=(-\infty,\boldsymbol{x}]$ for $\boldsymbol{x}\in(0,\infty)^d$ with $\min(x_1,\ldots,x_d)<\infty$, and if assumption (3) is satisfied, then (4) is valid. A further consequence here is that $\mathbf{Z}$ is as well *multivariate subexponential* in the sense of [52], since by their definition, a random vector $\mathbf{Z}$ is multivariate subexponential if (4) holds for sets of the form $A=(-\infty,\boldsymbol{x}]$ where $\boldsymbol{x}\in(0,\infty)^d$ with $\min(x_1,\ldots,x_d)<\infty$. An alternative definition of multivariate subexponential distributions was introduced by Samorodnitsky and Sun [56] so that (4) holds for a broader class of sets given by

$$\mathcal{A}:=\{A\subseteq\mathbb{R}^d: A \text{ open, increasing }, \ A^c \text{ convex }, \ 0\notin\overline{A}\},$$

see [56, Corollary 4.10]. Naturally, the class of multivariate subexponential distributions in [56] turns out to be a strict subclass of [52], and it also contains the classical MRV distributions with tail equivalent margins (cf. [56, Proposition 4.14]). Hence, we may observe here that, not quite surprisingly, extending the principle of 'one large jump' to more general sets also leads to shrinking the class of possible distributions that satisfy the principle; see also the related paper [42].

Nevertheless, even when $\mathbf{Z}\in\mathcal{MRV}(\alpha,b,\mu,\mathbb{R}_+^d\setminus\{\mathbf{0}\})$ with tail-equivalent margins, one often encounters examples where for a large class of sets A, the value of $\mu(A)$ is equal to zero and in this case,

$$\lim_{t\to\infty}b^{\leftarrow}(t)\mathbb{P}(\mathbf{Z}^{(1)}+\cdots+\mathbf{Z}^{(n)}\in tA)=0=\lim_{t\to\infty}b^{\leftarrow}(t)\mathbb{P}(\mathbf{Z}\in tA),$$

so that we are not able to conclude (4) from that. For example, if the elements of $\mathbf{Z}=(Z_1,\ldots,Z_d)$ are themselves i.i.d. (with regularly varying marginal tail distributions) and we consider the sets

$$A=\left\{\boldsymbol{z}\in\mathbb{R}_+^d: z_j>x_j \text{ for all } j\in S\right\}, \tag{5}$$

for some index $S\subseteq\mathbb{I}:=\{1,\ldots,d\}$ with $|S|\geq 2$, $x_j>0$ for $j\in S$, then under classical MRV assumptions we have $\mu(A)=0$, since for $\mathbf{Z}\in tA$ to hold, at least two components of $\mathbf{Z}$ need to be large together (cf. Example 4). In fact, the components of $\mathbf{Z}$ need not be independent at all, a Gaussian dependence among variables with tail equivalent regularly varying marginal distributions and pairwise correlations less than one will ensure $\mu(A)=0$ as well; see Section 5.2

However, for various applications, it is important to know the exact rate at which $\mathbb{P}(\mathbf{Z}^{(1)}+\cdots+\mathbf{Z}^{(n)}\in tA)$ decreases for any set A. For example consider $\mathbf{Z}^{(1)}+\cdots+\mathbf{Z}^{(n)}$ to be the first n aggregated d-dimensional losses of d business lines and let A be a 'ruin set'. Then A of the form (5) reflects that all business lines in set S are ruined if n losses occur. Ruin sets may also be much more complex, for example, one may be interested in the event that at least i business lines are ruined, where $1\leq i\leq d$. Estimating such ruin probabilities is relevant for the companies and quite often for such types of sets we may observe $\mu(A)=0$.

The aim of the present paper is twofold. First, we derive sufficient criteria for a classical MRV random vector $\mathbf{Z}\in\mathcal{MRV}(\alpha,b,\mu,\mathbb{R}_+^d\setminus\{\mathbf{0}\})$ and Borel sets A bounded away from $\mathbf{0}$ to satisfy the 'one large jump principle' in (4) even if $\mu(A)=0$. To this end, we use the concept of MRV on various subcones on $\mathbb{R}_+^d$ and show the presence of the MRV property for the aggregate $\sum_{k=1}^n \boldsymbol{Z}^{(k)}$ on these subcones under quite general assumptions. Furthermore, we derive the exact rate of decrease of the tail probability $\mathbb{P}(\boldsymbol{Z}^{(1)}+\cdots+\boldsymbol{Z}^{(n)} \in tA)$ as $t \to \infty$, which depends on the structure of the set A.

Second, we also show that the presence of MRV on various subcones of $\mathbb{R}_+^d$ may result in a behavior which is quite different from (4). For example, a phenomenon investigated for sets A as defined in (5) is that

$$\lim_{t\to\infty} \frac{\mathbb{P}\big(\boldsymbol{Z}^{(1)}+\cdots+\boldsymbol{Z}^{(n)} \in tA\big)}{\mathbb{P}(\boldsymbol{Z}\in tA)} = C_{n,|S|} \tag{6}$$

(cf. Remark 9), where $C_{n,|S|}>0$ is not necessarily equal to n, and depends not only on the number of summands n, but also on the cardinality of the set S (cf. (5)), and of course on the distribution of $\boldsymbol{Z}$. Here, the tail event $\boldsymbol{Z}^{(1)}+\cdots+\boldsymbol{Z}^{(n)} \in tA$ may be determined by threshold crossings in different coordinates of S, by different variables $\boldsymbol{Z}^{(k)}$. Thus, an aggregation of random vectors leads to a tail event with a 'few large jumps' and these jumps occur either together in one random vector $\boldsymbol{Z}^{(k)}$, or separately in a few different random vectors; the associated results are discussed in Proposition 2 and Section 4.2. Interestingly, we note that the phenomenon of 'a few large jumps' for such aggregation of heavy-tailed vectors in fact occurs under the more general setup of *adapted multivariate regular variation* (adapted-MRV) for the underlying random vectors. This is a new concept that we have introduced in this paper, which provides significant flexibility in dealing with random vectors that have regularly varying margins but which do not necessarily exhibit MRV on various subcones of $\mathbb{R}_+^d$. Hence, although the traditional methods of finding tail estimates for the random vectors and aggregates do not work for adapted-MRV random vectors, yet we are able to calculate the exact tail rates for their aggregates; see Example 1 for one such case.

Moving beyond aggregation of random vectors, a key contribution of this paper is in characterizing the tail probabilities of multi-dimensional regularly varying Lévy processes which are a natural generalization of finite aggregation of regularly varying random vectors and have inherent applications to stochastic storage processes including insurance claims, inventory management, financial modeling, and more (cf. [3, 5, 8, 14, 53]). In an insurance company with multiple business lines, a typical model for the multivariate risk process is a multivariate compound Poisson process or, more generally, a multivariate Lévy process, and the ruin probability is a multivariate tail event (cf. [33, 43] in the one-dimensional case and [6, 7, 30, 56] in the multivariate setup). In finance, a bank's total operational risk can be modeled by a multivariate compound Poisson process where the components are again the different business lines (cf. [11, 38]).

Recall that a (multivariate) Lévy process $\boldsymbol{L}=(\boldsymbol{L}(s))_{s\geq 0}$, is a stochastic process with $\boldsymbol{L}(0)=\boldsymbol{0}$ $\mathbb{P}$-almost surely, has stationary and independent increments, and has càdlàg sample paths (cf. [57]). Consequently, $\boldsymbol{L}(s)$ is infinitely divisible and has the same distribution as sums of i.i.d. random vectors; following the basic premise of this paper. A Lévy process $\boldsymbol{L}$ is characterized by its Lévy measure $\Pi(A)$, which measures the expected number of jumps of $\boldsymbol{L}$ in [0, 1] whose jump sizes are in A (cf. Section 6). The principle of 'one large jump' has been illustrated for MRV Lévy processes by Hult and Lindskog [29, 31] and the asymptotic rates of further hidden jumps have been characterized in [47] (for the univariate case). Our work addresses the case where the results of [31] hold, including and specifically addressing cases with negligible probability approximation for a tail event. In particular, a conclusion of [31, Proposition 3.1] is that $\boldsymbol{L}(1) \in \mathcal{MRV}(\alpha, b, \mu, \mathbb{R}_+^d \setminus \{\mathbf{0}\})$ if and only if $\Pi \in \mathcal{MRV}(\alpha, b, \mu, \mathbb{R}_+^d \setminus \{\mathbf{0}\})$ and then, for any Borel set A bounded away from $\mathbf{0}$ satisfying (3) we have

$$\mathbb{P}(\boldsymbol{L}(s) \in tA) \sim \frac{s}{b^{\leftarrow}(t)} \mu(A) \sim s\Pi(tA) \quad \text{as } t \to \infty,$$

resulting in

$$\lim_{t\to\infty} \frac{\mathbb{P}(\boldsymbol{L}(s) \in tA)}{\Pi(tA)} = s \tag{7}$$

(see [23] for the one-dimensional case). Naturally, if $\mu(A) = 0$ then we are not able to conclude (7) anymore from classical MRV. This again happens, for example, if we consider $\boldsymbol{L}$ to be composed of d i.i.d. one-dimensional regularly varying Lévy processes and A is defined as in (5). If the Lévy measure Π is multivariate regular varying on relevant subcones of $\mathbb{R}_+^d$, under quite general conditions, we show that $\boldsymbol{L}(s)$ is multivariate regularly varying on that subcone as well. As a consequence, we still observe (7) holding for sets A with $\mu(A) = 0$. Furthermore, we also find conditions where the asymptotics are quite different and we observe

$$\lim_{t\to\infty} \frac{\mathbb{P}(\boldsymbol{L}(s) \in tA)}{\Pi(tA)} = \phi(s), \tag{8}$$

where ϕ is a function for which $\phi(s) = s$, for all s may not hold, defying the linearity property in the time s often observed for Lévy processes as in (7), thus generalizing the phenomenon of ‘one large jump’ to ‘a few large jumps’. Here we present examples of Lévy processes and tail sets where $\mathbb{P}(\boldsymbol{L}(s) \in tA)$ and the Lévy measure $\Pi(tA)$ decrease at different rates as $t \to \infty$, which is contrary to the common belief regarding classical MRV of Lévy processes where these two quantities are tail equivalent. Our newly defined framework of adapted-MRV aids in this regard. These results on the tail probabilities of heavy-tailed Lévy processes have obvious implications on risk and ruin problems, especially in the context of insurance and finance, where logarithmic asset prices and claim amount processes can be modeled as Lévy processes.

This paper is organized as follows. In Section 2, we provide necessary preliminary results and background for our work; we discuss the basic framework in terms of MRV on subcones of $\mathbb{R}_+^d$ using $\mathbb{M}$-convergence. Then, in Section 3, we derive alternative characterizations of MRV on subcones and introduce the new concept of *adapted-MRV* as a generalization of MRV. Our main result, Theorem 1, appears in Section 4 which is used to obtain results on both ‘one large jump’ of the form (4) and its generalization to ‘a few large jumps’ of the form (6) under a variety of assumptions. In Section 5.1, the results on finite convolutions are extended to random sums. Our results are complemented with examples of finite sums and compound Poisson sums in Section 5.2, where the dependence structure of the underlying i.i.d. jumps is modeled by diverse copula models. Finally, an application to assess the tail behavior in regularly varying Lévy processes is given in Section 6. We conclude in Section 7 with indications toward future work. The proofs of the paper are relegated to the appendices.

2. Preliminaries

For investigating the asymptotic behavior of probabilities of tail events in a multivariate setting, we employ the framework of $\mathbb{M}$-convergence used to define MRV on subcones of $\mathbb{R}_+^d = [0, \infty)^d$ (cf. [19, 47]); we briefly recall and discuss these concepts in the current section.

Unless otherwise specified, for the rest of the paper, all random vectors are assumed to lie on the positive orthant $\mathbb{E}_d := \mathbb{R}_+^d$. Notationally, vector operations are understood componentwise, e.g., for vectors $\boldsymbol{z} = (z_1, \ldots, z_d)$ and $\boldsymbol{x} = (x_1, \ldots, x_d)$, $\boldsymbol{x} \le \boldsymbol{z}$ means $x_i \le z_i$ for all i. Moreover, for a constant $t > 0$ and a set $A \subseteq \mathbb{R}_+^d$, we denote by $tA := \{t\boldsymbol{z} \in \mathbb{R}_+^d : \boldsymbol{z} \in A\}$. Finally, for $x \in \mathbb{R}$, $(x)_+ := \max(x, 0)$.

2.1. Regular variation

The theory of regular variation provides a systematic framework to discuss heavy-tailed distributions; see [10, 55] for details. Here we briefly discuss regular varying functions and MRV of measures and random vectors on Euclidean cones with $\mathbb{M}$-convergence. A measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is *regularly varying* at infinity if for all $x > 0$, we have $\lim_{t\to\infty} f(tx)/f(t) = x^\beta$ for some fixed $\beta \in \mathbb{R}$. We write $f \in \mathcal{RV}_\beta$; and if $\beta = 0$ then the function f is called *slowly varying*. A real-valued random variable Z with distribution function F, denoted by $Z \sim F$, is regularly varying (at $+\infty$) if $\overline{F} := 1 - F \in \mathcal{RV}_{-\alpha}$ for some $\alpha > 0$. Equivalently, $Z \sim F$ is regularly varying with index $-\alpha < 0$ if there exists a measurable function $b : \mathbb{R}_+ \to \mathbb{R}_+$ with $b(t) \to \infty$ as $t \to \infty$ such that

$$t\,\mathbb{P}(Z > b(t)x) = t\,\overline{F}(b(t)x) \xrightarrow{t\to\infty} x^{-\alpha} \quad \text{for all } x > 0.$$

We write $\overline{F} \in \mathcal{RV}_{-\alpha}(b)$. As a consequence $b(t) \in \mathcal{RV}_{1/\alpha}$ and a canonical choice for b is $b(t) = F^{\leftarrow}(1 - 1/t) = \overline{F}^{\leftarrow}(1/t)$ where $F^{\leftarrow}(x) = \inf\{y \in \mathbb{R} : F(y) \ge x\}$.

Measures μ and ν defined on $\mathcal{B}(\mathbb{R}_+)$ are *(right) tail equivalent* if $\lim_{t\to\infty} \mu((t, \infty))/\nu((t, \infty)) = c$ for some $c > 0$. Naturally, the same holds for probability measures and, hence, distribution functions F and G are *(right) tail equivalent* if

$$\lim_{t\to\infty} \frac{\overline{F}(t)}{\overline{G}(t)} = \lim_{t\to\infty} \frac{1 - F(t)}{1 - G(t)} = c,$$

for some $c > 0$. We call measures μ and ν (respectively, distributions F and G) *completely* tail equivalent if $c = 1$. We often assume that components of the random vectors considered in this paper are tail equivalent (if not identically distributed, or completely tail equivalent).

In what follows, (multivariate) regular variation on Euclidean subspaces of $\mathbb{E}_d = \mathbb{R}_+^d$ is introduced using $\mathbb{M}$-convergence of measures which differs from vague convergence, the traditional notion used for MRV. See [19, 32, 47] for further details and the preference for this notion over vague convergence; moreover, see [9] for a broader notion of vague convergence. Consider the space $\mathbb{E}_d$ endowed with the sup-norm metric $d(\boldsymbol{x}, \boldsymbol{y}) = ||\boldsymbol{x} - \boldsymbol{y}||_\infty$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{E}_d$. A *cone* $\mathbb{C} \subset \mathbb{E}_d$ is a set that is closed under scalar multiplication: if $\boldsymbol{z} \in \mathbb{C}$ then $t\boldsymbol{z} \in \mathbb{C}$ for $t > 0$; a *closed cone* is a cone which is a closed set in $\mathbb{E}_d$. Regular variation is defined using $\mathbb{M}$-convergence on a closed cone $\mathbb{C} \subset \mathbb{E}_d$ with a closed cone $\mathbb{C}_0 \subset \mathbb{C}$ deleted. We say that a subset $A \subset \mathbb{C} \setminus \mathbb{C}_0$ is *bounded away from* $\mathbb{C}_0$ if $d(A, \mathbb{C}_0) = \inf\{d(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{x} \in A, \boldsymbol{y} \in \mathbb{C}_0\} > 0$. Denote by $\mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0)$ the class of Borel measures on $\mathbb{C} \setminus \mathbb{C}_0$ assigning finite measures to all Borel sets $A \subset \mathbb{C} \setminus \mathbb{C}_0$, which are bounded away from $\mathbb{C}_0$. We often refer to a subspace of the closed cone $\mathbb{E}_d$ which is a cone, as a *subcone*. Next, we define $\mathbb{M}$-convergence first and subsequently use it to define regular variation on $\mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0)$ where $\mathbb{C}_0 \subset \mathbb{C} \subset \mathbb{R}_+^d$ are closed cones containing $\boldsymbol{0}$.

Definition 1. ($\mathbb{M}$-*convergence.*) Let μ, μ_n, $n \ge 1$ be Borel measures on $\mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0)$. Suppose $\int f \, \mathrm{d}\mu_n \to \int f \, \mathrm{d}\mu$ as $n \to \infty$ for any bounded, continuous, real-valued function f whose support is bounded away from $\mathbb{C}_0$, then we say μ_n *converges* to μ in $\mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0)$, and write $\mu_n \to \mu$ in $\mathbb{M}(\mathbb{C}\setminus\mathbb{C}_0)$.

Next we define regular variation of measures on $\mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0)$ which is an extension of the definition found in [32] for measures in $\mathbb{M}(\mathbb{R}_+^d \setminus \{\mathbf{0}\})$.

Definition 2. A Borel measure Π on $\mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0)$ is regularly varying on $\mathbb{C} \setminus \mathbb{C}_0$ if there exists a regularly varying function $b(t) \in \mathcal{RV}_{1/\alpha}$, $\alpha > 0$, called the scaling *function* and a non-null (Borel) measure $\mu(\cdot) \in \mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0)$ called the *limit or tail measure* such that as $t \to \infty$,

$$t\,\Pi(b(t)\cdot) \to \mu(\cdot),$$

in $\mathbb{M}(\mathbb{C} \setminus \mathbb{C}_0)$. Similarly, a random vector $\mathbf{Z} \in \mathbb{R}_+^d$ is multivariate regularly varying on $\mathbb{C} \setminus \mathbb{C}_0$ if the probability measure $\Pi(\cdot) := \mathbb{P}(\mathbf{Z} \in \cdot)$ is regularly varying on $\mathbb{C} \setminus \mathbb{C}_0$.

We write $\Pi \in \mathcal{MRV}(\alpha, b, \mu, \mathbb{C} \setminus \mathbb{C}_0)$ and $\mathbf{Z} \in \mathcal{MRV}(\alpha, b, \mu, \mathbb{C} \setminus \mathbb{C}_0)$, respectively; one or more parameters are often dropped according to convenience.

Remark 1. Since $b(t) \in \mathcal{RV}_{1/\alpha}$, we observe that the limit measure $\mu(\cdot)$ has the scaling property $\mu(t\,\cdot) = t^{-\alpha}\mu(\cdot)$ for $t > 0$. Hence, if the measure or the random vector is $\mathcal{MRV}(\alpha, b, \mu, \mathbb{C} \setminus \mathbb{C}_0)$, we often refer to $-\alpha < 0$ as its tail index (in the subspace $\mathbb{C} \setminus \mathbb{C}_0$).

2.2. Coordinate subcones of the positive orthant

Equipped with the notion of $\mathbb{M}$-convergence and regular variation, we proceed to discuss regular variation on a particular set of subcones of $\mathbb{R}_+^d$ and also provide equivalent conditions for the same (cf. [50] and [19, Section 2]). For $z \in \mathbb{R}_+^d$ we write $z = (z_1, \ldots, z_d)$ and denote the (decreasing) order statistics of z by $z_{(1)} \geq z_{(2)} \geq \cdots \geq z_{(d)}$. For $0 \leq i \leq d-1$ let

$$\mathbb{CA}_d^{(i)} := \bigcup_{1 \leq j_1 < \cdots < j_{d-i} \leq d} \left\{z \in \mathbb{R}_+^d : z_{j_1} = 0, \ldots, z_{j_{d-i}} = 0\right\} = \left\{z \in \mathbb{R}_+^d : z_{(i+1)} = 0\right\},$$

and $\mathbb{CA}_d^{(d)} := \{z \in \mathbb{R}_+^d : z_{(d)} > 0\}$. For any $i = 1, \ldots, d$, the closed cone $\mathbb{CA}_d^{(i)}$ represents the union of all i-dimensional coordinate hyperplanes in $\mathbb{R}_+^d$. Define the following sequence of subcones of $\mathbb{R}_+^d$ where we investigate regular variation (when it exists):

$$\mathbb{E}_d^{(i)} := \mathbb{R}_+^d \setminus \mathbb{CA}_d^{(i-1)} = \left\{z \in \mathbb{R}_+^d : z_{(i)} > 0\right\}, \quad 1 \leq i \leq d.$$

We call the subsets $\mathbb{E}_d^{(i)}$ *coordinate subcones* since they are cones obtained from $\mathbb{R}_+^d$ by removing particular coordinate hyperplanes. Here $\mathbb{E}_d^{(1)}$ is the positive orthant with $\{\mathbf{0}\} = \mathbb{CA}_d^{(0)}$ removed, $\mathbb{E}_d^{(2)}$ is the positive orthant with all one-dimensional coordinate axes removed, $\mathbb{E}_d^{(3)}$ is the positive orthant with all two-dimensional coordinate hyperplanes removed, and so on. Clearly, $\mathbb{E}_d^{(1)} \supset \mathbb{E}_d^{(2)} \supset \cdots \supset \mathbb{E}_d^{(d)} = \mathbb{CA}_d^{(d)}$.

3. Adapted-MRV and regular variation on $\mathbb{E}_d^{(i)}$

Adding random vectors which exhibit MRV on the various coordinate subcones $\mathbb{E}_d^{(i)}$ often result in random vectors which exhibit MRV on the same subcone; interestingly enough though, for the sum to be MRV, it is not necessary for the summands to be strictly MRV on that subcone. This is an important observation, especially while dealing with random vectors which may exhibit classical MRV (on $\mathbb{E}_d^{(1)}$) but not on one or more of the other subcones $\mathbb{E}_d^{(i)}$, $i = 2, \ldots, d$. Hence, we begin with defining a new notion of *adapted-MRV* in Section 3.1, extending the classical notion of MRV, in order to allow for a broader class of models and examples. Later, in Section 3.2, we also characterize MRV on the subcones $\mathbb{E}_d^{(i)}$ of $\mathbb{R}_+^d$, since extreme events often occur in tail subsets of such cones; we also derive necessary and sufficient conditions for convergence in $\mathbb{M}(\mathbb{E}_d^{(i)})$ in Proposition 1.

3.1. Adapted-MRV

Since our interest is in the tail behavior of aggregates over independent regularly varying random vectors, when considering joint exceedances of sums of such vectors, regular variation on various combinations of subcones becomes quite important. The following definition provides a framework in which such random vectors possess a property that is more general than MRV while still being tractable for assessing tail asymptotics with MRV under aggregation.

Definition 3. Suppose $\boldsymbol{Z} \in \mathbb{R}_+^d$ is a random vector such that the following hold.

(1) We have $\boldsymbol{Z} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, \Delta$ for some $\Delta \leq d$.

(2) If $\Delta < d$ then additionally assume that there exists a $\gamma > 0$, such that for $i = \Delta + 1, \ldots, d$ and any $A \in \mathcal{R}^{(i)}$, we have

$$\lim_{t\to\infty} t\,\mathbb{P}\left(\frac{\boldsymbol{Z}}{b_i(t)} \in A\right) = 0, \tag{9}$$

where $b_i(t) := t^{1/\alpha_i^*}$ with $\alpha_i^* = (\alpha_\Delta - (\Delta+1)\alpha_1)_+ + i(\alpha_1 + \gamma)$, i.e. with the rate b_i, we have convergence to zero. We refer to (9) as *null convergence* and write $\boldsymbol{Z} \in \mathcal{NC}(b_i, \mathbb{E}_d^{(i)})$.

Then we say $\boldsymbol{Z}$ is *adapted multivariate regular varying* on $\mathbb{R}_+^d$ and write $\{\boldsymbol{Z} \in \mathcal{MRV}^*(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)}); i = 1, \ldots, d; \Delta\}$ where $\alpha_i = \infty,\ \mu_i \equiv 0$ for $i = \Delta + 1, \ldots, d$.

Remark 2. The following remarks regarding Definition 3 highlight the importance of this new concept vis-a-vis the notion of classical MRV on subcones $\mathbb{E}_d^{(i)},\ i = 1, \ldots, d$.

(i) For $i = 1, \ldots, \Delta$, $\mathcal{MRV}^*(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ and $\mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ are equivalent.

(ii) For $i = \Delta + 1, \ldots, d$, the notation $\mathcal{MRV}^*(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ means that for some $\alpha_i^* > 0$ as defined, we have $\boldsymbol{Z} \in \mathcal{NC}(b_i(t) = t^{1/\alpha_i^*}, \mathbb{E}_d^{(i)})$ and $\alpha_i = \infty,\ \mu_i \equiv 0$. The constant γ is chosen to be the same for all $i = \Delta + 1, \ldots, d$ and the value of $\Delta = \arg\max_i\{\alpha_i : \alpha_i < \infty\}$ is implicit.

(iii) Note that an alternative representation for $\alpha_{\Delta+1}^*$ is

$$\alpha_{\Delta+1}^* = \max(\alpha_\Delta, (\Delta+1)\alpha_1) + (\Delta+1)\gamma.$$

First, this implies that $\alpha_\Delta < \alpha_{\Delta+1}^* < \cdots < \alpha_d^*$ and, hence, $b_{i+1}^{\leftarrow}(t) = o(b_i^{\leftarrow}(t))$ as $t \to \infty$ for $i = \Delta, \ldots, d-1$. Second, we also have $\alpha_{\Delta+1}^* > (\Delta+1)\alpha_1$.

(iv) The steps for finding (hidden) regular variation on $\mathbb{E}_d^{(i)}$ have been discussed in [19, Section 2.1], so we will not repeat them here. Nevertheless, the same procedure can be applied for adapted-MRV. In addition, methods to estimate the index of MRV on the different subcones are proposed in [16, Section 4]; the tail indices pertaining to adapted-MRV in the various cones can be found in a similar manner.

The class of adapted-MRV distributions is highly flexible and encompasses most MRV distributions on $\mathbb{E}_d^{(1)}$. Hence, for practical applications, it is sufficient to consider this class. In the following remark we discuss how the notion of adapted-MRV enriches the model class of heavy-tailed distributions in the multivariate setting.

Remark 3.

(a) If $\boldsymbol{Z} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, d$, then clearly

$$\left\{\boldsymbol{Z} \in \mathcal{MRV}^*\left(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)}\right); i = 1, \ldots, d; \Delta = d\right\}.$$

Thus, condition (9) provides us with more flexibility in case we fail to have MRV on subcone $\mathbb{E}_d^{(j)}$ for $j > \Delta$ for some $\Delta = 2, \ldots, d$.

(b) Condition (9) is satisfied if $\mathbb{E}(Z_{(i)})^{(\alpha_\Delta - (\Delta+1)\alpha_1)_+ + i(\alpha_1 + \widetilde{\gamma}))} < \infty$ for some $\widetilde{\gamma} > \gamma$; here $Z_{(1)} \geq \cdots \geq Z_{(d)}$ are the order statistics of the elements of $\boldsymbol{Z} = (Z_1, \ldots, Z_d)$. In particular, one such example is when $Z_{(j)} = 0$ for $j > \Delta$, see Section 4.2. For further examples of multivariate heavy-tailed distributions exhibiting such a property see [15, 20].

(c) If $\Delta < d$ then (9) still allows for MRV to hold on $\mathbb{E}_d^{(i)}$, $i = \Delta + 1, \ldots, d$ albeit with a lighter regularly varying tail rate than $(\alpha_\Delta - (\Delta + 1)\alpha_1)_+ + i(\alpha_1 + \gamma)$.

Example 1. Let $X^{(1)}, X^{(2)}$ be i.i.d. random variables with $\mathbb{P}(X^{(1)} > x) = x^{-\alpha}$, $x > 1$ for some $\alpha > 0$. Let $B^{(1)}, B^{(2)}$ be i.i.d. random variables with $\mathbb{P}(B^{(1)} = 1) = 0.5 = \mathbb{P}(B^{(1)} = 0)$. Define for $k = 1, 2$,

$$\boldsymbol{Z}^{(k)} := B^{(k)}(X^{(k)}, 0) + (1 - B^{(k)})(0, X^{(k)}).$$

Then $\boldsymbol{Z}^{(1)}, \boldsymbol{Z}^{(2)}$ are i.i.d. with $\boldsymbol{Z}^{(1)} \in \mathcal{MRV}(\alpha, b(t) = t^{1/\alpha}, \mu_1, \mathbb{E}_2^{(1)})$ where

$$\mu_1(([0, x_1] \times [0, x_2])^c) = 0.5x_1^{-\alpha} + 0.5x_2^{-\alpha}, \quad x_1, x_2 > 0.$$

Clearly, $\boldsymbol{Z}^{(1)} \in \mathcal{NC}(t^{1/(2(\alpha+\gamma))}, \mathbb{E}_2^{(2)})$ for any $\gamma > 0$ and does not possess MRV on $\mathbb{E}_2^{(2)}$. Hence, $\boldsymbol{Z}^{(1)}, \boldsymbol{Z}^{(2)}$ are adapted-MRV with $\Delta = 1$.

Now we can check that $\boldsymbol{Z}^{(1)} + \boldsymbol{Z}^{(2)} \in \mathcal{MRV}(\alpha, b_1(t) = t^{1/\alpha}, 2\mu_1, \mathbb{E}_2^{(1)})$ and $\boldsymbol{Z}^{(1)} + \boldsymbol{Z}^{(2)} \in \mathcal{MRV}(2\alpha, b_2(t) = t^{1/(2\alpha)}, \mu_2, \mathbb{E}_2^{(2)})$ where

$$\mu_2((x_1, \infty) \times (x_2, \infty)) = 0.5(x_1x_2)^{-\alpha}, \quad x_1, x_2 > 0.$$

Most examples of regularly varying vectors on $\mathbb{E}_d^{(1)}$ are in fact adapted-MRV. Later, in Section 5.2, we provide examples of joint distributions for which our framework is useful and use copulas to represent joint dependence. This allows us to demonstrate our findings through a variety of examples.

3.2. Characterization of regular variation on $\mathbb{E}_d^{(i)}$

In the rest of this section, we characterize a particular family of sets $\mathcal{R}^{(i)}$ (defined in (11)), proving that it is an $\mathbb{M}$-convergence determining class on $\mathbb{E}_d^{(i)}$ and, hence, it provides a necessary and sufficient criterion for MRV on $\mathbb{E}_d^{(i)}$ and, finally, for adapted-MRV. The particular tail sets appear in multivariate risk and reliability problems where the quantity of interest is a finite or a random sum of identically distributed vectors.

Let $\mathcal{B} := \mathcal{B}(\mathbb{R}_+^d)$ denote the Borel σ-algebra on $\mathbb{R}_+^d$. For any $i \in \{1, \ldots, d\} =: \mathbb{I}$, $\mathbb{E}_d^{(i)}$ is a subspace of $\mathbb{R}_+^d$ and we denote its induced σ-algebra by

$$\mathcal{B}^{(i)} := \mathcal{B}\big(\mathbb{E}_d^{(i)}\big) = \big\{A \in \mathcal{B} : A \subseteq \mathbb{E}_d^{(i)}\big\}. \tag{10}$$

A *rectangular set* in $\mathbb{E}_d^{(i)}$ is defined as any set $A = \{z \in \mathbb{R}_+^d : z_j > x_j \text{ for all } j \in S\}$ where $S \subseteq \mathbb{I}$, $|S| \geq i$, and $x_j > 0$ for all $j \in S$. Let us denote the collection

$$\mathcal{R}^{(i)} := \big\{A \in \mathcal{B}^{(i)} : A \text{ is a rectangular set in } \mathbb{E}_d^{(i)}\big\}. \tag{11}$$

Lemma 1. *We have that $\mathcal{R}^{(i)}$ is a π-system and $\sigma(\mathcal{R}^{(i)}) = \mathcal{B}^{(i)}$.*

The proofs of Lemma 1 as well as other subsequent results in this section are given in Appendix A. The following proposition shows that for verifying convergence of measures in $\mathbb{E}_d^{(i)}$, we can restrict to testing convergence in sets belonging to $\mathcal{R}^{(i)}$. The result and the proof are in the spirit of [55, Lemma 6.1].

Proposition 1. *Let μ, $\mu_t \in \mathbb{M}(\mathbb{E}_d^{(i)})$ for all $t > 0$ and some fixed $i \in \mathbb{I}$. Then as $t \to \infty$,*

$$\mu_t \to \mu \quad \text{in } \mathbb{M}\big(\mathbb{E}_d^{(i)}\big) \tag{12}$$

if and only if

$$\lim_{t\to\infty} \mu_t(A) \to \mu(A), \qquad \text{for all } A \in \mathcal{R}^{(i)} \tag{13}$$

with $\mu(\partial A) = 0$ (μ-continuity set), where $\mathcal{R}^{(i)}$ is the collection of sets as defined in (11).

Remark 4. In light of Proposition 1, when we seek regular variation (or any measure convergence) in the space $\mathbb{E}_d^{(i)}$ using $\mathbb{M}$-convergence (as in Definition 2), we can equivalently show this only for rectangular sets in $\mathcal{R}^{(i)}$ (which are also continuity sets with respect to the limit measure).

We wrap this section up with an extension of the so-called ‘heavier tail wins’ phenomenon, in the context of MRV on a subcone $\mathbb{E}_d^{(i)}$. It is useful for many of the proofs in this paper.

Lemma 2. *Suppose $\boldsymbol{X} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ for some fixed $i \in \mathbb{I}$, and $\boldsymbol{X}$ is independent of the $\mathbb{R}^d$-valued random vector $\boldsymbol{Y}$ with $\mathbb{E}\|\boldsymbol{Y}\|^{\alpha_i+\gamma} < \infty$ for some $\gamma > 0$. Then*

$$\boldsymbol{X} + \boldsymbol{Y} \in \mathcal{MRV}\big(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)}\big).$$

4. Aggregating regularly varying random vectors

In the introduction, we discussed the principle of ‘one large jump’ determining the behavior of aggregates of MRV random vectors in the classical framework. The novelty of this work lies in the fact that, on one hand, we are able to extend the idea to a more general class of tail events and, on the other hand, we show that the more general phenomenon of ‘a few large jumps’ occurs as well, in particular under the assumption of adapted-MRV; hence, it is also not necessary to restrict to assuming MRV on the different cones.

We start by assuming that individual random vectors have tail equivalent margins and they admit adapted-MRV (see Definition 3). In our first result, Theorem 1, we consider two independent random vectors that are not necessarily identically distributed, and we assess the asymptotic probability of the sum belonging to various tail sets. This theorem forms the basis of many subsequent results for finite sums of i.i.d. random vectors in Section 4.1. Furthermore, we demonstrate that the sum is indeed adapted-MRV as well.

Theorem 1. *Let* $\mathbf{Z}^{(1)}, \mathbf{Z}^{(2)} \in \mathbb{R}_+^d$ *be independent random vectors, each with tail equivalent marginal distributions and* $\{\mathbf{Z}^{(k)} \in \mathcal{MRV}^*(\alpha_i^{(k)}, b_i^{(k)}, \mu_i^{(k)}, \mathbb{E}_d^{(i)}),\ \ i=1, \ldots, d; \Delta_k\}$ *for* $k=1, 2$, *i.e. they are adapted-MRV on* $\mathbb{R}_+^d$. *Define* $\alpha_0^{(k)}=0$, $b_0^{(k)\leftarrow}(t) \equiv 1$, $\mu_0^{(k)} \equiv 1$, *for* $k=1, 2$, *and*

$$I(i) := \operatorname{argmax}_{j \in \{0, \ldots, i\}} \left\{ \bar{c}_j^{(i)} : \bar{c}_j^{(i)} < \infty \right\},$$

where

$$\bar{c}_j^{(i)} := \max_{0 \le m \le i} \left\{ \limsup_{t \to \infty} \frac{b_j^{(1)\leftarrow}(t) b_{i-j}^{(2)\leftarrow}(t)}{b_m^{(1)\leftarrow}(t) b_{i-m}^{(2)\leftarrow}(t)} \right\}.$$

Define for $i=1, \ldots, d$, *and* $m=0, \ldots, i$*:*

$$c_m^{I(i)} := \lim_{t \to \infty} \frac{b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t)}{b_m^{(1)\leftarrow}(t) b_{i-m}^{(2)\leftarrow}(t)}.$$

Suppose that for $k=1, 2$, *and each* $m=1, \ldots, i-1$, *either* $\lim_{t\to\infty} b_m^{(k)\leftarrow}(t)/b_{m+1}^{(k)\leftarrow}(t)=0$ *or* $c_m^{I(i)}=0$. *Then*

$$\left\{ \mathbf{Z}^{(1)}+\mathbf{Z}^{(2)} \in \mathcal{MRV}^*\left(\alpha_i, b_i, \mu_i^{\oplus}, \mathbb{E}_d^{(i)}\right),\ \ i=1, \ldots, d; \Delta^{\oplus} \right\}$$

with $\alpha_i=\alpha_{I(i)}^{(1)}+\alpha_{i-I(i)}^{(2)}$, $b_i^{\leftarrow}(t)=b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t)$, *and*

$$\mu_i^{\oplus}(A)=\sum_{m=0}^{i} c_m^{I(i)} \mu_{m,i}^*(A) \quad \text{for } A \in \mathcal{B}\left(\mathbb{E}_d^{(i)}\right),$$

where $\mu_{m,i}^*$ *is the measure which is uniquely defined on* $\mathcal{R}^{(i)}$ *as follows: for* $A=\{z \in \mathbb{R}_+^d : z_j > x_j \text{ for all } j \in S\} \in \mathcal{R}^{(i)}$ *with* $S \subseteq \mathbb{I}$, $|S| \ge i$, *and* $x_j > 0$ for all $j \in S$ *we have*

$$\mu_{m,i}^*(A)=\sum_{\substack{J \subseteq S \cup \{\varnothing\} \\ |J|=m}} \mu_m^{(1)}\left(\left\{z \in \mathbb{E}_d^{(m)} : z_j > x_j \text{ for all } j \in J\right\}\right) \mu_{i-m}^{(2)}\left(\left\{z \in \mathbb{E}_d^{(i-m)} : z_j > x_j \text{ for all } j \in S \backslash J\right\}\right).$$

Moreover, $\Delta^{\oplus} \in \{\max(\Delta_1+1, \Delta_2+1), \ldots, \min(\Delta_1+\Delta_2, d)\}$.

The proof is quite involved requiring a meticulous investigation of several components. A fundamental ingredient needed was a proper characterization of $\mathbb{M}$-convergence in the subspace $\mathbb{E}_d^{(i)}$ via rectangular sets, which we derived in Proposition 1 (building on Lemma 1). Another key part of the proof of Theorem 1 was in identifying which components (of the many possible) contribute to the limit measure and which do not (Lemmas 3–5); these required quite technically involved calculations. Therefore, we have relegated the proofs of this theorem and the results of this section to Appendix B.

Remark 5. Since the output of argmax may contain multiple elements, $I(i)$ is not defined uniquely; hence, a value for $I(i)$ is often chosen from these outputs according to convenience.

We have refrained from stating a general result akin to Theorem 1 for adding n random vectors since the parameters of the limit model become notationally cumbersome without providing additional insight; on the other hand, for a variety of joint dependence behaviors, we often observe nicer structures appearing. In the rest of the section, we discuss the consequences of Theorem 1 on the finite sum of i.i.d. random vectors under various assumptions.

4.1. All subcones exhibit regular variation

First, we investigate the case where we add i.i.d. random vectors which are MRV on all relevant cones. The results, as we show, are direct consequences of Theorem 1. We begin with a well-known model where all components of each vector are also i.i.d. random variables; see Example 4 for the exact structure of the limit measure in this case. The following proposition provides a slightly more general version of this case.

Proposition 2. (Nearly independent case.) *Let* $\mathbf{Z}^{(1)}, \ldots, \mathbf{Z}^{(n)}$ *be i.i.d. random vectors in* $\mathbb{R}_+^d$ *with tail equivalent marginal distributions and* $\mathbf{Z}^{(1)} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ *for* $i = 1, \ldots, d$ *where* $b_i^{\leftarrow}(t) = (b_1^{\leftarrow}(t))^i$ *and* $b_1(t) \in \mathcal{RV}_{1/\alpha}$. *Then* $\alpha_i = i\alpha$ *and*

$$\sum_{k=1}^{n} \mathbf{Z}^{(k)} \in \mathcal{MRV}\big(\alpha_i, b_i, \mathbb{E}_d^{(i)}\big) \quad \text{for } i = 1, \ldots, d. \tag{14}$$

Now if for some $\kappa_j > 0, j = 1, \ldots, d,$

$$\mu_i\big(\big\{z \in \mathbb{E}_d^{(i)} : z_j > x_j \text{ for all } j \in S\big\}\big) = \prod_{j \in S} \kappa_j x_j^{-\alpha} \tag{15}$$

for $S \subseteq \mathbb{I}$ *with* $|S| = i$, $x_j > 0$ for all $j \in S$ *and* $\mu_i(\mathbb{E}_d^{(i+1)}) = 0$, $i = 1, \ldots, d$, *then*

$$\sum_{k=1}^{n} \mathbf{Z}^{(k)} \in \mathcal{MRV}\big(i\alpha, b_i, n^i \mu_i, \mathbb{E}_d^{(i)}\big) \quad \text{for } i = 1, \ldots, d.$$

Remark 6. If all components of the random vectors $\mathbf{Z}^{(1)}, \ldots, \mathbf{Z}^{(n)}$ are completely tail equivalent, then $\kappa_1 = \kappa_2 = \cdots = \kappa_d$.

Although condition (15) seem rather restrictive, the result obtained in (14), i.e. if $\mathbf{Z}^{(1)} \in \mathcal{MRV}(\alpha_i, b_i, \mathbb{E}_d^{(i)})$ then $\sum_{k=1}^{n} \mathbf{Z}^{(k)} \in \mathcal{MRV}(\alpha_i, b_i, \mathbb{E}_d^{(i)})$, holds under much weaker assumptions. The particular assumption (15) only helps to calculate the exact form of the limit measure. In the following result, we discuss another case with a different set of conditions which allows us to create many examples. In the special case of (15), tail events of the form (5) occur by threshold crossings of all possible combinations of coordinates of the n summands which gives the ‘few large jumps’ phenomenon instead of the well-known ‘one large jump’ phenomenon.

Proposition 3. *Let* $\mathbf{Z}^{(1)}, \ldots, \mathbf{Z}^{(n)}$ *be i.i.d. random vectors in* $\mathbb{R}_+^d$ *with tail equivalent marginal distributions and* $\mathbf{Z}^{(1)} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ *for* $i = 1, \ldots, d$. *Moreover, assume that* $\alpha_i < \alpha_m + \alpha_{i-m}$ *for all* $m = 1, \ldots, i-1$ *and* $i = 2, \ldots, d$. *Then*

$$\sum_{k=1}^{n} \mathbf{Z}^{(k)} \in \mathcal{MRV}\big(\alpha_i, b_i, n\mu_i, \mathbb{E}_d^{(i)}\big) \quad \text{for } i = 1, \ldots, d.$$

Remark 7. Clearly, a sufficient condition for Proposition 3 to hold would be to assume that $c_m^{I(i)} = 0$, $m = 1, \ldots, i-1$ and $i = 2, \ldots, d$ instead of $\alpha_i < \alpha_m + \alpha_{i-m}$ for all $m = 1, \ldots, i-1$ and $i = 2, \ldots, d$. This requires the notation of Theorem 1, and we prefer the latter in lieu of interpretability.

Remark 8. In both Propositions 2 and 3, we observe that while adding finitely many random vectors $\mathbf{Z}^{(1)}, \ldots, \mathbf{Z}^{(n)}$, we obtain $\sum_{k=1}^{n} \mathbf{Z}^{(k)} \in \mathcal{MRV}(\alpha_i, b_i, \mu_{i,n}^{\oplus}, \mathbb{E}_d^{(i)})$. The indices of regular variation α_i and the scaling parameter b_i remain the same no matter how many vectors we add although the measure $\mu_{i,n}^{\oplus}$ are quite different for different values of n. Note the following.

(i) Under the assumptions of Proposition 2, we have $\alpha_i = \alpha_m + \alpha_{i-m}$ for $m = 0, \ldots, i$, and $\mu_{i,n}^{\oplus} = n^i \mu_i$. Interestingly, $\alpha_i = \alpha_m + \alpha_{i-m}$ for $m = 0, \ldots, i$ does not necessarily imply that $\mu_{i,n}^{\oplus} = n^i \mu_i$.

(ii) Under the assumptions of Proposition 3, we have $\alpha_i < \alpha_m + \alpha_{i-m}$ for $m = 1, \ldots, i-1$, which turns out to be a sufficient condition for $\mu_{i,n}^{\oplus} = n\mu_i$.

The two extreme cases of dependence considered in general are the case of fully independent components for $\mathbf{Z}^{(1)}$, which is covered in Proposition 2, and the case where the components of $\mathbf{Z}^{(1)}$ are dependent such that $\mathbf{Z}^{(1)} \in \mathcal{MRV}(\alpha, b, \mu, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, d$. The following corollary addresses the latter case.

Corollary 1. (Dependent case, corollary to Proposition 3.) *Let* $\mathbf{Z}^{(1)}, \ldots, \mathbf{Z}^{(n)}$ *be i.i.d. random vectors in* $\mathbb{R}_+^d$ *with tail equivalent marginal distributions and* $\mathbf{Z}^{(1)} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ *for* $i = 1, \ldots, d$. *Moreover,* $(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)}) = (\alpha, b, \mu, \mathbb{E}_d^{(i)})$ *for* $i = 1, \ldots, i^*$ *for some* $i^* \leq d$. *Then*

$$\sum_{k=1}^{n} \mathbf{Z}^{(k)} \in \mathcal{MRV}\big(\alpha, b, n\mu, \mathbb{E}_d^{(i)}\big) \quad \text{for } i = 1, \ldots, i^*.$$

In all the propositions and corollaries of this section, we observe that if $\mathbf{Z}^{(1)} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$, then their finite sum $\sum_{k=1}^{n} \mathbf{Z}^{(k)} \in \mathcal{MRV}(\alpha_i, b_i, C_{n,i}\mu_i, \mathbb{E}_d^{(i)})$ for some constant $C_{n,i} > 0$. The 'one large jump' phenomenon is observed here, in the sense of (5) with $C_{n,i} = n$ in Proposition 3. But $C_{n,i}$ is not necessarily n, as for example in Proposition 2, and this is a case which we think of as a phenomenon of 'more than one large jump' or 'a few large jumps'. Note that in both cases mentioned here, we did assume $\mathbf{Z}^{(1)}$ to have MRV on all cones, but not to be strictly adapted-MRV. In the following Section 4.2, we illustrate that a similar principle holds, even under the assumption of adapted-MRV, although the characterizing jumps are now of the form (6) which relates to 'a few large jumps' phenomenon.

4.2. Not all subcones necessarily exhibit regular variation

In certain contexts, we may be interested in adding random vectors which are not necessarily MRV on all relevant cones. For example, we may have a sequence of i.i.d. random vectors for which not all components are non-zero in each realization. In this section, we concentrate on a few such examples. The general structure for the limit measures in such problems is often not quite apparent.

Proposition 4. *Let* $\mathbf{Z}^{(1)}, \ldots, \mathbf{Z}^{(n)} \in \mathbb{R}_+^d$ *be i.i.d. random vectors with tail equivalent marginal distributions which are* $\mathcal{RV}_{-\alpha}$ *and let*

$$\left\{\mathbf{Z}^{(1)} \in \mathcal{MRV}^*\left(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)}\right),\ i=1, \ldots, d;\ \Delta=1\right\}$$

with $\alpha_1=\alpha$ *and* $b_i(t)=t^{1/i(\alpha+\gamma_0)}$ *for some fixed* $\gamma_0>0$ *and* $i=2, \ldots, d$. *Then*

$$\left\{\sum_{k=1}^n \mathbf{Z}^{(k)} \in \mathcal{MRV}^*\left(\alpha_{i,n}, b_{i,n}, \mu_{i,n}^{\oplus}, \mathbb{E}_d^{(i)}\right),\ i=1, \ldots, d;\ \Delta=\min\{d, n\}\right\}.$$

Specifically, for $i=1, \ldots, d$ *we have the following.*

(a) *If* $n \geq i$ *then*

$$\sum_{k=1}^n \mathbf{Z}^{(k)} \in \mathcal{MRV}\left(\alpha_{i,n}=i\alpha, b_{i,n}, \mu_{i,n}^{\oplus}, \mathbb{E}_d^{(i)}\right), \tag{16}$$

where $b_{i,n}^{\leftarrow}(t)=(b_1^{\leftarrow}(t))^i$ *and*

$$\mu_{i,n}^{\oplus}\left(\left\{z \in \mathbb{E}_d^{(i)}: z_j>x_j \text{ for all } j \in S\right\}\right)=\frac{n!}{(n-i)!} \prod_{j \in S} \mu_1\left(\left\{z \in \mathbb{E}_d^{(i)}: z_j>x_j\right\}\right), \tag{17}$$

for $S \subseteq \mathbb{I}$ *with* $|S|=i$, $x_j>0$ *for* $j \in S$ *and* $\mu_{i,n}^{\oplus}(\mathbb{E}_d^{(i+1)})=0$.

(b) *If* $1 \leq n<i$ *then*

$$\sum_{k=1}^n \mathbf{Z}^{(k)} \in \mathcal{NC}\left(b_{i,n}(t)=t^{i(\alpha+\gamma_i)}, \mathbb{E}_d^{(i)}\right) \tag{18}$$

with $\gamma_i:=\gamma_0/i!$ *and, in particular,* $\sum_{k=1}^n \mathbf{Z}^{(k)} \in \mathcal{NC}(t^{1/i(\alpha+\gamma_d)}, \mathbb{E}_d^{(i)})$.

Remark 9. In Proposition 4, if the marginal distributions are completely tail equivalent with distribution functions $F_j,\ j=1, \ldots, d$ and $b_1(t)=\overline{F}_1^{\leftarrow}(1/t)$, then $\mu_{i,n}^{\oplus}$ in (17) is given by

$$\mu_{i,n}^{\oplus}\left(\left\{z \in \mathbb{E}_d^{(i)}: z_j>x_j \text{ for all } j \in S\right\}\right)=\frac{n!}{(n-i)!} \prod_{j \in S} x_j^{-\alpha}.$$

Remark 10. For the conclusion of Proposition 4 to hold, the random variables $\mathbf{Z}^{(1)}, \ldots, \mathbf{Z}^{(n)}$ need not be identically distributed as long as they are independent and are all adapted-MRV with the same sets of parameters. The proof follows by similar arguments as the proof of Proposition 4 and is skipped.

The phenomenon of a 'few large jumps' holds here too, as illustrated next. Assume that in Proposition 4, the marginal distributions are completely tail equivalent as in Remark 9; and

$\boldsymbol{Z} = (Z_1, \ldots, Z_d) \sim \boldsymbol{Z}^{(1)}$. Without loss of generality let tA be the tail event of interest where $A = \{z \in \mathbb{E}_d^{(i)} : z_j > x_j \text{ for all } j \in \{1, \ldots, i\}\}$. Note that from (16) and (17), we can infer that, in fact, as $t \to \infty$,

$$\begin{aligned}\mathbb{P}(\boldsymbol{Z}^{(1)} + \cdots + \boldsymbol{Z}^{(n)} \in tA) &\sim \frac{1}{(b_1^{\leftarrow}(t))^i} \frac{n!}{(n-i)!} \prod_{j=1}^{i} x_j^{-\alpha} \\ &= C_{n,i} \prod_{j=1}^{i} \frac{x_j^{-\alpha}}{b_1^{\leftarrow}(t)} \\ &\sim C_{n,i} \prod_{j=1}^{i} \mathbb{P}(Z_j > tx_j),\end{aligned}$$

where $C_{n,i} = (n!)/((n-i)!)$. Hence, $\boldsymbol{Z}^{(1)} + \cdots + \boldsymbol{Z}^{(n)} \in tA$ occurs at the same rate with which i independent univariate marginals cross their respective thresholds, indicating i many large jumps. The constant $C_{n,i}$ gives the number of possible choices of independent jumps, here the marginal jumps counted are all from different variables $\boldsymbol{Z}^{(k)}$.

In the rest of the section, we provide examples exhibiting Proposition 4 and its possible generalization.

Example 2. Let $(X^{(k)})_{k\in\mathbb{N}}$ be a sequence of i.i.d. random variables with distribution function F and $\overline{F} \in \mathcal{RV}_{-\alpha}$, $\alpha > 0$. Let $(\boldsymbol{B}^{(k)})_{k\in\mathbb{N}}$ be i.i.d. random vectors taking values in $\{e_1, \ldots, e_d\}$ with $\mathbb{P}(\boldsymbol{B}^{(1)} = e_l) = p_l \geq 0$, $\sum_l p_l = 1$, and $e_l = (0, \ldots, 0, 1, 0, \ldots, 0) \in \{0, 1\}^d$ where the only non-zero entry 1 is at the lth place. Define $\boldsymbol{Y}^{(k)} := X^{(k)}\boldsymbol{B}^{(k)}$, $k \in \mathbb{N}$. Moreover, let $(\boldsymbol{\varepsilon}^{(k)})_{k\in\mathbb{N}}$ be an i.i.d. sequence of random vectors with $\mathbb{E}\|\boldsymbol{\varepsilon}^{(k)}\|^{d(\alpha+\theta)} < \infty$ for some $\theta > 0$. Finally, also assume that $X^{(1)}, X^{(2)}, \ldots, \boldsymbol{B}^{(1)}, \boldsymbol{B}^{(2)}, \ldots, \boldsymbol{\varepsilon}^{(1)}, \boldsymbol{\varepsilon}^{(2)}, \ldots$ are independent. Then $\boldsymbol{Z}^{(k)} := \boldsymbol{Y}^{(k)} + \boldsymbol{\varepsilon}^{(k)}$, $k \in \mathbb{N}$, are a sequence of i.i.d. adapted-MRV random vectors with $\Delta = 1$ (cf. Lemma 2) and, hence, Proposition 4 (along with Remark 10) provides the tail asymptotic behavior of $\sum_{k=1}^{n} \boldsymbol{Z}^{(k)}$ for any $n \geq 1$.

The neat expressions for limit measures and tail indices as obtained by Proposition 4 in aggregating i.i.d. adapted-MRV random vectors with $\Delta = 1$ do not extend as nicely for $\Delta > 1$. Nevertheless, we may still be able to find a pattern in certain cases and our next example with $\Delta = 2$ elaborates on this.

Example 3. The setting is similar to Example 2. Let $X^{(1)}, X^{(2)}, \ldots, \widetilde{X}^{(1)}, \widetilde{X}^{(2)}, \ldots \sim F_\alpha$ be i.i.d. random variables with $\overline{F}_\alpha \in \mathcal{RV}_{-\alpha}$, $\alpha > 0$. Let $\boldsymbol{B}^{(1)}, \boldsymbol{B}^{(2)}, \ldots, \widetilde{\boldsymbol{B}}^{(1)}, \widetilde{\boldsymbol{B}}^{(2)}, \ldots$ be i.i.d. random vectors taking values in $\{e_1, \ldots, e_d\}$ as defined in Example 2. Also assume that $X^{(1)}, X^{(2)}, \ldots, \widetilde{X}^{(1)}, \widetilde{X}^{(2)}, \ldots, \boldsymbol{B}^{(1)}, \boldsymbol{B}^{(2)}, \ldots, \widetilde{\boldsymbol{B}}^{(1)}, \widetilde{\boldsymbol{B}}^{(2)}, \ldots$ are independent. Then $\boldsymbol{Y}^{(k)} := 2^{-1/\alpha}X^{(k)}\boldsymbol{B}^{(k)} + 2^{-1/\alpha}\widetilde{X}^{(k)}\widetilde{\boldsymbol{B}}^{(k)}$, $k \in \mathbb{N}$, are i.i.d. adapted-MRV random vectors with $\Delta = 2$. Specifically for any $k \geq 1$:

(i) $\boldsymbol{Y}^{(k)} \in \mathcal{MRV}(\alpha, b_1, \mu_1, \mathbb{E}_d^{(1)})$ where $b_1(t) = \overline{F}_\alpha^{\leftarrow}(1/t)$ and with $A_1 = \{z \in \mathbb{E}_d^{(i)} : z_j > x_j\} \in \mathcal{R}^{(1)}$ for some $j \in \mathbb{I}$, $\mu_1(A_1) = p_j x_j^{-\alpha}$;

(ii) $\boldsymbol{Y}^{(k)} \in \mathcal{MRV}(2\alpha, b_2, \mu_2, \mathbb{E}_d^{(2)})$ where $b_2^{\leftarrow}(t) = (b_1^{\leftarrow}(t))^2$ and with $A_2 = \{z \in \mathbb{E}_d^{(2)} : z_j > x_j, z_\ell > x_\ell\} \in \mathcal{R}^{(2)}$ for some $j, \ell \in \mathbb{I}, j \neq \ell$ we have $\mu_2(A_2) = \frac{1}{2} p_j p_\ell (x_j x_\ell)^{-\alpha}$;

(iii) for $i = 3, \ldots, d$ and some $\gamma > 0$, we have $\boldsymbol{Y}^{(k)} \in \mathcal{NC}(t^{1/(i(\alpha+\gamma))}, \mathbb{E}_d^{(i)})$.

Applying Theorem 1, and following the proof of Proposition 4, we can show that for $n \geq i$,

$$\sum_{k=1}^{n} \boldsymbol{Y}^{(k)} \in \mathcal{MRV}\big(i\alpha, b_i, \mu_{i,n}^{*\oplus}, \mathbb{E}_d^{(i)}\big),$$

where $b_i^{\leftarrow}(t) = (b_1^{\leftarrow}(t))^i$ and

$$\begin{aligned}\mu_{i,n}^{*\oplus}\big(\{z \in \mathbb{E}_d^{(i)} : z_j > x_j \text{ for all } j \in S\}\big) &= f_i(n) \prod_{j \in S} \mu_1\big(\{z \in \mathbb{E}_d^{(i)} : z_j > x_j\}\big) \\ &= f_i(n) \prod_{j \in S} p_j x_j^{-\alpha}, \end{aligned} \tag{19}$$

for $S \subseteq \mathbb{I}$ with $|S| = i$, $x_j > 0$ for $j \in S$ and some function $f_i : \mathbb{N} \to \mathbb{R}_+$ where

$$\frac{n!}{(n-i)!} \leq f_i(n) \leq n^i, \quad n \geq i, \; i \in \mathbb{I}.$$

Furthermore, $\mu_{i,n}^{*\oplus}(\mathbb{E}_d^{(i+1)}) = 0$. In particular, we can check that

$$\begin{aligned} &f_1(n) = n, \quad n \geq 1, && f_2(n) = n(n-1/2), \quad n \geq 1, \\ &f_3(n) = n(n-1/2)(n-1), \quad n \geq 2, && f_4(n) = n(n-1/2)(n-1)(n-3/2), \quad n \geq 2. \end{aligned}$$

A pattern in the value of f emerges for this example, but it depends on the limit measures of the underlying variables $\boldsymbol{Y}^{(k)}$. Examples in the same spirit can be computed for $\Delta \geq 3$ involving some careful combinatorial accounting.

Remark 11. It is easy to extend Example 3 in the spirit of Example 2. Suppose $\boldsymbol{Y}^{(k)}$, $k \in \mathbb{N}$, are the same random vectors as in Example 3 and $(\boldsymbol{\varepsilon}^{(k)})_{k\in\mathbb{N}}$ are i.i.d. random vectors with $\mathbb{E}\|\boldsymbol{\varepsilon}^{(k)}\|^{d(\alpha+\theta)} < \infty$ for some $\theta > 0$, which are also independent of the sequence $(\boldsymbol{Y}^{(k)})_{k\in\mathbb{N}}$. Then $\boldsymbol{Z}^{(k)} := \boldsymbol{Y}^{(k)} + \boldsymbol{\varepsilon}^{(k)}$, $k \in \mathbb{N}$, is an adapted-MRV sequence of random vectors with $\Delta = 2$. All the conclusions for $(\boldsymbol{Y}^{(k)})_{k\in\mathbb{N}}$, and $\sum_{k=1}^{n} \boldsymbol{Y}^{(k)}$, $n \geq i$, in Example 3 also hold for $(\boldsymbol{Z}^{(k)})_{k\in\mathbb{N}}$, and $\sum_{k=1}^{n} \boldsymbol{Z}^{(k)}$, $n \geq i$, by an application of Lemma 2.

5. Regular variation of compound Poisson sums

Having investigated finite sums in the previous section, we now study random sums, and in particular compound sums, in this section. For compound sums, the number of summands is assumed to be Poisson distributed and independent of the summands, as is the case for compound Poisson processes at fixed time points. Compound Poisson processes are typical models for yearly aggregated insurance or operational risks. In Section 5.1, we develop the general theory behind the MRV of random sums on subcones $\mathbb{E}_d^{(i)}$ under the broad model Assumption 1, which is satisfied by all of our examples. A classical approach to modeling dependence in a compound sum is to use a copula model for the i.i.d. summands (risks). Subsequently, using this approach, in Section 5.2 we exhibit our results in several examples of compound Poisson sums. Our examples cover the cases of independent risks (cf. Example 4), mutually asymptotically independent risks (cf. Example 5), pairwise asymptotically independent risks (cf. Example 6) but also strongly dependent risks (cf. Example 7); see [18] for the explicit definitions of pairwise and mutually asymptotic independence.

5.1. Random sums of regularly varying random vectors

We observed in Section 4 that the tail behavior of a finite sum may take various forms even when the sums are still MRV. Hence, for convenience, for the rest of the paper, we assume that the following is satisfied.

Assumption 1. *Let* $\mathbf{Z}$, $(\mathbf{Z}^{(k)})_{k\in\mathbb{N}}$ *be a sequence of i.i.d. random vectors in* $\mathbb{R}_+^d$. *Assume that for all* $i=1,\dots,d$ *there exists a measurable function* $f_i:\mathbb{N}\to\mathbb{R}_+$ *and a non-null measure* $\mu_i\in\mathbb{M}(\mathbb{E}_d^{(i)})$ *such that for any* $n\in\mathbb{N}$,

$$\left\{\sum_{k=1}^{n}\mathbf{Z}^{(k)}\in\mathcal{MRV}^*\big(\alpha_{i,n},\,b_{i,n},\,\mu_{i,n}^{\oplus}=f_i(n)\mu_i,\,\mathbb{E}_d^{(i)}\big);\,i=1,\dots,d;\,\Delta_n\right\}$$

and $\Delta_n=d$ *for* $n\ge d$. *Furthermore, assume that for all* $i=1,\dots,d$, *there exist a finite constant* $\alpha_i>0$ *and a regularly varying function* $b_i(t)\in\mathcal{RV}_{1/\alpha_i}$ *such that if* $f_i(n)\neq 0$ *we have* $\alpha_{i,n}=\alpha_i$ *and* $b_{i,n}=b_i$.

Remark 12.

(a) In general, the structure of the function f_i can be quite complex and often requires an involved combinatorial accounting procedure, see Example 3; nevertheless in several examples we do observe that $f_i(n)=n$ and in all our examples $0\le f_i(n)\le n^i$. Assumption 1 allows us the flexibility not to be involved in the computation of f_i.

(b) For $\Delta_n<d$ we have $f_i(n)=0$ and $\alpha_{i,n}=\infty$ for $i=\Delta_n+1,\dots,d$, and, hence, we have null convergence. On the other hand, for $i=1,\dots,\Delta_n$ we have $0<f_i(n)<\infty$, $\alpha_{i,n}=\alpha_i$ and $b_{i,n}=b_i$. For the examples considered in Section 4.2 this happens to be the case.

(c) Suppose $\Delta_1=d$, then Assumption 1 implies that $I(i)$ as defined in Theorem 1 can be chosen to be i and, hence, $\alpha_m+\alpha_{i-m}\ge\alpha_i$ for every $m=0,\dots,i$ and $i=1,\dots,d$. On the other hand, $\alpha_m+\alpha_{i-m}>\alpha_i$ for every $m=1,\dots,i-1$ is a sufficient condition for $I(i)=i$.

Remark 13. Under Assumption 1, define $\mathbf{Z}^{(\oplus,k)}:=\sum_{l=(k-1)d+1}^{kd}\mathbf{Z}^{(l)}$, $k\in\mathbb{N}$. Also let $\mathbf{Z}^{\oplus}:=(Z_1^{\oplus},\dots,Z_d^{\oplus})\sim\mathbf{Z}^{(\oplus,1)}$ and $Z_{(1)}^{\oplus}\ge\dots Z_{(d)}^{\oplus}$ be the order statistics of $Z_1^{\oplus},\dots,Z_d^{\oplus}$.

(a) From Assumption 1 we have

$$\sum_{k=1}^{n}\mathbf{Z}^{(\oplus,k)}\in\mathcal{MRV}\big(\alpha_i,\,b_i,f_i(dn)\mu_i,\,\mathbb{E}_d^{(i)}\big)$$

with $0<f_i(dn)<\infty$ for $i=1,\dots,d$ and $n\in\mathbb{N}$. Now, a consequence of Remark 12(c) is that $I(i)$ (as defined in Theorem 1) for the random vector $\mathbf{Z}^{(\oplus,1)}$ (or, equivalently, $\mathbf{Z}^{\oplus}$) is equal to i and, hence, $\alpha_m+\alpha_{i-m}\ge\alpha_i$ for every $m=0,\dots,i$ and $i=1,\dots,d$. Now, $I(i)=i$ implies as well that there exists a finite constant $C^*>0$ such that

$$0\le\sup_{t>0}\sum_{i=1}^{d}\sum_{m=0}^{i}\frac{\mathbb{P}\big(Z_{(m)}^{\oplus}>t\big)\mathbb{P}\big(Z_{(i-m)}^{\oplus}>t\big)}{\mathbb{P}\big(Z_{(i)}^{\oplus}>t\big)}\le C^*. \tag{20}$$

(b) Note that the function $g_i(t) := \mathbb{P}(Z^{\oplus}_{(i)} > t) \in \mathcal{RV}_{-\alpha_i}$ for any $i \in \mathbb{I}$. Hence, using Potter's bound [21, Proposition B.1.19(5)] there exists a finite constant $C^{**} > 0$ such that

$$\sup_{i\in\mathbb{I}} \sup_{t>0} \frac{\mathbb{P}\big(Z^{\oplus}_{(i)} > t/2\big)}{\mathbb{P}\big(Z^{\oplus}_{(i)} > t\big)} \leq C^{**}. \tag{21}$$

Theorem 2. *Let Assumption 1 hold and let the i.i.d. sequence* $(\mathbf{Z}^{(k)})_{k\in\mathbb{N}}$ *be independent of the* $\mathbb{N}_0$*-valued random variable* τ *with* $\mathbb{E}(\kappa^{\tau}) < \infty$ *for all* $\kappa > 0$. *Then for* $i = 1, \ldots, d$,

$$\sum_{k=1}^{\tau} \mathbf{Z}^{(k)} \in \mathcal{MRV}\big(\alpha_i, b_i, \mathbb{E}(f_i(\tau))\mu_i, \mathbb{E}_d^{(i)}\big).$$

The proof is in Appendix C. Note that the examples considered in both Section 4.1 and Section 4.2 all satisfy Assumption 1 (as well as (20)). Hence, for any $\mathbb{N}_0$-valued random variable τ whose moment generating function exists on the positive real line, we can compute the tail probability of a random sum of τ many i.i.d. MRV random vectors using Theorem 2.

5.2. Copula models and compound Poisson sums

In the discussion henceforth, examples of d-dimensional MRV random vectors and their compound Poisson sums are explored, which are often used to model risks/claim sizes and aggregated risks, respectively, in operational and insurance risk models [11, 30, 38, 56]. Following common practice, we model the marginal distributions of the risks separately from the dependence structure and hence, resort to using *copulas* (cf. [36, 51]). To this end, for all examples in this section, we consider random vectors $\mathbf{Z} = (Z_1, \ldots, Z_d)$ with identically distributed continuous marginal components with distribution function F_α where $\overline{F}_\alpha \in \mathcal{RV}_{-\alpha}$ with $\alpha > 1$, and the dependence is given by the particular (survival) copula. Moreover, we fix $b_\alpha(t) = \overline{F}_\alpha^{\leftarrow}(1/t)$, $t > 1$. Note that assuming tail equivalent marginals would lead to similar conclusions but the notation becomes cumbersome. Our interest is in tail sets, hence we will often use *survival copulas* along with copulas which we recall briefly here. For a random vector $\mathbf{Z} = (Z_1, \ldots, Z_d) \sim F$ with continuous marginal distributions $F_1, \ldots, F_d$, the copula $C : [0, 1]^d \to [0, 1]$ and the survival copula $\widehat{C} : [0, 1]^d \to [0, 1]$ are distribution functions such that for $(x_1, \ldots, x_d) \in \mathbb{R}^d$:

$$\begin{aligned}
\mathbb{P}(Z_1 \leq x_1, \ldots, Z_d \leq x_d) &= C(F_1(x_1), \ldots, F_d(x_d)),\\
\mathbb{P}(Z_1 > x_1, \ldots, Z_d > x_d) &= \widehat{C}(\overline{F}_1(x_1), \ldots, \overline{F}_d(x_d)),
\end{aligned}$$

where $\overline{F}_j = 1 - F_j$ for all $j \in \mathbb{I}$. Finally, for aggregated models we suppose that $\mathbf{Z}^{(1)}, \ldots, \mathbf{Z}^{(n)}$ are i.i.d. random vectors with distribution of $\mathbf{Z}$ and independent of the Poisson random variable $N(s)$ with intensity λs for some $\lambda, s > 0$.

Example 4. (Independence copula.) A widely used copula to exhibit asymptotic tail independence and (hidden) regular variation is the independence copula. The independence copula $C_\perp$ and the survival copula $\widehat{C}_\perp$ are given by

$$C_\perp(u_1, \ldots, u_d) = \widehat{C}_\perp(u_1, \ldots, u_d) = u_1 u_2, \ldots, u_d, \quad 0 < u_i < 1. \tag{22}$$

Let $\mathbf{Z} \sim F$ with identical (continuous) marginals F_α as defined above and dependence given by $C_\perp$ (or $\widehat{C}_\perp$). Then with $b_i^{\leftarrow}(t) = (b_\alpha^{\leftarrow}(t))^i$

$$\mathbf{Z} \in \mathcal{MRV}\big(i\alpha, b_i, \mu_i, \mathbb{E}_d^{(i)}\big) \tag{23}$$

where

$$\mu_i\big(\{z \in \mathbb{E}_d^{(i)} : z_j > x_j \text{ for all } j \in S\}\big) = \prod_{j\in S} x_j^{-\alpha} \tag{24}$$

for $S \subseteq \mathbb{I}$ with $|S| = i$, $i = 1, \ldots, d$ (cf. [19]) and $\mu_i(\mathbb{E}_d^{(i+1)}) = 0$. Clearly $\boldsymbol{Z}$ and $\sum_{k=1}^{n} \boldsymbol{Z}^{(k)}$, which has as well the independence copula $C_\perp$, because the margins are independent, exhibit regular variation on all subcones $\mathbb{E}_d^{(i)}$ with respect to MRV on $\mathbb{E}_d^{(i-1)}$ for $i = 2, \ldots, d$. However, the margins of $\sum_{k=1}^{N(s)} \boldsymbol{Z}^{(k)}$ are not independent anymore, so the copula is not the independence copula, but still, due to Theorem 2,

$$\sum_{k=1}^{N(s)} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}\big(i\alpha, b_i, s\lambda\mu_i, \mathbb{E}_d^{(i)}\big) \quad \text{for } i = 1, \ldots, d.$$

Example 5 (*Marshall–Olkin copula.*) In reliability theory, the Marshall–Olkin distribution provides an elegant mechanism to capture the dependence between the failure of subsystems in an entire system. We focus on a particular structure of the Marshall–Olkin survival copula as given in [45, (2.4), p. 58]. Assume that for all $\varnothing \neq S \subseteq \mathbb{I}$ there exists a parameter $\lambda_S > 0$. Consider then the generalized Marshall–Olkin survival copula given by

$$\widehat{C}_{\mathrm{MO}}(u_1, \ldots, u_d) = \prod_{i=1}^{d} \prod_{|S|=i} \bigwedge_{j\in S} u_j^{\eta_j^S}, \quad 0 < u_j < 1,$$

where

$$\eta_j^S = \frac{\lambda_S}{\sum\limits_{J \supseteq \{j\}} \lambda_J}, \quad j \in S \subseteq \mathbb{I}. \tag{25}$$

A typographical error in the formula for η_j^S in [45, eq. (2.4), p. 58] is corrected in (25). We consider two particular choices of the parameters λ_S for our examples.

(a) *Equal parameter for all sets:* Let $\lambda_S = \lambda > 0$ for all $\varnothing \neq S \subseteq \mathbb{I}$; hence, from (25) we have

$$\eta_j^S = 2^{-(d-1)} =: \beta. \tag{26}$$

Therefore,

$$\begin{aligned}\mathbb{P}(Z_1 > x_1, \ldots, Z_d > x_d) &= \widehat{C}(\overline{F}_\alpha(x_1), \ldots, \overline{F}_\alpha(x_d)) \\ &= \prod_{i=1}^{d} \prod_{|S|=i} \bigwedge_{j\in S} (\overline{F}_\alpha(x_j))^\beta \\ &= \prod_{j=1}^{d} (\overline{F}_\alpha(x_{(j)}))^{2^{d-j}\beta} = \prod_{j=1}^{d} (\overline{F}_\alpha(x_{(j)}))^{2^{-(j-1)}},\end{aligned}$$

where $x_{(1)} \geq \cdots \geq x_{(d)}$ denote the decreasing order statistics of $x_1, \ldots, x_d$. Now, we can check that for $i = 1, \ldots, d$,

$$\boldsymbol{Z} \in \mathcal{MRV}\big(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)}\big)$$

where

$$\begin{aligned}\alpha_i &= (2-2^{-(i-1)})\alpha,\\ b_i^{\leftarrow}(t) &= (b_\alpha^{\leftarrow}(t))^{\alpha_i/\alpha} = (b_\alpha^{\leftarrow}(t))^{(2-2^{-(i-1)})},\\ \mu_i\big(\{z\in\mathbb{E}_d^{(i)} : z_j > x_j \text{ for all } j\in S\}\big) &= \prod_{j=1}^{i}(x_{(j)})^{-\alpha 2^{-(j-1)}},\end{aligned}$$

where $S\subseteq\mathbb{I}$, $|S|=i$ with $x_j>0$ for $j\in S$, and $x_{(1)}\geq\cdots\geq x_{(i)}$ denote the decreasing order statistics of $(x_j)_{j\in S}$ and $\mu_i(\mathbb{E}_d^{(i+1)})=0$. Now observe that for fixed $i=1,\ldots,d$, and $m=1,\ldots,i-1$,

$$\begin{aligned}\alpha_m+\alpha_{i-m} &= (2-2^{-(m-1)})\alpha+(2-2^{-(i-m-1)})\alpha\\ &\geq 2\alpha\\ &> (2-2^{-(i-1)})\alpha=\alpha_i.\end{aligned}\tag{27}$$

Hence, by Proposition 3 and Theorem 2, we have for $i=1,\ldots,d$,

$$\sum_{k=1}^{n}\boldsymbol{Z}^{(k)}\in\mathcal{MRV}\big(\alpha_i, b_i, n\mu_i, \mathbb{E}_d^{(i)}\big)\quad\text{and}\quad\sum_{k=1}^{N(s)}\boldsymbol{Z}^{(k)}\in\mathcal{MRV}\big(\alpha_i, b_i, s\lambda\mu_i, \mathbb{E}_d^{(i)}\big).$$

(b) *Parameters proportional to cardinality of the sets:* Let $\lambda_S=|S|\,\lambda$ where $\lambda>0$ for all $\varnothing\neq S\subseteq\mathbb{I}$. From (25) we have $\eta_j^S=|S|(d+1)2^{-(d-1)}=|S|(d+1)\beta$ using the definition in (26). Following a similar logic as in part (a), we obtain in this case

$$\mathbb{P}(Z_1>x_1,\ldots,Z_d>x_d)=\prod_{j=1}^{d}(\overline{F}_\alpha(x_{(j)}))^{(1-(j-1)/(d+1))2^{-(j-1)}}.$$

Again we can check that for $i=1,\ldots,d$,

$$\boldsymbol{Z}\in\mathcal{MRV}\big(\alpha_i^*, b_i^*, \mu_i^*, \mathbb{E}_d^{(i)}\big)$$

where

$$\begin{aligned}\alpha_i^* &= \sum_{j=1}^{i}\left(1-\frac{j-1}{d+1}\right)\frac{\alpha}{2^{j-1}}=\alpha_i\frac{d}{d+1}+\frac{i\alpha}{(d+1)2^{i-1}},\\ b_i^{*\leftarrow}(t) &= (b_\alpha^{\leftarrow}(t))^{\alpha_i^*/\alpha},\\ \mu_i^*\big(\{z\in\mathbb{E}_d^{(i)} : z_j>x_j \text{ for all } j\in S\}\big) &= \prod_{j=1}^{i}(x_{(j)})^{-\alpha(1-(j-1)/(d+1))2^{-(j-1)}},\end{aligned}$$

where $|S|=i$ with $x_j>0$ for $j\in S$ and $\mu_i^*(\mathbb{E}_d^{(i+1)})=0$. Again note that for fixed $i=1,\ldots,d$, $m=1,\ldots,i-1$, and α_i as in (5),

$$\begin{aligned}\alpha_m^*+\alpha_{i-m}^* &= \alpha_m\frac{d}{d+1}+\frac{m\alpha}{(d+1)2^{m-1}}+\alpha_{i-m}\frac{d}{d+1}+\frac{(i-m)\alpha}{(d+1)2^{i-m-1}}\\ &= (\alpha_m+\alpha_{i-m})\frac{d}{d+1}+\frac{\alpha}{(d+1)}\left(\frac{m}{2^{m-1}}+\frac{i-m}{2^{i-m-1}}\right)\\ &> \alpha_i\frac{d}{d+1}+\frac{\alpha}{(d+1)}\left(\frac{m}{2^{i-1}}+\frac{i-m}{2^{i-1}}\right)\quad\text{(using (27))}\\ &= \alpha_i^*.\end{aligned}$$

Hence, by Proposition 3 and Theorem 2 we have for $i = 1, \dots, d$,

$$\sum_{k=1}^{n} \mathbf{Z}^{(k)} \in \mathcal{MRV}\big(\alpha_i^*, b_i^*, n\mu_i^*, \mathbb{E}_d^{(i)}\big) \quad \text{and} \quad \sum_{k=1}^{N(s)} \mathbf{Z}^{(k)} \in \mathcal{MRV}\big(\alpha_i^*, b_i^*, s\lambda\mu_i^*, \mathbb{E}_d^{(i)}\big).$$

In both examples of the Marshall–Olkin copula dependence (with identical regularly varying margins), $\mathbf{Z}$, $\sum_{k=1}^{n} \mathbf{Z}^{(k)}$, and $\sum_{k=1}^{N(s)} \mathbf{Z}^{(k)}$ exhibit regular variation on all cones $\mathbb{E}_d^{(i)}$, $i = 2, \dots, d$.

Example 6. (*Archimedean copula (ACIG).*) This Archimedean copula example based on the Laplace transform of the inverse gamma distribution, called ACIG copula in short, appears in [27] with its (hidden) regular variation discussed in [28, Example 4.4]. Suppose $\mathbf{Z} = (Z_1, \dots, Z_d)$ has an ACIG copula with dependence parameter $1 < \beta < 2$ and identical margins $\overline{F}_\alpha \in \mathcal{RV}_{-\alpha}$, $i = 1, \dots, d$. Then $\mathbf{Z} \in \mathcal{MRV}(\alpha, b_\alpha, \mu_1, \mathbb{E}_d^{(1)})$ and $\mathbf{Z} \in \mathcal{MRV}$ $(\alpha\beta, b_2, \mu_2, \mathbb{E}_d^{(i)})$ for $i = 2, \dots, d$ where $b_2^{\leftarrow}(t) = (b_\alpha^{\leftarrow}(t))^\beta$. In this particular example $\mathbf{Z}$ exhibits regular variation on $\mathbb{E}_d^{(2)}$ with respect to MRV on $\mathbb{E}_d^{(1)}$ but no further (multivariate) regular variation at any subsequent coordinate subcone $\mathbb{E}_d^{(i)}$, $i \geq 3$. Now, clearly the conditions of Proposition 3 are satisfied and, hence, we have

$$\sum_{k=1}^{n} \mathbf{Z}^{(k)} \in \mathcal{MRV}\big(\alpha, b_\alpha, n\mu_1, \mathbb{E}_d^{(1)}\big),$$

and

$$\sum_{k=1}^{n} \mathbf{Z}^{(k)} \in \mathcal{MRV}\big(\alpha\beta, b_2, n\mu_2, \mathbb{E}_d^{(i)}\big) \quad \text{for } i = 2, \dots, d,$$

and similarly by Theorem 2,

$$\sum_{k=1}^{N(s)} \mathbf{Z}^{(k)} \in \mathcal{MRV}\big(\alpha, b_\alpha, s\lambda\mu_1, \mathbb{E}_d^{(1)}\big)$$

and

$$\sum_{k=1}^{N(s)} \mathbf{Z}^{(k)} \in \mathcal{MRV}\big(\alpha\beta, b_2, s\lambda\mu_2, \mathbb{E}_d^{(i)}\big) \quad \text{for } i = 2, \dots, d.$$

Example 7. (*Asymptotically tail-dependent copula.*) In the previous examples, we observed distributions with regularly varying marginals and copulas exhibiting *asymptotic tail independence* leading to MRV with different indices on different spaces, see [17] for an appropriate definition. But some distributions exhibit so-called *asymptotic tail dependence* which would lead to MRV with the same index, rate function and limit measure on all subcones $\mathbb{E}_d^{(i)}$; see [26] for general examples in dimension $d = 2$ and [12] for higher-dimensional Archimedean copulas exhibiting asymptotic tail dependence. We illustrate this with one example. Let $\mathbf{Z} \sim F$ such that for $\alpha > 1$,

$$F(\boldsymbol{x}) = 1 - \left(1 + \sum_{j=1}^{d} x_j^\alpha\right)^{-1}, \quad \boldsymbol{x} \in \mathbb{R}_+^d.$$

We can check that the marginals are identically Pareto distributed with index $-\alpha$ and, hence, the tails are $\mathcal{RV}_{-\alpha}$. Moreover, $\boldsymbol{Z} \in \mathcal{MRV}(\alpha, b(t) = t^{1/\alpha}, \mu, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, d$ where

$$\mu(\{\boldsymbol{z} \in \mathbb{E}_d : z_j > x_j \text{ for all } j \in S\}) = \sum_{j=1}^{|S|} (-1)^{j+1} \sum_{\substack{k_1 < \cdots < k_j \\ k_1, \ldots, k_j \in S}} \left(\sum_{l=1}^{j} x_{k_l}^{\alpha} \right)^{-1} \tag{28}$$

for $S \subseteq \mathbb{I}$. Hence, we can compute tail asymptotics of its convolution using Corollary 1 and by Theorem 2,

$$\sum_{k=1}^{N(s)} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}\big(\alpha, b, s\lambda\mu, \mathbb{E}_d^{(i)}\big) \quad \text{for } i = 1, \ldots, d.$$

Remark 14. (*Gaussian copula.*) Gaussian copulas have been widely considered as a key example of asymptotic tail independence, for which coefficients of tail dependence, tail order and (hidden) regular variation have been studied, especially in the bivariate case; see [25, 28, 44]. The recent paper [16] studied the case $d \geq 3$ in detail and provided the appropriate order of MRV on the different subcones. Indeed, we have *pairwise asymptotic independence* but we do not necessarily have *mutual asymptotic independence* as explored in [18].

6. Regular variation of multivariate Lévy processes

As a natural extension of regular variation for random sums and compound Poisson processes, this section examines regular variation of multivariate Lévy processes $\boldsymbol{L} = (\boldsymbol{L}(s))_{s \geq 0}$ on the different subcones $\mathbb{E}_d^{(i)}$, $i = 1, \ldots, d$, and relates it to MRV of the corresponding Lévy measure. Note that a Lévy process is characterized by its *Lévy–Khinchine representation* $\mathbb{E}(\mathrm{e}^{i\langle \Theta, \boldsymbol{L}(s)\rangle}) = \exp(-s\,\Psi(\Theta))$ for $\Theta \in \mathbb{R}^d$, where

$$\Psi(\Theta) = -i\langle \gamma, \Theta\rangle + \frac{1}{2}\langle \Theta, \Sigma\, \Theta\rangle + \int_{\mathbb{R}^d} \left(1 - \mathrm{e}^{i\langle \Theta, \boldsymbol{x}\rangle} + i\langle \boldsymbol{x}, \Theta\rangle\right) \Pi(d\boldsymbol{x})$$

with $\gamma \in \mathbb{R}^d$, Σ a non-negative definite matrix in $\mathbb{R}^{d\times d}$ and a Borel measure Π on $\mathbb{R}^d$, called the Lévy measure which satisfies $\int_{\mathbb{R}^d} \min\{\|\boldsymbol{x}\|^2, 1\}\, \Pi(d\boldsymbol{x}) < \infty$ and $\Pi(\boldsymbol{0}) = 0$. Moreover, $\langle \cdot, \cdot\rangle$ denotes the inner product in $\mathbb{R}^d$. The Lévy measure $\Pi(A)$ measures the expected number of jumps of the Lévy process in the interval $[0, 1]$ which lies in the set A. We denote by Π_j for $j = 1, \ldots, d$ the marginal Lévy measures. The compound Poisson process is also a special kind of Lévy process which is represented by a random sum at every fixed time point. In this paper, we restrict to Lévy processes in $\mathbb{R}_+^d$, i.e. the marginal Lévy processes are *subordinators*, which are increasing Lévy processes, and used as models for multivariate claim amounts in operational risks (cf. [11, 38]) and insurance risks (cf. [6, 7, 56]). For more details on Lévy processes see [4, 58].

Regular variation in multivariate Lévy processes, especially characterizing complex tail events, including but not restricted to (5), can happen in a variety of ways. We may observe regular variation for the Lévy process itself or the Lévy measure, and they may have different implications depending on the dependence structure of the Lévy process. In the following three subsections, we investigate this in detail; the proofs of the associated results are provided in Appendix D.

6.1. The Lévy measure admits regular variation on all subcones

Hereafter, we assume that the Lévy measure is multivariate regularly varying on all subcones $\mathbb{E}_d^{(i)}$, $i = 1, \ldots, d$ and show that the same is true for the Lévy process, in fact, they are *tail equivalent* (in a multivariate sense) as we exhibit next.

Proposition 5. (Extending Proposition 3.) *Let* $(\boldsymbol{L}(s))_{s\geq 0}$ *be a Lévy process in* $\mathbb{R}_+^d$ *with Lévy measure* $\Pi \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ *for* $i = 1, \ldots, d$ *whose univariate marginal Lévy measures are tail equivalent. Moreover, assume that* $\alpha_i < \alpha_m + \alpha_{i-m}$ *for all* $m = 1, \ldots, i-1$ *and* $i = 2, \ldots, d$. *For* $s > 0$, *we have then*

$$\boldsymbol{L}(s) \in \mathcal{MRV}\big(\alpha_i, b_i, s\mu_i, \mathbb{E}_d^{(i)}\big) \quad \text{for } i = 1, \ldots, d.$$

A direct consequence of Proposition 5 is the tail equivalence of the Lévy measure of the set tA and the probability measure of the Lévy process belonging to tA, for Borel sets $A \in \mathcal{B}^{(i)}$ bounded away from $\mathbb{CA}_d^{(i-1)}$ with $\mu_i(A) > 0$ and $\mu_i(\partial A) = 0$ such that

$$\mathbb{P}(\boldsymbol{L}(s) \in tA) \sim s\, \mathbb{P}(\boldsymbol{L}(1) \in tA) \sim s\, \Pi(tA) \sim \frac{s}{b_i^{\leftarrow}(t)}\, \mu_i(A) \quad \text{as } t \to \infty.$$

Although the tail equivalence of the Lévy process and the Lévy measure holds for a variety of sets, the tail rate differs depending on which subcone $\mathbb{E}_d^{(i)}$ the set A belongs to. A similar conclusion was shown in [31], but only for sets A with $\mu_1(A) > 0$. However, in many situations this is not the case as we see in the following examples.

Example 8. Suppose $L_\alpha^{(j)}$, $j = 1, 2, 3$ are i.i.d. Lévy processes in $\mathbb{R}_+$ with Lévy measure $\Pi_\alpha \in \mathcal{MRV}(\alpha, b_\alpha, \mu_\alpha, (0, \infty))$, $L_\beta^{(j)}, j = 1, 2$ are i.i.d. Lévy processes in $\mathbb{R}_+$ with Lévy measure $\Pi_\beta \in \mathcal{MRV}(\beta, b_\beta, \mu_\beta, (0, \infty))$ and L_γ is a Lévy process in $\mathbb{R}_+$ with Lévy measure $\Pi_\gamma \in \mathcal{MRV}(\gamma, b_\gamma, \mu_\gamma, (0, \infty))$. A typical example for L_α is an α-stable Lévy process with Lévy measure $\Pi_\alpha(\mathrm{d}x) = \alpha x^{-\alpha-1}\mathbf{1}_{(0,\infty)}(x)\,\mathrm{d}x$. Furthermore, assume all processes are independent and $\alpha < \beta < \gamma < 2\alpha$. Then the three-dimensional Lévy process

$$\boldsymbol{L}(s) = \big(L_\alpha^{(1)}(s) + L_\beta^{(1)}(s) + L_\gamma(s), L_\alpha^{(2)}(s) + L_\beta^{(1)}(s) + L_\gamma(s), L_\alpha^{(3)}(s) + L_\beta^{(2)}(s) + L_\gamma(s)\big),$$

has Lévy measure

$$\Pi(A) = \sum_{j=1}^{3} \Pi_\alpha(A_j) + \Pi_\beta(A_1 \cap A_2) + \Pi_\beta(A_3) + \Pi_\gamma(A_1 \cap A_2 \cap A_3),$$

where $A_1 = \{z \in \mathbb{R}_+ : (z, 0, 0) \in A\}$, $A_2 = \{z \in \mathbb{R}_+ : (0, z, 0) \in A\}$, and $A_3 = \{z \in \mathbb{R}_+ : (0, 0, z) \in A\}$. Of course, Π satisfies the assumptions of Proposition 5 with $(\alpha_1, \alpha_2, \alpha_3) = (\alpha, \beta, \gamma)$, $(b_1, b_2, b_3) = (b_\alpha, b_\beta, b_\gamma)$, and

$$\mu_1(A) = \sum_{j=1}^{3} \mu_\alpha(A_j), \quad \mu_2(A) = \mu_\beta(A_1 \cap A_2) \quad \text{and} \quad \mu_3(A) = \mu_\gamma(A_1 \cap A_2 \cap A_3).$$

Finally, $\boldsymbol{L}(s) \in \mathcal{MRV}(\alpha_i, b_i, s\mu_i, \mathbb{E}_d^{(i)})$ for $i = 1, 2, 3$.

In Example 8, $\boldsymbol{L}(s)$ is multivariate regularly varying on $\mathbb{E}_d^{(1)}$ and $\mathbb{E}_d^{(2)}$ but with different indices and, hence, $\mu_1(\mathbb{E}_d^{(2)}) = 0$. This implies that the components of $\boldsymbol{L}(s)$ are asymptotically pairwise tail independent. In the special case where the components of $\boldsymbol{L}(s)$ are (strongly) dependent, the next result follows directly from Proposition 5.

Corollary 2. (Extending Corollary 1: dependent case.) *Let* $(\boldsymbol{L}(s))_{s\geq 0}$ *be a Lévy process in* $\mathbb{R}_+^d$ *with Lévy measure* $\Pi \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ *for* $i = 1, \ldots, d$ *whose univariate marginal Lévy measures are tail equivalent. Moreover,* $(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)}) = (\alpha, b, \mu, \mathbb{E}_d^{(i)})$ *for* $i = 1, \ldots, i^*$ *for some* $i^* \leq d$*. Then for* $s > 0$ *we have*

$$\boldsymbol{L}(s) \in \mathcal{MRV}\big(\alpha, b, s\mu, \mathbb{E}_d^{(i)}\big) \quad \text{for } i = 1, \ldots, i^*.$$

Example 9.

(a) Completely dependent case: Let $L_1 = (L_1(s))_{s\geq 0}$ be a Lévy process in $\mathbb{R}_+$ with univariate marginal Lévy measure $\Pi_\alpha \in \mathcal{MRV}(\alpha, b_\alpha, \mu_\alpha, (0, \infty))$ and $\boldsymbol{L} = (L_1, \ldots, L_1)$. Then the Lévy measure of $\boldsymbol{L}$ is given by

$$\Pi(A) = \min_{j\in S} \Pi_\alpha((x_j, \infty))$$

for a rectangular set $A = \{z \in \mathbb{R}_+^d : z_j > x_j \text{ for all } j \in S\}$ with $S \subseteq \mathbb{I}$ and $x_j > 0$ for $j \in S$. In this case, we are in the setting of Corollary 2 with $i^* = d$ and $\mu(A) = \min_{j\in S} x_j^{-\alpha}$ for a rectangular set A as above.

More generally, if the marginal tail Lévy measures are not necessarily identical but are completely tail equivalent satisfying $\Pi_j \in \mathcal{MRV}(\alpha, b_\alpha, \mu_\alpha, (0, \infty))$ for $j = 1, \ldots, d$ and

$$\Pi(A) = \min_{j\in S} \Pi_j(x_j, \infty),$$

then $\Pi \in \mathcal{MRV}(\alpha, b_\alpha, \mu, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, d$ as well and the assumptions of Corollary 2 are satisfied. Indeed, this is a Lévy measure, it is constructed by the *completely dependent Lévy copula* (cf. [37]).

(b) Suppose L_j, $j = 1, \ldots, d$ are Lévy processes in $\mathbb{R}_+$ with univariate completely tail equivalent marginal Lévy measures $\Pi_j \in \mathcal{MRV}(\alpha, b_\alpha, \mu_\alpha, (0, \infty))$ and $\boldsymbol{L}(s) = (L_1(s), \ldots, L_d(s))$ is a d-dimensional Lévy process with Lévy measure

$$\Pi(A) = \left(\sum_{j\in S} (\Pi_j((x_j, \infty)))^{-\theta} \right)^{1/\theta}$$

for some $\theta > 0$, where $A = \{z \in \mathbb{R}_+^d : z_j > x_j \text{ for all } j \in S\}$ is a rectangular set with $S \subseteq \mathbb{I}$ and $x_j > 0$ for $j \in S$. This Lévy measure is constructed using the Clayton Lévy copula (cf. [37]). Let the measure μ on $\mathcal{B}^{(1)}$ be defined as

$$\mu(A) = \left(\sum_{j\in S} x_j^{-\alpha\theta} \right)^{-1/\theta}$$

for a rectangular set A as above. Then $\Pi \in \mathcal{MRV}(\alpha, b_\alpha, \mu, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, d$ and, hence, due to Corollary 2, $\boldsymbol{L}(s) \in \mathcal{MRV}(\alpha, b_\alpha, s\mu, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, d$ as well.

(c) Another (dependent) example of a Lévy process can be constructed by a compound Poisson process where the jump sizes have the distribution F as in Example 7.

Remark 15. Regular variation of the Lévy measure on different subcones $\mathbb{E}_d^{(i)}$ can be related to regular variation of the Lévy copula and Pareto Lévy copula, respectively, on these different subcones; cf. [22, 41] for classical regular variation of such Lévy measure on $\mathbb{E}_d^{(1)}$. This work is under investigation by the authors.

6.2. The Lévy process is asymptotically tail independent

In Proposition 5 and subsequently Corollary 2, the underlying Lévy measure admits regular variation on all relevant subcones, but this may not necessarily be the case in general. The next result includes the case where the Lévy measure is adapted-MRV, i.e. MRV need not exist in all the relevant subcones.

Proposition 6. (Extending Proposition 4.) *Let* $(\boldsymbol{L}(s))_{s\geq 0}$ *be a Lévy process in* $\mathbb{R}_+^d$ *with Lévy measure* Π *such that* $\{\Pi \in \mathcal{MRV}^*(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)}),\ \ i=1, \ldots, d;\ \ \Delta = 1\}$ *and* Π *has tail equivalent univariate marginal Lévy measures in* $\mathcal{RV}_{-\alpha}$. *Then for* $s>0$ *we have*

$$\boldsymbol{L}(s) \in \mathcal{MRV}\big(i\alpha_1, b_i, s^i\mu_i^L, \mathbb{E}_d^{(i)}\big) \quad \text{for } i=1, \ldots, d,$$

with $b_i^{\leftarrow}(t) = (b_1^{\leftarrow}(t))^i$ *and*

$$\mu_i^L\left(\bigcap_{j\in S}\{z\in\mathbb{E}_d^{(i)} : z_j > x_j\}\right) = \prod_{j\in S}\mu_1\big(\{z\in\mathbb{E}_d^{(i)} : z_j > x_j\}\big)$$

for $S\subseteq\mathbb{I}$ *with* $|S|=i$, $x_j>0$ *for* $j\in S$, *and* $\mu_i^L(\mathbb{E}_d^{(i+1)})=0$.

Interestingly, for rectangular sets $A\in\mathcal{R}^{(i)}$ as in (5) with $|S|=i$, we observe that

$$\mathbb{P}(\boldsymbol{L}(s)\in tA) \sim s^i\,\mathbb{P}(\boldsymbol{L}(1)\in tA) \quad \text{as } t\to\infty. \tag{29}$$

Hence, the linearity property of $\mathbb{P}(\boldsymbol{L}(s)\in tA)\sim s\,\mathbb{P}(\boldsymbol{L}(1)\in tA)$ as $t\to\infty$, which we had noticed in the dependent cases of Proposition 5 and Corollary 2, respectively, vanishes here making this an unusual phenomenon for Lévy processes. Moreover, although Π is multivariate regularly varying on $\mathbb{E}_d^{(1)}$, for sets $A\in\mathcal{R}^{(2)}$ the tail measures $\Pi(tA)$ and $\mathbb{P}(\boldsymbol{L}(1)\in tA)$ are not tail equivalent anymore, in contrast to the common wisdom for regular variation of Lévy processes on $\mathbb{E}_d^{(1)}$.

Example 10.

(a) Suppose the marginal Lévy processes $L_1, \ldots, L_d$ of $\boldsymbol{L}=(L_1, \ldots, L_d)$ are independent with tail equivalent univariate marginal Lévy measures Π_j which are regularly varying with tail index $-\alpha<0$. Then the Lévy measure of $\boldsymbol{L}$ is

$$\Pi(A) = \sum_{j=1}^d \Pi_j(A_j)$$

for $A_j=\{z\in\mathbb{R}_+ : (0, \ldots, 0, z, 0, \ldots, 0)\in A\}$, where z appears in the jth coordinate. This Lévy measure has mass only on the coordinate axes. Hence, the assumptions of Proposition 6 are satisfied and this proposition can be applied to show MRV of $\boldsymbol{L}$ on various subcones. In particular, it satisfies (29).

(b) A compound Poisson process with jumps sizes $(\boldsymbol{Z}^{(k)})_{k\in\mathbb{N}}$ as in Example 2 satisfies the assumptions of Proposition 6 as well, providing an example for the same.

6.3. The Lévy measure is asymptotically tail independent

A special case of the next proposition is a compound Poisson process where the marginal distributions of the jump sizes are independent. The observed phenomena are again different from Proposition 6, which covers as a special case a compound Poisson process with independent marginal Lévy processes.

Proposition 7. (Extending Proposition 2.) *Let* $(\boldsymbol{L}(s))_{s\geq 0}$ *be a Lévy process in* $\mathbb{R}_+^d$ *with Lévy measure* $\Pi\in\mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ *for* $i=1,\ldots,d$ *where* $b_1(t)\in\mathcal{RV}_{1/\alpha}$, $b_i^{\leftarrow}(t)=(b_1^{\leftarrow}(t))^i$, *and for some* $\kappa_j>0$, $j=1,\ldots,d$,

$$\mu_i\left(\bigcap_{j\in S}\left\{z\in\mathbb{E}_d^{(i)} : z_j>x_j\right\}\right)=\prod_{j\in S}\kappa_j x_j^{\alpha}$$

for $S\subseteq\mathbb{I}$ *with* $|S|=i$, $x_j>0$ *for* $j\in S$ *and* $\mu_i(\mathbb{E}_d^{(i+1)})=0$, $i=1,\ldots,d$, *and* Π *has tail equivalent univariate marginal Lévy measures. Let* $(N^*(s))_{s\geq 0}$ *denote a Poisson process with intensity 1. Then for* $s>0$ *we have*

$$\boldsymbol{L}(s)\in\mathcal{MRV}\big(i\alpha, b_i, \mathbb{E}(N^*(s)^i)\mu_i, \mathbb{E}_d^{(i)}\big)\quad \text{for } i=1,\ldots,d.$$

As a consequence of Proposition 7, for any rectangular set $A\in\mathcal{R}^{(i)}$ as in (5) with $|S|=i$ we obtain

$$\mathbb{P}(\boldsymbol{L}(s)\in tA)\sim\frac{\mathbb{E}(N^*(s)^i)}{\mathbb{E}(N^*(1)^i)}\mathbb{P}(\boldsymbol{L}(1)\in tA)\quad \text{as } t\to\infty,$$

where $\mathbb{E}(N^*(s)^i)$ is a polynomial of order i in s, and $s^i\leq\mathbb{E}(N^*(s)^i)$.

Remark 16.

(a) We can verify that indeed the result in Proposition 7 is in accordance with Theorem 1. From Proposition 7 we get that for the i.i.d. random vectors

$$\boldsymbol{L}(1) \text{ and } \boldsymbol{L}(2)-\boldsymbol{L}(1)\in\mathcal{MRV}\big(i\alpha, b_i, \mathbb{E}(N^*(1)^i)\mu_i, \mathbb{E}_d^{(i)}\big)\quad \text{for } i=1,\ldots,d.$$

Applying Theorem 1 gives

$$\boldsymbol{L}(2)=\boldsymbol{L}(1)+[\boldsymbol{L}(2)-\boldsymbol{L}(1)]\in\mathcal{MRV}\big(i\alpha, b_i, \mu_i^{\oplus}, \mathbb{E}_d^{(i)}\big)\quad \text{for } i=1,\ldots,d$$

with

$$\begin{aligned}\mu_i^{\oplus}&=\sum_{m=0}^{i}\binom{i}{m}\mathbb{E}(N^*(1)^m)\mathbb{E}([N^*(2)-N^*(1)]^{i-m})\mu_i\\&=\mathbb{E}\left(\sum_{m=0}^{i}\binom{i}{m}N^*(1)^m[N^*(2)-N^*(1)]^{i-m}\right)\mu_i\\&=\mathbb{E}(N^*(1)+[N^*(2)-N^*(1)])^i\mu_i=\mathbb{E}(N^*(2)^i)\mu_i,\end{aligned}$$

which is also a consequence of Proposition 7.

(b) Suppose $(\boldsymbol{L}(s))_{s\geq 0}$ is a compound Poisson process with Lévy measure $\Pi \in \mathcal{MRV}(\alpha_i, b_i, K\mu_i, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, d$ with $\alpha_i, b_i,$ and μ_i as in Proposition 7 and $K > 0$ is some positive constant. Furthermore, suppose the marginal Lévy measures of Π are tail-equivalent. Let $(\widetilde{\boldsymbol{L}}(s))_{s\geq}$ be another compound Poisson process with Lévy measure $\widetilde{\Pi} = \Pi/K$. Then $\widetilde{\Pi} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, d$ and due to Proposition 2, we have

$$\widetilde{\boldsymbol{L}}(s) \in \mathcal{MRV}\big(i\alpha, b_i, \mathbb{E}(N^*(s)^i)\mu_i, \mathbb{E}_d^{(i)}\big) \quad \text{for } i = 1, \ldots, d.$$

But $\boldsymbol{L}(s) \overset{\mathrm{d}}{=} \widetilde{\boldsymbol{L}}(Ks)$ and, hence, we have

$$\boldsymbol{L}(s) \in \mathcal{MRV}\big(i\alpha, b_i, \mathbb{E}\big(N^*(Ks)^i\big)\mu_i, \mathbb{E}_d^{(i)}\big) \quad \text{for } i = 1, \ldots, d.$$

Example 11. Consider the compound Poisson process

$$\boldsymbol{L}(s) = \sum_{k=1}^{N^*(s)} \big(Z_1^{(k)}, \ldots, Z_d^{(k)}\big)$$

where $(N^*(s))_{s\geq 0}$ is a Poisson process with intensity 1, which is independent of the i.i.d. sequence of jump sizes $(Z_1^{(k)}, \ldots, Z_d^{(k)})_{k\in\mathbb{N}}$. Suppose $(Z_1^{(k)})_{k\in\mathbb{N}}, \ldots, (Z_d^{(k)})_{k\in\mathbb{N}}$ are as well independent of each other with tail equivalent marginal distributions F_j and $\overline{F}_j \in \mathcal{RV}_{-\alpha}$. Then for a rectangular set $A = \{z \in \mathbb{R}_+^d : z_j > x_j \text{ for all } j \in S\}$ with $S \subseteq \mathbb{I}$ and $x_j > 0$ for $j \in S$ we have

$$\Pi(A) = \prod_{j\in S}^{d} \overline{F}_j(x_j).$$

Thus, the assumptions of Proposition 7 are again satisfied and hence, can be applied here.

7. Conclusion

In this paper, we have provided a general framework for finding tail probabilities of risk vectors or multivariate risk processes pertaining to the aggregation of heavy-tailed random vectors; these range from finite sums to random sums and eventually to multivariate Lévy processes as well. We discovered that while under certain conditions, the well-known ‘one large jump’ phenomenon still holds, there are scenarios, especially under the broader framework of adapted-MRV, where we may observe a more general phenomenon of ‘a few large jumps’. Notably, a new discovery for classical MRV Lévy processes is that there are tail sets for which the probability measure and the Lévy measure are no longer tail-equivalent.

Furthermore, classical MRV random vectors with tail equivalent margins are known to be multivariate subexponential in the sense of [52, 56]. In comparison to these definitions of multivariate subexponentiality, in our work, we have been able to extend the collection of tail sets that exhibit the ‘one large jump’ principle.

In the direction of future work, first note that only a few popular copula models have been used to demonstrate our results; clearly, these can be extended to many other useful models depending on the domain of application. Another clear application of this work will be in finding risk contagion in operational risk and ruin probabilities for multivariate risk processes, which is under active investigation by the authors. Finally, given particular types of multivariate tail events, our work could also be applied to finding optimal portfolios for hedging against such extreme events.

Appendix A. Proofs of the results in Section 3

Proof of Lemma 1. The set $(1, \infty)^d \in \mathcal{R}^{(i)}$ and, hence, it is non-empty. Now, let A and B be two arbitrary sets in $\mathcal{R}^{(i)}$. Then for some $m, n \geq i$, with $x_j > 0,\ j \in \{k_1, \ldots, k_m\} =: S_1 \subseteq \mathbb{I}$ and $y_j > 0,\ j \in \{\ell_1, \ldots, \ell_n\} =: S_2 \subseteq \mathbb{I}$ we have

$$A = \left\{z \in \mathbb{R}_+^d : z_j > x_j \text{ for all } j \in S_1\right\}, \quad B = \left\{z \in \mathbb{R}_+^d : z_j > y_j \text{ for all } j \in S_2\right\}.$$

For $j \in S^* := S_1 \cup S_2$, define

$$w_j = \begin{cases} x_j, & \text{if } j \in S_1 \cap {S_2}^c, \\ y_j & \text{if } j \in {S_1}^c \cap S_2, \\ \max\{x_j, y_j\} & \text{if } j \in S_1 \cap S_2. \end{cases}$$

Thus, $A \cap B = \{z \in \mathbb{R}_+^d : z_j > w_j \text{ for all } j \in S^*\}$ where $|S^*| \geq \max(m, n) \geq i$ and $w_j > 0$ for all $j \in S^*$. Hence, $A \cap B \in \mathcal{R}^{(i)}$ and $\mathcal{R}^{(i)}$ is a π-system. It can also be checked that $\sigma(\mathcal{R}^{(i)}) = \mathcal{B}^{(i)}$. ☐

Proof of Proposition 1.

(12) ⇒ (13): Using [47, Theorem 2.1], if (12) holds then any set of the form $A \in \mathcal{R}^{(i)}$ with $\mu(\partial A) = 0$ is clearly bounded away from $\mathbb{CA}_i$ and belongs to the σ-algebra $\mathcal{B}^{(i)}$ as defined in (10). Hence, (13) holds as $t \to \infty$.

(13) ⇒ (12): Now assume (13) holds for all μ-continuity sets $A \in \mathcal{R}^{(i)}$. Denote by M, the collection

$$M = \{\mu_t : t > 0\} \subset \mathbb{M}\big(\mathbb{E}_d^{(i)}\big).$$

For any $r > 0$, let $\nu^{(r)}$ be the restriction of $\nu \in \mathbb{M}(\mathbb{E}_d^{(i)})$ to $\mathbb{E}_d^{(i)} \setminus \mathbb{CA}_i^{(r)}$ where $\mathbb{CA}_i^{(r)} = \{z \in \mathbb{R}_+^d : d(z, \mathbb{CA}_i) < r\}$. Let $M^{(r)} := \{\nu^{(r)} : \nu \in M\}$ and let $\{r_\ell\}$ be a sequence $r_\ell \downarrow 0$ as $\ell \to \infty$. Note that $M^{(r)} \subset \mathbb{M}(\mathbb{E}_d^{(i)} \setminus \mathbb{CA}_i^{(r)})$ which is a class of finite Borel measures.

Denote by $\mathcal{C}_i^{(r)}$ = all real-valued, bounded continuous functions f on $\mathbb{E}_d^{(i)}$ which vanishes on $\mathbb{CA}_i^{(r)}$. Fix $\ell \geq 1$ and pick any $f \in \mathcal{C}_i^{(r_\ell)}$ which is uniformly continuous. Then by definition, the support of f lies on a finite union of rectangular sets given by

$$A = \bigcup_{\substack{S \subset \mathbb{I} \\ |S| = i}} A_S,$$

where $A_S = \{z \in \mathbb{R}_+^d : z_j > r^* \text{ for all } j \in S\} \in \mathcal{R}^{(i)}$ for some $0 < r^* < r_\ell$. Without loss of generality the sets A_S can be assumed to be μ-continuity sets using [47, Lemma 2.5]. Now, using (13) we have convergence on the sets A_S and, therefore,

$$\begin{aligned} \sup_{\nu \in M^{(r_\ell)}} \nu(f) = \sup_t \mu_t^{(r_\ell)}(f) &\leq \sup_{z \in \mathbb{R}_+^d} f(z) \sup_t \mu_t(A) \\ &\leq \sup_{z \in \mathbb{R}_+^d} f(z) \sum_{\substack{S \subset \mathbb{I} \\ |S| = i}} \sup_t \mu_t^{(r_\ell)}(A_S) < \infty. \end{aligned}$$

Hence, for any sequence of measures $\{\nu_n\}_{n \geq 1} \in M^{(r_\ell)}$, the sequence $\nu_n(f)$ has a convergent subsequence. Since this is true for any uniformly continuous $f \in \mathcal{C}_i^{(r_\ell)}$, it is true for any $f \in \mathcal{C}_{i,K}^{(r_\ell)}$, which are compactly supported functions in $\mathcal{C}_i^{(r_\ell)}$. Since $\mathcal{C}_{i,K}^{(r_\ell)}$ is separable, using a countable dense collection $\{f_j\}_{j\geq 1}\in\mathcal{C}_{i,K}^{(r_\ell)}$, and a diagonal argument we can show that any sequence of measures $\{\nu_n\}_{n\geq 1}\in M^{(r_\ell)}$ has a subsequence $\{\nu_{n_k}\}$ such that $\lim_{n_k\to\infty}\nu_{n_k}(g)=\nu(g)$ for any $g\in\mathcal{C}_{i,K}^{(r_\ell)}$ and, hence, for all uniformly continuous functions $f\in\mathcal{C}_i^{(r_\ell)}$ (using a sequence $g_n\to f$ where $g_n\in\mathcal{C}_{i,K}^{(r_\ell)}$). Thus, $M^{(r_\ell)}=\{\mu_t^{(r_\ell)}:t>0\}$ is relatively compact; cf. [55, (3.16), p. 51)]; and this holds for a sequence $\{r_\ell\}$ where $r_\ell\downarrow 0$. In addition, $M^{(r_\ell)}\subset M$. Hence, by [47, Theorem 2.4] we have M is relatively compact.

Suppose as $t\to\infty$, μ_t has two different sequential limits μ_1 and μ_2, then by assumption they clearly agree on all sets $A\in\mathcal{R}^{(i)}$. By Lemma 1 such rectangular sets form a π-system generating the σ-algebra $\mathcal{B}^{(i)}$. Hence, $\mu_1=\mu_2=\mu$ on $\mathbb{E}_d^{(i)}$. □

Proof of Lemma 2. Let $A=\{\boldsymbol{z}\in\mathbb{R}_+^d : z_j>x_j \text{ for all } j\in S\}$ where $S\subseteq\mathbb{I}$, $|S|\geq i$, $x_j>0$ for $j\in S$, and $\mu_i(\partial A)=0$. Furthermore, let $0<\varepsilon<\min_{j\in S}x_j$ and define the sets

$$\begin{aligned}A_\varepsilon^+ &:= \big\{\boldsymbol{z}\in\mathbb{R}_+^d : z_j>x_j-\varepsilon \text{ for all } j\in S\big\},\\ A_\varepsilon^- &:= \big\{\boldsymbol{z}\in\mathbb{R}_+^d : z_j>x_j+\varepsilon \text{ for all } j\in S\big\},\\ N_\varepsilon &:= \big\{\boldsymbol{z}\in\mathbb{R}_+^d : |z_j|\leq\varepsilon \text{ for all } j\in S\big\}.\end{aligned}$$

Suppose without loss of generality $\mu_i(\partial A_\varepsilon^+)=\mu_i(\partial A_\varepsilon^-)=0$ (otherwise choose ε appropriate). On the one hand,

$$\begin{aligned}\mathbb{P}(\boldsymbol{X}+\boldsymbol{Y}\in tA) &\leq \mathbb{P}\big(\boldsymbol{X}\in tA_\varepsilon^+\big)+\mathbb{P}\big(\boldsymbol{Y}\in tN_\varepsilon^c\big)\\ &\leq \mathbb{P}\big(\boldsymbol{X}\in tA_\varepsilon^+\big)+\mathbb{P}\big(\|\boldsymbol{Y}\|_\infty>\varepsilon t\big).\end{aligned}$$

Hence,

$$\begin{aligned}\limsup_{t\to\infty} b_i^{\leftarrow}(t)\mathbb{P}(\boldsymbol{X}+\boldsymbol{Y}\in tA) &\leq \limsup_{t\to\infty} b_i^{\leftarrow}(t)\mathbb{P}(\boldsymbol{X}\in tA_\varepsilon^+)+\limsup_{t\to\infty} b_i^{\leftarrow}(t)\mathbb{P}(\|\boldsymbol{Y}\|_\infty>\varepsilon t)\\ &\leq \mu_i(A_\varepsilon^+)+\limsup_{t\to\infty} b_i^{\leftarrow}(t)(\gamma t)^{-(\alpha_i+\gamma)}\mathbb{E}\|\boldsymbol{Y}\|^{\alpha_i+\gamma}\\ &= \mu_i(A_\varepsilon^+)\downarrow\mu_i(A) \quad \text{as } \varepsilon\downarrow 0, \end{aligned}\tag{A1}$$

since $\mu_i(\partial A)=0$. On the other hand,

$$\begin{aligned}\mathbb{P}(\boldsymbol{X}+\boldsymbol{Y}\in tA) &\geq \mathbb{P}(\boldsymbol{X}+\boldsymbol{Y}\in tA,\,\boldsymbol{Y}\in tN_\varepsilon)\\ &\geq \mathbb{P}(\boldsymbol{X}\in tA_\varepsilon^-,\,\boldsymbol{Y}\in tN_\varepsilon)\\ &= \mathbb{P}(\boldsymbol{X}\in tA_\varepsilon^-)\mathbb{P}(\boldsymbol{Y}\in tN_\varepsilon).\end{aligned}$$

Therefore,

$$\begin{aligned}\liminf_{t\to\infty} b_i^{\leftarrow}(t)\mathbb{P}(\boldsymbol{X}+\boldsymbol{Y}\in tA) &\geq \limsup_{t\to\infty} b_i^{\leftarrow}(t)\,\mathbb{P}(\boldsymbol{X}\in tA_\varepsilon^-)\mathbb{P}(\boldsymbol{Y}\in tN_\varepsilon)\\ &= \mu_i(A_\varepsilon^-)\uparrow\mu_i(A) \quad \text{as } \varepsilon\downarrow 0,\end{aligned}\tag{A2}$$

since $\mu_i(\partial A)=0$. Thus, (A1) and (A2) imply that

$$\lim_{t\to\infty} b_i^{\leftarrow}(t)\mathbb{P}(\boldsymbol{X}+\boldsymbol{Y}\in tA)=\mu_i(A),$$

and using Proposition 1 we can conclude the statement. □

Appendix B. Proofs of the results in Section 4

The following auxiliary lemmas are used to prove Theorem 1.

Lemma 3. *Let the assumptions of Theorem 1 hold. Then for any* $m = 0, \ldots, i-1$*:*

$$\lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t)\, \mathbb{P}\big(Z_{(m+1)}^{(1)} > t\big)\, \mathbb{P}\big(Z_{(i-m)}^{(2)} > t\big) = 0 \tag{B1}$$

$$\lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t)\, \mathbb{P}\big(Z_{(m+1)}^{(2)} > t\big)\, \mathbb{P}\big(Z_{(i-m)}^{(1)} > t\big) = 0. \tag{B2}$$

Proof. Let $\Gamma^{(k)} := \arg\max_i\{\alpha_i^{(k)} < \infty\}$, $k = 1, 2$. By definition, for $i \le \Gamma^{(k)}$, we have $\mathbb{P}(Z_{(i)}^{(k)} > t) = O(1/b_i^{(k)\leftarrow}(t))$; and for $i > \Gamma^{(k)}$, $\mathbb{P}(Z_{(i)}^{(k)} > t) = o(1/b_i^{(k)\leftarrow}(t))$. Hence, to prove (B1), we need only to show that

$$a_m := \limsup_{t\to\infty} \frac{b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t)}{b_{m+1}^{(1)\leftarrow}(t) b_{i-m}^{(2)\leftarrow}(t)} = 0, \qquad m = 0, \ldots, i-1.$$

For $m = 0$ we have

$$a_0 = c_0^{I(i)} \lim_{t\to\infty} 1/b_1^{(1)\leftarrow}(t) = 0.$$

Let $m \in \{1, \ldots, i-1\}$. Note that

$$a_m = c_m^{I(i)} \limsup_{t\to\infty} \frac{b_m^{(1)\leftarrow}(t)}{b_{m+1}^{(1)\leftarrow}(t)}.$$

Since $\limsup_{t\to\infty} b_m^{(1)\leftarrow}(t)/b_{m+1}^{(1)\leftarrow}(t) < \infty$, the last equality implies that $a_m = 0$ is only possible if either $c_m^{I(i)} = 0$ or $\limsup_{t\to\infty} b_m^{(1)\leftarrow}(t)/b_{m+1}^{(1)\leftarrow}(t) = 0$, which holds true by the assumptions in Theorem 1. The proof of (B2) is analogous.

Lemma 4. *Let the assumptions of Theorem 1 hold and*

$$A = \big\{z \in \mathbb{R}_+^d : z_j > x_j \text{ for all } j \in S\big\}$$

be a rectangular set with $S \subseteq \mathbb{I}$, $|S| = i$, $x_j > 0$ *for* $j \in S$ *and* $\mu_i^{\oplus}(\partial A) = 0$. *Then*

$$\lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) \mathbb{P}(\mathbf{Z}^{(1)} + \mathbf{Z}^{(2)} \in tA) = \mu_i^{\oplus}(A). \tag{B3}$$

Proof.

Step 1. First, we derive an upper bound for the left-hand side of (B3). Let $0 < \epsilon < 1$ such that $\epsilon x_j < (1-\epsilon)x_l$ for all $j, l \in S$. Then

$$\begin{aligned}
\mathbb{P}(\mathbf{Z}^{(1)} + \mathbf{Z}^{(2)} \in tA) = &\Bigg[\sum_{\substack{J_2 \subseteq S \cup \{\varnothing\} \\ J_1 = S\setminus J_2}} + \sum_{\substack{J_2 \subseteq S \cup \{\varnothing\} \\ J_1 \subsetneq S\setminus J_2 \cup \{\varnothing\}}}\Bigg] \mathbb{P}\Bigg(\bigcap_{j\in S}\big\{Z_j^{(1)} + Z_j^{(2)} > tx_j\big\} \\
&\cap \bigcap_{j\in J_1}\big\{Z_j^{(1)} \le t\epsilon x_j\big\} \cap \bigcap_{j\in S\setminus J_1}\big\{Z_j^{(1)} > t(1-\epsilon)x_j\big\} \\
&\cap \bigcap_{j\in J_2}\big\{Z_j^{(2)} \le t\epsilon x_j\big\} \cap \bigcap_{j\in S\setminus J_2}\big\{Z_j^{(2)} > t(1-\epsilon)x_j\big\}\Bigg) \\
=: &\, M_1(t,\epsilon) + M_2(t,\epsilon).
\end{aligned} \tag{B4}$$

Note that in the case $J_1 \cap J_2 \neq \varnothing$ these probabilities are zero since it results in computing probabilities of empty sets. Next, we find an upper bound for $M_1(t, \epsilon)$. Note that

$$\begin{aligned} M_1(t,\epsilon) &\leq \sum_{\substack{J_2 \subseteq S \cup \{\varnothing\} \\ J_1 = S \setminus J_2}} \mathbb{P}\left(\bigcap_{j \in S \setminus J_1} \{Z_j^{(1)} > t(1-\epsilon)x_j\} \cap \bigcap_{j \in S \setminus J_2} \{Z_j^{(2)} > t(1-\epsilon)x_j\} \right) \\ &= \sum_{J_2 \subseteq S \cup \{\varnothing\}} \mathbb{P}\left(\bigcap_{j \in J_2} \{Z_j^{(1)} > t(1-\epsilon)x_j\} \right) \mathbb{P}\left(\bigcap_{j \in S \setminus J_2} \{Z_j^{(2)} > t(1-\epsilon)x_j\} \right). \end{aligned}$$

Let $A_\epsilon := \{z \in \mathbb{R}_+^d : z_j > (1-\epsilon)x_j \text{ for all } j \in S\}$ and choose $\epsilon > 0$ such that $\mu_i^{\oplus}(\partial A_\epsilon) = 0$. Then

$$\limsup_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) M_1(t,\epsilon) \leq \mu_i^{\oplus}(A_\epsilon). \tag{B5}$$

Define $x^* := \min_{j \in S} x_j$. Following a similar argument for $M_2(t, \epsilon)$ we get the upper bound

$$\begin{aligned} M_2(t,\epsilon) &\leq \sum_{\substack{J_2 \subseteq S \cup \{\varnothing\} \\ J_1 \subsetneq S \setminus J_2 \cup \{\varnothing\}}} \mathbb{P}\left(\bigcap_{j \in J_2 \cup S \setminus J_1} \{Z_j^{(1)} > t(1-\epsilon)x_j\} \right) \mathbb{P}\left(\bigcap_{j \in J_1 \cup S \setminus J_2} \{Z_j^{(2)} > t(1-\epsilon)x_j\} \right) \\ &\leq \sum_{\substack{J_2 \subseteq S \cup \{\varnothing\} \\ J_1 \subsetneq S \setminus J_2 \cup \{\varnothing\}}} \mathbb{P}\big(Z^{(1)}_{(|J_2 \cup S \setminus (J_1 \cup J_2)|)} > t(1-\epsilon)x^*\big)\, \mathbb{P}\big(Z^{(2)}_{(|S \setminus J_2|)} > t(1-\epsilon)x^*\big) \\ &\leq \sum_{m=0}^{i-1} \sum_{l=1}^{i-m} \sum_{\substack{J_2 \subseteq S \cup \{\varnothing\} \\ |J_2| = m}} \sum_{\substack{J_1 \subsetneq S \setminus J_2 \cup \{\varnothing\} \\ |S \setminus (J_1 \cup J_2)| = l}} \mathbb{P}\big(Z^{(1)}_{(m+l)} > t(1-\epsilon)x^*\big)\, \mathbb{P}\big(Z^{(2)}_{(i-m)} > t(1-\epsilon)x^*\big). \end{aligned}$$

Finally, an application of Lemma 3 yields

$$\begin{aligned} &\limsup_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) M_2(t,\epsilon) \\ &\quad \leq \sum_{m=0}^{i-1} \sum_{l=1}^{i-m} \sum_{\substack{J_2 \subseteq S \cup \{\varnothing\} \\ |J_2| = m}} \sum_{\substack{J_1 \subsetneq S \setminus J_2 \cup \{\varnothing\} \\ |S \setminus (J_1 \cup J_2)| = l}} \limsup_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) \\ &\qquad \times \mathbb{P}\big(Z^{(1)}_{(m+1)} > t(1-\epsilon)x^*\big)\, \mathbb{P}\big(Z^{(2)}_{(i-m)} > t(1-\epsilon)x^*\big) = 0. \end{aligned} \tag{B6}$$

Now from (B4), (B5), and (B6) we have

$$\limsup_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t)\, \mathbb{P}(\mathbf{Z}^{(1)} + \mathbf{Z}^{(2)} \in tA) \leq \mu_i^{\oplus}(A_\epsilon) \downarrow \mu_i^{\oplus}(A) \quad \text{as } \epsilon \downarrow 0,$$

where in the last step we use the fact that $\mu_i^{\oplus}(\partial A) = 0$.

Step 2. Next, we derive a lower bound for the asymptotic limit. There are a total of $2^{|S|}$ subsets of S which we order as $J(1), \ldots, J(2^{|S|})$ (in any way). Now, define the sets

$$C_l := \bigcap_{j \in J(l)} \{Z_j^{(1)} > tx_j\} \cap \bigcap_{j \in S \setminus J(l)} \{Z_j^{(2)} > tx_j\}, \qquad l = 1, \ldots, 2^{|S|}.$$

Then,

$$\{\mathbf{Z}^{(1)} + \mathbf{Z}^{(2)} \in tA\} \supseteq \bigcup_{l=1}^{2^{|S|}} C_l,$$

and using the inclusion–exclusion principle we have

$$\mathbb{P}(\mathbf{Z}^{(1)}+\mathbf{Z}^{(2)}\in tA)\geq \mathbb{P}\left(\bigcup_{l=1}^{2^{|S|}} C_l\right)\geq \sum_{l=1}^{2^{|S|}}\mathbb{P}(C_l)-\sum_{1\leq l_1<l_2\leq 2^{|S|}}\mathbb{P}(C_{l_1}\cap C_{l_2}). \tag{B7}$$

Now, on the one hand,

$$\lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t)b_{i-I(i)}^{(2)\leftarrow}(t)\sum_{l=1}^{2^{|S|}}\mathbb{P}(C_l)=\mu_i^{\oplus}(A), \tag{B8}$$

and on the other hand, for any $1\leq l_1<l_2\leq 2^{|S|}$ the inequality

$$0\leq \mathbb{P}(C_{l_1}\cap C_{l_2})\leq \mathbb{P}\big(Z^{(1)}_{(|J(l_1)\cup J(l_2)|)}>tx^*\big)\mathbb{P}\big(Z^{(2)}_{(|S\setminus(J(l_1)\cap J(l_2))|)}>tx^*\big)$$

holds. Define $m:=|J(l_1)\cap J(l_2)|$. Since $J(l_1)\neq J(l_2)$ we have $|J(l_1)\cup J(l_2)|\geq |J(l_1)\cap J(l_2)|+1=m+1$. Hence, a conclusion of Lemma 3 is that

$$\begin{aligned}&\limsup_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t)b_{i-I(i)}^{(2)\leftarrow}(t)\,\mathbb{P}(C_{l_1}\cap C_{l_2})\\ &\quad\leq \limsup_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t)b_{i-I(i)}^{(2)\leftarrow}(t)\mathbb{P}\big(Z^{(1)}_{(m+1)}>tx^*\big)\mathbb{P}\big(Z^{(2)}_{(i-m)}>tx^*\big)=0\end{aligned} \tag{B9}$$

for any $1\leq l_1<l_2\leq 2^{|S|}$. Then (B7), (B8), and (B9) result in the lower bound

$$\liminf_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t)b_{i-I(i)}^{(2)\leftarrow}(t)\,\mathbb{P}(\mathbf{Z}^{(1)}+\mathbf{Z}^{(2)}\in tA)\geq \mu_i^{\oplus}(A),$$

and together with the upper bound in Step 1 the lemma is proven. ☐

Lemma 5. *Let the assumptions of Theorem 1 hold and*

$$A=\left\{z\in\mathbb{R}_+^d: z_j>x_j \text{ for all } j\in S\right\}$$

be a rectangular set with $S\subseteq\mathbb{I}$, $|S|>i$, *and* $x_j>0$ *for* $j\in S$ *where* $\mu_i^{\oplus}(\partial A)=0$. *Then*

$$\mu_i^{\oplus}(A)=c_i^{I(i)}\mu_i^{(1)}(A)+c_0^{I(i)}\mu_i^{(2)}(A).$$

Proof. Since

$$\mu_i^{\oplus}(A)=\sum_{m=0}^{i}c_m^{I(i)}\mu_{m,i}^*(A)$$

we have to show that $\sum_{m=1}^{i-1}c_m^{I(i)}\mu_{m,i}^*(A)=0$. But

$$\begin{aligned}0&\leq \sum_{m=1}^{i-1}c_m^{I(i)}\mu_{m,i}^*(A)\\ &=\lim_{t\to\infty}\sum_{m=1}^{i-1}b_{I(i)}^{(1)\leftarrow}(t)b_{i-I(i)}^{(2)\leftarrow}(t)\sum_{\substack{J\subseteq S\cup\{\varnothing\}\\ |J|=m}}\mathbb{P}\left(\bigcap_{j\in J}\{Z_j^{(1)}>tx_j\}\right)\mathbb{P}\left(\bigcap_{j\in S\setminus J}\{Z_j^{(2)}>tx_j\}\right)\\ &\leq 2^i\lim_{t\to\infty}\sum_{m=1}^{i-1}b_{I(i)}^{(1)\leftarrow}(t)b_{i-I(i)}^{(2)\leftarrow}(t)\mathbb{P}\Big(Z^{(1)}_{(m)}>t\min_{j\in S}x_j\Big)\mathbb{P}\Big(Z^{(2)}_{|S|-m}>t\min_{j\in S}x_j\Big)\\ &\leq 2^i\sum_{m=1}^{i-1}\lim_{t\to\infty}b_{I(i)}^{(1)\leftarrow}(t)b_{i-I(i)}^{(2)\leftarrow}(t)\mathbb{P}\Big(Z^{(1)}_{((m-1)+1)}>t\min_{j\in S}x_j\Big)\mathbb{P}\Big(Z^{(2)}_{i-(m-1)}>t\min_{j\in S}x_j\Big).\end{aligned}$$

The right-hand side is equal to zero due to Lemma 3. ☐

Proof of Theorem 1. Due to Proposition 1 it is sufficient to study the convergence on the rectangular sets $A = \{z \in \mathbb{R}_+^d : z_j > x_j \text{ for all } j \in S\}$ where $S \subseteq \mathbb{I}$, $|S| \geq i$, and $x_j > 0$ for $j \in S$ with $\mu_i^{\oplus}(\partial A) = 0$. If $|S| = i$, a consequence of Lemma 4 is that

$$\lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) \mathbb{P}(\mathbf{Z}^{(1)} + \mathbf{Z}^{(2)} \in tA) = \mu_i^{\oplus}(A).$$

If $|S| > i$ then $i \leq d-1$. Thus, using Lemma 3 and similar elaborate calculations as in the proof of Lemma 4 (cf. the proof of Lemma 5) we can show that

$$\begin{aligned}
&\lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) \mathbb{P}(\mathbf{Z}^{(1)} + \mathbf{Z}^{(2)} \in tA) \\
&\quad = \lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) \mathbb{P}(\mathbf{Z}^{(1)} \in tA) + \lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) \mathbb{P}(\mathbf{Z}^{(2)} \in tA) \\
&\quad = c_i^{I(i)} \mu_{|S|}^{(1)}(A) \lim_{t\to\infty} \frac{b_i^{(1)\leftarrow}(t)}{b_{|S|}^{(1)\leftarrow}(t)} + c_0^{I(i)} \mu_{|S|}^{(2)}(A) \lim_{t\to\infty} \frac{b_i^{(2)\leftarrow}(t)}{b_{|S|}^{(2)\leftarrow}(t)}.
\end{aligned}$$

In case $\lim_{t\to\infty} b_i^{(1)\leftarrow}(t)/b_{|S|}^{(1)\leftarrow}(t) = 0$ we have $\mu_i^{(1)}(A) = 0$. Otherwise,

$$\mu_{|S|}^{(1)}(A) \lim_{t\to\infty} \frac{b_i^{(1)\leftarrow}(t)}{b_{|S|}^{(1)\leftarrow}(t)} = \mu_i^{(1)}(A).$$

In summary,

$$\lim_{t\to\infty} b_{I(i)}^{(1)\leftarrow}(t) b_{i-I(i)}^{(2)\leftarrow}(t) \mathbb{P}(\mathbf{Z}^{(1)} + \mathbf{Z}^{(2)} \in tA) = c_i^{I(i)} \mu_i^{(1)}(A) + c_0^{I(i)} \mu_i^{(2)}(A) = \mu_i^{\oplus}(A)$$

where the final equality is due to Lemma 5. ☐

Proof of Proposition 2. Note that $\alpha_i = i\alpha$ is immediate from $b_i(t) \in \mathcal{RV}_{1/(i\alpha)}$. Using Theorem 1, it is sufficient to prove the statements for $n = 2$ and the rest follows by induction (which is direct and omitted here). Using the notation of Theorem 1, for any $i = 1, \ldots, d$, we have $I(i) = i$ and $c_m^{I(i)} = 1$ and $\lim_{t\to\infty} b_m^{\leftarrow}(t)/b_{m+1}^{\leftarrow}(t) = 0$ for $m = 0, \ldots, i$ and for any $i = 1, \ldots, d$. Thus,

$$\mathbf{Z}^{(1)} + \mathbf{Z}^{(2)} \in \mathcal{MRV}\big(i\alpha, b_i, \mathbb{E}_d^{(i)}\big)$$

and (14) follows by induction. Now if (15) is satisfied then, for $n = 2$, using the notation of Theorem 1, we have $\mu_{m,i}^*(\cdot) = \binom{i}{m} \mu_i(\cdot)$ and, hence, for any $A \in \mathcal{B}(\mathbb{E}_d^{(i)})$ with $\mu_i(\partial A) = 0$ we get

$$\mu_i^{\oplus}(A) = \sum_{m=0}^{i} \binom{i}{m} \mu_i(A) = 2^i \mu_i(A)$$

implying that

$$\mathbf{Z}^{(1)} + \mathbf{Z}^{(2)} \in \mathcal{MRV}\big(i\alpha, b_i, 2^i \mu_i, \mathbb{E}_d^{(i)}\big).$$

Now (15) follows by induction using Theorem 1. ☐

Proof of Proposition 3. Consider $n = 2$ and the notation of Theorem 1. Fix some $i \in \{1, \ldots, d\}$. We have $I(i) = i$, $c_m^{I(i)} = 0$, $m = 1, \ldots, i-1$ and $c_0^{I(i)} = c_i^{I(i)} = 1$. Clearly, $\mu_{0,i}^* = \mu_{i,i}^* = \mu_i$ and, hence, $\mu_i^{\oplus} = \mu_{0,i}^* + \mu_{i,i}^* = 2\mu_i$. Now by Theorem 1, we get

$$\mathbf{Z}^{(1)} + \mathbf{Z}^{(2)} \in \mathcal{MRV}\big(\alpha_i, b_i, 2\mu_i, \mathbb{E}_d^{(i)}\big).$$

The final result can now be derived using induction (which we skip here). ☐

Proof of Corollary 1. Since $\alpha_i = \alpha < \alpha + \alpha = \alpha_m + \alpha_{i-m}$, this holds as a direct consequence of Proposition 3. $\square$

Proof of Proposition 4. By Definition 3, $\Delta = 1$, $b_1(t) \in \mathcal{RV}_{1/\alpha}$ and for some fixed $\gamma_0 > 0$, $b_i(t) = t^{1/(i(\alpha+\gamma_0))}$, $i = 2, \ldots, d$. Then $b_1^{\leftarrow}(t) \in \mathcal{RV}_\alpha$ and let $b_1^{\leftarrow}(t) = t^\alpha \ell(t)$ where $\ell(t)$ is some slowly varying function. Also define $S^{(n)} := \sum_{k=1}^n \mathbf{Z}^{(k)}$. We prove the statement by induction.

(1) Consider $i = 1$. By definition, (16) holds for $n = i = 1$. Using classical MRV results [46, Theorem 1.30], [34, Example 3.2], we obtain $S^{(n)} \in \mathcal{MRV}(\alpha, b_1, n\mu_1, \mathbb{E}_d^{(1)})$. Thus (16) holds for all $n \ge i = 1$; the form of the limit measure is (17) with $|S| = i = 1$. For $n = 2$, $\mu^{\oplus}_{1,2}(\mathbb{E}_d^{(2)}) = 0$ follows using Theorem 1 and then $\mu^{\oplus}_{1,n}(\mathbb{E}_d^{(n)}) = 0$ for $n \ge 2$ follows using induction.

(2) Fix $i_0 \in \{2, \ldots, d\}$. By way of induction, assume that for any $i \in \{1, \ldots, i_0 - 1\}$, (16) holds for all $n \ge i$ and (18) holds for $n < i$ with $\gamma_{i_0-1} = \gamma_0/(i_0-1)!$. Note that the induction base case holds for $i_0 = 2$. Moreover, for $n = 1$, (18) holds for $i = 2, \ldots, d$ with γ_0 by the adapted-MRV assumption on $\mathbf{Z}^{(1)}$. First, in part (i) we show that for $i = i_0$, (18) holds for $n < i = i_0$ with $\gamma_{i_0} = \gamma_0/i_0!$. Hence, this will imply that (18) will hold with γ_{i_0} for any $i \in \{2, \ldots, i_0\}$ and $n < i_0$. Then, we show (16) holds for $n = i = i_0$ in part (ii) and for all $n > i = i_0$ in part (iii) assuming (18) holds with γ_{i_0} for $n < i = i_0$.

(i) Here in addition to the induction assumption of (2), assume that for $i = i_0$, (18) holds for all $n = 1, \ldots, n_0 - 1$ where $n_0 < i_0$. If $i_0 = 2$ then the only choice of n is $n = n_0 = 1$ and (18) holds since $S^{(n_0)} = \mathbf{Z}^{(1)} \in \mathcal{NC}(t^{1/(2(\alpha+\gamma_0))}, \mathbb{E}_d^{(2)})$. Thus, assume $i_0 \ge 3$. Note that $S^{(n_0)} = S^{(n_0-1)} + \mathbf{Z}^{(n_0)}$. Now by the adapted-MRV assumption on $\mathbf{Z}^{(k)}$, we have $\mathbf{Z}^{(n_0)} \in \mathcal{NC}(t^{1/(i(\alpha+\gamma_0))}, \mathbb{E}_d^{(i)})$ and, hence, $\mathbf{Z}^{(n_0)} \in \mathcal{NC}(t^{1/(i(\alpha+\gamma_{i_0-1}))}, \mathbb{E}_d^{(i)})$ for $i = 2, \ldots, d$ since $\gamma_{i_0-1} \le \gamma_0$; thus we can and do choose $b_i(t) = t^{1/(i(\alpha+\gamma_{i_0-1}))}$. Using the notation of Theorem 1, for $j = 0, \ldots, n_0 - 2$,

$$\begin{aligned}
\bar{c}^{(i_0)}_{j,n_0} &:= \max_{0\le m\le i_0}\left\{\limsup_{t\to\infty}\frac{b^{\leftarrow}_{j,n_0-1}(t)\,b^{\leftarrow}_{i_0-j}(t)}{b^{\leftarrow}_{m,n_0-1}(t)\,b^{\leftarrow}_{i_0-m}(t)}\right\}\\
&\ge \limsup_{t\to\infty}\frac{b^{\leftarrow}_{j,n_0-1}(t)\,b^{\leftarrow}_{i_0-j}(t)}{b^{\leftarrow}_{n_0-1,n_0-1}(t)\,b^{\leftarrow}_{i_0-(n_0-1)}(t)}\\
&= \limsup_{t\to\infty}\frac{(t^\alpha\ell(t))^j\cdot t^{(i_0-j)(\alpha+\gamma_{i_0-1})}}{(t^\alpha\ell(t))^{n_0-1}\cdot t^{(i_0-(n_0-1))(\alpha+\gamma_{i_0-1})}} = \infty,
\end{aligned}$$

since $b_{j,n_0-1}(t) = (b_1^{\leftarrow}(t))^{n_0-1}$ for $j = 0, \ldots, n_0 - 1$ from (2). Similarly, for $j = n_0, \ldots, i_0$ we obtain

$$\begin{aligned}
\bar{c}^{(i_0)}_{j,n_0} &= \max_{0\le m\le i_0}\left\{\limsup_{t\to\infty}\frac{b^{\leftarrow}_{j,n_0-1}(t)\,b^{\leftarrow}_{i_0-j}(t)}{b^{\leftarrow}_{m,n_0-1}(t)\,b^{\leftarrow}_{i_0-m}(t)}\right\}\\
&\ge \limsup_{t\to\infty}\frac{b^{\leftarrow}_{j,n_0-1}(t)\,b^{\leftarrow}_{i_0-j}(t)}{b^{\leftarrow}_{n_0-1,n_0-1}(t)\,b^{\leftarrow}_{i_0-(n_0-1)}(t)}\\
&= \limsup_{t\to\infty}\frac{t^{j(\alpha+\gamma_{i_0-1})}\cdot t^{(i_0-j)(\alpha+\gamma_{i_0-1})}}{(t^\alpha\ell(t))^{n_0-1}\cdot t^{(i_0-(n_0-1))(\alpha+\gamma_{i_0-1})}} = \infty.
\end{aligned}$$

Due to the calculations above we can also conclude that $\bar{c}^{(i_0)}_{n_0-1,n_0} = 1$. Hence, $I(i_0) = n_0 - 1$ with

$$c^{I(i_0)}_{m,n_0} = \begin{cases} 1, & m = n_0 - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, by Theorem 1, $S^{(n_0)} \in \mathcal{MRV}^*(\alpha_{i_0,n_0}, \widetilde{b}_{i_0,n_0}, \mu^{\oplus}_{i_0,n_0}, \mathbb{E}^{(i_0)}_d)$ where

$$\begin{aligned} \alpha_{i_0,n_0} &= \alpha_{n_0-1,n_0-1} + \alpha_{i_0-(n_0-1)} = (n_0-1)\alpha + \infty = \infty, \qquad \text{(recall } i_0 \geq 3\text{)}, \\ \widetilde{b}^{\leftarrow}_{i_0,n_0}(t) &= (t^{\alpha}\ell(t))^{n_0-1} \cdot t^{(i_0-(n_0-1))(\alpha+\gamma_{i_0-1})}, \end{aligned}$$

and $\mu^{\oplus}_{i_0,n_0} \equiv 0$. Defining $b^{\leftarrow}_{i_0,n_0}(t) = t^{i_0(\alpha+\gamma_{i_0})}$, we have $\widetilde{b}_{i_0,n_0}(t) = o(b_{i_0,n_0}(t))$ as $t \to \infty$, since

$$\begin{aligned} i_0(\alpha + \gamma_{i_0}) &= i_0(\alpha + \gamma_0/i_0!) \\ &< i_0\left(\alpha + \frac{i_0 - n_0}{i_0}\frac{\gamma_0}{(i_0-1)!}\right) = i_0\alpha + (i_0 - (n_0-1)\gamma_{i_0-1}) \end{aligned}$$

and, hence, $S^{(n_0)} \in \mathcal{NC}(b_{i_0,n_0}(t), \mathbb{E}^{(i_0)}_d)$. Here we have shown that (18) holds for $i = i_0$ and $n = n_0$. Therefore, by induction, (18) holds for all $n < i_0$.

(ii) Now, we show (16) holds for $n = n_0 = i_0$. Note that from 2(i) we can choose $b_i(t) = t^{1/(i(\alpha+\gamma_{i_0}))}$ as $t \to \infty$ for $i = 2, \ldots, d$. Then for $j = 0, \ldots, i_0 - 2$,

$$\begin{aligned} \bar{c}^{(i_0)}_{j,n_0} = \bar{c}^{(i_0)}_{j,i_0} &:= \max_{0\leq m\leq i_0}\left\{\limsup_{t\to\infty}\frac{b^{\leftarrow}_{j,i_0-1}(t)b^{\leftarrow}_{i_0-j}(t)}{b^{\leftarrow}_{m,i_0-1}(t)b^{\leftarrow}_{i_0-m}(t)}\right\} \\ &\geq \limsup_{t\to\infty}\frac{b^{\leftarrow}_{j,i_0-1}(t)b^{\leftarrow}_{i_0-j}(t)}{b^{\leftarrow}_{i_0-1,i_0-1}(t)b^{\leftarrow}_{1}(t)} \\ &= \limsup_{t\to\infty}\frac{(t^{\alpha}\ell(t))^{j}\cdot t^{(i_0-j)(\alpha+\gamma_{i_0})}}{(t^{\alpha}\ell(t))^{(i_0-1)}\cdot t^{\alpha}\ell(t)} = \infty, \end{aligned}$$

since by the induction assumption $b^{\leftarrow}_{j,n_0-1}(t) = b^{\leftarrow}_{j,i_0-1}(t) = (b^{\leftarrow}_1(t))^j$ for all $j \leq n_0 - 1$. Similarly for $j = i_0$, we have

$$\bar{c}_{i_0,n_0} = \bar{c}_{i_0,i_0} = \infty$$

since we have $b^{\leftarrow}_{i_0,i_0-1}(t) = t^{i_0(\alpha+\gamma_{i_0})}$ from part 2(i) of the proof. For $j = i_0 - 1$,

$$\begin{aligned} \bar{c}^{(i_0)}_{i_0-1,n_0} = \bar{c}^{(i_0)}_{i_0-1,i_0} &:= \max_{0\leq m\leq i_0}\left\{\limsup_{t\to\infty}\frac{b^{\leftarrow}_{i_0-1,i_0-1}(t)b^{\leftarrow}_{1}(t)}{b^{\leftarrow}_{m,i_0-1}(t)b^{\leftarrow}_{i_0-m}(t)}\right\} \\ &= \max_{0\leq m\leq i_0}\left\{\limsup_{t\to\infty}\frac{(t^{\alpha}\ell(t))^{(i_0-1)}\cdot t^{\alpha}\ell(t)}{b^{\leftarrow}_{m,i_0-1}(t)b^{\leftarrow}_{i_0-m}(t)}\right\} = 1. \end{aligned}$$

Finally, for $j = i_0$,

$$\begin{aligned} \bar{c}^{(i_0)}_{i_0,n_0} = \bar{c}^{(i_0)}_{i_0,i_0} &:= \max_{0\leq m\leq i_0}\left\{\limsup_{t\to\infty}\frac{b^{\leftarrow}_{i_0,i_0-1}(t)b^{\leftarrow}_{0}(t)}{b^{\leftarrow}_{m,i_0-1}(t)b^{\leftarrow}_{i_0-m}(t)}\right\} \\ &\geq \limsup_{t\to\infty}\frac{t^{i_0(\alpha+\gamma_{i_0})}}{b^{\leftarrow}_{i_0-1,i_0-1}(t)b^{\leftarrow}_{1}(t)} = \infty. \end{aligned}$$

Hence, $I(i_0)=i_0-1$ with

$$c^{I(i_0)}_{m,n_0}=c^{I(i_0)}_{m,i_0}=\begin{cases}1, & m=i_0-1,\\ 0, & \text{otherwise.}\end{cases}$$

Therefore, by Theorem 1, $S^{(n_0)}=S^{(i_0)}\in\mathcal{MRV}(\alpha_{i_0,n_0}, b_{i_0,n_0}, \mu^{\oplus}_{i_0,n_0}, \mathbb{E}^{(i_0)}_d)$ where

$$\alpha_{i_0,n_0}=\alpha_{i_0,i_0}=\alpha_{i_0-1,i_0-1}+\alpha_1=(i_0-1)\alpha+\alpha=i_0\alpha,$$

$b^{\leftarrow}_{i_0,n_0}(t)=b^{\leftarrow}_{i_0,i_0}(t)b^{\leftarrow}_1(t)=(t^{\alpha}\ell(t))^{i_0}$ and

$$\mu^{\oplus}_{i_0,n_0}=\mu^{\oplus}_{i_0,i_0}=\sum_{m=0}^{i_0}c^{I(i_0)}_{m,i_0}\mu^{*}_{m,i_0,i_0}=\mu_{i_0-1,i_0,i_0},$$

where for $A=\{z\in\mathbb{R}^d_+ : z_j>x_j \text{ for all } j\in S\}\in\mathcal{R}^{(i_0)}$ with $|S|=i_0$, $x_j>0$ for $j\in S$

$$\begin{aligned}\mu_{i_0-1,i_0,i_0}(A)&=\sum_{\substack{J\subseteq S\\|J|=i_0-1}}\mu^{\oplus}_{i_0-1,i_0-1}\big(\big\{z\in\mathbb{E}^{(i_0-1)}_d : z_j>x_j \text{ for all } j\in J\big\}\big)\\&\qquad\times\mu_1\big(\big\{z\in\mathbb{E}^{(1)}_d : z_j>x_j \text{ for all } j\in S\setminus J\big\}\big)\\&=\sum_{\substack{J\subseteq S\\|J|=i_0-1}}\Big[(i_0-1)!\prod_{j\in J}\mu_1\big(\big\{z\in\mathbb{E}^{(i)}_d : z_j>x_j\big\}\big)\Big]\\&\qquad\times\mu_1\big(\big\{z\in\mathbb{E}^{(i)}_d : z_j>x_j \text{ for all } j\in S\setminus J\big\}\big)\\&=\sum_{j\in S}(i_0-1)!\Big[\prod_{k\in S\setminus\{j\}}\mu_1\big(\big\{z\in\mathbb{E}^{(i)}_d : z_k>x_k\big\}\big)\Big]\mu_1\big(\big\{z\in\mathbb{E}^{(i)}_d : z_j>x_j\big\}\big)\\&=i_0!\prod_{j\in S}\mu_1\big(\big\{z\in\mathbb{E}^{(i)}_d : z_j>x_j\big\}\big).\end{aligned}$$

Hence, (16) holds for $n_0=i_0$.

(iv) Here we show (16) holds for all $n\geq i_0$. By way of induction (additionally) assume that for $i=i_0$, (16) holds for all $n\in\{i_0, i_0+1, \ldots, n_0\}$. We show that then it also holds for $n=n_0+1$. By part 2(ii), we know that it holds for $n_0=i_0$. For $j=0, \ldots, i_0-2$,

$$\begin{aligned}\bar{c}^{(i_0)}_{j,n_0+1}&:=\max_{0\leq m\leq i_0}\left\{\limsup_{t\to\infty}\frac{b^{\leftarrow}_{j,n_0}(t)b^{\leftarrow}_{i_0-j}(t)}{b^{\leftarrow}_{m,n_0}(t)b^{\leftarrow}_{i_0-m}(t)}\right\}\\&\geq\limsup_{t\to\infty}\frac{b^{\leftarrow}_{j,n_0}(t)b^{\leftarrow}_{i_0-j}(t)}{b^{\leftarrow}_{i_0,n_0}(t)b^{\leftarrow}_0(t)}\\&=\limsup_{t\to\infty}\frac{(t^{\alpha}\ell(t))^{j}\cdot t^{(i_0-j)(\alpha+\gamma_{i_0})}}{(t^{\alpha}\ell(t))^{i_0}\cdot 1}=\limsup_{t\to\infty}t^{(i_0-j)\gamma_{i_0}}(\ell(t))^{(j-i_0)}=\infty,\end{aligned}$$

since by the induction assumption $b^{\leftarrow}_{j,n_0}(t)=(b^{\leftarrow}_1(t))^{j}$ for all $j\leq n_0$ and from parts 2(i) and (ii) we may assume $b_i(t)=t^{1/(i(\alpha+\gamma_{i_0}))}$, $i=2, \ldots, d$. By similar arguments we have for

$$\bar{c}^{(i_0)}_{i_0-1,n_0+1}=\bar{c}^{(i_0)}_{i_0,n_0+1}=1.$$

Hence, $I(i_0) = i_0 - 1$, and

$$c^{I(i_0)}_{m,n_0+1} = \begin{cases} 1, & m \in \{i_0 - 1, i_0\}, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore, by Theorem 1, $S^{(n_0+1)} \in \mathcal{MRV}(\alpha_{i_0,n_0+1}, b_{i_0,n_0+1}, \mu^{\oplus}_{i_0,n_0+1}, \mathbb{E}^{(i_0)}_d)$ where

$$\begin{aligned} \alpha_{i_0,n_0+1} &= \alpha_{i_0-1,n_0} + \alpha_1 = (i_0 - 1)\alpha + \alpha = i_0\alpha, \\ b^{\leftarrow}_{i_0,n_0+1}(t) &= b^{\leftarrow}_{i_0-1,n_0}(t) b^{\leftarrow}_1(t) = (t^{\alpha}\ell(t))^{i_0}, \\ \mu^{\oplus}_{i_0,n_0+1} &= \sum_{m=0}^{i_0} c^{I(i_0)}_{m,n_0+1} \mu^{*}_{m,i_0,n_0+1} = \mu_{i_0-1,i_0,n_0+1} + \mu_{i_0,i_0,n_0+1}, \end{aligned}$$

where for $A = \{z \in \mathbb{R}^d_+ : z_j > x_j \text{ for all } j \in S\} \in \mathcal{R}^{(i_0)}$ with $|S| = i_0$, $x_j > 0$ for $j \in S$

$$\begin{aligned} &\mu_{i_0-1,i_0,n_0+1}(A) \\ &= \sum_{\substack{J \subseteq S \\ |J| = i_0 - 1}} \mu^{\oplus}_{i_0-1,n_0}\big(\{z \in \mathbb{E}^{(i_0-1)}_d : z_j > x_j \text{ for all } j \in J\}\big) \mu_1\big(\{z \in \mathbb{E}^{(1)}_d : z_j > x_j \text{ for all } j \in S \setminus J\}\big) \\ &= \sum_{\substack{J \subseteq S \\ |J| = i_0 - 1}} \left[\frac{n_0!}{(n_0 - i_0 + 1)!} \prod_{j \in J} \mu_1\big(\{z \in \mathbb{E}^{(i)}_d : z_j > x_j\}\big)\right] \mu_1\big(\{z \in \mathbb{E}^{(i)}_d : z_j > x_j, j \in S \setminus J\}\big) \\ &= \sum_{j \in S} \frac{n_0!}{(n_0 - i_0 + 1)!} \left[\prod_{k \in S \setminus \{j\}} \mu_1\big(\{z \in \mathbb{E}^{(i)}_d : z_k > x_k\}\big)\right] \mu_1\big(\{z \in \mathbb{E}^{(i)}_d : z_j > x_j\}\big) \\ &= i_0 \cdot \frac{n_0!}{(n_0 - i_0 + 1)!} \prod_{j \in S} \mu_1\big(\{z \in \mathbb{E}^{(i)}_d : z_j > x_j\}\big), \end{aligned}$$

and

$$\begin{aligned} \mu_{i_0,i_0,n_0+1}(A) &= \sum_{\substack{J \subseteq S \\ |J| = i_0}} \mu^{\oplus}_{i_0,n_0}\big(\{z \in \mathbb{E}^{(i_0-1)}_d : z_j > x_j \text{ for all } j \in J\}\big) \\ &= \frac{n_0!}{(n_0 - i_0)!} \prod_{j \in J} \mu_1\big(\{z \in \mathbb{E}^{(i_0)}_d : z_j > x_j\}\big). \end{aligned}$$

The measures $\mu^{\oplus}_{i_0-1,n_0}$, $\mu^{\oplus}_{i_0,n_0}$, μ_1 are obtained from our assumptions and induction hypothesis. Now,

$$\begin{aligned} \mu^{\oplus}_{i_0,n_0+1}(A) &= \mu_{i_0-1,i_0,n_0+1}(A) + \mu_{i_0,i_0,n_0+1}(A) \\ &= \left[i_0 \cdot \frac{n_0!}{(n_0 - i_0 + 1)!} + \frac{n_0!}{(n_0 - i_0)!}\right] \prod_{j \in S} \mu_1\big(\{z \in \mathbb{E}^{(i)}_d : z_j > x_j\}\big) \\ &= \frac{(n_0 + 1)!}{(n_0 + 1 - i_0)!} \prod_{j \in S} \mu_1\big(\{z \in \mathbb{E}^{(i)}_d : z_j > x_j\}\big). \end{aligned}$$

Hence, (16) holds for $i = i_0$ and $n = n_0 + 1$, thus by induction it holds for all $n \geq i_0$. □

Appendix C. Proofs of the results in Section 5

For the proof of Theorem 2, we require some auxiliary results.

Lemma 6. *Let the assumptions of Theorem 2 hold. Define*

$$\mathbf{Z}^{\oplus} := (Z_1^{\oplus}, \dots, Z_d^{\oplus}) := \sum_{k=1}^{d} \mathbf{Z}^{(k)}$$

and denote by $Z_{(1)}^{\oplus} \geq \cdots \geq Z_{(d)}^{\oplus}$ *the order statistics of* $Z_1^{\oplus}, \dots, Z_d^{\oplus}$*. In addition, let* $Z_{(1)}^{(k)} \geq \cdots \geq Z_{(d)}^{(k)})$ *be the order statistics of the elements of* $\mathbf{Z}^{(k)} = (Z_1^{(k)}, \dots, Z_d^{(k)})$ *for any* $k \geq 1$*. Furthermore, for* $n \in \mathbb{N}$ *and* $i = 1, \dots, d$ *define*

$$\alpha_{i,n} := \sup_{S \subseteq \mathbb{I},\, |S| \leq i} \sup\nolimits_{t>0} \frac{\mathbb{P}\Big(\bigcap_{j\in S}\Big\{\sum_{k=1}^{n} Z_j^{(k)} > t\Big\}\Big)}{\mathbb{P}(Z_{(|S|)}^{\oplus} > t)}.$$

Then there exists a finite constant $K_i > 0$ *such that for any* $n \in \mathbb{N}$:

$$\alpha_{i,n+1} \leq K_i^n.$$

For one-dimensional random variables with $i = d = 1$, a stronger result holds: for any $\epsilon > 0$ there exists a constant $K > 0$ such that the left-hand side is bounded by $K(1+\epsilon)^n$ (cf. [24, Lemma 1.3.5]).

Proof. First, we show recursively that for any $i \in \mathbb{I}$ there exists a constant $K_i > 0$ such that $\alpha_{i,n+1} \leq K_i \alpha_{i,n}$ for any $n \geq d$. Let $S \subseteq \mathbb{I}$ with $|S| \leq i$ and $n \geq d$. Then

$$\begin{aligned}
&\mathbb{P}\left(\bigcap_{j\in S}\left\{\sum_{k=1}^{n+1} Z_j^{(k)} > t\right\}\right)\\
&= \sum_{\substack{J\subseteq S\\ J\neq\varnothing}} \mathbb{P}\left(\bigcap_{j\in S}\left\{\sum_{k=1}^{n+1} Z_j^{(k)} > t\right\} \cap \bigcap_{j\in J}\{Z_j^{(n+1)} > t/2\} \cap \bigcap_{j\in S\setminus J}\{Z_j^{(n+1)} \leq t/2\}\right)\\
&\quad + \mathbb{P}\left(\bigcap_{j\in S}\left\{\sum_{k=1}^{n+1} Z_j^{(k)} > t\right\} \cap \bigcap_{j\in S}\{Z_j^{(n+1)} \leq t/2\}\right)\\
&=: J_{n,1}(t,S) + J_{n,2}(t,S).
\end{aligned} \tag{C1}$$

We investigate the two terms separately. First,

$$\begin{aligned}
J_{n,1}(t,S) &\leq \sum_{\substack{J\subseteq S\\ J\neq\varnothing}} \mathbb{P}\left(\bigcap_{j\in S\setminus J}\left\{\sum_{k=1}^{n+1} Z_j^{(k)} > t\right\} \cap \bigcap_{j\in J}\{Z_j^{(n+1)} > t/2\}\right)\\
&\leq \sum_{\substack{J\subseteq S\\ J\neq\varnothing}} \sum_{K\subseteq S\setminus J\cup\{\varnothing\}} \mathbb{P}\left(\bigcap_{j\in S\setminus(J\cup K)}\left\{\sum_{k=1}^{n} Z_j^{(k)} > t/2\right\} \cap \bigcap_{j\in J}\{Z_j^{(n+1)} > t/2\} \cap \bigcap_{j\in K}\{Z_j^{(n+1)} > t/2\}\right)\\
&\leq \sum_{\substack{J\subseteq S\\ J\neq\varnothing}} \sum_{K\subseteq S\setminus J\cup\{\varnothing\}} \mathbb{P}\left(\bigcap_{j\in S\setminus(J\cup K)}\left\{\sum_{k=1}^{n} Z_j^{(k)} > t/2\right\}\right) \mathbb{P}\left(\bigcap_{j\in J\cup K}\{Z_j^{(n+1)} > t/2\}\right).
\end{aligned}$$

Since the set $S\backslash J\cup K$ has at most $i-1$ elements and by definition $\alpha_{i-1,n}\le\alpha_{i,n}$, we have that

$$J_{n,1}(t,S)\le\alpha_{i,n}\sum_{\substack{J\subseteq S\\ J\neq\varnothing}}\ \sum_{K\subseteq S\backslash J\cup\{\varnothing\}}\mathbb{P}(Z^{\oplus}_{(|S\backslash(J\cup K)|)}>t/2)\mathbb{P}\big(Z^{(n+1)}_{(|J\cup K|)}>t/2\big).$$

Now applying (20) and (21) we have

$$\begin{aligned}\sup_{\substack{S\subseteq\mathbb{I}\\ |S|\le i}}\ \sup\nolimits_{t>0}\frac{J_{n,1}(t,S)}{\mathbb{P}\big(Z^{\oplus}_{(|S|)}>t\big)}&\le\alpha_{i,n}\sup_{\substack{S\subseteq\mathbb{I}\\ |S|\le i}}\sum_{\substack{J\subseteq S\\ J\neq\varnothing}}\ \sum_{K\subseteq S\backslash J\cup\{\varnothing\}}C^*C^{**}\\ &\le\alpha_{i,n}2^{2i}C^*C^{**}=\alpha_{i,n}\widetilde{K}_i\end{aligned}\tag{C2}$$

with $\widetilde{K}_i:=2^{2i}C^*C^{**}$. Next, for the second term in (C1) and $S=\{j_1,\ldots,j_i\}$ we have

$$\begin{aligned}&\sup_{\substack{S\subseteq\mathbb{I}\\ |S|\le i}}\ \sup\nolimits_{t>0}\frac{J_{n,2}(t,S)}{\mathbb{P}\big(Z^{\oplus}_{|S|}>t\big)}\\ &\quad=\sup_{\substack{S\subseteq\mathbb{I}\\ |S|\le i}}\ \sup\nolimits_{t>0}\int_0^{t/2}\cdots\int_0^{t/2}\frac{\mathbb{P}\big(\bigcap_{j\in S}\big\{\sum_{k=1}^n Z_j^{(k)}>t-y_j\big\}\big)}{\mathbb{P}\big(Z^{\oplus}_{|S|}>t/2\big)}\\ &\qquad\cdot\frac{\mathbb{P}\big(Z^{\oplus}_{|S|}>t/2\big)}{\mathbb{P}\big(Z^{\oplus}_{|S|}>t\big)}F_{Z_{j_1},\ldots,Z_{j_i}}(\mathrm{d}y_1,\ldots,\mathrm{d}y_i)\\ &\quad\le C^{**}\sup_{\substack{S\subseteq\mathbb{I}\\ |S|\le i}}\ \sup\nolimits_{t>0}\int_0^{t/2}\cdots\int_0^{t/2}\frac{\mathbb{P}\big(\bigcap_{j\in S}\big\{\sum_{k=1}^n Z_j^{(k)}>t/2\big\}\big)}{\mathbb{P}\big(Z^{\oplus}_{(|S|)}>t/2\big)}F_{Z_{j_1},\ldots,Z_{j_i}}(\mathrm{d}y_1,\ldots,\mathrm{d}y_i),\end{aligned}$$

where we applied (21) once more. Now the last term above is bounded by $C^{**}\alpha_{i,n}$ and, hence, we have

$$\sup_{\substack{S\subseteq\mathbb{I}\\ |S|\le i}}\ \sup\nolimits_{t>0}\frac{J_{n,2}(t,S)}{\mathbb{P}\big(Z^{\oplus}_{|S|}>t\big)}\le C^{**}\alpha_{i,n}.\tag{C3}$$

Now from (C1), (C2), and (C3) we get

$$\alpha_{i,n+1}\le\alpha_{i,n}\widetilde{K}_i+\alpha_{i,n}C^{**}=(\widetilde{K}_i+C^{**})\alpha_{i,n}.\tag{C4}$$

Note that $\alpha_{i,d}\le 1$ for $i=1,\ldots,d$. Thus, applying (C4) recursively we obtain $\alpha_{i,n+1}\le(\widetilde{K}_i+C^{**})^{n-d}$ for $n\ge d$, $i=1,\ldots,d$. But for $n\le d$ we have of course $\alpha_{i,n}\le 1$ for $i=1,\ldots,d$. Thus, with $K_i=\max(1,\widetilde{K}_i+C^{**})$ the statement of the lemma is satisfied.

Proof of Theorem 2. Define $\mathbf{Z}^{\oplus}:=(Z_1^{\oplus},\ldots,Z_d^{\oplus}):=\sum_{k=1}^d\mathbf{Z}^{(k)}$ and denote by $Z^{\oplus}_{(1)}\ge\cdots\ge Z^{\oplus}_{(d)}$ the order statistics of $Z_1^{\oplus},\ldots,Z_d^{\oplus}$. Let $A=\{z\in\mathbb{R}_+^d: z_j>x_j \text{ for all } j\in S\}$ be a rectangular set in $\mathbb{E}_d^{(i)}$ where $S\subseteq\mathbb{I}$, $|S|\ge i$, and $x_j>0$, for all $j\in S$ with $\mu_i(\partial A)=0$. Suppose

$\widetilde{S} \subseteq S$ with $|\widetilde{S}| = i$. Then

$$\lim_{t\to\infty} \frac{\mathbb{P}\big(\sum_{k=1}^{\tau} \mathbf{Z}^{(k)} \in tA\big)}{\mathbb{P}\big(Z^{\oplus}_{(i)} > t\big)} = \lim_{t\to\infty} \sum_{n=0}^{\infty} \mathbb{P}(\tau = n) \frac{\mathbb{P}\big(\bigcap_{j\in S} \big\{ \sum_{k=1}^{n} Z_j^{(k)} > tx_j \big\}\big)}{\mathbb{P}\big(Z^{\oplus}_{(i)} > t\big)}. \tag{C5}$$

But for any $n \in \mathbb{N}$ we have

$$\begin{aligned} 0 &\le \sup_{t>0} \frac{\mathbb{P}\big(\bigcap_{j\in S} \big\{ \sum_{k=1}^{n} Z_j^{(k)} > tx_j \big\}\big)}{\mathbb{P}\big(Z^{\oplus}_{(i)} > t\big)} \\ &\le \sup_{t>0} \frac{\mathbb{P}\big(\bigcap_{j\in \widetilde{S}} \big\{ \sum_{k=1}^{n} Z_j^{(k)} > t \min_{j\in S} x_j \big\}\big)}{\mathbb{P}\big(Z^{\oplus}_{(i)} > t \min_{j\in S} x_j\big)} \frac{\mathbb{P}\big(Z^{\oplus}_{(i)} > t \min_{j\in S} x_j\big)}{\mathbb{P}\big(Z^{\oplus}_{(i)} > t\big)} \\ &\le \alpha_{i,n} \sup_{t>0} \frac{\mathbb{P}\big(Z^{\oplus}_{(i)} > t \min_{j\in S} x_j\big)}{\mathbb{P}\big(Z^{\oplus}_{(i)} > t\big)}. \end{aligned} \tag{C6}$$

Since $\mathbf{Z}^{(\oplus)} \in \mathcal{MRV}(\alpha_i, b_i, f_i(d)\mu_i, \mathbb{E}_d^{(i)})$ and $f_i(d)\mu_i(\{z \in \mathbb{R}_+^d : z_{(i)} > 1\}) > 0$, we have

$$1 \le \frac{\mathbb{P}\big(Z^{\oplus}_{(i)} > t \min_{j\in S} x_j\big)}{\mathbb{P}\big(Z^{\oplus}_{(i)} > t\big)} \overset{t\to\infty}{\to} \big(\min_{j\in S} x_j\big)^{-\alpha_i} < \infty.$$

Hence, there exists a finite constant $C > 0$ such that

$$\sup_{t>0} \frac{\mathbb{P}\big(Z^{\oplus}_{(i)} > t \min_{j\in S} x_j\big)}{\mathbb{P}\big(Z^{\oplus}_{(i)} > t\big)} \le C. \tag{C7}$$

Then an application of Lemma 6 and (C6), (C7) yields

$$0 \le \sup_{t>0} \frac{\mathbb{P}\big(\bigcap_{j\in S} \big\{ \sum_{k=1}^{n} Z_j^{(k)} > tx_j \big\}\big)}{\mathbb{P}\big(Z^{\oplus}_{(i)} > t\big)} \le CK_i^n, \quad n \in \mathbb{N}.$$

Thus, there exists a uniform finite upper bound of the right-hand side of (C5) such that due to Pratt's theorem we are allowed to exchange the limit and the sum. A conclusion of Assumption 1 is then

$$\lim_{t\to\infty} \frac{\mathbb{P}\big(\sum_{k=1}^{\tau} \mathbf{Z}^{(k)} \in tA\big)}{\mathbb{P}\big(Z^{\oplus}_{(i)} > t\big)} = \sum_{n=0}^{\infty} \mathbb{P}(\tau = n) f_i(n) \frac{\mu_i(A)}{f_i(d)\mu_i\big(\big\{z \in \mathbb{R}_+^d : z_{(i)} > 1\big\}\big)}.$$

Then Proposition 1 and $\mathbf{Z}^{\oplus} \in \mathcal{MRV}(\alpha_i, b_i, f_i(d)\mu_i, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, d$ result in $\sum_{k=1}^{\tau} \mathbf{Z}^{(k)} \in \mathcal{MRV}(\alpha_i, b_i, \mathbb{E}(f_i(\tau))\mu_i, \mathbb{E}_d^{(i)})$. □

Appendix D. Proofs of the results in Section 6

Proof of Proposition 5.

Step 1. To begin with, let $(\boldsymbol{L}(s))_{s\geq 0}$ be a compound Poisson process with intensity $\lambda > 0$ and jump size distribution $\mathbb{P}_{\boldsymbol{Z}} = \Pi/\lambda$, which is a proper probability measure on $\mathbb{R}^d_+$. Let us also assume that $(N(s))_{s\geq 0}$ is a Poisson process with intensity λ and $\boldsymbol{Z}^{(1)}, \boldsymbol{Z}^{(2)}, \ldots$ are i.i.d. with distribution $\mathbb{P}_{\boldsymbol{Z}}$. Then $\mathbb{P}_{\boldsymbol{Z}} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i/\lambda, \mathbb{E}_d^{(i)})$. Since $\mathbb{E}(N(s)) = \lambda s$, using Proposition 3 and Theorem 2 we have

$$\boldsymbol{L}(s) \stackrel{\mathrm{d}}{=} \sum_{k=1}^{N(s)} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}\big(\alpha_i, b_i, s\mu_i, \mathbb{E}_d^{(i)}\big) \quad \text{for } i = 1, \ldots, d.$$

Step 2. Now let $(\boldsymbol{L}(s))_{s\geq 0}$ be a general Lévy process. Define

$$D_{a,\infty} := \{z \in \mathbb{R}^d : a < \|z\| < \infty\} \quad \text{for any } a > 0.$$

Due to the Lévy–Itô decomposition (see [58, Theorems 19.2 and 19.3]) we can decompose $\boldsymbol{L}$ into two independent Lévy processes $\boldsymbol{L}_1 = (\boldsymbol{L}_1(s))_{s\geq 0}$ and $\boldsymbol{L}_2 = (\boldsymbol{L}_2(s))_{s\geq 0}$ such that

$$\boldsymbol{L}(s) = \boldsymbol{L}_1(s) + \boldsymbol{L}_2(s), \quad s \geq 0,$$

where $\boldsymbol{L}_1$ is a compound Poisson process with Lévy measure $\Pi(\,\cdot \cap D_{a,\infty})/\Pi(D_{a,\infty})$ and Poisson intensity $\Pi(D_{a,\infty})$, whereas $\boldsymbol{L}_2$ satisfies $\mathbb{E}\|\boldsymbol{L}_2(s)\|^\theta < \infty$ for any $\theta > 0$ (see [46, Lemma 2.2 and proof of Theorem 2.3]). Thus, the Lévy measure of $\boldsymbol{L}_1$ is $\Pi(\,\cdot \cap D_{a,\infty}) \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, d$ and by step 1 we have

$$\boldsymbol{L}_1(s) \in \mathcal{MRV}\big(\alpha_i, b_i, s\mu_i, \mathbb{E}_d^{(i)}\big) \quad \text{for } i = 1, \ldots, d.$$

Then an application of Lemma 2 and $\boldsymbol{L}(s) = \boldsymbol{L}_1(s) + \boldsymbol{L}_2(s)$ gives us the result. □

Proof of Proposition 6. As in Proposition 5 it is sufficient to investigate compound Poisson processes $(\boldsymbol{L}(s))_{s\geq 0} = (\sum_{k=1}^{N(s)} \boldsymbol{Z}^{(k)})_{s\geq 0}$ with intensity $\lambda > 0$ and jumps size distribution $\mathbb{P}_{\boldsymbol{Z}} = \Pi/\lambda$. Then for the jump size distribution we have $\mathbb{P}_{\boldsymbol{Z}} = \Pi/\lambda \in \mathcal{MRV}(\alpha_i, b_i, \mu_i/\lambda, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, d$. Since $\mathbb{E}(N(s)(N(s)-1)(N(s)-i+1)) = (\lambda s)^i$, $f_i(n) = 0$ for $n < i$, $f_i(n) = n!/(n-i)!$ for $n \geq i$, Proposition 4 and Theorem 2 result in

$$\boldsymbol{L}(s) = \sum_{k=1}^{N(s)} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}\big(i\alpha_1, b_i, s^i \mu_i^L, \mathbb{E}_d^{(i)}\big) \quad \text{for } i = 1, \ldots, d,$$

which is the statement. □

Proof of Theorem 7. Suppose $(\boldsymbol{L}(s))_{s\geq 0}$ is a compound Poisson process with $\Pi(\mathbb{R}^d_+) \leq 1$. Let $\boldsymbol{Z}^{(1)}, \boldsymbol{Z}^{(2)}, \ldots$ be a sequence of i.i.d. random vectors with distribution $\mathbb{P}(\boldsymbol{Z}^{(1)} = \boldsymbol{0}) = 1 - \Pi(\mathbb{R}^d_+)$ and $\mathbb{P}(\boldsymbol{Z}^{(1)} \in A \backslash \{\boldsymbol{0}\}) = \Pi(A \backslash \{\boldsymbol{0}\})$ for all sets $A \in \mathcal{B}(\mathbb{R}^d_+)$. Then $\boldsymbol{Z}^{(1)} \in \mathcal{MRV}(\alpha_i, b_i, \mu_i, \mathbb{E}_d^{(i)})$ for $i = 1, \ldots, d$. Due to Proposition 2 and Theorem 2 we receive

$$\boldsymbol{L}(s) \stackrel{\mathrm{d}}{=} \sum_{k=1}^{N^*(s)} \boldsymbol{Z}^{(k)} \in \mathcal{MRV}\big(i\alpha, b_i, \mathbb{E}(N^*(s)^i)\mu_i, \mathbb{E}_d^{(i)}\big).$$

We extend this result to general Lévy measures and Lévy processes as in Proposition 5 by choosing a large enough so that $\Pi(D_{a,\infty}) \leq 1$. □

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