The Low-Degree Bounded Cohomology of Classical Groups

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Notation and Conventions

Basics: We write $X \subset Y$ for the relation of every element of the set X being an element of the set Y. We denote the natural numbers, integers, rational numbers, real numbers, complex numbers, and quaternions by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, respectively, where we convene that $0 \in \mathbb{N}$. If X and Y are sets, then the set of maps from X to Y is denoted by $\mathrm{Map}(X, Y)$. By a Banach space we mean a real Banach space. The class of a vector $x \in V \setminus \{0\}$ in $\mathbb{P}(V)$ is denoted by $[x]$.

Groups: The identity of a group G is usually denoted by e_G or e. If H and G are groups, then we denote the relation of H being a subgroup of G by $H < G$. The center of a group G is denoted by $\mathrm{Z}(G)$. We convene that every (locally) compact group is automatically assumed to be Hausdorff. By a Lie group we mean a real Lie group.

Group Actions: We exclusively consider left G-actions. If G is a group and X is a G-set, then the action of $g \in G$ on $x \in X$ is denoted by $g.x$ or gx; the set of G-invariants of X is denoted by X^G; we say that X is a trivial G-set if $X^G = X$. The orbit space of an action $G \curvearrowright X$ is denoted by $G \backslash X$. An action $G \curvearrowright X$ is called continuous if the corresponding action map $G \times X \to X$ is continuous, in which case X is referred to as a G-space. However, a smooth action is **not** defined to be a group action with smooth action map (see Subsection 2.2.1). Unless stated otherwise, if X is a G-set, then we consider X^n as a G-set using the diagonal action $g(x_1, \ldots, x_n) \coloneqq (gx_1, \ldots, gx_n)$.

Introduction

This thesis is concerned with certain nice matrix groups—namely, classical groups—acting isometrically on nice non-positively curved spaces—namely, symmetric spaces of non-compact type. In order to understand such an action, it is helpful to compare its associated geometric invariants to its associated asymptotic invariants; we will see that this comparison boils down to the description of *rigidity phenomena* for the Laplace equation $\Delta f = 0$.

Since the invariants we consider will be rather abstract, we first build up some intuition by considering the most well-known examples.

Poincaré's Upper Half-Plane: We equip Poincaré's upper half-plane

$$\mathbb{H}^2 = \{x + \mathrm{i}y \in \mathbb{C} \mid x, y \in \mathbb{R},\ y > 0\}$$

with its usual hyperbolic metric defined by

$$d(z_1, z_2) = 2\mathrm{artanh}\left(\frac{|z_1 - z_2|}{|z_1 - \overline{z_2}|}\right) \quad (z_1, z_2 \in \mathbb{H}^2).$$

Let us fix $G = \mathrm{SL}(2, \mathbb{R})$. Recall that the action of G on $\mathbb{H}^2$ by Möbius transformations is isometric, i.e. the map

$$\mathbb{H}^2 \to \mathbb{H}^2,\ z \mapsto \frac{az + b}{cz + d}$$

is an isometry for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

We denote the vector space of differential n-forms on $\mathbb{H}^2$ by $\Omega^n(\mathbb{H}^2)$. Then the subspace $\Omega^2(\mathbb{H}^2)^G$ of G-invariant differential 2-forms is spanned by the area form $\mathrm{d}\mu$ on $\mathbb{H}^2$. This produces a G-invariant continuous function $J(\mathrm{d}\mu)\colon (\mathbb{H}^2)^3 \to \mathbb{R}$ by setting

$$J(\mathrm{d}\mu)(z_0, z_1, z_2) := \int_{\Delta(z_0, z_1, z_2)} \mathrm{d}\mu,$$

where $\Delta(z_0, z_1, z_2) \subset \mathbb{H}^2$ denotes the geodesic triangle with end points z_0, z_1, z_2. Linear extension yields the map $J\colon \Omega^2(\mathbb{H}^2)^G \to C((\mathbb{H}^2)^3)^G$ defined by

$$J(\omega)(z_0, z_1, z_2) := \int_{\Delta(z_0, z_1, z_2)} \omega \qquad (\omega \in \Omega^2(\mathbb{H}^2)^G).$$

One can show that J is injective; let us now describe the image of J. Stokes' theorem yields

$$\int_{\partial\Delta(z_0, \ldots, z_3)} \omega = \int_{\Delta(z_0, \ldots, z_3)} \mathrm{d}\omega.$$

Since $\mathbb{H}^2$ is two-dimensional, we have $\Omega^3(\mathbb{H}^2) = \{0\}$, which implies $\mathrm{d}\omega = 0$ and

$$\begin{aligned} 0 &= \int_{\partial\Delta(z_0,\ldots,z_3)} \omega \\ &= \int_{\Delta(z_1,z_2,z_3)} \omega - \int_{\Delta(z_0,z_2,z_3)} \omega + \int_{\Delta(z_0,z_1,z_3)} \omega - \int_{\Delta(z_0,z_1,z_2)} \omega \\ &= J(\omega)(z_1,z_2,z_3) - J(\omega)(z_0,z_2,z_3) + J(\omega)(z_0,z_1,z_3) - J(\omega)(z_0,z_1,z_2). \end{aligned}$$

We say that $J(\omega)$ satisfies the *cocycle condition*; this motivates the consideration of the *homogeneous* 2*-differential* $d^2\colon C((\mathbb{H}^2)^3)^G \to C((\mathbb{H}^2)^4)^G$ defined by

$$d^2(f)(z_0,\ldots,z_3) \coloneqq f(z_1,z_2,z_3) - f(z_0,z_2,z_3) + f(z_0,z_1,z_3) - f(z_0,z_1,z_2),$$

so that we have $\mathrm{im}(J) \subset \ker(d^2)$.

One can show that $J\colon \Omega^2(\mathbb{H}^2)^G \to \ker(d^2)$ is "close to surjective" in the sense that for every $f \in \ker(d^2)$ there exists a G-invariant continuous function $g\colon (\mathbb{H}^2)^2 \to \mathbb{R}$ such that the function defined by

$$(z_0,z_1,z_2) \mapsto f(z_0,z_1,z_2) + g(z_1,z_2) - g(z_0,z_2) + g(z_0,z_1)$$

is in the image of J. This motivates the consideration of the *homogeneous* 1*-differential* $d^1\colon C((\mathbb{H}^2)^2)^G \to C((\mathbb{H}^2)^3)^G$ defined by

$$d^1(g)(z_0,z_1,z_2) \coloneqq g(z_1,z_2) - g(z_0,z_2) + g(z_0,z_1).$$

One can check that $\mathrm{im}(d^1) \subset \ker(d^2)$; hence the quotient vector space

$$\mathrm{H}^2_{\mathrm{c}}(G) \coloneqq \ker(d^2)/\mathrm{im}(d^1)$$

is well-defined. It is possible to show that J induces an isomorphism $\Omega^2(\mathbb{H}^2)^G \xrightarrow{\cong} \mathrm{H}^2_{\mathrm{c}}(G)$; we call $\mathrm{H}^2_{\mathrm{c}}(G)$ the *second continuous cohomology* of the isometric action $G \curvearrowright \mathbb{H}^2$. Continuous cohomology is the main geometric invariant that we will consider. Let us now discuss its asymptotic counterpart.

Writing $\mathbb{D}^2 = \{z \in \mathbb{C} : |z| < 1\}$, we can consider the Cayley transform

$$\mathbb{H}^2 \to \mathbb{D}^2,\ z \mapsto \frac{z - \mathrm{i}}{z + \mathrm{i}},$$

which is clearly a diffeomorphism; thus, we can transport the hyperbolic metric on $\mathbb{H}^2$, as well as the G-action, to $\mathbb{D}^2$ to obtain *Poincaré's disk model* of the hyperbolic plane. Clearly, the (topological) boundary of $\mathbb{D}^2$ in $\mathbb{C}$ is given by $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$.

We can extend the action of G on $\mathbb{D}^2$ to an action on $\mathbb{S}^1$. Note however that $\mathbb{S}^1$ neither admits a G-invariant metric nor a G-invariant measure with respect to this action. Indeed, this action only preserves the Lebesgue *measure class* $[\tau]$ of $\mathbb{S}^1$; such an action $G \curvearrowright (\mathbb{S}^1, [\tau])$ is often called a *non-singular action*.

In the following we identify $\mathbb{D}^2$ and $\mathbb{S}^1$ with the corresponding subsets of $\mathbb{R}^2$. We call a function $f\colon \mathbb{D}^2 \to \mathbb{R}$ *harmonic* if it solves the Laplace equation $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.

Denoting the vector space of bounded harmonic functions on $\mathbb{D}^2$ by $\mathcal{H}^\infty(\mathbb{D}^2)$, the *Poisson transform* gives an isomorphism

$$\mathcal{P}\colon L^\infty(\mathbb{S}^1) \xrightarrow{\cong} \mathcal{H}^\infty(\mathbb{D}^2),\ \mathcal{P}(f)(x) := \int_{\mathbb{S}^1} f(\xi)K(x,\xi)\mathrm{d}\tau(\xi),$$

where $K(x,\xi)$ denotes the *Poisson kernel* given by

$$K(x,\xi) := \frac{1-\|x\|^2}{\|\xi - x\|^2} \qquad (x \in \mathbb{D}^2,\ \xi \in \mathbb{S}^1).$$

Using product measures and products of Poisson kernels, we also obtain an isomorphism

$$\mathcal{P}^n\colon L^\infty((\mathbb{S}^1)^{n+1}) \xrightarrow{\cong} \mathcal{H}^\infty((\mathbb{D}^2)^{n+1})$$

for all $n \in \mathbb{N}$. Note that $\mathcal{P}^n$ maps G-invariant functions to G-invariant functions. Since every harmonic function is automatically smooth and hence continuous, we have the inclusion $\mathcal{H}^\infty((\mathbb{D}^2)^{n+1}) \hookrightarrow C((\mathbb{D}^2)^{n+1})$. Now it is natural to ask how the continuous cohomology $\mathrm{H}^2_\mathrm{c}(G)$ relates to the Poisson transform.

Note that the definition of the homogeneous differentials d^2 and d^1 also works in the setting of L^∞-functions, i.e. we can consider the maps $d^2_\mathrm{b}\colon L^\infty((\mathbb{S}^1)^3)^G \to L^\infty((\mathbb{S}^1)^4)^G$ and $d^1_\mathrm{b}\colon L^\infty((\mathbb{S}^1)^2)^G \to L^\infty((\mathbb{S}^1)^3)^G$ defined by the same alternating sum conditions. In this context, the quotient vector space

$$\mathrm{H}^2_\mathrm{mb}(G) := \ker\big(d^2_\mathrm{b}\big)/\mathrm{im}\big(d^1_\mathrm{b}\big)$$

is called the *second (measurable) bounded cohomology* of the measure class preserving action $G \curvearrowright (\mathbb{S}^1, [\tau])$.

One can check that $\mathcal{P}^2$ maps $\ker\big(d^2_\mathrm{b}\big)$ to $\ker(d^2)$ and $\mathrm{im}\big(d^1_\mathrm{b}\big)$ to $\mathrm{im}(d^1)$, which means that $\mathcal{P}^2$ induces a map

$$\overline{\mathcal{P}^2}\colon \mathrm{H}^2_\mathrm{mb}(G) \to \mathrm{H}^2_\mathrm{c}(G),$$

which in the context of bounded cohomology is known as the *comparison map*. We emphasize that $\overline{\mathcal{P}^2}$ measures the difference between G-invariant differential 2-forms on $\mathbb{H}^2$ (the right-hand side) and harmonic representatives thereof (the left-hand side).

Theorem (Burger–Monod [25])**.** *The map* $\overline{\mathcal{P}^2}\colon \mathrm{H}^2_\mathrm{mb}(G) \to \mathrm{H}^2_\mathrm{c}(G)$ *is an isomorphism.*

This theorem can be viewed as a *rigidity phenomenon* for the Laplace equation on $\mathbb{H}^2$. Concretely, the surjectivity part of this theorem means that every G-invariant continuous function f on $(\mathbb{H}^2)^3$ satisfying the cocycle condition $d^2(f) = 0$ is "close" to a solution of the elliptic partial differential equation $\Delta f = 0$ in the following sense: There exists some G-invariant continuous function g on $(\mathbb{H}^2)^2$ such that $f + d^1(g)$ is a solution of this PDE, i.e. a harmonic function. The injectivity part of this theorem means that for every G-invariant bounded harmonic function f on $(\mathbb{H}^2)^3$ that can be written as $f = d^1(g)$ for some G-invariant continuous function g on $(\mathbb{H}^2)^2$ there exists some G-invariant bounded harmonic function h on $(\mathbb{H}^2)^2$ such that $f = d^1(h)$.

From this point of view, the above theorem can be seen as an analogue of the *Hodge theorem*, which we now state as a reminder. We say that a differential n-form $\omega \in \Omega^n(M)$ on a Riemannian manifold M is *harmonic* if $\Delta\omega = 0$, where $\Delta\colon \Omega^n(M) \to \Omega^n(M)$ denotes the *Laplace–Beltrami operator*; the vector space of all such forms is denoted by $\mathcal{H}^n(M)$.

The Hodge Theorem (see e.g. [127, Ch. 6]). *Let M be a closed Riemannian manifold. Then the inclusion $\mathcal{H}^n(M) \hookrightarrow \Omega^n(M)$ induces an isomorphism between $\mathcal{H}^n(M)$ and the de Rham cohomology $\mathrm{H}^n(M)$ of M.*

We mention in passing that $\mathrm{H}^2_c(G)$ can *naturally* be identified with the second de Rham cohomology of the *compact dual* of $\mathbb{H}^2$, i.e. with $\mathrm{H}^2(\mathbb{S}^2)$, where $\mathbb{S}^2$ denotes the 2-sphere.

In Remark 2.4.4 we provide another geometric interpretation of the comparison map by discussing extensions of differential forms to the boundary without using the Poisson transform. Using this point of view, we explain in Example 2.4.5 how the surjectivity of $\overline{\mathcal{P}^2}$ follows from the *negative curvature* of $\mathbb{H}^2$; proving injectivity requires more work.

Having discussed the degree-two case, let us explain the extension to higher degrees. It is natural to generalize the cocycle condition by considering the *homogeneous n-differential* $d^n\colon C((\mathbb{H}^2)^{n+1})^G \to C((\mathbb{H}^2)^{n+2})^G$ defined by

$$d^n(f)(z_0,\dots,z_{n+1}) \coloneqq \sum_{i=0}^{n+1}(-1)^i f(z_0,\dots,z_{i-1},z_{i+1},\dots,z_{n+1}).$$

One can check that $\mathrm{im}(d^{n-1}) \subset \ker(d^n)$, so we can define the *$n$-th continuous cohomology* $\mathrm{H}^n_c(G) \coloneqq \ker(d^n)/\mathrm{im}(d^{n-1})$ of the isometric action $G \curvearrowright \mathbb{H}^2$. Using the same alternating sum formula, we obtain the map $d^n_b\colon L^\infty((\mathbb{S}^1)^{n+1})^G \to L^\infty((\mathbb{S}^1)^{n+2})^G$, so we can define the *$n$-th (measurable) bounded cohomology* $\mathrm{H}^n_{\mathrm{mb}}(G) \coloneqq \ker\big(d^n_b\big)/\mathrm{im}\big(d^{n-1}_b\big)$ of the measure class preserving action $G \curvearrowright (\mathbb{S}^1,[\tau])$. Again, the Poisson transform induces a linear map $\overline{\mathcal{P}^n}\colon \mathrm{H}^n_{\mathrm{mb}}(G) \to \mathrm{H}^n_c(G)$. Keeping the Hodge theorem in mind, we are led to ask the following question:

Question. Is $\overline{\mathcal{P}^n}\colon \mathrm{H}^n_{\mathrm{mb}}(G) \to \mathrm{H}^n_c(G)$ an isomorphism for all $n \geq 3$?

As in the degree-two case, one can show that the map

$$J\colon \Omega^n(\mathbb{H}^2)^G \to C((\mathbb{H}^2)^{n+1})^G,\ J(\omega)(z_0,\dots,z_n) \coloneqq \int_{\Delta(z_0,\dots,z_n)} \omega$$

induces an isomorphism $\Omega^n(\mathbb{H}^2)^G \xrightarrow{\cong} \mathrm{H}^n_c(G)$. Since $\Omega^n(\mathbb{H}^2) = \{0\}$ for all $n \geq 3$, the above question is equivalent to the triviality of $\mathrm{H}^n_{\mathrm{mb}}(G)$ for all $n \geq 3$.

This question has seen very limited progress in the last 25 years. Burger and Monod [27] solved the $n = 3$ case by explicitly considering the associated functional equation—the Spence–Abel equation—and its unique solution—Roger's dilogarithm; Hartnick and Ott [71] solved the $n = 4$ case by explicitly considering the associated PDE problem. Close to nothing, not even finite-dimensionality of $\mathrm{H}^n_{\mathrm{mb}}(G)$, is known in degrees $n > 4$.

In this thesis, instead of increasing the degree considered, we will consider this question in degrees $n = 2, 3, 4$ for symmetric spaces generalizing $\mathbb{H}^2$.

Siegel's Upper Half-Spaces: One of the most well-known infinite families of symmetric spaces generalizing Poincaré's upper half-plane is the family of *Siegel's upper half-spaces*

$$\mathbb{H}_r \coloneqq \big\{X + \mathrm{i}Y \in \mathbb{C}^{r\times r} \;\big|\; X, Y \in \mathbb{R}^{r\times r} \text{ symmetric, } Y \text{ is positive definite}\big\} \quad (r \geq 1).$$

We clearly have $\mathbb{H}_1 = \mathbb{H}^2$. The metric on $\mathbb{H}_r$ is defined by setting

$$d(Z_1, Z_2) := \sqrt{\sum_{j=1}^{r} (2\operatorname{artanh}(\sigma_j))^2},$$

where $\sigma_1, \ldots, \sigma_r$ are the singular values of the matrix $(Z_1 - Z_2)\big(Z_1 - \overline{Z_2}\big)^{-1}$. One can show that $\mathbb{H}_r$ is a symmetric space of dimension $r(r+1)$ and rank r, i.e. r is the maximum dimension of a subspace of the tangent space (of any point) on which the curvature is identically zero. Actually, $\mathbb{H}_r$ is even a *Kähler manifold*. Note that, in contrast to the negatively curved hyperbolic plane, $\mathbb{H}_r$ for $r > 1$ only has *non-positive curvature*.

The isometry group of $\mathbb{H}_r$ is (the projectivization of) the *real symplectic group*

$$\mathrm{Sp}(2r, \mathbb{R}) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\middle|\; AB^\top = BA^\top,\ CD^\top = DC^\top,\ AD^\top - BC^\top = \mathrm{I}_r \right\} \subset \mathbb{R}^{2r \times 2r}.$$

Let us now fix $G_r = \mathrm{Sp}(2r, \mathbb{R})$. Then G_r acts isometrically on $\mathbb{H}_r$ by the generalized Möbius transformations

$$\mathbb{H}_r \to \mathbb{H}_r,\ Z \mapsto (AZ + B)(CZ + D)^{-1}$$

for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_r$.

In complete analogy to the case of the hyperbolic plane, we can define the continuous cohomology $\mathrm{H}^n_{\mathrm{c}}(G_r)$ of the isometric action $G_r \curvearrowright \mathbb{H}_r$ by considering equivalence classes of G_r-invariant continuous functions on $\mathbb{H}_r^{n+1}$, and integrate G_r-invariant differential n-forms over geodesic coning n-simplices in $\mathbb{H}_r$ to obtain the isomorphism $\Omega^n(\mathbb{H}_r)^{G_r} \cong \mathrm{H}^n_{\mathrm{c}}(G_r)$. Again, we can consider the space $\mathcal{H}^\infty(\mathbb{H}_r)$ of bounded functions f on $\mathbb{H}_r$ satisfying the Laplace equation $\Delta f = 0$.

In contrast to the hyperbolic plane, it is not as straightforward to define a suitable boundary for $\mathbb{H}_r$. Because of the non-positive curvature of $\mathbb{H}_r$, geodesics still diverge, and we still obtain the *visual boundary* $\partial_\infty(\mathbb{H}_r)$ of $\mathbb{H}_r$, which, again, neither admits a G_r-invariant metric nor a G_r-invariant measure, but only a G_r-invariant measure class.

However, the visual boundary $\partial_\infty(\mathbb{H}_r)$ is "too big" to yield a generalized Poisson transform $L^\infty(\partial_\infty(\mathbb{H}_r)) \xrightarrow{\cong} \mathcal{H}^\infty(\mathbb{H}_r)$; indeed, we need to restrict to the well-studied *Furstenberg boundary* $\partial_{\mathrm{F}}(\mathbb{H}_r) \subset \partial_\infty(\mathbb{H}_r)$ of $\mathbb{H}_r$ which, similarly, only admits a G_r-invariant measure class $[\tau]$. In [56] Furstenberg proved that the generalized Poisson transform

$$\mathcal{P} \colon L^\infty(\partial_{\mathrm{F}}(\mathbb{H}_r)) \to \mathcal{H}^\infty(\mathbb{H}_r),\ \mathcal{P}(f)(x) := \int_{\partial_{\mathrm{F}}(\mathbb{H}_r)} f(\xi) K(x, \xi) \mathrm{d}\tau(\xi)$$

is an isomorphism, where $K(x, \xi)$ denotes a generalized Poisson kernel.

Now we can define the bounded cohomology $\mathrm{H}^n_{\mathrm{mb}}(G_r)$ of the measure class preserving action $G_r \curvearrowright (\partial_{\mathrm{F}}(\mathbb{H}_r), [\tau])$ by considering equivalence classes of G_r-invariant L^∞-functions on $\partial_{\mathrm{F}}(\mathbb{H}_r)^{n+1}$, and note that the generalized Poisson transform induces a linear map $\overline{\mathcal{P}^n} \colon \mathrm{H}^n_{\mathrm{mb}}(G_r) \to \mathrm{H}^n_{\mathrm{c}}(G_r)$.

Theorem (Burger–Monod [25])**.** *The map* $\overline{\mathcal{P}^2} \colon \mathrm{H}^2_{\mathrm{mb}}(G_r) \to \mathrm{H}^2_{\mathrm{c}}(G_r)$ *is an isomorphism.*

Question. Is $\overline{\mathcal{P}^n}\colon \mathrm{H}^n_{\mathrm{mb}}(G_r) \to \mathrm{H}^n_{\mathrm{c}}(G_r)$ an isomorphism for all $n \geq 3$?

We can now state the main result of the present thesis; this result is proved using methods developed by De la Cruz Mengual and Hartnick [40], [44], [43], [42].

Main Theorem ($G_r = \mathrm{Sp}(2r, \mathbb{R})$). *The map $\overline{\mathcal{P}^3}\colon \mathrm{H}^3_{\mathrm{mb}}(G_r) \to \mathrm{H}^3_{\mathrm{c}}(G_r)$ is an isomorphism for all $r \geq 1$.*

Again, $\mathrm{H}^3_{\mathrm{c}}(G_r)$ can *naturally* be identified with the third de Rham cohomology of the compact dual of $\mathbb{H}_r$, which is the *complex Lagrangian Grassmannian* $\Lambda_r = \mathrm{Sp}(r)/\mathrm{U}(r)$; it is known that $\mathrm{H}^3(\Lambda_r) = 0$. Hence we will prove vanishing of $\mathrm{H}^3_{\mathrm{mb}}(G_r)$, which is equivalent to the following rigidity result for the Laplace equation on $\mathbb{H}_r$:

Corollary ($G_r = \mathrm{Sp}(2r, \mathbb{R})$). *For every G_r-invariant bounded harmonic function f on $\mathbb{H}_r^4$ satisfying the cocycle condition $d^3(f) = 0$ there exists a G_r-invariant bounded harmonic function g on $\mathbb{H}_r^3$ such that $f = d^2(g)$.*

We will also prove partial results for different generalizations of $\mathbb{H}^2$ in degrees $n = 2, 3, 4$. Let us now introduce the full generality of groups and spaces with which we will work.

The Classical Groups: We will consider the ten well-known families of *(semisimple) classical groups*, which are all given as determinant-one matrix subgroups of automorphism groups of formed vector spaces (see Section 1.1). These families are given as follows:

(i) $\mathrm{SL}(n, \mathbb{C})$, the complex special linear groups

(ii) $\mathrm{SO}(n, \mathbb{C})$, the complex special orthogonal groups

(iii) $\mathrm{Sp}(2n, \mathbb{C})$, the complex symplectic groups

(iv) $\mathrm{SL}(n, \mathbb{R})$, the real special linear groups

(v) $\mathrm{SU}(p, q)$, the special unitary groups

(vi) $\mathrm{SO}_0(p, q)$, the (identity components of) the real special orthogonal groups

(vii) $\mathrm{Sp}(2n, \mathbb{R})$, the real symplectic groups

(viii) $\mathrm{SL}(n, \mathbb{H})$, the quaternionic special linear groups

(ix) $\mathrm{Sp}(p, q)$, the quaternionic symplectic groups

(x) $\mathrm{SO}^*(2n)$, the quaternionic special orthogonal groups

The groups in the families (i)–(iii) admit the structures of complex Lie groups, while the groups in the other families are real Lie groups which do not admit a complex structure (except for $\mathrm{SO}_0(3, 1)$).

Let G be any group in the list above. Then the symmetric space with which we will work is the homogeneous space G/K for a maximal compact subgroup K of G; in the case $G = \mathrm{Sp}(2r, \mathbb{R})$ we can take the unitary group $K = \mathrm{U}(r)$.

As above, we can define the n-th continuous cohomology of the isometric action $G \curvearrowright G/K$ by considering equivalence classes of G-invariant continuous functions on

$(G/K)^{n+1}$. Again, integrating G-invariant differential n-forms over geodesic coning n-simplices in G/K yields an isomorphism $\Omega^n(G/K)^G \cong \mathrm{H}^n_{\mathrm{c}}(G)$, and we can consider the space $\mathcal{H}^\infty((G/K)^{n+1})$ of bounded harmonic functions on $(G/K)^{n+1}$, i.e. bounded functions f satisfying $\Delta f = 0$.

In this setting, the Furstenberg boundary $\partial_{\mathrm{F}}(G/K)$ of G/K turns out to be isomorphic to the homogeneous space G/P, where P is a *minimal parabolic subgroup* of G; in the case $G = \mathrm{SL}(2,\mathbb{R})$, one can take

$$P = \left\{ \begin{pmatrix} \lambda & w \\ 0 & \lambda^{-1} \end{pmatrix} \,\middle|\, \lambda \in \mathbb{R}^\times,\ w \in \mathbb{R} \right\}.$$

Considering equivalence classes of G-invariant L^∞-functions on $(G/P)^{n+1}$ yields the n-th bounded cohomology $\mathrm{H}^n_{\mathrm{mb}}(G)$ of the measure class preserving action $G \curvearrowright G/P$.

In [56] Furstenberg even showed that the Poisson transform can be generalized to an isomorphism

$$\mathcal{P}^n \colon L^\infty((G/P)^{n+1}) \xrightarrow{\cong} \mathcal{H}^\infty((G/K)^{n+1}),$$

which again yields the induced *comparison map* $\overline{\mathcal{P}^n} \colon \mathrm{H}^n_{\mathrm{mb}}(G) \to \mathrm{H}^n_{\mathrm{c}}(G)$. The bijectivity of this map was first conjectured by Monod [102] in his ICM address in 2006 and has since become known in the field of bounded cohomology as the *isomorphism conjecture.*

The Isomorphism Conjecture (Monod [102])**.** The comparison map

$$\overline{\mathcal{P}^n} \colon \mathrm{H}^n_{\mathrm{mb}}(G) \to \mathrm{H}^n_{\mathrm{c}}(G)$$

is an isomorphism.

A conjecture implying surjectivity had previously been conjectured by Dupont [48], who proved the former in the $n = 2$ case. We will only consider degrees ≤ 4, where, by results of Gromov [65], Bucher [10], Hartnick–Ott [71], Monod [101], Bucher–Burger–Iozzi [15] and De la Cruz Mengual [42], the comparison map is always known to be surjective. See Subsection 2.5.2 for further cases in which surjectivity is known.

While the continuous cohomology of classical groups is known in all cases, computing bounded cohomology or proving injectivity of the comparison map is notoriously hard. To this day, the isomorphism conjecture is only known to hold in degrees $n \leq 2$, with additional results in degrees $n = 3$ and $n = 4$ for specific groups only. See Subsection 2.5.1 for the cases in which injectivity is known.

Note that classical groups and their associated symmetric spaces are particularly well-behaved with respect to the Poisson transform, and that there exist non-positively curved examples where the comparison map is neither surjective nor injective. For a counterexample to surjectivity, one can show that $\mathrm{H}^n_{\mathrm{mb}}(\mathbb{R}^k) = 0$ and that $\mathrm{H}^n_{\mathrm{c}}(\mathbb{R}^k)$ is isomorphic to the exterior power $\wedge^n \mathbb{R}^k$.

Measurable Bounded Cohomology: The theory of measurable bounded cohomology was introduced by Burger and Monod (see [25], [100], [26]) as a measurable generalization of discrete bounded cohomology, which was independently introduced by Johnson [83], Trauber (unpublished), and Gromov [65]. The most complete account of the theory of bounded cohomology is Monod's book [100]. In Chapter 2 we will formally define bounded cohomology and discuss those aspects which are relevant for the isomorphism conjecture.

One can show that $\mathrm{H}^n_{\mathrm{mb}}(G)$ can also be computed by considering equivalence classes of continuous bounded functions on G^{n+1} or L^∞-functions on G^{n+1}, instead of L^∞-functions on $(G/P)^{n+1}$. Indeed, it is more common to denote $\mathrm{H}^n_{\mathrm{mb}}(G)$ by $\mathrm{H}^n_{\mathrm{cb}}(G)$ and to speak of the *continuous bounded cohomology* of G. Similarly, $\mathrm{H}^n_{\mathrm{c}}(G)$ can be computed by considering equivalence classes of continuous functions on G^{n+1} or L^0-functions on G^{n+1}, instead of continuous functions on $(G/K)^{n+1}$. Viewed this way, the comparison map $\mathrm{H}^n_{\mathrm{cb}}(G) \to \mathrm{H}^n_{\mathrm{c}}(G)$ is induced both by the map

$$C_{\mathrm{bounded}}(G^{n+1})^G \hookrightarrow C(G^{n+1})^G,$$

as well as by the map

$$L^\infty(G^{n+1})^G \hookrightarrow L^0(G^{n+1})^G,$$

i.e. by just forgetting boundedness.

The Isomorphism Conjecture in Low Degree: In low degree continuous cohomology is closely connected to the existence of certain additional structures on the group or the associated symmetric space; this makes studying the boundedness problem posed by the isomorphism conjecture in these degrees especially significant and intriguing.

In degree two, $\Omega^2(G/K)^G$ is non-zero if and only if G/K admits the structure of a Kähler manifold. In this case, $\Omega^2(G/K)^G$ is one-dimensional and spanned by the corresponding *Kähler form* on G/K. Boundedness of this form has been applied in rigidity theory, see [46], [37], [24], [23].

In degree three, $\Omega^3(G/K)^G$ is non-zero if and only if G admits the structure of a complex Lie group. In this case, $\Omega^3(G/K)^G$ is one-dimensional and spanned by the corresponding *Borel form* on G/K.

In degree four, $\Omega^4(G/K)^G$ is non-zero if and only if G/K is at least four-dimensional and G/K admits the structure of a Kähler manifold or the structure of a quaternion-Kähler manifold. In this case, $\Omega^4(G/K)^G$ is at most four-dimensional and is spanned by (possibly) the cup product of the corresponding Kähler form with itself as well as (possibly) the corresponding quaternion-Kähler forms.

By work of Burger and Monod [25], the isomorphism conjecture is known to hold in degree two. In degree three, the isomorphism conjecture has been proved for several groups, including $\mathrm{SL}(n,\mathbb{R})$ (see [27] and [101]), $\mathrm{SL}(n,\mathbb{C})$ (see [62], [5], [101], [15]), the complex symplectic groups $\mathrm{Sp}(2n,\mathbb{C})$ (see [40]), and the complex orthogonal groups $\mathrm{SO}(2n+1,\mathbb{C})$ (see [44], [43], [42]).

In Table 1 and Table 2 we list the continuous cohomology of all classical groups in degrees $k = 2, 3, 4$, in rank one (which we do not tackle in this thesis) and in higher rank, respectively; additionally, information on whether the isomorphism conjecture is (partially) solved is provided.

Group/Degree	$k=2$	$k=3$	$k=4$
$\mathrm{SO}_0(2,1),\ \mathrm{SL}(2,\mathbb{R})$	$\mathbb{R}$ ✓	0 ✓	0 ✓
$\mathrm{SO}_0(3,1),\ \mathrm{SL}(2,\mathbb{C})$	0 ✓	$\mathbb{R}$ ✓	0 ✓
$\mathrm{SO}_0(4,1),\ \mathrm{Sp}(1,1)$	0 ✓	0 ✓	$\mathbb{R}$ ✓
$\mathrm{SO}_0(n,1)\ (n \geq 5)$	0 ✓	0 ✓	0 ✓
$\mathrm{SU}(n,1)$	$\mathbb{R}$ ✓	0 ✓	$\mathbb{R}$?
$\mathrm{Sp}(n,1)\ (n \geq 2)$	0 ✓	0 ?	$\mathbb{R}$?

Table 1.: Continuous cohomology of rank one classical groups and current state of the isomorphism conjecture. Green: isomorphism conjecture known to hold. Orange: surjectivity of the comparison map known; injectivity thereof unknown.

Our further results are summarized by the following theorem.

Rank Reduction Theorem. *(i) The isomorphism conjecture holds in degree three for all groups of the form* $\mathrm{SO}_0(r+1,r)$*,* $r \geq 1$*, if and only if* $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}_0(4,3)) = 0$*.*

(ii) Let $d \geq 3$ *be odd. Then the isomorphism conjecture holds in degree three for all groups of the form* $\mathrm{SO}_0(r+d,r)$*,* $r \geq 1$*, if and only if* $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}_0(3+d,3)) = \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}_0(2+d,2)) = 0$*.*

(iii) Let $d \geq 0$*. Then the isomorphism conjecture holds in degree three for all groups of the form* $\mathrm{SU}(r+d,r)$*,* $r \geq 1$*, if and only if* $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SU}(3+d,3)) = \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SU}(2+d,2)) = 0$*.*

(iv) Let $d \geq 1$ *be odd. Then the isomorphism conjecture holds in degree four for all groups of the form* $\mathrm{SO}_0(r+d,r)$*,* $r \geq 5$*, if and only if* $\mathrm{H}^4_{\mathrm{cb}}(\mathrm{SO}_0(5+d,5)) = 0$*.*

(v) The isomorphism conjecture holds in degree four for all groups of the form $\mathrm{Sp}(2r,\mathbb{C})$*,* $r \geq 1$*, if and only if* $\mathrm{H}^4_{\mathrm{cb}}(\mathrm{Sp}(4,\mathbb{C})) = 0$*.*

(vi) The isomorphism conjecture holds in degree four for all groups of the form $\mathrm{SO}(2r+1,\mathbb{C})$*,* $r \geq 1$*, if and only if* $\mathrm{H}^4_{\mathrm{cb}}(\mathrm{SO}(5,\mathbb{C})) = 0$*.*

That's Odd: We do not obtain similar rank reduction results for groups of the form $\mathrm{SO}_0(r+d,r)$ for *even* d. The reason for this is that we actually prove rank reduction statements for the *full* automorphism groups $\mathrm{O}(r+d,r)$; these results only directly translate to the groups $\mathrm{SO}_0(r+d,r)$ if the determinant homomorphism $\mathrm{O}(r+d,r) \to \{\pm 1\}$ admits a homomorphic section, which is the case if and only if d is odd (a similar statement holds for $\mathrm{O}(n,\mathbb{C})$ and $\mathrm{SO}(n,\mathbb{C})$). We do, however, obtain the following vanishing result.

Orthogonal Vanishing Theorem. *We have* $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(r,r)) = 0$ *for all* $r \geq 1$*.*

Unfortunately, since $\mathrm{O}(r,r)$ is not connected, an interpretation of this vanishing result in terms of harmonic functions is not available.

Group/Degree	$k=2$	$k=3$	$k=4$
$\mathrm{SL}(n,\mathbb{C})$ $(n\geq 3)$	0 ✓	$\mathbb{R}$ ✓	0 (✓)
$\mathrm{SO}(5,\mathbb{C})$	0 ✓	$\mathbb{R}$ ✓	0 ?
$\mathrm{SO}(2n+1,\mathbb{C})$ $(n\geq 3)$	0 ✓	$\mathbb{R}$ ✓	0 (✓)
$\mathrm{Sp}(4,\mathbb{C})$	0 ✓	$\mathbb{R}$ ✓	0 ?
$\mathrm{Sp}(2n,\mathbb{C})$ $(n\geq 3)$	0 ✓	$\mathbb{R}$ ✓	0 (✓)
$\mathrm{SO}(4,\mathbb{C})$	0 ✓	$\mathbb{R}^2$ ✓	0 ?
$\mathrm{SO}(2n,\mathbb{C})$ $(n\geq 3)$	0 ✓	$\mathbb{R}$?	0 ?
$\mathrm{SL}(n,\mathbb{R})$ $(n\geq 3)$	0 ✓	0 ✓	0 (✓)
$\mathrm{SU}(p,2)$ $(p\geq 2)$	$\mathbb{R}$ ✓	0 ?	$\mathbb{R}^2$?
$\mathrm{SU}(p,3)$ $(p\geq 3)$	$\mathbb{R}$ ✓	0 ?	$\mathbb{R}$?
$\mathrm{SU}(p,q)$ $(p,q\geq 4)$	$\mathbb{R}$ ✓	0 (✓)	$\mathbb{R}$ (✓)
$\mathrm{SO}_0(3,2)$	$\mathbb{R}$ ✓	0 ✓	$\mathbb{R}$?
$\mathrm{SO}_0(2+d,2)$ $(d\geq 3$ odd$)$	$\mathbb{R}$ ✓	0 ?	$\mathbb{R}$?
$\mathrm{SO}_0(4,3)$	0 ✓	0 ?	$\mathbb{R}^2$?
$\mathrm{SO}_0(3+d,3)$ $(d\geq 3$ odd$)$	0 ✓	0 ?	0 ?
$\mathrm{SO}_0(4+d,4)$ $(d\geq 1$ odd$)$	0 ✓	0 (✓)	$\mathbb{R}^2$?
$\mathrm{SO}_0(r+d,r)$ $(r\geq 5,\ d\geq 1$ odd$)$	0 ✓	0 (✓)	0 (✓)
$\mathrm{Sp}(2n,\mathbb{R})$ $(n\geq 2)$	$\mathbb{R}$ ✓	0 ✓	0 (✓)
$\mathrm{SO}_0(2,2)$	$\mathbb{R}^2$ ✓	0 ✓	$\mathbb{R}$?
$\mathrm{SO}_0(4,2)$	$\mathbb{R}$ ✓	0 ?	$\mathbb{R}^2$?
$\mathrm{SO}_0(2+d,2)$ $(d\geq 4$ even$)$	$\mathbb{R}$ ✓	0 ?	$\mathbb{R}$?
$\mathrm{SO}_0(3,3)$	0 ✓	0 ✓	0 ?
$\mathrm{SO}_0(3+d,3)$ $(d\geq 2$ even$)$	0 ✓	0 ?	0 ?
$\mathrm{SO}_0(4,4)$	0 ✓	0 ?	$\mathbb{R}^3$?
$\mathrm{SO}_0(4+d,4)$ $(d\geq 2$ even$)$	0 ✓	0 ?	$\mathbb{R}^2$?
$\mathrm{SO}_0(r+d,r)$ $(r\geq 5,\ d\geq 0$ even$)$	0 ✓	0 ?	0 ?
$\mathrm{SL}(n,\mathbb{H})$ $(n\geq 3)$	0 ✓	0 ✓	0 (✓)
$\mathrm{Sp}(p,q)$ $(p,q\geq 2)$	0 ✓	0 ?	0 ?
$\mathrm{SO}^*(2n)$ $(n\geq 4)$	$\mathbb{R}$ ✓	0 ?	$\mathbb{R}$?

Table 2.: Continuous cohomology of higher rank classical groups and current state of the isomorphism conjecture. Green: isomorphism conjecture known to hold. Yellow: rank reduction known to be possible (for sufficiently high rank). Orange: surjectivity of the comparison map known; injectivity thereof unknown. Blue: new results obtained in this thesis (with check mark: new instances of the isomorphism conjecture proved; with check mark in parentheses: improvements to rank reduction; with question mark: full calculations or rank reduction improvements for full automorphism groups, into which the relevant semisimple classical groups embed with finite index). Gray: mistake in the literature partially corrected in this thesis (with the injectivity remaining unknown).

Let us now explain the proofs of our Main Theorem, our Rank Reduction Theorem, and our Orthogonal Vanishing Theorem. The first observation is that our groups embed into each other, e.g. we have a natural block embedding

$$\iota_r \colon \mathrm{Sp}(2r, \mathbb{R}) \hookrightarrow \mathrm{Sp}(2r+2, \mathbb{R}).$$

In the following, we will use these embeddings systematically.

Bounded-Cohomological Stabilization: While the methods employed to calculate the bounded cohomology of rank one groups vary widely, by far the most successful method in higher rank is to prove *bounded-cohomological stability* (see Section 2.3). Let $\varphi_r \colon G_r \to G_{r+1}$ be continuous group homomorphisms, $r \in \mathbb{N}$. The sequence $(G_r, \varphi_r)_{r\in\mathbb{N}}$ is called *bounded-cohomologically stable* if there exists a function $\rho \colon \mathbb{N} \to \mathbb{N}$ such that

$$\mathrm{H}^q_{\mathrm{cb}}\big(\varphi_{\rho(q)+s}\big) \colon \mathrm{H}^q_{\mathrm{cb}}\big(G_{\rho(q)+s+1}\big) \to \mathrm{H}^q_{\mathrm{cb}}\big(G_{\rho(q)+s}\big)$$

is an isomorphism for all $s \in \mathbb{N}$. This kind of stability is usually proved by using an analogue of Quillen's stability criterion (see [110] and [44]).

In [101] Monod proved bounded-cohomological stability of $(\mathrm{SL}(r+1, \mathbb{K}), \iota_r)_r$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, where ι_r denotes the left corner inclusion. Explicitly, in degree three he obtained the following injections/isomorphisms:

$$\ldots \xrightarrow{\cong} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(5, \mathbb{K})) \xrightarrow{\cong} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(4, \mathbb{K})) \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(3, \mathbb{K})) \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(2, \mathbb{K}))$$

Together with the previously known results in rank one, namely $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(2, \mathbb{R})) = 0$ and $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(2, \mathbb{C})) = \mathbb{R}$ (see [27] and [5]), we can see that $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(r+1, \mathbb{R})) = 0$ for all $r \geq 1$ and that $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(r+1, \mathbb{C}))$ is at most one-dimensional for all $r \geq 1$.

In [43] De la Cruz Mengual and Hartnick proved bounded-cohomological stabilization for most other families of classical groups, including the families $\mathrm{SO}(2n+1, \mathbb{C})$, $\mathrm{Sp}(2n, \mathbb{C})$, $\mathrm{SU}(p, q)$, $\mathrm{SO}_0(r+d, r)$ (d odd), and $\mathrm{Sp}(2n, \mathbb{R})$. Their stability ranges have been improved to linear stability ranges for most split classical groups by Kastenholz and Sroka [85]. In contrast to Monod's result, the stability ranges obtained are too high to yield a reduction to a group whose third bounded cohomology is known.

Secondary Stability: To circumvent the issue of stability ranges not being low enough, we employ the concept of *secondary cohomological stability.* This concept was introduced in the setting of discrete groups by Galatius–Kupers–Randal-Williams [57] and was subsequently used in the context of the isomorphism conjecture in bounded cohomology by De la Cruz Mengual [40], [42] for complex classical groups in degree 3.

The idea is to prove that the "stability-failure" of a family $(G_r, \iota_r)_r$ is stable, i.e. to prove that $(\ker(\mathrm{H}^n_{\mathrm{cb}}(\iota_r)))_r$ or $(\mathrm{coker}(\mathrm{H}^n_{\mathrm{cb}}(\iota_r)))_r$ is stable. The stability of the kernels can then be used to decrease the injectivity range, while the stability of the cokernels can be used to decrease the surjectivity range.

To prove stability of the relevant kernels or cokernels, we will use an argument involving a spectral sequence to prove that these kernels or cokernels are isomorphic to the *action cohomology* $\mathrm{H}^k_{\mathrm{mb}}(G_r \curvearrowright \mathcal{P}_r)$; this cohomology is defined by equivalence classes of G_r-invariant L^∞-functions on $\mathcal{P}_r^{k+1}$, where $\mathcal{P}_r$ denotes the *isotropic projective variety* corresponding to G_r (see Section 1.1 for its definition).

Secondary Stability Theorem (see Theorem 3.1.6). *For every $k \geq 1$ we have a linear isomorphism*

$$\mathrm{H}^k_{\mathrm{mb}}(G_{r+1} \curvearrowright \mathcal{P}_{r+1}) \cong \mathrm{H}^k_{\mathrm{mb}}(G_r \curvearrowright \mathcal{P}_r)$$

for all $r \geq s_k$, where s_k grows linearly in k.

In degree three for $G_r = \mathrm{Sp}(2r, \mathbb{R})$, the spectral sequence argument yields an isomorphism $\ker(\mathrm{H}^3_{\mathrm{cb}}(\iota_r)) \cong \mathrm{H}^3_{\mathrm{mb}}(G_{r+1} \curvearrowright \mathcal{P}_{r+1})$ for all $r \geq 1$. By the "primary" stability results of [43], there exists some $s \geq 1$ such that $\ker(\mathrm{H}^3_{\mathrm{cb}}(\iota_s)) = 0$. Our Secondary Stability Theorem now implies

$$\ker(\mathrm{H}^3_{\mathrm{cb}}(\iota_r)) \cong \mathrm{H}^3_{\mathrm{mb}}(G_{r+1} \curvearrowright \mathcal{P}_{r+1}) \cong \mathrm{H}^3_{\mathrm{mb}}(G_{s+1} \curvearrowright \mathcal{P}_{s+1}) \cong \ker(\mathrm{H}^3_{\mathrm{cb}}(\iota_s)) = 0,$$

which yields the improved stability range

$$\ldots \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}(6,\mathbb{R})) \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}(4,\mathbb{R})) \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}(2,\mathbb{R})).$$

Since $\mathrm{Sp}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$, we have

$$\mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}(2r,\mathbb{R})) \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}(2,\mathbb{R})) = \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(2,\mathbb{R})) = 0$$

for all $r \geq 1$, which yields our Main Theorem.

We will prove the secondary stability theorem by parametrizing the configuration spaces $G_r \backslash \mathcal{P}^k_r$ for all $k \geq 1$. Inside of our stability range $G_r \backslash \mathcal{P}^k_r$ is, in a suitable sense, isomorphic to the product of a Euclidean space (generalizing the classical *cross-ratio*) and a compact space of dimension at most one (generalizing *Cartan's angular invariant*), see Chapter 3.

We will also show that $\mathcal{P}_r$ is isomorphic to the homogeneous space G_r/P, where P is a *maximal* parabolic subgroup of G_r. Thus, our Secondary Stability Theorem can be viewed in the context of reductive Lie groups acting on *generalized flag manifolds*, i.e. as a result in *parabolic geometry*.

Unfortunately, even our Secondary Stability Theorem does not yield stability ranges low enough to give any more direct evidence for the isomorphism conjecture for families different from $\mathrm{Sp}(2r,\mathbb{R})$; our method gives partial results, leaving "gaps" consisting of medium-rank groups, which would need to be treated using a different technique.

Reformulation of Results and Further Corollaries: Let us now reformulate our Rank Reduction Theorem; the proofs of the following theorems and corollaries, as well as the proof of our Orthogonal Vanishing Theorem, can be found in Section 6.2.

Theorem A. The induced maps

$$\begin{array}{llll} \ldots \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{U}(5+d,5)) \xrightarrow{\mathrm{H}^3_{\mathrm{cb}}(\iota_4)} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{U}(4+d,4)) \xrightarrow{\mathrm{H}^3_{\mathrm{cb}}(\iota_3)} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{U}(3+d,3)) & (d \in \mathbb{N}), \\ \ldots \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(5+d,5)) \xrightarrow{\mathrm{H}^3_{\mathrm{cb}}(\iota_4)} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(4+d,4)) \xrightarrow{\mathrm{H}^3_{\mathrm{cb}}(\iota_3)} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(3+d,3)) & (d \in \mathbb{N}), \\ \ldots \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}(6,\mathbb{R})) \xrightarrow{\mathrm{H}^3_{\mathrm{cb}}(\iota_2)} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}(4,\mathbb{R})) \xrightarrow{\mathrm{H}^3_{\mathrm{cb}}(\iota_1)} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}(2,\mathbb{R})) & \end{array}$$

are injective.

Proof of Our Main Theorem. Theorem A yields

$$\mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}(2r,\mathbb{R})) \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}(2,\mathbb{R})) = \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(2,\mathbb{R})) = 0 = \mathrm{H}^3_{\mathrm{c}}(\mathrm{Sp}(2r,\mathbb{R})). \qquad \square$$

Theorem B. The induced maps

$$\ldots \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(6+d,6)) \xrightarrow{\mathrm{H}^3_{\mathrm{cb}}(\iota_5)} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(5+d,5)) \xrightarrow{\mathrm{H}^3_{\mathrm{cb}}(\iota_4)} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(4+d,4)) \qquad (d \in \mathbb{N})$$

are surjective.

Theorem C. The induced maps

$$\ldots \to \mathrm{H}^4_{\mathrm{cb}}(\mathrm{O}(6+d,6)) \xrightarrow{\mathrm{H}^4_{\mathrm{cb}}(\iota_5)} \mathrm{H}^4_{\mathrm{cb}}(\mathrm{O}(5+d,5)) \xrightarrow{\mathrm{H}^4_{\mathrm{cb}}(\iota_4)} \mathrm{H}^4_{\mathrm{cb}}(\mathrm{O}(4+d,4)) \qquad (d \in \mathbb{N})$$

are injective.

Theorem D. The induced maps

$$\ldots \to \mathrm{H}^4_{\mathrm{cb}}(\mathrm{O}(9,\mathbb{C})) \xrightarrow{\mathrm{H}^4_{\mathrm{cb}}(\iota_3)} \mathrm{H}^4_{\mathrm{cb}}(\mathrm{O}(7,\mathbb{C})) \xrightarrow{\mathrm{H}^4_{\mathrm{cb}}(\iota_2)} \mathrm{H}^4_{\mathrm{cb}}(\mathrm{O}(5,\mathbb{C})),$$

$$\ldots \to \mathrm{H}^4_{\mathrm{cb}}(\mathrm{Sp}(8,\mathbb{C})) \xrightarrow{\mathrm{H}^4_{\mathrm{cb}}(\iota_3)} \mathrm{H}^4_{\mathrm{cb}}(\mathrm{Sp}(6,\mathbb{C})) \xrightarrow{\mathrm{H}^4_{\mathrm{cb}}(\iota_2)} \mathrm{H}^4_{\mathrm{cb}}(\mathrm{Sp}(4,\mathbb{C})),$$

$$\ldots \to \mathrm{H}^4_{\mathrm{cb}}(\mathrm{O}(8,\mathbb{C})) \xrightarrow{\mathrm{H}^4_{\mathrm{cb}}(\iota_3)} \mathrm{H}^4_{\mathrm{cb}}(\mathrm{O}(6,\mathbb{C})) \xrightarrow{\mathrm{H}^4_{\mathrm{cb}}(\iota_2)} \mathrm{H}^4_{\mathrm{cb}}(\mathrm{O}(4,\mathbb{C}))$$

are injective.

Various theorems in the theory of bounded cohomology now imply our Orthogonal Vanishing Theorem, as well as the following corollaries (see Subsection 6.2.4 for proofs).

Corollary A. The induced maps

$$\ldots \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SU}(5+d,5)) \xrightarrow{\mathrm{H}^3_{\mathrm{cb}}(\iota_4)} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SU}(4+d,4)) \xrightarrow{\mathrm{H}^3_{\mathrm{cb}}(\iota_3)} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SU}(3+d,3)) \quad (d \in \mathbb{N}),$$

$$\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}_0(4+d,4)) \xrightarrow{\mathrm{H}^3_{\mathrm{cb}}(\iota_3)} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}_0(3+d,3)) \ (d \text{ odd}),$$

$$\ldots \to \mathrm{H}^4_{\mathrm{cb}}(\mathrm{SO}_0(6+d,6)) \xrightarrow{\mathrm{H}^4_{\mathrm{cb}}(\iota_5)} \mathrm{H}^4_{\mathrm{cb}}(\mathrm{SO}_0(5+d,5)) \xrightarrow{\mathrm{H}^4_{\mathrm{cb}}(\iota_4)} \mathrm{H}^4_{\mathrm{cb}}(\mathrm{SO}_0(4+d,4)) \ (d \text{ odd}),$$

$$\ldots \to \mathrm{H}^4_{\mathrm{cb}}(\mathrm{SO}(9,\mathbb{C})) \xrightarrow{\mathrm{H}^4_{\mathrm{cb}}(\iota_3)} \mathrm{H}^4_{\mathrm{cb}}(\mathrm{SO}(7,\mathbb{C})) \xrightarrow{\mathrm{H}^4_{\mathrm{cb}}(\iota_2)} \mathrm{H}^4_{\mathrm{cb}}(\mathrm{SO}(5,\mathbb{C}))$$

are injective and the induced maps

$$\ldots \to \mathrm{H}^2_{\mathrm{cb}}(\mathrm{SU}(5+d,5)) \xrightarrow{\mathrm{H}^2_{\mathrm{cb}}(\iota_4)} \mathrm{H}^2_{\mathrm{cb}}(\mathrm{SU}(4+d,4)) \xrightarrow{\mathrm{H}^2_{\mathrm{cb}}(\iota_3)} \mathrm{H}^2_{\mathrm{cb}}(\mathrm{SU}(3+d,3)) \quad (d \in \mathbb{N}),$$

$$\ldots \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}_0(6+d,6)) \xrightarrow{\mathrm{H}^3_{\mathrm{cb}}(\iota_5)} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}_0(5+d,5)) \xrightarrow{\mathrm{H}^3_{\mathrm{cb}}(\iota_4)} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}_0(4+d,4)) \ (d \text{ odd})$$

are isomorphisms.

Corollary B. The isomorphism conjecture holds in degree three for $\mathrm{SO}_0(3,2)$. In particular, we have $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}_0(3,2)) = 0$.

The following corollary is due to a theorem by Monod [103].

Corollary C. Let Γ be an irreducible lattice in $\mathrm{Sp}(2r,\mathbb{R})$, $r \geq 2$. Then $\mathrm{H}^3_{\mathrm{b}}(\Gamma) = 0$.

The following corollary follows from Gromov's mapping theorem, which is a classical and deep result in bounded cohomology.

Corollary D. Let X be a topological space whose fundamental group is an irreducible lattice in $\mathrm{Sp}(2r,\mathbb{R})$ for some $r \geq 2$. Then the third bounded cohomology of X with trivial real coefficients (see [55, Ch. 5] for a precise definition) vanishes.

Example A. Recall that the symmetric space X_r associated to $\mathrm{Sp}(2r,\mathbb{R})$ is known as the *Siegel upper half-space*. The *Siegel modular variety* $\mathcal{A}_{e_1,\dots,e_g}$, which parametrizes principally polarized abelian varieties of type $(e_1,\dots,e_g)$, is a locally symmetric space locally isometric to X_r for some $r \geq 2$. Hence the fundamental group of $\mathcal{A}_{e_1,\dots,e_g}$ is an irreducible lattice in $\mathrm{Sp}(2r,\mathbb{R})$ for some $r \geq 2$, which implies $\mathrm{H}^3_{\mathrm{b}}(\mathcal{A}_{e_1,\dots,e_g}) = 0$. In the same way we have $\mathrm{H}^3_{\mathrm{b}}(\mathcal{A}_{e_1,\dots,e_g}(n)) = 0$ for the Siegel modular variety $\mathcal{A}_{e_1,\dots,e_g}(n)$, which parametrizes principally polarized abelian varieties of type $(e_1,\dots,e_g)$ with level n-structure. For more information on Siegel modular varieties see [79].

Bounded cohomology has been extended to the setting of *measurable groupoids* by Sarti and Savini [113], [114]. Hartnick and Sarti [72] recently proved a cohomological induction statement in this setting, which implies the following corollary.

Corollary E. Let (X,μ,Y) be an ergodic transverse $\mathrm{Sp}(2r,\mathbb{R})$-system for some $r \geq 2$. Then the third measurable bounded cohomology of the associated transverse measured groupoid vanishes.

Lacking references, we will also prove the following (probably well-known) statement.

Proposition A. The induced maps

$$\dots \to \mathrm{H}^2_{\mathrm{cb}}(\mathrm{U}(4+d,4)) \xrightarrow{\mathrm{H}^2_{\mathrm{cb}}(\iota_3)} \mathrm{H}^2_{\mathrm{cb}}(\mathrm{U}(3+d,3)) \xrightarrow{\mathrm{H}^2_{\mathrm{cb}}(\iota_2)} \mathrm{H}^2_{\mathrm{cb}}(\mathrm{U}(2+d,2)) \qquad (d \in \mathbb{N}),$$
$$\dots \to \mathrm{H}^2_{\mathrm{cb}}(\mathrm{Sp}(6,\mathbb{R})) \xrightarrow{\mathrm{H}^2_{\mathrm{cb}}(\iota_2)} \mathrm{H}^2_{\mathrm{cb}}(\mathrm{Sp}(4,\mathbb{R})) \xrightarrow{\mathrm{H}^2_{\mathrm{cb}}(\iota_1)} \mathrm{H}^2_{\mathrm{cb}}(\mathrm{Sp}(2,\mathbb{R}))$$

are isomorphisms, and $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{O}(r+1+d,r+1)) \to \mathrm{H}^2_{\mathrm{cb}}(\mathrm{O}(r+d,r))$ is the zero map for all $r,d \in \mathbb{N}$.

Structure of the Thesis:

(i) Part I provides the necessary foundations. Chapter 1 details the structure of formed spaces and their associated automorphism groups, and provides explicit root space decompositions of classical Lie algebras. Chapter 2 reviews the requisite cohomology theories, including discrete, continuous, and (continuous) bounded cohomology, and formally introduces and discusses the isomorphism conjecture.

Part I consists mostly of previously known results. References for the smooth version of Witt's lemma (Theorem 1.1.6), the root space decompositions of Section 1.3

(in coordinates corresponding to a Witt basis) and the proof of base point independence of cohomology (Lemma 2.1.1) are hard to find in the literature, which is why they are explicated here. Furthermore, in Proposition 2.5.13 we partially correct a mistake in [42].

(ii) Part II forms the technical core of our geometric arguments. We develop parametrizations of configuration spaces of isotropic projective varieties, using as invariants generalizations of the cross-ratio and Cartan's angular invariant.

The theorems proved in Part II are new, except for some cases that were already treated in [42]. While the criterion of Lemma 3.3.1 to obtain isomorphisms of Lebesgue spaces was implicitly used in [40] and [42], the stabilizer criterion of Lemma 3.3.2 seems to be new.

(iii) Part III applies the calculations of the previous chapters to derive the stability results listed above. We employ a spectral sequence associated with the L^∞-double complex of a classical group to prove the injectivity and surjectivity statements.

Section 5.1 consists almost entirely of previously known results, the sole exception being Lemma 5.1.6, which is an extension of an argument by Monod [104]. The results of Section 5.2 were largely known for complex groups, while the results for non-complex groups are mostly new. Proposition 5.2.7 is an extension of an extremely useful argument by Monod [104]. While bounded-cohomological stability was known for all families of classical groups considered, the ranges obtained in Chapter 6 are strict improvements in all cases.

Outlook: Let us highlight two open questions (see Section 6.3 for a more detailed discussion).

(i) Can the geometric methods employed for rank one groups, as well as the cohomological stabilization techniques for groups of higher rank, be extended to yield additional evidence for the isomorphism conjecture for the quaternionic classical groups $\mathrm{Sp}(p,q)$ and $\mathrm{SO}^*(2n)$?

(ii) Our results often leave gaps for medium-rank groups. Can the methods of [71] or a better understanding of the functional equations corresponding to cross-ratios be used to fill these gaps?

Part I.

Background Material

1. The Classical Groups

1.1. Formed Spaces

In this section we define the classical formed spaces in a unified way and state a strong version of Witt's lemma.

Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $\sigma\colon \mathbb{K} \to \mathbb{K}$ be a continuous field automorphism, i.e. either the identity or, if $\mathbb{K} = \mathbb{C}$, complex conjugation. Let V and V' be finite-dimensional $\mathbb{K}$-vector spaces.

We call $f\colon V \to V'$ a *σ-linear* map if f is additive and

$$f(\lambda v) = \sigma(\lambda) f(v) \qquad (\lambda \in \mathbb{K},\ v \in V).$$

We call $\omega\colon V \times V \to \mathbb{K}$ a *σ-sesquilinear* form on V if ω is additive, $\omega(-, v)$ is linear and $\omega(v, -)$ is σ-linear for all $v \in V$. Let ω be a σ-sesquilinear form on V and let ω' be a σ-sesquilinear form on V'. A *morphism* from (V, ω) to (V', ω') is a linear map $f\colon V \to V'$ such that

$$\omega'(f(v), f(w)) = \omega(v, w) \qquad (v, w \in V).$$

We call ω *reflexive* if for all $v, w \in V$ the statements $\omega(v, w) = 0$ and $\omega(w, v) = 0$ are equivalent. In this case, we call (V, ω) a *formed space*. We obtain the category of formed spaces, in which the automorphisms of a formed space (V, ω) are the bijective self-morphisms of (V, ω); the group of automorphisms of (V, ω) is denoted by $\mathrm{Aut}(V, \omega)$.

Let (V, ω) be a formed space and let X be a subset of V. The $\mathbb{K}$-subspace

$$X^\perp := \{v \in V \mid \omega(v, x) = 0\ \forall x \in X\}$$

of V is called the space *orthogonal* to X. A subspace W of V is called *totally isotropic* if $W \subset W^\perp$. We call (V, ω) or ω *non-degenerate* if $V^\perp$ is trivial. The induced map

$$\overline{\omega}\colon \left(V/V^\perp\right) \times \left(V/V^\perp\right) \to \mathbb{K},\ \left(v + V^\perp, w + V^\perp\right) \mapsto \omega(v, w)$$

is well-defined, and $\left(V/V^\perp, \overline{\omega}\right)$ is a non-degenerate formed space.

Let (V, ω) and (V', ω') be formed spaces. We define a σ-sesquilinear form $\omega \sqcup \omega'$ on the direct sum $V \sqcup V'$ of V and V' by setting

$$(\omega \sqcup \omega')((v, v'), (w, w')) := \omega(v, w) + \omega'(v', w') \qquad (v, w \in V,\ v', w' \in V').$$

One can easily check that $(V, \omega) \sqcup (V', \omega') := (V \sqcup V', \omega \sqcup \omega')$, together with the inclusions of V and V' into $V \sqcup V'$, defines a coproduct of V and V' in the category of formed spaces.

Remark 1.1.1. We have $(V,\omega) \cong \big(V/V^{\perp},\overline{\omega}\big) \sqcup \big(V^{\perp},0\big)$, where 0 denotes the constant zero form. Hence we restrict ourselves to non-degenerate forms in the following.

Theorem 1.1.2 (Birkhoff–von Neumann [4])**.** *Let (V,ω) be a non-degenerate formed space, where $\dim(V) > 1$. Then exactly one of the following statements holds:*

(i) ω *is* symmetric*, i.e.* $\sigma = \mathrm{id}_{\mathbb{K}}$ *and* $\omega(v,w) = \omega(w,v)$ *for all* $v, w \in V$*,*

(ii) ω *is* skew-symmetric*, i.e.* $\sigma = \mathrm{id}_{\mathbb{K}}$ *and* $\omega(v,w) = -\omega(w,v)$ *for all* $v, w \in V$*,*

(iii) $\mathbb{K} = \mathbb{C}$, $\sigma = \overline{\cdot}$ *is complex conjugation, and* (V,ω) *is isomorphic to a formed space* (V,ω')*, where* ω' *is* Hermitian*, i.e.* $\omega'(v,w) = \overline{\omega'(w,v)}$ *for all* $v, w \in V$*.*

Proof. See [109, Proposition 6.1, Proposition 7.13], or [45, Ch. I, §6]. ☐

Remark 1.1.3. By choosing bases, we can identify V and $\mathrm{End}_{\mathbb{K}}(V)$ with $\mathbb{K}^n$ and $\mathbb{K}^{n^2}$, respectively, where $n = \dim(V)$; this turns V and $\mathrm{End}_{\mathbb{K}}(V)$ into smooth manifolds, even into affine $\mathbb{K}$-algebraic varieties. The restriction of the smooth structure of $\mathrm{End}_{\mathbb{K}}(V)$ to the general linear group $\mathrm{GL}_{\mathbb{K}}(V)$ of V turns the open subset $\mathrm{GL}_{\mathbb{K}}(V) = \det^{-1}(\mathbb{K}^{\times})$ into a smooth manifold. On the other hand, we can view $\mathrm{GL}_{\mathbb{K}}(V)$ as an affine $\mathbb{K}$-algebraic variety in $\mathbb{K}^{n^2+1}$. Cramer's rule implies that $\mathrm{GL}_{\mathbb{K}}(V)$ is a linear Lie group, even a linear $\mathbb{K}$-algebraic group. Furthermore, $\mathrm{GL}_{\mathbb{K}}(V) \curvearrowright V$ yields a natural smooth and even $\mathbb{K}$-algebraic action map.

The automorphism group $\mathrm{Aut}(V,\omega)$ of a formed space (V,ω) is a linear $\mathbb{R}$-algebraic subgroup of $\mathrm{GL}_{\mathbb{K}}(V)$ and the natural action $\mathrm{Aut}(V,\omega) \curvearrowright V$ is $\mathbb{R}$-algebraic. In particular, $\mathrm{Aut}(V,\omega)$ is a closed subgroup of the Lie group $\mathrm{GL}_{\mathbb{K}}(V)$, hence, by Cartan's closed subgroup theorem, $\mathrm{Aut}(V,\omega)$ admits a unique smooth structure that turns it into a Lie group embedded in $\mathrm{GL}_{\mathbb{K}}(V)$.

We call a non-degenerate formed space (V,ω) a(n)

- *orthogonal geometry* if ω is symmetric. In this case we call $\mathrm{O}(V,\omega) := \mathrm{Aut}(V,\omega)$ the *orthogonal group* of (V,ω),
- *symplectic geometry* if ω is skew-symmetric. In this case we call $\mathrm{Sp}(V,\omega) := \mathrm{Aut}(V,\omega)$ the *symplectic group* of (V,ω),
- *unitary geometry* if ω is Hermitian. In this case we call $\mathrm{U}(V,\omega) := \mathrm{Aut}(V,\omega)$ the *unitary group* of (V,ω).

We call (V,ω) a $(\mathbb{K},\sigma)$-*ordinary geometry* or just an *ordinary geometry* if (V,ω) is of any of these three types. If (V,ω) is an ordinary geometry, there exists $\varepsilon_\omega \in \{\pm 1\}$ such that $\omega(v,w) = \varepsilon_\omega \sigma(\omega(w,v))$ for all $v, w \in V$, and $(\sigma, \varepsilon_\omega) \in \{(\mathrm{id}_{\mathbb{K}}, 1), (\mathrm{id}_{\mathbb{K}}, -1), (\overline{\cdot}, 1)\}$.

Theorem 1.1.4 (Witt's Lemma)**.** *Let (V,ω) be a non-degenerate formed space, let $(W,\omega|_{W\times W})$ be a formed subspace and let $f\colon W \to V$ be an injective morphism. Then there exists $g \in \mathrm{Aut}(V,\omega)$ such that $g|_W = f$.*

Proof. See [120, Theorem 7.4]. ☐

Remark 1.1.5. It follows that any two maximal totally isotropic subspaces of V have the same dimension. This dimension is called the *rank* of (V,ω).

If (V,ω) is a non-degenerate formed space, then for all $k \leq \dim_{\mathbb{K}}(V)$ we define the open conull submanifold

$$V^{(k)} := \{v = (v_1,\dots,v_k) \in V^k \mid v_1,\dots,v_k \text{ are linearly independent}\}$$

of V^k. Now we consider the subspace

$$M^k(V,\omega) := \big\{(v,w) \in V^{(k)} \times V^{(k)} \,\big|\, \text{the map } v_i \mapsto w_i \text{ is a morphism}\big\},$$

which by the preimage theorem is an embedded smooth submanifold of $V^{(k)} \times V^{(k)}$. Clearly, $\operatorname{Aut}(V,\omega) \curvearrowright M^k(V,\omega)$ yields a natural smooth action map. Looking at the proof of [120, Theorem 7.4], we can conclude the following strong version of Witt's lemma.

Theorem 1.1.6 (Witt's Lemma, Strong Version)**.** *Let (V,ω) be a non-degenerate formed space. Then for all $k \leq \dim_{\mathbb{K}}(V)$ there exists a map $M^k(V,\omega) \to \operatorname{Aut}(V,\omega)$ sending each $(v,w) \in M^k(V,\omega)$ to an extension of $v_i \mapsto w_i$ in $\operatorname{Aut}(V,\omega)$, which is smooth on an $\operatorname{Aut}(V,\omega)$-invariant open conull submanifold of $M^k(V,\omega)$.*

We call $(x,y) \in V \times V$ a *hyperbolic pair* with respect to ω if $\omega(x,x) = 0 = \omega(y,y)$ and $\omega(x,y) = 1$. A *hyperbolic plane* is a formed space spanned by a hyperbolic pair. We call a formed space (V,ω) *totally anisotropic* if $\omega(v,v) \neq 0$ for all $v \in V \setminus \{0\}$.

Theorem 1.1.7 (Witt)**.** *Let (V,ω) be a non-degenerate formed space of rank r. There exist hyperbolic planes $H_1,\dots,H_r$ and an up to isomorphism unique totally anisotropic formed space A such that $(V,\omega) \cong \left(\bigsqcup_{i=1}^r H_i\right) \sqcup A$.*

Proof. See [120, Chapter 7, Section "Flags and Frames"]. □

In particular, there exist linearly independent vectors $e_r,\dots,e_1,f_1,\dots,f_r \in V$ such that $\omega(e_i,e_j) = \omega(f_i,f_j) = 0$ and $\omega(e_i,f_j) = \delta_{ij}$, $i,j = 1,\dots,r$. We say that $(e_r,\dots,e_1,f_1,\dots,f_r)$ is *adapted* to ω. The ordered standard basis of $\mathbb{K}^{2r}$ is adapted to the form

$$\omega^r_{\sigma,\varepsilon}((a_r,\dots,a_1,b_1,\dots,b_r),(a'_r,\dots,a'_1,b'_1,\dots,b'_r)) = \sum_{i=1}^r \sigma(a_i)b'_i + \varepsilon a'_i\sigma(b_i)$$

for all $(\sigma,\varepsilon) \in \{(\mathrm{id}_{\mathbb{K}},1),(\mathrm{id}_{\mathbb{K}},-1),(\overline{\cdot},1)\}$.

Remark 1.1.8 (Classification of Symplectic Geometries). Let (V,ω) be a symplectic geometry of rank r. Then $\omega(v,v) = 0$ for all $v \in V$, so $(V,\omega) \cong \bigsqcup_{i=1}^r H_i \cong \big(\mathbb{K}^{2r},\omega^r_{\mathrm{id}_{\mathbb{K}},-1}\big)$. Thus, for every $r \in \mathbb{N}$ there exists an up to isomorphism unique symplectic geometry of rank r. We call $\big(\mathbb{K}^{2r},\omega^r_{\mathrm{id}_{\mathbb{K}},-1}\big)$ the *standard symplectic space* of rank r. We call $\operatorname{Sp}(2r,\mathbb{K}) := \operatorname{Sp}\big(\mathbb{K}^{2r},\omega^r_{\mathrm{id}_{\mathbb{K}},-1}\big)$ the *symplectic group* of rank r over $\mathbb{K}$.

Lemma 1.1.9 (Hyperbolic Gram–Schmidt)**.** *Let (V,ω) be an ordinary geometry of rank r and let $e_1,f_1 \in V$ satisfy $\omega(e_1,f_1) = 1$. There exist $e_r,\dots,e_2,f_2,\dots,f_r \in V$ such that $\omega(e_i,e_j) = \omega(f_i,f_j) = 0$, $i,j = 1,\dots,r$, and $\omega(e_i,f_j) = \delta_{ij}$, $i,j = 1,\dots,r$.*

Proof. The claim is trivial if $r = 1$. If $r > 1$, then we assume that $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$, $i, j = 1, \dots, r-1$, and $\omega(e_i, f_j) = \delta_{ij}$, $i, j = 1, \dots, r-1$. Let $v \in V$ be linearly independent from $e_{r-1}, \dots, e_1, f_1, \dots, f_{r-1}$ and set

$$e_r = v - \varepsilon_\omega \sum_{i=1}^{r-1} \omega(f_i, v)e_i + \sigma(\omega(v, e_i))f_i.$$

Then $\omega(e_r, e_r) = 0$ and $\omega(e_r, e_i) = \omega(e_r, f_i) = 0$, $i = 1, \dots, r-1$. Since ω is non-degenerate, there exists $f'_r \in V$ such that $\omega(e_r, f'_r) = 1$. Now set

$$f_r = f'_r - \varepsilon_\omega \sum_{i=1}^{r-1} \omega(f_i, f'_r)e_i + \sigma(\omega(f'_r, e_i))f_i.$$

Then $\omega(e_i, f_r) = \omega(f_i, f_r) = 0$, $i = 1, \dots, r-1$, and $\omega(e_r, f_r) = \omega(e_r, f'_r) = 1$. □

Remark 1.1.10. If (V, ω) is symplectic, then the vectors above can be chosen to form a basis adapted to ω.

Lemma 1.1.11 (Anisotropic Gram–Schmidt)**.** *Let (A, ω) be a totally anisotropic ordinary geometry, $d = \dim(A)$. Then there exists a basis $h_1, \dots, h_d$ of A such that $\omega(h_k, h_l) = 0$, $k, l = 1, \dots, d$, $k \neq l$.*

Proof. The claim is trivial if $d = 1$. If $d > 1$, let $h_1, \dots, h_{d-1}, h'_d$ be a basis of A and assume that $\omega(h_k, h_l) = 0$, $k, l = 1, \dots, d-1$, $k \neq l$. Set

$$h_d = h'_d - \sum_{k=1}^{d-1} \frac{\omega(h_k, h'_d)}{\omega(h_k, h_k)} h_k.$$

Then $h_1, \dots, h_d$ is a basis of A such that $\omega(h_k, h_l) = 0$, $k, l = 1, \dots, d$, $k \neq l$. □

Corollary 1.1.12. *Every ordinary geometry $(V, \omega) \cong \left(\bigsqcup_{i=1}^r H_i\right) \sqcup A$ admits a* Witt basis*, i.e. a basis $(e_r, \dots, e_1, h_1, \dots, h_d, f_1, \dots, f_r)$ such that H_i is a hyperbolic plane spanned by e_i and f_j satisfying $\omega(e_i, f_j) = \delta_{ij}$, $i, j = 1, \dots, r$, and $h_1, \dots, h_d$ span A and satisfy $\omega(h_k, h_l) = 0$, $k, l = 1, \dots, d$, $k \neq l$.*

Remark 1.1.13. If $\mathbb{K} = \mathbb{C}$, then the basis above can be chosen to satisfy $\omega(h_k, h_k) = 1$, $k = 1, \dots, d$. If $\mathbb{K} = \mathbb{R}$, then it can be chosen to satisfy $\omega(h_k, h_k) = \pm 1$, $k = 1, \dots, d$.

Let $\lambda_1, \dots, \lambda_d \in \mathbb{K}$. We define the form

$$\langle \lambda_1, \dots, \lambda_d \rangle \colon \mathbb{K}^d \times \mathbb{K}^d \to \mathbb{K},\ ((a_1, \dots, a_d), (b_1, \dots, b_d)) \mapsto \sum_{k=1}^d \lambda_k \sigma(a_k) b_k.$$

Remark 1.1.14 (Classification of Complex Orthogonal Geometries). Let $\mathbb{K} = \mathbb{C}$, let (V, ω) be an orthogonal geometry of rank r with totally anisotropic part A, and let $d = \dim(A)$. Assume that $d \geq 2$. Then there exist $h_1, h_2 \in A$ such that $\omega(h_k, h_l) = \delta_{kl}$, $k, l = 1, 2$, but $\omega(h_1 + \mathrm{i}h_2, h_1 + \mathrm{i}h_2) = 0$, which is a contradiction. Hence $d \leq 1$. If $d = 0$, then (V, ω)

is isomorphic to $\big(\mathbb{C}^{2r}, \omega^{r,0}_{\mathrm{id}_\mathbb{C},1}\big) := \big(\mathbb{C}^{2r}, \omega^{r}_{\mathrm{id}_\mathbb{C},1}\big)$. If $d = 1$, then there exists $h \in A \setminus \{0\}$ such that $\omega(h,h) = 1$, so (V,ω) is isomorphic to $\big(\mathbb{C}^{2r+1}, \omega^{r,1}_{\mathrm{id}_\mathbb{C},1}\big) \cong \big(\mathbb{C}^{2r}, \omega^{r}_{\mathrm{id}_\mathbb{C},1}\big) \sqcup (\mathbb{C}, \langle 1\rangle)$, where

$$\begin{aligned}&\omega^{r,1}_{\mathrm{id}_\mathbb{C},1}((a_r,\dots,a_1,c,b_1,\dots,b_r),(a'_r,\dots,a'_1,c',b'_1,\dots,b'_r))\\ =&\left(\sum_{i=1}^{r} a_i b'_i + a'_i b_i\right) + cc'.\end{aligned}$$

Thus, for every $r \in \mathbb{N}$ there exist up to isomorphism exactly two complex orthogonal geometries of rank r. We call $\left(\mathbb{C}^{2r+d}, \omega^{r,d}_{\mathrm{id}_\mathbb{C},1}\right)$ the *standard orthogonal space* of rank r and parameter d. We call $\mathrm{O}(2r+d,\mathbb{C}) := \mathrm{O}\left(\mathbb{C}^{2r+d}, \omega^{r,d}_{\mathrm{id}_\mathbb{C},1}\right)$ the *complex orthogonal group* of rank r and parameter d.

Remark 1.1.15 (Classification of Real Orthogonal and Complex Unitary Geometries). Let either $\mathbb{K} = \mathbb{R}$ and (V,ω) be an orthogonal geometry, or $\mathbb{K} = \mathbb{C}$ and (V,ω) be a unitary geometry. Let r be the rank of (V,ω) and A its totally anisotropic part. We have $\omega(v,v) \in \mathbb{R}$ for all $v \in V$. By the intermediate value theorem, either $\omega(v,v) > 0$ for all $v \in A$ or $\omega(v,v) < 0$ for all $v \in A$. Let $h_1,\dots,h_d$ be a basis of A such that $\omega(h_k,h_l) = \varepsilon\delta_{kl}$, where $\varepsilon \in \{\pm 1\}$, and $\varepsilon = 1$ if $\mathbb{K} = \mathbb{C}$. Then (V,ω) is isomorphic to $\left(\mathbb{K}^{2r+d}, \omega^{r,d,\varepsilon}_{\sigma,1}\right) \cong \big(\mathbb{K}^{2r}, \omega^{r}_{\sigma,1}\big) \sqcup (\mathbb{K}^d, \langle \varepsilon,\dots,\varepsilon\rangle)$, where

$$\begin{aligned}&\omega^{r,d,\varepsilon}_{\sigma,1}((a_r,\dots,a_1,c_1,\dots,c_d,b_1,\dots,b_r),(a'_r,\dots,a'_1,c'_1,\dots,c'_d,b'_1,\dots,b'_r))\\ =&\left(\sum_{i=1}^{r} \sigma(a_i) b'_i + a'_i \sigma(b_i)\right) + \sum_{k=1}^{d} \varepsilon\sigma(c_k)c'_k.\end{aligned}$$

Thus, for every $r \in \mathbb{N}$, $d \geq 1$ there exist up to isomorphism exactly two real orthogonal or complex unitary geometries of rank r and parameter d, namely $\left(\mathbb{K}^{2r+d}, \omega^{r,d,-1}_{\sigma,1}\right)$ and $\left(\mathbb{K}^{2r+d}, \omega^{r,d,1}_{\sigma,1}\right)$. If $d = 0$, then for every $r \in \mathbb{N}$ there exists an up to isomorphism unique real orthogonal or complex unitary geometry of rank r and parameter d, namely $\big(\mathbb{K}^{2r}, \omega^{r,0,\pm 1}_{\sigma,1}\big)$. We call $\left(\mathbb{R}^{2r+d}, \omega^{r,d,\varepsilon}_{\sigma,1}\right)$ the *standard orthogonal space* of rank r, parameter d and sign ε, and we call $\mathrm{O}(r, d+r) := \mathrm{O}\left(\mathbb{R}^{2r+d}, \omega^{r,d,-1}_{\sigma,1}\right)$, respectively $\mathrm{O}(r+d, r) := \mathrm{O}\left(\mathbb{R}^{2r+d}, \omega^{r,d,1}_{\sigma,1}\right)$, the *real orthogonal group* of rank r, parameter d and sign -1, respectively sign 1. We call $\left(\mathbb{C}^{2r+d}, \omega^{r,d,\varepsilon}_{\sigma,1}\right)$ the *standard unitary space* of rank r, parameter d and sign ε, and we call $\mathrm{U}(r, r+d) := \mathrm{U}\left(\mathbb{C}^{2r+d}, \omega^{r,d,-1}_{\sigma,1}\right)$, respectively $\mathrm{U}(r+d, r) := \mathrm{U}\left(\mathbb{C}^{2r+d}, \omega^{r,d,1}_{\sigma,1}\right)$, the *unitary group* of rank r, parameter d and sign -1, respectively sign 1.

To unify notation, we also write $\omega^{r,0,1}_{\mathrm{id}_\mathbb{K},-1} := \omega^{r}_{\mathrm{id}_\mathbb{K},-1}$ and $\omega^{r,d,1}_{\mathrm{id}_\mathbb{C},1} := \omega^{r,d}_{\mathrm{id}_\mathbb{C},1}$.

Definition 1.1.16. We say that a formed space (V,ω) of rank r is a *classical space* if there exists $d \in \mathbb{N}$ and $\varepsilon \in \{\pm 1\}$ such that $(V,\omega) = \big(\mathbb{K}^{2r+d}, \omega^{r,d,\varepsilon}_{\sigma,\varepsilon_\omega}\big)$.

Corollary 1.1.17. *Any non-degenerate formed space of dimension* > 1 *is isomorphic to a classical space.*

Let (V_r, ω) be a classical space of rank $r \geq 1$, with automorphism group $G_r := \operatorname{Aut}(V_r, \omega)$ and Witt basis $(e_r, \dots, e_1, h_1, \dots, h_d, f_1, \dots, f_r)$. We define its *associated quadratic form* $q\colon V_r \to \mathbb{K}$, $v \mapsto \omega(v, v)$ and its *isotropic projective variety*

$$\mathcal{P}_r := \{[v] \in \mathbb{P}(V_r) \mid q(v) = 0\}.$$

Note that the diagonal action $G_r \curvearrowright \mathcal{P}_r^k$ yields a smooth and even $\mathbb{R}$-algebraic action map for every $k \geq 1$. One of our goals will be to understand the orbit spaces $G_r \backslash \mathcal{P}_r^k$, $k \geq 1$, up to G_r-invariant null sets. Hence we consider the G_r-invariant open conull submanifolds

$$\mathcal{P}_r^{(k)} := \left\{ v = [v_1, \dots, v_k] \in \mathcal{P}_r^k \;\middle|\; \omega(v_i, v_j) \neq 0,\ i \neq j \right\},$$
$$\mathcal{P}_r^{\{k\}} := \left\{ v \in \mathcal{P}_r^{(k)} \;\middle|\; v \text{ is in general position} \right\}$$

of $\mathcal{P}_r^k$, where we say that $[v_1, \dots, v_k] \in \mathcal{P}_r^{(k)}$ is *in general position* if every m-subtuple of $(v_1, \dots, v_k)$ spans an m-dimensional subspace of V_r, where $m := \min(\dim_{\mathbb{K}}(V_r), k)$.

Next, we define the G_r-invariant *pseudo-sphere* $S_{r,\rho} := \{v \in V_r \setminus \{0\} \mid q(v) = \rho\}$ of pseudo-radius $\rho \in \mathbb{R}$. Note that $S_{r,0} = V_r \setminus \{0\}$ if ω is alternating.

Lemma 1.1.18. *(i) The action $G_r \curvearrowright S_{r,\rho}$ is transitive for all $\rho \in \mathbb{R}$. In particular, the action $G_r \curvearrowright \mathcal{P}_r$ is transitive.*

(ii) The action $G_r \curvearrowright \mathcal{P}_r^{(2)}$ is transitive.

Proof. (i) Let $v, w \in S_{r,\rho}$. By Witt's lemma, the isometric embedding $v \mapsto w$ extends to an element of G_r.

(ii) Let $[v_0, v_1] \in \mathcal{P}_r^{(2)}$. By Witt's lemma, the isometric embedding $v_0 \mapsto \omega(v_0, v_1)e_r$, $v_1 \mapsto f_r$ extends to an element of G_r. □

Corollary 1.1.19. *The orbit spaces $G_r \backslash \mathcal{P}_r$ and $G_r \backslash \mathcal{P}_r^{(2)}$ are trivial.*

Notation 1.1.20. For a classical space (V_r, ω) of rank $r \geq 1$ we will always use the following notation: We have a field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, a continuous automorphism σ of $\mathbb{K}$, a σ-sesquilinear form ω on the finite-dimensional $\mathbb{K}$-vector space V_r such that $(V_r, \omega) = \left(\mathbb{K}^{2r+d}, \omega_{\sigma,\varepsilon_\omega}^{r,d,\varepsilon}\right)$ for some $d \in \mathbb{N}$ and $\varepsilon \in \{\pm 1\}$, and $(e_r, \dots, e_1, h_1, \dots, h_d, f_1, \dots, f_r)$ is a Witt basis for (V, ω); the automorphism group of (V_r, ω) is denoted by G_r, the corresponding quadratic form is denoted by q, the corresponding isotropic projective variety is denoted by $\mathcal{P}_r$, and the pairwise non-orthogonal k-tuples of $\mathcal{P}_r$ in general position are denoted by $\mathcal{P}_r^{\{k\}}$.

1.2. The Classical Groups and their Embeddings

The inclusion

$$\mathbb{K}^{2r+d} \hookrightarrow \mathbb{K}^{2(r+1)+d},$$
$$(a_r, \dots, a_1, c_1, \dots, c_d, b_1, \dots, b_r) \mapsto (0, a_r, \dots, a_1, c_1, \dots, c_d, b_1, \dots, b_r, 0)$$

induces monomorphisms

$$\begin{array}{rcl}\left(\mathbb{K}^d,\omega^{0,d,\varepsilon}_{\sigma,\varepsilon_\omega}\right)\hookrightarrow & \left(\mathbb{K}^{2+d},\omega^{1,d,\varepsilon}_{\sigma,\varepsilon_\omega}\right)\hookrightarrow & \left(\mathbb{K}^{4+d},\omega^{2,d,\varepsilon}_{\sigma,\varepsilon_\omega}\right)\hookrightarrow\dots\\ \operatorname{Aut}\left(\mathbb{K}^d,\omega^{0,d,\varepsilon}_{\sigma,\varepsilon_\omega}\right)\hookrightarrow & \operatorname{Aut}\left(\mathbb{K}^{2+d},\omega^{1,d,\varepsilon}_{\sigma,\varepsilon_\omega}\right)\hookrightarrow & \operatorname{Aut}\left(\mathbb{K}^{4+d},\omega^{2,d,\varepsilon}_{\sigma,\varepsilon_\omega}\right)\hookrightarrow\dots\end{array}$$

We obtain the embeddings

$$1\hookrightarrow \mathrm{Sp}(2,\mathbb{C})\hookrightarrow \mathrm{Sp}(4,\mathbb{C})\hookrightarrow\dots \tag{1.2.1}$$

$$1\hookrightarrow \mathrm{O}(2,\mathbb{C})\hookrightarrow \mathrm{O}(4,\mathbb{C})\hookrightarrow\dots \tag{1.2.2}$$

$$\{\pm1\}\hookrightarrow \mathrm{O}(3,\mathbb{C})\hookrightarrow \mathrm{O}(5,\mathbb{C})\hookrightarrow\dots \tag{1.2.3}$$

$$1\hookrightarrow \mathrm{Sp}(2,\mathbb{R})\hookrightarrow \mathrm{Sp}(4,\mathbb{R})\hookrightarrow\dots \tag{1.2.4}$$

$$\mathrm{O}(d,0)\hookrightarrow \mathrm{O}(1+d,1)\hookrightarrow \mathrm{O}(2+d,2)\hookrightarrow\dots \tag{1.2.5}$$

$$\mathrm{U}(d,0)\hookrightarrow \mathrm{U}(1+d,1)\hookrightarrow \mathrm{U}(2+d,2)\hookrightarrow\dots \tag{1.2.6}$$

$$\mathrm{O}(0,d)\hookrightarrow \mathrm{O}(1,d+1)\hookrightarrow \mathrm{O}(2,d+2)\hookrightarrow\dots \tag{1.2.7}$$

$$\mathrm{U}(0,d)\hookrightarrow \mathrm{U}(1,d+1)\hookrightarrow \mathrm{U}(2,d+2)\hookrightarrow\dots \tag{1.2.8}$$

of Lie groups, where the groups in the sequences (1.2.1), (1.2.2), and (1.2.3) admit structures of complex Lie groups. The sequence (1.2.7) can be treated analogously to sequence (1.2.5), and (1.2.8) analogously to (1.2.6), so we will usually ignore the sequences (1.2.7) and (1.2.8).

The groups in the sequences (1.2.1)–(1.2.8) are called *(reductive) classical groups* and the corresponding formed spaces are exactly the classical spaces.

Together with the families of general linear groups $\mathrm{GL}(r+1,\mathbb{R})$, $\mathrm{GL}(r+1,\mathbb{C})$, $\mathrm{GL}(r+1,\mathbb{H})$, the *quaternion orthogonal groups* $\mathrm{O}^*(4r+2d)$, and the *indefinite symplectic groups* $\mathrm{Sp}(r+d,r),\mathrm{Sp}(r,d+r)$, we arrive at the well-known complete list of (reductive) classical groups. All of these groups are indeed reductive and can be realized as automorphism groups of more general formed spaces, see [109].

To exhibit the classical groups as matrix groups, we define the matrices

$$J_r=\begin{pmatrix}0&&1\\&\iddots&\\1&&0\end{pmatrix}\in\mathfrak{gl}(r,\mathbb{R}),$$

$$B_{r,d}=\begin{pmatrix}0&0&J_r\\0&\mathrm{I}_d&0\\J_r&0&0\end{pmatrix}\in\mathfrak{gl}(2r+d,\mathbb{R}),$$

$$C_r=\begin{pmatrix}0&J_r\\-J_r&0\end{pmatrix}\in\mathfrak{gl}(2r,\mathbb{R}).$$

Using the respective Witt bases to identify automorphisms of classical spaces with the corresponding matrices, we have

$$\begin{aligned}\mathrm{O}(2r+d,\mathbb{C})&=\left\{g\in\mathrm{GL}(2r+d,\mathbb{C})\mid g^\top B_{r,d}g^{-1}=B_{r,d}\right\},\\ \mathrm{Sp}(2r,\mathbb{C})&=\left\{g\in\mathrm{GL}(2r,\mathbb{C})\mid g^\top C_r g=C_r\right\},\\ \mathrm{U}(r+d,r)&=\left\{g\in\mathrm{GL}(2r+d,\mathbb{C})\mid g^* B_{r,d}g=B_{r,d}\right\},\\ \mathrm{O}(r+d,r)&=\left\{g\in\mathrm{GL}(2r+d,\mathbb{R})\mid g^\top B_{r,d}g=B_{r,d}\right\},\\ \mathrm{Sp}(2r,\mathbb{R})&=\left\{g\in\mathrm{GL}(2r,\mathbb{R})\mid g^\top C_r g=C_r\right\}.\end{aligned}$$

Restricting to the respective determinant 1 subgroups, we obtain the *semisimple classical groups*:

Type A:	$\mathrm{SL}(r+1,\mathbb{C})$,		
Type B:	$\mathrm{SO}(2r+1,\mathbb{C})$,		
Type C:	$\mathrm{Sp}(2r,\mathbb{C})$,		
Type D:	$\mathrm{SO}(2r,\mathbb{C})$,		
Type A I:	$\mathrm{SL}(r+1,\mathbb{R})$,		
Type A II:	$\mathrm{SL}(r+1,\mathbb{H})$,		
Type A III:	$\mathrm{SU}(r+d,r)$,	$\mathrm{SU}(r,d+r)$,	
Type B I:	$\mathrm{SO}(r+d,r)$,	$\mathrm{SO}(r,d+r)$,	(d odd),
Type C I:	$\mathrm{Sp}(2r,\mathbb{R})$,		
Type C II:	$\mathrm{Sp}(r+d,r)$,	$\mathrm{Sp}(r,d+r)$,	
Type D I:	$\mathrm{SO}(r+d,r)$,	$\mathrm{SO}(r,d+r)$,	(d even),
Type D III:	$\mathrm{SO}^*(4r+2d)$.		

Up to local isomorphism, these exhaust all but finitely many simple Lie groups with finitely many connected components. The naming convention according to their "type" is due to Cartan; we will also use this convention for the corresponding families of reductive groups. For a complete list of local isomorphisms between these families see Section 1.4.

All of the groups in the list above, except for $\mathrm{SO}(r+d,r)$ and $\mathrm{SO}(r,d+r)$, are connected. The corresponding identity components are denoted by $\mathrm{SO}_0(r+d,r)$ and $\mathrm{SO}_0(r,d+r)$.

If G is a classical group, then we denote its *projectivization* by $\mathrm{P}G := G/\mathrm{Z}(G)$. Let us now list some facts concerning the projectivizations of classical groups.

Lemma 1.2.1. *We have the isomorphisms*

$$\begin{aligned}
&\mathrm{PSL}(r+1,\mathbb{C}) \xrightarrow{\cong} \mathrm{PGL}(r+1,\mathbb{C}), \\
&\mathrm{PSL}(r+1,\mathbb{R}) \xrightarrow{\cong} \mathrm{PGL}(r+1,\mathbb{R}) && (r \text{ even}), \\
&\mathrm{PSL}(r+1,\mathbb{H}) \xrightarrow{\cong} \mathrm{PGL}(r+1,\mathbb{H}), \\
&\mathrm{PSU}(r+d,r) \xrightarrow{\cong} \mathrm{PU}(r+d,r) && (d \in \mathbb{N}), \\
&\mathrm{PSO}(2r+1,\mathbb{C}) \xrightarrow{\cong} \mathrm{PO}(2r+1,\mathbb{C}), \\
&\mathrm{PSO}(r+d,r) \xrightarrow{\cong} \mathrm{PO}(r+d,r) && (d \text{ odd}).
\end{aligned}$$

Furthermore, we have the finite index inclusions

$$\begin{aligned}
&\mathrm{PSL}(r+1,\mathbb{R}) \hookrightarrow \mathrm{PGL}(r+1,\mathbb{R}) && (r \text{ odd}), \\
&\mathrm{PSO}(2r,\mathbb{C}) \hookrightarrow \mathrm{PO}(2r,\mathbb{C}), \\
&\mathrm{PSO}(r+d,r) \hookrightarrow \mathrm{PO}(r+d,r) && (d \text{ even}), \\
&\mathrm{PSO}^*(4r+2d) \hookrightarrow \mathrm{PO}^*(4r+2d) && (d=0,1).
\end{aligned}$$

Proof. The statements for the families $\mathrm{GL}(r+1,\mathbb{C})$ and $\mathrm{GL}(r+1,\mathbb{R})$ are easy to verify. The statements for the families $\mathrm{U}(r+d,r)$, $\mathrm{O}(2r+d,\mathbb{C})$, and $\mathrm{O}(r+d,r)$ are proved in [43, Lemma 5.2]. The statements for the families $\mathrm{GL}(r+1,\mathbb{H})$ and $\mathrm{O}^*(4r+2d)$ can be verified using the matrix descriptions in e.g. [64, Section 1.1.4]. □

1.3. Root Space Decompositions of Classical Lie Algebras

In this section we will determine (restricted) root space decompositions of the *classical Lie algebras*, i.e. of the Lie algebras of the classical groups. Such a Lie algebra will be denoted by the corresponding Fraktur letters, i.e. the Lie algebra of $\mathrm{Sp}(2r,\mathbb{R})$ will be denoted by $\mathfrak{sp}(2r,\mathbb{R})$, etc. Note that projectivizing does not change the Lie algebra, and that we have $\mathfrak{o}(2r+d,\mathbb{C}) = \mathfrak{so}(2r+d,\mathbb{C})$ and $\mathfrak{o}(r+d,r) = \mathfrak{so}(r+d,r) = \mathfrak{so}_0(r+d,r)$.

1.3.1. Root Space Decompositions of Complex Classical Lie Algebras

Let $d \in \{0,1\}$. For every complex reductive Lie algebra $\mathfrak{g} \in \{\mathfrak{o}(2r+d,\mathbb{C}), \mathfrak{sp}(2r,\mathbb{C})\}$ we will determine

(i) a block matrix description of $\mathfrak{g}$,

(ii) a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$,

(iii) a basis $\mathcal{B}^*$ of $\mathfrak{h}^*$,

(iv) the root system Δ of $\mathfrak{g}$ with respect to $\mathfrak{h}$,

(v) a set Δ^+ of positive roots,

(vi) the set Π of simple roots with respect to Δ^+,

(vii) the root spaces $\mathfrak{g}_\lambda$, $\lambda \in \Delta$.

The Complex Orthogonal Lie Algebras: We have

$$
\begin{aligned}
\mathfrak{g} &= \mathfrak{o}(2r+d,\mathbb{C}) = \mathfrak{so}(2r+d,\mathbb{C}) \\
&= \left\{ X \in \mathfrak{gl}(2r+d,\mathbb{C}) \;\middle|\; X^\top B_{r,d} + B_{r,d} X = 0 \right\} \\
&= \left\{ \begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & 0 & -X_2^\top J_r \\ X_6 & -J_r X_4^\top & -J_r X_1^\top J_r \end{pmatrix} \;\middle|\; \begin{matrix} X_3^\top = -J_r X_3 J_r, \\ X_6^\top = -J_r X_6 J_r \end{matrix} \right\}, \\
\mathfrak{h} &= \{\mathrm{diag}(a_1,\dots,a_r,0,\dots,0,-a_r,\dots,-a_1)\},
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{B}^* = \{\varepsilon_1,\dots,\varepsilon_r\}, \text{ where} \\
&\varepsilon_k(\mathrm{diag}(a_1,\dots,a_r,0,-a_r,\dots,-a_1)) = a_k,
\end{aligned}
$$

$$\begin{aligned}\Delta &= \{\varepsilon_k - \varepsilon_l \mid k \neq l\} \cup \{\varepsilon_k \mid k = 1, \ldots, r\} \cup \{\varepsilon_k + \varepsilon_l \mid k \leq l\} \\ &\quad \cup \{-\varepsilon_k \mid k = 1, \ldots, r\} \cup \{-\varepsilon_k - \varepsilon_l \mid k \geq l\}, \\ \Delta^+ &= \{\varepsilon_k - \varepsilon_l \mid k < l\} \cup \{\varepsilon_k \mid k = 1, \ldots, r\} \cup \{\varepsilon_k + \varepsilon_l \mid k \leq l\}, \\ \Pi &= \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{r-1} - \varepsilon_r, \varepsilon_r\},\end{aligned}$$

$$\begin{aligned}\mathfrak{g}_{\varepsilon_k - \varepsilon_l} &= \mathbb{C}(\mathrm{E}_{k,l} - \mathrm{E}_{2r+d+1-l,2r+d+1-k}) && (k, l = 1, \ldots, r,\ k \neq l), \\ \mathfrak{g}_{\varepsilon_k} &= \mathbb{C}(\mathrm{E}_{k,r+1} - \mathrm{E}_{r+1,2r+2-k}) && (k = 1, \ldots, r,\ d = 1), \\ \mathfrak{g}_{\varepsilon_k + \varepsilon_l} &= \mathbb{C}(\mathrm{E}_{k,2r+d+1-l} - \mathrm{E}_{l,2r+d+1-k}) && (k, l = 1, \ldots, r,\ k \leq l), \\ \mathfrak{g}_{-\varepsilon_k} &= \mathbb{C}(\mathrm{E}_{r+1,k} - \mathrm{E}_{2r+2-k,r+1}) && (k = 1, \ldots, r,\ d = 1), \\ \mathfrak{g}_{-\varepsilon_k - \varepsilon_l} &= \mathbb{C}(\mathrm{E}_{2r+d+1-k,l} - \mathrm{E}_{2r+d+1-l,k}) && (k, l = 1, \ldots, r,\ k \geq l).\end{aligned}$$

The Complex Symplectic Lie Algebras: We have

$$\begin{aligned}\mathfrak{g} &= \mathfrak{sp}(2r, \mathbb{C}) \\ &= \left\{X \in \mathfrak{gl}(2r, \mathbb{C}) \;\middle|\; X^\top C_r + C_r X = 0\right\} \\ &= \left\{\begin{pmatrix} X_1 & X_2 \\ X_3 & -J_r X_1^\top J_r \end{pmatrix} \;\middle|\; \begin{aligned} X_2^\top &= J_r X_2 J_r, \\ X_3^\top &= J_r X_3 J_r \end{aligned}\right\}, \\ \mathfrak{h} &= \{\mathrm{diag}(a_1, \ldots, a_r, -a_r, \ldots, -a_1)\},\end{aligned}$$

$$\begin{aligned}&\mathcal{B}^* = \{\varepsilon_1, \ldots, \varepsilon_r\}, \text{ where} \\ &\varepsilon_k(\mathrm{diag}(a_1, \ldots, a_r, -a_r, \ldots, -a_1)) = a_k,\end{aligned}$$

$$\begin{aligned}\Delta &= \{\varepsilon_k - \varepsilon_l \mid k \neq l\} \cup \{\varepsilon_k + \varepsilon_l \mid k \leq l\} \cup \{-\varepsilon_k - \varepsilon_l \mid k \geq l\}, \\ \Delta^+ &= \{\varepsilon_k - \varepsilon_l \mid k < l\} \cup \{\varepsilon_k + \varepsilon_l \mid k \leq l\}, \\ \Pi &= \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{r-1} - \varepsilon_r, 2\varepsilon_r\},\end{aligned}$$

$$\begin{aligned}\mathfrak{g}_{\varepsilon_k - \varepsilon_l} &= \mathbb{C}(\mathrm{E}_{k,l} - \mathrm{E}_{2r+1-l,2r+1-k}) && (k, l = 1, \ldots, r,\ k \neq l), \\ \mathfrak{g}_{\varepsilon_k + \varepsilon_l} &= \mathbb{C}(\mathrm{E}_{k,2r+1-l} + \mathrm{E}_{l,2r+1-k}) && (k, l = 1, \ldots, r,\ k \leq l), \\ \mathfrak{g}_{-\varepsilon_k - \varepsilon_l} &= \mathbb{C}(\mathrm{E}_{2r+1-k,l} + \mathrm{E}_{2r+1-l,k}) && (k, l = 1, \ldots, r,\ k \geq l).\end{aligned}$$

1.3.2. Restricted Root Space Decompositions of Non-Complex Classical Lie Algebras

Let $d \in \mathbb{N}$. For every reductive Lie algebra $\mathfrak{g} \in \{\mathfrak{u}(r+d, r), \mathfrak{o}(r+d, r), \mathfrak{sp}(2r, \mathbb{R})\}$ we will determine

(i) a block matrix description of $\mathfrak{g}$,

(ii) a block matrix description of $\mathfrak{k}$ and $\mathfrak{p}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$ with respect to the Cartan involution $X \mapsto -X^*$,

(iii) a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$,

(iv) the centralizer $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathfrak{k}$,

(v) a basis $\mathcal{B}^*$ of $\mathfrak{a}^*$,

(vi) the restricted root system Δ of $\mathfrak{g}$ with respect to $\mathfrak{a}$,

(vii) a set Δ^+ of positive restricted roots,

(viii) the set Π of simple restricted roots with respect to Δ^+,

(ix) the restricted root spaces $\mathfrak{g}_\lambda$, $\lambda \in \Delta$,

(x) a minimal parabolic subalgebra $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\lambda\in\Delta^+} \mathfrak{g}_\lambda$.

The Unitary Lie Algebras: We have

$$\begin{aligned}
\mathfrak{g} &= \mathfrak{u}(r+d, r) \\
&= \{X \in \mathfrak{gl}(2r+d, \mathbb{C}) \mid X^* B_{r,d} + B_{r,d} X = 0\} \\
&= \left\{ \begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & -X_2^* J_r \\ X_6 & -J_r X_4^* & -J_r X_1^* J_r \end{pmatrix} \middle| \begin{array}{l} X_3^* = -J_r X_3 J_r, \\ X_5^* = -X_5, \\ X_6^* = -J_r X_6 J_r \end{array} \right\}, \\
\mathfrak{k} &= \left\{ \begin{pmatrix} X_1 & X_2 & X_3 \\ -X_2^* & X_5 & -X_2^* J_r \\ -X_3^* & J_r X_2 & J_r X_1 J_r \end{pmatrix} \middle| \begin{array}{l} X_1^* = -X_1, \\ X_3^* = -J_r X_3 J_r, \\ X_5^* = -X_5 \end{array} \right\}, \\
\mathfrak{p} &= \left\{ \begin{pmatrix} X_1 & X_2 & X_3 \\ X_2^* & 0 & -X_2^* J_r \\ X_3^* & -J_r X_2 & -J_r X_1 J_r \end{pmatrix} \middle| \begin{array}{l} X_1^* = X_1, \\ X_3^* = -J_r X_3 J_r \end{array} \right\}, \\
\mathfrak{a} &= \{\mathrm{diag}(a_1, \ldots, a_r, 0, \ldots, 0, -a_r, \ldots, -a_1) \mid a_1, \ldots, a_r \in \mathbb{R}\}, \\
\mathfrak{m} &= \left\{ \begin{pmatrix} \mathrm{diag}(b_1 \mathrm{i}, \ldots, b_r \mathrm{i}) & 0 & 0 \\ 0 & X_5 & 0 \\ 0 & 0 & -\mathrm{diag}(b_r \mathrm{i}, \ldots, b_1 \mathrm{i}) \end{pmatrix} \middle| \begin{array}{l} b_1, \ldots, b_r \in \mathbb{R}, \\ X_5^* = -X_5 \end{array} \right\},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{B}^* = \{\varepsilon_1, \ldots, \varepsilon_r\}, \text{ where} \\
&\varepsilon_k(\mathrm{diag}(a_1, \ldots, a_r, 0, \ldots, 0, -a_r, \ldots, -a_1)) = a_k,
\end{aligned}$$

$$\begin{aligned}
\Delta &= \{\varepsilon_k - \varepsilon_l \mid k \neq l\} \cup \{\varepsilon_k \mid k = 1, \ldots, r\} \cup \{\varepsilon_k + \varepsilon_l \mid k \leq l\} \\
&\quad \cup \{-\varepsilon_k \mid k = 1, \ldots, r\} \cup \{-\varepsilon_k - \varepsilon_l \mid k \geq l\}, \\
\Delta^+ &= \{\varepsilon_k - \varepsilon_l \mid k < l\} \cup \{\varepsilon_k \mid k = 1, \ldots, r\} \cup \{\varepsilon_k + \varepsilon_l \mid k \leq l\}, \\
\Pi &= \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{r-1} - \varepsilon_r, \varepsilon_r\},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{g}_{\varepsilon_k-\varepsilon_l} &= \mathbb{R}(\mathrm{E}_{k,l} - \mathrm{E}_{2r+d+1-l,2r+d+1-k}) \\
&\quad \oplus \mathbb{R}\mathrm{i}(\mathrm{E}_{k,l} + \mathrm{E}_{2r+d+1-l,2r+d+1-k}) \qquad (k,l = 1,\dots,r,\ k \neq l), \\
\mathfrak{g}_{\varepsilon_k} &= \bigoplus_{l=1,\dots,d} \mathbb{R}(\mathrm{E}_{k,r+l} - \mathrm{E}_{r+l,2r+d+1-k}) \\
&\quad \oplus \bigoplus_{l=1,\dots,d} \mathbb{R}\mathrm{i}(\mathrm{E}_{k,r+l} + \mathrm{E}_{r+l,2r+d+1-k}) \qquad (k = 1,\dots,r), \\
\mathfrak{g}_{\varepsilon_k+\varepsilon_l} &= \mathbb{R}(\mathrm{E}_{k,2r+d+1-l} - \mathrm{E}_{l,2r+d+1-k}) \\
&\quad \oplus \mathbb{R}\mathrm{i}(\mathrm{E}_{k,2r+d+1-l} + \mathrm{E}_{l,2r+d+1-k}) \qquad (k,l = 1,\dots,r,\ k \leq l), \\
\mathfrak{g}_{-\varepsilon_k} &= \bigoplus_{l=1,\dots,d} \mathbb{R}(\mathrm{E}_{r+l,k} - \mathrm{E}_{2r+d+1-k,r+l}) \\
&\quad \oplus \bigoplus_{l=1,\dots,d} \mathbb{R}\mathrm{i}(\mathrm{E}_{r+l,k} + \mathrm{E}_{2r+d+1-k,r+l}) \qquad (k = 1,\dots,r), \\
\mathfrak{g}_{-\varepsilon_k-\varepsilon_l} &= \mathbb{R}(\mathrm{E}_{2r+d+1-k,l} - \mathrm{E}_{2r+d+1-l,k}) \\
&\quad \oplus \mathbb{R}\mathrm{i}(\mathrm{E}_{2r+d+1-k,l} + \mathrm{E}_{2r+d+1-l,k}) \qquad (k,l = 1,\dots,r,\ k \geq l),
\end{aligned}$$

$$\mathfrak{q} = \left\{ \begin{pmatrix} \mathrm{diag}(a_1,\dots,a_r) & X_2 & X_3 \\ 0 & X_5 & -X_2^\top J_r \\ 0 & 0 & -\mathrm{diag}(a_r,\dots,a_1) \end{pmatrix} \middle| \begin{array}{l} a_1,\dots,a_r \in \mathbb{C}, \\ X_3^* = -J_r X_3 J_r, \\ X_5^* = -X_5 \end{array} \right\}.$$

The Real Orthogonal Lie Algebras: We have

$$\begin{aligned}
\mathfrak{g} &= \mathfrak{o}(r+d,r) = \mathfrak{so}(r+d,r) \\
&= \left\{ X \in \mathfrak{gl}(2r+d,\mathbb{R}) \,\middle|\, X^\top B_{r,d} + B_{r,d} X = 0 \right\} \\
&= \left\{ \begin{pmatrix} X_1 & X_2 & X_3 \\ X_4 & X_5 & -X_2^\top J_r \\ X_6 & -J_r X_4^\top & -J_r X_1^\top J_r \end{pmatrix} \middle| \begin{array}{l} X_3^\top = -J_r X_3 J_r, \\ X_5^\top = -X_5, \\ X_6^\top = -J_r X_6 J_r \end{array} \right\}, \\
\mathfrak{k} &= \left\{ \begin{pmatrix} X_1 & X_2 & X_3 \\ -X_2^\top & X_5 & -X_2^\top J_r \\ -X_3^\top & J_r X_2 & J_r X_1 J_r \end{pmatrix} \middle| \begin{array}{l} X_1^\top = -X_1, \\ X_3^\top = -J_r X_3 J_r, \\ X_5^\top = -X_5 \end{array} \right\},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{p} &= \left\{ \begin{pmatrix} X_1 & X_2 & X_3 \\ X_2^\top & 0 & -X_2^\top J_r \\ X_3^\top & -J_r X_2 & -J_r X_1 J_r \end{pmatrix} \middle| \begin{array}{l} X_1^\top = X_1, \\ X_3^\top = -J_r X_3 J_r \end{array} \right\}, \\
\mathfrak{a} &= \{\mathrm{diag}(a_1,\dots,a_r,0,\dots,0,-a_r,\dots,-a_1)\}, \\
\mathfrak{m} &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & X_5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| X_5^\top = -X_5 \right\},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{B}^* = \{\varepsilon_1,\dots,\varepsilon_r\}, \text{ where} \\
&\varepsilon_k(\mathrm{diag}(a_1,\dots,a_r,0,\dots,0,-a_r,\dots,-a_1)) = a_k,
\end{aligned}$$

$$\begin{aligned}
\Delta &= \{\varepsilon_k - \varepsilon_l \mid k \neq l\} \cup \{\varepsilon_k \mid k = 1, \ldots, r\} \cup \{\varepsilon_k + \varepsilon_l \mid k \leq l\} \\
&\quad \cup \{-\varepsilon_k \mid k = 1, \ldots, r\} \cup \{-\varepsilon_k - \varepsilon_l \mid k \geq l\}, \\
\Delta^+ &= \{\varepsilon_k - \varepsilon_l \mid k < l\} \cup \{\varepsilon_k \mid k = 1, \ldots, r\} \cup \{\varepsilon_k + \varepsilon_l \mid k \leq l\}, \\
\Pi &= \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{r-1} - \varepsilon_r, \varepsilon_r\},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{g}_{\varepsilon_k - \varepsilon_l} &= \mathbb{R}(\mathrm{E}_{k,l} - \mathrm{E}_{2r+d+1-l,2r+d+1-k}) && (k, l = 1, \ldots, r,\ k \neq l), \\
\mathfrak{g}_{\varepsilon_k} &= \bigoplus_{l=1,\ldots,d} \mathbb{R}(\mathrm{E}_{k,r+l} - \mathrm{E}_{r+l,2r+d+1-k}) && (k = 1, \ldots, r), \\
\mathfrak{g}_{\varepsilon_k + \varepsilon_l} &= \mathbb{R}(\mathrm{E}_{k,2r+d+1-l} - \mathrm{E}_{l,2r+d+1-k}) && (k, l = 1, \ldots, r,\ k \leq l), \\
\mathfrak{g}_{-\varepsilon_k} &= \bigoplus_{l=1,\ldots,d} \mathbb{R}(\mathrm{E}_{r+l,k} - \mathrm{E}_{2r+d+1-k,r+l}) && (k = 1, \ldots, r), \\
\mathfrak{g}_{-\varepsilon_k - \varepsilon_l} &= \mathbb{R}(\mathrm{E}_{2r+d+1-k,l} - \mathrm{E}_{2r+d+1-l,k}) && (k, l = 1, \ldots, r,\ k \geq l),
\end{aligned}$$

$$\mathfrak{q} = \left\{ \begin{pmatrix} \mathrm{diag}(a_1, \ldots, a_r) & X_2 & X_3 \\ 0 & X_5 & -X_2^\top J_r \\ 0 & 0 & -\mathrm{diag}(a_r, \ldots, a_1) \end{pmatrix} \middle| \begin{array}{l} X_3^\top = -J_r X_3 J_r, \\ X_5^\top = -X_5 \end{array} \right\}.$$

The Real Symplectic Lie Algebras: We have

$$\begin{aligned}
\mathfrak{g} &= \mathfrak{sp}(2r, \mathbb{R}) \\
&= \left\{ X \in \mathfrak{gl}(2r, \mathbb{R}) \;\middle|\; X^\top C_r + C_r X = 0 \right\} \\
&= \left\{ \begin{pmatrix} X_1 & X_2 \\ X_3 & -J_r X_1^\top J_r \end{pmatrix} \middle| \begin{array}{l} X_2^\top = J_r X_2 J_r, \\ X_3^\top = J_r X_3 J_r \end{array} \right\}, \\
\mathfrak{k} &= \left\{ \begin{pmatrix} X_1 & X_2 \\ -X_2^\top & J_r X_1 J_r \end{pmatrix} \middle| \begin{array}{l} X_1^\top = -X_1, \\ X_2^\top = J_r X_2 J_r \end{array} \right\}, \\
\mathfrak{p} &= \left\{ \begin{pmatrix} X_1 & X_2 \\ X_2^\top & -J_r X_1 J_r \end{pmatrix} \middle| \begin{array}{l} X_1^\top = X_1, \\ X_2^\top = J_r X_2 J_r \end{array} \right\}, \\
\mathfrak{a} &= \{\mathrm{diag}(a_1, \ldots, a_r, -a_r, \ldots, -a_1)\}, \\
\mathfrak{m} &= \{0\},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{B}^* = \{\varepsilon_1, \ldots, \varepsilon_r\}, \text{ where} \\
&\varepsilon_k(\mathrm{diag}(a_1, \ldots, a_r, -a_r, \ldots, -a_1)) = a_k,
\end{aligned}$$

$$\begin{aligned}
\Delta &= \{\varepsilon_k - \varepsilon_l \mid k \neq l\} \cup \{\varepsilon_k + \varepsilon_l \mid k \leq l\} \cup \{-\varepsilon_k - \varepsilon_l \mid k \geq l\}, \\
\Delta^+ &= \{\varepsilon_k - \varepsilon_l \mid k < l\} \cup \{\varepsilon_k + \varepsilon_l \mid k \leq l\}, \\
\Pi &= \{\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{r-1} - \varepsilon_r, 2\varepsilon_r\},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{g}_{\varepsilon_k - \varepsilon_l} &= \mathbb{R}(\mathrm{E}_{k,l} - \mathrm{E}_{2r+1-l,2r+1-k}) && (k, l = 1, \ldots, r,\ k \neq l), \\
\mathfrak{g}_{\varepsilon_k + \varepsilon_l} &= \mathbb{R}(\mathrm{E}_{k,2r+1-l} + \mathrm{E}_{l,2r+1-k}) && (k, l = 1, \ldots, r,\ k \leq l), \\
\mathfrak{g}_{-\varepsilon_k - \varepsilon_l} &= \mathbb{R}(\mathrm{E}_{2r+1-k,l} + \mathrm{E}_{2r+1-l,k}) && (k, l = 1, \ldots, r,\ k \geq l),
\end{aligned}$$

$$\mathfrak{q} = \{X \in \mathfrak{g} \text{ upper triangular}\}.$$

1.4. Sporadic Isogenies

In low rank some of the (restricted) root systems of semisimple classical Lie algebras coincide. Such coincidences yield isomorphisms between the respective classical Lie algebras, which yield local isomorphisms between the respective classical groups. These local isomorphisms can be chosen to yield global two-to-one maps or even global isomorphisms; in this section we list all such coincidences on the (global) group level.

By [58] and [98] there exist surjective morphisms

$$\begin{aligned} \mathrm{SL}(2,\mathbb{C}) &\twoheadrightarrow \mathrm{SO}(3,\mathbb{C}), & \mathrm{SU}(2) &\twoheadrightarrow \mathrm{SO}(3), \\ \mathrm{SL}(2,\mathbb{C})^2 &\twoheadrightarrow \mathrm{SO}(4,\mathbb{C}), & \mathrm{SU}(2)^2 &\twoheadrightarrow \mathrm{SO}(4), \\ \mathrm{Sp}(4,\mathbb{C}) &\twoheadrightarrow \mathrm{SO}(5,\mathbb{C}), & \mathrm{Sp}(2) &\twoheadrightarrow \mathrm{SO}(5), \\ \mathrm{SL}(4,\mathbb{C}) &\twoheadrightarrow \mathrm{SO}(6,\mathbb{C}), & \mathrm{SU}(4) &\twoheadrightarrow \mathrm{SO}(6), \end{aligned}$$

$$\begin{aligned} \mathrm{SL}(2,\mathbb{R}) &\twoheadrightarrow \mathrm{SO}_0(2,1), \\ \mathrm{SL}(2,\mathbb{C}) &\twoheadrightarrow \mathrm{SO}_0(3,1), \\ \mathrm{Sp}(1,1) &\twoheadrightarrow \mathrm{SO}_0(4,1), \\ \mathrm{SL}(2,\mathbb{H}) &\twoheadrightarrow \mathrm{SO}_0(5,1), \\ \mathrm{SL}(2,\mathbb{R})^2 &\twoheadrightarrow \mathrm{SO}_0(2,2), \\ \mathrm{Sp}(4,\mathbb{R}) &\twoheadrightarrow \mathrm{SO}_0(3,2), \\ \mathrm{SU}(2,2) &\twoheadrightarrow \mathrm{SO}_0(4,2), \\ \mathrm{SO}^*(8) &\twoheadrightarrow \mathrm{SO}_0(6,2), \\ \mathrm{SL}(4,\mathbb{R}) &\twoheadrightarrow \mathrm{SO}_0(3,3), \end{aligned}$$

$$\begin{aligned} \mathrm{SU}(2)\times\mathrm{SL}(2,\mathbb{R}) &\twoheadrightarrow \mathrm{SO}^*(4), \\ \mathrm{SU}(3,1) &\twoheadrightarrow \mathrm{SO}^*(6) \end{aligned}$$

with kernels isomorphic to $\mathbb{Z}/2\mathbb{Z}$. These maps are known as *sporadic isogenies*. Together with the equalities and isomorphisms

$$\begin{aligned} \mathrm{Sp}(2,\mathbb{C}) &= \mathrm{SL}(2,\mathbb{C}), \\ \mathrm{Sp}(2,\mathbb{R}) &= \mathrm{SL}(2,\mathbb{R}) \cong \mathrm{SU}(1,1), \\ \mathrm{Sp}(1) &\cong \mathrm{SU}(2) \end{aligned}$$

all special isomorphisms of semisimple Lie algebras induced by coincidences of (restricted) root systems are exhausted (see [75, Ch. X, §6, 4]). We point out that $\mathrm{SO}_0(3,1)$ is the only "seemingly non-complex" connected simple classical group which admits the structure of a complex Lie group.

2. Bounded Cohomology and Related Cohomology Theories

2.1. Some Cohomology Theories

The goal of this section is to define and discuss some of the cohomology theories that will be used in the sequel. We emphasize the geometric viewpoint of continuous cohomology via differential forms and its connection to Dupont's Question 2.1.14 in Subsection 2.1.3, and give a concrete description of the low-degree continuous cohomology of connected simple Lie groups with finite center in terms of geometric structures in Subsection 2.1.4.

2.1.1. Group Cohomology

Group cohomology was first defined by Eilenberg–Mac Lane [52], following earlier work by Hurewicz [80] and Hopf [77]; see [96] for comments on the historical development. A basic reference in this subject is the book by Brown [9].

Let G be a group, let X be a G-set, and let A be a G-module[1]. Then $\mathrm{Map}(X^n, A)$ becomes a G-module for all $n \in \mathbb{N}$ by setting

$$(gf)(x_1, \dots, x_n) \coloneqq gf\bigl(g^{-1}(x_1, \dots, x_n)\bigr) \qquad (g \in G,\ f \in \mathrm{Map}(X^n, A)). \tag{2.1.1}$$

Recall that the *simplicial resolution* or the *homogeneous resolution* of $G \curvearrowright X$ with coefficients in A is defined as $(C^n(G \curvearrowright X; A), d^n)_{n\in\mathbb{N}}$, where we define the $\mathbb{Z}$-module $C^n(G \curvearrowright X; A)$ of *n-cochains* by setting

$$C^n(G \curvearrowright X; A) \coloneqq \mathrm{Map}\bigl(X^{n+1}, A\bigr)^G,$$

and the $\mathbb{Z}$-linear *coboundary map* $d^n \colon C^n(G \curvearrowright X; A) \to C^{n+1}(G \curvearrowright X; A)$ by setting

$$d^n(f)(x_0, \dots, x_{n+1}) \coloneqq \sum_{j=0}^{n+1} (-1)^j f(x_0, \dots, \widehat{x_j}, \dots, x_{n+1}), \tag{2.1.2}$$

where, as usual, $\widehat{\ }$ over an argument means that the argument should be omitted. The $\mathbb{Z}$-modules $\ker(d^n)$ and $\mathrm{im}(d^{n-1})$ are called *n-cocycles* and *n-coboundaries*, respectively. We have $d^n \circ d^{n-1} = 0$ (where we interpret d^{-1} as the zero map), and we call the $\mathbb{Z}$-module

$$\begin{aligned} \mathrm{H}^n(G \curvearrowright X; A) &\coloneqq \mathrm{H}^n\Bigl(0 \to C^0(G \curvearrowright X; A) \xrightarrow{d^0} C^1(G \curvearrowright X; A) \xrightarrow{d^1} \dots\Bigr) \\ &\coloneqq \ker(d^n)/\mathrm{im}\bigl(d^{n-1}\bigr) \end{aligned}$$

[1] also known as a $\mathbb{Z}G$-module

the n-th *action cohomology* of $G \curvearrowright X$ with coefficients in A. Equipping G with the left multiplication action on itself, we write $C^n(G; A) := C^n(G \curvearrowright G; A)$ and obtain the n-th *cohomology*

$$\mathrm{H}^n(G; A) := \mathrm{H}^n(G \curvearrowright G; A)$$

of G with coefficients in A.

Action cohomology is functorial in all three variables. Indeed, let $\varphi\colon H \to G$ be a group homomorphism; setting $hx := \varphi(h)x$ and $ha := \varphi(h)a$ for all $h \in H$, $x \in X$, $a \in A$ turns X into an H-set and A into an H-module. Now let Y be an H-set, $\phi\colon Y \to X$ an H-map, B an H-module, and $\psi\colon A \to B$ an H-linear map. Then we obtain a chain map

$$\begin{aligned} &\theta\colon C^n(G \curvearrowright X; A) \to C^n(H \curvearrowright Y; B), \\ &\theta(f)(y_0, \dots, y_n) := \psi(f(\phi(y_0), \dots, \phi(y_n))), \end{aligned} \tag{2.1.3}$$

which yields the $\mathbb{Z}$-linear map $\mathrm{H}^n(\varphi \curvearrowright \phi; \psi)\colon \mathrm{H}^n(G \curvearrowright X; A) \to \mathrm{H}^n(H \curvearrowright Y; B)$. We also write $\mathrm{H}^n(\varphi; \psi) := \mathrm{H}^n(\varphi \curvearrowright \varphi; \psi)$ and $\mathrm{H}^n(\varphi) := \mathrm{H}^n(\varphi; \mathrm{id})$.

Let $x \in X$. Then the chain map

$$\iota_x\colon C^n(G \curvearrowright X; A) \to C^n(G; A),\ \iota_x(f)(g_0, \dots, g_n) := f(g_0 x, \dots, g_n x) \tag{2.1.4}$$

induces the $\mathbb{Z}$-linear *evaluation map* $\mathrm{ev}_x\colon \mathrm{H}^n(G \curvearrowright X; A) \to \mathrm{H}^n(G; A)$.

Since it is hard to locate the proof of the following well-known lemma in the literature, we explicate it here.

Lemma 2.1.1. *The map* $\mathrm{ev}_x\colon \mathrm{H}^n(G \curvearrowright X; A) \to \mathrm{H}^n(G; A)$ *is independent of the base point* x *chosen.*

Proof. Let c be a homogeneous n-cocycle on X with coefficients in A and consider the *prism operator*

$$p\colon X^{2n} \to A,\ (x_0, \dots, x_{n-1}, y_0, \dots, y_{n-1}) \mapsto \sum_{i=0}^{n-1} (-1)^i c(x_0, \dots, x_i, y_i, \dots, y_{n-1}).$$

Let $x, y \in X$. We define

$$\beta\colon G^n \to A,\ (g_0, \dots, g_{n-1}) \mapsto p(g_0 x, \dots, g_{n-1} x, g_0 y, \dots, g_{n-1} y).$$

Then

$$\begin{aligned} \mathrm{d}^n(\beta)(g_0, \dots, g_n) &= \sum_{j=0}^{n} (-1)^j \beta(g_0, \dots, \widehat{g_j}, \dots, g_n) \\ &= \sum_{j=0}^{n} (-1)^j p(g_0 x, \dots, \widehat{g_j x}, \dots, g_n x, g_0 y, \dots, \widehat{g_j y}, \dots, g_n y) \\ &= \sum_{j=0}^{n} \sum_{i=0}^{j-1} (-1)^{i+j} c(g_0 x, \dots, g_i x, g_i y, \dots, \widehat{g_j y}, \dots, g_n y) \\ &+ \sum_{j=0}^{n} \sum_{i=j}^{n-1} (-1)^{i+j} c(g_0 x, \dots, \widehat{g_j x}, \dots, g_{i+1} x, g_{i+1} y, \dots, g_n y). \end{aligned}$$

Let us write $\Omega_k = (g_0x, \dots, g_kx, g_ky, \dots, g_ny) \in X^{n+2}$, $k = 0, \dots, n$. We see that the summand $(g_0x, \dots, g_ix, g_iy, \dots, \widehat{g_jy}, \dots, g_ny)$ in the first part of the sum is equal to $\partial_{g_jy}(\Omega_i)$, and that the summand $(g_0x, \dots, \widehat{g_jx}, \dots, g_{i+1}x, g_{i+1}y, \dots, g_ny)$ in the second part of the sum is equal to $\partial_{g_jx}(\Omega_{i+1})$. Sorting both of these parts by the corresponding Ω_k, we can use the cocycle identity $\mathrm{d}(c)(\Omega_k) = 0$ to reduce $\mathrm{d}^n(\beta)(g_0, \dots, g_n)$ to a k-indexed sum of two summands, yielding a telescoping sum with boundary terms $\iota_x(c)(g_0, \dots, g_n) = c(g_0x, \dots, g_nx)$ and $\iota_y(c)(g_0, \dots, g_n) = c(g_0y, \dots, g_ny)$, which yields the claim. □

Hence we will write $\mathrm{ev} \coloneqq \mathrm{ev}_x \colon \mathrm{H}^n(G \curvearrowright X; A) \to \mathrm{H}^n(G; A)$.

The *bar resolution* or the *inhomogeneous resolution* of G with coefficients in A is defined as $(\mathrm{Map}(G^n, A), \delta^n)_{n\in\mathbb{N}}$, where we define the coboundary $\delta^n \colon \mathrm{Map}(G^n, A) \to \mathrm{Map}(G^{n+1}, A)$ by setting

$$\delta^n(f)(g_0, \dots, g_n) \coloneqq f(g_1, \dots, g_n) - \sum_{j=0}^{n} (-1)^{j+1} f(g_0, \dots, g_j g_{j+1}, \dots, g_n) + (-1)^{n+1} f(g_0, \dots, g_{n-1}). \tag{2.1.5}$$

Clearly,

$$\begin{aligned} &\varphi \colon C^n(G; A) \to \mathrm{Map}(G^n, A),\ \varphi(f)(g_1, \dots, g_n) \coloneqq f(e, g_1, g_1g_2, \dots, g_1 \cdot \dots \cdot g_n), \\ &\psi \colon \mathrm{Map}(G^n, A) \to C^n(G; A),\ \psi(f)(g_0, \dots, g_n) \coloneqq f\big(g_0^{-1}g_1, \dots, g_{n-1}^{-1}g_n\big) \end{aligned} \tag{2.1.6}$$

define mutually inverse chain isomorphisms. In particular, the homogeneous and the inhomogeneous resolution of G define the same cohomology, i.e.

$$\mathrm{H}^n(G; A) = \mathrm{H}^n\Big(0 \to \mathrm{Map}(G^0, A) \xrightarrow{\delta^0} \mathrm{Map}(G^1, A) \xrightarrow{\delta^1} \dots\Big).$$

Let us equip $\mathbb{R}$ with the trivial G-module structure. We define the *alternation map* $\mathrm{Alt}_n \colon C^n(G \curvearrowright X; \mathbb{R}) \to C^n(G \curvearrowright X; \mathbb{R})$ by setting

$$\mathrm{Alt}_n(f)(x_0, \dots, x_n) \coloneqq \frac{1}{(n+1)!} \sum_{\sigma \in \mathrm{Sym}_{n+1}} \mathrm{sgn}(\sigma) f\big(x_{\sigma(0)}, \dots, x_{\sigma(n)}\big), \tag{2.1.7}$$

and define the *alternating* cochains $C^n_{\mathrm{alt}}(G \curvearrowright X; \mathbb{R}) \coloneqq \mathrm{im}(\mathrm{Alt}_n)$ and the *non-alternating* cochains $C^n_{\mathrm{n-alt}}(G \curvearrowright X; \mathbb{R}) \coloneqq \ker(\mathrm{Alt}_n)$. Since Alt_n is idempotent, we have

$$C^n(G \curvearrowright X; \mathbb{R}) = C^n_{\mathrm{alt}}(G \curvearrowright X; \mathbb{R}) \oplus C^n_{\mathrm{n-alt}}(G \curvearrowright X; \mathbb{R}).$$

One can check that Alt_n commutes with the coboundary map (2.1.2). Denoting the resulting cohomologies by $\mathrm{H}^n_{\mathrm{alt}}(G \curvearrowright X; \mathbb{R})$ and $\mathrm{H}^n_{\mathrm{n-alt}}(G \curvearrowright X; \mathbb{R})$, we obtain the decomposition

$$\mathrm{H}^n(G \curvearrowright X; \mathbb{R}) = \mathrm{H}^n_{\mathrm{alt}}(G \curvearrowright X; \mathbb{R}) \oplus \mathrm{H}^n_{\mathrm{n-alt}}(G \curvearrowright X; \mathbb{R}).$$

2.1.2. Continuous and Measurable Cohomology

Continuous group cohomology was first defined by Hu [78] and van Est [122], following earlier work by Cartan [30] (see also [108, §6]) and Chevalley–Eilenberg [35]. Basic references in this subject are the books by Guichardet [66] and Borel–Wallach [8].

Let G be a locally compact, second countable group, equipped with its Haar σ-algebra. Let us denote the $\mathbb{R}$-vector space of continuous maps from G^n to $\mathbb{R}$ by $C(G^n)$. Note that the coboundary map (2.1.5) restricts to a linear map $\delta^n_{\mathrm{c}} \colon C(G^n) \to C(G^{n+1})$. Hence we can define the n-th *continuous cohomology* $\mathrm{H}^n_{\mathrm{c}}(G) := \ker(\delta^n_{\mathrm{c}})/\mathrm{im}\big(\delta^{n-1}_{\mathrm{c}}\big)$ of G (with trivial real coefficients).

Let us denote the $\mathbb{R}$-vector space of equivalence classes of Haar-a.e. measurable maps from G^n to $\mathbb{R}$ by $L^0(G^n)$. Note that the coboundary map (2.1.5) induces a linear map $\delta^n_{\mathrm{m}} \colon L^0(G^n) \to L^0(G^{n+1})$. Hence we can define the n-th *measurable cohomology* $\mathrm{H}^n_{\mathrm{m}}(G) := \ker(\delta^n_{\mathrm{m}})/\mathrm{im}\big(\delta^{n-1}_{\mathrm{m}}\big)$ of G (with trivial real coefficients).

We have a natural chain inclusion $C(G^n) \to L^0(G^n)$, which yields the natural linear *comparison map* $c^n_{\mathrm{c,m}} \colon \mathrm{H}^n_{\mathrm{c}}(G) \to \mathrm{H}^n_{\mathrm{m}}(G)$.

Theorem 2.1.2 (Austin–Moore [2, Theorem A])**.** *Let G be a locally compact, second countable group. Then $c^n_{\mathrm{c,m}} \colon \mathrm{H}^n_{\mathrm{c}}(G) \to \mathrm{H}^n_{\mathrm{m}}(G)$ is a linear isomorphism for all $n \in \mathbb{N}$.*

Let $\varphi \colon H \to G$ be a continuous group homomorphism between locally compact, second countable groups. Then Equation (2.1.3) restricts to a map $C(G^{n+1})^G \to C(H^{n+1})^H$, which induces a continuous linear map $\mathrm{H}^n_{\mathrm{c}}(\varphi) \colon \mathrm{H}^n_{\mathrm{c}}(G) \to \mathrm{H}^n_{\mathrm{c}}(H)$.

Let H be a closed subgroup of G. Then the action (2.1.1) induces G-actions on $C((G/H)^n)$ and $L^0((G/H)^n)$, and the coboundary map (2.1.2) induces linear maps

$$C((G/H)^{n+1})^G \to C((G/H)^{n+2})^G, \qquad L^0((G/H)^{n+1})^G \to L^0((G/H)^{n+2})^G.$$

We define the n-th *continuous action cohomology*

$$\mathrm{H}^n_{\mathrm{c}}(G \curvearrowright G/H) := \mathrm{H}^n\big(0 \to C(G/H)^G \to C((G/H)^2)^G \to \dots\big)$$

of $G \curvearrowright G/H$ (with trivial real coefficients) and the n-th *measurable action cohomology*

$$\mathrm{H}^n_{\mathrm{m}}(G \curvearrowright G/H) := \mathrm{H}^n\big(0 \to L^0(G/H)^G \to L^0((G/H)^2)^G \to \dots\big)$$

of $G \curvearrowright G/H$ (with trivial real coefficients).

The isomorphisms in (2.1.6) induce a chain isomorphism between $C(G^{n+1})^G$ and $C(G^n)$, and between $L^0(G^{n+1})^G$ and $L^0(G^n)$, respectively, implying

$$\mathrm{H}^n_{\mathrm{c}}(G) \cong \mathrm{H}^n_{\mathrm{c}}(G \curvearrowright G), \qquad \mathrm{H}^n_{\mathrm{m}}(G) \cong \mathrm{H}^n_{\mathrm{m}}(G \curvearrowright G).$$

Looking at the proof of Lemma 2.1.1, the map (2.1.4) induces the natural linear map $\mathrm{ev_c} \colon \mathrm{H}^n_{\mathrm{c}}(G \curvearrowright G/H) \to \mathrm{H}^n_{\mathrm{c}}(G)$, called the *continuous evaluation map*, and the natural linear map $\mathrm{ev_m} \colon \mathrm{H}^n_{\mathrm{m}}(G \curvearrowright G/H) \to \mathrm{H}^n_{\mathrm{m}}(G)$, called the *measurable evaluation map.*

Proposition 2.1.3. *Let K be a compact subgroup of G. Then the evaluation map $\mathrm{ev_c} \colon \mathrm{H}^n_{\mathrm{c}}(G \curvearrowright G/K) \to \mathrm{H}^n_{\mathrm{c}}(G)$ is an isomorphism.*

Proof. By [66, Ch. III, Proposition 2.3], we have $\mathrm{H}^n_{\mathrm{c}}(G \curvearrowright G/K) \cong \mathrm{H}^n_{\mathrm{c}}(G)$. Since ev_{c} extends the identity, [76, §2] yields the claim. □

Now let G be a connected semisimple Lie group with finite center and let K be a maximal compact subgroup of G. Then G/K, equipped with a G-invariant Riemannian metric, is a symmetric space of non-compact type. Recall that the *Furstenberg boundary* of G/K is (up to conjugacy) defined as G/P, where P is a minimal parabolic subgroup of G.

Theorem 2.1.4 (Monod [104, Theorem B]). *Let G be a connected semisimple Lie group with finite center, let P be a minimal parabolic subgroup of G, let $A < P$ be a maximal split torus, and let w_0 be a representative of the longest element of the Weyl group corresponding to A. Then $\mathrm{ev}_{\mathrm{m}} \colon \mathrm{H}^n_{\mathrm{m}}(G \curvearrowright G/P) \to \mathrm{H}^n_{\mathrm{m}}(G)$ is surjective for all $n \in \mathbb{N}$ and we have a short exact sequence*

$$0 \to \mathrm{H}^{n-2}_{\mathrm{m}}(A)^{w_0} \to \ker(\mathrm{ev}_{\mathrm{m}}) \to \mathrm{H}^{n-1}_{\mathrm{m}}(A)^{w_0} \to 0$$

for all $n \geq 2$.

Recall that $A \cong \mathbb{R}^r$ for $r = \mathrm{rk}_{\mathbb{R}}(G)$; the measurable cohomology of $\mathbb{R}^r$ is determined explicitly in Example 2.1.10. If w_0 acts as -1 on the Lie algebra of A, then we have $\mathrm{H}^{2k}_{\mathrm{m}}(A)^{w_0} = \mathrm{H}^{2k}_{\mathrm{m}}(A)$ and $\mathrm{H}^{2k+1}_{\mathrm{m}}(A)^{w_0} = 0$. Hence we obtain a clear understanding of the difference between $\mathrm{H}^n_{\mathrm{m}}(G \curvearrowright G/P)$ and $\mathrm{H}^n_{\mathrm{m}}(G)$ in this case.

One can refine the above theorem by considering alternating and non-alternating cochains separately. Indeed, the alternation map (2.1.7) induces decompositions

$$\begin{aligned} L^0((G/P)^{n+1})^G &= L^0_{\mathrm{alt}}((G/P)^{n+1})^G \oplus L^0_{\mathrm{n-alt}}((G/P)^{n+1})^G, \\ \mathrm{H}^n_{\mathrm{m}}(G \curvearrowright G/P) &= \mathrm{H}^n_{\mathrm{m,alt}}(G \curvearrowright G/P) \oplus \mathrm{H}^n_{\mathrm{m,n-alt}}(G \curvearrowright G/P). \end{aligned}$$

Theorem 2.1.5 (Bucher–Savini [17, Theorems 2–3]). *Using the notation of Theorem 2.1.4, we have*

$$\begin{aligned} &\mathrm{H}^n_{\mathrm{m,alt}}(G \curvearrowright G/P) \cong \mathrm{H}^n_{\mathrm{m}}(G), && n \leq 1, \\ &\mathrm{H}^n_{\mathrm{m,alt}}(G \curvearrowright G/P) \cong \mathrm{H}^n_{\mathrm{m}}(G) \oplus \mathrm{H}^{n-1}_{\mathrm{m}}(A)^{w_0}, && n \geq 2, \\ &\mathrm{H}^n_{\mathrm{m,n-alt}}(G \curvearrowright G/P) = 0, && n \leq 2, \\ &\mathrm{H}^n_{\mathrm{m,n-alt}}(G \curvearrowright G/P) \cong \mathrm{H}^{n-2}_{\mathrm{m}}(A)^{w_0}, && n \geq 3. \end{aligned}$$

Remark 2.1.6. Theorem 2.1.4 can also be generalized by considering $\mathrm{ev}_{\mathrm{m}} \colon \mathrm{H}^n_{\mathrm{m}}(G \curvearrowright G/L) \to \mathrm{H}^n_{\mathrm{m}}(G)$ for a closed subgroup $L < P$ such that the stabilizer of almost every pair of points in G/L is compact; this works for example in the case of L being the unipotent radical of P or one of the commutator subgroups of the former, as well as in the case $L = A$, see [19].

Of course, the alternation map (2.1.7) also induces decompositions

$$\begin{aligned} C((G/P)^{n+1})^G &= C_{\mathrm{alt}}((G/P)^{n+1})^G \oplus C_{\mathrm{n-alt}}((G/P)^{n+1})^G, \\ \mathrm{H}^n_{\mathrm{c}}(G \curvearrowright G/P) &= \mathrm{H}^n_{\mathrm{c,alt}}(G \curvearrowright G/P) \oplus \mathrm{H}^n_{\mathrm{c,n-alt}}(G \curvearrowright G/P). \end{aligned}$$

Clearly, the natural chain inclusion $C_{\mathrm{alt}}((G/P)^{n+1})^G \to L^0_{\mathrm{alt}}((G/P)^{n+1})^G$ induces a natural linear map $c^n_{\mathrm{c,m,alt}}\colon \mathrm{H}^n_{\mathrm{c,alt}}(G \curvearrowright G/P) \to \mathrm{H}^n_{\mathrm{m,alt}}(G \curvearrowright G/P)$, and, in analogy to Theorem 2.1.2, one may ask whether this comparison map is an isomorphism. The injectivity is unknown, while the surjectivity has recently been proved:

Theorem 2.1.7 (Bucher–Savini [18, Theorem 1])**.** *The map $c^n_{\mathrm{c,m,alt}}\colon \mathrm{H}^n_{\mathrm{c,alt}}(G \curvearrowright G/P) \to \mathrm{H}^n_{\mathrm{m,alt}}(G \curvearrowright G/P)$ is surjective for all $n \in \mathbb{N}$.*

2.1.3. The van Est Isomorphism

Let us now discuss how to explicitly compute the continuous cohomology of a Lie group. Surveys of the theory surrounding this section's theorems include [118], [69], [70, Section 2.1], and [40, Appendix A].

We denote the $\mathbb{R}$-vector space of differential n-forms on a smooth manifold M by $\Omega^n(M)$. Let G be a Lie group with finitely many connected components and let K be a maximal compact subgroup of G.

Proposition 2.1.8 (see [8, Ch. IX, Proposition 5.5])**.** *We have an isomorphism*

$$\mathrm{H}^n_{\mathrm{c}}(G) \cong \mathrm{H}^n\big(0 \to \Omega^0(G/K)^G \to \Omega^1(G/K)^G \to \ldots\big).$$

Let $\mathfrak{g}$ be an $\mathbb{R}$-Lie algebra and let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$. Let us write

$$C^n(\mathfrak{g},\mathfrak{k}) := \left\{ f \in (\wedge^n(\mathfrak{g}/\mathfrak{k}))^* \;\middle|\; \sum_{j=1}^n f(X_1,\ldots,[X,X_j],\ldots,X_n) = f(X_1,\ldots,X_n)\ \forall X \in \mathfrak{k} \right\}$$

and consider the linear coboundary map $d^n\colon C^n(\mathfrak{g},\mathfrak{k}) \to C^{n+1}(\mathfrak{g},\mathfrak{k})$ defined by

$$\begin{aligned} d^n(f)(X_0,\ldots,X_n) &= \sum_{j=0}^n (-1)^j f\Big(X_0,\ldots,\widehat{X_j},\ldots,X_n\Big) \\ &\quad + \sum_{i<j} (-1)^{i+j} f\Big([X_i,X_j],X_0,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_n\Big). \end{aligned}$$

We have $d^n \circ d^{n-1} = 0$, and we call the $\mathbb{R}$-vector space $\mathrm{H}^n(\mathfrak{g},\mathfrak{k}) := \ker(d^n)/\mathrm{im}(d^{n-1})$ the n-th *Lie algebra cohomology* of $\mathfrak{g}$ relative to $\mathfrak{k}$ (with trivial real coefficients).

Now let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{k}$ be the Lie algebra of K. We write $C^n(\mathfrak{g},K) := C^n(\mathfrak{g},\mathfrak{k})^{K/K_0}$, where K acts via the adjoint representation. This yields a subcomplex of $C^\bullet(\mathfrak{g},\mathfrak{k})$, whose cohomology we denote by $\mathrm{H}^\bullet(\mathfrak{g},K)$. If K is connected, then we clearly have $C^n(\mathfrak{g},K) = C^n(\mathfrak{g},\mathfrak{k})$ for all $n \in \mathbb{N}$.

One can show that G-invariant extension yields a chain isomorphism $C^\bullet(\mathfrak{g},K) \cong \Omega^\bullet(G/K)^G$, which implies $\mathrm{H}^n_{\mathrm{c}}(G) \cong \mathrm{H}^n(\mathfrak{g},K)$ for all $n \in \mathbb{N}$. This isomorphism is known as the *van Est isomorphism*, see [123, Theorem 2] or [8, Ch. IX, Corollary 5.6].

Remark 2.1.9. (i) We see that $\mathrm{H}^n_{\mathrm{c}}(G)$ is finite-dimensional for all $n \in \mathbb{N}$ and vanishes for $n > \dim(\mathfrak{g}/\mathfrak{k})$.

(ii) If K is connected, then $\mathrm{H}^n_{\mathrm{c}}(G) \cong \mathrm{H}^n(\mathfrak{g},\mathfrak{k})$.

(iii) Lie algebra cohomology can be shown to obey the following *Künneth rule* (see [8, Ch. I, 1.3]):

$$\mathrm{H}^n(\mathfrak{g}_1 \times \mathfrak{g}_2, \mathfrak{k}_1 \times \mathfrak{k}_2) = \bigoplus_{p+q=n} \mathrm{H}^p(\mathfrak{g}_1, \mathfrak{k}_1) \otimes \mathrm{H}^q(\mathfrak{g}_2, \mathfrak{k}_2)$$

Example 2.1.10. Let $G = \mathbb{R}^k$. Then $K = \{0\}$ and $\mathfrak{g} = \mathbb{R}^k$, $\mathfrak{k} = \{0\}$. From the definition it easily follows that $\mathrm{H}^n_c(\mathbb{R}^k) \cong \mathrm{H}^n(\mathbb{R}^k, \{0\}) = (\wedge^n(\mathbb{R}^k))^*$ for all $n \in \mathbb{N}$.

From now on let G be a connected semisimple Lie group with finite center and let K be a maximal compact subgroup of G. By a lemma of Cartan, every invariant differential form on a symmetric space is closed; in particular, we have an isomorphism $\mathrm{H}^n_c(G) \cong \Omega^n(G/K)^G$. Our next goal will be to understand this isomorphism on the level of cochains. Let $o \in G/K$ be a base point. For all $(g_1, \ldots, g_n) \in G^n$ we define the *geodesic coning n-simplex* $\Delta(g_1, \ldots, g_n) \subset G/K$ inductively by setting $\Delta(g_1) := \{g_1.o\}$ and by letting $\Delta(g_1, \ldots, g_n)$ be the geodesic cone with apex $g_n.o$ and base $\Delta(g_1, \ldots, g_{n-1})$.

Proposition 2.1.11 (Dupont [47, Proposition 5.1]; see also [76, Theorem 6.1], and [116, Proposition 3.5])**.** *The map $I\colon \Omega^n(G/K)^G \to C(G^{n+1})^G$ defined by*

$$I(\omega)(g_0, \ldots, g_n) := \int_{\Delta(g_0,\ldots,g_n)} \omega$$

induces a natural isomorphism $\overline{I}\colon \Omega^n(G/K)^G \xrightarrow{\cong} \mathrm{H}^n_c(G)$, which is independent of the choice of base point o.

Proof. The map I clearly extends the identity, and Stokes' theorem implies that I is a chain map. [76, §2] now yields the claim. □

Remark 2.1.12. Analogously, one can see that the map $J\colon \Omega^n(G/K)^G \to C((G/K)^{n+1})^G$ defined by

$$J(\omega)(g_0K, \ldots, g_nK) := \int_{\Delta(g_0,\ldots,g_n)} \omega$$

yields a natural isomorphism $\overline{J}\colon \Omega^n(G/K)^G \xrightarrow{\cong} \mathrm{H}^n_c(G \curvearrowright G/K)$.

Example 2.1.13. Let $G = \mathrm{SO}_0(2,1)$. Then $\Omega^2(G/K)^G$ is generated by the volume form. Hence I is given by integration over geodesic triangles in the hyperbolic plane $G/K \cong \mathbb{H}^2$ (see Figure 1). Clearly, $\|I(\omega)\|_\infty$ is bounded for all $\omega \in \Omega^2(G/K)^G$.

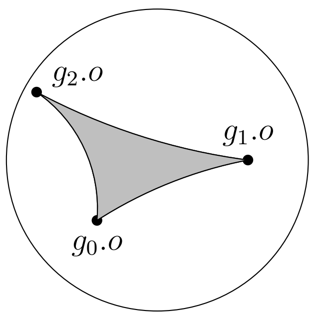

Figure 1.: A geodesic triangle in the Poincaré disk model of the hyperbolic plane.

Question 2.1.14 (Dupont [48, Remark 3]). Let $n \geq 2$. Is $\|I(\omega)\|_\infty$ bounded for all $\omega \in \Omega^n(G/K)^G$?

Remark 2.1.15. The above question has been answered affirmatively in the cases

(i) $n = 2$ (see [48, Theorem 4.1]);

(ii) any n, where G is a product of real rank one groups (the proof is an extension of Example 2.1.13, see [65, Section 0.4] and [81, Theorem 1]).

In some of these cases even the exact number $\|I(\omega)\|_\infty$ is known for all $\omega \in \Omega^n(G/K)^G$. Indeed,

(i) for $n = 2$ and G of Hermitian type we have $\|I(\kappa)\|_\infty = \mathrm{rk}_{\mathbb{R}}(G) \cdot \pi$, where $\kappa \in \Omega^2(G/K)^G$ denotes the *Kähler form* on G/K and the minimal holomorphic sectional curvature of G/K is normalized to be -1 (see [46] and [37]; for applications of this boundedness result to rigidity theory see [23], [24]);

(ii) for $n = 3$ and $G = \mathrm{SO}_0(3,1)$ we have $\|I(\mathrm{vol}_{\mathbb{H}^3})\|_\infty = \frac{3}{2}\sum_{k=1}^\infty \frac{1}{k^2}\sin\left(\frac{2\pi k}{3}\right)$ (see [121, Theorem 7.2.1]; see also [68], where it is shown that $\|I(\mathrm{vol}_{\mathbb{H}^n})\|_\infty$ behaves asymptotically like $\frac{\sqrt{n}}{n!}\mathrm{e}$; precise values for $\|I(\mathrm{vol}_{\mathbb{H}^n})\|_\infty$ are known for $n \leq 6$);

(iii) for $n = 4$ and $G = \mathrm{SO}_0(2,1)^2$ we have $\|I(\mathrm{vol}_{\mathbb{H}^2\times\mathbb{H}^2})\|_\infty = \frac{2}{3}\pi^2$ (see [14]);

The author is not aware of other cases of this question having been decided; nonetheless, the following conjecture has been solved in many more cases (see Subsection 2.5.2).

Conjecture 2.1.16 (see e.g. [10, Conjecture 7]). Every element in $\mathrm{H}^n_{\mathrm{c}}(G)$ admits a bounded representative in the homogeneous resolution.

Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{k}$ be the Lie algebra of K. Let G_u denote the (up to isomorphism) unique 1-connected compact semisimple Lie group with Lie algebra $\mathfrak{g}$, where K is the (up to conjugacy) unique connected subgroup of G_u with Lie algebra $\mathfrak{k}$. We consider the de Rham cohomology

$$\mathrm{H}^n(G_u/K) \coloneqq \mathrm{H}^n\big(0 \to \Omega^0(G_u/K) \to \Omega^1(G_u/K) \to \dots\big)$$

of the *compact dual* G_u/K of G/K. This cohomology is well-known by work of H. Cartan [32] and Borel [6]; see [99, Ch. III, §6] for a textbook account.

A standard averaging argument yields the following classical theorem.

Theorem 2.1.17 (Chevalley–Eilenberg [35, Theorem 12.1]). *The inclusion of G_u-invariant differential forms $\Omega^n(G_u/K)^{G_u} \hookrightarrow \Omega^n(G_u/K)$ induces the isomorphisms*

$$\mathrm{H}^n(G_u/K) \cong \mathrm{H}^n\big(0 \to \Omega^0(G_u/K)^{G_u} \to \Omega^1(G_u/K)^{G_u} \to \dots\big) \cong \Omega^n(G_u/K)^{G_u}.$$

Remark 2.1.18. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Notice that $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ implies that every differential $d^n\colon C^n(\mathfrak{g},\mathfrak{k}) \to C^{n+1}(\mathfrak{g},\mathfrak{k})$ vanishes. Restriction to the base point eK yields an isomorphism

$$\Omega^n(G_u/K)^{G_u} \xrightarrow{\cong} ((\wedge^n \mathrm{i}\mathfrak{p})^*)^K.$$

Combining this with the flip isomorphism

$$\iota\colon ((\wedge^n \mathrm{i}\mathfrak{p})^*)^K \xrightarrow{\cong} ((\wedge^n \mathfrak{p})^*)^K,\ \iota(f)(X_1,\dots,X_n) := f(\mathrm{i}X_1,\dots,\mathrm{i}X_n) \qquad (X_1,\dots,X_n \in \mathfrak{p}),$$

we conclude that $\mathrm{H}^n_{\mathrm{c}}(G) \cong \Omega^n(G/K)^G \cong \mathrm{H}^n(\mathfrak{g},K)$ is isomorphic to $\mathrm{H}^n(G_u/K)$.

Example 2.1.19. Let $G = \mathrm{SO}_0(k,1)$. Then $G/K \cong \mathbb{H}^k$ is hyperbolic k-space and $G_u/K \cong S^k$ is the k-sphere. By Poincaré duality we have

$$\mathrm{H}^n_{\mathrm{c}}(G) \cong \mathrm{H}^n(S^k) \cong \mathrm{H}_{k-n}(S^k),$$

which is one-dimensional for $n \in \{0,k\}$ and trivial otherwise.

Remark 2.1.20. One can show that $\mathrm{H}^\bullet_{\mathrm{c}}(G)$ is isomorphic to the tensor product of a truncated polynomial algebra of even-degree generators and an exterior algebra of odd-degree generators [50, Theorem 5.2].

2.1.4. The Continuous Cohomology of Simple Groups

Let us now discuss the low-degree continuous cohomology of a connected simple Lie group G with finite center. In degree zero, we have $\mathrm{H}^0_{\mathrm{c}}(G) = \mathbb{R}$. In degree one, we have $\mathrm{H}^1_{\mathrm{c}}(G) = 0$ (see e.g. [40, Section 1.5.1]).

The next three theorems describe the degree-two, degree-three, and degree-four continuous cohomology of G, respectively; they follow from the classification of simple Lie groups (see [75, Ch. X]), together with the knowledge of the de Rham cohomology of the compact duals of their associated symmetric spaces (see [99, Ch. III, §6]). The first of these theorems follows more directly, without referring to the classification of simple Lie groups, from [67], where an explicit bounded cocycle is given without using Proposition 2.1.11 (but see also [49], where it is shown that the two constructions yield the same cohomology classes).

Recall that a connected simple Lie group is of Hermitian type if and only if it is locally isomorphic to one of the groups

$$\mathrm{SU}(r+d,r),\ \ \mathrm{SO}_0(n,2)\ (n\neq 2),\ \ \mathrm{Sp}(2r,\mathbb{R}),\ \ \mathrm{SO}^*(4r+2d),\ \ \mathrm{E}_{6(-14)},\ \ \mathrm{E}_{7(-25)},$$

which is the case if and only if the center of its maximal compact subgroup is isomorphic to the circle group, see [75, Ch. X, §6, 3]. In this case, there exists a Kähler form on the associated symmetric space.

Theorem 2.1.21. *The following statements are equivalent:*

(i) *G is of Hermitian type.*

(ii) $\mathrm{H}^2_{\mathrm{c}}(G) \neq 0$.

(iii) $\dim(\mathrm{H}^2_{\mathrm{c}}(G)) = 1$.

If one of these statements holds, then $\mathrm{H}^2_{\mathrm{c}}(G) \cong \Omega^2(G/K)^G$ *is generated by the corresponding Kähler form.*

Recall that a connected simple Lie group admits the structure of a complex Lie group if and only if it is locally isomorphic to one of the groups

$$\mathrm{SL}(r+1,\mathbb{C}),\ \ \mathrm{SO}(n,\mathbb{C})\ (n\neq 4),\ \ \mathrm{Sp}(2r,\mathbb{C}),\ \ \mathrm{E}_r^{\mathbb{C}}\ (r=6,7,8),\ \ \mathrm{F}_4^{\mathbb{C}},\ \ \mathrm{G}_2^{\mathbb{C}},$$

which is the case if and only if its Lie algebra admits a complex structure. In this case, there exists an invariant differential 3-form on the associated symmetric space, which is known as the *Borel form*.

The next theorem can also be obtained without referring to the classification of simple Lie groups (see [41, Theorem 1]).

Theorem 2.1.22. *The following statements are equivalent:*

(i) *G admits the structure of a complex Lie group.*

(ii) $\mathrm{H}_\mathrm{c}^3(G)\neq 0$.

(iii) $\dim(\mathrm{H}_\mathrm{c}^3(G))=1$.

If one of these statements holds, then $\mathrm{H}_\mathrm{c}^3(G)\cong\Omega^3(G/K)^G$ *is generated by the corresponding Borel form.*

Recall that a connected simple Lie group is of quaternion-Kähler type if and only if it is locally isomorphic to one of the groups

$$\mathrm{SU}(k,2),\ \ \mathrm{SO}_0(n,4)\ (n\neq 2),\ \ \mathrm{Sp}(1+d,1),\ \ \mathrm{E}_{6(2)},\ \ \mathrm{E}_{7(-5)},\ \ \mathrm{E}_{8(-24)},\ \ \mathrm{F}_{4(4)},\ \ \mathrm{G}_{2(2)},$$

which is the case if and only if its maximal compact subgroup contains a factor locally isomorphic to SU(2), see [74, Table 1]. In this case, there exist at most k linearly independent invariant differential 4-forms on the associated symmetric space, which are known as *quaternion-Kähler forms*, where k is the number of factors of a maximal compact subgroup locally isomorphic to SU(2).

Theorem 2.1.23. *The following statements are equivalent:*

(i) *G is of Hermitian type or of quaternion-Kähler type, and G is not locally isomorphic to* $\mathrm{SO}_0(2,1)$.

(ii) $\mathrm{H}_\mathrm{c}^4(G)\neq 0$.

(iii) $\dim(\mathrm{H}_\mathrm{c}^4(G))\in\{1,2,3\}$.

If one of these statements holds, then $\mathrm{H}_\mathrm{c}^4(G)\cong\Omega^4(G/K)^G$ *is generated by the corresponding cup product of the Kähler form with itself (if G is of Hermitian type), together with the quaternion-Kähler forms (if G is of quaternion-Kähler type).*

Note that the theorem above excludes $\mathrm{SO}_0(2,1)$ because of the low-dimensionality of its associated symmetric space—indeed, the cup product of the corresponding Kähler form with itself vanishes.

There is no such clear geometric description of $\mathrm{H}_\mathrm{c}^n(G)$ for $n\geq 5$.

We conclude this section with vanishing results concerning the continuous cohomology of certain totally disconnected groups.

Theorem 2.1.24 (Casselman–Wigner [33, Corollary 1]). *Let* $\mathbf{G}$ *be a* 1*-connected, isotropic, simple algebraic* $\mathbb{Q}_p$*-group and equip* $\mathbf{G}(\mathbb{Q}_p)$ *with its Hausdorff topology. Then we have* $\mathrm{H}^n_\mathrm{c}(\mathbf{G}(\mathbb{Q}_p)) = 0$ *for all* $n \geq 1$.

Theorem 2.1.25 (see [8, Ch. X, Lemma 1.12]). *Let* G *act properly and strongly transitively by automorphisms on a locally finite Euclidean building. Then we have* $\mathrm{H}^n_\mathrm{c}(G) = 0$ *for all* $n \geq 1$.

2.2. Bounded Cohomology

The most complete account of the theory of bounded cohomology is Monod's book [100]. Other accounts, mainly focusing on discrete groups, include [95], [55], and [28]. The book [22] deals with the categorical and algebraic foundations of the theory.

After defining the necessary technical tools (Lebesgue G-spaces, Banach G-modules, and coefficient G-modules), we define bounded cohomology in Subsection 2.2.3, where we discuss the most important resolutions.

2.2.1. Lebesgue G-Spaces

Let G be a locally compact, second countable group. A *Lebesgue space* is a standard Borel space equipped with the measure class of a σ-finite Borel measure; if the measure class is non-zero, then there exists a Borel probability measure representing this class. A *morphism* of Lebesgue spaces is an equivalence class (up to null sets) of measure class preserving Borel maps. A *Lebesgue G-space* is a Lebesgue space equipped with a Borel G-action such that the measure class is G-invariant.[2] A *morphism* of Lebesgue G-spaces is a G-equivariant morphism of Lebesgue spaces.

Example 2.2.1. A smooth n-manifold M, equipped with its Borel structure, admits a canonical σ-finite Borel measure class defined by taking the Lebesgue measure class on $\mathbb{R}^n$ for each chart in a collection of charts covering M. This measure class is called the *Lebesgue measure class* of M; it is independent of the charts chosen (see [92, Lemma 6.6]).

If G acts on M by diffeomorphisms, then M becomes a Lebesgue G-space.

Let us now discuss quotients of Lebesgue G-spaces. Recall that a Borel space X is said to be *countably separated* if there exists a countable set $\mathcal{A}$ of Borel subsets of X such that for all $x, y \in X$, $x \neq y$, there exists $E \in \mathcal{A}$ such that $x \in E \not\ni y$. Standard Borel spaces are clearly countably separated. If X is countably separated, then we say that a Borel G-action $G \curvearrowright X$ is *smooth* if the quotient Borel structure on $G\backslash X$ is countably separated.

Theorem 2.2.2 (Glimm–Effros [60], [51]). *Let* X *be a Polish space, equipped with its standard Borel structure, and let* $G \curvearrowright X$ *be a continuous action. Then the following statements are equivalent:*

[2] Monod ([100, Definition 2.1.1]) defines the analogous notion of *regular G-space* using an additional continuity requirement. In our setting of locally compact, second countable groups it turns out to be superfluous, see [22, Appendix D, Remark 1.2.4]; hence our Lebesgue G-spaces are regular G-spaces in the sense of Monod.

(i) $G \curvearrowright X$ *is smooth.*

(ii) Every orbit of $G \curvearrowright X$ *is locally closed in* X*.*

(iii) The map $G/\mathrm{Stab}_G(x) \to Gx$*,* $g\mathrm{Stab}_G(x) \mapsto gx$ *is a homeomorphism for all* $x \in X$*.*

(iv) $G\backslash X$ *is standard Borel.*

If $\pi\colon X \to G\backslash X$ *denotes the canonical projection, if one of the above statements holds, and if additionally* $[\mu]$ *is a measure class on* X *turning* $(X, [\mu])$ *into a Lebesgue G-space, then setting* $\widetilde{\mu}(E) := \mu(\pi^{-1}(E))$ *turns* $(G\backslash X, [\widetilde{\mu}])$ *into a Lebesgue space.*

Definition 2.2.3. In the setting above, $(G\backslash X, [\widetilde{\mu}])$ is called the *quotient Lebesgue space* of the smooth action $G \curvearrowright X$.

The following theorem provides the main source of examples of quotient Lebesgue spaces.

Theorem 2.2.4 (Borel–Serre [7])**.** *Let* k *be a local field of characteristic* 0*, let* $\mathbf{G}$ *be an algebraic* k*-group acting* k*-algebraically on a* k*-variety* $\mathbf{V}$*, and equip* $\mathbf{V}(k)$ *with its Hausdorff topology. Then* $\mathbf{G}(k) \curvearrowright \mathbf{V}(k)$ *is smooth.*

Example 2.2.5. Let (V_r, ω) be a classical space with automorphism group G_r. We equip V_r^k with its Lebesgue measure class $[\mu]$. Since $G_r \curvearrowright V_r^k$ is $\mathbb{R}$-algebraic, $(G_r\backslash V_r^k, [\widetilde{\mu}])$ is a Lebesgue space.

Lemma 2.2.6 (see [54, 214L])**.** *(i) The category of Lebesgue G-spaces admits finite coproducts.*

(ii) Let $(X, [\mu])$ *be a Lebesgue G-space with a decomposition* $X = \bigcup_{i=1}^n X_i$ *into G-invariant Borel subsets and equip every* X_i *with its subspace measure class* $[\mu|_{X_i}]$*. Then we have* $(X, [\mu]) \cong \bigsqcup_{i=1}^n (X_i, [\mu|_{X_i}])$ *as Lebesgue G-spaces. If some* $[\mu|_{X_j}]$ *is the zero measure class, then we clearly have* $(X, [\mu]) \cong \bigsqcup_{i\in\{1,\dots,n\},\ i\neq j} (X_i, [\mu|_{X_i}])$ *as Lebesgue G-spaces.*

If X is a Lebesgue G-space, then we define the Lebesgue G-space $X^{\sqcup k}$ inductively by setting $X^{\sqcup 1} := X$ and $X^{\sqcup k} := X^{\sqcup k-1} \sqcup X$.

2.2.2. Banach G-Modules and Coefficient G-Modules

Let G be a locally compact, second countable group. A *Banach G-module* is a Banach space equipped with a G-action by linear isometries.[3] A *G-morphism* is a G-equivariant, continuous (not necessarily isometric) linear map between Banach G-modules. A Banach G-module E is *continuous* if the G-action on E is continuous. If E is a Banach G-module, then we denote by $E^\sharp$ its topological dual, equipped with the weak-$*$ topology and the G-action $(gf)(x) := f(g^{-1}x)$, $g \in G$, $f \in E^\sharp$, $x \in E$.

[3] Note that according to our conventions, we consider real Banach spaces, while Monod [100] exclusively considers complex Banach spaces. All of Monod's theorems that we use in this thesis work, mutatis mutandis, also in our setting of real Banach spaces (see e.g. [22]).

A *coefficient G-module* is a pair $(E, E^\sharp)$, where E is a separable, continuous Banach G-module. A *morphism of coefficient G-modules* $(E, E^\sharp) \to (F, F^\sharp)$ is a pair $(\Phi, \Phi^\sharp)$, where $\Phi\colon E \to F$ is a continuous linear map with G-equivariant dual map $\Phi^\sharp\colon F^\sharp \to E^\sharp$.

If $(E^\flat, E)$ is a coefficient G-module, then the only role of $E^\flat$ is to define a weak-$*$ topology on E. Hence we will usually omit $E^\flat$ from our notation and refer to E as a coefficient G-module. In this way a morphism between coefficient G-modules is a G-equivariant linear map $E \to F$ which is continuous with respect to the weak-$*$ topologies induced by the omitted pre-duals of E and F.

Example 2.2.7. Let $(X, [\mu])$ be a Lebesgue G-space and consider the G-action on $L^1(X, \mu)$ defined by

$$(gf)(x) := \rho_\mu\big(g^{-1}, x\big) f\big(g^{-1}x\big) \qquad (g \in G,\ f \in L^1(X, \mu)),$$

where $\rho_\mu(g, x) := \mathrm{d}(g_*\mu)/\mathrm{d}\mu(x)$ denotes the Radon–Nikodym derivative. Then the pair $(L^1(X, \mu), L^\infty(X, [\mu]))$ is a coefficient G-module (see [22, Appendix D, Corollary 1.2.3]), which, up to isomorphism, is independent of the representative of $[\mu]$.

Let $(X, [\mu])$ be a Lebesgue G-space and let E be a coefficient G-module. Let us denote the Banach space of equivalence classes of a.e. measurable, essentially bounded maps from X to E, equipped with the essential supremum norm, by $L^\infty((X, [\mu]), E)$ or $L^\infty(X, E)$. Then the action (2.1.1) induces a G-action on $L^\infty(X, E)$, turning the latter into a Banach G-module. Actually, $L^\infty(X, E)$ is even a coefficient G-module, being the dual of the Bochner space $L^1(X, \mu, E^\flat)$, where $E^\flat$ denotes the pre-dual of E, see [100, Proposition 2.3.1].

Remark 2.2.8. Let E be a coefficient G-module and let $\varphi\colon Y \to X$ be a morphism of Lebesgue G-spaces. Then the map

$$L^\infty(\varphi, E)\colon L^\infty(X, E) \to L^\infty(Y, E),\ f \mapsto f \circ \varphi$$

is a morphism of coefficient G-modules with pre-dual operator

$$L^1(\varphi, E^\flat)\colon L^1(Y, \nu, E^\flat) \to L^1(X, \varphi_*\nu, E^\flat),\ L^1(\varphi, E^\flat)(f)(x) := \mathrm{d}(\varphi_*(f \cdot \nu))/\mathrm{d}(\varphi_*\nu).$$

Clearly, $L^\infty(-, E)$ yields a contravariant functor from the category of Lebesgue G-spaces to the category of coefficient G-modules. One can check that this functor maps finite coproducts to products, i.e. for all Lebesgue G-spaces X_1 and X_2 we have

$$L^\infty(X_1 \sqcup X_2, E) \cong L^\infty(X_1, E) \times L^\infty(X_2, E)$$

as coefficient G-modules.

Furthermore, if E is a trivial coefficient G-module and X is a Polish space equipped with a smooth and continuous G-action, then we have a canonical isomorphism

$$L^\infty((G\backslash X, [\widetilde{\mu}]), E) \cong L^\infty((X, [\mu]), E)^G$$

of trivial coefficient G-modules.

2.2.3. Continuous Bounded and Measurable Bounded Cohomology

Let G be a locally compact, second countable group, let X be a G-space, and let E be a Banach G-module. Let us denote the Banach G-module of continuous bounded maps from X^n to E, equipped with the supremum norm and the restriction of the action (2.1.1), by $C_{\mathrm{b}}(X^n, E)$. Note that the coboundary map (2.1.2) restricts to a continuous linear map $d^n_{\mathrm{cb}} \colon C_{\mathrm{b}}(X^{n+1}, E)^G \to C_{\mathrm{b}}(X^{n+2}, E)^G$. Hence we can define the n-th *continuous bounded action cohomology* $\mathrm{H}^n_{\mathrm{cb}}(G \curvearrowright X; E) \coloneqq \ker(d^n_{\mathrm{cb}})/\mathrm{im}\big(d^{n-1}_{\mathrm{cb}}\big)$ of $G \curvearrowright X$ with coefficients in E, which, equipped with the quotient seminorm, becomes a seminormed vector space. Equipping G with the left multiplication action on itself, we obtain the n-th *continuous bounded cohomology* $\mathrm{H}^n_{\mathrm{cb}}(G; E) \coloneqq \mathrm{H}^n_{\mathrm{cb}}(G \curvearrowright G; E)$ of G with coefficients in E. Looking at the proof of Lemma 2.1.1, we also obtain the natural continuous linear map $\mathrm{ev}_{\mathrm{cb}} \colon \mathrm{H}^n_{\mathrm{cb}}(G \curvearrowright X; E) \to \mathrm{H}^n_{\mathrm{cb}}(G; E)$, called the *continuous bounded evaluation map*, and write $\mathrm{H}^n_{\mathrm{cb}}(G \curvearrowright X) \coloneqq \mathrm{H}^n_{\mathrm{cb}}(G \curvearrowright X; \mathbb{R})$ and $\mathrm{H}^n_{\mathrm{cb}}(G) \coloneqq \mathrm{H}^n_{\mathrm{cb}}(G; \mathbb{R})$. For a (not necessarily countable) discrete group Γ we can similarly define the *n-th bounded cohomology* $\mathrm{H}^n_{\mathrm{b}}(\Gamma; E) \coloneqq \mathrm{H}^n_{\mathrm{cb}}(\Gamma; E)$ of Γ with coefficients in E.

Theorem 2.2.9 (Monod [100])**.** *Let X be a locally compact, second countable, proper G-space and let E be a Banach G-module. Then the evaluation map $\mathrm{ev}_{\mathrm{cb}} \colon \mathrm{H}^n_{\mathrm{cb}}(G \curvearrowright X; E) \to \mathrm{H}^n_{\mathrm{cb}}(G; E)$ is an isomorphism.*

Proof. By [100, Theorem 7.4.5] we have $\mathrm{H}^n_{\mathrm{cb}}(G \curvearrowright X; E) \cong \mathrm{H}^n_{\mathrm{cb}}(G; E)$. Since $\mathrm{ev}_{\mathrm{cb}}$ extends the identity, the proof of [100, Corollary 7.2.7] yields the claim. ☐

Example 2.2.10. Let G be a connected semisimple Lie group with finite center and let K be a maximal compact subgroup of G. Then $\mathrm{ev}_{\mathrm{cb}} \colon \mathrm{H}^n_{\mathrm{cb}}(G \curvearrowright G/K; E) \to \mathrm{H}^n_{\mathrm{cb}}(G; E)$ is an isomorphism for all $n \in \mathbb{N}$.

Now let X be a Lebesgue G-space and let E be a coefficient G-module. Then the coboundary map (2.1.2) induces a continuous linear map $d^n_{\mathrm{mb}} \colon L^\infty(X^{n+1}, E)^G \to L^\infty(X^{n+2}, E)^G$. Hence we can define the n-th *measurable bounded action cohomology* $\mathrm{H}^n_{\mathrm{mb}}(G \curvearrowright X; E) \coloneqq \ker(d^n_{\mathrm{mb}})/\mathrm{im}\big(d^{n-1}_{\mathrm{mb}}\big)$ of $G \curvearrowright X$ with coefficients in E, which, equipped with the quotient seminorm, becomes a seminormed vector space. Equipping G with the left multiplication action on itself, we obtain the n-th *measurable bounded cohomology* $\mathrm{H}^n_{\mathrm{mb}}(G; E) \coloneqq \mathrm{H}^n_{\mathrm{mb}}(G \curvearrowright G; E)$ of G with coefficients in E. Looking at the proof of Lemma 2.1.1, we also obtain the natural continuous linear map $\mathrm{ev}_{\mathrm{mb}} \colon \mathrm{H}^n_{\mathrm{mb}}(G \curvearrowright X; E) \to \mathrm{H}^n_{\mathrm{mb}}(G; E)$, called the *measurable bounded evaluation map*, and write $\mathrm{H}^n_{\mathrm{mb}}(G \curvearrowright X) \coloneqq \mathrm{H}^n_{\mathrm{mb}}(G \curvearrowright X; \mathbb{R})$ and $\mathrm{H}^n_{\mathrm{mb}}(G) \coloneqq \mathrm{H}^n_{\mathrm{mb}}(G; \mathbb{R})$.

We also have a natural isometric chain embedding $C_{\mathrm{b}}(G^n, E)^G \to L^\infty(G^n, E)^G$, which yields the natural continuous linear *comparison map* $c^n_{\mathrm{cb,mb}} \colon \mathrm{H}^n_{\mathrm{cb}}(G; E) \to \mathrm{H}^n_{\mathrm{mb}}(G; E)$.

Theorem 2.2.11 (Monod [100, Proposition 7.5.1])**.** *Let G be a locally compact, second countable group and let E be a coefficient G-module. Then the comparison map $c^n_{\mathrm{cb,mb}} \colon \mathrm{H}^n_{\mathrm{cb}}(G; E) \to \mathrm{H}^n_{\mathrm{mb}}(G; E)$ is an isometric isomorphism for all $n \in \mathbb{N}$.*

Remark 2.2.12. One reason why one might prefer working in the measurable setting is the *exponential law* for L^∞-spaces, which says that for all Lebesgue G-spaces X_1, X_2

and every coefficient G-module E there exists a canonical isometric coefficient G-module isomorphism

$$L^\infty(X_1 \times X_2, E) \cong L^\infty(X_1, L^\infty(X_2, E)),$$

see [100, Corollary 2.3.3]. The corresponding statement for continuous bounded functions fails, as for some G-spaces X_1 and X_2 and some Banach G-module E the inclusion

$$C_{\mathrm{b}}(X_1, C_{\mathrm{b}}(X_2, E)) \subset C_{\mathrm{b}}(X_1 \times X_2, E)$$

is in general strict.

A useful application of the exponential law is given by the next lemma.

Lemma 2.2.13. *Let E be a coefficient G-module and assume that $\mathrm{H}^n_{\mathrm{cb}}(G; E) = 0$. Then for any Lebesgue G-space X on which G acts trivially we have $\mathrm{H}^n_{\mathrm{cb}}(G; L^\infty(X, E)) = 0$.*

Proof. Writing out the respective homogeneous resolutions, this result follows directly from an application of Fubini's theorem. □

Our next goal is to describe how we obtain an isometric isomorphism $\mathrm{H}^n_{\mathrm{mb}}(G; E) \cong \mathrm{H}^n_{\mathrm{mb}}(G \curvearrowright S; E)$ for every amenable G-space S. Thus, let us recall the definition of an amenable G-space.[4]

Let S be a Lebesgue G-space. A *conditional expectation* $\mathfrak{m}\colon L^\infty(G \times S) \to L^\infty(S)$ is an isometric linear map such that

(i) $\mathfrak{m}(\mathbf{1}_{G\times S}) = \mathbf{1}_S$,

(ii) for all $f \in L^\infty(G \times S)$ and every measurable subset A of S we have $\mathfrak{m}(f \cdot \mathbf{1}_{G\times A}) = \mathfrak{m}(f) \cdot \mathbf{1}_A$.

We call S an *amenable* G-space if there exists a G-equivariant conditional expectation $L^\infty(G \times S) \to L^\infty(S)$.

Example 2.2.14. Let H be a closed subgroup of G.

(i) The left multiplication action of G turns G/H into an amenable G-space if and only if H is an amenable group. In particular, G is an amenable G-space.

(ii) If G is a connected semisimple Lie group and P is a minimal parabolic subgroup of G, then P is amenable; hence the Furstenberg boundary G/P is an amenable G-space.

(iii) If S is an amenable G-space, then restriction turns S into an amenable H-space. In particular, G is an amenable H-space.

Let S be an amenable G-space with G-invariant conditional expectation $\mathfrak{m}$. Then we have a map

$$L^\infty(G) \hookrightarrow L^\infty(G \times S) \xrightarrow{\mathfrak{m}} L^\infty(S),$$

which extends to a contracting chain map $\mathfrak{m}^n\colon L^\infty(G^n, E)^G \to L^\infty(S^n, E)^G$ (see [100, Lemma 7.5.6]). This yields the map $c^n_{\mathfrak{m}}\colon \mathrm{H}^n_{\mathrm{mb}}(G; E) \to \mathrm{H}^n_{\mathrm{mb}}(G \curvearrowright S; E)$.

[4] We exclusively consider amenable G-spaces in the sense of Zimmer [131].

Theorem 2.2.15 (Monod [100, Theorem 7.5.3])**.** *Let G be a locally compact, second countable group, S an amenable G-space with G-invariant conditional expectation $\mathfrak{m}$, and E a coefficient G-module. Then the map $c^n_{\mathfrak{m}}\colon \mathrm{H}^n_{\mathrm{mb}}(G;E) \to \mathrm{H}^n_{\mathrm{mb}}(G \curvearrowright S;E)$ is an isometric isomorphism with inverse $\mathrm{ev}_{\mathrm{mb}}\colon \mathrm{H}^n_{\mathrm{mb}}(G \curvearrowright S;E) \to \mathrm{H}^n_{\mathrm{mb}}(G;E)$ for all $n \in \mathbb{N}$. In particular, $c^n_{\mathfrak{m}}$ is independent of $\mathfrak{m}$.*

Proof. By [100, Lemma 7.5.6], $\mathfrak{m}^n$ extends the identity, and $c^n_{\mathfrak{m}}$ is an isomorphism by [100, Theorem 7.5.3]. Since $\mathrm{ev}_{\mathrm{mb}}$ also extends the identity, the proof of [100, Corollary 7.2.7] yields the claim. □

Example 2.2.16. Let G be a connected semisimple Lie group with finite center and let P be a minimal parabolic subgroup of G. Then $\mathrm{ev}_{\mathrm{mb}}\colon \mathrm{H}^n_{\mathrm{mb}}(G \curvearrowright G/P;E) \to \mathrm{H}^n_{\mathrm{mb}}(G;E)$ is an isomorphism for all $n \in \mathbb{N}$.

Let p be a probability measure on G such that some convolution power of p is absolutely continuous with respect to the (left) Haar measure on G and such that the support of p generates G as a semigroup. Let us consider the space of *p-harmonic* measurable bounded functions

$$\mathcal{H}^\infty(G^n) := \left\{ f\colon G^n \to \mathbb{R} \text{ measurable, bounded} \;\middle|\; f(g) = \int_{G^n} f(gg')\mathrm{d}p^{\otimes n}(g') \ \forall g \in G^n \right\}.$$

One can show that $\mathcal{H}^\infty(G^n)$ naturally embeds into $C(G^n)$. There exists a Borel measure τ on G/P, which belongs to the Lebesgue measure class, such that the *Poisson transform*

$$\mathcal{P}\colon L^\infty((G/P)^n, \tau^{\otimes n}) \to \mathcal{H}^\infty(G^n),\ \mathcal{P}(f)(g) := \int_{(G/P)^n} f(g\xi)\mathrm{d}\tau^{\otimes n}(\xi)$$

is a G-equivariant isometric isomorphism. Consequently, we also have an isometric isomorphism

$$\mathrm{H}^n_{\mathrm{mb}}(G) \cong \mathrm{H}^n_{\mathrm{mb}}(G \curvearrowright G/P) \cong \mathrm{H}^n\big(0 \to \mathcal{H}^\infty(G)^G \to \mathcal{H}^\infty(G^2)^G \to \dots\big)$$

for all $n \in \mathbb{N}$. In [56] Furstenberg proved the existence of a G-equivariant isometric isomorphism $L^\infty((G/P)^n) \xrightarrow{\cong} \mathcal{H}^\infty((G/K)^n)$, where K denotes a maximal compact subgroup of G, which implies

$$\mathrm{H}^n_{\mathrm{mb}}(G) \cong \mathrm{H}^n\big(0 \to \mathcal{H}^\infty(G/K)^G \to \mathcal{H}^\infty((G/K)^2)^G \to \dots\big).$$

See the Introduction for more discussion of this statement.

Let A be a Banach G-module and let $\varphi\colon H \to G$ be a continuous group homomorphism; setting $ha := \varphi(h)a$ for all $h \in H$, $a \in A$ turns A into a Banach H-module. Now let B be a Banach H-module and let $\psi\colon A \to B$ be a morphism of Banach H-modules. Then Equation (2.1.3) restricts to a map

$$\theta\colon C_{\mathrm{b}}(G^{n+1}, A)^G \to C_{\mathrm{b}}(H^{n+1}, B)^H,$$

which yields the continuous linear map $\mathrm{H}^n_{\mathrm{cb}}(\varphi;\psi)\colon \mathrm{H}^n_{\mathrm{cb}}(G;A) \to \mathrm{H}^n_{\mathrm{cb}}(H;B)$ of seminorm at most $\|\psi\|$. We also write $\mathrm{H}^n_{\mathrm{cb}}(\varphi) := \mathrm{H}^n_{\mathrm{cb}}(\varphi;\mathrm{id})$.

The following corollary follows directly from Theorem 2.2.15.

Corollary 2.2.17. *Let G be a locally compact, second countable group, let N be an amenable normal subgroup of G, and let E be a coefficient G-module. Then the canonical projection $G \to G/N$ and the canonical inclusion $E^N \to E$ induce an isometric isomorphism $\mathrm{H}^n_{\mathrm{cb}}(G/N; E^N) \xrightarrow{\cong} \mathrm{H}^n_{\mathrm{cb}}(G; E)$ for all $n \in \mathbb{N}$.*

Let us now apply the corollary above to the classical groups.

Corollary 2.2.18. *We have isometric isomorphisms*

$$\begin{aligned}
&\mathrm{H}^n_{\mathrm{cb}}(\mathrm{GL}(k,\mathbb{C})) &&\cong \mathrm{H}^n_{\mathrm{cb}}(\mathrm{SL}(k,\mathbb{C})), \\
&\mathrm{H}^n_{\mathrm{cb}}(\mathrm{GL}(2k+1,\mathbb{R})) &&\cong \mathrm{H}^n_{\mathrm{cb}}(\mathrm{SL}(2k+1,\mathbb{R})), \\
&\mathrm{H}^n_{\mathrm{cb}}(\mathrm{GL}(k,\mathbb{H})) &&\cong \mathrm{H}^n_{\mathrm{cb}}(\mathrm{SL}(k,\mathbb{H})), \\
&\mathrm{H}^n_{\mathrm{cb}}(\mathrm{U}(r+d,r)) &&\cong \mathrm{H}^n_{\mathrm{cb}}(\mathrm{SU}(r+d,r)) && (d \in \mathbb{N}), \\
&\mathrm{H}^n_{\mathrm{cb}}(\mathrm{O}(2r+1,\mathbb{C})) &&\cong \mathrm{H}^n_{\mathrm{cb}}(\mathrm{SO}(2r+1,\mathbb{C})), \\
&\mathrm{H}^n_{\mathrm{cb}}(\mathrm{O}(r+d,r)) &&\cong \mathrm{H}^n_{\mathrm{cb}}(\mathrm{SO}(r+d,r)) && (d \text{ odd})
\end{aligned}$$

for all $n \in \mathbb{N}$.

Proof. This follows from the isomorphisms in Lemma 1.2.1. □

Corollary 2.2.19. *Every special isogeny (see Section 1.4) induces an isomorphism in bounded cohomology.*

Proposition 2.2.20 (Monod [100, Proposition 8.6.2])**.** *Let H be a closed subgroup of G and let E be a continuous Banach G-module. If G/H admits a finite invariant measure, then the restriction*

$$\mathrm{res}\colon \mathrm{H}^n_{\mathrm{cb}}(G; E) \to \mathrm{H}^n_{\mathrm{cb}}(H; E)$$

is isometrically injective for all $n \in \mathbb{N}$.

The finite index inclusions in Lemma 1.2.1 now yield the following corollary.

Corollary 2.2.21. *We have isometrically injective maps*

$$\begin{aligned}
&\mathrm{H}^n_{\mathrm{cb}}(\mathrm{GL}(2k,\mathbb{R})) &&\hookrightarrow \mathrm{H}^n_{\mathrm{cb}}(\mathrm{SL}(2k,\mathbb{R})), \\
&\mathrm{H}^n_{\mathrm{cb}}(\mathrm{O}(2r,\mathbb{C})) &&\hookrightarrow \mathrm{H}^n_{\mathrm{cb}}(\mathrm{SO}(2r,\mathbb{C})), \\
&\mathrm{H}^n_{\mathrm{cb}}(\mathrm{O}(r+d,r)) &&\hookrightarrow \mathrm{H}^n_{\mathrm{cb}}(\mathrm{SO}(r+d,r)) && (d \text{ even}), \\
&\mathrm{H}^n_{\mathrm{cb}}(\mathrm{O}^*(2k)) &&\hookrightarrow \mathrm{H}^n_{\mathrm{cb}}(\mathrm{SO}^*(2k))
\end{aligned}$$

for all $n \in \mathbb{N}$.

Lemma 2.2.22 (De la Cruz Mengual–Hartnick [43, Lemma 5.3])**.** *Let $n < 2r$. Then the restriction*

$$\mathrm{res}\colon \mathrm{H}^n_{\mathrm{cb}}(\mathrm{SO}(r+d,r)) \to \mathrm{H}^n_{\mathrm{cb}}(\mathrm{SO}_0(r+d,r))$$

is an isomorphism.

Corollary 2.2.23. *We have isometric isomorphisms*

$$\begin{aligned} \mathrm{H}^2_{\mathrm{cb}}(\mathrm{O}(r+d,r)) &\cong \mathrm{H}^2_{\mathrm{cb}}(\mathrm{SO}_0(r+d,r)) \qquad (r \geq 2), \\ \mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(r+d,r)) &\cong \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}_0(r+d,r)) \qquad (r \geq 2), \\ \mathrm{H}^4_{\mathrm{cb}}(\mathrm{O}(r+d,r)) &\cong \mathrm{H}^4_{\mathrm{cb}}(\mathrm{SO}_0(r+d,r)) \qquad (r \geq 3) \end{aligned}$$

for all odd $d \in \mathbb{N}$.

Let us now discuss how to induce the bounded cohomology of a “small” group with coefficients in a “small” module to the bounded cohomology of a “big” group with coefficients in a “big” module. This is known as the Eckmann–Shapiro-lemma or *Eckmann–Shapiro-induction* (see [72] for a recent extension to the setting of transverse measured groupoids).

Let G be a locally compact, second countable group, H a closed subgroup, X an amenable G-space, and E a coefficient G-module. We define

$$i^n_X \colon L^\infty(X^{n+1}, E)^H \to L^\infty(X^{n+1}, L^\infty(G/H, E))^G, \; i^n_X(f)(x)(gH) \coloneqq gf(g^{-1}x).$$

Proposition 2.2.24 (Monod [100, Proposition 10.1.3])**.** *The maps i^n_X induce an isomorphism*

$$\mathrm{H}^\bullet_{\mathrm{mb}}(H; E) \xrightarrow{\cong} \mathrm{H}^\bullet_{\mathrm{mb}}(G; L^\infty(G/H, E)),$$

which is natural and independent of X.

2.3. (Co-)Homological Stability

The concept of cohomological stability will be our method of choice to obtain further evidence for the isomorphism conjecture, which will be introduced in Section 2.4 below. (Co-)Homological stability has first been used by Quillen [110], [111].

Let $\varphi_r \colon G_r \to G_{r+1}$ be group homomorphisms, $r \in \mathbb{N}$. The sequence $(G_r, \varphi_r)_{r\in\mathbb{N}}$ is called *homologically stable* if there exists a function $\rho \colon \mathbb{N} \to \mathbb{N}$ such that

$$\mathrm{H}_q\big(\varphi_{\rho(q)+s}\big) \colon \mathrm{H}_q\big(G_{\rho(q)+s}\big) \to \mathrm{H}_q\big(G_{\rho(q)+s+1}\big)$$

is an isomorphism for all $s \in \mathbb{N}$.

A useful criterion to prove homological stability is *Quillen's stability criterion* [110] (see also [3], [112], [117]), which can be stated in the following way: Let $X(r)$ be a Δ-complex, equipped with a simplicial G_r-action, and let $\gamma(r)$ and $\tau(r)$ be natural numbers such that

(Q1) $X(r)$ is $\gamma(r)$-*acyclic*, i.e. the reduced homology $\widetilde{H}_\bullet(X(r))$ vanishes up to degree $\gamma(r)$;

(Q2) $X(r)$ is $\tau(r)$-*transitive*, i.e. there is only one G_r-orbit of l-simplices for all $l \in \{0, \dots, \tau(r)\}$;

(Q3) the complexes are $\tau(r)$-*compatible*, i.e. the stabilizer of an l-simplex in $X(r)$ is isomorphic to G_{r-l-1} for every $l < \tau(r)$ and these isomorphisms are, in a certain sense, compatible with the inclusions of stabilizers (see [3, §2] for a precise statement).

Then $(G_r)_{r\in\mathbb{N}}$ is homologically stable if $\gamma(r) \to \infty$ and $\tau(r) \to \infty$ for $r \to \infty$.

Let us now discuss the results known for bounded cohomology. The following theorems are proved using a bounded-cohomological analogue of Quillen's stability criterion (see [44]).

Theorem 2.3.1 (Monod [101]; see also [44, Example 1.4, Example 1.6])**.** *Let* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ *and* $q \geq 3$. *Then the left corner inclusions* $\mathrm{GL}(n, \mathbb{K}) \hookrightarrow \mathrm{GL}(n+1, \mathbb{K})$ *induce isomorphisms/injections*

$$\begin{aligned}\dots \xrightarrow{\cong} \mathrm{H}^q_{\mathrm{cb}}(\mathrm{GL}(2q-1, \mathbb{K})) \xrightarrow{\cong} \mathrm{H}^q_{\mathrm{cb}}&(\mathrm{GL}(2q-2, \mathbb{K})) \\ &\hookrightarrow \mathrm{H}^q_{\mathrm{cb}}(\mathrm{GL}(2q-3, \mathbb{K})) \hookrightarrow \mathrm{H}^q_{\mathrm{cb}}(\mathrm{GL}(2q-4, \mathbb{K}))\end{aligned}$$

and the left corner inclusions $\mathrm{SL}(n, \mathbb{K}) \hookrightarrow \mathrm{SL}(n+1, \mathbb{K})$ *induce isomorphisms/injections*

$$\begin{aligned}\dots \xrightarrow{\cong} \mathrm{H}^q_{\mathrm{cb}}(\mathrm{SL}(2q, \mathbb{K})) \xrightarrow{\cong} \mathrm{H}^q_{\mathrm{cb}}&(\mathrm{SL}(2q-1, \mathbb{K})) \\ &\xrightarrow{\cong} \mathrm{H}^q_{\mathrm{cb}}(\mathrm{SL}(2q-2, \mathbb{K})) \hookrightarrow \mathrm{H}^q_{\mathrm{cb}}(\mathrm{GL}(2q-3, \mathbb{K})).\end{aligned}$$

Theorem 2.3.2 (De la Cruz Mengual–Hartnick [43, Theorem A])**.** *Let* $(G_r, \iota_r)_r$ *be a family of automorphism groups of classical spaces with block inclusions* $\iota_r \colon G_r \hookrightarrow G_{r+1}$. *Then for all* $n \in \mathbb{N}$ *there exists* $r \in \mathbb{N}$ *such that* $\mathrm{H}^n_{\mathrm{cb}}(\iota_s) \colon \mathrm{H}^n_{\mathrm{cb}}(G_{s+1}) \to \mathrm{H}^n_{\mathrm{cb}}(G_s)$ *is an isomorphism for all* $s \geq r$.

Remark 2.3.3. De la Cruz Mengual and Hartnick even provide an explicit stability range for all families above. Kastenholz and Sroka have even improved this stability range to a linear one for the families $\mathrm{O}(2r, \mathbb{C})$, $\mathrm{O}(r, r)$, and $\mathrm{Sp}(2r, \mathbb{K})$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ (see [85]). Unfortunately, these ranges are not low enough to (directly) yield evidence for the isomorphism conjecture.

Let us now discuss the conceptual utility of (co-)homological stability. To compute the (co-)homology of a family $(G_r, \varphi_r)_{r\in\mathbb{N}}$ one can follow the following steps:

(i) Prove (co-)homological stability of $(G_r, \varphi_r)_{r\in\mathbb{N}}$.

(ii) Compute the (co-)homology of the colimit group $\mathrm{colim}_{r\to\infty} G_r$.

(iii) Use (i) and (ii) to determine the (co-)homology of G_r for large enough $r \in \mathbb{N}$.

By the universal property of (co-)limits, there exist natural maps

$$\begin{aligned}\mathrm{H}_n\Big(\operatorname*{colim}_{r\to\infty} G_r\Big) &\to \operatorname*{colim}_{r\to\infty} \mathrm{H}_n(G_r), \\ \mathrm{H}^n\Big(\operatorname*{colim}_{r\to\infty} G_r\Big) &\to \lim_{r\to\infty} \mathrm{H}^n(G_r),\end{aligned}$$

which are isomorphisms because (co-)homology commutes with filtered (co-)limits. Hence, for ordinary group (co-)homology, (iii) is immediate once (i) and (ii) are completed.

The above situation is more complicated for bounded cohomology, however.

Theorem 2.3.4 (Campagnolo–Fournier-Facio–Lodha–Moraschini [29, Corollary 1.9])**.** *Let R be a ring with identity. Then we have* $\mathrm{H}_{\mathrm{b}}^{n}\Big(\operatorname*{colim}_{r\to\infty} G_r\Big) = 0$ *for* $(G_r)_{r\in\mathbb{N}} \in \{(\mathrm{GL}(r,R))_{r\in\mathbb{N}}, (\mathrm{SL}(r,R))_{r\in\mathbb{N}}, (\mathrm{Sp}(2r,R))_{r\in\mathbb{N}}\}$ *and all* $n \geq 1$.

Corollary 2.3.5 (see also [84, Section 3] and [40, Corollary 8.16])**.** *The natural map*

$$\mathrm{H}_{\mathrm{b}}^{n}\Big(\operatorname*{colim}_{r\to\infty} G_r\Big) \to \lim_{r\to\infty} \mathrm{H}_{\mathrm{b}}^{n}(G_r)$$

is not surjective for $n = 2$ *and* $G_r = \mathrm{Sp}(2r,\mathbb{Z})$*, or* $n = 3$ *and* $G_r = \mathrm{SL}(r,\mathbb{Z}[i])$ *or* $G_r = \mathrm{Sp}(2r,\mathbb{Z}[i])$*.*

Nevertheless, one may ask whether the image of the map

$$\mathrm{H}_{\mathrm{b}}^{n}\Big(\operatorname*{colim}_{r\to\infty} G_r\Big) \to \lim_{r\to\infty} \mathrm{H}_{\mathrm{b}}^{n}(G_r)$$

is the space of classes whose norm is uniformly bounded on all G_r, $r \in \mathbb{N}$ (see [85, Question 1.4]).

For some results on the related question of when $\mathrm{H}_{\mathrm{b}}^{n}(G_r) = 0$ for all $r \in \mathbb{N}$ implies $\mathrm{H}_{\mathrm{b}}^{n}\Big(\operatorname*{colim}_{r\to\infty} G_r\Big) = 0$ see [53, Section 4.4].

2.4. The Isomorphism Conjecture

2.4.1. Equivalent Formulations

We have a natural isometric chain embedding $C_{\mathrm{b}}(G^n,E)^G \to C(G^n,E)^G$, which yields the natural continuous linear *comparison map* $c^n_{\mathrm{cb,c}} \colon \mathrm{H}^n_{\mathrm{cb}}(G;E) \to \mathrm{H}^n_{\mathrm{c}}(G;E)$. We can equivalently restate Conjecture 2.1.16 as follows.

Conjecture 2.4.1. Let G be a connected semisimple Lie group with finite center. Then the comparison map $c^n_{\mathrm{cb,c}} \colon \mathrm{H}^n_{\mathrm{cb}}(G) \to \mathrm{H}^n_{\mathrm{c}}(G)$ is surjective.

One can also ask whether this comparison map is injective, which is equivalent to every continuous bounded cocycle with continuous primitive admitting a continuous bounded primitive. Indeed, this question has been raised by Monod in his ICM address in 2006 and has since become known in the field of bounded cohomology as the *isomorphism conjecture*; it is the main problem of this thesis.

Isomorphism Conjecture 2.4.2 (Monod [102, Problem A]; see also [34, Conjecture 16.1])**.** Let G be a connected semisimple Lie group with finite center. Then the comparison map $c^n_{\mathrm{cb,c}} \colon \mathrm{H}^n_{\mathrm{cb}}(G) \to \mathrm{H}^n_{\mathrm{c}}(G)$ is an isomorphism.

Proving this conjecture is challenging because, even though it is computed by many resolutions, the continuous bounded cohomology of a group is usually very hard to determine explicitly.

Let G be a connected semisimple Lie group with finite center. We have a natural continuous chain inclusion $L^\infty(G^n)^G \to L^0(G^n)^G$, which yields the natural continuous linear *comparison map* $c^n_{\mathrm{mb,m}} \colon \mathrm{H}^n_{\mathrm{mb}}(G) \to \mathrm{H}^n_{\mathrm{m}}(G)$.

Remark 2.4.3. By [104, Remark 1.3], the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^n_{\mathrm{cb}}(G) & \xrightarrow{c^n_{\mathrm{cb,c}}} & \mathrm{H}^n_{\mathrm{c}}(G) \\ {\scriptstyle c^n_{\mathrm{cb,mb}}}\big\downarrow & & \big\downarrow{\scriptstyle c^n_{\mathrm{c,m}}} \\ \mathrm{H}^n_{\mathrm{mb}}(G) & \xrightarrow[c^n_{\mathrm{mb,m}}]{} & \mathrm{H}^n_{\mathrm{m}}(G) \end{array}$$

Thus, to prove the Isomorphism Conjecture 2.4.2, one could equivalently prove that $c^n_{\mathrm{mb,m}}$ is an isomorphism.

Remark 2.4.4. Let G be a connected semisimple Lie group with finite center, let K be a maximal compact subgroup of G, and let P be a minimal parabolic subgroup of G. Then we have the diagram

$$\begin{array}{ccccc} & & \Omega^n(G/K)^G & & \\ & \overset{?}{\swarrow} & & \overset{J}{\searrow} & \\ L^\infty((G/P)^{n+1})^G & & & & C((G/K)^{n+1})^G \\ \big\downarrow & & & & \big\downarrow \\ L^\infty(G^{n+1})^G & & & & C(G^{n+1})^G \\ & \searrow & & \swarrow & \\ & & L^0(G^{n+1})^G & & \end{array}$$

where J is the Dupont map from Remark 2.1.12 and the vertical maps are evaluations at a base point. In cohomology we obtain the following diagram:

$$\begin{array}{ccccc} & & \Omega^n(G/K)^G & & \\ & \overset{?}{\swarrow} & & \overset{\overline{J}}{\searrow} & \\ \mathrm{H}^n_{\mathrm{mb}}(G \curvearrowright G/P) & & & & \mathrm{H}^n_{\mathrm{c}}(G \curvearrowright G/K) \\ {\scriptstyle \mathrm{ev}_{\mathrm{mb}}}\big\downarrow & & & & \big\downarrow{\scriptstyle \mathrm{ev}_{\mathrm{c}}} \\ \mathrm{H}^n_{\mathrm{mb}}(G) & & & & \mathrm{H}^n_{\mathrm{c}}(G) \\ & \underset{c^n_{\mathrm{mb,m}}}{\searrow} & & \underset{c^n_{\mathrm{c,m}}}{\swarrow} & \\ & & \mathrm{H}^n_{\mathrm{m}}(G) & & \end{array}$$

We say that a cocycle $\overline{\alpha} \in L^\infty((G/P)^{n+1})^G$ *extends* a differential form $\omega \in \Omega^n(G/K)^G$ or a cocycle $\alpha = J(\omega) \in C((G/K)^{n+1})^G$ if the classes of the images of $\overline{\alpha}$ and α coincide in $\mathrm{H}^n_{\mathrm{m}}(G)$. Thus, Conjecture 2.4.1 asks whether every continuous cocycle on G/K can be extended to a measurable bounded cocycle on G/P; the Isomorphism Conjecture 2.4.2 additionally asks whether such an extension is unique (up to coboundaries), or, equivalently, whether every measurable bounded extension to G/P of a continuous cocycle on G/K admitting a continuous primitive admits a measurable bounded primitive.

Example 2.4.5. Let $G = \mathrm{SO}_0(2,1)$; then $G/K = \mathbb{H}^2$ is the hyperbolic plane. In Example 2.1.13 we have seen that $I(\omega)$ is bounded for all $\omega \in \Omega^2(G/K)^G$, which implies that $\mathrm{H}^2_{\mathrm{cb}}(G) \to \mathrm{H}^2_{\mathrm{c}}(G)$ is surjective.

We denote the *geodesic compactification* of $\mathbb{H}^2$ by $\overline{\mathbb{H}^2}$. Then G/P can be considered to be the visual boundary of $\overline{\mathbb{H}^2}$. For all $g_0, g_1, g_2 \in G$ let us denote the *ideal triangle* with end points $g_0P, g_1P, g_2P \in G/P$ by $\overline{\Delta}(g_0, g_1, g_2) \subset \overline{\mathbb{H}^2}$. We consider the map $\widetilde{I}\colon \Omega^2(G/K)^G \to \mathrm{Map}((G/P)^3, \mathbb{R})$ defined by

$$\widetilde{I}(\omega)(g_0P, g_1P, g_2P) := \int_{\overline{\Delta}(g_0,g_1,g_2)} \omega.$$

Let $\mathrm{dvol}_{\mathbb{H}^2}$ denote the volume form on G/K and let us write $\alpha := J(\mathrm{dvol}_{\mathbb{H}^2}) \in C((G/K)^3)^G$. Note that $\overline{\alpha} := \widetilde{I}(\mathrm{dvol}_{\mathbb{H}^2})$ only takes the values $\{-\pi, 0, \pi\}$; critically, $\overline{\alpha}$ is bounded because of the negative curvature of $\mathbb{H}^2$. Note also that $\overline{\alpha}$ is not continuous, although it is continuous on an open conull subset of $(G/P)^3$, c.f. Theorem 2.4.7. Using arguments similar to the proof of Lemma 2.1.1, one can show that $\overline{\alpha}$ is an extension of α in the sense of Remark 2.4.4 (see Figure 2).

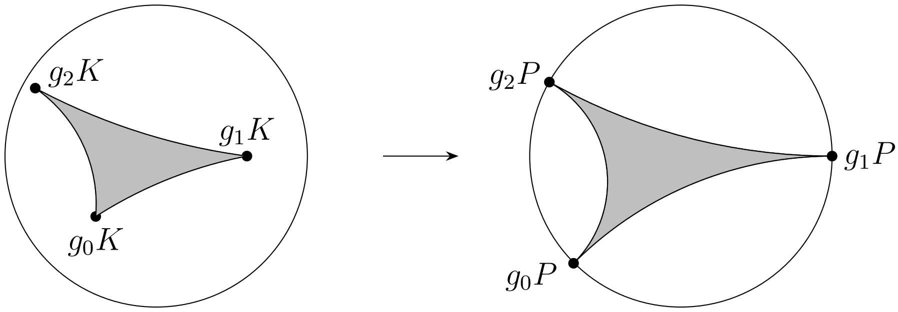

Figure 2.: Extending a geodesic 2-simplex to an ideal triangle in the geodesic compactification of the hyperbolic plane.

Let P be a minimal parabolic subgroup of G. We have a natural continuous chain inclusion $L^\infty((G/P)^n)^G \to L^0((G/P)^n)^G$, which yields the natural continuous linear *comparison map* $c^n_{\mathrm{mb,m},P}\colon \mathrm{H}^n_{\mathrm{mb}}(G \curvearrowright G/P) \to \mathrm{H}^n_{\mathrm{m}}(G \curvearrowright G/P)$.

Remark 2.4.6. Let us consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^n_{\mathrm{mb}}(G \curvearrowright G/P) & \xrightarrow{\mathrm{ev}_{\mathrm{mb}}} & \mathrm{H}^n_{\mathrm{mb}}(G) \\ \Big\downarrow c^n_{\mathrm{mb,m},P} & & \Big\downarrow c^n_{\mathrm{mb,m}} \\ \mathrm{H}^n_{\mathrm{m}}(G \curvearrowright G/P) & \xrightarrow[\mathrm{ev}_{\mathrm{m}}]{} & \mathrm{H}^n_{\mathrm{m}}(G) \end{array}$$

Note that $c^n_{\mathrm{mb,m}}$ is surjective if $c^n_{\mathrm{mb,m},P}$ is surjective, and that $c^n_{\mathrm{mb,m}}$ is injective if $c^n_{\mathrm{mb,m},P}$ is injective and $\ker(\mathrm{ev}_{\mathrm{m}}) \cap \mathrm{im}\big(c^n_{\mathrm{mb,m},P}\big) = 0$ (cf. Theorem 2.1.4 and Theorem 2.5.2).

Again, the alternation map (2.1.7) induces decompositions

$$\begin{aligned} C_{\mathrm{b}}((G/P)^{n+1})^G &= C_{\mathrm{b,alt}}((G/P)^{n+1})^G \oplus C_{\mathrm{b,n-alt}}((G/P)^{n+1})^G, \\ \mathrm{H}^n_{\mathrm{cb}}(G \curvearrowright G/P) &= \mathrm{H}^n_{\mathrm{cb,alt}}(G \curvearrowright G/P) \oplus \mathrm{H}^n_{\mathrm{cb,n-alt}}(G \curvearrowright G/P), \\ L^\infty((G/P)^{n+1})^G &= L^\infty_{\mathrm{alt}}((G/P)^{n+1})^G \oplus L^\infty_{\mathrm{n-alt}}((G/P)^{n+1})^G, \\ \mathrm{H}^n_{\mathrm{mb}}(G \curvearrowright G/P) &= \mathrm{H}^n_{\mathrm{mb,alt}}(G \curvearrowright G/P) \oplus \mathrm{H}^n_{\mathrm{mb,n-alt}}(G \curvearrowright G/P), \end{aligned}$$

and the inclusions $C_{\mathrm{b,alt}}((G/P)^{n+1})^G \to C_{\mathrm{alt}}((G/P)^{n+1})^G$ and $C_{\mathrm{b,alt}}((G/P)^{n+1})^G \to L^\infty_{\mathrm{alt}}((G/P)^{n+1})^G$ induce natural continuous linear *comparison maps* $c^n_{\mathrm{cb,c,alt}} \colon \mathrm{H}^n_{\mathrm{cb,alt}}(G \curvearrowright G/P) \to \mathrm{H}^n_{\mathrm{c,alt}}(G \curvearrowright G/P)$ and $c^n_{\mathrm{cb,mb,alt}} \colon \mathrm{H}^n_{\mathrm{cb,alt}}(G \curvearrowright G/P) \to \mathrm{H}^n_{\mathrm{mb,alt}}(G \curvearrowright G/P)$.

Theorem 2.4.7 (Bucher–Savini [18, Theorem 2])**.** *The map $c^n_{\mathrm{cb,mb,alt}} \colon \mathrm{H}^n_{\mathrm{cb,alt}}(G \curvearrowright G/P) \to \mathrm{H}^n_{\mathrm{mb,alt}}(G \curvearrowright G/P)$ is an isomorphism for all $n \in \mathbb{N}$.*

Remark 2.4.8. Let us use the notation of Theorem 2.1.4. The proofs of Theorem 2.1.7 and Theorem 2.4.7 use spectral sequences and imply the existence of a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^n_{\mathrm{cb}}(G) & \xrightarrow{\cong} & \mathrm{H}^n_{\mathrm{cb,alt}}(G \curvearrowright G/P) \\
\downarrow{\scriptstyle c^n_{\mathrm{cb,c}}} & & \downarrow{\scriptstyle c^n_{\mathrm{cb,c,alt}}} \\
\mathrm{H}^n_{\mathrm{c}}(G) & & \mathrm{H}^n_{\mathrm{c,alt}}(G \curvearrowright G/P) \\
& \searrow & \downarrow{\scriptstyle \mathrm{pr}} \\
& & \mathrm{H}^n_{\mathrm{c,alt}}(G \curvearrowright G/P)/\mathrm{H}^{n-1}_{\mathrm{c}}(A)^{w_0}
\end{array}$$

from which we see that the comparison map $c^n_{\mathrm{cb,c}}$ is injective if and only if $c^n_{\mathrm{cb,c,alt}}$ is injective and $\mathrm{H}^{n-1}_{\mathrm{c}}(A)^{w_0} \cap \mathrm{im}\big(c^n_{\mathrm{cb,c,alt}}\big) = 0$ (see [18, Proposition 3]). The latter condition is fulfilled if $n > \mathrm{rk}_{\mathbb{R}}(G)$, or if n is even and w_0 acts as -1 on the Lie algebra of A.

2.4.2. Beyond Connected Semisimple Lie Groups

Let us now discuss what the Isomorphism Conjecture 2.4.2 would mean for a general connected locally compact, second countable group G.

Theorem 2.4.9 (Gleason–Yamabe [59], [128]; see also [119, Ch. 6])**.** *Let H be a locally compact group. There exists an open subgroup H' of H such that for every open neighborhood U of the identity of H' there exists a compact normal subgroup $K \subset U$ of H' such that H'/K is isomorphic to a Lie group.*

Recall that if N is an amenable normal subgroup of G, then we have an isomorphism

$$\mathrm{H}^n_{\mathrm{mb}}(G) \cong \mathrm{H}^n_{\mathrm{mb}}(G \curvearrowright G/N) = \mathrm{H}^n_{\mathrm{mb}}(G/N).$$

Theorem 2.4.9 implies the existence of a compact normal subgroup K of G such that G/K is isomorphic to a Lie group. Since compact groups are amenable, we obtain $\mathrm{H}^n_{\mathrm{mb}}(G) \cong \mathrm{H}^n_{\mathrm{mb}}(G/K)$. Recall that the universal covering $\widetilde{G/K}$ of G/K is a central extension of the latter (see [87, Proposition 1.101]). Hence $\mathrm{H}^n_{\mathrm{mb}}(G/K) \cong \mathrm{H}^n_{\mathrm{mb}}\Big(\widetilde{G/K}\Big)$. By Levi's theorem (see [124, Theorem 3.18.13]) we have an isomorphism of Lie groups

$$\widetilde{G/K} \cong R \rtimes (S_1 \times S_2),$$

where R denotes the solvable radical of $\widetilde{G/K}$, S_1 is a connected compact semisimple Lie group and S_2 is a connected semisimple Lie group without compact factors. We obtain $\mathrm{H}^n_{\mathrm{mb}}\Big(\widetilde{G/K}\Big) \cong \mathrm{H}^n_{\mathrm{mb}}(S_2/\mathrm{Z}(S_2))$ and conclude that G has the same bounded cohomology as the centerless semisimple Lie group without compact factors $S_2/\mathrm{Z}(S_2)$. Hence the Isomorphism Conjecture 2.4.2 and the computability of continuous cohomology would imply that, in principle, the bounded cohomology of every connected locally compact, second countable group was known.

Remark 2.4.10. To prove the Isomorphism Conjecture 2.4.2 it clearly suffices to prove that the comparison map is an isomorphism for all centerless connected semisimple Lie groups without compact factors. We point out that no Künneth formula is known for bounded cohomology; hence it does not suffice to restrict oneself to simple groups. We also point out that the Isomorphism Conjecture 2.4.2 is false in the extended setting of infinite center groups. Indeed, letting G denote the universal covering of $\mathrm{SL}(2,\mathbb{R})$, the comparison map $c^2_{\mathrm{cb,c}}\colon \mathrm{H}^2_{\mathrm{cb}}(G) \to \mathrm{H}^2_{\mathrm{c}}(G)$ is not injective, while $c^3_{\mathrm{cb,c}}\colon \mathrm{H}^3_{\mathrm{cb}}(G) \to \mathrm{H}^3_{\mathrm{c}}(G)$ is not surjective, see [100, Example 9.3.11]. The conjecture also fails for different coefficient modules, even for complex irreducible Hilbert modules, see [27].

We conclude this subsection with vanishing results concerning the bounded cohomology of certain totally disconnected groups (cf. Theorem 2.1.24 and Theorem 2.1.25).

Theorem 2.4.11 (Monod [105, Theorem A]). *Let* $\mathbf{G}$ *be an algebraic* k*-group, where* k *is a non-Archimedean local field, and equip* $\mathbf{G}(k)$ *with its Hausdorff topology. Then we have* $\mathrm{H}^n_{\mathrm{cb}}(\mathbf{G}(k)) = 0$ *for all* $n \geq 1$.

Theorem 2.4.12 (Monod [105, Theorem B]; see also Bucher–Monod [16, Theorem 1]). *Let* G *act properly and strongly transitively by automorphisms on a locally finite Euclidean building. Then we have* $\mathrm{H}^n_{\mathrm{cb}}(G) = 0$ *for all* $n \geq 1$.

Zhao [130] extended one case of the theorem above by determining $\mathrm{H}^n_{\mathrm{cb}}(\mathrm{Aut}(T);\mathcal{H})$ for every regular tree T and every complex irreducible Hilbert $\mathrm{Aut}(T)$-module $\mathcal{H}$.

2.5. Evidence for the Isomorphism Conjecture

In this section we list evidence for the Isomorphism Conjecture 2.4.2. To establish surjectivity, injectivity in rank one, and injectivity in higher rank, diverse and interrelated methods have been used; the goal of the present thesis is to establish further evidence for injectivity in the higher rank case.

Let G be a connected semisimple Lie group with finite center. It is easy to see that $\mathrm{H}^0_{\mathrm{cb}}(G) = \mathbb{R}$ and $\mathrm{H}^1_{\mathrm{cb}}(G) = 0$, and that $c^0_{\mathrm{cb,c}}\colon \mathrm{H}^0_{\mathrm{cb}}(G) \to \mathrm{H}^0_{\mathrm{c}}(G)$ is an isomorphism, which implies the validity of the Isomorphism Conjecture 2.4.2 in degrees 0 and 1.

2.5.1. Injectivity Results

Theorem 2.5.1 (Burger–Monod [25, Lemma 6.1]). *Let* G *be a connected semisimple Lie group with finite center. Then the comparison map* $\mathrm{H}^2_{\mathrm{cb}}(G) \to \mathrm{H}^2_{\mathrm{c}}(G)$ *is injective. In particular, the Isomorphism Conjecture 2.4.2 holds in degrees* ≤ 2.

For higher degrees the proofs of injectivity of the comparison map distinguish between rank one and higher rank. Let us start with what is known in rank one. Using the notation of Theorem 2.1.4, we denote by $(G/P)^{(3)}$ the subspace of triples of distinct points in G/P.

Theorem 2.5.2 (Bucher–Savini [21, Theorem 10]; see also [107, Corollary 1.2])**.** *Let G be a product of groups of type* $\mathrm{PO}_0(n,1)$ *(with possibly differing n). Then the comparison map* $c^3_{\mathrm{mb,m}} \colon \mathrm{H}^3_{\mathrm{mb}}(G) \to \mathrm{H}^3_{\mathrm{m}}(G)$ *is injective. In particular,*

$$\begin{aligned} \mathrm{H}^3_{\mathrm{mb}}(\mathrm{SO}_0(n,1)) &= 0 \qquad (n \neq 3),\\ \mathrm{H}^3_{\mathrm{mb}}(\mathrm{SO}_0(3,1)) &= \mathbb{R}, \end{aligned}$$

and the Isomorphism Conjecture 2.4.2 holds in degree 3 *for* $G = \mathrm{SO}_0(n,1)$*,* $n \geq 2$*.*

Sketch of Proof. We use the proof strategy described in Remark 2.4.6; the injectivity of $c^3_{\mathrm{mb,m},P}$ easily follows from the fact that the orbit space $G\backslash((G/P)^{(3)})$ is finite, while the statement $\mathrm{im}(c^3_{\mathrm{mb,m},P}) \cap \ker(\mathrm{ev_m}) = 0$ follows from a hard computation showing that some explicit measurable 3-cocycle on G/P is unbounded (see [21, Proposition 24]). This yields the claimed injectivity.

Since $\mathrm{SO}_0(n,1)$ admits the structure of a complex Lie group if and only if $n = 3$, we conclude $\mathrm{H}^3_{\mathrm{mb}}(\mathrm{SO}_0(n,1)) = 0$ for $n \neq 3$. Since $c^3_{\mathrm{mb,m}}$ is surjective for $\mathrm{PSL}(2,\mathbb{C}) \cong \mathrm{PO}_0(3,1)$ (see Subsection 2.5.2, (viii)), we obtain $\mathrm{H}^3_{\mathrm{mb}}(\mathrm{SO}_0(3,1)) = \mathbb{R}$. □

Remark 2.5.3. This result was previously known in the cases $n = 2$ (see [27]) and $n = 3$ (see [5]) by different methods.

Theorem 2.5.4 (Bucher–Savini [18, Corollary 7])**.** *The Isomorphism Conjecture 2.4.2 holds in degree* 3 *for* $G = \mathrm{SU}(n,1)$*,* $n \geq 1$*. In particular,* $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SU}(n,1)) = 0$*,* $n \geq 1$*.*

Sketch of Proof. We use the proof strategy described in Remark 2.4.8; the injectivity of $c^3_{\mathrm{cb,c,alt}}$ easily follows from the fact that the orbit space $G\backslash((G/P)^{(3)})$ is compact (see [18, Proposition 6]), while the other statement is implied by $\mathrm{H}^2_{\mathrm{c}}(A) = \mathrm{H}^2_{\mathrm{c}}(\mathbb{R}) = 0$. Since G does not admit the structure of a complex Lie group, we obtain the result. □

Theorem 2.5.5 (Bucher–Savini [18, Theorem 5])**.** *The Isomorphism Conjecture 2.4.2 holds in degree* 4 *for* $G = \mathrm{SO}_0(n,1)$*,* $n \geq 2$*. In particular,*

$$\begin{aligned} \mathrm{H}^4_{\mathrm{cb}}(\mathrm{SO}_0(n,1)) &= 0 \qquad (n \neq 4),\\ \mathrm{H}^4_{\mathrm{cb}}(\mathrm{SO}_0(4,1)) &= \mathbb{R}. \end{aligned}$$

Sketch of Proof. We use the proof strategy described in Remark 2.4.8; the injectivity of $c^3_{\mathrm{cb,c,alt}}$ follows from a clever estimation involving the 5-term functional equation corresponding to $(G/P)^{(3)}$ (see [18, Theorem 4]), while the other statement is implied by $\mathrm{H}^2_{\mathrm{c}}(A) = \mathrm{H}^2_{\mathrm{c}}(\mathbb{R}) = 0$. Since Conjecture 2.4.1 holds for G (see Subsection 2.5.2, (vii)), [99, Ch. III, §6] yields the result. □

Remark 2.5.6. The case $n = 2$ of the above theorem has first been proved by Hartnick–Ott [71] by completely different methods, namely by constructing explicit primitives using partial differential equations. See [106] for some further developments.

Remark 2.5.7. The proofs of the three theorems above rely heavily on the structure of $G\backslash((G/P)^{(3)})$. Let $P = MAN$ be a Langlands decomposition of P (see Proposition 3.1.1). Then $(G/P)^{(3)}$ is diffeomorphic to an open dense subspace N_{opp} of N, and the G-action on $(G/P)^{(3)}$ corresponds to the action of MA on N_{opp} by conjugation (see [20, Section 2]). Consequently, we have a homeomorphism $G\backslash((G/P)^{(3)}) \cong (MA)\backslash N_{\mathrm{opp}}$, where the latter orbit space is usually easier to handle.

Remark 2.5.8. As evidenced by the three theorems above, the methods of Remark 2.4.6 and Remark 2.4.8 work very well in rank 1. In higher rank, the only result known thus far, using one of these methods, is in degree 3 for $\mathrm{SL}(3, \mathbb{K})$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ (see [21, Theorem 11]), which has previously also been established using bounded-cohomological stabilization; $(MA)\backslash N_{\mathrm{opp}}$ is non-compact for these two groups. Indeed, $(MA)\backslash N_{\mathrm{opp}}$ seems to be compact if and only if G is a product of rank 1 groups. A simple heuristic for this claim is the fact that, when increasing the rank in a family of classical groups, the dimension of N grows quadratically, while the dimension of MA only grows linearly (see Section 1.3); since N is topologically Euclidean, MA may be "too low-dimensional" to make N_{opp} compact. Indeed, $(MA)\backslash N_{\mathrm{opp}}$ is non-compact for $G = \mathrm{Sp}(4, \mathbb{K})$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Let us now discuss the known results in higher rank, where the main tool is bounded-cohomological stabilization.

Corollary 2.5.9. *The Isomorphism Conjecture 2.4.2 holds in degree* 3 *for* $G = \mathrm{SL}(r+1, \mathbb{R})$ *and* $G = \mathrm{SL}(r+1, \mathbb{H})$*,* $r \geq 1$*; for these groups we have* $\mathrm{H}^3_{\mathrm{cb}}(G) = 0$.

Proof. We have $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}(2, \mathbb{K})) \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(2, \mathbb{K}))$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{H}\}$. Now

$$\begin{aligned} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(n, \mathbb{R})) &\hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(2, \mathbb{R})) = \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}_0(2,1)) = 0, \\ \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(n, \mathbb{H})) &\hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(2, \mathbb{H})) = \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}_0(5,1)) = 0, \end{aligned}$$

which hold by Theorem 2.3.1, Section 1.4, and Theorem 2.5.2, yield the claim. □

For the case $G = \mathrm{SL}(r+1, \mathbb{C})$ more work is needed, see Subsection 2.5.2, (viii).

Theorem 2.5.10 (De la Cruz Mengual [42])**.** *The induced maps*

$$\begin{aligned} \ldots \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(7, \mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(5, \mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(3, \mathbb{C})), \\ \ldots \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}(6, \mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}(4, \mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}(2, \mathbb{C})) \end{aligned}$$

are isomorphisms. In particular, the Isomorphism Conjecture 2.4.2 holds in degree 3 *for these groups, whose third continuous bounded cohomology is one-dimensional.*

See Subsection 2.5.2, (ix) for some further information on this result.

Remark 2.5.11. The proof of the theorem above works by using secondary bounded-cohomological stabilization (c.f. Remark 6.2.1). The main goal of the present thesis is to adapt this proof strategy in order to prove the Isomorphism Conjecture 2.4.2 in degree 3 for $G = \mathrm{Sp}(2r, \mathbb{R})$, $r \geq 1$.

Remark 2.5.12. In [42, Theorem 3], it is claimed that the restriction $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(2n, \mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(2n, \mathbb{C}))$ was a linear isomorphism for all $n \geq 3$, which would imply the Isomorphism Conjecture 2.4.2 in degree 3 for $\mathrm{SO}(2n, \mathbb{C})$. This claim relies on [42, Lemma 6.13], which is false. Indeed, in the proof of this lemma, one can not argue for the left face of the cube as for the bottom face or the back face. Indeed, one only obtains the results listed in Proposition 2.5.13 below.

2.5.2. Surjectivity Results

Beyond trivial degrees, the following is known in the context of Conjecture 2.4.1:

(i) The conjecture clearly holds in all cases in which Question 2.1.14 is solved, i.e. in the cases listed in Remark 2.1.15.

(ii) The conjecture holds in *top degree*, i.e. in degree $\dim(G/K)$. The proof works by replacing the geodesic coning simplices of Proposition 2.1.11 by barycentrically straightened simplices and determining a uniform bound for the Jacobians (see [39], [89], [12]; note that the proof in [115] for the $\mathrm{SL}(n,\mathbb{R})$ case is incorrect, see [11]; see also [13] for the connection to Gromov's proportionality principle).

(iii) As an extension of (ii), if G has no factor locally isomorphic to $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{SL}(4,\mathbb{R})$, then the conjecture holds in every degree higher than or equal to $\dim(G/K) - \mathrm{rk}_{\mathbb{R}}(G) + 2$ (see [90]; see also [126] for the limits of this approach).

(iv) As an extension of (iii), if G also has no factor locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{Sp}(4,\mathbb{R})$, or $G_{2(2)}$, then the conjecture holds in every degree higher than or equal to $\mathrm{srk}(G) + 2$, where $\mathrm{srk}(G)$ denotes the *splitting rank* of G, i.e. the maximal dimension of a totally geodesic subspace of G/K which splits isometrically as a non-trivial product (see [125], where a table of the values of $\mathrm{srk}(G)$ is given for every case; note also that $\mathrm{srk}(G) \leq \dim(G/K) - \mathrm{rk}_{\mathbb{R}}(G)$, where we have equality if and only if G is locally isomorphic to products of type $\mathrm{SL}(n,\mathbb{R})$).

(v) Goncharov [63] established surjectivity in degree 5 for the group $\mathrm{SL}(3,\mathbb{C})$ by constructing an explicit bounded cocycle representing the generator of $\mathrm{H}^5_{\mathrm{c}}(\mathrm{SL}(3,\mathbb{C}))$, which he defined using a globally bounded variant of the trilogarithm. The proof that this construction actually satisfies the cocycle condition relies crucially on the highly non-trivial functional equations of the trilogarithm.

(vi) Primary characteristic classes of real algebraic subgroups of $\mathrm{GL}(n,\mathbb{R})$ can be represented by cocycles having finite value sets on singular simplices (see [65, Section 1.3] and [10, Theorem 4]).

(vii) As an extension of (vi), it is shown in [70] that the primary characteristic classes correspond precisely to the truncated polynomial algebra part of $\mathrm{H}^\bullet_{\mathrm{c}}(G)$ in Remark 2.1.20. Hence all even generators are bounded. This implies the conjecture for all groups without exterior algebra part, which are all groups that have no factor locally isomorphic to a complex classical group, $\mathrm{SL}(n,\mathbb{R})$ for $n \geq 3$, $\mathrm{SL}(n,\mathbb{H})$ for $n \geq 2$, $\mathrm{SO}_0(p,q)$ for $p \cdot q$ odd, $\mathrm{E}_{6(6)}$, or $\mathrm{E}_{6(-26)}$.

(viii) The Isomorphism Conjecture 2.4.2 holds in degree 3 for $\mathrm{SL}(n,\mathbb{C})$, where $n \geq 2$ [62], [5], [101], [15]. Indeed, $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(n,\mathbb{C}))$ is generated by the *bounded Borel class* β_n, which has Gromov norm $\frac{n(n^2-1)}{6} v_3$, where v_3 denotes the volume of a maximal ideal tetrahedron in $\mathbb{H}^3$ (c.f. Remark 2.1.15), and the left corner inclusion $\mathrm{SL}(n,\mathbb{C}) \hookrightarrow \mathrm{SL}(n+1,\mathbb{C})$ induces a linear isomorphism $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(n+1,\mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(n,\mathbb{C}))$.

(ix) The restrictions $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(2n+1,\mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(2n+1,\mathbb{C}))$ and $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(2n,\mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}(2n,\mathbb{C}))$ are isometric isomorphisms [40], [42]. This implies that the respective block inclusions induce linear isomorphisms in degree-three continuous bounded cohomology, further implying the Isomorphism Conjecture 2.4.2 in degree 3 for $\mathrm{SO}(2n+1,\mathbb{C})$ and $\mathrm{Sp}(2n,\mathbb{C})$, $n \geq 1$.

Proposition 2.5.13 (De la Cruz Mengual [41], [42]). *The following is known about* $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(2n,\mathbb{C}))$*:*

(i) The comparison map $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(2n,\mathbb{C})) \to \mathrm{H}^3_{\mathrm{c}}(\mathrm{SO}(2n,\mathbb{C}))$ *is surjective for all* $n \geq 3$.

(ii) The restriction map $\mathrm{res}_n \colon \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(2n,\mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(2n,\mathbb{C}))$ *is injective for all* $n \geq 2$*, with* res_2 *having operator norm* $\frac{2}{5}$ *and* $\mathrm{res}_2(\beta_4)$ *having Gromov norm* $4v_3$*, and* res_3 *being a linear isomorphism.*

(iii) The restriction map $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(8,\mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(6,\mathbb{C}))$ *is surjective and the restriction map* $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(6,\mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(4,\mathbb{C}))$ *is injective.*

(iv) The restriction maps $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(2(n+1),\mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(2n,\mathbb{C}))$ *and* $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}(2n,\mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(2n,\mathbb{C}))$ *are linear isomorphisms between one-dimensional* $\mathbb{R}$*-vector spaces for all* $n \geq 3$*; they are injective for* $n = 2$.

(v) We have $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(4,\mathbb{C})) \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(4,\mathbb{C})) \cong \mathbb{R}^2$.

Proof. (i) We consider the diagram

$$\begin{array}{ccc} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(2n,\mathbb{C})) & \xrightarrow{\cong} & \mathrm{H}^3_{\mathrm{c}}(\mathrm{SL}(2n,\mathbb{C})) \\ {\scriptstyle \mathrm{res}_n}\downarrow & & \downarrow \\ \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(2n,\mathbb{C})) & \longrightarrow & \mathrm{H}^3_{\mathrm{c}}(\mathrm{SO}(2n,\mathbb{C})) \end{array}$$

and claim that the restriction $\mathrm{H}^3_{\mathrm{c}}(\mathrm{SL}(2n,\mathbb{C})) \to \mathrm{H}^3_{\mathrm{c}}(\mathrm{SO}(2n,\mathbb{C}))$ is an isomorphism for all $n \geq 3$, which implies (i). To prove the claim, we argue as in Section 6.1: The symmetric space of $\mathrm{SO}(2n,\mathbb{C})$ can be smoothly embedded into the symmetric space of $\mathrm{SL}(2n,\mathbb{C})$. Since $\mathrm{H}^3_{\mathrm{c}}(\mathrm{SL}(2n,\mathbb{C}))$ and $\mathrm{H}^3_{\mathrm{c}}(\mathrm{SO}(2n,\mathbb{C}))$ are both one-dimensional, it suffices to prove that the restriction of some differential 3-form generating $\mathrm{H}^3_{\mathrm{c}}(\mathrm{SL}(2n,\mathbb{C}))$ to the symmetric space of $\mathrm{SO}(2n,\mathbb{C})$ is non-zero. This follows from the fact that the Killing form $B_{\mathfrak{g}}$ of $\mathfrak{g} \in \{\mathfrak{sl}(2n,\mathbb{C}), \mathfrak{so}(2n,\mathbb{C})\}$ is proportional to the trace form of $\mathfrak{g}$ (see [75, Ch. III, §8]), and [41, Formula (1.1)], which reads

$$\omega(X,Y,Z) \coloneqq B_{\mathfrak{g}}(X, J[Y,Z]) \qquad (X,Y,Z \in \mathfrak{p}),$$

where K is a maximal compact subgroup of $G \in \{\mathrm{SL}(2n,\mathbb{C}), \mathrm{SO}(2n,\mathbb{C})\}$ with Lie algebra $\mathfrak{k}$, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$, J is the complex structure of $\mathfrak{g}$, and ω is a generator of $((\wedge^3\mathfrak{p})^*)^K \cong \mathrm{H}^3_{\mathrm{c}}(G)$. Indeed, since the complex structure of $\mathfrak{sl}(2n,\mathbb{C})$ restricts to the complex structure of $\mathfrak{so}(2n,\mathbb{C})$, restricting this generator of $\mathrm{H}^3_{\mathrm{c}}(\mathrm{SL}(2n,\mathbb{C}))$ yields a generator of $\mathrm{H}^3_{\mathrm{c}}(\mathrm{SO}(2n,\mathbb{C}))$.

(ii) The argument above also implies injectivity of res_n for all $n \geq 2$. By [42, Proposition 6.12], res_2 has operator norm $\frac{2}{5}$ and $\mathrm{res}_2(\beta_4)$ has Gromov norm $4v_3$. By Section 1.4 and Corollary 2.2.19 we have

$$\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(6,\mathbb{C})) \cong \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(4,\mathbb{C})) \cong \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(6,\mathbb{C})) \cong \mathbb{R},$$

which implies that res_3 is a linear isomorphism.

(iii) The claim follows from the commutativity of the following diagram:

$$\begin{array}{ccc} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(8,\mathbb{C})) & \hookrightarrow & \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(8,\mathbb{C})) \\ \cong\downarrow & & \downarrow \\ \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(6,\mathbb{C})) & \xrightarrow{\cong} & \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(6,\mathbb{C})) \\ \cong\downarrow & & \downarrow \\ \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(4,\mathbb{C})) & \hookrightarrow & \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(4,\mathbb{C})) \end{array}$$

(iv) By Corollary 2.2.21 the restriction $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(2n,\mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(2n,\mathbb{C}))$ is an isometric injection. Considering the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}(6,\mathbb{C})) & \longrightarrow & \mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(6,\mathbb{C})) \\ \cong\downarrow & & \downarrow \\ \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}(6,\mathbb{C})) & \xrightarrow[\cong]{} & \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(6,\mathbb{C})) \end{array}$$

we first conclude that $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(6,\mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(6,\mathbb{C}))$ is a linear isomorphism, which implies that $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}(6,\mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(6,\mathbb{C}))$ is a linear isomorphism. Inductively, the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}(2(n+1),\mathbb{C})) & \longrightarrow & \mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(2(n+1),\mathbb{C})) \\ \cong\downarrow & & \downarrow \\ \mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}(2n,\mathbb{C})) & \longrightarrow & \mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(2n,\mathbb{C})) \end{array}$$

where the injectivity of $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(2(n+1),\mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(2n,\mathbb{C}))$ is due to [42, Key Lemma], implies that $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}(2n,\mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(2n,\mathbb{C}))$ and $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(2(n+1),\mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(2n,\mathbb{C}))$ are linear isomorphisms for all $n \geq 3$. Arguing as above, it is easy to see that the restriction $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}(4,\mathbb{C})) \to \mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(4,\mathbb{C}))$ is injective.

(v) By Section 1.4, Theorem 2.5.2, Remark 2.1.9, and Remark 2.1.15 we have

$$\mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(4,\mathbb{C})) \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SO}(4,\mathbb{C})) \cong \mathrm{H}^3_{\mathrm{cb}}\big(\mathrm{SO}_0(3,1)^2\big) \cong \mathrm{H}^3_{\mathrm{c}}\big(\mathrm{SO}_0(3,1)^2\big) \cong \mathrm{H}_3((S^3)^2) = \mathbb{R}^2.$$

□

Remark 2.5.14. We have seen that sporadic isogenies and the interplay between the (semi-)simple families and the reductive families can be helpful to compute continuous bounded cohomology. Indeed, using Section 1.4 and the results of this section, we also obtain $\mathrm{H}^3_{\mathrm{cb}}(G) = 0$ for

$$G \in \{\mathrm{Sp}(1,1), \mathrm{SO}_0(2,2), \mathrm{SO}_0(3,3), \mathrm{SO}^*(4), \mathrm{SO}^*(6)\},$$

and $\mathrm{H}^4_{\mathrm{cb}}(G) = 0$ for

$$G \in \{\mathrm{SL}(2,\mathbb{R}), \mathrm{SL}(2,\mathbb{C}), \mathrm{SL}(2,\mathbb{H}), \mathrm{SO}(3,\mathbb{C}), \mathrm{SO}^*(4)\},$$

as well as $\mathrm{H}^4_{\mathrm{cb}}(\mathrm{Sp}(1,1)) = \mathbb{R}^2$. Some of these results could be used as the base case of the induction when proving the Isomorphism Conjecture 2.4.2 for the corresponding families.

Part II.

Calculations for Classical Groups

3. Stabilizers and Invariants

Our goal for the next two chapters is to understand the configuration spaces $G_r\backslash\mathcal{P}_r^k$ for every classical space (V_r, ω) of rank $r \geq 1$, where we use Notation 1.1.20 throughout. Recall that in Corollary 1.1.19 we have seen that $G_r\backslash\mathcal{P}_r$ and $G_r\backslash\mathcal{P}_r^{(2)}$ are trivial.

3.1. Statement of Results

First, let us quickly recall the concept of a Langlands decomposition.

Proposition 3.1.1 (see [87, Proposition 7.83])**.** *Let P be a parabolic subgroup of a reductive Lie group G. Then there exists a reductive subgroup M, an abelian subgroup A, and a nilpotent subgroup N such that $P = MAN$ and $M \times A \times N \to P$ is a diffeomorphism.*

A decomposition $P = MAN$ as above is called a *Langlands decomposition*.

Let (V_r, ω) be a classical space, where we use Notation 1.1.20. We will prove the following three theorems.

Theorem 3.1.2. *Let $(x, y) \in V_r \times V_r$ be a hyperbolic pair. Then*

(i) $P \coloneqq \mathrm{Stab}_{G_r}([x])$ is a maximal parabolic subgroup of G_r;

(ii) $L \coloneqq \mathrm{Stab}_{G_r}([x, y])$ is a Levi factor of P;

(iii) if $P = MAN$ is a Langlands decomposition of P, then $M \cong G_{r-1}$ and $A \cong \mathbb{K}^\times$;

(iv) N is at most two-step nilpotent, and N is abelian if and only if ω is symmetric, or ω is alternating and $r = 1$.

In particular, $\mathcal{P}_r \cong G_r/P$ can be considered a generalized flag manifold, and L is reductive.

Corollary 3.1.3. *There exist surjective morphisms $P \twoheadrightarrow G_{r-1}$, $L \twoheadrightarrow G_{r-1}$ with solvable kernel.*

Proof. This follows from N being a normal subgroup of P and from the Lie group isomorphism $L \cong M \times A$. □

Definition 3.1.4. Let $k \geq 3$. We define the *stability range* r_k and the *weak stability range* r'_k of $G_r \curvearrowright \mathcal{P}_r^k$ to be

$$r_k \coloneqq \begin{cases} \left\lfloor \frac{k}{2} \right\rfloor, & G_r = \mathrm{Sp}(2r, \mathbb{K}), \\ \left\lceil \frac{k-d}{2} \right\rceil, & G_r = \mathrm{O}(2r+d, \mathbb{C}), \\ k-1, & G_r = \mathrm{O}(r+d, r), \mathrm{U}(r+d, r), \end{cases} \qquad r'_k \coloneqq \begin{cases} \left\lceil \frac{k}{2} \right\rceil, & G_r = \mathrm{Sp}(2r, \mathbb{K}), \\ r_k, & G_r \neq \mathrm{Sp}(2r, \mathbb{K}). \end{cases}$$

In Section 3.4 we will introduce a generalized angular Cartan invariant, which is one of the concepts needed for the following parametrizations.

Theorem 3.1.5. *(i)* $G_r \backslash \mathcal{P}_r^k$ *is a Lebesgue space for all* $k \geq 1$*;*

(ii) the generalized angular Cartan invariant yields an isomorphism $G_r \backslash \mathcal{P}_r^{\{3\}} \cong \mathcal{C}^{(3)}$ *of Lebesgue spaces, where* $\mathcal{C}^{(3)}$ *is a compact manifold of dimension at most* 1*;*

(iii) for every $k \geq 3$ *and every* $r \geq r_k$ *we have an isomorphism* $G_r \backslash \mathcal{P}_r^k \cong \mathcal{C}^{(3)} \times \mathbb{K}^{m_k}$ *as Lebesgue spaces, where* $m_k = \frac{k(k-3)}{2}$*;*

(iv) for every $r \geq r_k$ *there exists* $c_k \leq 2^{k-2}$ *and trivial Lebesgue* G_r*-spaces* $\mathcal{C}_i^{(k)}$, $i = 1, \ldots, c_k$*, such that we have an isomorphism*

$$\coprod_{i=1}^{c_k} \Big((G_r / H_{k-1,i}) \times \mathcal{C}_i^{(k)} \Big) \xrightarrow{\cong} \mathcal{P}_r^k$$

of Lebesgue G_r*-spaces, where the* $H_{k-1,i}$ *are stabilizers of certain points in* $\mathcal{P}_r^k$*.*

The stability range in Theorem 3.1.5, (iii) is optimal in the sense that $G_r \backslash \mathcal{P}_r^k \ncong G_{r+1} \backslash \mathcal{P}_{r+1}^k$ for every $r < r_k$.

Theorem 3.1.5, (iii) is the main ingredient for the proof of our Secondary Stability Theorem, which we now reformulate to make it precise.

Theorem 3.1.6. *We have a linear isomorphism*

$$\mathrm{H}^k_{\mathrm{mb}}(G_{r+1} \curvearrowright \mathcal{P}_{r+1}) \cong \mathrm{H}^k_{\mathrm{mb}}(G_r \curvearrowright \mathcal{P}_r)$$

in the following cases:

(i) in the case $r \geq r'_{k+2}$*;*

(ii) in the case $r = 2$ *and* $k = 3$ *for* $G_r = \mathrm{Sp}(2r, \mathbb{K})$*.*

Theorem 3.1.2 is proved in Section 3.2; Theorem 3.1.5, (i) is proved in Section 3.3; Theorem 3.1.5, (ii) proved in Section 3.4; the proof of Theorem 3.1.5, (iii) is spread over Section 3.5 (for quadruples in the symplectic case) and Chapter 4 (for all other cases); Theorem 3.1.5, (iv) and Theorem 3.1.6 are proved in Section 4.5.

3.2. The Stabilizer of an Isotropic Point

The purpose of this section is to prove Theorem 3.1.2. Let (V_r, ω) be a classical space, where we use Notation 1.1.20, and let $(x, y) \in V_r \times V_r$ be a hyperbolic pair. We will now determine an explicit Langlands decomposition of $\mathrm{Stab}_{G_r}([x])$.

Definition 3.2.1. We define

$$\begin{aligned} U_r^{x,y} &:= \{ v \in V_r \mid \omega(x, v) = 0 = \omega(y, v) \}, \\ G_r^{x,y} &:= \mathrm{Aut}(U_r^{x,y}, \omega|_{U_r^{x,y} \times U_r^{x,y}}), \\ W_r^{x,y} &:= \{ w \in V_r \mid \omega(x, w) = 0 = q(y + w) \}. \end{aligned}$$

Let $\psi \in G_r^{x,y}$, $\lambda \in \mathbb{K}^\times$, $w \in W_r^{x,y}$. For all $v \in U_r^{x,y}$ we define

$$\begin{aligned} m_\psi(x) &= x, & m_\psi(y) &= y, & m_\psi(v) &= \psi(v), \\ a_\lambda(x) &= \lambda x, & a_\lambda(y) &= \sigma\big(\lambda^{-1}\big)y, & a_\lambda(v) &= v, \\ n_w(x) &= x, & n_w(y) &= y + w, & n_w(v) &= v - \omega(v,w)x. \end{aligned}$$

We also define the subsets

$$M_r^{x,y} := \{m_\psi \mid \psi \in G_r^{x,y}\}, \qquad A_r^{x,y} := \{a_\lambda \mid \lambda \in \mathbb{K}^\times\}, \qquad N_r^{x,y} := \{n_w \mid w \in W_r^{x,y}\}$$

of $\mathrm{Stab}_{G_r}([x])$.

The identities

$$m_{\psi'} m_\psi = m_{\psi'\psi}, \qquad a_{\lambda'} a_\lambda = a_{\lambda'\lambda}, \qquad n_{w'} n_w = n_{w'+w-\omega(w,w')x} \tag{3.2.1}$$

imply that $M_r^{x,y}$, $A_r^{x,y}$, and $N_r^{x,y}$ are subgroups of $\mathrm{Stab}_{G_r}([x])$, and that $M_r^{x,y} \cong G_{r-1}$ and $A_r^{x,y} \cong \mathbb{K}^\times$.

Proposition 3.2.2. *Let $(x,y) \in V_r \times V_r$ be a hyperbolic pair. Then the multiplication maps*

$$\begin{aligned} M_r^{x,y} \times A_r^{x,y} \times N_r^{x,y} &\to \mathrm{Stab}_{G_r}([x]), \\ M_r^{x,y} \times N_r^{x,y} &\to \mathrm{Stab}_{G_r}(x), \\ M_r^{x,y} \times A_r^{x,y} &\to \mathrm{Stab}_{G_r}([x,y]), \\ M_r^{x,y} &\to \mathrm{Stab}_{G_r}(x,y) \end{aligned}$$

are diffeomorphisms, where the latter two maps are even Lie group isomorphisms.

Proof. We prove that for every $\varphi \in \mathrm{Stab}_{G_r}([x])$ there exist unique $\psi \in G_r^{x,y}$, $\lambda \in \mathbb{K}^\times$, $w \in W_r^{x,y}$ such that $\varphi = m_\psi a_\lambda n_w$. We write $w = \mu x + w_0$ for some $w_0 \in U_r^{x,y}$. For all $v \in U_r^{x,y}$ we have

$$\begin{aligned} m_\psi a_\lambda n_w(x) &= \lambda x, \\ m_\psi a_\lambda n_w(y) &= \lambda\mu x + \sigma\big(\lambda^{-1}\big)y + \psi(w_0), \\ m_\psi a_\lambda n_w(v) &= -\lambda\omega(v,w)x + \psi(v). \end{aligned}$$

Using these formulas, it is easy to see that a decomposition $\varphi = m_\psi a_\lambda n_w$ as above is unique. It remains to prove existence.

Let $\varphi \in \mathrm{Stab}_{G_r}([x])$. There exists $\lambda \in \mathbb{K}^\times$ such that $a_\lambda^{-1}\varphi(x) = x$. It is easy to see that $w := a_\lambda^{-1}\varphi(y) - y \in W_r^{x,y}$, so we have $n_w^{-1}a_\lambda^{-1}\varphi(y) = y$. Since also $n_w^{-1}a_\lambda^{-1}\varphi(x) = x$, there exists $\psi \in G_r^{x,y}$ such that $\varphi = a_\lambda n_w m_\psi$. It now suffices to note that $m_{\psi^{-1}}(w) \in W_r^{x,y}$ and $a_\lambda n_w m_\psi = m_\psi a_\lambda n_{m_{\psi^{-1}}(w)}$. This implies that $M_r^{x,y} \times A_r^{x,y} \times N_r^{x,y} \to \mathrm{Stab}_{G_r}([x])$ is a diffeomorphism; the other statements claimed follow similarly. $\square$

Remark 3.2.3. The proof above shows that the decomposition

$$\mathrm{Stab}_{G_r}([x]) = A_r^{x,y} N_r^{x,y} M_r^{x,y}$$

is more natural. We will, however, exclusively use the decomposition $\mathrm{Stab}_{G_r}([x]) = M_r^{x,y} A_r^{x,y} N_r^{x,y}$; it leads to simpler formulas in the following sections.

Lemma 3.2.4. *$N_r^{x,y}$ is a normal subgroup of* $\mathrm{Stab}_{G_r}([x])$.

Proof. Let $w \in W_r^{x,y}$. We have $m_\psi(w) \in W_r^{x,y}$ and $m_\psi n_w m_\psi^{-1} = n_{m_\psi(w)}$ for all $\psi \in G_r^{x,y}$. Similarly, we have $\sigma(\lambda)a_\lambda(w) \in W_r^{x,y}$ and $a_\lambda n_w a_\lambda^{-1} = n_{\sigma(\lambda)a_\lambda(w)}$ for all $\lambda \in \mathbb{K}^\times$. □

Lemma 3.2.5. *$N_r^{x,y}$ is a nilpotent group of nilpotence-degree at most two; it is abelian if and only if ω is symmetric, or ω is alternating and $r = 1$.*

Proof. **Case 1:** ω is symmetric. Then by Equation (3.2.1), $N_r^{x,y}$ is clearly abelian.
Case 2: ω is alternating. Then by Equation (3.2.1), the center of $N_r^{x,y}$ is given by $\mathrm{Z}(N_r^{x,y}) = \{n_w \mid \omega(w, w') = 0 \ \forall w' \in W_r^{x,y}\} \cong \mathbb{K}x \cong \mathbb{K}$, and we have $N_r^{x,y}/\mathrm{Z}(N_r^{x,y}) \cong V_{r-1}$. Hence we have the central extension

$$1 \to \mathbb{K} \to N_r^{x,y} \to V_{r-1} \to 1,$$

so $N_r^{x,y} \cong \mathrm{Z}(N_r^{x,y}) \times_{\omega'} N_r^{x,y}/\mathrm{Z}(N_r^{x,y})$ is a nilpotent group of nilpotence-degree at most 2, where $\omega' \colon (N_r^{x,y}/\mathrm{Z}(N_r^{x,y}))^2 \to \mathrm{Z}(N_r^{x,y})$ is defined by

$$\omega'(n_w \mathrm{Z}(N_r^{x,y}), n_{w'} \mathrm{Z}(N_r^{x,y})) := -\omega(w, w')x.$$

Note that $N_r^{x,y}$ is abelian if and only if $\mathrm{Z}(N_r^{x,y}) = N_r^{x,y}$, which is the case if and only if $r = 1$.
Case 3: $(\mathbb{K}, \sigma) = (\mathbb{C}, \bar{\cdot})$ and ω is Hermitian. Then by Equation (3.2.1), the center of $N_r^{x,y}$ is given by $\mathrm{Z}(N_r^{x,y}) = \{n_w \mid \mathrm{Im}(\omega(w, w')) = 0 \ \forall w' \in W_r^{x,y}\}$, and $N_r^{x,y}/\mathrm{Z}(N_r^{x,y})$ is clearly abelian. Hence $N_r^{x,y} \cong \mathrm{Z}(N_r^{x,y}) \times_{\omega'} N_r^{x,y}/\mathrm{Z}(N_r^{x,y})$ is a nilpotent group of nilpotence-degree at most two, where, again, $\omega' \colon (N_r^{x,y}/\mathrm{Z}(N_r^{x,y}))^2 \to \mathrm{Z}(N_r^{x,y})$ is defined by

$$\omega'(n_w \mathrm{Z}(N_r^{x,y}), n_{w'} \mathrm{Z}(N_r^{x,y})) := -\omega(w, w')x.$$

Note that $\mathrm{Z}(N_r^{x,y})$ is a proper subgroup of $N_r^{x,y}$; hence $N_r^{x,y}$ is not abelian in this case. □

Notation 3.2.6. In the case $(x, y) = (e_r, f_r)$ we drop the upper indices of $U_r^{x,y}, W_r^{x,y}, \ldots$ but, to avoid ambiguity, not of $G_r^{x,y}$. Thus, we have the decompositions

$$\begin{aligned} \mathrm{Stab}_{G_r}([e_r]) &= M_r A_r N_r, \\ \mathrm{Stab}_{G_r}(e_r) &= M_r N_r, \\ \mathrm{Stab}_{G_r}([e_r, f_r]) &= M_r A_r, \\ \mathrm{Stab}_{G_r}(e_r, f_r) &= M_r. \end{aligned}$$

Note that we have *canonical* isomorphisms $U_r \cong V_{r-1}$ and $G_r^{e_r,f_r} \cong G_{r-1}$.

Lemma 3.2.7. *Let $(x, y) \in V_r \times V_r$ be a hyperbolic pair. Then* $\mathrm{Stab}_{G_r}([x])$ *is a maximal parabolic subgroup of the reductive Lie group G_r with Langlands decomposition* $\mathrm{Stab}_{G_r}([x]) = M_r^{x,y} A_r^{x,y} N_r^{x,y}$.

Proof. By Lemma 1.1.18, (i), to prove that $\mathrm{Stab}_{G_r}([x])$ is a maximal parabolic subgroup of G_r, it suffices to consider the case $(x, y) = (e_r, f_r)$. We use the notation of Section 1.3.

In the case that G_r is a complex Lie group, respectively a non-complex Lie group, a standard calculation shows that $\mathrm{Stab}_{G_r}([e_r])$ is the normalizer of

$$\mathfrak{q}_{\max} = \mathfrak{h} \oplus \bigoplus_{\lambda\in\Gamma} \mathfrak{g}_\lambda,$$

respectively of

$$\mathfrak{q}_{\max} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\lambda\in\Gamma} \mathfrak{g}_\lambda,$$

in G_r, where in both cases we have

$$\Gamma = \Delta^+ \cup \{\beta \in \Delta \mid \beta \in \mathrm{span}(\Pi \setminus \{\varepsilon_1 - \varepsilon_2\})\}.$$

Let us make this more explicit in the case $G_r = \mathrm{Sp}(2r, \mathbb{R})$. Then we have $\Gamma = \Delta \setminus \{\varepsilon_2 - \varepsilon_1, \ldots, \varepsilon_r - \varepsilon_1\}$, implying

$$\mathfrak{q}_{\max} = \left\{ \begin{pmatrix} a & B^\top & c \\ 0 & D & E \\ 0 & 0 & f \end{pmatrix} \in \mathfrak{sp}(2r,\mathbb{R}) \;\middle|\; a,c,f \in \mathbb{R},\ B,E \in \mathbb{R}^{(2r-2)\times 1},\ D \in \mathfrak{gl}_{2r-2}(\mathbb{R}) \right\}.$$

Representing $\mathrm{Stab}_{G_r}([e_r]) = M_r A_r N_r$ with respect to the standard Witt basis, we see that $m_\psi a_\lambda n_w$ corresponds to the matrix

$$\begin{pmatrix} \lambda & \omega(w,-) & \omega(w,f_r) \\ 0 & \psi & \omega(w,-)' \\ 0 & 0 & \lambda^{-1} \end{pmatrix},$$

where we interpret ψ as a matrix in $\mathrm{Sp}(2r-2,\mathbb{R})$, and where

$$\begin{aligned} \omega(w,-) &= (\omega(w,e_{r-1}), \ldots, \omega(w,e_1), \omega(w,f_1), \ldots, \omega(w,f_{r-1})) \in \mathbb{R}^{1\times(2r-2)}, \\ \omega(w,-)' &= (\omega(w,f_{r-1}), \ldots, \omega(w,f_1), \omega(e_1,w), \ldots, \omega(e_{r-1},w))^\top \in \mathbb{R}^{(2r-2)\times 1}. \end{aligned}$$

Having established these matrix representations, it is not hard to see that $\mathrm{Stab}_{G_r}([e_r])$ is the normalizer of $\mathfrak{q}_{\max}$ in G_r.

Let us return to the general case. Using [87, Proposition 7.83 (b)], $\mathrm{Stab}_{G_r}([e_r])$ is the parabolic subgroup of G_r corresponding to the maximal subset $\Pi \setminus \{\varepsilon_1 - \varepsilon_2\}$ of Π, i.e. a maximal parabolic subgroup of G_r. The results of Section 3.2 imply that a Langlands decomposition of $\mathrm{Stab}_{G_r}([e_r])$ is given by $\mathrm{Stab}_{G_r}([x]) = M_r^{x,y} A_r^{x,y} N_r^{x,y}$. □

Proof of Theorem 3.1.2. (i) follows from Lemma 3.2.7; (ii) then follows from the decomposition $\mathrm{Stab}_{G_r}([x,y]) = M_r^{x,y} A_r^{x,y}$ in Proposition 3.2.2; (iii) now follows immediately; (iv) is proved in Lemma 3.2.5. □

Remark 3.2.8. If $x \in V_r$ is anisotropic with respect to ω, then

$$\begin{aligned} \mathrm{Stab}_{G_r}([x]) &\cong \mathrm{Aut}\big(\{x\}^\perp, \omega|_{\{x\}^\perp\times\{x\}^\perp}\big) \times \mathbb{K}^\times, \\ \mathrm{Stab}_{G_r}(x) &\cong \mathrm{Aut}\big(\{x\}^\perp, \omega|_{\{x\}^\perp\times\{x\}^\perp}\big). \end{aligned}$$

In this case, $\mathrm{Stab}_{G_r}(x)$ is isomorphic to a classical group. Consequently, Witt's lemma holds for $\mathrm{Stab}_{G_r}(x)$ acting on $\{x\}^\perp$.

3.3. Identifying Lebesgue Spaces

The purpose of this section is to prove Theorem 3.1.5, (i) and to explain the strategy that we will employ to prove the other parts of this theorem. Let (V_r, ω) be a classical space, where we use Notation 1.1.20.

Proof of Theorem 3.1.5, (i). We have pointed out in Remark 1.1.3 that $G_r \curvearrowright V_r$ is an $\mathbb{R}$-algebraic action; consequently also $G_r \curvearrowright \mathcal{P}_r^k$ is $\mathbb{R}$-algebraic. By Theorem 2.2.2, equipping $\mathcal{P}_r^k$ with its Lebesgue measure class $[\mu]$ turns $(G_r \backslash \mathcal{P}_r^k, [\widetilde{\mu}])$ into a Lebesgue space. □

Let us now discuss how to obtain an isomorphism of Lebesgue spaces. We will use the following direct criterion to prove parts (ii) and (iv) of Theorem 3.1.5.

Lemma 3.3.1. *Let G be a Lie group, let $\mathcal{P}$ and $\mathcal{C}$ be smooth manifolds, let $G \curvearrowright \mathcal{P}$ be a smooth action by diffeomorphisms with canonical projection $\pi\colon \mathcal{P} \to G\backslash\mathcal{P}$, and let $f\colon \mathcal{P} \to \mathcal{C}$ be a G-invariant smooth map. If f admits a smooth section $h\colon \mathcal{C} \to \mathcal{P}$ such that $\pi \circ h\colon \mathcal{C} \to G\backslash\mathcal{P}$ is surjective, then $\pi \circ h$ is an isomorphism of Lebesgue spaces with inverse the map $G\backslash\mathcal{P} \to \mathcal{C}$ induced by f.*

Proof. Let $\widetilde{f}\colon G\backslash\mathcal{P} \to \mathcal{C}$ denote the map induced by f. Then we have

$$\widetilde{f}(\pi(h(c))) = f(h(c)) = c$$

for all $c \in C$, so $\widetilde{f} \circ (\pi \circ h) = \mathrm{id}_{\mathcal{C}}$. Since $\pi \circ h$ is surjective, for all $p \in \mathcal{P}$ there exists $c \in \mathcal{C}$ such that $Gp = Gh(c)$, which implies

$$\pi\Big(h\Big(\widetilde{f}(Gp)\Big)\Big) = \pi\Big(h\Big(\widetilde{f}(Gh(c))\Big)\Big) = \pi(h(f(h(c)))) = \pi(h(c)) = Gh(c) = Gp,$$

i.e. $(\pi \circ h) \circ \widetilde{f} = \mathrm{id}_{G\backslash\mathcal{P}}$. Hence $\pi \circ h$ is an isomorphism of Borel spaces with inverse $\widetilde{f}$.

It remains to show that these isomorphisms preserve null sets. Let $N \subset \mathcal{C}$ be a null set. One can show that f is a smooth submersion. Recall that the preimage of a null set under a smooth submersion is a null set. Hence $\pi^{-1}\Big(\widetilde{f}^{-1}(N)\Big) = f^{-1}(N) \subset \mathcal{P}$ is a null set, which, by the definition of the quotient measure class on $G\backslash\mathcal{P}$, implies that $\widetilde{f}^{-1}(N) \subset G\backslash\mathcal{P}$ is a null set.

Conversely, let $N \subset G\backslash\mathcal{P}$ be a null set. Then $\pi^{-1}(N) \subset \mathcal{P}$ is a null set. By smoothness of f and [92, Theorem 6.9] also $f(\pi^{-1}(N)) \subset \mathcal{C}$ is a null set and one can check that $f(\pi^{-1}(N)) = \widetilde{f}(N)$.

Thus, $\pi \circ h$ and $\widetilde{f}$ are even isomorphisms of Lebesgue spaces. □

In Chapter 4 we will use the following criterion to prove Theorem 3.1.5, (iii), which, compared to the criterion above, sidesteps the required a priori knowledge of a G-invariant smooth map f in the product case. Although its formulation is rather lengthy, the reader should convince himself or herself that this criterion is very intuitive.

Lemma 3.3.2. *Let G be a Lie group, let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{C}_1, \mathcal{C}_2$ be smooth manifolds, let $G \curvearrowright \mathcal{P}_1$ and $G \curvearrowright \mathcal{P}_2$ be actions by diffeomorphisms such that $G \curvearrowright \mathcal{P}_1$ and the diagonal action $G \curvearrowright \mathcal{P}_1 \times \mathcal{P}_2$ are smooth, and let $\pi_1 \colon \mathcal{P}_1 \to G\backslash\mathcal{P}_1$ and $\pi \colon \mathcal{P}_1 \times \mathcal{P}_2 \to G\backslash(\mathcal{P}_1 \times \mathcal{P}_2)$ denote the canonical projections. Let $h_1 \colon \mathcal{C}_1 \to \mathcal{P}_1$ be a smooth map such that $\pi_1 \circ h_1 \colon \mathcal{C}_1 \to G\backslash\mathcal{P}_1$ is an isomorphism of Lebesgue spaces, let $h_2 \colon \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{P}_2$ be a smooth map, and define*

$$h \colon \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{P}_1 \times \mathcal{P}_2, \ (c_1, c_2) \mapsto (h_1(c_1), h_2(c_1, c_2)).$$

Assume that

(i) for all $c_1 \in \mathcal{C}_1$, $p_2 \in \mathcal{P}_2$ there exists some $g \in G$ and a unique $c_2 \in \mathcal{C}_2$ such that $h_1(c_1) = gp_1$ for some $p_1 \in \mathcal{P}_1$ and $h_2(c_1, c_2) \in \mathrm{Stab}_G(h_1(c_1))gp_2$;

(ii) the map $f \colon \mathcal{P}_1 \times \mathcal{P}_2 \to \mathcal{C}_1 \times \mathcal{C}_2$ defined by assigning to every (p_1, p_2) the unique element (c_1, c_2) such that $(h_1(c_1), h_2(c_1, c_2)) \in G(p_1, p_2)$ is smooth.

Then f is G-invariant with section h, and $\pi \circ h \colon \mathcal{C}_1 \times \mathcal{C}_2 \to G\backslash(\mathcal{P}_1 \times \mathcal{P}_2)$ is an isomorphism of Lebesgue spaces with inverse the map $G\backslash(\mathcal{P}_1 \times \mathcal{P}_2) \to \mathcal{C}_1 \times \mathcal{C}_2$ induced by f.

Proof. Condition (i) implies that f is well-defined and G-invariant, and that h is a smooth section of f such that $\pi \circ h$ is surjective. Condition (ii) and Lemma 3.3.1 now yield the claim. □

3.4. The Generalized Angular Cartan Invariant

In this section we introduce the (generalized) angular Cartan invariant and use it to prove Theorem 3.1.5, (ii).

The Classical Angular Cartan Invariant: Let (V_1, ω) be a classical space of rank $r = 1$ and dimension ≥ 3, where we use Notation 1.1.20. One can prove that, if $G_1 \neq \mathrm{U}(1+d, 1)$, then the configuration space $G_1\backslash\mathcal{P}_1^{\{3\}}$ is trivial. In the case $G_1 = \mathrm{U}(1+d, 1)$ let us write

$$\mathcal{C}^{(3)} \coloneqq \{z \in \mathbb{C} : |z| = 1, \ \mathrm{Re}(z) \leq 0\}$$

and for all $z \in \mathcal{C}^{(3)}$ let us write

$$x_z \coloneqq \left[e_1, f_1, e_1 + zf_1 + \sqrt{-2\mathrm{Re}(z)}h_1\right] \in \mathcal{P}_1^3.$$

Note that $x_z \in \mathcal{P}_1^{(3)}$ for all $z \in \mathcal{C}^{(3)}$, and that $x_z \in \mathcal{P}_1^{\{3\}}$ if and only if $\mathrm{Re}(z) < 0$.

For any $[v_0, v_1, v_2] \in \mathcal{P}_1^{(3)}$ let us denote the argument of $\frac{\omega(v_2, v_0)}{\omega(v_2, v_1)\omega(v_1, v_0)}$, as an element of $S^1 \subset \mathbb{C}$, by $z[v_0, v_1, v_2]$. A quick calculation yields $z[v_0, v_1, v_2] \in \mathcal{C}^{(3)}$; we call $z[v_0, v_1, v_2]$ the *angular Cartan invariant* of $[v_0, v_1, v_2]$.

Proposition 3.4.1 (Cartan [31])**.** *Let (V_1, ω) be a classical space of rank $r = 1$, where $G_1 = \mathrm{U}(1+d, 1)$ for some $d \geq 1$. Then the map*

$$\mathcal{P}_1^{(3)} \to \mathcal{C}^{(3)}, \ [v_0, v_1, v_2] \mapsto z[v_0, v_1, v_2]$$

is smooth and G_1-invariant, and admits the smooth section

$$\mathcal{C}^{(3)} \to \mathcal{P}_1^{(3)}, \ z \mapsto x_z,$$

inducing an isomorphism $\mathcal{C}^{(3)} \cong \mathrm{U}(1+d,1)\backslash\mathcal{P}_1^{(3)}$ of Lebesgue spaces.

For more discussion of this invariant see [61, Section 7.1]. This invariant has also been considered in the quaternion and octonion rank 1 cases, see [1]. For connections to the Maslov triple index see [36].

The Angular Cartan Invariant in Higher Rank: Let $r \geq 2$ and let (V_r, ω) be a classical space, where we use Notation 1.1.20. Our goal is to understand the Lebesgue space $G_r\backslash\mathcal{P}_r^3$. To this end, we define the smooth manifold

$$\begin{aligned} \mathcal{C}^{(3)} &:= \{1\} && \text{if } G_r \text{ is a complex Lie group,} \\ \mathcal{C}^{(3)} &:= \{z \in \mathbb{K} : |z| = 1\} && \text{if } G_r \text{ is a non-complex Lie group,} \end{aligned}$$

and equip $\mathcal{C}^{(3)}$ with its Lebesgue measure class. In this section we prove that there exists an isomorphism $G_r\backslash\mathcal{P}_r^3 \cong \mathcal{C}^{(3)}$ of Lebesgue spaces. Let us now write down the maps which induce this isomorphism. Recall that the spaces $\mathcal{P}_r^{\{k\}} \subset \mathcal{P}_r^{(k)}$ consist of tuples of points having pairwise non-orthogonal representatives.

If G_r is a non-complex Lie group and $[v_0, v_1, v_2] \in \mathcal{P}_r^{\{3\}}$, then we use the polar decomposition and write

$$\frac{\omega(v_2, v_0)}{\omega(v_2, v_1)\omega(v_1, v_0)} = r\mathrm{e}^{\mathrm{i}\theta}.$$

There exists a unique $\lambda = \lambda(v_0, v_1, v_2) \in \mathbb{R}_{>0}$ such that $\lambda^2 = r$; writing $z[v_0, v_1, v_2] := \mathrm{e}^{\mathrm{i}\theta}$, we have

$$\lambda^2 z[v_0, v_1, v_2] = \frac{\omega(v_2, v_0)}{\omega(v_2, v_1)\omega(v_1, v_0)}.$$

If G_r is a complex Lie group, then we just write $z[v_0, v_1, v_2] := 1$ and choose a measurable map $(v_0, v_1, v_2) \mapsto \lambda(v_0, v_1, v_2) \in \mathbb{C}^\times$ such that

$$\lambda(v_0, v_1, v_2)^2 = \frac{\omega(v_2, v_0)}{\omega(v_2, v_1)\omega(v_1, v_0)}$$

for all $[v_0, v_1, v_2] \in \mathcal{P}_r^{\{3\}}$; our result will turn out to be independent of this choice of measurable map.

In both cases we call $z[v_0, v_1, v_2]$ the *generalized angular Cartan invariant* of $[v_0, v_1, v_2]$. We will now show that this invariant classifies the G_r-orbits of $\mathcal{P}_r^{\{3\}}$. For all $z \in \mathcal{C}^{(3)}$ let us write

$$x_z := [e_r, f_r, e_r + zf_r + e_{r-1} - zf_{r-1}] \in \mathcal{P}_r^{\{3\}}.$$

Proposition 3.4.2. *Let $r \geq 2$ and let (V_r, ω) be a classical space. Then the map*

$$\mathcal{P}_r^{\{3\}} \to \mathcal{C}^{(3)}, \ [v_0, v_1, v_2] \mapsto z[v_0, v_1, v_2]$$

is smooth and G_r-invariant, and admits the smooth section

$$\mathcal{C}^{(3)} \to \mathcal{P}_r^{\{3\}}, \; z \mapsto x_z = [e_r, f_r, e_r + zf_r + e_{r-1} - zf_{r-1}],$$

inducing an isomorphism $\theta_r^3 \colon \mathcal{C}^{(3)} \to G_r \backslash \mathcal{P}_r^{\{3\}}$ of Lebesgue spaces.

Theorem 3.1.5, (ii) is clearly an immediate corollary of the proposition above. Our proof of this proposition consists of three elementary but rather technical lemmas.

Notation 3.4.3. Let us now fix $[v_0, \dots, v_n] \in \mathcal{P}_r^{\{n+1\}}$, and write $z_0 = z[v_0, v_1, v_2]$ and $\lambda = \lambda(v_0, v_1, v_2)$. Let $\varphi_0 \in G_r$ denote the morphism defined by $v_0 \mapsto \omega(v_0, v_1)e_r$, $v_1 \mapsto f_r$.

Lemma 3.4.4. *We have*

$$\varphi_0(v_k) = \omega(v_k, v_1)e_r + \frac{\omega(v_k, v_0)}{\omega(v_1, v_0)} f_r + v_k'$$

for some $v_k' \in V_{r-1}$, $k = 0, \dots, n$, and

$$\omega(v_k', v_l') = \omega(v_k, v_l) - \frac{\omega(v_k, v_1)\omega(v_0, v_l)}{\omega(v_0, v_1)} - \frac{\omega(v_k, v_0)\omega(v_1, v_l)}{\omega(v_1, v_0)}, \qquad k, l = 0, \dots, n.$$

Proof. Let us write $\varphi_0(v_k) = \alpha_k e_r + \sigma(\beta_k) f_r + v_k'$ for some $v_k' \in V_{r-1}$. Then

$$\begin{aligned}\omega(v_0, v_k) &= \omega(\varphi_0(v_0), \varphi_0(v_k)) \\ &= \omega(\omega(v_0, v_1)e_r, \alpha_k e_r + \sigma(\beta_k) f_r + v_k') \\ &= \omega(v_0, v_1)\beta_k\end{aligned}$$

implies

$$\sigma(\beta_k) = \frac{\sigma(\omega(v_0, v_k))}{\sigma(\omega(v_0, v_1))} = \frac{\varepsilon\omega(v_k, v_0)}{\varepsilon\omega(v_1, v_0)} = \frac{\omega(v_k, v_0)}{\omega(v_1, v_0)}.$$

Furthermore, we have

$$\begin{aligned}\omega(v_k, v_1) &= \omega(\varphi_0(v_k), \varphi_0(v_1)) \\ &= \omega(\alpha_k e_r + \sigma(\beta_k) f_r + v_k', f_r) \\ &= \alpha_k,\end{aligned}$$

which implies $\varphi_0(v_k) = \omega(v_k, v_1)e_r + \frac{\omega(v_k, v_0)}{\omega(v_1, v_0)} f_r + v_k'$, as claimed. Now

$$\begin{aligned}\omega(v_k, v_l) &= \omega(\varphi_0(v_k), \varphi_0(v_l)) \\ &= \omega\left(\omega(v_k, v_1)e_r + \frac{\omega(v_k, v_0)}{\omega(v_1, v_0)} f_r + v_k', \omega(v_l, v_1)e_r + \frac{\omega(v_l, v_0)}{\omega(v_1, v_0)} f_r + v_l'\right) \\ &= \omega(v_k, v_1)\frac{\sigma(\omega(v_l, v_0))}{\sigma(\omega(v_1, v_0))} + \varepsilon\frac{\omega(v_k, v_0)}{\omega(v_1, v_0)}\sigma(\omega(v_l, v_1)) + \omega(v_k', v_l') \\ &= \frac{\omega(v_k, v_1)\omega(v_0, v_l)}{\omega(v_0, v_1)} + \frac{\omega(v_k, v_0)\omega(v_1, v_l)}{\omega(v_1, v_0)} + \omega(v_k', v_l')\end{aligned}$$

implies $\omega(v_k', v_l') = \omega(v_k, v_l) - \frac{\omega(v_k, v_1)\omega(v_0, v_l)}{\omega(v_0, v_1)} - \frac{\omega(v_k, v_0)\omega(v_1, v_l)}{\omega(v_1, v_0)}$. □

Lemma 3.4.5. *We have*

$$a_\lambda\varphi_0(v_k) \in \left[e_r + z_0\frac{\omega(v_k,v_0)\omega(v_2,v_1)}{\omega(v_k,v_1)\omega(v_2,v_0)}f_r + \frac{\lambda^{-1}}{\omega(v_k,v_1)}v'_k\right].$$

Proof. By Lemma 3.4.4 we have

$$\begin{aligned} a_\lambda\varphi_0(v_k) &= \lambda\omega(v_k,v_1)e_r + \lambda^{-1}\frac{\omega(v_k,v_0)}{\omega(v_1,v_0)}f_r + v'_k \\ &\in \left[e_r + \frac{\lambda^{-2}\omega(v_k,v_0)}{\omega(v_k,v_1)\omega(v_1,v_0)}f_r + \frac{\lambda^{-1}}{\omega(v_k,v_1)}v'_k\right] \\ &= \left[e_r + z_0\frac{\omega(v_k,v_0)\omega(v_2,v_1)}{\omega(v_k,v_1)\omega(v_2,v_0)}f_r + \frac{\lambda^{-1}}{\omega(v_k,v_1)}v'_k\right]. \end{aligned}$$

□

Lemma 3.4.6. *There exists $\psi_0 \in G_{r-1}$ such that*

$$m_{\psi_0}a_\lambda\varphi_0(v_2) \in [e_r + z_0f_r + e_{r-1} - z_0f_{r-1}].$$

In particular, $x_{z_0} \in G_r.[v_0,v_1,v_2]$.

Proof. We have

$$\begin{aligned} &\omega\left(\frac{\lambda^{-1}}{\omega(v_k,v_1)}v'_k, \frac{\lambda^{-1}}{\omega(v_l,v_1)}v'_l\right) \\ &= \frac{\lambda^{-2}}{\omega(v_k,v_1)\sigma(\omega(v_l,v_1))}\omega(v'_k,v'_l) \\ &= \frac{\varepsilon z_0\omega(v_2,v_1)\omega(v_1,v_0)}{\omega(v_2,v_0)\omega(v_k,v_1)\omega(v_1,v_l)}\omega(v'_k,v'_l) \\ &= \frac{\varepsilon z_0\omega(v_2,v_1)\omega(v_1,v_0)}{\omega(v_2,v_0)\omega(v_k,v_1)\omega(v_1,v_l)}\left(\omega(v_k,v_l) - \frac{\omega(v_k,v_1)\omega(v_0,v_l)}{\omega(v_0,v_1)} - \frac{\omega(v_k,v_0)\omega(v_1,v_l)}{\omega(v_1,v_0)}\right) \\ &= \varepsilon z_0\left(\frac{\omega(v_2,v_1)\omega(v_1,v_0)\omega(v_k,v_l)}{\omega(v_2,v_0)\omega(v_k,v_1)\omega(v_1,v_l)} - \frac{\omega(v_2,v_1)\omega(v_1,v_0)\omega(v_0,v_l)}{\omega(v_2,v_0)\omega(v_0,v_1)\omega(v_1,v_l)} - \frac{\omega(v_k,v_0)\omega(v_2,v_1)}{\omega(v_k,v_1)\omega(v_2,v_0)}\right). \end{aligned}$$

In particular, we have

$$\begin{aligned} q\left(\frac{\lambda^{-1}}{\omega(v_2,v_1)}v'_2\right) &= -\varepsilon z_0\left(\frac{\omega(v_2,v_1)\omega(v_1,v_0)}{\omega(v_2,v_0)}\frac{\omega(v_0,v_2)}{\omega(v_0,v_1)\omega(v_1,v_2)} - 1\right) \\ &= -\varepsilon z_0\left(\left(\lambda^2 z_0\right)^{-1}\left(\varepsilon\sigma\left(\lambda^2 z_0\right)\right) + 1\right) \\ &= -\varepsilon z_0\left(\varepsilon\sigma(z_0)^2 + 1\right) \\ &= -(\sigma(z_0) + \varepsilon z_0) \\ &= q(e_{r-1} - z_0f_{r-1}). \end{aligned}$$

By Witt's lemma and Lemma 3.4.5, there exists $\psi_0 \in G_{r-1}$ such that

$$\begin{aligned} m_{\psi_0}a_\lambda\varphi_0(v_2) &\in \left[e_r + z_0f_r + \psi_0\left(\frac{\lambda^{-1}}{\omega(v_2,v_1)}v'_2\right)\right] \\ &= [e_r + z_0f_r + e_{r-1} - z_0f_{r-1}]. \end{aligned}$$

We conclude that $m_{\psi_0}a_\lambda\varphi_0[v_0,v_1,v_2] = x_{z_0}$. □

Proof of Proposition 3.4.2. Lemma 3.4.6 implies that the composition

$$G_r\backslash\mathcal{P}_r^{\{3\}} \to \mathcal{C}^{(3)} \to G_r\backslash\mathcal{P}_r^{\{3\}}$$

is the identity, which means that $\mathcal{C}^{(3)} \to G_r\backslash\mathcal{P}_r^{\{3\}}$ is surjective. The calculation

$$\frac{\omega(e_r + zf_r + e_{r-1} - zf_{r-1}, e_r)}{\omega(e_r + zf_r + e_{r-1} - zf_{r-1}, f_r)\omega(f_r, e_r)} = \frac{\varepsilon z}{1 \cdot \varepsilon} = z$$

shows that $\mathcal{C}^{(3)} \to \mathcal{P}_r^{\{3\}}$ is a smooth section. Lemma 3.3.1 now yields the claim. □

3.5. Cross-Ratios and Parametrization of Quadruples in the Symplectic Case

In this section we prove Theorem 3.1.5, (iii) in the easiest case, namely for quadruples in the symplectic case, where we again use Lemma 3.3.1 to obtain the desired isomorphism; all other cases of this part of the theorem, which are treated in Chapter 4, will be proved using Lemma 3.3.2.

Let $r \geq 2$ and let (V_r, ω) be a classical space, where we use Notation 1.1.20. Our next goal is to understand the space $G_r\backslash\mathcal{P}_r^4$. De la Cruz Mengual proved that if G_r is a complex Lie group, then there exists an isomorphism $G_r\backslash\mathcal{P}_r^4 \cong \mathbb{C}^2$ of Lebesgue spaces, which is induced by certain *cross-ratios* $\mathrm{cr}_1, \mathrm{cr}_2\colon \mathcal{P}_r^{(4)} \to \mathbb{C}^\times$ (see [42, Proposition 3.8]).

Remark 3.5.1. These cross-ratios have previously been considered both in the rank 1 case ([88], see also [61, Section 7.2]), as well as in the higher rank case ([38]); for the group $\mathrm{Sp}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})$ the classical cross-ratio is recovered. One can even define a more abstract notion of cross-ratio for quadruples in G/P, where G is any semisimple Lie group and P is any parabolic subgroup of G, see [61, Section 7.2.3]. For other generalizations of the classical cross-ratio see [73] and the references therein.

In our unified setting including non-complex Lie groups, the cross-ratios are the G_r-invariant smooth maps $\mathrm{cr}_1, \mathrm{cr}_2\colon \mathcal{P}_r^{(4)} \to \mathbb{K}^\times$ defined by setting

$$\begin{aligned}
\mathrm{cr}_1[v_0, \dots, v_3] &:= \frac{\omega(v_3, v_0)\omega(v_2, v_1)}{\omega(v_3, v_1)\omega(v_2, v_0)}, \\
\mathrm{cr}_2[v_0, \dots, v_3] &:= \frac{\omega(v_3, v_2)\omega(v_2, v_1)\omega(v_1, v_0)}{\omega(v_3, v_1)\omega(v_1, v_2)\omega(v_2, v_0)}.
\end{aligned}$$

We will prove the existence of an isomorphism

$$G_r\backslash\mathcal{P}_r^{\{4\}} \cong \mathcal{C}^{(4)}, \quad \text{where} \quad \mathcal{C}^{(4)} := \mathcal{C}^{(3)} \times \mathbb{K}^2,$$

and where $\mathcal{C}^{(4)}$ is equipped with the Lebesgue measure class corresponding to its smooth manifold structure.

In the symplectic case we will prove that this isomorphism is induced by the G_r-invariant smooth map

$$\mathcal{P}_r^{\{4\}} \to \mathcal{C}^{(4)},\ [v_0, \dots, v_3] \mapsto (z[v_0, v_1, v_2], \mathrm{cr}_1[v_0, \dots, v_3], \mathrm{cr}_2[v_0, \dots, v_3]).$$

The following lemma, which is proved by straightforward calculations, explains why the angular Cartan invariants and cross-ratios of permuted triples or quadruples of points are not needed for this parametrization.

Lemma 3.5.2. *Let (V_r, ω) be a classical space. Then we have*

$$\begin{aligned}\mathrm{cr}_1[v_0, v_1, v_2, v_3] &= \mathrm{cr}_1[v_1, v_0, v_3, v_2],\\ &= \mathrm{cr}_1[v_0, v_1, v_3, v_2]^{-1},\\ &= \sigma(\mathrm{cr}_1[v_3, v_2, v_1, v_0]),\\ &= \mathrm{cr}_1[v_0, v_2, v_1, v_3]\cdot \mathrm{cr}_2[v_0, v_1, v_2, v_3],\\ &= z[v_3, v_0, v_2]\cdot \mathrm{cr}_2[v_3, v_0, v_1, v_2].\end{aligned}$$

The first step of De la Cruz Mengual's proof can be given uniformly for all classical groups. Let us now use Notation 3.4.3. By Lemma 3.4.6, there exists $\psi_0 \in G_{r-1}$ such that $m_{\psi_0} a_\lambda \varphi_0[v_0, v_1, v_2] = x_{z_0}$.

Lemma 3.5.3. *We have*

$$\psi_0(v_k') = \gamma_k(e_{r-1} - \varepsilon\sigma(z_0) f_{r-1}) + \lambda^{-1}\frac{\omega(v_k', v_2')}{\omega(v_1, v_2)} f_{r-1} + v_k'' \tag{3.5.1}$$

for some $\gamma_k \in \mathbb{K}$ and some $v_k'' \in V_{r-2}$ such that

$$\omega(v_k'', v_l'') = \omega(v_k', v_l') - \lambda^{-1}\left(\gamma_k \frac{\omega(v_2', v_l')}{\omega(v_2, v_1)} + \varepsilon\sigma(\gamma_l)\frac{\omega(v_k', v_2')}{\omega(v_1, v_2)}\right) - \gamma_k\sigma(\gamma_l)(\sigma(z_0) + \varepsilon z_0) \tag{3.5.2}$$

for all $k, l = 3, \dots, n$.

Proof. Let us write $\psi_0(v_k') = \gamma_k e_r + \sigma(\beta_k) f_r + v_k''$ for some $v_k'' \in V_{r-2}$. Then

$$\begin{aligned}\omega(v_2', v_k') &= \omega(\psi_0(v_2'), \psi_0(v_k'))\\ &= \omega(\lambda\omega(v_2, v_1)(e_{r-1} - z_0 f_{r-1}), \gamma_k e_{r-1} + \sigma(\delta_k) f_{r-1} + v_k'')\\ &= \lambda\omega(v_2, v_1)(\delta_k - \varepsilon z_0 \sigma(\gamma_k))\end{aligned}$$

implies $\delta_k = \lambda^{-1}\frac{\omega(v_2', v_k')}{\omega(v_2, v_1)} + \varepsilon z_0 \sigma(\gamma_k)$, i.e. $\sigma(\delta_k) = \lambda^{-1}\frac{\omega(v_k', v_2')}{\omega(v_1, v_2)} + \varepsilon\sigma(z_0)\gamma_k$, which implies $\psi_0(v_k') = \gamma_k(e_{r-1} + \varepsilon\sigma(z_0) f_{r-1}) + \lambda^{-1}\frac{\omega(v_k', v_2')}{\omega(v_1, v_2)} f_{r-1} + v_k''$, as claimed. Now

$$\begin{aligned}&\omega(v_k', v_l')\\ &= \omega(\psi_0(v_k'), \psi_0(v_l'))\\ &= \omega\Bigg(\gamma_k e_{r-1} + \left(\lambda^{-1}\frac{\omega(v_k', v_2')}{\omega(v_1, v_2)} + \varepsilon\sigma(z_0)\gamma_k\right) f_{r-1} + v_k'',\\ &\qquad \gamma_l e_{r-1} + \left(\lambda^{-1}\frac{\omega(v_l', v_2')}{\omega(v_1, v_2)} + \varepsilon\sigma(z_0)\gamma_l\right) f_{r-1} + v_l''\Bigg)\\ &= \gamma_k\sigma\left(\lambda^{-1}\frac{\omega(v_l', v_2')}{\omega(v_1, v_2)} + \varepsilon\sigma(z_0)\gamma_l\right) + \varepsilon\left(\lambda^{-1}\frac{\omega(v_k', v_2')}{\omega(v_1, v_2)} + \varepsilon\sigma(z_0)\gamma_k\right)\sigma(\gamma_l) + \omega(v_k'', v_l'')\\ &= \lambda^{-1}\left(\gamma_k\frac{\omega(v_2', v_l')}{\omega(v_2, v_1)} + \varepsilon\sigma(\gamma_l)\frac{\omega(v_k', v_2')}{\omega(v_1, v_2)}\right) + \gamma_k\sigma(\gamma_l)(\sigma(z_0) + \varepsilon z_0) + \omega(v_k'', v_l'')\end{aligned}$$

implies $\omega(v_k'', v_l'') = \omega(v_k', v_l') - \lambda^{-1}\left(\gamma_k\frac{\omega(v_2', v_l')}{\omega(v_2, v_1)} + \varepsilon\sigma(\gamma_l)\frac{\omega(v_k', v_2')}{\omega(v_1, v_2)}\right) - \gamma_k\sigma(\gamma_l)(\sigma(z_0) + \varepsilon z_0)$. □

Trying to extend De la Cruz Mengual's proof strategy uniformly to all classical groups becomes extremely challenging and notationally cumbersome at this stage. Thus, we will only pursue this strategy in the symplectic case, i.e. for the families $\mathrm{Sp}(2r,\mathbb{C})$ and $\mathrm{Sp}(2r,\mathbb{R})$; the remaining families $\mathrm{O}(n,\mathbb{C})$, $\mathrm{U}(r+d,r)$, $\mathrm{O}(r+d,r)$ will be handled, using explicit descriptions of stabilizers, in Chapter 4.

Parametrization of Quadruples in the Symplectic Case: From now on we assume that ω is alternating, i.e. that we have $G_r = \mathrm{Sp}(2r,\mathbb{K})$. For all $(z,\lambda_1,\lambda_2) \in \mathcal{C}^{(4)}$ we write

$$x_{z,\lambda_1,\lambda_2} := [x_z, e_r + z\lambda_1 f_r + z(\lambda_2 - \lambda_1 + 1) f_{r-1}] \in \mathcal{P}_r^4.$$

Clearly, $x_{z,\lambda_1,\lambda_2} \in \mathcal{P}_r^{\{4\}}$ if and only if $\lambda_1, \lambda_2 \neq 0$ and $\lambda_1 \neq \lambda_2 + 1$. Thus, we define the open conull submanifold

$$\mathcal{C}^{\{4\}} := \{(z,\lambda_1,\lambda_2) \in \mathcal{C}^{(4)} \mid \lambda_1, \lambda_2 \neq 0,\ \lambda_1 \neq \lambda_2 + 1\}$$

of $\mathcal{C}^{(4)}$.

Proposition 3.5.4. *Let $r \geq 2$ and let (V_r,ω) be a classical space, where ω is alternating. Then the map*

$$\mathcal{P}_r^{\{4\}} \to \mathcal{C}^{\{4\}},\ [v_0,\ldots,v_3] \mapsto (z[v_0,v_1,v_2], \mathrm{cr}_1[v_0,\ldots,v_3], \mathrm{cr}_2[v_0,\ldots,v_3])$$

is G_r-invariant and smooth, and admits the smooth section

$$\mathcal{C}^{\{4\}} \to \mathcal{P}_r^{\{4\}},\ (z,\lambda_1,\lambda_2) \mapsto x_{z,\lambda_1,\lambda_2},$$

inducing an isomorphism $\theta_r^4 \colon \mathcal{C}^{\{4\}} \to G_r \backslash \mathcal{P}_r^{\{4\}}$ of Lebesgue spaces.

First, note that in our alternating setting Equation (3.5.1) and Equation (3.5.2) simplify to

$$\psi_0(v_k') = \gamma_k(e_{r-1} + z_0 f_{r-1}) + \lambda^{-1}\frac{\omega(v_k',v_2')}{\omega(v_1,v_2)} f_{r-1} + v_k''$$

and

$$\omega(v_k'',v_l'') = \omega(v_k',v_l') - \lambda^{-1}\left(\gamma_k \frac{\omega(v_2',v_l')}{\omega(v_2,v_1)} - \gamma_l \frac{\omega(v_k',v_2')}{\omega(v_1,v_2)}\right),$$

respectively. Hence we can choose ψ_0 such that $\gamma_3 = 0$ and $v_3'' = 0$, i.e. such that

$$\psi_0(v_3') = \lambda^{-1}\frac{\omega(v_3',v_2')}{\omega(v_1,v_2)} f_{r-1}.$$

Lemma 3.5.5. *We have*

$$m_{\psi_0} a_\lambda \varphi_0(v_3) \in [e_r + z_0 \mathrm{cr}_1[v_0,\ldots,v_3] f_r + z_0(\mathrm{cr}_2 - \mathrm{cr}_1 + 1)[v_0,\ldots,v_3] f_{r-1}].$$

In particular, $x_{z_0,\mathrm{cr}_1,\mathrm{cr}_2} \in G_r.[v_0,\ldots,v_3]$.

Proof. We have $\frac{\lambda^{-1}}{\omega(v_3,v_1)}\psi_0(v_3') = \frac{\lambda^{-2}\omega(v_3',v_2')}{\omega(v_3,v_1)\omega(v_1,v_2)} f_{r-1}$ and

$$\begin{aligned}
&\frac{\lambda^{-2}\omega(v_3',v_2')}{\omega(v_3,v_1)\omega(v_1,v_2)}\\
&= \frac{z_0\omega(v_2,v_1)\omega(v_1,v_0)}{\omega(v_3,v_1)\omega(v_1,v_2)\omega(v_2,v_0)}\left(\omega(v_3,v_2) - \frac{\omega(v_3,v_1)\omega(v_0,v_2)}{\omega(v_0,v_1)} - \frac{\omega(v_3,v_0)\omega(v_1,v_2)}{\omega(v_1,v_0)}\right)\\
&= z_0\left(\frac{\omega(v_3,v_2)\omega(v_2,v_1)\omega(v_1,v_0)}{\omega(v_3,v_1)\omega(v_1,v_2)\omega(v_2,v_0)} + 1 - \frac{\omega(v_3,v_0)\omega(v_2,v_1)}{\omega(v_3,v_1)\omega(v_2,v_0)}\right)\\
&= z_0(\mathrm{cr}_2[v_0,\dots,v_3] + 1 - \mathrm{cr}_1[v_0,\dots,v_3])\\
&= z_0(\mathrm{cr}_2 - \mathrm{cr}_1 + 1)[v_0,\dots,v_3].
\end{aligned}$$

We conclude that $m_{\psi_0}a_\lambda\varphi_0[v_0,\dots,v_3] = x_{z_0,\mathrm{cr}_1,\mathrm{cr}_2}$. □

Proof of Proposition 3.5.4. Lemma 3.5.5 implies that the composition

$$G_r\backslash\mathcal{P}_r^{\{4\}} \to \mathcal{C}^{\{4\}} \to G_r\backslash\mathcal{P}_r^{\{4\}}$$

is well-defined and equal to the identity, which means that $\mathcal{C}^{\{4\}} \to G_r\backslash\mathcal{P}_r^{\{4\}}$ is surjective. The calculations $\mathrm{cr}_1(x_{z,\lambda_1,\lambda_2}) = \lambda_1$ and $\mathrm{cr}_2(x_{z,\lambda_1,\lambda_2}) = \lambda_2$ show that $\mathcal{C}^{\{4\}} \to \mathcal{P}_r^{\{4\}}$ is a smooth section. Lemma 3.3.1 now yields the claim. □

Remark 3.5.6. Interestingly, Proposition 3.5.4 implies that for $[v_0,\dots,v_3] \in \mathcal{P}_r^{(4)}$ we have

$$\begin{aligned}
[v_0,\dots,v_3] \in \mathcal{P}_r^{\{4\}} &\iff \mathrm{cr}_1[v_0,\dots,v_3] \neq \mathrm{cr}_2[v_0,\dots,v_3] + 1\\
&\iff \omega(v_3,v_2)\omega(v_1,v_0) + \omega(v_3,v_0)\omega(v_2,v_1) \neq \omega(v_3,v_1)\omega(v_2,v_0).
\end{aligned}$$

Lemma 3.5.7. *The unique map D^2 making the diagram*

$$\begin{array}{ccc}
L^\infty(\mathcal{P}_r^3)^{G_r} & \xrightarrow{d_r^2} & L^\infty(\mathcal{P}_r^4)^{G_r}\\
{\scriptstyle(\theta_r^3)^*}\downarrow & & \downarrow{\scriptstyle(\theta_r^4)^*}\\
L^\infty(\mathcal{C}^{(3)}) & \xrightarrow[D^2]{} & L^\infty(\mathcal{C}^{(4)})
\end{array}$$

commute for all $r \geq 2$ is given by $D^2 = 0$ if $\mathbb{K} = \mathbb{C}$ and

$$D^2(f)(z,\lambda_1,\lambda_2) = f(-z\cdot\mathrm{sgn}(\lambda_2)) - f(-z\cdot\mathrm{sgn}(\lambda_1\lambda_2)) + f(z\cdot\mathrm{sgn}(\lambda_1)) - f(z)$$

for almost every $(z,\lambda_1,\lambda_2) \in \mathcal{C}^{(4)}$ if $\mathbb{K} = \mathbb{R}$.

Proof. For $\mathbb{K} = \mathbb{C}$ the claim is trivial. For $\mathbb{K} = \mathbb{R}$ the claim follows directly from

$$\begin{aligned}
z[e_r, f_r, e_r + z_0\lambda_1 f_r + z_0(\lambda_2 - \lambda_1 + 1)f_{r-1}] &= z_0\cdot\mathrm{sgn}(\lambda_1),\\
z[e_r, e_r + z_0 f_r + e_{r-1} - z_0 f_{r-1}, e_r + z_0\lambda_1 f_r + z_0(\lambda_2 - \lambda_1 + 1)f_{r-1}] &= -z_0\cdot\mathrm{sgn}(\lambda_1\lambda_2),\\
z[f_r, e_r + z_0 f_r + e_{r-1} - z_0 f_{r-1}, e_r + z_0\lambda_1 f_r + z_0(\lambda_2 - \lambda_1 + 1)f_{r-1}] &= -z_0\cdot\mathrm{sgn}(\lambda_2).
\end{aligned}$$ □

Remark 3.5.8. If $\mathbb{K} = \mathbb{C}$, then $L^\infty(\mathcal{C}^{(3)}) \cong \mathbb{R}$. If $\mathbb{K} = \mathbb{R}$, then $\ker(D^2)$ consists precisely of the constant functions on $\mathcal{C}^{(3)}$, which implies that $\mathrm{im}(D^2)$ is one-dimensional; in this case, the equation $D^2(f) = 0$ can be considered a lower-degree analogue of the Spence–Abel functional equation (see [129, Section 2]).

4. Parametrizations Using Stabilizers

Let (V_r, ω) be a classical space, where we use Notation 1.1.20 throughout. In Sections 4.1–4.4 we parametrize the configuration space $G_r \backslash \mathcal{P}_r^k$, for $k = 5, 6$ in the case $G_r = \mathrm{Sp}(2r, \mathbb{K})$, for $k = 4, 5, 6$ in the case $G_r = \mathrm{O}(2r + d, \mathbb{C})$, $d \in \{0, 1\}$, and for $k = 4, 5$ in the case $G_r = \mathrm{U}(r+d, r), \mathrm{O}(r+d, r)$, $d \in \mathbb{N}$. This yields a proof of Theorem 3.1.5, (iii) for the cases considered. In Section 4.5 we extend the proof of Theorem 3.1.5, (iii) to arbitrarily high k. In contrast to Section 3.4 and Section 3.5, we use Lemma 3.3.2 to obtain our desired parametrizations. Also in Section 4.5, we prove Theorem 3.1.5, (iv) using Lemma 3.3.1, and prove Theorem 3.1.6, completing the proof of the theorems listed in Section 3.1.

Note that when using Lemma 3.3.2, we can restrict to G_r-invariant open conull submanifolds. Concretely, to prove the existence of an isomorphism $\mathcal{C}^{(k)} \xrightarrow{\cong} G_r \backslash \mathcal{P}_r^k$ of Lebesgue spaces, we will follow the steps

(S0) determine a smooth manifold $\mathcal{C}^{(k-1)}$ and a map $\mathcal{C}^{(k-1)} \to \mathcal{P}_r^{k-1}$, $c \mapsto x_c$ which is smooth on an open conull submanifold of $\mathcal{C}^{(k-1)}$ and induces an isomorphism $\mathcal{C}^{(k-1)} \xrightarrow{\cong} G_r \backslash \mathcal{P}_r^{k-1}$ of Lebesgue spaces;

(S1) determine an explicit description of $\mathrm{Stab}_{G_r}(x_c)$ for every c in an open conull submanifold of $\mathcal{C}^{(k-1)}$;

(S2) determine a smooth manifold $\mathcal{D}^{(k)}$ such that, setting $\mathcal{C}^{(k)} := \mathcal{C}^{(k-1)} \times \mathcal{D}^{(k)}$, there exists a smooth map

$$(c, d) \mapsto [x_c, y_{c,d}] \in \mathcal{P}_r^k \tag{$\star$}$$

defined on an open conull submanifold of $\mathcal{C}^{(k)}$ such that for every $[x_c, v]$ in a G_r-invariant open conull submanifold of $\mathcal{P}_r^k$ there exists $d \in \mathcal{D}^{(k)}$ such that $[y_{c,d}] \in \mathrm{Stab}_{G_r}(x_c).[v]$;

(S3) prove that $[y_{c,d}] \in \mathrm{Stab}_{G_r}(x_c).[y_{c,d'}]$ implies $d = d'$ for every d, d' in an open conull submanifold of $\mathcal{D}^{(k)}$;

(S4) prove that the map defined on a G_r-invariant open conull submanifold of $\mathcal{P}_r^k$ assigning to almost every $[v] \in \mathcal{P}_r^k$ the unique element $(c, d) \in \mathcal{C}^{(k)}$ such that $[x_c, y_{c,d}] \in G_r.[v]$ is smooth.

Then, by Lemma 3.3.2, the map ($\star$) induces an isomorphism $\mathcal{C}^{(k)} \xrightarrow{\cong} G_r \backslash \mathcal{P}_r^k$ of Lebesgue spaces.

In (S1) by an "explicit" description of the stabilizer we mean a description such that it is possible to accomplish (S2) and (S3).

Note that (S0) has been accomplished for $k = 4$ in Proposition 3.4.2 for all classical groups, and for $k = 5$ in Proposition 3.5.4 for $G_r = \mathrm{Sp}(2r, \mathbb{K})$. Since we will proceed iteratively, step (S0) will always be obvious. Let us also point out that for each family under consideration we will either use Witt's lemma or rational functions in step (S2). Using the strong version of Witt's lemma (Theorem 1.1.6), step (S4) will also be obvious (except in the case of sextuples for $G_r = \mathrm{Sp}(2r, \mathbb{K})$, see Remark 4.1.4), which is why we will only write out steps (S1)–(S3).

Notation. Let $\psi \in G_r$. We define $m_\psi^{(k)} \in G_{r+k}$ inductively by setting $m_\psi^{(1)} := m_\psi$ and $m_\psi^{(k)} := m_{m_\psi^{(k-1)}}$.

4.1. Parametrization for the Symplectic Groups

Let $r \geq 2$ and $G_r = \mathrm{Sp}(2r, \mathbb{K})$. We use the notation of Section 3.5.

4.1.1. Parametrization of Quintuples

Step (S1): Let $(z, \lambda_1, \lambda_2) \in \mathcal{C}^{\{4\}}$. Then we have

$$\mathrm{Stab}_{G_r}(x_{z,\lambda_1,\lambda_2}) = \left\{ \lambda m_\psi^{(2)} \;\middle|\; \lambda \in \{\pm 1\},\ \psi \in G_{r-2} \right\}.$$

Step (S2): For all $(z, \lambda_1, \dots, \lambda_5) \in \mathcal{C}^{(5)} := \mathcal{C}^{\{4\}} \times \mathbb{K}^3$ we write

$$x_{z,\lambda_1,\dots,\lambda_5} := [x_{z,\lambda_1,\lambda_2}, e_r + \lambda_5 f_r + \lambda_4 e_{r-1} + \lambda_3 f_{r-1} + e_{r-2}] \in \mathcal{P}_r^5,$$

where we set $e_0 := 0$ in the case $r = 2$ (c.f. Lemma 4.5.2). One can easily check that

$$\mathcal{C}^{\{5\}} := \left\{ (z, \lambda_1, \dots, \lambda_5) \in \mathcal{C}^{(5)} \;\middle|\; x_{z,\lambda_1,\dots,\lambda_5} \in \mathcal{P}_r^{\{5\}} \right\}$$

is an open conull submanifold of $\mathcal{C}^{(5)}$.

Proposition 4.1.1. *The map*

$$\mathcal{C}^{\{5\}} \to \mathcal{P}_r^{\{5\}},\ (z, \lambda_1, \dots, \lambda_5) \mapsto x_{z,\lambda_1,\dots,\lambda_5}$$

induces a surjection $\theta_r^5 \colon \mathcal{C}^{\{5\}} \to G_r \backslash \mathcal{P}_r^{\{5\}}$.

Proof. Let $v = \alpha e_r + v_1 + v_0$, where $v_1 \in \mathrm{span}_{\mathbb{K}}(e_{r-1}, f_{r-1}, f_r)$ and $v_0 \in V_{r-2}$, satisfy $[x_{z,\lambda_1,\lambda_2}, v] \in \mathcal{P}_r^{\{5\}}$. By Witt's lemma, there exists $\psi \in G_{r-2}$ such that $\psi(v_0) = \alpha e_{r-2}$. Hence there exist $\lambda_3, \lambda_4, \lambda_5 \in \mathbb{K}$ such that $x_{z,\lambda_1,\dots,\lambda_5} \in G_r.[x_{z,\lambda_1,\lambda_2}, v]$. □

Step (S3): This is obvious from the above.

Corollary 4.1.2. *The map* $\theta_r^5 \colon \mathcal{C}^{\{5\}} \to G_r \backslash \mathcal{P}_r^{\{5\}}$ *is an isomorphism of Lebesgue spaces for all* $r \geq 2$.

4.1.2. Parametrization of Sextuples

Step (S1): Let $r \geq 3$ and let $(z, \lambda_1, \ldots, \lambda_5) \in \mathcal{C}^{\{5\}}$. Then we have

$$\mathrm{Stab}_{G_r}(x_{z,\lambda_1,\ldots,\lambda_5}) = \Big\{\lambda m^{(2)}_{m_\psi n_w} \;\Big|\; \lambda \in \{\pm 1\},\ \psi \in G_{r-3},\ w \in W_{r-2}\Big\},$$
$$\mathrm{Stab}_{G_2}(x_{z,\lambda_1,\ldots,\lambda_5}) = \{\pm \mathrm{id}\}.$$

Step (S2): Let $r \geq 3$; let us write

$$\mathcal{C}^{(6)}_r := \mathcal{C}^{\{5\}} \times \mathbb{K}^4,$$
$$\mathcal{C}^{(6)}_2 := \mathcal{C}^{\{5\}} \times \mathbb{K}^3.$$

For all $(z, \lambda_1, \ldots, \lambda_9) \in \mathcal{C}^{(6)}_r$ we write

$$x_{z,\lambda_1,\ldots,\lambda_9} := [x_{z,\lambda_1,\ldots,\lambda_5}, e_r + \lambda_9 f_r + \lambda_8 e_{r-1} + \lambda_7 f_{r-1} + \lambda_6 f_{r-2}] \in \mathcal{P}^6_r.$$

For all $(z, \lambda_1, \ldots, \lambda_8) \in \mathcal{C}^{(6)}_2$ we write

$$x_{z,\lambda_1,\ldots,\lambda_8} := [x_{z,\lambda_1,\ldots,\lambda_8}, e_2 + \lambda_8 f_2 + \lambda_7 e_1 + \lambda_6 f_1] \in \mathcal{P}^6_2.$$

One can easily check that

$$\mathcal{C}^{\{6\}}_r := \Big\{(z, \lambda_1, \ldots, \lambda_9) \in \mathcal{C}^{(6)}_r \;\Big|\; x_{z,\lambda_1,\ldots,\lambda_9} \in \mathcal{P}^{\{6\}}_r\Big\},$$
$$\mathcal{C}^{\{6\}}_2 := \Big\{(z, \lambda_1, \ldots, \lambda_8) \in \mathcal{C}^{(6)}_2 \;\Big|\; x_{z,\lambda_1,\ldots,\lambda_8} \in \mathcal{P}^{\{6\}}_2\Big\}$$

are open conull submanifolds of $\mathcal{C}^{(6)}_r$ and $\mathcal{C}^{(6)}_2$, respectively.

Proposition 4.1.3. *(i) Let $r \geq 3$. Then the map*

$$\mathcal{C}^{\{6\}}_r \to \mathcal{P}^{\{6\}}_r,\ (z, \lambda_1, \ldots, \lambda_9) \mapsto x_{z,\lambda_1,\ldots,\lambda_9}$$

induces an essential surjection $\theta^6_r \colon \mathcal{C}^{\{6\}}_r \to G_r \backslash \mathcal{P}^{\{6\}}_r$.

(ii) The map

$$\mathcal{C}^{\{6\}}_2 \to \mathcal{P}^{\{6\}}_2,\ (z, \lambda_1, \ldots, \lambda_8) \mapsto x_{z,\lambda_1,\ldots,\lambda_8}$$

induces a surjection $\theta^6_2 \colon \mathcal{C}^{\{6\}}_2 \to G_2 \backslash \mathcal{P}^{\{6\}}_2$.

Proof. Statement (ii) is obvious. For the proof of statement (i) let $v = v_1 + \alpha e_{r-2} + \beta f_{r-2} + v_0$, where $v_1 \in \mathrm{span}_{\mathbb{K}}(e_r, e_{r-1}, f_{r-1}, f_r)$, $\beta \neq 0$, and $v_0 \in V_{r-3}$, satisfy $[x_{z,\lambda_1,\ldots,\lambda_5}, v] \in \mathcal{P}^{\{6\}}_r$. We set $w = -\frac{1}{\beta}(\alpha e_{r-2} + v_0) \in W_{r-2}$. Then

$$m^{(2)}_{n_w}(v) = v_1 + \beta f_{r-2}.$$

Hence there exist $\lambda_6, \ldots, \lambda_9 \in \mathbb{K}$ such that

$$[e_r + \lambda_9 f_r + \lambda_8 e_{r-1} + \lambda_7 f_{r-1} + \lambda_6 f_{r-2}] \in \mathrm{Stab}_{G_r}(x_{z,\lambda_1,\ldots,\lambda_5}).[v]. \qquad \square$$

Remark 4.1.4. We point out that the set of elements $[v_0, \ldots, v_5] \in \mathcal{P}^{\{6\}}_r$ such that $[x_{z,\lambda_1,\ldots,\lambda_5}, v] \in G_r.[v_0, \ldots, v_5]$ for some $(z, \lambda_1, \ldots, \lambda_5) \in \mathcal{C}^{\{5\}}_r$ and some v as in the proof above, i.e. satisfying $\omega(e_{r-2}, v) \neq 0$, is open and conull in $\mathcal{P}^6_r$. In step (S3) we will see that this set is also G_r-invariant, which we will show by proving that the set of elements $[v] \in \mathcal{P}_r$ satisfying $\omega(e_{r-2}, v) \neq 0$ is $\mathrm{Stab}_{G_r}(x_{z,\lambda_1,\ldots,\lambda_5})$-invariant for every $(z, \lambda_1, \ldots, \lambda_5) \in \mathcal{C}^{\{5\}}$.

Step (S3): For $r = 2$ this is obvious from the above. For $r \geq 3$ we assume that

$$m^{(2)}_{m_\psi n_w}(e_r + \lambda_9 f_r + \lambda_8 e_{r-1} + \lambda_7 f_{r-1} + \lambda_6 f_{r-2}) = \nu(e_r + \mu_9 f_r + \mu_8 e_{r-1} + \mu_7 f_{r-1} + \mu_6 f_{r-2})$$

for some $\nu, \mu_6, \dots, \mu_9 \in \mathbb{K}$ and some $\psi \in G_{r-3}$, $w \in W_{r-2}$. The left hand side of this equation is given by

$$e_r + \lambda_9 f_r + \lambda_8 e_{r-1} + \lambda_7 f_{r-1} + \lambda_6 (f_{r-2} + m_\psi(w)),$$

from which we can immediately deduce $\nu = 1$ and $\lambda_k = \mu_k$ for $k = 7, 8, 9$. From the definition of W_{r-2} we see that

$$\omega(e_{r-2}, m_\psi(w)) = \omega(m_\psi(e_{r-2}), m_\psi(w)) = \omega(e_{r-2}, w) = 0,$$

which yields $\lambda_6 = \mu_6$.

Corollary 4.1.5. *The map $\theta^6_r \colon \mathcal{C}^{\{6\}}_r \to G_r \backslash \mathcal{P}^{\{6\}}_r$ is an isomorphism of Lebesgue spaces for all $r \geq 2$.*

4.2. Parametrization for the Complex Orthogonal Groups

Let $r \geq 2$ and $G_r = G_{r,d} = \mathrm{O}(2r + d, \mathbb{C})$, $d \in \{0, 1\}$. We use the notation of Section 3.3.

4.2.1. Parametrization of Quadruples

Step (S1): We have

$$\mathrm{Stab}_{G_r}(x_1) = \bigl\{\lambda m_\psi \bigm| \lambda \in \{\pm 1\},\ \psi \in \mathrm{Stab}_{G_{r-1}}(e_{r-1} - f_{r-1})\bigr\}.$$

Step (S2): Let us write

$$\mathcal{C}^{(4)} := \mathbb{C}^2.$$

For all $(\lambda_1, \lambda_2) \in \mathcal{C}^{(4)}$ we write

$$x_{\lambda_1,\lambda_2} := \bigl[x_1, e_r + \bigl(\lambda_2^2 - \lambda_1^2\bigr) f_r + \lambda_2(e_{r-1} - f_{r-1}) + \lambda_1(e_{r-1} + f_{r-1})\bigr] \in \mathcal{P}^4_r.$$

One can easily check that

$$\mathcal{C}^{\{4\}} := \bigl\{(\lambda_1, \lambda_2) \in \mathcal{C}^{(4)} \bigm| x_{\lambda_1,\lambda_2} \in \mathcal{P}^{\{4\}}_r\bigr\}$$

is an open conull submanifold of $\mathcal{C}^{(4)}$.

Proposition 4.2.1. *The map*

$$\mathcal{C}^{\{4\}} \to \mathcal{P}^{\{4\}}_r,\ (\lambda_1, \lambda_2) \mapsto x_{\lambda_1,\lambda_2}$$

induces a surjection $\theta^4_r \colon \mathcal{C}^{\{4\}} \to G_r \backslash \mathcal{P}^{\{4\}}_r$.

Proof. Let $v = v_1 + v_0$, where $v_1 \in \mathrm{span}_{\mathbb{C}}(e_r, f_r, e_{r-1} - f_{r-1})$ and $v_0 \in \mathbb{C}(e_{r-1} + f_{r-1}) \oplus V_{r-2}$, satisfy $[x_1, v] \in \mathcal{P}_r^{\{4\}}$. By Witt's lemma there exist $\psi \in \mathrm{Stab}_{G_{r-1}}(e_{r-1} - f_{r-1})$ and $\lambda_1 \in \mathbb{C}$ such that

$$\psi(v_0) = \lambda_1(e_{r-1} + f_{r-1}).$$

Hence there exists $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$ such that

$$[e_r + \lambda_3 f_r + \lambda_2(e_{r-1} - f_{r-1}) + \lambda_1(e_{r-1} + f_{r-1})] \in \mathrm{Stab}_{G_r}(x_1).[v].$$

The condition $q(v) = 0$ implies $\lambda_3 = \lambda_2^2 - \lambda_1^2$. Hence there exists $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ such that

$$\left[e_r + \left(\lambda_2^2 - \lambda_1^2\right) f_r + \lambda_2(e_{r-1} - f_{r-1}) + \lambda_1(e_{r-1} + f_{r-1})\right] \in \mathrm{Stab}_{G_r}(x_1).[v]. \qquad \square$$

Step (S3): It is sufficient to note that

$$\begin{aligned} &\left[e_r + \left(\lambda_2^2 - \lambda_1^2\right) f_r + \lambda_2(e_{r-1} - f_{r-1}) + \lambda_1 \psi(e_{r-1} + f_{r-1})\right] \\ = &\left[e_r + \left(\mu_2^2 - \mu_1^2\right) f_r + \mu_2(e_{r-1} - f_{r-1}) + \mu_1(e_{r-1} + f_{r-1})\right] \end{aligned}$$

for some $\psi \in \mathrm{Stab}_{G_{r-1}}(e_{r-1} - f_{r-1})$ implies $\lambda_1^2 = \mu_1^2$ and $\lambda_2 = \mu_2$.

Corollary 4.2.2. *The map $\theta_r^4 \colon \mathcal{C}^{\{4\}} \to G_r \backslash \mathcal{P}_r^{\{4\}}$ is an isomorphism of Lebesgue spaces for all $r \geq 2$.*

4.2.2. Parametrization of Quintuples

Step (S1): Let $r \geq 3$ and let $(\lambda_1, \lambda_2) \in \mathcal{C}^{\{4\}}$. Then we have

$$\mathrm{Stab}_{G_r}(x_{\lambda_1, \lambda_2}) = \left\{ \lambda m_\psi^{(2)} \;\middle|\; \lambda \in \{\pm 1\},\ \psi \in G_{r-2} \right\}.$$

Step (S2): Let $r \geq 3$; let us write

$$\begin{aligned} \mathcal{C}_{r,d}^{(5)} &\coloneqq \mathcal{C}^{\{4\}} \times \mathbb{C}^3, \\ \mathcal{C}_{2,1}^{(5)} &\coloneqq \mathcal{C}^{\{4\}} \times \mathbb{C}^3, \\ \mathcal{C}_{2,0}^{(5)} &\coloneqq \mathcal{C}^{\{4\}} \times \mathbb{C}^2. \end{aligned}$$

For all $(\lambda_1, \ldots, \lambda_5) \in \mathcal{C}_{r,d}^{(5)}$ we write

$$\begin{aligned} x_{\lambda_1, \ldots, \lambda_5} \coloneqq \big[x_{\lambda_1, \lambda_2}, &e_r + \left(\lambda_5^2 - \lambda_4^2 - \lambda_3^2\right) f_r + \lambda_5(e_{r-1} - f_{r-1}) \\ &+ \lambda_4(e_{r-1} + f_{r-1}) + \lambda_3(e_{r-2} - f_{r-2})\big] \in \mathcal{P}_r^5. \end{aligned}$$

For all $(\lambda_1, \ldots, \lambda_5) \in \mathcal{C}_{2,1}^{(5)}$ we write

$$x_{\lambda_1, \ldots, \lambda_5} \coloneqq \left[x_{\lambda_1, \lambda_2}, e_2 + \left(\lambda_5^2 - \lambda_4^2 - \frac{1}{2}\lambda_3^2\right) f_2 + \lambda_5(e_1 - f_1) + \lambda_4(e_1 + f_1) + \lambda_3 h_1\right] \in \mathcal{P}_r^5.$$

For all $(\lambda_1, \ldots, \lambda_4) \in \mathcal{C}_{2,0}^{(5)}$ we write

$$x_{\lambda_1, \ldots, \lambda_4} \coloneqq \left[x_{\lambda_1, \lambda_2}, e_2 + \left(\lambda_4^2 - \lambda_3^2\right) f_2 + \lambda_4(e_1 - f_1) + \lambda_3(e_1 + f_1)\right] \in \mathcal{P}_r^5.$$

One can easily check that

$$\begin{aligned}
\mathcal{C}_{r,d}^{\{5\}} &:= \left\{(\lambda_1,\ldots,\lambda_5)\in\mathcal{C}_{r,d}^{(5)} \;\middle|\; x_{\lambda_1,\ldots,\lambda_5}\in\mathcal{P}_r^{\{5\}}\right\},\\
\mathcal{C}_{2,1}^{\{5\}} &:= \left\{(\lambda_1,\ldots,\lambda_5)\in\mathcal{C}_{2,1}^{(5)} \;\middle|\; x_{\lambda_1,\ldots,\lambda_5}\in\mathcal{P}_2^{\{5\}}\right\},\\
\mathcal{C}_{2,0}^{\{5\}} &:= \left\{(\lambda_1,\ldots,\lambda_4)\in\mathcal{C}_{2,0}^{(5)} \;\middle|\; x_{\lambda_1,\ldots,\lambda_4}\in\mathcal{P}_2^{\{5\}}\right\}
\end{aligned}$$

are open conull submanifolds of $\mathcal{C}_{r,d}^{(5)}$, $\mathcal{C}_{2,1}^{(5)}$, and $\mathcal{C}_{2,0}^{(5)}$, respectively.

Proposition 4.2.3. *(i) Let $r\geq 3$. Then the map*

$$\mathcal{C}_{r,d}^{\{5\}}\to\mathcal{P}_r^{\{5\}},\ (\lambda_1,\ldots,\lambda_5)\mapsto x_{\lambda_1,\ldots,\lambda_5}$$

induces a surjection $\theta_r^5\colon \mathcal{C}_{r,d}^{\{5\}}\to \mathrm{O}(2r+d,\mathbb{C})\backslash\mathcal{P}_r^{\{5\}}$.

(ii) The map

$$\mathcal{C}_{2,1}^{\{5\}}\to\mathcal{P}_2^{\{5\}},\ (\lambda_1,\ldots,\lambda_5)\mapsto x_{\lambda_1,\ldots,\lambda_5}$$

induces a surjection $\theta_2^5\colon \mathcal{C}_{2,1}^{\{5\}}\to \mathrm{O}(5,\mathbb{C})\backslash\mathcal{P}_2^{\{5\}}$.

(iii) The map

$$\mathcal{C}_{2,0}^{\{5\}}\to\mathcal{P}_2^{\{5\}},\ (\lambda_1,\ldots,\lambda_4)\mapsto x_{\lambda_1,\ldots,\lambda_4}$$

induces a surjection $\theta_2^5\colon \mathcal{C}_{2,0}^{\{5\}}\to \mathrm{O}(4,\mathbb{C})\backslash\mathcal{P}_2^{\{5\}}$.

Proof. Statements (ii) and (iii) are obvious. Statement (i) is analogous to the proof of Proposition 4.2.1. □

Step (S3): This is completely analogous to Step (S3) of Subsection 4.2.1.

Note that for $r\geq 3$ or $(r,d)=(2,1)$ there exists an isomorphism $\mathcal{C}_{r,d}^{\{5\}}\cong\mathcal{C}^{(3)}\times\mathbb{C}^5$ of Lebesgue spaces.

Corollary 4.2.4. *The map $\theta_r^5\colon\mathcal{C}_{r,d}^{\{5\}}\to G_r\backslash\mathcal{P}_r^{\{5\}}$ is an isomorphism of Lebesgue spaces for all $r\geq 2$.*

4.2.3. Parametrization of Sextuples

Step (S1): Let $r\geq 3$ and let $(\lambda_1,\ldots,\lambda_5)\in\mathcal{C}^{\{5\}}$. Then we have

$$\begin{aligned}
\mathrm{Stab}_{G_{r,d}}(x_{\lambda_1,\ldots,\lambda_5}) &= \left\{\lambda m_\psi^{(2)} \;\middle|\; \lambda\in\{\pm1\},\ \psi\in\mathrm{Stab}_{G_{r-2}}(e_{r-2}-f_{r-2})\right\},\\
\mathrm{Stab}_{G_{2,1}}(x_{\lambda_1,\ldots,\lambda_5}) &= \left\{\lambda m_\psi^{(2)} \;\middle|\; \lambda\in\{\pm1\},\ \psi\in\mathrm{Stab}_{G_{r-2}}(h_1)\right\}.
\end{aligned}$$

Step (S2): Let us write

$$\mathcal{C}_{r,d}^{(6)} \coloneqq \mathcal{C}_{r,d}^{\{5\}} \times \mathbb{C}^4,$$
$$\mathcal{C}_{2,1}^{(6)} \coloneqq \mathcal{C}_{2,1}^{\{5\}} \times \mathbb{C}^3.$$

For all $(\lambda_1, \ldots, \lambda_9) \in \mathcal{C}_{r,d}^{(6)}$ we write

$$x_{\lambda_1,\ldots,\lambda_9} \coloneqq \big[x_{\lambda_1,\ldots,\lambda_5}, e_r + \big(\lambda_9^2 - \lambda_8^2 + \lambda_7^2 - \lambda_6^2\big) f_r + \lambda_9(e_{r-1} - f_{r-1}) + \lambda_8(e_{r-1} + f_{r-1}) + \lambda_7(e_{r-2} - f_{r-2}) + \lambda_6(e_{r-2} + f_{r-2})\big] \in \mathcal{P}_r^6.$$

For all $(\lambda_1, \ldots, \lambda_8) \in \mathcal{C}_{2,1}^{(6)}$ we write

$$x_{\lambda_1,\ldots,\lambda_8} \coloneqq \left[x_{\lambda_1,\ldots,\lambda_5}, e_2 + \left(\lambda_8^2 - \lambda_7^2 - \frac{1}{2}\lambda_6^2\right) f_2 + \lambda_8(e_1 - f_1) + \lambda_7(e_1 + f_1) + \lambda_6 h_1\right] \in \mathcal{P}_2^6.$$

One can easily check that

$$\mathcal{C}_{r,d}^{\{6\}} \coloneqq \left\{(\lambda_1, \ldots, \lambda_9) \in \mathcal{C}_{r,d}^{(6)} \;\middle|\; x_{\lambda_1,\ldots,\lambda_9} \in \mathcal{P}_r^{\{6\}}\right\},$$
$$\mathcal{C}_{2,1}^{\{6\}} \coloneqq \left\{(\lambda_1, \ldots, \lambda_8) \in \mathcal{C}_{2,1}^{(6)} \;\middle|\; x_{\lambda_1,\ldots,\lambda_8} \in \mathcal{P}_2^{\{6\}}\right\}$$

are open conull submanifolds of $\mathcal{C}_{r,d}^{(6)}$ and $\mathcal{C}_{2,1}^{(6)}$, respectively.

Proposition 4.2.5. *(i) Let $r \geq 3$. Then the map*

$$\mathcal{C}_{r,d}^{\{6\}} \to \mathcal{P}_r^{\{6\}}, \; (\lambda_1, \ldots, \lambda_9) \mapsto x_{\lambda_1,\ldots,\lambda_9}$$

induces a surjection $\theta_r^6 \colon \mathcal{C}_{r,d}^{\{6\}} \to \mathrm{O}(2r+d, \mathbb{C}) \backslash \mathcal{P}_r^{\{6\}}$.

(ii) The map

$$\mathcal{C}_{2,1}^{\{6\}} \to \mathcal{P}_2^{\{6\}}, \; (\lambda_1, \ldots, \lambda_8) \mapsto x_{\lambda_1,\ldots,\lambda_8}$$

induces a surjection $\theta_2^6 \colon \mathcal{C}_{2,1}^{\{6\}} \to \mathrm{O}(5, \mathbb{C}) \backslash \mathcal{P}_r^{\{6\}}$.

Proof. Statement (ii) is obvious. Statement (i) is analogous to the proof of Proposition 4.2.1. □

Step (S3): This is completely analogous to Step (S3) of Subsection 4.2.1.

Note that for $r \geq 3$ there exists an isomorphism $\mathcal{C}_{r,d}^{\{6\}} \cong \mathcal{C}^{(3)} \times \mathbb{C}^9$ of Lebesgue spaces.

Corollary 4.2.6. *The map $\theta_r^6 \colon \mathcal{C}_{r,d}^{\{6\}} \to G_r \backslash \mathcal{P}_r^{\{6\}}$ is an isomorphism of Lebesgue spaces for all $r \geq 3$.*

4.3. Parametrization for the Unitary Groups

Let $r \geq 3$ and $G_r = \mathrm{U}(r+d, r)$. We use the notation of Section 3.3.

4.3.1. Parametrization of Quadruples

Step (S1): We consider the open conull submanifold $\mathcal{C}^{\{3\}} := \mathcal{C}^{(3)} \setminus \{\pm \mathrm{i}\}$ of $\mathcal{C}^{(3)}$. Let $z \in \mathcal{C}^{\{3\}}$. Then we have

$$\operatorname{Stab}_{G_r}(x_z) = \big\{\lambda m_\psi \mid \lambda \in \mathcal{C}^{(3)},\ \psi \in \operatorname{Stab}_{G_{r-1}}(e_{r-1} - z f_{r-1})\big\}.$$

Step (S2): Let us write

$$\mathcal{B}^{(4)} := \mathcal{C}^{\{3\}} \times \mathbb{R}_{\geq 0} \times \mathbb{C} \times \mathbb{R}.$$

For all $(z, \lambda_1, \lambda_2, \lambda_3) \in \mathcal{B}^{(4)}$ we write

$$\begin{aligned} x^{(1)}_{z,\lambda_1,\lambda_2,\lambda_3} := \big[x_z, &e_r + \big(\operatorname{Re}(z)\big(|\lambda_2|^2 - \lambda_1^2\big) + \lambda_3 \mathrm{i}\big) f_r \\ &+ \lambda_2(e_{r-1} - z f_{r-1}) + \lambda_1(e_{r-1} + \overline{z} f_{r-1})\big] \in \mathcal{P}_r^4, \\ x^{(2)}_{z,\lambda_1,\lambda_2,\lambda_3} := \big[x_z, &e_r + \big(\operatorname{Re}(z)\big(|\lambda_2|^2 + \lambda_1^2\big) + \lambda_3 \mathrm{i}\big) f_r \\ &+ \lambda_2(e_{r-1} - z f_{r-1}) + \lambda_1(e_{r-2} - z f_{r-2})\big] \in \mathcal{P}_r^4. \end{aligned}$$

One can easily check that

$$\mathcal{B}^{\{4\}} := \Big\{(z, \lambda_1, \lambda_2, \lambda_3) \in \mathcal{B}^{(4)} \,\Big|\, x^{(i)}_{z,\lambda_1,\lambda_2,\lambda_3} \in \mathcal{P}_r^{\{4\}},\ i = 1, 2\Big\}$$

is an open conull submanifold of $\mathcal{B}^{(4)}$.

Proposition 4.3.1. *The maps*

$$\mathcal{B}^{\{4\}} \to \mathcal{P}_r^{\{4\}},\ (z, \lambda_1, \lambda_2, \lambda_3) \mapsto x^{(i)}_{z,\lambda_1,\lambda_2,\lambda_3}, \qquad i = 1, 2,$$

induce a surjection $\theta_r^4 \colon \mathcal{B}^{\{4\}} \sqcup \mathcal{B}^{\{4\}} \to G_r \backslash \mathcal{P}_r^{\{4\}}$.

Proof. Let $v = v_1 + v_0$, where $v_1 \in \operatorname{span}_{\mathbb{C}}(e_r, f_r, e_{r-1} - z f_{r-1})$ and $v_0 \in \mathbb{C}(e_{r-1} + \overline{z} f_{r-1}) \oplus V_{r-2}$, satisfy $[x_z, v] \in \mathcal{P}_r^{\{4\}}$.

Case 1: $\operatorname{Re}(z) q(v_0) \geq 0$. By Witt's lemma there exists $\psi \in \operatorname{Stab}_{G_{r-1}}(e_{r-1} - z f_{r-1})$ such that

$$\psi(v_0) = \sqrt{\frac{q(v_0)}{2\operatorname{Re}(z)}}(e_{r-1} + \overline{z} f_{r-1}).$$

Hence there exists $(\lambda_1, \lambda_2, \lambda_3') \in \mathbb{R}_{\geq 0} \times \mathbb{C}^2$ such that

$$[e_r + \lambda_3' f_r + \lambda_2(e_{r-1} - z f_{r-1}) + \lambda_1(e_{r-1} + \overline{z} f_{r-1})] \in \operatorname{Stab}_{G_r}(x_z).[v].$$

The condition $q(v) = 0$ implies $\operatorname{Re}(\lambda_3') = \operatorname{Re}(z)(|\lambda_2|^2 - \lambda_1^2)$. Hence there exists $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}_{\geq 0} \times \mathbb{C} \times \mathbb{R}$ such that

$$\big[e_r + \big(\operatorname{Re}(z)\big(|\lambda_2|^2 - \lambda_1^2\big) + \lambda_3 \mathrm{i}\big) f_r + \lambda_2(e_{r-1} - z f_{r-1}) + \lambda_1(e_{r-1} + \overline{z} f_{r-1})\big] \in \operatorname{Stab}_{G_r}(x_z).[v].$$

Case 2: $\operatorname{Re}(z) q(v_0) < 0$. By Witt's lemma there exists $\psi \in \operatorname{Stab}_{G_{r-1}}(e_{r-1} - z f_{r-1})$ such that

$$\psi(v_0) = \sqrt{\frac{q(v_0)}{-2\operatorname{Re}(z)}}(e_{r-2} - z f_{r-2}).$$

Hence there exists $(\lambda_1,\lambda_2,\lambda_3')\in\mathbb{R}_{\geq 0}\times\mathbb{C}^2$ such that

$$[e_r+\lambda_3' f_r+\lambda_2(e_{r-1}-zf_{r-1})+\lambda_1(e_{r-2}-zf_{r-2})]\in \mathrm{Stab}_{G_r}(x_z).[v].$$

The condition $q(v)=0$ implies $\mathrm{Re}(\lambda_3')=\mathrm{Re}(z)(|\lambda_2|^2+\lambda_1^2)$. Hence there exists $(\lambda_1,\lambda_2,\lambda_3)\in\mathbb{R}_{\geq 0}\times\mathbb{C}\times\mathbb{R}$ such that

$$\big[e_r+\big(\mathrm{Re}(z)\big(|\lambda_2|^2+\lambda_1^2\big)+\lambda_3\mathrm{i}\big)f_r+\lambda_2(e_{r-1}-zf_{r-1})+\lambda_1(e_{r-2}-zf_{r-2})\big]\in \mathrm{Stab}_{G_r}(x_z).[v].\ \square$$

Step (S3): It is sufficient to note that

$$\begin{aligned}&\big[e_r+\big(\mathrm{Re}(z)\big(|\lambda_2|^2-\lambda_1^2\big)+\lambda_3\mathrm{i}\big)f_r+\lambda_2(e_{r-1}-zf_{r-1})+\lambda_1\psi(e_{r-1}+\overline{z}f_{r-1})\big]\\ =&\big[e_r+\big(\mathrm{Re}(z)\big(|\mu_2|^2-\mu_1^2\big)+\mu_3\mathrm{i}\big)f_r+\mu_2(e_{r-1}-zf_{r-1})+\mu_1(e_{r-1}+\overline{z}f_{r-1})\big]\end{aligned}$$

or

$$\begin{aligned}&\big[e_r+\big(\mathrm{Re}(z)\big(|\lambda_2|^2+\lambda_1^2\big)+\lambda_3\mathrm{i}\big)f_r+\lambda_2(e_{r-1}-zf_{r-1})+\lambda_1\psi(e_{r-2}-zf_{r-2})\big]\\ =&\big[e_r+\big(\mathrm{Re}(z)\big(|\mu_2|^2+\mu_1^2\big)+\mu_3\mathrm{i}\big)f_r+\mu_2(e_{r-1}-zf_{r-1})+\mu_1(e_{r-2}-zf_{r-2})\big]\end{aligned}$$

for some $\psi\in\mathrm{Stab}_{G_{r-1}}(e_{r-1}-zf_{r-1})$ implies $(\lambda_1,\lambda_2,\lambda_3)=(\mu_1,\mu_2,\mu_3)$, while

$$\begin{aligned}&\big[e_r+\big(\mathrm{Re}(z)\big(|\lambda_2|^2-\lambda_1^2\big)+\lambda_3\mathrm{i}\big)f_r+\lambda_2(e_{r-1}-zf_{r-1})+\lambda_1\psi(e_{r-1}+\overline{z}f_{r-1})\big]\\ =&\big[e_r+\big(\mathrm{Re}(z)\big(|\mu_2|^2+\mu_1^2\big)+\mu_3\mathrm{i}\big)f_r+\mu_2(e_{r-1}-zf_{r-1})+\mu_1(e_{r-2}-zf_{r-2})\big]\end{aligned}$$

for some $\psi\in\mathrm{Stab}_{G_{r-1}}(e_{r-1}-zf_{r-1})$ implies $\lambda_1=0=\mu_1$ and $\lambda_2=\mu_2$, $\lambda_3=\mu_3$.

Let us now set $\mathcal{C}^{\{4\}}:=\mathcal{B}^{\{4\}}\sqcup\mathcal{B}^{\{4\}}$. Note that there exists an isomorphism $\mathcal{C}^{\{4\}}\cong\mathcal{C}^{(3)}\times\mathbb{C}^2$ of Lebesgue spaces.

Corollary 4.3.2. *The map $\theta_r^4\colon\mathcal{C}^{\{4\}}\to G_r\backslash\mathcal{P}_r^{\{4\}}$ is an isomorphism of Lebesgue spaces for all $r\geq 3$.*

4.3.2. Parametrization of Quintuples

Step (S1): Let $(z,\lambda_1,\lambda_2,\lambda_3)\in\mathcal{B}^{\{4\}}$. Then we have

$$\begin{aligned}\mathrm{Stab}_{G_r}\Big(x^{(1)}_{z,\lambda_1,\lambda_2,\lambda_3}\Big)&=\Big\{\lambda m^{(2)}_\psi\ \Big|\ \lambda\in\mathcal{C}^{(3)},\ \psi\in G_{r-2}\Big\},\\ \mathrm{Stab}_{G_r}\Big(x^{(2)}_{z,\lambda_1,\lambda_2,\lambda_3}\Big)&=\big\{\lambda m_\psi\ \big|\ \lambda\in\mathcal{C}^{(3)},\ \psi\in\mathrm{Stab}_{G_{r-1}}(e_{r-1}-zf_{r-1},e_{r-2}-zf_{r-2})\big\}.\end{aligned}$$

Step (S2): Let us write

$$\begin{aligned}\mathcal{B}^{(5)}&:=\mathcal{B}^{\{4\}}\times\mathbb{R}_{\geq 0}\times\mathbb{C}^2\times\mathbb{R},\\ \mathcal{A}^{(5)}&:=\big\{(-1,\lambda_1,\dots,\lambda_7)\in\mathcal{B}^{(5)}\big\}.\end{aligned}$$

For all $(z,\lambda_1,\dots,\lambda_7)\in\mathcal{B}^{(5)}$ we write

$$\begin{aligned}
x^{(1,1)}_{z,\lambda_1,\dots,\lambda_7} &:= \Big[x^{(1)}_{z,\lambda_1,\lambda_2,\lambda_3}, e_r + \big(\mathrm{Re}(z)\big(|\lambda_6|^2-|\lambda_5|^2-\lambda_4^2\big)+\lambda_7\mathrm{i}\big)f_r \\
&\qquad + \lambda_6(e_{r-1}-zf_{r-1}) + \lambda_5(e_{r-1}+\overline{z}f_{r-1}) + \lambda_4(e_{r-2}+\overline{z}f_{r-2})\Big],\\
x^{(1,2)}_{z,\lambda_1,\dots,\lambda_7} &:= \Big[x^{(1)}_{z,\lambda_1,\lambda_2,\lambda_3}, e_r + \big(\mathrm{Re}(z)\big(|\lambda_6|^2-|\lambda_5|^2+\lambda_4^2\big)+\lambda_7\mathrm{i}\big)f_r \\
&\qquad + \lambda_6(e_{r-1}-zf_{r-1}) + \lambda_5(e_{r-1}+\overline{z}f_{r-1}) + \lambda_4(e_{r-2}-zf_{r-2})\Big],\\
x^{(2,1)}_{z,\lambda_1,\dots,\lambda_7} &:= \Big[x^{(2)}_{z,\lambda_1,\lambda_2,\lambda_3}, e_r + \big(\mathrm{Re}(z)\big(|\lambda_6|^2+|\lambda_5|^2-\lambda_4^2\big)+\lambda_7\mathrm{i}\big)f_r \\
&\qquad + \lambda_6(e_{r-1}-zf_{r-1}) + \lambda_5(e_{r-2}-zf_{r-2}) + \lambda_4(e_{r-2}+\overline{z}f_{r-2})\Big],\\
x^{(2,2)}_{z,\lambda_1,\dots,\lambda_7} &:= \Big[x^{(2)}_{z,\lambda_1,\lambda_2,\lambda_3}, e_r + \big(\mathrm{Re}(z)\big(|\lambda_6|^2+|\lambda_5|^2+\lambda_4^2\big)+\lambda_7\mathrm{i}\big)f_r \\
&\qquad + \lambda_6(e_{r-1}-zf_{r-1}) + \lambda_5(e_{r-2}-zf_{r-2}) + \lambda_4(e_{r-3}-zf_{r-3})\Big].
\end{aligned}$$

One can easily check that

$$\begin{aligned}
\mathcal{B}^{\{5\}} &:= \Big\{(z,\lambda_1,\dots,\lambda_7)\in\mathcal{B}^{(5)} \;\Big|\; x^{(i,j)}_{z,\lambda_1,\dots,\lambda_7}\in\mathcal{P}^{\{5\}}_r,\ i,j=1,2\Big\},\\
\mathcal{A}^{\{5\}} &:= \mathcal{A}^{(5)}\cap\mathcal{B}^{\{5\}}
\end{aligned}$$

are open conull submanifolds of $\mathcal{B}^{(5)}$ and $\mathcal{A}^{(5)}$, respectively. We obtain smooth maps

$$\theta^{5,(i,j)}_r\colon \mathcal{B}^{\{5\}}\to\mathcal{P}^{\{5\}}_r,\ (z,\lambda_1,\dots,\lambda_7)\mapsto x^{(i,j)}_{z,\lambda_1,\dots,\lambda_7},\qquad i,j=1,2,$$

which restrict to smooth maps $\mathcal{A}^{\{5\}}\to\mathcal{P}^{\{5\}}_r$.

Proposition 4.3.3. *(i) Let $r\geq 4$. Then the maps $\theta^{5,(i,j)}_r$, $i,j=1,2$, induce a surjection $\theta^5_r\colon \big(\mathcal{B}^{\{5\}}\big)^{\sqcup 4}\to \mathrm{U}(r+d,r)\backslash\mathcal{P}^{\{5\}}_r$.*

(ii) Let $d\geq 1$. Then the maps $\theta^{5,(1,1)}_3$, $\theta^{5,(1,2)}_3$, $\theta^{5,(2,1)}_3$, and the restriction of $\theta^{5,(2,2)}_3$ to $\mathcal{A}^{\{5\}}$, induce a surjection $\theta^5_3\colon \big(\mathcal{B}^{\{5\}}\big)^{\sqcup 3}\sqcup\mathcal{A}^{\{5\}}\to \mathrm{U}(3+d,3)\backslash\mathcal{P}^{\{5\}}_3$.

(iii) The maps $\theta^{5,(1,1)}_3$, $\theta^{5,(1,2)}_3$, $\theta^{5,(2,1)}_3$ induce a surjection $\theta^5_3\colon \big(\mathcal{B}^{\{5\}}\big)^{\sqcup 3}\to \mathrm{U}(3,3)\backslash\mathcal{P}^{\{5\}}_3$.

Proof. The proof that the orbits of elements $\Big[x^{(1)}_{z,\lambda_1,\lambda_2,\lambda_3}, v\Big]\in\mathcal{P}^{\{5\}}_r$ are parametrized by $\theta^{5,(1,1)}_r$ and $\theta^{5,(1,2)}_r$ is analogous to the proof of Proposition 4.3.1.

Let $v=v_1+v_0$, where $v_1\in\mathrm{span}_{\mathbb{C}}(e_r,f_r,e_{r-1}-zf_{r-1},e_{r-2}-zf_{r-2})$ and $v_0\in\mathrm{span}_{\mathbb{C}}(e_{r-1}+\overline{z}f_{r-1},e_{r-2}+\overline{z}f_{r-2})\oplus V_{r-3}$, satisfy $\Big[x^{(2)}_{z,\lambda_1,\lambda_2,\lambda_3}, v\Big]\in\mathcal{P}^{\{5\}}_r$.

Case 1: $\mathrm{Re}(z)q(v_0)\geq 0$. The proof that the orbits of such elements $\Big[x^{(2)}_{z,\lambda_1,\lambda_2,\lambda_3}, v\Big]\in\mathcal{P}^{\{5\}}_r$ are parametrized by $\theta^{5,(2,1)}_r$ is analogous to the proof of Proposition 4.3.1.

Case 2: $\mathrm{Re}(z)q(v_0)<0$. We distinguish between the three cases above.

(i) Let $r \geq 4$. By Witt's lemma there exists $\psi \in \mathrm{Stab}_{G_{r-1}}(e_{r-1} - zf_{r-1}, e_{r-2} - zf_{r-2})$ such that

$$\psi(v_0) = \sqrt{\frac{q(v_0)}{-2\mathrm{Re}(z)}}(e_{r-3} - zf_{r-3}).$$

Hence there exists $(\lambda_4, \lambda_5, \lambda_6, \lambda_7') \in \mathbb{R}_{\geq 0} \times \mathbb{C}^3$ such that

$$\begin{aligned}&[e_r + \lambda_7' f_r + \lambda_6(e_{r-1} - zf_{r-1}) + \lambda_5(e_{r-2} - zf_{r-2}) + \lambda_4(e_{r-3} - zf_{r-3})]\\ &\in \mathrm{Stab}_{G_r}\left(x^{(2)}_{z,\lambda_1,\lambda_2,\lambda_3}\right).[v].\end{aligned}$$

The condition $q(v) = 0$ implies $\mathrm{Re}(\lambda_7') = \mathrm{Re}(z)(|\lambda_6|^2 + |\lambda_5|^2 + \lambda_4^2)$. Hence there exists $(\lambda_4, \dots, \lambda_7) \in \mathbb{R}_{\geq 0} \times \mathbb{C}^2 \times \mathbb{R}$ such that

$$\begin{aligned}&\big[e_r + \big(\mathrm{Re}(z)\big(|\lambda_6|^2 + |\lambda_5|^2 + \lambda_4^2\big) + \lambda_7 \mathrm{i}\big) f_r + \lambda_6(e_{r-1} - zf_{r-1})\\ &\quad + \lambda_5(e_{r-2} - zf_{r-2}) + \lambda_4(e_{r-3} - zf_{r-3})\big] \in \mathrm{Stab}_{G_r}\left(x^{(2)}_{z,\lambda_1,\lambda_2,\lambda_3}\right).[v].\end{aligned}$$

(ii) Let $r = 3$, $d \geq 1$. Then $z = -1$. By Witt's lemma there exists $\psi \in \mathrm{Stab}_{G_2}(e_2 + f_2, e_1 + f_1)$ such that

$$\psi(v_0) = \sqrt{q(v_0)}h_d.$$

Hence there exists $(\lambda_4, \lambda_5, \lambda_6, \lambda_7') \in \mathbb{R}_{\geq 0} \times \mathbb{C}^3$ such that

$$[e_3 + \lambda_7' f_3 + \lambda_6(e_2 + f_2) + \lambda_5(e_1 + f_1) + \lambda_4 h_d] \in \mathrm{Stab}_{G_3}\left(x^{(2)}_{-1,\lambda_1,\lambda_2,\lambda_3}\right).[v].$$

The condition $q(v) = 0$ implies $\mathrm{Re}(\lambda_7') = -\big(|\lambda_6|^2 + |\lambda_5|^2 + \frac{1}{2}\lambda_4^2\big)$. Hence there exists $(\lambda_4, \dots, \lambda_7) \in \mathbb{R}_{\geq 0} \times \mathbb{C}^2 \times \mathbb{R}$ such that

$$\begin{aligned}&\left[e_3 + \left(-\left(|\lambda_6|^2 + |\lambda_5|^2 + \frac{1}{2}\lambda_4^2\right) + \lambda_7 \mathrm{i}\right) f_3 + \lambda_6(e_2 + f_2) + \lambda_5(e_1 + f_1) + \lambda_4 h_d\right]\\ &\in \mathrm{Stab}_{G_3}\left(x^{(2)}_{-1,\lambda_1,\lambda_2,\lambda_3}\right).[v].\end{aligned}$$

(iii) It is easy to see that $\mathrm{Re}(z)q(v_0) < 0$ is impossible in the case $r = 3$, $d = 0$. □

Step (S3): This is completely analogous to Step (S3) of Subsection 4.3.1.

Let us now set $\mathcal{C}_r^{\{5\}} := \big(\mathcal{B}^{\{5\}}\big)^{\sqcup 4}$ for $r \geq 4$, $\mathcal{C}_3^{\{5\}} := \big(\mathcal{B}^{\{5\}}\big)^{\sqcup 3} \sqcup \mathcal{A}^{\{5\}}$ for $d \geq 1$, and $\mathcal{C}_3^{\{5\}} := \big(\mathcal{B}^{\{5\}}\big)^{\sqcup 3}$ for $d = 0$. Note that for $r \geq 4$ there exists an isomorphism $\mathcal{C}_r^{\{5\}} \cong \mathcal{C}^{(3)} \times \mathbb{C}^5$ of Lebesgue spaces.

Corollary 4.3.4. *The map $\theta_r^5 \colon \mathcal{C}_r^{\{5\}} \to G_r \backslash \mathcal{P}_r^{\{5\}}$ is an isomorphism of Lebesgue spaces for all $r \geq 3$.*

4.4. Parametrization for the Real Orthogonal Groups

Let $r \geq 3$ and $G_r = \mathrm{O}(r + d, r)$. We use the notation of Section 3.3.

4.4.1. Parametrization of Quadruples

Step (S1): Let $z \in \mathcal{C}^{(3)}$. Then we have

$$\mathrm{Stab}_{G_r}(x_z) = \{\lambda m_\psi \mid \lambda \in \mathcal{C}^{(3)},\ \psi \in \mathrm{Stab}_{G_{r-1}}(e_{r-1} - z f_{r-1})\}.$$

Step (S2): Let us write

$$\mathcal{B}^{(4)} := \mathcal{C}^{(3)} \times \mathbb{R}_{\geq 0} \times \mathbb{R}.$$

For all $(z, \lambda_1, \lambda_2) \in \mathcal{B}^{(4)}$ we write

$$\begin{aligned} x^{(1)}_{z,\lambda_1,\lambda_2} := \big[x_z, &e_r + z\big(\lambda_2^2 - \lambda_1^2\big) f_r \\ &+ \lambda_2(e_{r-1} - z f_{r-1}) + \lambda_1(e_{r-1} + z f_{r-1})\big] \in \mathcal{P}^4_r, \\ x^{(2)}_{z,\lambda_1,\lambda_2} := \big[x_z, &e_r + z\big(\lambda_2^2 + \lambda_1^2\big) f_r \\ &+ \lambda_2(e_{r-1} - z f_{r-1}) + \lambda_1(e_{r-2} - z f_{r-2})\big] \in \mathcal{P}^4_r. \end{aligned}$$

One can easily check that

$$\mathcal{B}^{\{4\}} := \Big\{(z, \lambda_1, \lambda_2) \in \mathcal{B}^{(4)} \ \Big|\ x^{(i)}_{z,\lambda_1,\lambda_2} \in \mathcal{P}^{\{4\}}_r,\ i = 1, 2\Big\}$$

is an open conull submanifold of $\mathcal{B}^{(4)}$.

Proposition 4.4.1. *The maps*

$$\mathcal{B}^{\{4\}} \to \mathcal{P}^{\{4\}}_r,\ (z, \lambda_1, \lambda_2) \mapsto x^{(i)}_{z,\lambda_1,\lambda_2}, \qquad i = 1, 2,$$

induce a surjection $\theta^4_r \colon \mathcal{B}^{\{4\}} \sqcup \mathcal{B}^{\{4\}} \to G_r \backslash \mathcal{P}^{\{4\}}_r$.

Proof. Let $v = v_1 + v_0$, where $v_1 \in \mathrm{span}_{\mathbb{R}}(e_r, f_r, e_{r-1} - z f_{r-1})$ and $v_0 \in \mathbb{R}(e_{r-1} + z f_{r-1}) \oplus V_{r-2}$, satisfy $[x_z, v] \in \mathcal{P}^{\{4\}}_r$.

Case 1: $zq(v_0) \geq 0$. By Witt's lemma there exists $\psi \in \mathrm{Stab}_{G_{r-1}}(e_{r-1} - z f_{r-1})$ such that

$$\psi(v_0) = \sqrt{\frac{q(v_0)}{2z}}(e_{r-1} + z f_{r-1}).$$

Hence there exists $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2$ such that

$$[e_r + \lambda_3 f_r + \lambda_2(e_{r-1} - z f_{r-1}) + \lambda_1(e_{r-1} + z f_{r-1})] \in \mathrm{Stab}_{G_r}(x_z).[v].$$

The condition $q(v) = 0$ implies $\lambda_3 = z(\lambda_2^2 - \lambda_1^2)$. Hence there exists $(\lambda_1, \lambda_2) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$ such that

$$\big[e_r + z\big(\lambda_2^2 - \lambda_1^2\big) f_r + \lambda_2(e_{r-1} - z f_{r-1}) + \lambda_1(e_{r-1} + z f_{r-1})\big] \in \mathrm{Stab}_{G_r}(x_z).[v].$$

Case 2: $zq(v_0) < 0$. By Witt's lemma there exists $\psi \in \mathrm{Stab}_{G_{r-1}}(e_{r-1} - z f_{r-1})$ such that

$$\psi(v_0) = \sqrt{\frac{q(v_0)}{-2z}}(e_{r-2} - z f_{r-2}).$$

Hence there exists $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2$ such that

$$[e_r + \lambda_3 f_r + \lambda_2(e_{r-1} - z f_{r-1}) + \lambda_1(e_{r-2} - z f_{r-2})] \in \mathrm{Stab}_{G_r}(x_z).[v].$$

The condition $q(v) = 0$ implies $\lambda_3 = z(\lambda_2^2 + \lambda_1^2)$. Hence there exists $(\lambda_1, \lambda_2) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$ such that

$$\big[e_r + z\big(\lambda_2^2 + \lambda_1^2\big) f_r + \lambda_2(e_{r-1} - z f_{r-1}) + \lambda_1(e_{r-2} - z f_{r-2})\big] \in \mathrm{Stab}_{G_r}(x_z).[v]. \qquad \square$$

Step (S3): It is sufficient to note that

$$\begin{aligned}&\big[e_r + z\big(\lambda_2^2 - \lambda_1^2\big) f_r + \lambda_2(e_{r-1} - z f_{r-1}) + \lambda_1(e_{r-1} + z f_{r-1})\big]\\ &= \big[e_r + z\big(\mu_2^2 - \mu_1^2\big) f_r + \mu_2(e_{r-1} - z f_{r-1}) + \mu_1(e_{r-1} + z f_{r-1})\big]\end{aligned}$$

or

$$\begin{aligned}&\big[e_r + z\big(\lambda_2^2 + \lambda_1^2\big) f_r + \lambda_2(e_{r-1} - z f_{r-1}) + \lambda_1(e_{r-2} - z f_{r-2})\big]\\ &= \big[e_r + z\big(\mu_2^2 + \mu_1^2\big) f_r + \mu_2(e_{r-1} - z f_{r-1}) + \mu_1(e_{r-2} - z f_{r-2})\big]\end{aligned}$$

for some $\psi \in \mathrm{Stab}_{G_{r-1}}(e_{r-1} - z f_{r-1})$ implies $(\lambda_1, \lambda_2) = (\mu_1, \mu_2)$, while

$$\begin{aligned}&\big[e_r + z\big(\lambda_2^2 - \lambda_1^2\big) f_r + \lambda_2(e_{r-1} - z f_{r-1}) + \lambda_1(e_{r-1} + z f_{r-1})\big]\\ &= \big[e_r + z\big(\mu_2^2 + \mu_1^2\big) f_r + \mu_2(e_{r-1} - z f_{r-1}) + \mu_1(e_{r-2} - z f_{r-2})\big]\end{aligned}$$

for some $\psi \in \mathrm{Stab}_{G_{r-1}}(e_{r-1} - z f_{r-1})$ implies $\lambda_1 = 0 = \mu_1$ and $\lambda_2 = \mu_2$.

Let us now set $\mathcal{C}^{\{4\}} := \mathcal{B}^{\{4\}} \sqcup \mathcal{B}^{\{4\}}$. Note that there exists an isomorphism $\mathcal{C}^{\{4\}} \cong \mathcal{C}^{(3)} \times \mathbb{R}^2$ of Lebesgue spaces.

Corollary 4.4.2. *The map $\theta_r^4 \colon \mathcal{C}^{\{4\}} \to G_r \backslash \mathcal{P}_r^{\{4\}}$ is an isomorphism of Lebesgue spaces for all $r \geq 3$.*

4.4.2. Parametrization of Quintuples

Step (S1): Let $(z, \lambda_1, \lambda_2) \in \mathcal{B}^{\{4\}}$. Then we have

$$\begin{aligned}\mathrm{Stab}_{G_r}\Big(x^{(1)}_{z,\lambda_1,\lambda_2}\Big) &= \Big\{\lambda m^{(2)}_\psi \;\Big|\; \lambda \in \mathcal{C}^{(3)},\ \psi \in G_{r-2}\Big\},\\ \mathrm{Stab}_{G_r}\Big(x^{(2)}_{z,\lambda_1,\lambda_2}\Big) &= \big\{\lambda m_\psi \mid \lambda \in \mathcal{C}^{(3)},\ \psi \in \mathrm{Stab}_{G_{r-1}}(e_{r-1} - z f_{r-1}, e_{r-2} - z f_{r-2})\big\}.\end{aligned}$$

Step (S2): Let us write

$$\begin{aligned}\mathcal{B}^{(5)} &:= \mathcal{B}^{\{4\}} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^2,\\ \mathcal{A}^{(5)} &:= \big\{(-1, \lambda_1, \ldots, \lambda_5) \in \mathcal{B}^{(5)}\big\}.\end{aligned}$$

For all $(z, \lambda_1, \ldots, \lambda_5) \in \mathcal{B}^{(5)}$ we write

$$\begin{aligned}x^{(1,1)}_{z,\lambda_1,\ldots,\lambda_5} &:= \Big[x^{(1)}_{z,\lambda_1,\lambda_2}, e_r + z\big(\lambda_5^2 - \lambda_4^2 - \lambda_3^2\big) f_r\\ &\qquad + \lambda_5(e_{r-1} - z f_{r-1}) + \lambda_4(e_{r-1} + z f_{r-1}) + \lambda_3(e_{r-2} + z f_{r-2})\Big] \in \mathcal{P}_r^5,\\ x^{(1,2)}_{z,\lambda_1,\ldots,\lambda_5} &:= \Big[x^{(1)}_{z,\lambda_1,\lambda_2}, e_r + z\big(\lambda_5^2 - \lambda_4^2 + \lambda_3^2\big) f_r\\ &\qquad + \lambda_5(e_{r-1} - z f_{r-1}) + \lambda_4(e_{r-1} + z f_{r-1}) + \lambda_3(e_{r-2} - z f_{r-2})\Big] \in \mathcal{P}_r^5,\\ x^{(2,1)}_{z,\lambda_1,\ldots,\lambda_5} &:= \Big[x^{(2)}_{z,\lambda_1,\lambda_2}, e_r + z\big(\lambda_5^2 + \lambda_4^2 - \lambda_3^2\big) f_r\\ &\qquad + \lambda_5(e_{r-1} - z f_{r-1}) + \lambda_4(e_{r-2} - z f_{r-2}) + \lambda_3(e_{r-2} + z f_{r-2})\Big] \in \mathcal{P}_r^5,\\ x^{(2,2)}_{z,\lambda_1,\ldots,\lambda_5} &:= \Big[x^{(2)}_{z,\lambda_1,\lambda_2}, e_r + z\big(\lambda_5^2 + \lambda_4^2 + \lambda_3^2\big) f_r\\ &\qquad + \lambda_5(e_{r-1} - z f_{r-1}) + \lambda_4(e_{r-2} - z f_{r-2}) + \lambda_3(e_{r-3} - z f_{r-3})\Big] \in \mathcal{P}_r^5.\end{aligned}$$

One can easily check that

$$\mathcal{B}^{\{5\}} := \left\{(z,\lambda_1,\dots,\lambda_5)\in\mathcal{B}^{(5)} \,\middle|\, x^{(i,j)}_{z,\lambda_1,\dots,\lambda_5}\in\mathcal{P}^{\{5\}}_r,\ i,j=1,2\right\},$$
$$\mathcal{A}^{\{5\}} := \mathcal{A}^{(5)}\cap\mathcal{B}^{\{5\}}$$

are open conull submanifolds of $\mathcal{B}^{(5)}$ and $\mathcal{A}^{(5)}$, respectively. We obtain smooth maps

$$\theta^{5,(i,j)}_r\colon \mathcal{B}^{\{5\}}\to\mathcal{P}^{\{5\}}_r,\ (z,\lambda_1,\dots,\lambda_5)\mapsto x^{(i,j)}_{z,\lambda_1,\dots,\lambda_5},\qquad i,j=1,2,$$

which restrict to smooth maps $\mathcal{A}^{\{5\}}\to\mathcal{P}^{\{5\}}_r$.

Proposition 4.4.3. *(i) Let $r\geq 4$. Then the maps $\theta^{5,(i,j)}_r$, $i,j=1,2$, induce a surjection $\theta^5_r\colon \left(\mathcal{B}^{\{5\}}\right)^{\sqcup 4}\to \mathrm{O}(r+d,r)\backslash\mathcal{P}^{\{5\}}_r$.*

(ii) Let $d\geq 1$. Then the maps $\theta^{5,(1,1)}_3$, $\theta^{5,(1,2)}_3$, $\theta^{5,(2,1)}_3$, and the restriction of $\theta^{5,(2,2)}_3$ to $\mathcal{A}^{\{5\}}$, induce a surjection $\theta^5_3\colon \left(\mathcal{B}^{\{5\}}\right)^{\sqcup 3}\sqcup\mathcal{A}^{\{5\}}\to\mathrm{O}(3+d,3)\backslash\mathcal{P}^{\{5\}}_3$.

(iii) The maps $\theta^{5,(1,1)}_3$, $\theta^{5,(1,2)}_3$, $\theta^{5,(2,1)}_3$ induce a surjection $\theta^5_3\colon\left(\mathcal{B}^{\{5\}}\right)^{\sqcup 3}\to\mathrm{O}(3,3)\backslash\mathcal{P}^{\{5\}}_3$.

Proof. The proof that the orbits of elements $\left[x^{(1)}_{z,\lambda_1,\lambda_2},v\right]\in\mathcal{P}^{\{5\}}_r$ are parametrized by $\theta^{5,(1,1)}_r$ and $\theta^{5,(1,2)}_r$ is analogous to the proof of Proposition 4.4.1.

Let $v=v_1+v_0$, where $v_1\in\operatorname{span}_{\mathbb{R}}(e_r,f_r,e_{r-1}-zf_{r-1},e_{r-2}-zf_{r-2})$ and $v_0\in\operatorname{span}_{\mathbb{R}}(e_{r-1}+zf_{r-1},e_{r-2}+zf_{r-2})\oplus V_{r-3}$, satisfy $\left[x^{(2)}_{z,\lambda_1,\lambda_2},v\right]\in\mathcal{P}^{\{5\}}_r$.

Case 1: $zq(v_0)\geq 0$. The proof that the orbits of such elements $\left[x^{(2)}_{z,\lambda_1,\lambda_2},v\right]\in\mathcal{P}^{\{5\}}_r$ are parametrized by $\theta^{5,(2,1)}_r$ is analogous to the proof of Proposition 4.4.1.

Case 2: $zq(v_0)<0$. We distinguish between the three cases above.

(i) Let $r\geq 4$. By Witt's lemma there exists $\psi\in\operatorname{Stab}_{G_{r-1}}(e_{r-1}-zf_{r-1},e_{r-2}-zf_{r-2})$ such that

$$\psi(v_0)=\sqrt{\frac{q(v_0)}{-2z}}(e_{r-3}-zf_{r-3}).$$

Hence there exists $(\lambda_3,\dots,\lambda_6)\in\mathbb{R}_{\geq 0}\times\mathbb{R}^3$ such that

$$\begin{aligned}&[e_r+\lambda_6 f_r+\lambda_5(e_{r-1}-zf_{r-1})+\lambda_4(e_{r-2}-zf_{r-2})+\lambda_3(e_{r-3}-zf_{r-3})]\\&\in\operatorname{Stab}_{G_r}\left(x^{(2)}_{z,\lambda_1,\lambda_2}\right).[v].\end{aligned}$$

The condition $q(v)=0$ implies $\lambda_6=z(\lambda_5^2+\lambda_4^2+\lambda_3^2)$. Hence there exists $(\lambda_3,\lambda_4,\lambda_5)\in\mathbb{R}_{\geq 0}\times\mathbb{R}^2$ such that

$$\begin{aligned}&\big[e_r+z\big(\lambda_5^2+\lambda_4^2+\lambda_3^2\big)f_r+\lambda_5(e_{r-1}-zf_{r-1})\\&\quad+\lambda_4(e_{r-2}-zf_{r-2})+\lambda_3(e_{r-3}-zf_{r-3})\big]\in\operatorname{Stab}_{G_r}\left(x^{(2)}_{z,\lambda_1,\lambda_2}\right).[v].\end{aligned}$$

(ii) Let $r=3$, $d\geq 1$. Then $z=-1$. By Witt's lemma there exists $\psi\in\operatorname{Stab}_{G_2}(e_2+f_2,e_1+f_1)$ such that

$$\psi(v_0)=\sqrt{q(v_0)}h_d.$$

Hence there exists $(\lambda_3,\dots,\lambda_6) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^3$ such that

$$[e_3 + \lambda_6 f_3 + \lambda_5(e_2+f_2) + \lambda_4(e_1+f_1) + \lambda_3 h_d] \in \operatorname{Stab}_{G_3}\left(x^{(2)}_{-1,\lambda_1,\lambda_2}\right).[v].$$

The condition $q(v) = 0$ implies $\lambda_6 = -\left(\lambda_5^2 + \lambda_4^2 + \frac{1}{2}\lambda_3^2\right)$. Hence there exists $(\lambda_3,\lambda_4,\lambda_5) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^2$ such that

$$\left[e_3 - \left(\lambda_5^2 + \lambda_4^2 + \frac{1}{2}\lambda_3^2\right) f_3 + \lambda_5(e_2+f_2) + \lambda_4(e_1+f_1) + \lambda_3 h_d\right]$$
$$\in \operatorname{Stab}_{G_3}\left(x^{(2)}_{-1,\lambda_1,\lambda_2}\right).[v].$$

(iii) It is easy to see that $zq(v_0) < 0$ is impossible in the case $r = 3$, $d = 0$. □

Step (S3): This is completely analogous to Step (S3) of Subsection 4.4.1.

Let us now set $\mathcal{C}_r^{\{5\}} := \left(\mathcal{B}^{\{5\}}\right)^{\sqcup 4}$ for $r \geq 4$, $\mathcal{C}_3^{\{5\}} := \left(\mathcal{B}^{\{5\}}\right)^{\sqcup 3} \sqcup \mathcal{A}^{\{5\}}$ for $d \geq 1$, and $\mathcal{C}_3^{\{5\}} := \left(\mathcal{B}^{\{5\}}\right)^{\sqcup 3}$ for $d = 0$. Note that for $r \geq 4$ there exists an isomorphism $\mathcal{C}_r^{\{5\}} \cong \mathcal{C}^{(3)} \times \mathbb{R}^5$ of Lebesgue spaces.

Corollary 4.4.4. *The map $\theta_r^5 \colon \mathcal{C}_r^{\{5\}} \to G_r\backslash\mathcal{P}_r^{\{5\}}$ is an isomorphism of Lebesgue spaces for all $r \geq 3$.*

4.5. Parametrization of Arbitrary Tuples

In this section we extend the parametrization results of the previous sections to parametrizations of $G_r\backslash\mathcal{P}_r^k$ for arbitrarily high k. Let us now explain the proofs of parts (iii) and (iv) of Theorem 3.1.5.

For all $k \geq 3$ we set

$$m_k := \sum_{i=2}^{k-2} i = \frac{k(k-3)}{2}$$

and for all $k \geq 4$ we set

$$\mathcal{C}^{(k)} := \mathcal{C}^{(3)} \times \mathbb{K}^{m_k}.$$

From now on let $k \geq 3$. Equipping $\mathcal{C}^{(k)}$ with its Lebesgue measure class and the trivial G_r-action turns the former into a Lebesgue G_r-space.

The Case $G_r = \mathrm{Sp}(2r,\mathbb{K})$: Inductively, we obtain an isomorphism

$$\theta_r^k \colon \mathcal{C}^{(k)} \xrightarrow{\cong} G_r\backslash\mathcal{P}_r^k,\ (z,\lambda_1,\dots,\lambda_{m_k}) \mapsto x_{z,\lambda_1,\dots,\lambda_{m_k}}$$

of Lebesgue spaces for all $r \geq \left\lfloor \frac{k}{2} \right\rfloor$. All stabilizers H_{k-1} of a point on a G_r-invariant open conull submanifold of $\mathcal{P}_r^k$ are *identical*. Indeed, we have

$$H_{2k-1} = \left\{\lambda m_\psi^{(k)} \;\middle|\; \lambda \in \{\pm 1\},\ \psi \in G_{r-k}\right\},$$
$$H_{2k} = \left\{\lambda m_{m_\psi n_w}^{(k)} \;\middle|\; \lambda \in \{\pm 1\},\ \psi \in G_{r-k-1},\ w \in W_{r-k}\right\}.$$

Using Lemma 3.3.1, we obtain an isomorphism

$$(G_r/H_{k-1}) \times \mathcal{C}^{(k)} \xrightarrow{\cong} \mathcal{P}_r^k,\ (gH_{k-1}, (z, \lambda_1, \dots, \lambda_{m_k})) \mapsto g.x_{z,\lambda_1,\dots,\lambda_{m_k}}$$

of Lebesgue G_r-spaces.

The Case $G_r = \mathrm{O}(2r+d, \mathbb{C})$: Inductively, we obtain an isomorphism

$$\theta_r^k \colon \mathcal{C}^{(k)} \xrightarrow{\cong} G_r \backslash \mathcal{P}_r^k,\ (1, \lambda_1, \dots, \lambda_{m_k}) \mapsto x_{\lambda_1,\dots,\lambda_{m_k}}$$

of Lebesgue spaces if $\dim_{\mathbb{C}}(V_r) \geq k$, i.e. for all $r \geq \frac{1}{2}(k-d)$. All stabilizers H_{k-1} of a point on a G_r-invariant open conull submanifold of $\mathcal{P}_r^k$ are *identical*. Indeed, we have

$$\begin{aligned} H_{2k-1} &= \left\{ \lambda m_\psi^{(k)} \;\middle|\; \lambda \in \{\pm 1\},\ \psi \in G_{r-k} \right\}, \\ H_{2k} &= \left\{ \lambda m_\psi^{(k)} \;\middle|\; \lambda \in \{\pm 1\},\ \psi \in \mathrm{Stab}_{G_{r-k}}(e_{r-k} - f_{r-k}) \right\}. \end{aligned}$$

Using Lemma 3.3.1, we obtain an isomorphism

$$(G_r/H_{k-1}) \times \mathcal{C}^{(k)} \xrightarrow{\cong} \mathcal{P}_r^k,\ (gH_{k-1}, (1, \lambda_1, \dots, \lambda_{m_k})) \mapsto g.x_{\lambda_1,\dots,\lambda_{m_k}}$$

of Lebesgue G_r-spaces.

The Case $G_r = \mathrm{O}(r+d, r)$: Inductively, we obtain an isomorphism

$$\theta_r^k \colon \mathcal{C}^{(k)} \xrightarrow{\cong} G_r \backslash \mathcal{P}_r^k$$

of Lebesgue spaces for all $r \geq k-1$. There exist $c_k \leq 2^{k-2}$ different stabilizers $H_{k-1,i}$, $i = 1, \dots, c_k$, of a point on a G_r-invariant open conull submanifold of $\mathcal{P}_r^k$. Let us decompose $\mathcal{C}^{(k)} \cong \bigsqcup_{i=1}^{c_k} \mathcal{C}_i^{(k)}$ accordingly; using Lemma 3.3.1, we obtain an isomorphism

$$\bigsqcup_{i=1}^{c_k} \Big((G_r/H_{k-1,i}) \times \mathcal{C}_i^{(k)} \Big) \to \mathcal{P}_r^k$$

of Lebesgue G_r-spaces.

The Case $G_r = \mathrm{U}(r+d, r)$: Inductively, we obtain an isomorphism

$$\theta_r^k \colon \mathcal{C}^{(k)} \xrightarrow{\cong} G_r \backslash \mathcal{P}_r^k$$

of Lebesgue spaces for all $r \geq k-1$.

Lemma 4.5.1. *There exist $c_k \leq 2^{k-2}$ conjugacy classes $[H_{k-1,i}]_{i=1,\dots,c_k}$ of stabilizers of a point on a G_r-invariant open conull submanifold of $\mathcal{P}_r^k$. Decomposing $\mathcal{C}^{(k)} \cong \bigsqcup_{i=1}^{c_k} \mathcal{C}_i^{(k)}$ accordingly, we obtain an isomorphism*

$$\bigsqcup_{i=1}^{c_k} \Big((G_r/H_{k-1,i}) \times \mathcal{C}_i^{(k)} \Big) \to \mathcal{P}_r^k$$

of Lebesgue G_r-spaces.

Proof. We spell out the proof in the case $k = 3$; the case $k > 3$ works analogously. We write $H_2^{(z)} = \mathrm{Stab}_{G_r}(x_z)$, $z \in \mathcal{C}^{(3)}$. Note that $H_2^{(z)} \cong H_2^{(z')}$ if and only if $\mathrm{sgn}(\mathrm{Re}(z)) = \mathrm{sgn}(\mathrm{Re}(z'))$. We write

$$\begin{aligned}\mathcal{C}_1^{(3)} &= \mathcal{C}^{(3)} \cap \mathrm{Re}^{-1}((0,1]),\\ \mathcal{C}_2^{(3)} &= \mathcal{C}^{(3)} \cap \mathrm{Re}^{-1}([-1,0)).\end{aligned}$$

For $z \in \mathcal{C}_1^{(3)}$ we define $\varphi_z \in G_r$ by setting

$$\begin{aligned}\varphi_z(e_{r-1} - f_{r-1}) &= \frac{1}{\sqrt{\mathrm{Re}(z)}}(e_{r-1} - z f_{r-1}),\\ \varphi_z(e_{r-1} + f_{r-1}) &= \frac{1}{\sqrt{\mathrm{Re}(z)}}(e_{r-1} + \sigma(z) f_{r-1}),\\ \varphi_z|_{\{e_{r-1}, f_{r-1}\}^\perp} &= \mathrm{id}.\end{aligned}$$

Then we have $\varphi_z^{-1} H_2^{(z)} \varphi_z = H_2^{(1)}$. For $z \in \mathcal{C}_2^{(3)}$ we define $\varphi_z \in G_r$ by setting

$$\begin{aligned}\varphi_z(e_{r-1} + f_{r-1}) &= \frac{1}{\sqrt{-\mathrm{Re}(z)}}(e_{r-1} - z f_{r-1}),\\ \varphi_z(e_{r-1} - f_{r-1}) &= \frac{1}{\sqrt{-\mathrm{Re}(z)}}(e_{r-1} + \sigma(z) f_{r-1}),\\ \varphi_z|_{\{e_{r-1}, f_{r-1}\}^\perp} &= \mathrm{id}.\end{aligned}$$

Then we have $\varphi_z^{-1} H_2^{(z)} \varphi_z = H_2^{(-1)}$. Using Lemma 3.3.1, the smooth maps

$$\begin{aligned}\left(G_r/H_2^{(1)}\right) \times \mathcal{C}_1^{(3)} \to \mathcal{P}_r^3,\ \left(gH_2^{(1)}, z\right) &\mapsto g\varphi_z^{-1}.x_z,\\ \left(G_r/H_2^{(-1)}\right) \times \mathcal{C}_2^{(3)} \to \mathcal{P}_r^3,\ \left(gH_2^{(-1)}, z\right) &\mapsto g\varphi_z^{-1}.x_z\end{aligned}$$

induce an isomorphism

$$\left(\left(G_r/H_2^{(1)}\right) \times \mathcal{C}_1^{(3)}\right) \sqcup \left(\left(G_r/H_2^{(-1)}\right) \times \mathcal{C}_2^{(3)}\right) \xrightarrow{\cong} \mathcal{P}_r^3$$

of Lebesgue G_r-spaces. □

This completes the proof of Theorem 3.1.5.

The General Case: Let us return to the general case of G_r being the automorphism group of a classical space (V_r, ω). Our last goal for this chapter is the proof of Theorem 3.1.6.

By part (iii) of Theorem 3.1.5, we have an isomorphism

$$\kappa_r^k := \theta_{r+1}^k \circ \left(\theta_r^k\right)^{-1} \colon G_r \backslash \mathcal{P}_r^k \to G_{r+1} \backslash \mathcal{P}_{r+1}^k$$

of Lebesgue spaces for all $r \geq r_k$. Let $i_r^k \colon \mathcal{P}_r^k \to \mathcal{P}_{r+1}^k$ denote the inclusion induced by the isometric embedding

$$V_r \hookrightarrow V_{r+1},\ e_i \mapsto e_{i+1},\ f_i \mapsto f_{i+1},\ h_k \mapsto h_k.$$

Lemma 4.5.2. *The diagram*

$$\begin{array}{ccc} \mathcal{P}_r^k & \xrightarrow{i_r^k} & \mathcal{P}_{r+1}^k \\ \big\downarrow & & \big\downarrow \\ G_r\backslash\mathcal{P}_r^k & \xrightarrow[\kappa_r^k]{} & G_{r+1}\backslash\mathcal{P}_{r+1}^k \end{array}$$

commutes for all $r \geq r'_k$.

Proof. Directly follows from the explicit description of θ_r^k. □

Now let $G_r = \mathrm{Sp}(2r, \mathbb{K})$. Then the isometric embedding

$$V_r \hookrightarrow V_{r+1},\ (\alpha_r, \ldots, \alpha_1, \beta_1, \ldots, \beta_r) \mapsto (\alpha_r, \ldots, \alpha_1, \alpha_r, 0, \beta_1, \ldots, \beta_r)$$

induces an inclusion $j_r^k \colon \mathcal{P}_r^k \hookrightarrow \mathcal{P}_{r+1}^k$.

Lemma 4.5.3. *Let* $G_r = \mathrm{Sp}(2r, \mathbb{K})$*. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{P}_r^{2r+1} & \xrightarrow{j_r^{2r+1}} & \mathcal{P}_{r+1}^{2r+1} \\ \big\downarrow & & \big\downarrow \\ G_r\backslash\mathcal{P}_r^{2r+1} & \xrightarrow[\kappa_r^{2r+1}]{} & G_{r+1}\backslash\mathcal{P}_{r+1}^{2r+1} \end{array}$$

Proof. Directly follows from the explicit description of θ_r^{2r+1}, see step (S2) of Subsection 4.1.1. □

Corollary 4.5.4. *The diagram*

$$\begin{array}{ccccc} L^\infty\big(\mathcal{P}_{r+1}^k\big)^{G_{r+1}} & \xrightarrow{d_{r+1}^{k-1}} & L^\infty\big(\mathcal{P}_{r+1}^{k+1}\big)^{G_{r+1}} & \xrightarrow{d_{r+1}^{k}} & L^\infty\big(\mathcal{P}_{r+1}^{k+2}\big)^{G_{r+1}} \\ {\scriptstyle (\kappa_r^k)^*}\big\downarrow & & \big\downarrow{\scriptstyle (\kappa_r^{k+1})^*} & & \big\downarrow{\scriptstyle (\kappa_r^{k+2})^*} \\ L^\infty\big(\mathcal{P}_r^k\big)^{G_r} & \xrightarrow[d_r^{k-1}]{} & L^\infty\big(\mathcal{P}_r^{k+1}\big)^{G_r} & \xrightarrow[d_r^{k}]{} & L^\infty\big(\mathcal{P}_r^{k+2}\big)^{G_r} \end{array}$$

commutes in the following cases:

(i) in the case $r \geq r'_{k+2}$*;*

(ii) in the case $r = 2$ *and* $k = 3$ *for* $G_r = \mathrm{Sp}(2r, \mathbb{K})$.

Proof. Part (i) follows immediately from Lemma 4.5.2. For part (ii) we use Lemma 4.5.3 and

$$G_3.i_2^4[v_0, \ldots, v_3] = G_3.j_2^4[v_0, \ldots, v_3] \qquad \Big([v_0, \ldots, v_3] \in \mathcal{P}_2^{\{4\}}\Big),$$

which holds by Proposition 3.5.4, since $i_2^4[v_0, \ldots, v_3]$ and $j_2^4[v_0, \ldots, v_3]$ have the same angular Cartan invariant and cross-ratios. □

Proof of Theorem 3.1.6. Corollary 4.5.4 yields the relevant chain isomorphisms. □

We have now proved all the theorems listed in Section 3.1.

Part III.

Bounded-Cohomological Stability

5. Spectral Sequences

5.1. Spectral Sequences Associated to Double Complexes

Spectral sequences, introduced by Leray [93], [94], are powerful computational tools in geometry, topology, and algebra. A basic reference in this subject is the book by McCleary [97], whose notation we adopt in the following. Our goal is to prove an extremely useful vanishing criterion for differentials on higher pages of spectral sequences associated to double complexes (Lemma 5.1.6). We exclusively use the real numbers $\mathbb{R}$ as a base field.

A *(descending) filtration* $F = (F^n A)_{n\in\mathbb{N}}$ of a (real) vector space A consists of subspaces

$$\ldots \subset F^2A \subset F^1A \subset F^0A = A.$$

A *filtered differential graded module* $(F, A^n, d^n)_{n\in\mathbb{N}}$ consists of vector spaces A^n, linear maps $d^n\colon A^n \to A^{n+1}$ satisfying $d^n \circ d^{n-1} = 0$, and a filtration F of $A = \bigoplus_{n\in\mathbb{N}} A^n$ such that d^n restricts to a map $d^n\colon F^pA^n \to F^pA^{n+1}$ for all $p \in \mathbb{N}$. Let $(F, A^n, d^n)_{n\in\mathbb{N}}$ be a filtered differential graded module. We write $\mathrm{H}^n(A, d) := \ker(d^n)/\mathrm{im}(d^{n-1})$, and say that $(F, A^n, d^n)_{n\in\mathbb{N}}$ is *bounded* if for all $n \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that $F^pA^n = 0$.

A *(cohomological) spectral sequence* $(\mathrm{E}_r^{p,q}, \mathrm{d}_r^{p,q})_{r,p,q\in\mathbb{N}}$ consists of vector spaces $\mathrm{E}_r^{p,q}$ and linear maps $\mathrm{d}_r^{p,q}\colon \mathrm{E}_r^{p,q} \to \mathrm{E}_r^{p+r,q-r+1}$ such that

$$\mathrm{E}_{r+1}^{p,q} \cong \ker(\mathrm{d}_r^{p,q})/\mathrm{im}\big(\mathrm{d}_r^{p-r,q+r-1}\big).$$

Let $(\mathrm{E}_r^{p,q}, \mathrm{d}_r^{p,q})_{r,p,q\in\mathbb{N}}$ be a spectral sequence. The *limit terms* of $(\mathrm{E}_r^{p,q}, \mathrm{d}_r^{p,q})_{r,p,q\in\mathbb{N}}$ are defined by

$$\mathrm{E}_\infty^{p,q} := \left(\bigcap_{r\in\mathbb{N}} \ker(\mathrm{d}_r^{p,q})\right)\Big/\left(\bigcup_{r\in\mathbb{N}} \mathrm{im}\big(\mathrm{d}_r^{p-r,q+r-1}\big)\right).$$

We say that $(\mathrm{E}_r^{p,q}, \mathrm{d}_r^{p,q})_{r,p,q\in\mathbb{N}}$ *converges* to a graded vector space $(H^n)_{n\in\mathbb{N}}$ if there exists a filtration F of $H = \bigoplus_{n\in\mathbb{N}} H^n$ such that $\mathrm{E}_\infty^{p,q} \cong F^pH^{p+q}/F^{p+1}H^{p+q}$, where we set $F^pH^n := F^pH \cap H^n$.

Theorem 5.1.1 (see [97, Theorem 2.6])**.** *Let $(F, A^n, d^n)_{n\in\mathbb{N}}$ be a filtered differential graded module. Then there exists a spectral sequence $(\mathrm{E}_r^{p,q}, \mathrm{d}_r^{p,q})_{r,p,q\in\mathbb{N}}$ such that*

$$\mathrm{E}_1^{p,q} \cong \mathrm{H}^{p+q}\big(0 \to F^pA^0/F^{p+1}A^0 \to F^pA^1/F^{p+1}A^1 \to \ldots\big).$$

If $(F, A^n, d^n)_{n\in\mathbb{N}}$ is bounded, then $(\mathrm{E}_r^{p,q}, \mathrm{d}_r^{p,q})_{r,p,q\in\mathbb{N}}$ converges to $(\mathrm{H}^n(A, d))_{n\in\mathbb{N}}$.

Sketch of Proof. We define

$$\begin{aligned} \mathrm{Z}_r^{p,q} &:= F^p A^{p+q} \cap \left(d^{p+q}\right)^{-1}\left(F^{p+r} A^{p+q+1}\right), \\ \mathrm{B}_r^{p,q} &:= F^p A^{p+q} \cap d^{p+q-1}\left(F^{p-r} A^{p+q-1}\right), \\ \mathrm{E}_r^{p,q} &:= \mathrm{Z}_r^{p,q} / \left(\mathrm{Z}_{r-1}^{p+1,q-1} + \mathrm{B}_{r-1}^{p,q}\right). \end{aligned}$$

Then d^{p+q} restricts to a map $\mathrm{Z}_r^{p,q} \to \mathrm{Z}_r^{p+r,q-r+1}$ and induces a map $\mathrm{d}_r^{p,q} \colon \mathrm{E}_r^{p,q} \to \mathrm{E}_r^{p+r,q-r+1}$ for all $r \in \mathbb{N}$. One can check that the composition

$$\mathrm{Z}_{r+1}^{p,q} \hookrightarrow \mathrm{Z}_r^{p,q} \twoheadrightarrow \mathrm{E}_r^{p,q}$$

has image $\ker(\mathrm{d}_r^{p,q})$ and that the surjective map $\mathrm{Z}_{r+1}^{p,q} \twoheadrightarrow \ker(\mathrm{d}_r^{p,q})/\mathrm{im}\left(\mathrm{d}_r^{p-r,q+r-1}\right)$ has kernel $\mathrm{Z}_r^{p+1,q-1} + \mathrm{B}_r^{p,q}$. Hence $(\mathrm{E}_r^{p,q}, \mathrm{d}_r^{p,q})_{r,p,q\in\mathbb{N}}$ is a spectral sequence. The rest is a standard verification. □

In the following we will keep using the notation of the sketch of proof above.

Lemma 5.1.2. *Let $(F, A^n, d^n)_{n\in\mathbb{N}}$ be a filtered differential graded module with associated spectral sequence $(\mathrm{E}_r^{p,q}, \mathrm{d}_r^{p,q})_{r,p,q\in\mathbb{N}}$. Then the vanishing of $\mathrm{d}_r^{p,q}$ is equivalent to $\mathrm{B}_r^{p+r,q-r+1} \subset \mathrm{Z}_{r-2}^{p+r+1,q-r} + \mathrm{B}_{r-1}^{p+r,q-r+1}$.*

Proof. Let ${}^{\mathrm{I}}\mathrm{d}_r^{p,q}$ be the unique map such that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{E}_r^{p,q} & \xrightarrow{\mathrm{d}_r^{p,q}} & \mathrm{E}_r^{p+r,q-r+1} \\ \cong\downarrow & & \downarrow\cong \\ \ker\left(\mathrm{d}_{r-1}^{p,q}\right)/\mathrm{im}\left(\mathrm{d}_{r-1}^{p-r+1,q+r-2}\right) & \xrightarrow[{}^{\mathrm{I}}\mathrm{d}_r^{p,q}]{} & \ker\left(\mathrm{d}_{r-1}^{p+r,q-r+1}\right)/\mathrm{im}\left(\mathrm{d}_{r-1}^{p+1,q-1}\right) \end{array}$$

A direct verification shows that ${}^{\mathrm{I}}\mathrm{d}_r^{p,q}$ vanishes if and only if $\mathrm{B}_r^{p+r,q-r+1} \subset \mathrm{Z}_{r-2}^{p+r+1,q-r} + \mathrm{B}_{r-1}^{p+r,q-r+1}$. □

Definition 5.1.3. A *double complex* $(A^{p,q}, d_{\mathrm{H}}^{p,q}, d_{\mathrm{V}}^{p,q})_{p,q\in\mathbb{N}}$ consists of vector spaces $A^{p,q}$ and linear maps $d_{\mathrm{H}}^{p,q} \colon A^{p,q} \to A^{p+1,q}$ and $d_{\mathrm{V}}^{p,q} \colon A^{p,q} \to A^{p,q+1}$ such that

$$\begin{aligned} d_{\mathrm{H}}^{p+1,q} \circ d_{\mathrm{H}}^{p,q} &= 0, \\ d_{\mathrm{V}}^{p,q+1} \circ d_{\mathrm{V}}^{p,q} &= 0, \\ d_{\mathrm{H}}^{p,q+1} \circ d_{\mathrm{V}}^{p,q} + d_{\mathrm{V}}^{p+1,q} \circ d_{\mathrm{H}}^{p,q} &= 0. \end{aligned}$$

Let $(A^{p,q}, d_{\mathrm{H}}^{p,q}, d_{\mathrm{V}}^{p,q})_{p,q\in\mathbb{N}}$ be a double complex. We write $\mathrm{H}_{\mathrm{V}}^{p,q} := \ker(\mathrm{d}_{\mathrm{V}}^{p,q})/\mathrm{im}\left(\mathrm{d}_{\mathrm{V}}^{p,q-1}\right)$ and $\mathrm{H}_{\mathrm{H}}^{p,q} := \ker(\mathrm{d}_{\mathrm{H}}^{p,q})/\mathrm{im}\left(\mathrm{d}_{\mathrm{H}}^{p-1,q}\right)$. Note that $d_{\mathrm{H}}^{p,q}$ induces a map $\overline{d_{\mathrm{H}}^{p,q}} \colon \mathrm{H}_{\mathrm{V}}^{p,q} \to \mathrm{H}_{\mathrm{V}}^{p+1,q}$ and that $d_{\mathrm{V}}^{p,q}$ induces a map $\overline{d_{\mathrm{V}}^{p,q}} \colon \mathrm{H}_{\mathrm{H}}^{p,q} \to \mathrm{H}_{\mathrm{H}}^{p,q+1}$.

Setting $A^n := \bigoplus_{p+q=n} A^{p,q}$, $F^p A^n := \bigoplus_{r\geq p} A^{r,n-r}$ and $d_{\mathrm{T}}^{p+q}|_{A^{p,q}} := d_{\mathrm{H}}^{p,q} + d_{\mathrm{V}}^{p,q}$ turns $(F, A^n, d_{\mathrm{T}}^n)_{n\in\mathbb{N}}$ into a filtered differential graded module.

Theorem 5.1.4 (see [97, Theorem 2.15]). *Let* $(A^{p,q}, d_{\mathrm{H}}^{p,q}, d_{\mathrm{V}}^{p,q})_{p,q\in\mathbb{N}}$ *be a double complex with associated filtered differential graded module* $(F, A^n, d_{\mathrm{T}}^n)_{n\in\mathbb{N}}$ *and associated spectral sequence* $(\mathrm{E}_r^{p,q}, \mathrm{d}_r^{p,q})_{r,p,q\in\mathbb{N}}$. *Then we have the following commutative diagrams:*

$$\begin{array}{ccc} \mathrm{E}_0^{p,q} & \xrightarrow{\mathrm{d}_0^{p,q}} & \mathrm{E}_0^{p,q+1} \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ A^{p,q} & \xrightarrow[d_{\mathrm{V}}^{p,q}]{} & A^{p,q+1} \end{array} \qquad \begin{array}{ccc} \mathrm{E}_1^{p,q} & \xrightarrow{\mathrm{d}_1^{p,q}} & \mathrm{E}_1^{p+1,q} \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \mathrm{H}_{\mathrm{V}}^{p,q} & \xrightarrow[\overline{d_{\mathrm{H}}^{p,q}}]{} & \mathrm{H}_{\mathrm{V}}^{p+1,q} \end{array}$$

Furthermore, $(\mathrm{E}_r^{p,q}, \mathrm{d}_r^{p,q})_{r,p,q\in\mathbb{N}}$ *converges to* $(\mathrm{H}^n(A, d_{\mathrm{T}}))_{n\in\mathbb{N}}$.

Remark 5.1.5. If $(A^{p,q}, d_{\mathrm{H}}^{p,q}, d_{\mathrm{V}}^{p,q})_{p,q\in\mathbb{N}}$ is a double complex, then we can also apply Theorem 5.1.4 to its *transpose* double complex $(A^{q,p}, d_{\mathrm{V}}^{q,p}, d_{\mathrm{H}}^{q,p})_{p,q\in\mathbb{N}}$. This yields a spectral sequence $\left({}^{\top}\mathrm{E}_r^{p,q}, {}^{\top}\mathrm{d}_r^{p,q}\right)_{r,p,q\in\mathbb{N}}$ such that the following diagrams commute:

$$\begin{array}{ccc} {}^{\top}\mathrm{E}_0^{p,q} & \xrightarrow{{}^{\top}\mathrm{d}_0^{p,q}} & {}^{\top}\mathrm{E}_0^{p,q+1} \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ A^{q,p} & \xrightarrow[d_{\mathrm{H}}^{q,p}]{} & A^{q+1,p} \end{array} \qquad \begin{array}{ccc} {}^{\top}\mathrm{E}_1^{p,q} & \xrightarrow{{}^{\top}\mathrm{d}_1^{p,q}} & {}^{\top}\mathrm{E}_1^{p+1,q} \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \mathrm{H}_{\mathrm{H}}^{q,p} & \xrightarrow[\overline{d_{\mathrm{V}}^{q,p}}]{} & \mathrm{H}_{\mathrm{H}}^{q,p+1} \end{array}$$

Note that $\left({}^{\top}\mathrm{E}_r^{p,q}, {}^{\top}\mathrm{d}_r^{p,q}\right)_{r,p,q\in\mathbb{N}}$ also converges to $(\mathrm{H}^n(A, d_{\mathrm{T}}))_{n\in\mathbb{N}}$.

The following lemma is non-standard and extremely useful; it will be used in Chapter 6 to prove the main theorems of this thesis. Its proof is, essentially, an extension of [104, Proposition 6.3].

Lemma 5.1.6. *Let* $(A^{p,q}, d_{\mathrm{H}}^{p,q}, d_{\mathrm{V}}^{p,q})_{p,q\in\mathbb{N}}$ *be a double complex with associated filtered differential graded module* $(F, A^n, d_{\mathrm{T}}^n)_{n\in\mathbb{N}}$ *and associated spectral sequence* $(\mathrm{E}_r^{p,q}, \mathrm{d}_r^{p,q})_{r,p,q\in\mathbb{N}}$. *Then a sufficient criterion for the vanishing of* $\mathrm{d}_r^{p,q}$, $r \geq 2$, *is that the class of* $d_{\mathrm{H}}^{p+r-1,q-r+1}(a)$ *vanishes in* $\mathrm{H}_{\mathrm{V}}^{p+r,q-r+1}$ *for all* $a \in A^{p+r-1,q-r+1}$.

Proof. By Lemma 5.1.2, we have to show that $\mathrm{B}_r^{p+r,q-r+1} \subset \mathrm{Z}_{r-2}^{p+r+1,q-r} + \mathrm{B}_{r-1}^{p+r,q-r+1}$. We will now drop the upper indices of $d_{\mathrm{H}}^{p,q}$ and $d_{\mathrm{V}}^{p,q}$, and convene that the elements $a_{p,q}, b_{p,q}, \ldots$ are automatically assumed to be in $A^{p,q}$. We have

$$\begin{aligned} \mathrm{B}_r^{p+r,q-r+1} = \{&(d_{\mathrm{H}}(a_{p+r-1,q-r+1}) + d_{\mathrm{V}}(a_{p+r,q-r}), \ldots, \\ &d_{\mathrm{H}}(a_{p+q-1,1}) + d_{\mathrm{V}}(a_{p+q,0}), d_{\mathrm{H}}(a_{p+q,0})) \mid \\ &\exists a_{p,q}, \ldots, a_{p+r-2,q-r+2} \colon\ d_{\mathrm{V}}(a_{p,q}) = 0, \\ &d_{\mathrm{H}}(a_{p,q}) + d_{\mathrm{V}}(a_{p+1,q-1}) = 0, \ldots, \\ &d_{\mathrm{H}}(a_{p+r-2,q-r+2}) + d_{\mathrm{V}}(a_{p+r-1,q-r+1}) = 0\}, \end{aligned}$$

$$\begin{aligned} \mathrm{B}_{r-1}^{p+r,q-r+1} = \{&(d_{\mathrm{H}}(b_{p+r-1,q-r+1}) + d_{\mathrm{V}}(b_{p+r,q-r}), \ldots, \\ &d_{\mathrm{H}}(b_{p+q-1,1}) + d_{\mathrm{V}}(b_{p+q,0}), d_{\mathrm{H}}(b_{p+q,0})) \mid \\ &\exists b_{p+1,q-1}, \ldots, b_{p+r-2,q-r+2} \colon\ d_{\mathrm{V}}(b_{p+1,q-1}) = 0, \\ &d_{\mathrm{H}}(b_{p+1,q-1}) + d_{\mathrm{V}}(b_{p+2,q-2}) = 0, \ldots, \\ &d_{\mathrm{H}}(b_{p+r-2,q-r+2}) + d_{\mathrm{V}}(b_{p+r-1,q-r+1}) = 0\}, \end{aligned}$$

$$\begin{aligned}\mathrm{Z}_{r-2}^{p+r+1,q-r} = \{(c_{p+r+1,q-r},\dots,c_{p+q+1,0}) \mid\ & d_{\mathrm{V}}(c_{p+r+1,q-r}) = 0,\\ & d_{\mathrm{H}}(c_{p+r+1,q-r}) + d_{\mathrm{V}}(c_{p+r+2,q-r-1}) = 0,\dots,\\ & d_{\mathrm{H}}(c_{p+2r-1,q-2r+4}) + d_{\mathrm{V}}(c_{p+2r-2,q-2r+3}) = 0\}.\end{aligned}$$

This yields the system of $r-1$ equations

$$\begin{aligned}& d_{\mathrm{H}}(a_{p+2r-3,q-2r+3}) + d_{\mathrm{V}}(a_{p+2r-2,q-2r+2})\\ ={}& d_{\mathrm{H}}(b_{p+2r-3,q-2r+3}) + d_{\mathrm{V}}(b_{p+2r-2,q-2r+2}) + c_{p+2r-2,q-2r+3},\\ & \vdots\\ & d_{\mathrm{H}}(a_{p+r,q-r}) + d_{\mathrm{V}}(a_{p+r+1,q-r-1})\\ ={}& d_{\mathrm{H}}(b_{p+r,q-r}) + d_{\mathrm{V}}(b_{p+r+1,q-r-1}) + c_{p+r+1,q-r},\\ & d_{\mathrm{H}}(a_{p+r-1,q-r+1}) + d_{\mathrm{V}}(a_{p+r,q-r})\\ ={}& d_{\mathrm{H}}(b_{p+r-1,q-r+1}) + d_{\mathrm{V}}(b_{p+r,q-r}),\end{aligned}$$

where for all a's we have to find suitable b's and c's. We have

$$\begin{aligned}d_{\mathrm{V}}(d_{\mathrm{H}}(a_{p+r-1,q-r+1})) &= -d_{\mathrm{H}}(d_{\mathrm{V}}(a_{p+r-1,q-r+1}))\\ &= d_{\mathrm{H}}(d_{\mathrm{H}}(a_{p+r-2,q-r+2}))\\ &= 0,\end{aligned}$$

which implies $d_{\mathrm{H}}(a_{p+r-1,q-r+1}) \in \ker\big(d_{\mathrm{V}}^{p+r,q-r+1}\big)$. Assume there exists $b'_{p+r,q-r} \in A^{p+r,q-r}$ such that

$$d_{\mathrm{H}}(a_{p+r-1,q-r+1}) = d_{\mathrm{V}}\big(b'_{p+r,q-r}\big).$$

Then setting

$$\begin{aligned}b_{p+1,q-1} &:= 0,\\ &\vdots\\ b_{p+r-1,q-r+1} &:= 0,\\ b_{p+r,q-r} &:= a_{p+r,q-r} + b'_{p+r,q-r},\\ b_{p+r+1,q-r-1} &:= 0,\\ &\vdots\\ b_{p+2r-2,q-2r+2} &:= 0,\\ c_{p+r+1,q-r} &:= d_{\mathrm{V}}(a_{p+r+1,q-r-1}) - d_{\mathrm{H}}\big(b'_{p+r,q-r}\big),\\ c_{p+r+2,q-r-1} &:= d_{\mathrm{H}}(a_{p+r+1,q-r-1}) + d_{\mathrm{V}}(a_{p+r+2,q-r-2}),\\ &\vdots\\ c_{p+2r-2,q-2r+3} &:= d_{\mathrm{H}}(a_{p+2r-3,q-2r+3}) + d_{\mathrm{V}}(a_{p+2r-2,q-2r+2})\end{aligned}$$

yields a suitable solution to the equations above. Hence, to prove vanishing of $\mathrm{d}_r^{p,q}$, it suffices to show that the class of $d_{\mathrm{H}}(a_{p+r-1,q-r+1})$ vanishes in $\mathrm{H}_{\mathrm{V}}^{p+r,q-r+1} = \ker\big(d_{\mathrm{V}}^{p+r,q-r+1}\big)/\mathrm{im}\big(d_{\mathrm{V}}^{p+r,q-r}\big)$ for every $a_{p+r-1,q-r+1} \in A^{p+r-1,q-r+1}$. □

5.2. The L^∞-Double Complex of a Classical Group

In this section we introduce the double complex that will be our main technical tool to establish the main results of this thesis.

Let (V_r, ω) be a classical space, where we use Notation 1.1.20. Then Equation (2.1.1) turns $L^\infty\big(G_r^{p+1}, L^\infty(\mathcal{P}_r^q)\big)$ into a coefficient G_r-module for all $p, q \in \mathbb{N}$. We define

$$\begin{aligned}
&L^{p,q} := L^\infty\big(G_r^{p+1}, L^\infty(\mathcal{P}_r^q)\big)^{G_r},\\
&d_{\mathrm{H}}^{p,q}\colon L^{p,q} \to L^{p+1,q},\\
&d_{\mathrm{H}}^{p,q}(f)(g_0, \dots, g_{p+1}) := \sum_{i=0}^{p+1} (-1)^i f(g_0, \dots, \widehat{g_i}, \dots, g_{p+1}),\\
&d_r^{q-1}\colon L^\infty(\mathcal{P}_r^q)^{G_r} \to L^\infty\big(\mathcal{P}_r^{q+1}\big)^{G_r},\\
&d_r^{q-1}(f)([v_0, \dots, v_q]) := \sum_{i=0}^{q} (-1)^i f([v_0, \dots, \widehat{v_i}, \dots, v_q]),\\
&d_{\mathrm{V}}^{p,q}\colon L^{p,q} \to L^{p,q+1},\ f \mapsto (-1)^q\big(d_r^{q-1} \circ f\big).
\end{aligned}$$

The following lemma is easily proved.

Lemma 5.2.1. $(L^{p,q}, d_{\mathrm{H}}^{p,q}, d_{\mathrm{V}}^{p,q})_{p,q\in\mathbb{N}}$ *is a double complex.*

Note that Theorem 2.2.11 implies

$$\mathrm{H}_{\mathrm{cb}}^q(G_r; L^\infty(\mathcal{P}_r^p)) \cong \ker(d_{\mathrm{H}}^{q,p})/\mathrm{im}\big(d_{\mathrm{H}}^{q-1,p}\big) = \mathrm{H}_{\mathrm{H}}^{q,p}.$$

Theorem 5.1.4 and Remark 5.1.5 yield spectral sequences $\mathrm{E}_\bullet^{\bullet,\bullet}$ and ${}^\top\mathrm{E}_\bullet^{\bullet,\bullet}$. By [40, Lemma 5.15], $\mathrm{E}_\bullet^{\bullet,\bullet}$ converges to 0, so ${}^\top\mathrm{E}_\bullet^{\bullet,\bullet}$ also converges to 0.

Actually, we only needed the spectral sequence $\mathrm{E}_\bullet^{\bullet,\bullet}$ above to prove that ${}^\top\mathrm{E}_\bullet^{\bullet,\bullet}$ converges to zero. From now on, we forget about the former spectral sequence and denote the latter by $\mathrm{E}_\bullet^{\bullet,\bullet}$. We collect the results of our discussion in the following proposition.

Proposition 5.2.2 (De la Cruz Mengual [40, Proposition 5.16])**.** *There exists a spectral sequence* $\mathrm{E}_\bullet^{\bullet,\bullet}$ *that converges to zero with first page*

$$\mathrm{E}_1^{p,q} = \mathrm{H}_{\mathrm{cb}}^q(G_r; L^\infty(\mathcal{P}_r^p)), \qquad \mathrm{d}_1^{p,q} = \mathrm{H}_{\mathrm{cb}}^q\big(G_r; d_r^{p-1}\big)\colon \mathrm{E}_1^{p,q} \to \mathrm{E}_1^{p+1,q}. \tag{5.2.1}$$

Until the end of this thesis $(\mathrm{E}_\bullet^{\bullet,\bullet}, \mathrm{d}_\bullet^{\bullet,\bullet})$ denotes the spectral sequence (5.2.1). We will now calculate some entries of the first two pages of $\mathrm{E}_\bullet^{\bullet,\bullet}$. Let us write $H_0 = \mathrm{Stab}_{G_r}([e_r])$ and $H_1 = \mathrm{Stab}_{G_r}([e_r, f_r])$.

Lemma 5.2.3. *We have*

$$\begin{aligned}
\mathrm{E}_1^{0,q} &\cong \mathrm{H}_{\mathrm{cb}}^q(G_r),\\
\mathrm{E}_1^{1,q} &\cong \mathrm{H}_{\mathrm{cb}}^q(G_{r-1}),\\
\mathrm{E}_1^{2,q} &\cong \mathrm{H}_{\mathrm{cb}}^q(G_{r-1}).
\end{aligned}$$

Proof. The first isomorphism is obvious. By Lemma 1.1.18, Proposition 2.2.24, Corollary 3.1.3, and Corollary 2.2.17 we have

$$\begin{aligned} \mathrm{E}_1^{1,q} &= \mathrm{H}^q_{\mathrm{cb}}(G_r; L^\infty(\mathcal{P}_r)) \cong \mathrm{H}^q_{\mathrm{cb}}(G_r; L^\infty(G_r/H_0)) \cong \mathrm{H}^q_{\mathrm{cb}}(H_0) \cong \mathrm{H}^q_{\mathrm{cb}}(G_{r-1}), \\ \mathrm{E}_1^{2,q} &= \mathrm{H}^q_{\mathrm{cb}}(G_r; L^\infty(\mathcal{P}_r^2)) \cong \mathrm{H}^q_{\mathrm{cb}}(G_r; L^\infty(G_r/H_1)) \cong \mathrm{H}^q_{\mathrm{cb}}(H_1) \cong \mathrm{H}^q_{\mathrm{cb}}(G_{r-1}). \end{aligned}$$

□

Lemma 5.2.4. *The following diagram commutes:*

$$\begin{array}{ccccc} \mathrm{E}_1^{0,q} & \xrightarrow{\mathrm{d}_1^{0,q}} & \mathrm{E}_1^{1,q} & \xrightarrow{\mathrm{d}_1^{1,q}} & \mathrm{E}_1^{2,q} \\ \cong\downarrow & & \downarrow\cong & & \downarrow\cong \\ \mathrm{H}^q_{\mathrm{cb}}(G_r) & \xrightarrow[\mathrm{H}^q_{\mathrm{cb}}(\iota_{r-1})]{} & \mathrm{H}^q_{\mathrm{cb}}(G_{r-1}) & \xrightarrow[0]{} & \mathrm{H}^q_{\mathrm{cb}}(G_{r-1}) \end{array}$$

Proof. Let $\varphi \in G_r$ be defined by $\varphi(e_r) = \varepsilon f_r$, $\varphi(f_r) = e_r$, $\varphi|_{V_{r-1}} = \mathrm{id}$. We set $w_1 = \mathrm{id}_{G_r}$, $w_0 = \varphi^{-1}$. We obtain maps

$$\mathrm{Int}(w_i)\colon H_1 \to H_0,\ h \mapsto w_i h w_i^{-1}, \qquad i = 0, 1,$$

see [44, Lemma 3.4]. By [44, Lemma 3.7], the diagram

$$\begin{array}{ccccc} \mathrm{E}_1^{0,q} & \xrightarrow{\mathrm{d}_1^{0,q}} & \mathrm{E}_1^{1,q} & \xrightarrow{\mathrm{H}^q_{\mathrm{cb}}(G_r,\delta^i)} & \mathrm{E}_1^{2,q} \\ \cong\downarrow & & \downarrow\cong & & \downarrow\cong \\ \mathrm{H}^q_{\mathrm{cb}}(G_r) & \xrightarrow[\mathrm{H}^q_{\mathrm{cb}}(\iota)]{} & \mathrm{H}^q_{\mathrm{cb}}(H_0) & \xrightarrow[\mathrm{H}^q_{\mathrm{cb}}(\mathrm{Int}(w_i))]{} & \mathrm{H}^q_{\mathrm{cb}}(H_1) \end{array}$$

commutes, where $\iota\colon H_0 \to G_r$ denotes the inclusion. Let $\pi_0\colon H_0 \to G_{r-1}$ and $\pi_1\colon H_1 \to G_{r-1}$ denote the canonical projections from Corollary 3.1.3. We claim that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}^q_{\mathrm{cb}}(G_r) & \xrightarrow{\mathrm{H}^q_{\mathrm{cb}}(\iota_{r-1})} & \mathrm{H}^q_{\mathrm{cb}}(G_{r-1}) & \xrightarrow{\mathrm{id}} & \mathrm{H}^q_{\mathrm{cb}}(G_{r-1}) \\ \mathrm{id}\downarrow & & \downarrow\mathrm{H}^q_{\mathrm{cb}}(\pi_0) & & \downarrow\mathrm{H}^q_{\mathrm{cb}}(\pi_1) \\ \mathrm{H}^q_{\mathrm{cb}}(G_r) & \xrightarrow[\mathrm{H}^q_{\mathrm{cb}}(\iota)]{} & \mathrm{H}^q_{\mathrm{cb}}(H_0) & \xrightarrow[\mathrm{H}^q_{\mathrm{cb}}(\mathrm{Int}(w_i))]{} & \mathrm{H}^q_{\mathrm{cb}}(H_1) \end{array}$$

Indeed, consider the continuous homomorphic sections $\sigma_0\colon G_{r-1} \to H_0$, $\psi \mapsto m_\psi$ and $\sigma_1\colon G_{r-1} \to H_1$, $\psi \mapsto m_\psi$ of π_0 and π_1, respectively. Clearly, the following diagrams commute:

$$\begin{array}{ccc} H_1 & \xrightarrow{\mathrm{Int}(w_i)} & H_0 \\ \sigma_1\uparrow & & \downarrow\pi_0 \\ G_{r-1} & \xrightarrow[\mathrm{id}]{} & G_{r-1} \end{array} \qquad\qquad \begin{array}{ccc} H_0 & \xrightarrow{\iota} & G_r \\ \sigma_0\uparrow & & \downarrow\mathrm{id} \\ G_{r-1} & \xrightarrow[\iota_{r-1}]{} & G_r \end{array}$$

The claim now follows from the application of the functor $\mathrm{H}^q_{\mathrm{cb}}$. □

Lemma 5.2.5. *We have* $\mathrm{E}_1^{p,1} = 0$ *for all* $p \in \mathbb{N}$.

Proof. Observe that for $p = 0, 1, 2$ the claim follows from Lemma 5.2.3. We use the notation of Section 4.5 and prove that $\mathrm{E}_1^{k,1} = \mathrm{H}^1_{\mathrm{cb}}(G_r; L^\infty(\mathcal{P}^k_r))$ vanishes if $r \geq r_k$. This is the only case for which we will use this lemma in Chapter 6; the case $r < r_k$ can be proved analogously but requires a more tedious description of the occurring stabilizers. Recall that there exist finitely many conjugacy classes $[H_{k-1,i}]_{i=1,\ldots,c_k}$ of stabilizers of a point on a G_r-invariant open conull submanifold of $\mathcal{P}^k_r$. Theorem 3.1.5, (iv) and Proposition 2.2.24 yield

$$\begin{aligned}
\mathrm{H}^n_{\mathrm{cb}}\big(G_r; L^\infty(\mathcal{P}^k_r)\big) &= \mathrm{H}^n_{\mathrm{cb}}\left(G_r; L^\infty\left(\bigsqcup_{i=1}^{c_k}\Big((G_r/H_{k-1,i}) \times \mathcal{C}^{(k)}_i\Big)\right)\right)\\
&= \mathrm{H}^n_{\mathrm{cb}}\left(G_r; \prod_{i=1}^{c_k} L^\infty\Big((G_r/H_{k-1,i}) \times \mathcal{C}^{(k)}_i\Big)\right)\\
&= \prod_{i=1}^{c_k} \mathrm{H}^n_{\mathrm{cb}}\Big(G_r; L^\infty\Big(G_r/H_{k-1,i}, L^\infty\Big(\mathcal{C}^{(k)}_i\Big)\Big)\Big)\\
&= \prod_{i=1}^{c_k} \mathrm{H}^n_{\mathrm{cb}}\Big(H_{k-1,i}; L^\infty\Big(\mathcal{C}^{(k)}_i\Big)\Big).
\end{aligned}$$

Note that $L^\infty\Big(\mathcal{C}^{(k)}_i\Big)$ is a trivial coefficient $H_{k-1,i}$-module. Hence $L^\infty\Big(\mathcal{C}^{(k)}_i\Big)$ is semi-separable in the sense of [103], which by [103, Lemma 5.8] implies

$$\mathrm{H}^1_{\mathrm{cb}}\Big(H_{k-1,i}; L^\infty\Big(\mathcal{C}^{(k)}_i\Big)\Big) = 0,$$

yielding the claim. ☐

We can conclude that the first page of $\mathrm{E}^{\bullet,\bullet}_\bullet$ is conjugated to

$$\begin{array}{ccccccccc}
\mathrm{H}^4_{\mathrm{cb}}(G_r) & \xrightarrow{\mathrm{H}^4_{\mathrm{cb}}(\iota_{r-1})} & \mathrm{H}^4_{\mathrm{cb}}(G_{r-1}) & \xrightarrow{0} & \mathrm{H}^4_{\mathrm{cb}}(G_{r-1}) & \longrightarrow & * & \longrightarrow & * \\
\mathrm{H}^3_{\mathrm{cb}}(G_r) & \xrightarrow{\mathrm{H}^3_{\mathrm{cb}}(\iota_{r-1})} & \mathrm{H}^3_{\mathrm{cb}}(G_{r-1}) & \xrightarrow{0} & \mathrm{H}^3_{\mathrm{cb}}(G_{r-1}) & \longrightarrow & * & \longrightarrow & * \\
\mathrm{H}^2_{\mathrm{cb}}(G_r) & \xrightarrow{\mathrm{H}^2_{\mathrm{cb}}(\iota_{r-1})} & \mathrm{H}^2_{\mathrm{cb}}(G_{r-1}) & \xrightarrow{0} & \mathrm{H}^2_{\mathrm{cb}}(G_{r-1}) & \longrightarrow & * & \longrightarrow & * \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
\mathbb{R} & \xrightarrow{\mathrm{id}} & \mathbb{R} & \xrightarrow{0} & \mathbb{R} & \hookrightarrow & L^\infty(\mathcal{P}^3_r)^{G_r} & \xrightarrow{d^2_r} & L^\infty(\mathcal{P}^4_r)^{G_r}
\end{array}$$

Here, we applied the definitions to obtain the $q = 0$ row, Lemma 5.2.5 to obtain the $q = 1$ row, and Lemma 5.2.4 to obtain the $p = 0, 1, 2$ columns.

Remark 5.2.6. Using Theorem 2.2.11 it is easy to see that we have a linear isomorphism $\mathrm{E}_2^{p,0} \cong \mathrm{H}_{\mathrm{mb}}^{p-1}(G_r \curvearrowright \mathcal{P}_r)$.

The proof of the following proposition is adapted from [104, Proof of Proposition 4.1] (see also [130] for several proofs in the same spirit).

Proposition 5.2.7. *We have* $\mathrm{d}_2^{0,q} = 0$ *for all* $q \in \mathbb{N}$.

Proof. By Lemma 5.1.6, it suffices to show that for all $f \in L^\infty(G_r^q, L^\infty(\mathcal{P}_r))^{G_r}$ the class of $\alpha = d_{\mathrm{V}}(f)$ vanishes in $\mathrm{H}_{\mathrm{cb}}^{q-1}(G_r; L^\infty(\mathcal{P}_r^2))$. We follow α through the following isomorphisms:

$$\begin{array}{ccc}
\mathrm{H}_{\mathrm{cb}}^\bullet(G_r; L^\infty(\mathcal{P}_r^2)) & = & \mathrm{H}^\bullet\big(0 \to L^\infty(G_r, L^\infty(\mathcal{P}_r^2))^{G_r} \to \dots\big) \\
\downarrow \cong & & \\
\mathrm{H}_{\mathrm{cb}}^\bullet(G_r; L^\infty(G_r/H_1)) & = & \mathrm{H}^\bullet\big(0 \to L^\infty(G_r, L^\infty(G_r/H_1))^{G_r} \to \dots\big) \\
\downarrow \cong & & \\
\mathrm{H}_{\mathrm{cb}}^\bullet(H_1) & = & \mathrm{H}^\bullet\big(0 \to L^\infty(G_r)^{H_1} \to \dots\big) \\
\downarrow \cong & & \\
\mathrm{H}_{\mathrm{cb}}^\bullet(G_{r-1}) & = & \mathrm{H}^\bullet\big(0 \to L^\infty(G_r)^{G_{r-1}} \to \dots\big)
\end{array}$$

Here the isomorphisms are induced on the cochain level by

$$\begin{aligned}
&\Phi_1 \colon L^\infty(G_r^q, L^\infty(\mathcal{P}_r^2))^{G_r} \to L^\infty(G_r^q, L^\infty(G_r/H_1))^{G_r}, \\
&\Phi_1(\beta)(g_0, \dots, g_{q-1})(gH_1) := \beta(g_0, \dots, g_{q-1})(g.[e_r, f_r]),
\end{aligned}$$

$$\begin{aligned}
&\Phi_2 \colon L^\infty(G_r^q, L^\infty(G_r/H_1))^{G_r} \to L^\infty(G_r^q)^{H_1}, \\
&\Phi_2(\gamma)(g_0, \dots, g_{q-1}) := \gamma(g_0, \dots, g_{q-1})(H_1),
\end{aligned}$$

and the inclusion $L^\infty(G_r^q)^{H_1} \to L^\infty(G_r^q)^{G_{r-1}}$. Thus, α is mapped to $\alpha' \in L^\infty(G_r^q)^{G_{r-1}}$, given by

$$\begin{aligned}
\alpha'(g_0, \dots, g_{q-1}) &= \alpha(g_0, \dots, g_{q-1})([e_r, f_r]) \\
&= f(g_0, \dots, g_{q-1})([f_r]) - f(g_0, \dots, g_{q-1})([e_r]).
\end{aligned}$$

Let $w \in G_r$ be the element defined by $w(e_i) = f_i$, $w(f_i) = \varepsilon e_i$, $w(h_k) = h_k$ for all $i = 1, \dots, r$, $k = 1, \dots, d$. By G_r-invariance of f we have

$$\alpha'(g_0, \dots, g_{q-1}) = \big(f\big(w^{-1}g_0, \dots, w^{-1}g_{q-1}\big) - f(g_0, \dots, g_{q-1})\big)([e_r]).$$

By base point invariance of bounded cohomology, which can be established as in [66, Ch. I, §7], α' is cohomologous to α'' defined by

$$\alpha''(g_0, \dots, g_{q-1}) := \big(f\big(w^{-1}g_0 w, \dots, w^{-1}g_{q-1}w\big) - f(g_0, \dots, g_{q-1})\big)([e_r]).$$

Note that we can write $w = w_1 w_2$ for $w_1 \in \mathrm{Z}_{G_r}(\iota_{r-1}(G_{r-1}))$ defined by $w_1|_{\mathbb{K}e_r \oplus \mathbb{K}f_r} = w|_{\mathbb{K}e_r \oplus \mathbb{K}f_r}$, $w_1|_{V_{r-1}} = \mathrm{id}$ and $w_2 = \iota_{r-1}\big(w|_{V_{r-1}}\big)$. Thus, conjugation by w is trivial on $\mathrm{H}^\bullet_{\mathrm{cb}}(G_{r-1})$. On the other hand, $w^{-1} = \varepsilon w$ implies

$$\begin{aligned}
(w.\alpha'')(g_0, \ldots, g_{q-1}) &= \big(f(g_0, \ldots, g_{q-1}) - f\big(wg_0w^{-1}, \ldots, wg_{q-1}w^{-1}\big)\big)([e_r]) \\
&= (f(g_0, \ldots, g_{q-1}) - f(\varepsilon w g_0 w, \ldots, \varepsilon w g_{q-1} w))([e_r]) \\
&= \big(f(g_0, \ldots, g_{q-1}) - f\big(w^{-1}g_0w, \ldots, w^{-1}g_{q-1}w\big)\big)([e_r]) \\
&= -\alpha''(g_0, \ldots, g_{q-1}).
\end{aligned}$$

Hence α'' is trivial in $\mathrm{H}^{q-1}_{\mathrm{cb}}(G_{r-1})$. □

Remark 5.2.8. We can conclude that the second page of $\mathrm{E}^{\bullet,\bullet}_{\bullet}$ is conjugated to

$$\begin{tikzcd}
\ker(\mathrm{H}^4_{\mathrm{cb}}(\iota_{r-1})) \arrow[rrd, "0"] & \operatorname{coker}(\mathrm{H}^4_{\mathrm{cb}}(\iota_{r-1})) & \ker\big(\mathrm{d}_1^{2,4}\big) & * & * \\
\ker(\mathrm{H}^3_{\mathrm{cb}}(\iota_{r-1})) \arrow[rrd, "0"] & \operatorname{coker}(\mathrm{H}^3_{\mathrm{cb}}(\iota_{r-1})) & \ker\big(\mathrm{d}_1^{2,3}\big) & * & * \\
\ker(\mathrm{H}^2_{\mathrm{cb}}(\iota_{r-1})) & \operatorname{coker}(\mathrm{H}^2_{\mathrm{cb}}(\iota_{r-1})) & \ker\big(\mathrm{d}_1^{2,2}\big) & * & * \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathrm{H}^2_{\mathrm{mb}}(G_r \curvearrowright \mathcal{P}_r) & \mathrm{H}^3_{\mathrm{mb}}(G_r \curvearrowright \mathcal{P}_r)
\end{tikzcd}$$

6. Secondary Stability of Classical Groups

6.1. Degree-Two Stability of Classical Groups

In this section we will prove the following proposition.

Proposition 6.1.1. *(i) Let* $\iota_1 \colon \mathrm{SO}_0(2,1) \hookrightarrow \mathrm{SO}_0(3,2)$ *denote the block inclusion. Then* $\mathrm{H}^2_\mathrm{c}(\iota_1)$ *is the zero map.*

(ii) Let $\iota_r \colon \mathrm{Sp}(2r,\mathbb{R}) \hookrightarrow \mathrm{Sp}(2(r+1),\mathbb{R})$ *denote the block inclusion. Then* $\mathrm{H}^2_\mathrm{c}(\iota_r)$ *is an isomorphism for all* $r \geq 1$.

Let $(G_r, \iota_r)_{r\in\mathbb{N}}$ be a family of connected semisimple classical groups. We consider the involutive automorphism $\vartheta \colon G_r \to G_r$, $g \mapsto (g^*)^{-1}$ and write $K_r = G_r^\vartheta$. Let $\mathfrak{g}_r = \mathfrak{k}_r \oplus \mathfrak{p}_r$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}_r$ of G_r with respect to $\theta = d\vartheta$. Then the inner product

$$\langle X, Y\rangle \coloneqq \mathrm{tr}(XY) \qquad (X, Y \in \mathfrak{p}_r \cong \mathrm{T}_0(G_r/K_r))$$

can be extended to a G_r-invariant Riemannian metric g_r on the symmetric space $X_r = G_r/K_r$. It is easy to check that $\iota_r^{-1}(K_{r+1}) = K_r$, which implies that ι_r induces a topological embedding $\iota_{r,*} \colon X_r \hookrightarrow X_{r+1}$. By the equivariant rank theorem (see [92, Theorem 7.25]), $\iota_{r,*}$ is a smooth embedding.

Lemma 6.1.2 (see [91, Section 8.2.1])**.** *The induced map*

$$\iota_r^* = \Omega^2(\iota_{r,*}) \colon \Omega^2(X_{r+1})^{G_{r+1}} \to \Omega^2(X_r)^{G_r}$$

is given by restriction.

Now assume that G_r is of Hermitian type. By [75, Ch. VIII, Proposition 4.1], X_r admits a complex structure $\mathcal{J}_r$ such that $(X_r, \mathcal{J}_r, g_r)$ is a Kähler manifold. Explicitly, the center $\mathrm{Z}(K_r)$ of K_r is isomorphic to the circle group, so we can find an element $j \in \mathrm{Z}(K_r)$ such that $\mathrm{Ad}_{G_r}(j^2)|_{\mathfrak{p}_r} = -\mathrm{id}_{\mathfrak{p}_r}$. Then the corresponding Kähler form $\omega_r \in \Omega^2(X_r)^{G_r}$ is given by extending

$$\mathfrak{p}_r \times \mathfrak{p}_r \to \mathbb{R},\ (X, Y) \mapsto \mathrm{tr}(\mathrm{Ad}_{G_r}(j)(X)Y),$$

see [75, Ch. VIII, Proposition 6.1]. Using the commutative diagram

$$\begin{array}{ccc} \Omega^2(X_{r+1})^{G_{r+1}} & \xrightarrow{\iota_r^*} & \Omega^2(X_r)^{G_r} \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{H}^2_\mathrm{c}(G_{r+1}) & \xrightarrow[\mathrm{H}^2_\mathrm{c}(\iota_r)]{} & \mathrm{H}^2_\mathrm{c}(G_r) \end{array}$$

to prove Proposition 6.1.1 it thus suffices to show that the restriction to $\mathfrak{p}_r \times \mathfrak{p}_r$ of

$$\mathfrak{p}_{r+1} \times \mathfrak{p}_{r+1} \to \mathbb{R}, \ (X,Y) \mapsto \operatorname{tr}\big(\mathrm{Ad}_{G_{r+1}}(j)(X)Y\big)$$

is non-zero if $G_r = \mathrm{Sp}(2r,\mathbb{R})$, and zero if $r = 1$ and $G_1 = \mathrm{SO}_0(2,1)$. We now distinguish between these two cases.

The Real Special Orthogonal Groups: Block matrix descriptions of $\mathfrak{g}_r$, $\mathfrak{k}_r$, and $\mathfrak{p}_r$ can be found in Subsection 1.3.2. We have

$$\mathfrak{z}(\mathfrak{k}_1) = \mathbb{R}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathfrak{z}(\mathfrak{k}_2) = \mathbb{R}\begin{pmatrix} 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{pmatrix} \subset \mathfrak{sl}_{4+d}(\mathbb{R}),$$

$$\mathfrak{p}_1 = \left\{\begin{pmatrix} a_1 & a_2 & 0 \\ a_2 & 0 & -a_2 \\ 0 & -a_2 & -a_1 \end{pmatrix}\right\}, \qquad \mathfrak{p}_2 = \left\{\begin{pmatrix} a_1 & b_1 & a_2^\top & a_3 & 0 \\ b_1 & c_1 & b_2^\top & 0 & -a_3 \\ a_2 & b_2 & 0 & -b_2 & -a_2 \\ a_3 & 0 & -b_2^\top & -c_1 & -b_1 \\ 0 & -a_3 & -a_2^\top & -b_1 & -a_1 \end{pmatrix}\right\} \subset \mathfrak{sl}_{4+d}(\mathbb{R}).$$

Elements $j_1 \in \mathrm{Z}(K_1)$ and $j_2 \in \mathrm{Z}(K_2)$ of order 4 are given by

$$j_1 = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & -1 \\ -\sqrt{2} & 0 & -\sqrt{2} \\ -1 & \sqrt{2} & 1 \end{pmatrix} = \exp\left(\frac{\pi\sqrt{2}}{4}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\right),$$

$$j_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 2\mathrm{I}_d & 0 & 0 \\ 1 & 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 & 1 \end{pmatrix} = \exp\left(\frac{\pi}{4}\begin{pmatrix} 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 & 0 \end{pmatrix}\right).$$

A direct calculation shows that $\mathrm{Ad}_{G_1}(j_1^2)|_{\mathfrak{p}_1} = -\mathrm{id}_{\mathfrak{p}_1}$ and $\mathrm{Ad}_{G_2}(j_2^2)|_{\mathfrak{p}_2} = -\mathrm{id}_{\mathfrak{p}_2}$. We calculate

$$\operatorname{tr}\left(\mathrm{Ad}_{G_1}(j_1)\left(\begin{pmatrix} a_1 & a_2 & 0 \\ a_2 & 0 & -a_2 \\ 0 & -a_2 & -a_1 \end{pmatrix}\right)\begin{pmatrix} b_1 & b_2 & 0 \\ b_2 & 0 & -b_2 \\ 0 & -b_2 & -b_1 \end{pmatrix}\right) = 2\sqrt{2}(a_2b_1 - a_1b_2),$$

$$\operatorname{tr}\left(\mathrm{Ad}_{G_2}(j_2)\left(\begin{pmatrix} a_1 & b_1 & a_2^\top & a_3 & 0 \\ b_1 & c_1 & b_2^\top & 0 & -a_3 \\ a_2 & b_2 & 0 & -b_2 & -a_2 \\ a_3 & 0 & -b_2^\top & -c_1 & -b_1 \\ 0 & -a_3 & -a_2^\top & -b_1 & -a_1 \end{pmatrix}\right)\begin{pmatrix} x_1 & y_1 & x_2^\top & x_3 & 0 \\ y_1 & z_1 & y_2^\top & 0 & -x_3 \\ x_2 & y_2 & 0 & -y_2 & -x_2 \\ x_3 & 0 & -y_2^\top & -z_1 & -y_1 \\ 0 & -x_3 & -x_2^\top & -y_1 & -x_1 \end{pmatrix}\right)$$
$$= 2((b_1 - a_3)x_1 - (b_1 + a_3)z_1 + (c_1 + a_1)x_3 + (c_1 - a_1)y_1) + 4\big(b_2^\top x_2 - a_2^\top y_2\big).$$

We consider the inclusion

$$i\colon \mathfrak{p}_1 \to \mathfrak{p}_2, \quad \begin{pmatrix} a_1 & a_2 & 0 \\ a_2 & 0 & -a_2 \\ 0 & -a_2 & -a_1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & a_1 & a_2 & 0 & 0 \\ 0 & a_2 & 0 & -a_2 & 0 \\ 0 & 0 & -a_2 & -a_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now

$$\begin{aligned} &\mathrm{Ad}_{G_2}(j_2)\left(i\begin{pmatrix} a_1 & a_2 & 0 \\ a_2 & 0 & -a_2 \\ 0 & -a_2 & -a_1 \end{pmatrix}\right) i\begin{pmatrix} b_1 & b_2 & 0 \\ b_2 & 0 & -b_2 \\ 0 & -b_2 & -b_1 \end{pmatrix} \\ &= \left(\frac{1}{2}a_1b_1 + a_2b_2\right)\begin{pmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

has trace zero, which yields the claim.

The Real Symplectic Groups: Block matrix descriptions of $\mathfrak{g}_r$, $\mathfrak{k}_r$, and $\mathfrak{p}_r$ can be found in Subsection 1.3.2; we use the matrices J_r and C_r, which were defined in Section 1.3. We have

$$\mathfrak{z}(\mathfrak{k}_r) = \mathbb{R}C_r$$

and an element of order 8 in $\mathrm{Z}(K_r)$ is given by

$$j_r = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{I}_r & J_r \\ -J_r & \mathrm{I}_r \end{pmatrix} = \exp\left(\frac{\pi}{4}C_r\right).$$

A direct calculation shows that $\mathrm{Ad}_{G_r}(j_r^2)|_{\mathfrak{p}_r} = -\mathrm{id}_{\mathfrak{p}_r}$. We calculate

$$\mathrm{tr}\left(\mathrm{Ad}_{G_r}(j_r)\left(\begin{pmatrix} X_1 & X_2 \\ X_2^\top & -J_rX_1J_r \end{pmatrix}\right)\begin{pmatrix} Y_1 & Y_2 \\ Y_2^\top & -J_rY_1J_r \end{pmatrix}\right) = 2\mathrm{tr}((Y_1X_2 - X_1Y_2)J_r).$$

We consider the inclusion

$$\mathfrak{p}_r \to \mathfrak{p}_{r+1}, \quad \begin{pmatrix} X_1 & X_2 \\ X_2^\top & -J_rX_1J_r \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & X_1 & X_2 & 0 \\ 0 & X_2^\top & -J_rX_1J_r & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now the $\mathrm{Ad}_{G_{r+1}}(j_{r+1})$-invariance of $\mathfrak{p}_r$ yields the claim.

This concludes the proof of Proposition 6.1.1.

Remark 6.1.3. Using the same approach, it should be possible to prove that $\mathrm{H}^2_{\mathrm{c}}(\iota_r)$ is an isomorphism for the other families of simple classical groups of Hermitian type—namely, for the families $\mathrm{SU}(r+d, r)$ and $\mathrm{SO}^*(4r+2d)$. This is computationally challenging—at least when using matrix representations corresponding to Witt bases.

6.2. Secondary Stability of Classical Groups

The purpose of this section is to prove the results listed in the Introduction. Let (V_r, ω) be a classical space, where we use Notation 1.1.20.

Remark 6.2.1. The proofs of these statements work, essentially, by using the concept of *secondary cohomological stability*, introduced by Galatius–Kupers–Randal-Williams [57] in the discrete setting and subsequently employed in the context of the Isomorphism Conjecture 2.4.2 by De la Cruz Mengual [40], [42]. The idea is to prove the stability of the "stability-failure" of a family of groups $(G_r, \varphi_r)_r$, i.e. to prove that the sequences $(\ker(\mathrm{H}^n_{\mathrm{cb}}(\varphi_r)))_r$ or $(\mathrm{coker}(\mathrm{H}^n_{\mathrm{cb}}(\varphi_r)))_r$ are stable. Together with ordinary or "primary" stability of $(G_r, \varphi_r)_r$, the stability of $(\ker(\mathrm{H}^n_{\mathrm{cb}}(\varphi_r)))_r$ can be used to reduce the injectivity range of $(G_r, \varphi_r)_r$, while the stability of $(\mathrm{coker}(\mathrm{H}^n_{\mathrm{cb}}(\varphi_r)))_r$ can be used to reduce the surjectivity range of $(G_r, \varphi_r)_r$.

Using the results of Section 5.2, we can identify the relevant kernels and cokernels with certain entries of our spectral sequence. The fact that the latter converges to zero will make it possible to identify these entries with the action cohomology spaces $\mathrm{H}^n_{\mathrm{mb}}(G_r \curvearrowright \mathcal{P}_r)$, whose stability we proved in Theorem 3.1.6.

The justification for $\mathcal{P}_r$ being the G_r-space of choice is that we have

$$\mathcal{P}_r \cong G_r/\mathrm{Stab}_{G_r}([x]),$$

where $\mathrm{Stab}_{G_r}([x])$ is a maximal parabolic subgroup of G_r (see Lemma 3.2.7); using a *maximal* parabolic subgroup gives the strongest transitivity properties compared to other parabolic subgroups.

Remark 6.2.2. In Subsection 6.2.2 and Subsection 6.2.3 we determine certain entries of our spectral sequence $\mathrm{E}^{\bullet,\bullet}_{\bullet}$ explicitly. Even though we have information on many more entries, we replace every entry not directly relevant to our proofs by an asterisk.

6.2.1. Results in Degree Two

Lemma 6.2.3. *We have linear isomorphisms*

$$\ker\big(\mathrm{H}^2_{\mathrm{cb}}(\iota_r)\big) \cong \mathrm{H}^2_{\mathrm{mb}}(G_{r+1} \curvearrowright \mathcal{P}_{r+1}),$$
$$\mathrm{coker}\big(\mathrm{H}^2_{\mathrm{cb}}(\iota_r)\big) \cong \mathrm{H}^3_{\mathrm{mb}}(G_{r+1} \curvearrowright \mathcal{P}_{r+1})$$

for all $r \geq 1$.

Proof. Looking at the second page of $\mathrm{E}^{\bullet,\bullet}_{\bullet}$ (Remark 5.2.8), we immediately see that $\mathrm{d}^{0,2}_3$ is conjugated to an isomorphism $\ker\big(\mathrm{H}^2_{\mathrm{cb}}(\iota_r)\big) \to \mathrm{H}^2_{\mathrm{mb}}(G_{r+1} \curvearrowright \mathcal{P}_{r+1})$ and that $\mathrm{d}^{1,2}_3$ is conjugated to an isomorphism $\mathrm{coker}\big(\mathrm{H}^2_{\mathrm{cb}}(\iota_r)\big) \to \mathrm{H}^3_{\mathrm{mb}}(G_{r+1} \curvearrowright \mathcal{P}_{r+1})$. □

Recall that we denote the weak stability range of $G_r \curvearrowright \mathcal{P}^k_r$, as defined in Definition 3.1.4, by r'_k.

Proof of Proposition A. By Theorem 2.3.2, there exists some $s \geq r'_4 - 1$ such that $\ker(\mathrm{H}^2_{\mathrm{cb}}(\iota_s)) = 0$. Let $r \geq r'_4 - 1$. By Lemma 6.2.3 and Theorem 3.1.6, we have

$$\ker(\mathrm{H}^2_{\mathrm{cb}}(\iota_r)) = \mathrm{H}^2_{\mathrm{mb}}\big(G_{r'_4} \curvearrowright \mathcal{P}_{r'_4}\big) = \ker(\mathrm{H}^2_{\mathrm{cb}}(\iota_s)) = 0,$$

so $\mathrm{H}^2_{\mathrm{cb}}(\iota_r)$ is injective.

By Theorem 2.3.2, there also exists $s \geq r'_5 - 1$ such that $\operatorname{coker}(\mathrm{H}^2_{\mathrm{cb}}(\iota_s)) = 0$. Now let $r \geq r'_5 - 1$. By Lemma 6.2.3 and Theorem 3.1.6, we have

$$\operatorname{coker}(\mathrm{H}^2_{\mathrm{cb}}(\iota_r)) = \mathrm{H}^3_{\mathrm{mb}}\big(G_{r'_5} \curvearrowright \mathcal{P}_{r'_5}\big) = \operatorname{coker}(\mathrm{H}^2_{\mathrm{cb}}(\iota_s)) = 0,$$

so $\mathrm{H}^2_{\mathrm{cb}}(\iota_r)$ is surjective.

In the case $G_r = \mathrm{U}(r+d, r)$ surjectivity of $\mathrm{H}^2_{\mathrm{cb}}(\iota_2)$ follows from its injectivity, since $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{U}(2+d, 2))$ and $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{U}(3+d, 3))$ are both one-dimensional.

In the case $G_r = \mathrm{O}(r+d, r)$ we observe that Proposition 6.1.1, (i) yields vanishing in the only case in which $\mathrm{H}^2_{\mathrm{cb}}(\iota_r)$ is a map between two non-zero spaces. □

6.2.2. Results in Degree Three

Lemma 6.2.4. *Let G_r be a non-complex classical group. Then we have a linear isomorphism*

$$\ker\big(\mathrm{H}^3_{\mathrm{cb}}(\iota_r)\big) \cong \mathrm{H}^3_{\mathrm{mb}}(G_{r+1} \curvearrowright \mathcal{P}_{r+1}), \qquad r \geq r'_5 - 1.$$

Proof. By Proposition A the spectral sequence $\mathrm{E}^{\bullet,\bullet}_{\bullet}$ has first page conjugated to

$$\begin{array}{ccccccc}
\mathrm{H}^3_{\mathrm{cb}}(G_r) & \xrightarrow{\mathrm{H}^3_{\mathrm{cb}}(\iota_{r-1})} & \mathrm{H}^3_{\mathrm{cb}}(G_{r-1}) & \longrightarrow & * & \longrightarrow & * \\
\mathrm{H}^2_{\mathrm{cb}}(G_r) & \xhookrightarrow{\mathrm{H}^2_{\mathrm{cb}}(\iota_{r-1})} & \mathrm{H}^2_{\mathrm{cb}}(G_{r-1}) & \xrightarrow{0} & \mathrm{H}^2_{\mathrm{cb}}(G_{r-1}) & \longrightarrow & * \\
* & \longrightarrow & * & \longrightarrow & 0 & \longrightarrow & 0 \\
* & \longrightarrow & * & \longrightarrow & * & \longrightarrow & L^\infty\big(\mathcal{P}^3_r\big)^{G_r}
\end{array}$$

Hence we have $\mathrm{E}^{1,2}_2 = 0$. Note that by Proposition 5.2.7 we have $\mathrm{d}^{0,3}_2 = 0$. Hence the second page of $\mathrm{E}^{\bullet,\bullet}_{\bullet}$ is conjugated to

$$\begin{array}{ccccc}
\ker(\mathrm{H}^3_{\mathrm{cb}}(\iota_{r-1})) & * & * & * & * \\
& \overset{0}{\searrow} & & & \\
* & 0 & * & * & * \\
* & * & 0 & 0 & * \\
* & * & * & * & \mathrm{H}^3_{\mathrm{mb}}(G_r \curvearrowright \mathcal{P}_r)
\end{array}$$

Since $\mathrm{E}^{\bullet,\bullet}_{\bullet}$ converges to zero, $\mathrm{d}^{0,3}_4$ is an isomorphism conjugated to

$$\ker\big(\mathrm{H}^3_{\mathrm{cb}}(\iota_{r-1})\big) \to \mathrm{H}^3_{\mathrm{mb}}(G_r \curvearrowright \mathcal{P}_r), \qquad r \geq r'_5.$$ □

Proof of Theorem A. Using Lemma 6.2.4, this is analogous to the proof of Proposition A. □

Lemma 6.2.5. *Let $G_r = \mathrm{O}(r+d, r)$, $r \geq 5$. Then we have $\mathrm{H}^2_{\mathrm{cb}}\big(G_r; L^\infty\big(\mathcal{P}^3_r\big)\big) = 0$.*

Proof. By the proof of Lemma 5.2.5 we have

$$\mathrm{H}^2_{\mathrm{cb}}\big(G_r; L^\infty\big(\mathcal{P}^3_r\big)\big) \cong \mathrm{H}^2_{\mathrm{cb}}\Big(H_2^{(1)}\Big) \times \mathrm{H}^2_{\mathrm{cb}}\Big(H_2^{(-1)}\Big),$$

where $H_2^{(z)} = \mathrm{Stab}_{G_r}(x_z)$, $z \in \mathcal{C}^{(3)}$. By step (S1) of Subsection 4.4.1, $H_2^{(z)}$ has the same bounded cohomology as an automorphism group of a classical space of **Type BD I** of rank at least 3. In particular, $\mathrm{H}^2_{\mathrm{cb}}\Big(H_2^{(z)}\Big) = 0$, yielding the claim. □

Lemma 6.2.6. *Let $G_r = \mathrm{O}(r+d, r)$. Then we have a linear isomorphism*

$$\mathrm{coker}\big(\mathrm{H}^3_{\mathrm{cb}}(\iota_r)\big) \cong \mathrm{H}^4_{\mathrm{mb}}(G_{r+1} \curvearrowright \mathcal{P}_{r+1}), \qquad r \geq 4.$$

Proof. By Lemma 6.2.5, the spectral sequence $\mathrm{E}_\bullet^{\bullet,\bullet}$ has first page conjugated to

$$\begin{array}{ccccccccc}
\mathrm{H}^3_{\mathrm{cb}}(G_r) & \xrightarrow{\mathrm{H}^3_{\mathrm{cb}}(\iota_{r-1})} & \mathrm{H}^3_{\mathrm{cb}}(G_{r-1}) & \longrightarrow & * & \longrightarrow & * & \longrightarrow & * \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & * \\
* & \longrightarrow & * & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & * \\
* & \longrightarrow & * & \longrightarrow & * & \longrightarrow & L^\infty\big(\mathcal{P}^3_r\big)^{G_r} & \xrightarrow{d^2_r} & L^\infty\big(\mathcal{P}^4_r\big)^{G_r}
\end{array}$$

Its second page is conjugated to

$$\begin{array}{cccccc}
* & \mathrm{coker}(\mathrm{H}^3_{\mathrm{cb}}(\iota_{r-1})) & * & * & * & * \\
* & * & 0 & 0 & * & * \\
* & * & * & 0 & 0 & * \\
* & * & * & * & * & \mathrm{H}^4_{\mathrm{mb}}(G_r \curvearrowright \mathcal{P}_r)
\end{array}$$

Since $\mathrm{E}_\bullet^{\bullet,\bullet}$ converges to zero, $\mathrm{d}_4^{1,3}$ is an isomorphism conjugated to

$$\mathrm{coker}\big(\mathrm{H}^3_{\mathrm{cb}}(\iota_{r-1})\big) \to \mathrm{H}^4_{\mathrm{mb}}(G_r \curvearrowright \mathcal{P}_r), \qquad r \geq 5.$$ □

Proof of Theorem B. Using Lemma 6.2.6, this is analogous to the proof of Proposition A. □

6.2.3. Results in Degree Four

Lemma 6.2.7. *Let $G_r = \mathrm{O}(r+d, r)$. Then we have a linear isomorphism*

$$\ker(\mathrm{H}^4_{\mathrm{cb}}(\iota_r)) \cong \mathrm{H}^4_{\mathrm{mb}}(G_{r+1} \curvearrowright \mathcal{P}_{r+1}), \qquad r \geq 4.$$

Proof. The spectral sequence $\mathrm{E}_\bullet^{\bullet,\bullet}$ has first page conjugated to

$$\begin{array}{ccccccccc}
\mathrm{H}^4_{\mathrm{cb}}(G_r) & \xrightarrow{\mathrm{H}^4_{\mathrm{cb}}(\iota_{r-1})} & \mathrm{H}^4_{\mathrm{cb}}(G_{r-1}) & \longrightarrow & * & \longrightarrow & * & \longrightarrow & * \\
\mathrm{H}^3_{\mathrm{cb}}(G_r) & \xrightarrow{\mathrm{H}^3_{\mathrm{cb}}(\iota_{r-1})} & \mathrm{H}^3_{\mathrm{cb}}(G_{r-1}) & \longrightarrow & * & \longrightarrow & * & \longrightarrow & * \\
\mathrm{H}^2_{\mathrm{cb}}(G_r) & \longrightarrow & \mathrm{H}^2_{\mathrm{cb}}(G_{r-1}) & \longrightarrow & \mathrm{H}^2_{\mathrm{cb}}(G_{r-1}) & \longrightarrow & * & \longrightarrow & * \\
* & \longrightarrow & * & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & * \\
* & \longrightarrow & * & \longrightarrow & * & \longrightarrow & L^\infty\big(\mathcal{P}_r^3\big)^{G_r} & \xrightarrow{d_r^2} & L^\infty\big(\mathcal{P}_r^4\big)^{G_r}
\end{array}$$

Note that by Proposition 5.2.7 we have $\mathrm{d}_2^{0,4} = 0$. Hence for $r \geq 5$ the second page of $\mathrm{E}_\bullet^{\bullet,\bullet}$ is conjugated to

$$\begin{array}{cccccc}
\ker(\mathrm{H}^4_{\mathrm{cb}}(\iota_{r-1})) & * & * & * & * & * \\
 & \searrow^{0} & & & & \\
* & 0 & * & * & * & * \\
* & * & 0 & 0 & * & * \\
* & * & * & 0 & 0 & * \\
* & * & * & * & * & \mathrm{H}^4_{\mathrm{mb}}(G_r \curvearrowright \mathcal{P}_r)
\end{array}$$

Since $\mathrm{E}_\bullet^{\bullet,\bullet}$ converges to zero, $\mathrm{d}_5^{0,4}$ is an isomorphism conjugated to

$$\ker\big(\mathrm{H}^4_{\mathrm{cb}}(\iota_{r-1})\big) \to \mathrm{H}^4_{\mathrm{mb}}(G_r \curvearrowright \mathcal{P}_r), \qquad r \geq 5.$$ □

Proof of Theorem C. Using Lemma 6.2.7, this is analogous to the proof of Proposition A. □

Lemma 6.2.8. *Let G_r be a complex classical group. Then we have a linear isomorphism*

$$\ker\big(\mathrm{H}^4_{\mathrm{cb}}(\iota_r)\big) \cong \mathrm{H}^4_{\mathrm{mb}}(G_{r+1} \curvearrowright \mathcal{P}_{r+1}), \qquad r \geq 1.$$

Proof. The spectral sequence $\mathrm{E}_{\bullet}^{\bullet,\bullet}$ has first page conjugated to

$$\begin{array}{ccccccccccc}
\mathrm{H}^4_{\mathrm{cb}}(G_{r+1}) & \xrightarrow{\mathrm{H}^4_{\mathrm{cb}}(\iota_r)} & \mathrm{H}^4_{\mathrm{cb}}(G_r) & \longrightarrow & * & \longrightarrow & * & \longrightarrow & * & \longrightarrow & * \\
\mathrm{H}^3_{\mathrm{cb}}(G_{r+1}) & \xhookrightarrow{\mathrm{H}^3_{\mathrm{cb}}(\iota_r)} & \mathrm{H}^3_{\mathrm{cb}}(G_r) & \xrightarrow{0} & \mathrm{H}^3_{\mathrm{cb}}(G_r) & \longrightarrow & * & \longrightarrow & * & \longrightarrow & * \\
* & \longrightarrow & * & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & * & \longrightarrow & * \\
* & \longrightarrow & * & \longrightarrow & * & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & * \\
* & \longrightarrow & * & \longrightarrow & * & \longrightarrow & * & \longrightarrow & * & \longrightarrow & L^{\infty}\big(\mathcal{P}^5_{r+1}\big)^{G_{r+1}}
\end{array}$$

Note that by Proposition 5.2.7 we have $\mathrm{d}_2^{0,4} = 0$. Thus, its second page is conjugated to

$$\begin{array}{cccccc}
\ker(\mathrm{H}^4_{\mathrm{cb}}(\iota_r)) & * & * & * & * & * \\
 & \searrow^{0} & & & & \\
* & 0 & * & * & * & * \\
* & * & 0 & 0 & * & * \\
* & * & * & 0 & 0 & * \\
* & * & * & * & * & \mathrm{H}^4_{\mathrm{mb}}(G_{r+1} \curvearrowright \mathcal{P}_{r+1})
\end{array}$$

Since $\mathrm{E}_{\bullet}^{\bullet,\bullet}$ converges to zero, $\mathrm{d}_5^{0,4}$ is an isomorphism conjugated to

$$\ker\big(\mathrm{H}^4_{\mathrm{cb}}(\iota_r)\big) \xrightarrow{\cong} \mathrm{H}^4_{\mathrm{mb}}(G_{r+1} \curvearrowright \mathcal{P}_{r+1}). \qquad \square$$

Proof of Theorem D. Using Lemma 6.2.8, this is analogous to the proof of Proposition A. $\square$

6.2.4. Proofs of Our Orthogonal Vanishing Theorem and Our Corollaries

Proof of Our Orthogonal Vanishing Theorem. By Remark 2.5.14 we have $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(r,r)) = 0$ for $r = 1, 2, 3$. Theorem A now yields $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{O}(r,r)) = 0$ for all $r \geq 1$. $\square$

Proof of Corollary A. Follows from Theorem A, Corollary 2.2.18, and Corollary 2.2.23. $\square$

Remark 6.2.9. Note that $\mathrm{H}^n_{\mathrm{cb}}(\mathrm{SO}(5,\mathbb{C})) \cong \mathrm{H}^n_{\mathrm{cb}}(\mathrm{Sp}(4,\mathbb{C}))$. To establish the Isomorphism Conjecture 2.4.2 in degree four for the families $(\mathrm{SO}(2r+1,\mathbb{C}))_r$ and $(\mathrm{Sp}(2r,\mathbb{C}))_r$, it thus remains to show that one of the groups $\mathrm{SO}(5,\mathbb{C})$ or $\mathrm{Sp}(4,\mathbb{C})$ has vanishing fourth continuous bounded cohomology.

Proof of Corollary B. Follows from Theorem A and the sporadic isogeny $\mathrm{Sp}(4,\mathbb{R}) \twoheadrightarrow \mathrm{SO}_0(3,2)$. □

Proof of Corollary C. This follows from our Main Theorem and [103, Corollary 4.8]. □

Proof of Corollary D. This follows from Corollary C and Ivanov's extension of Gromov's mapping theorem ([82, Ch. 6]). □

Proof of Corollary E. This follows from our Main Theorem and [72, Corollary 4]. □

6.3. Outlook

Let us conclude this thesis with a discussion of possible directions for further research.

Questions 6.3.1. We start with questions concerning the recent work of Bucher and Savini [17], [18], [20], [21].

(i) In rank 1, the Isomorphism Conjecture 2.4.2 is open in degree 3 for the groups $\mathrm{Sp}(n,1)$ and $F_{4(-20)}$. In degree 4 this conjecture is additionally open for the groups $\mathrm{SU}(n,1)$. Can the methods of Remark 2.4.6 and Remark 2.4.8 yield injectivity in these cases?

(ii) For which groups G is $G\backslash((G/P)^{(3)})$ finite or compact (c.f. Remark 2.5.8)? We conjecture that this space is compact if and only if G is a product of rank 1 groups.

(iii) Can the methods of Remark 2.4.6 and Remark 2.4.8 yield injectivity for higher rank groups other than $\mathrm{SL}(3,\mathbb{K})$?

(iv) Can the estimations in the proof of Theorem 2.5.5 be extended to other functional equations, for example to [42, Equation (4.3)]?

Questions 6.3.2. We continue with questions concerning the work of De la Cruz Mengual, Hartnick, and Ott [42], [43], [44], [71].

(i) Is it possible to show bounded-cohomological stability for the quaternionic families $\mathrm{Sp}(r+d,r)$ and $\mathrm{O}^*(4r+2d)$ as in [43]?

(ii) Is it possible to salvage the proof of the Isomorphism Conjecture 2.4.2 in degree 3 for the family $\mathrm{SO}(2r,\mathbb{C})$ in [42]?

(iii) The methods of Bucher and Savini, together with the methods of the present thesis, often leave a gap for medium-rank groups (see Remark 6.2.9). Can the methods of [71] be used to fill these gaps?

Questions 6.3.3. We close with questions concerning the present thesis.

(i) If Question 6.3.2 (i) can be answered affirmatively, can the secondary stability methods of [42] and the present thesis be applied to these quaternionic families? We conjecture that this is the case; we further conjecture that the range obtained would not be low enough to yield additional cases of the Isomorphism Conjecture 2.4.2.

(ii) Do the block inclusions of the families $\mathrm{SU}(r+d, r)$ and $\mathrm{SO}^*(4r+2d)$ induce isomorphisms in second continuous cohomology? We conjecture that this can be proved as in Section 6.1.

(iii) Can the range of Theorem 3.1.6 be improved? We conjecture that this range can be reduced to $r \geq \lfloor \frac{k+2}{2} \rfloor$ in the symplectic case, which could be proved by establishing cross-ratio parametrizations of $G_r \backslash \mathcal{P}_r^k$ for $k \geq 5$. Equivalently, we conjecture that the diagram in Corollary 4.5.4 commutes for $r \geq r_{k+2}$ for *all* families of classical spaces.

(iv) Can the secondary stability methods of the present thesis be improved to further decrease the range of the theorems in this chapter? One way to do this would be to parametrize the orbit space $G_r \backslash \mathcal{P}_r^k$ using cross-ratios and to determine functional equations as in Lemma 3.5.7 to obtain the desired isomorphisms in cohomology without establishing isomorphisms, as in Corollary 4.5.4, on the cochain level (this seems rather difficult). Another possibility would be to replace $\mathcal{P}_r$ by another G_r-space which has better bounded-cohomological stabilization properties.

(v) Can the secondary stability methods of the present thesis be applied in higher degree? The author knows of three possibilities as to how this could be done. Firstly, one might be able to extend the idea of the proof of Proposition 5.2.7 to other maps on higher pages of the spectral sequence. Secondly, as an extension of [44, Lemma 3.7], one can show that for $G_r = \mathrm{Sp}(2r, \mathbb{K})$ the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{E}_1^{2,q} & \xrightarrow{\mathrm{d}_1^{2,q}} & \mathrm{E}_1^{3,q} & \xrightarrow{\mathrm{d}_1^{3,q}} & \mathrm{E}_1^{4,q} \\ \cong\downarrow & & \downarrow\cong & & \downarrow\cong \\ \mathrm{H}^q_{\mathrm{cb}}(G_{r-1}) & \xrightarrow[\mathrm{H}^q_{\mathrm{cb}}(\iota_{r-2}; D^1)]{} & \mathrm{H}^q_{\mathrm{cb}}(G_{r-2}; L^\infty(\mathcal{C}^{(3)})) & \xrightarrow[\mathrm{H}^q_{\mathrm{cb}}(\mathrm{id}; D^2)]{} & \mathrm{H}^q_{\mathrm{cb}}(G_{r-2}; L^\infty(\mathcal{C}^{(4)})) \end{array}$$

Here, $D^1 \colon \mathbb{R} \to L^\infty(\mathcal{C}^{(3)})$ is the inclusion of constants and D^2 is the map from Lemma 3.5.7. We conjecture that the lower row is exact at $\mathrm{H}^q_{\mathrm{cb}}(G_{r-2}; L^\infty(\mathcal{C}^{(3)}))$ and we expect that situation to be analogous both for other families of groups, as well as for the maps $\mathrm{d}_1^{p,q}$ for $p > 3$. This more explicit description of the differential could be used to prove vanishing of $\mathrm{E}_2^{p,q}$ for the relevant entries (p, q).

Both of these possibilities are limited by the lack of cross-ratio parametrizations of $G_r \backslash \mathcal{P}_r^k$ and the lacking understanding of the corresponding functional equations.

Thirdly, one could replace $\mathcal{P}_r$ by another homogeneous G_r-space which has better transitivity properties.

(vi) Lastly, what are the concrete algebro-geometric implications of the results in Example A? How does the bounded cohomology of these spaces behave when passing to some compactification?

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