



# Heterogeneous Dynamic Logic: Provability Modulo Program Theories

SAMUEL TEUBER, Karlsruhe Institute of Technology (KIT), Germany  
MATTIAS ULBRICH, Karlsruhe Institute of Technology (KIT), Germany  
ANDRÉ PLATZER, Karlsruhe Institute of Technology (KIT), Germany  
BERNHARD BECKERT, Karlsruhe Institute of Technology (KIT), Germany

Formally specifying, let alone verifying, properties of systems involving multiple programming languages is inherently challenging. We introduce *Heterogeneous Dynamic Logic* (HDL), a framework for combining reasoning principles from distinct (dynamic) program logics in a modular and compositional way. HDL mirrors the architecture of satisfiability modulo theories (SMT): Individual dynamic logics, along with their calculi, are treated as *dynamic theories* that can be combined to reason about heterogeneous systems whose components are verified using different program logics. HDL provides two key operations: *Lifting* extends an individual dynamic theory with new program constructs (e.g., the *havoc* operation or regular programs) and automatically augments its calculus with sound reasoning principles for the new constructs; and *Combination* enables cross-language reasoning in a *single modality* via *Heterogeneous Dynamic Theories*, facilitating the reuse of existing proof infrastructure. By lifting combined theories with regular programs, we obtain *heterogeneous* control structures that allow us to reason about intertwined cross-language behavior. We formalize dynamic theories, their lifting and combination, and prove the soundness of all proof rules in Isabelle. We also introduce a proof rule combining deductive DL-based reasoning with reasoning principles from Kleene Algebras with Tests. Furthermore, we prove *relative completeness* theorems for lifting and combination: Under usual assumptions, reasoning about lifted or combined theories is no harder than reasoning about the constituent dynamic theories and their common first-order structure (i.e., the “data theory”). We demonstrate HDL’s value by verifying an automotive case study where a Java controller (formalized in Java dynamic logic) steers a plant model (formalized in differential dynamic logic).

CCS Concepts: • **Theory of computation** → **Program verification**; **Logic and verification**; **Modal and temporal logics**; • **Software and its engineering** → **Software verification**.

Additional Key Words and Phrases: Dynamic Logic, Theory Combination, Relative Completeness, Isabelle

## ACM Reference Format:

Samuel Teuber, Mattias Ulbrich, André Platzer, and Bernhard Beckert. 2026. Heterogeneous Dynamic Logic: Provability Modulo Program Theories. *Proc. ACM Program. Lang.* 10, PLDI, Article 217 (June 2026), 26 pages. <https://doi.org/10.1145/3808295>

## 1 Introduction

Since its origins in Hoare logic [32] and Dijkstra’s predicate transformers [19], formal verification has grown into a mature discipline, supporting a wide range of languages, from bytecode [46, 67], C [16, 37], or Java [2, 11], to modern languages like Rust [4, 40] and even to hybrid programs for cyber-physical systems [49]. While individual “worlds” enjoy robust verification infrastructures, it remains challenging to *rigorously combine* verification results across languages. Yet, many modern

---

Authors’ Contact Information: Samuel Teuber, Karlsruhe Institute of Technology (KIT), Karlsruhe, Germany, [teuber@kit.edu](mailto:teuber@kit.edu); Mattias Ulbrich, Karlsruhe Institute of Technology (KIT), Karlsruhe, Germany, [ulbrich@kit.edu](mailto:ulbrich@kit.edu); André Platzer, Karlsruhe Institute of Technology (KIT), Karlsruhe, Germany, [platzer@kit.edu](mailto:platzer@kit.edu); Bernhard Beckert, Karlsruhe Institute of Technology (KIT), Karlsruhe, Germany, [beckert@kit.edu](mailto:beckert@kit.edu).



This work is licensed under a Creative Commons Attribution 4.0 International License.

© 2026 Copyright held by the owner/author(s).

ACM 2475-1421/2026/6-ART217

<https://doi.org/10.1145/3808295>

systems are inherently heterogeneous and would benefit from methodologies enabling *compositional* reasoning across program logics. This raises the fundamental questions of how semantics and proof principles of program logics can be combined to *specify* and *verify* heterogeneous systems. In practice, heterogeneous verification today often resorts to encoding one system (e.g., a software controller) in the language of another (e.g., hybrid programs) [27, 36, 38, 62]. However, such encodings are neither seamless nor intuitive: They obscure structure and fail to reuse existing infrastructure.

*Dynamic Logic.* We argue that Dynamic Logic (DL) provides a strong foundation for reasoning about heterogeneous systems due to its expressiveness and its structural properties. DL generalizes traditional verification techniques [19, 32] by treating programs as first-class citizens of the logic embedded within modalities (rather than residing outside the logic). This enables DL to specify a large variety of systems including imperative languages [30] or hybrid systems [49]. Crucially, DL’s treatment of programs gives rise to a fully compositional logic that naturally supports properties such as liveness [61] or incorrectness [55] as well as hyperproperties (e.g. refinement [41, 53, 68] or secure information flow [8]) *without necessitating specialized logics* (as necessary for, e.g., Hoare logic). However, prior DL frameworks are confined to a *single, fixed programming language*, and are thus ill-suited for verifying heterogeneous systems that combine components from multiple “worlds” (e.g. Java code and hybrid systems). As a lighter-weight counterpart to DL, Kleene Algebra with Tests [39] (KAT) is an effective methodology for (automated) program verification. Our approach also integrates with program rewriting via KAT to ease heterogeneous verification.

*Satisfiability Modulo Theories.* The success of satisfiability modulo theories (SMT) has transformed first-order reasoning across a wide range of applications. By requiring a *unified set of assumptions* (e.g., stable infiniteness [45]) from first-order theories, SMT solvers can delegate theory-specific queries, e.g., involving integers or arrays, to specialized sub-solvers. Users can freely combine theories within a single formula while theory combination mechanisms ensure soundness and, under suitable assumptions, even completeness. The seminal results by Shostak, Nelson and Oppen, and their subsequent extensions [6, 43, 45, 60, 65, 66] thus paved the way for the modular combination of first-order theories. This enables a *separation of concerns* between theory-specific reasoning and the logical structure connecting the theories. Inspired by this design, we ask: Can a similar, modular structure be applied to program logics? That is, can we reason about heterogeneous programs while preserving the semantics and proof rules established for individual languages? This paper answers the question in the affirmative, building on the rigorous foundations of dynamic logic.

*Contribution.* This paper introduces *Heterogeneous Dynamic Logic* (HDL): The foundational formal notions that allow DL to serve as a generalized, SMT-inspired vehicle to combine different *program logics* (as *dynamic theories*), just like first-order logic serves as a vehicle to combine different *data logics* (as first-order theories). Heterogeneous Dynamic Logic supports two key operations:

- *Lifting* (Section 6) extends a given *dynamic theory*  $\Delta$  with additional program constructs, such as nondeterministic choice ( $\Delta^{\text{nc}}$ , Section 6.1) or the closure over regular programs ( $\Delta^{\text{reg}}$ , Section 6.2). Lifting also automatically extends the dynamic theory’s calculus with sound proof rules to reason about the extended programming language.
- *Combination* (Section 7) merges two theories  $\Delta^{(0)}$  and  $\Delta^{(1)}$  (defined over programming languages  $\Lambda_p^{(0)}, \Lambda_p^{(1)}$ ) into a single *heterogeneous dynamic theory*  $\hat{\Delta}$ . This new dynamic theory supports *heterogeneous programs* in which programs from  $\Lambda_p^{(0)}$  and  $\Lambda_p^{(1)}$  can be composed freely using regular programs. Combination also merges and extends the individual proof calculi.

This enables rigorous, modular reasoning over *heterogeneous programs* with deeply intertwined behavior of programs stemming from individual logics. Along with a relatively complete proof

calculus for lifted/combined dynamic theories, we also present an axiom leveraging KAT-style equational rewriting – coupling KAT’s reasoning principles with DL’s proof system. We introduce a formalization of dynamic theories, their lifting and combination and their sound proof calculi in Isabelle in a reusable manner via its Locale infrastructure (formalized Lemmas/Theorems are in [cyan](#), see Appendix D). This formalization directly supports interactive theorem proving for heterogeneous systems with proof infrastructure reuse. Simultaneously, the SMT-like architecture of our proof calculus lays the groundwork for automated heterogeneous verification in the long term.

*Overview.* The remainder of this paper is structured as follows: Section 2 motivates the challenge of heterogeneous verification using a concrete case study. Section 3 surveys prior attempts to support general cross-language verification. Sections 4 and 5 introduce our notion of *dynamic theories*, a generalization of dynamic logic with reduced assumptions, and demonstrate that our assumptions suffice to recover well-known properties of “DL-like” logics. In this context, we also connect DL’s proof calculus to equational reasoning via KAT. Section 6 presents *lifting*, an approach to extend a given dynamic theory with additional program features such as regular programs along with deriving the necessary proof rules. To this end, we also prove that our regular program lifting results in a KAT which enables tool reuse. Section 7 proves that the *heterogeneous dynamic theory* over two dynamic theories is once again a *dynamic theory*. Corresponding additional proof rules are derived including additional KAT-style identities which ease verification in practice. Finally, Section 8 establishes conditions under which *relative completeness* transfers from a dynamic theory to its lifted and/or combined version. Section 9 showcases the utility of HDL on our automotive case study. Proofs can be found in the extended version on arXiv [64].

## 2 Motivation: Case Study

As example of a heterogeneous system we consider a car steered by a software controller written in Java while driving towards a stop sign. To effectively leverage existing proof infrastructures, we wish to model the time-continuous, physical system in Differential Dynamic Logic [49] ( $d\mathcal{L}$ ) using KeYmaera X [24, 48, 49, 52] while verifying the controller using Java Dynamic Logic [7] (JavaDL) using the KeY tool [2].

*Java Dynamic Logic.* JavaDL allows reasoning about Java programs [2, 7, 9] (including exceptions, heap state etc.). Consider the method in Figure 1a serving as controller for a car approaching a stop sign at distance  $p$  with velocity  $v$ . The method computes an acceleration via the *stateful* field `this.acc` and brakes in case the condition in line 3 is satisfied. The method is not idempotent due to `this.acc` and its computation also depends on static variables `amax`, `amin` and `T`. While JavaDL (and KeY) is capable of verifying Java programs, JavaDL cannot reason about physical dynamics, e.g. given as differential equations.

*Differential Dynamic Logic.* Differential Dynamic Logic [49] ( $d\mathcal{L}$ ) allows reasoning about *hybrid programs* whose states evolve either along discrete state transitions (variable assignments) or continuously along differential equations. Consider the hybrid program  $env$  in Figure 1b which

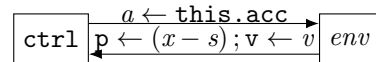
```

1 int x1 = v+1;
2 int x2 = x1*x1;
3 if(10000*x2 + (amax-amin)*T*
4   (amax*T + 200*x1) > -20000*amin*(p-1)) {
5   this.acc = amin;
6 } else {
7   this.acc += (p>10000) ? 10 : -10;
8   this.acc = (this.acc > amax) ? amax :
9             (this.acc < amin) ? amin : this.acc; }

```

(a) Java Controller `ctrl` that can be analyzed using JavaDL.

$env \equiv \left( t := 0; (x' = v, v' = a, t' = 1 \ \& \ t \leq T) \right);$   
 (b) Environment model that can be analyzed in Differential Dynamic Logic ( $d\mathcal{L}$ ).



(c) The heterogeneous system ( $\leftarrow$  denotes some kind of assignment; semantics see Section 9).

Fig. 1. Heterogeneous system case study

sets a clock variable  $t$  to 0 and evolves  $x$ ,  $v$  and  $t$  along a differential equation for a maximum time of  $T$ . The program describes the evolution of a car's position under constant acceleration  $a$ . Using the proof calculus implemented in the theorem prover KeYmaera X [24, 48, 49, 52], we can prove statements about hybrid programs. However, KeYmaera X cannot natively reason over Java programs which, e.g., encompass integer arithmetic with overflows and heap state invariants.

*Challenge.* Both of the introduced verification methodologies come with strengths and weaknesses. While they excel at verifying properties for their “native” domain of computation, they lack support for their respective counterpart. However, to verify the safety of the entire heterogeneous system, we require a formalism that allows us to reason about the *looped* heterogeneous system sketched in Figure 1c. We propose a *general* framework that gives a formal semantics to this sketch and provides a proof calculus that emphasizes the *reuse* of proof infrastructure. Our framework is not tied to the particular combination of languages discussed in this case study.

### 3 Related Work

*Dynamic Logic.* As summarized by Ahrendt *et al.* [3], DL has been instantiated for a wide range of behavioral languages (e.g., Java [2], hybrid programs [49, 50], byte code [67, 68] or abstract state machines [56]) allowing proofs of numerous properties (functional correctness [10, 17], safety [34, 51], liveness [61], incorrectness [55], refinement [41, 68] or information flow [8, 13]). However, the *combination* of behavioral languages has not previously been studied.

*Parameterized Dynamic Logic.* Parameterized Dynamic Logic [69] is a framework designed to build a dynamic logic and a calculus for a given programming language based on symbolic execution rules. While their approach also generalizes the concept of dynamic logic, it does not address the challenge to reason about *heterogeneous* programs which is at the core of this paper.

*Satisfiability Modulo Theories.* Research on SMT theory combination [6, 43, 45, 65, 66] is not readily extensible to modal/dynamic logic: Much of the theory is limited to quantifier-free first-order logic (which is even more restrictive than full first-order (dynamic) logic), and classic SMT theory combination has no way of dealing with a dynamic logic's program primitives.

*Intermediate Verification Languages.* To reason about programs or state transition systems across languages, an effective approach is to encode them into a unified intermediate representation. This paradigm is used in systems like Boogie [5], Why3 [21] and Viper [42], which enable language-agnostic reasoning. Similarly, in model checking, common input formats like SMV and exchange formats such as VMT [15] and Moxi [35] facilitate analyzing heterogeneous systems.

In contrast, heterogeneous dynamic logic deliberately avoids assuming a shared language or logic, aiming to integrate verification systems without a clear common ground. For instance, differential dynamic logic supports continuous value evolution through differential equations – a capability that cannot be naturally encoded within the languages mentioned earlier.

*Fibring.* In contrast to logic fibring [25, 26] (established for modal logics), Heterogeneous Dynamic Logic goes beyond the fibring of “truth bearing” entities that one could expect from a DL with multiple modalities. For example, fibring could produce formulas of the form  $\phi_1 \rightarrow [\alpha_1]_1 [\alpha_2]_2 \phi_2$  (see Section 7.1) where  $[\cdot]_1$  and  $[\cdot]_2$  represent modalities for distinct programming languages. While merging of truth-bearing entities can encode *some* behaviors observed in heterogeneous systems, there is a large class of behaviors that *cannot* be modeled by merely merging truth bearing entities since it ignores the programs. In particular, we cannot model looped execution.

*UTP and Heterogeneous Bisimulation.* The Unified Theories of Programming [33] (UTP) and its Isabelle formalization [23] provide a common framework for integrating diverse programming

languages into a common semantics with *shared state*. While this enables proofs on cross-language systems, it typically requires embedding existing program semantics within UTP. In contrast, HDL aims for modular and sound *reuse* of existing (external) verification infrastructure via the common interface provided by DL. For example, our case study natively leverages the existing DL-based proof infrastructures for Java [2] and hybrid programs [24] unchanged, whereas the UTP community has invested considerable effort (notably, Foster’s formalisation of hybrid relations [22]) to embed hybrid programs *within* UTP. This highlights HDL’s orthogonal philosophy in comparison to UTP. HDL also provides native support for a wide range of properties that can be encoded in DL (see also [3, Sec. 3]).

The range of properties also distinguishes HDL from heterogeneous (bi)simulation [44] which is designed for *comparison* of systems (e.g., a labeled transition system vs. a probabilistic system).

*Contract-based and Polyglot Verification.* Our approach provides three key advantages in comparison to contract-based techniques: First, in our framework all components, all communication, all invariants, and all desired properties can be modeled in a single integral formal system and can later be proved in a single formula. While language agnostic contract languages like Contract-LIB [20] aim to bridge the specification gap between different PLs, they cannot model heterogeneous systems in a single program. Instead, they require a meta-level argument on combination outside any program logic. Secondly, in its usual formulation, contracts *overapproximate* functional behavior and are hence only suitable for correctness/safety verification (box-properties) and not liveness/relational properties supported by DL. Finally, our relative completeness results in Section 8 ensure that we can prove all properties of interest using our calculus by reusing proofs for individual components.

Polyglot verification, specifically PolyVer [14], advocates for a contract-based verification approach for cross-language verification. While PolyVer can alleviate the issue of requiring meta-level arguments by providing an automaton-based semantics for the polyglot system, their approach does not provide a relative completeness guarantee. Hence, it is a priori unclear whether their specification language is sufficiently expressive to prove all safety properties. In contrast, PolyVer proposes an interesting, AI-based approach to synthesize auxiliary contracts. Evaluating similar techniques for heterogeneous dynamic logic could be an exciting direction of future work.

*Kleene Algebras.* Kleene Algebra Modulo Theories [29] (KAMT) is a conceptually related approach that proposes to turn a client theory into a Kleene Algebra with Tests (KAT). In contrast to KAMT, we derive a *dynamic logic* and emphasize *reuse* of proof infrastructure for combined theories. Kozen [39] motivated KAT with the observation that “many simple program manipulations [...] do not require the full power of [propositional DL]”; indeed, dynamic logics support a *wider* range of properties, including *quantifiers*, liveness, or properties with *alternating modalities*, which are typically beyond KA(M)T’s power. While DL’s expressivity precludes decidability, we prove relative completeness results for our calculus under theory lifting or combination (see Section 8). Moreover, as axiom (E) demonstrates, we can also integrate KA(M)T results into our reasoning where applicable. In complementary research, Kleene Algebra with Domain [18, 57, 58] extends KAT with (anti-)domain operators to recover propositional DL expressivity, but remains in the propositional, *quantifier-free* setting.

## 4 Dynamic Theories

The foundation of our approach is the idea behind First-Order Dynamic Logic (FODL) [30, 31], which extends first-order logic to a multi-modal logic where programs parameterize modalities. For instance, for an integer first-order theory, the formula

$$\underbrace{0 \leq v}_{\text{atom}} \rightarrow \underbrace{[w := v + 1]}_{\text{program}}, \underbrace{1 \leq w}_{\text{atom}} \quad (1)$$

states that after all execution of  $w := v + 1$ ,  $w$  is positive if  $v$  was non-negative. Unlike Hoare triples [32] or predicate transformers [19], Dynamic Logic treats programs as first-class citizens of formulas, enabling compositional reasoning about richer properties such as reachability ( $\langle w := v + 1 \rangle w > 0$ ) and hyper-properties ( $[p] \langle q \rangle F$ ).

Our contribution generalizes this principle: not only the first-order theory but also the underlying program language can be chosen freely. We introduce *dynamic theories* as a minimal semantic interface capturing the assumptions on programs and first-order reasoning. Concrete program logics (existing or new) are instantiations of a dynamic theory; multiple such theories can be composed into a heterogeneous dynamic theory that still satisfies the same interface. This composability allows “stacking” theories while retaining derived proof rules. By parameterizing over both programs and atoms, our framework unifies diverse program logics and provides a reusable foundation akin to SMT theory combination — ensuring that compliant dynamic logics can interoperate and inherit proof rules automatically.

#### 4.1 Syntax

To focus the presentation on the modal aspects of the logic, we abstract away from the term structure of a Dynamic Logic’s first-order fragment and instead begin our investigation directly at the level of first-order atoms. We formalize the syntactic material of a dynamic theory in the notion of a *dynamic signature*:

**Definition 4.1** (Dynamic Signature). *A dynamic signature  $\Lambda$  is given by a set of variables  $\Lambda_V$ , a set of first-order atoms  $\Lambda_A$ , and the set of programs  $\Lambda_P$  with all three sets pairwise disjoint. We denote this as  $\Lambda = (\Lambda_V, \Lambda_A, \Lambda_P)$*

Given a signature  $\Lambda$ , we can define the structure of formulas that are part of a dynamic theory in a manner that is similar to classical FODL while constraining programs and variables to all syntactical constructs contained in  $\Lambda$ :

**Definition 4.2** ( $\Lambda$ -formulas). *Let  $\Lambda = (\Lambda_V, \Lambda_A, \Lambda_P)$  be a dynamic signature. Given a variable  $v \in \Lambda_V$ , an atom  $a \in \Lambda_A$ , and a program  $p \in \Lambda_P$ ,  $\Lambda$ -formulas are defined by the following grammar:  $F, G ::= a \mid \neg F \mid F \wedge G \mid \forall v F \mid [p] F$*

The set of all  $\Lambda$ -Formulas is denoted as  $\text{Fml}_\Lambda$ . We denote the first-order fragment (i.e. all formulas without any box modality) as  $\text{FOL}_\Lambda \subset \text{Fml}_\Lambda$ . Further, we introduce syntactic sugar for the diamond modality, which we denote as  $\langle p \rangle F \equiv \neg [p] \neg F$ . Similarly, syntactic sugar can be introduced for existential quantifiers and other logical connectives ( $\vee, \rightarrow, \dots$ ).

**Example 4.1** (Syntax for Ordered Semirings). As a running example, consider an ordered semi-ring  $(\mathcal{R}, +, \cdot, \leq)$  and an arbitrary set of variables  $\Lambda_V^\mathcal{R} = \{v_1, v_2, \dots\}$ : We define terms as literals  $c \in L \subseteq \mathcal{R}$ , variables  $v \in \Lambda_V^\mathcal{R}$  and any composition of terms with  $+$  and  $\cdot$ . Atoms are any formulas of the form  $t_1 \leq t_2$  and programs have the form  $v := t_1$  (for  $t_1, t_2$  terms and  $v \in \Lambda_V^\mathcal{R}$ ). Together, these components form a dynamic signature containing variables, atoms and programs.

#### 4.2 Semantics

The previous section outlined the syntactic constructs necessary to define a dynamic theory. With formulas of our dynamic theory defined, we can now focus on the semantic constructs that are necessary to define a dynamic theory. To this end, we require five components: First, since we are considering a modal logic, we require a state space  $\mathcal{S}$  over which we can evaluate  $\Lambda$ -formulas. Secondly, we require functions to evaluate the values of variables ( $\mathcal{V}$ ) w.r.t. some universe ( $\mathcal{U}$ ), the truth-value of atoms ( $\mathcal{A}$ ), and the state-transition behavior of programs ( $\mathcal{P}$ ). Each of these

components must satisfy some conditions. Once these components are defined, we can define the evaluation of formulas. We now begin by defining state spaces and variable evaluations:

**Definition 4.3** (Universes, State Spaces and Variable Evaluation). *Given a dynamic signature  $\Lambda$  with variables  $\Lambda_V$  and a set  $\mathcal{U} \neq \emptyset$  called universe, a state space  $\mathcal{S} \neq \emptyset$  is defined as a set of states together with a variable evaluation function  $\mathcal{V} : \mathcal{S} \times \Lambda_V \rightarrow \mathcal{U}$  such that  $\mathcal{S}$  and  $\mathcal{V}$  satisfy:*

**(S INDEPENDENCE)** *For all  $\mu, \sigma \in \mathcal{S}$  and  $V \subseteq \Lambda_V$ , there exists an  $\omega \in \mathcal{S}$  such that for all  $v \in \Lambda_V$ :*  

$$\mathcal{V}(\omega, v) = \mathcal{V}(\mu, v) \text{ if } v \in V \text{ and else } \mathcal{V}(\omega, v) = \mathcal{V}(\sigma, v)$$

This definition of state spaces slightly deviates from popular definitions of Dynamic Logic that often consider the state space as the set of all mappings from  $\Lambda_V$  to  $\mathcal{U}$  which for us corresponds to defining  $\mathcal{V}(\mu, v) = \mu(v)$ . However, our set-based definition allows for more complex state spaces. For example, we can globally exclude the possibility of a variable  $v$  taking on certain values  $\delta \in \mathcal{U}$ , which can be leveraged for typed/sorted state spaces. Similarly, we can construct state spaces that are formed by composing states from other state spaces (an example of this can be found in Section 7). Our approach also circumvents formalization hurdles in Isabelle/HOL that would accompany a dependently typed mapping approach for state spaces. Importantly, the state independence property retains a notion of well-formedness of the state space which ensures that if a variable  $v \in \Lambda_V$  is assigned some value  $\delta \in \mathcal{U}$ , then any other state can also be modified so that  $v$  has that value  $\delta$ . For example, this can ensure that a variable which (in a more fine-grained view of the dynamic theory) is an integer variable can indeed be assigned all integer values independently of the remaining state. State independence is important for proving coincidence properties for formulas and programs which is crucial for proof rule derivation. For states  $\mu, \sigma \in \mathcal{S}$  we will say  $\mu$  and  $\sigma$  are equal on  $V \subseteq \Lambda_V$  (denoted  $\mu \equiv_V \sigma$ ) whenever for all  $v \in V$  it holds that  $\mathcal{V}(\mu, v) = \mathcal{V}(\sigma, v)$ .

For the semiring dynamic signature from Example 4.1 we can define the universe  $\mathcal{U} = \mathcal{R}$  and the state space as the set of all mappings  $\mu : \Lambda_V \rightarrow \mathcal{U}$ . The variable evaluation function then is  $\mathcal{V}^{\mathcal{R}}(\mu, v) = \mu(v)$ . For example, if  $\mathcal{R} = \mathbb{Z}$ , then a variable can take on any integer value.

A state can also be used for the evaluation of atoms which we define as follows:

**Definition 4.4** (Atom Evaluation). *Given a dynamic signature  $\Lambda$  with variables  $\Lambda_V$  and atoms  $\Lambda_A$  and a state space  $\mathcal{S}$  we define a free variable overapproximation  $\mathbf{FV}_A : \Lambda_A \rightarrow 2^{\Lambda_V}$  and an atom evaluation function  $\mathcal{A} : \Lambda_A \rightarrow 2^{\mathcal{S}}$  as any functions satisfying the following properties:*

**(FV<sub>A</sub> COVERAGE)** *For any atom  $a \in \Lambda_A$  and any states  $\mu, \sigma \in \mathcal{S}$*

*if  $\mu \equiv_{(\mathbf{FV}_A(a))} \sigma$  then  $\mu \in \mathcal{A}(a)$  iff  $\sigma \in \mathcal{A}(a)$*

**(FV<sub>A</sub> FINITE)**  *$\mathbf{FV}_A(a)$  is finite for all atoms  $a \in \Lambda_A$ .*

While the coverage property ensures that  $\mathbf{FV}_A$  covers all variables that influence the evaluation of a given atom, the finiteness property ensures that atoms are well-behaved in a sense that is classically expected in first-order logic (where only variables that syntactically occur in the finite formula can have an impact). These two properties are once again of particular importance for coincidence properties essential for deriving the dynamic theory proof rules.

Continuing our semiring dynamic signature from Example 4.1, we define the atom evaluation function that takes a state  $\mu$  and evaluates an atomic formula (and its resp. terms) w.r.t.  $\mu$  (i.e.  $\mu \in \mathcal{A}(t_1 \leq t_2)$  iff  $\mu(t_1) \leq \mu(t_2)$  in  $\mathbb{Z}$ , where  $\mu(t)$  evaluates the variables in term  $t$  w.r.t.  $\mu$ ). Concretely, a state  $\mu$  satisfies the atom  $0 \leq v$  iff  $\mathcal{V}^{\mathbb{Z}}(\mu, v) = \mu(v) \geq \mu(0) = 0$ . The definition of  $\mathbf{FV}_A^{\mathcal{R}}$  can then be given via straightforward syntactical analysis of terms and formulas.

Similarly to the semantics of atoms, the semantics of programs also have side conditions on how their behavior is influenced by variables which is captured by a program's free variables  $\mathbf{FV}_P$ . The side conditions also provide us with an (implicit) parameterization of programs w.r.t. their variables. Formally, we require the program evaluation function to satisfy the following constraints:

**Definition 4.5** (Program Evaluation). *Given a dynamic signature  $\Lambda$  with variables  $\Lambda_V$  and programs  $\Lambda_P$  and a state space  $\mathcal{S}$  we define a free variable overapproximation  $\mathbf{FV}_P : \Lambda_P \rightarrow 2^{\Lambda_V}$  and a program evaluation function  $\mathcal{P} : \Lambda_P \rightarrow 2^{\mathcal{S}^2}$  as any functions satisfying the following properties:*

**(FV<sub>P</sub> COVERAGE)** *For all programs  $p \in \Lambda_P$ , all states  $\mu, \sigma, \omega \in \mathcal{S}$ , and any set  $V \supseteq \mathbf{FV}_P(p)$ :  
if  $\mu \equiv_V \sigma$  and  $(\mu, \omega) \in \mathcal{P}(p)$ , then there exists a state  $\tilde{\omega}$  such that  
 $(\sigma, \tilde{\omega}) \in \mathcal{P}(p)$  and  $\omega \equiv_V \tilde{\omega}$*

**(FV<sub>P</sub> FINITE)** *For all programs  $p \in \Lambda_P$  the result of  $\mathbf{FV}_P(p)$  is finite.*

**(EXTENSIONALITY)** *For all programs  $p \in \Lambda_P$  and all states  $\mu, \tilde{\mu}, \sigma, \tilde{\sigma} \in \mathcal{S}$ :  
if  $\mu \equiv_{\Lambda_V} \tilde{\mu}$  and  $\sigma \equiv_{\Lambda_V} \tilde{\sigma}$ , then  $(\mu, \sigma) \in \mathcal{P}(p)$  iff  $(\tilde{\mu}, \tilde{\sigma}) \in \mathcal{P}(p)$*

The coverage property establishes that any variable possibly influencing the state transition of a program  $p$  must be part of  $\mathbf{FV}_P(p)$ . This is achieved by ensuring that if two states agree on a superset of  $\mathbf{FV}_P(p)$ , then there must exist “isomorphic” state transitions, i.e. state transitions in  $\mathcal{P}(p)$  where the resulting states equally agree on this set of variables. For the remainder of this paper, it is paramount that this property is not just satisfied for  $\mathbf{FV}_P(p)$ , but also for its supersets to ensure equivalent effects on any variables written by  $p$ . The second condition ensures that this new set of free variables is finite, too, which is relevant for central coincidence properties. Finally, the extensionality property ensures that program behavior is only dependent on variable assignment and not intensional differences between states which is also relevant for coincidence.

Recall the dynamic signature from Example 4.1, the program evaluation function for  $v := t_1$  induces a transition relation  $(\mu, \sigma) \in \mathcal{P}^{\mathcal{R}}(v := t_1)$  iff  $\sigma(v) = \mu(t_1)$  and  $\mu \equiv_{\{v\}^c} \sigma$  where  $\mu(t_1)$  is the evaluation of term  $t_1$  in state  $\mu$  (going forward we denote by  $V^c$  the complement set  $\Lambda_V \setminus V$  for  $V \subseteq \Lambda_V$ ).  $\mathbf{FV}_P$  is again obtained via syntactic analyses.

We summarize the semantics of a dynamic theory as a *domain of computation*, which together with its dynamic signature (syntax) fully specifies the theory:

**Definition 4.6** (Domain of Computation). *Given a dynamic signature  $\Lambda$ , a domain of computation of  $\Lambda$  (denoted as  $\mathcal{D} = (\mathcal{U}, \mathcal{S}, \mathcal{V}, \mathcal{A}, \mathbf{FV}_A, \mathcal{P}, \mathbf{FV}_P)$ ) consists of:*

- A universe  $\mathcal{U}$  (see Definition 4.3)
- A corresponding state space  $\mathcal{S}$  (see Definition 4.3)
- A variable evaluation function  $\mathcal{V}$  (see Definition 4.3)
- An atom evaluation function  $\mathcal{A}$  and free variable overapproximation  $\mathbf{FV}_A$  (see Definition 4.4)
- A program evaluation function  $\mathcal{P}$  and free variable overapproximation  $\mathbf{FV}_P$  (see Definition 4.5)

**Definition 4.7** (Dynamic Theory). *A dynamic theory is the combination of a dynamic signature  $\Lambda$  and a corresponding domain of computation  $\mathcal{D}$ . We denote a dynamic theory as  $\Delta = (\Lambda, \mathcal{D})$ .*

The semantics of  $\Lambda$ -formulas (Definition 4.2) for a dynamic theory  $\Delta = (\Lambda, \mathcal{D})$  are defined as follows:

**Definition 4.8** (Semantics of Formulas). *Consider a dynamic theory  $\Delta = (\Lambda, \mathcal{D})$ . We define the semantics of  $\Lambda$ -formulas as follows:*

$$\begin{aligned} \mathcal{D}[[a]] &= \mathcal{A}(a) & \mathcal{D}[[\neg F]] &= \mathcal{S} \setminus \mathcal{D}[[F]] & \mathcal{D}[[F \wedge G]] &= \mathcal{D}[[F]] \cap \mathcal{D}[[G]] \\ \mathcal{D}[[\forall v F]] &= \left\{ \mu \in \mathcal{S} \mid \text{for all } \sigma \in \mathcal{S} : \mu \equiv_{\{v\}^c} \sigma \text{ implies } \sigma \in \mathcal{D}[[F]] \right\} \\ \mathcal{D}[[[p] F]] &= \left\{ \mu \in \mathcal{S} \mid \text{for all } \sigma \in \mathcal{S} : (\mu, \sigma) \in \mathcal{P}(p) \text{ implies } \sigma \in \mathcal{D}[[F]] \right\} \end{aligned}$$

We say  $F$  is  $\Delta$ -valid (denoted as  $\models_{\Delta} F$ ) iff every state in  $\mathcal{S}$  satisfies  $F$ , i.e.  $\mathcal{S} = \mathcal{D}[[F]]$ . If a particular state  $\mu \in \mathcal{S}$  satisfies a formula  $F$  (i.e.,  $\mu \in \mathcal{D}[[F]]$ ) we denote this as  $\mu \models_{\Delta} F$  and say  $\mu$  satisfies  $F$  or that  $F$  is  $\Delta$ -satisfiable. For example, the formula  $F \wedge [p] G$  is true in a state  $\mu$  if  $\mu \models_{\Delta} F$  (i.e.  $\mu$  satisfies  $F$ ) and if for every  $(\mu, \sigma) \in \mathcal{D}[[p]$  it holds that  $\sigma \models_{\Delta} G$  (i.e. every such  $\sigma$  satisfies  $G$ ).

By now we have instantiated the semantics for the semiring dynamic signature  $(\mathcal{R}, +, \cdot, \leq)$  in Example 4.1. Lemma 4.2 shows that our construction yields a dynamic theory  $\Delta^{\mathcal{R}}$  satisfying all properties of Definition 4.6. A concrete example for a formula of this theory is Equation (1). Its program  $w := v + 1$  induces a state transition  $(\mu, \sigma) \in \mathcal{P}^{\mathbb{Z}}(w := v + 1)$  iff  $\sigma(w) = \mu(v + 1) = \mu(v) + 1$  (and  $\mu \equiv_{\{w\}\mathbb{C}} \sigma$ ). It is then easy to see that Equation (1) is indeed valid in  $\Delta^{\mathbb{Z}}$ : For any state with  $\mu(v) \geq 0$  and any state transition to  $\sigma$  such that  $\sigma(w) = \mu(v) + 1$  (and otherwise equal) it holds that  $\sigma(w) \geq 1$ .

*Relation to classical First-Order Dynamic Logic.* The syntax and semantics of classical first-order dynamic logic [31], are more restrictive in the supported structure of atoms and state spaces. State spaces are assumed to be simple valuations [31, p. 293] (and not more complex constructs admitting the evaluation of variables and atoms), and atoms are defined as predicates over terms [31, p. 103]. We demonstrate that axioms and proof rules derivable for classical, interpreted first-order dynamic logic are equally derivable using our smaller set of assumptions; Moreover, this relaxation, in turn, has two advantages: (1) The designer of a program logic, formalized as dynamic theory, can focus on the particularities of the program logic at hand and obtain extensions like regular programs or havoc operators for free (lifting, see Section 6), (2) Program logics formalized as dynamic theories can be combined arbitrarily in a manner that admits compositional reasoning (heterogeneous dynamic theories, see Section 7).

### 4.3 Static Semantics

For deriving proof rules, we need a precise notion of semantically free and bound variables, i.e. those that affect or are affected by a formula or program. Definition 4.6 already assumes overapproximations  $\mathbf{FV}_A$  and  $\mathbf{FV}_P$  of these sets that are typically defined syntactically. For example, in Example 4.1, variables in expressions are considered free, and those on the left of assignments are bound. Such approximations are not exact: For instance,  $x := x + y - y$  changes no state and hence has syntactically, but not semantically, free or bound variables. Exact computation of these sets is undecidable [54]. As we have no knowledge on atom and program structure, syntactic approximations are unavailable and we instead adopt the semantic approach from Platzer [48]:

**Definition 4.9** (Free and Bound Variables). *Let  $p$  be some program and  $F$  be some formula over a dynamic theory  $\Delta = (\Lambda, \mathcal{D})$ . Free variables  $\mathbf{FV}_\Delta$  and bound variables  $\mathbf{BV}_\Delta$  are defined as follows:*

$$\begin{aligned} \mathbf{FV}_\Delta(F) &= \left\{ v \in \Lambda_V \mid \text{exists } \mu, \tilde{\mu} \in \mathcal{S} \text{ s.t. } \mu \equiv_{\{v\}\mathbb{C}} \tilde{\mu} \text{ and } \mu \in \mathcal{D}[F] \not\equiv \tilde{\mu} \right\} \\ \mathbf{FV}_\Delta(p) &= \left\{ v \in \Lambda_V \mid \text{exists } \mu, \tilde{\mu}, \sigma \in \mathcal{S} \text{ s.t. } \mu \equiv_{\{v\}\mathbb{C}} \tilde{\mu} \text{ and } (\mu, \sigma) \in \mathcal{P}(p) \right. \\ &\quad \left. \text{but there is no } \tilde{\sigma} \text{ s.t. } \sigma \equiv_{\{v\}\mathbb{C}} \tilde{\sigma} \text{ and } (\tilde{\mu}, \tilde{\sigma}) \in \mathcal{P}(p) \right\} \\ \mathbf{BV}_\Delta(p) &= \{v \in \Lambda_V \mid \text{exists } \mu, \tilde{\mu} \in \mathcal{S} \text{ s.t. } (\mu, \tilde{\mu}) \in \mathcal{P}(p) \text{ and } \mathcal{V}(\mu, v) \neq \mathcal{V}(\tilde{\mu}, v)\} \end{aligned}$$

We denote  $\mathbf{Vars}_\Delta(p) = \mathbf{FV}_\Delta(p) \cup \mathbf{BV}_\Delta(p)$ . Going forward we will only use  $\mathbf{BV}_\Delta(p)$  and  $\mathbf{Vars}_\Delta(p)$  which combines both free and bound variables. The definition of free variables breaks down for atoms depending on an infinite set of variables (e.g., consider some atom that is true iff an infinite number of variables have value 1). However, we avoid this issue by assuming that  $\mathbf{FV}_A$  is finite for all atoms. Note that  $\mathbf{FV}_A(a)$  and  $\mathbf{FV}_P(p)$  resp. overapproximate  $\mathbf{FV}_\Delta(a)$  and  $\mathbf{FV}_\Delta(p)$  (see Lemma C.1).

### 4.4 Instantiations of Dynamic Theories

We have now introduced an elaborate definition of dynamic theories. While the proof-theoretic value of these definitions will become clear in the subsequent sections through the derivability of numerous proof rules from the proposed assumptions, we should first examine whether these definitions are sensible. To this end, we prove that three Dynamic Logics are instantiations of our definitions:

**Lemma 4.1 (PDL as Dynamic Theory).** *Propositional Dynamic Logic over arbitrary atomic programs and propositional atoms is an instantiation of a dynamic theory.*

**Lemma 4.2 (Semi-Ring First-Order Dynamic Logic as Dynamic Theory).** *Consider first-order atoms (equality and less than) over terms of an ordered semi-ring (e.g. natural numbers, see Example 4.1). The dynamic logic with programs of the form  $v := t$  where  $t$  is a term of the semi-ring is an instantiation of a dynamic theory.*

**PROOF SKETCH.** To prove that Semi-Ring First-Order Dynamic Logic is a dynamic theory, we prove that the state space, variables, atom and program evaluation, and their respective free variable definitions (see Example 4.1) satisfy the six requirements from Definitions 4.3 to 4.5. As this logic's state space is the set of all (unique) variable to universe mappings,  $\mathcal{S}$  INDEPENDENCE and EXTENSIONALITY is trivially satisfied.  $\mathbf{FV}_A$  COVERAGE and  $\mathbf{FV}_P$  COVERAGE can be achieved by constructing  $\mathbf{FV}_A$  and  $\mathbf{FV}_P$  as sound syntactical analyses.  $\mathbf{FV}_A$  FINITE and  $\mathbf{FV}_P$  FINITE are achieved through the syntactic construction of atoms/programs which only touch a finite set of variables.  $\square$

**Lemma 4.3 (Differential Dynamic Logic as Dynamic Theory).** *The variables, atoms, and programs of Differential Dynamic Logic as formalized in the Archive of Formal Proofs [12] form a dynamic theory.*

Consequently, all proof rules and results described below hold for these three dynamic theories. While the logic from Lemma 4.2 might at first seem overly simplistic, we show in Section 6 that we can lift this theory to support a wider range of programs. As discussed in Section 1, this paper is not only about existing dynamic theories, but also about the ability to construct *further* dynamic theories which are, by design, interoperable with the collection of existing dynamic theories.

## 4.5 Proof Calculi

From a program verification perspective, the objective of dynamic logics, and dynamic theories, is to prove that a program  $p$  satisfies formalized (functional) properties. For example, we might want to verify that for all inputs satisfying  $F$ , the program  $p$  satisfies the postcondition  $G$ . From a proof-theoretic perspective this corresponds to proving that dynamic theory formula  $F \rightarrow [p] G$  is  $\Delta$ -valid, i.e. that  $\models_{\Delta} F \rightarrow [p] G$ . This is usually achieved by *proof calculi* that, from axioms and proof rules, derive that a certain formula is valid.

For the purpose of this work, we consider a calculus to be a relation  $\vdash_{\Delta} \in 2^{\mathbf{Fml}_{\Delta}} \times \mathbf{Fml}_{\Delta}$  where  $(\Gamma, F) \in \vdash_{\Delta}$  (denoted  $\Gamma \vdash_{\Delta} F$  or  $\vdash_{\Delta} F$  iff  $\Gamma = \emptyset$ ) iff  $F$  can be derived from  $\Gamma$  using the proof rules and axioms of the calculus  $\vdash_{\Delta}$ . We are only interested in *sound* proof calculi:

**Definition 4.10 (Sound Calculus).** *For a dynamic theory  $\Delta$  a proof calculus  $\vdash_{\Delta}$  is called sound iff for all formulas  $F \in \mathbf{Fml}_{\Delta}$  it holds that  $\vdash_{\Delta} F$  implies  $\models_{\Delta} F$ .*

Soundness is a compositional property, i.e. a proof calculus is sound if all proof rules are sound and all instantiations of axioms are valid. Soundness is thus an inductive consequence of the soundness of individual calculus steps, because the calculus iteratively generates (valid) axiom instantiations and applies (sound) proof rules. As common in dynamic logics [9, 49, 67] and their practical implementations [2, 24], the remainder of this work assumes that all (elementary) dynamic theories  $\Delta$  are equipped with a sound proof calculus  $\vdash_{\Delta}$  that is complete w.r.t. propositional and uninterpreted first-order reasoning (e.g., by extending the Hilbert or Sequent calculus). Section 5 extends this calculus with additional proof rules that hold independently of a particular dynamic theory. Sections 6 and 7 extend and combine dynamic theories and provide sound calculi building on the given calculus for  $\Delta$ . This approach guarantees compositionality. Although our theories become more expressive as we lift and merge them with the proposed constructs, we can nonetheless reuse the calculus  $\vdash_{\Delta}$  of the original theory.

(G) $\frac{\vdash_{\Delta} \phi}{\vdash_{\Delta} [\alpha] \phi}$	(V) $\phi \rightarrow [\alpha] \phi$	(FV $_{\Delta}$ ( $\phi$ ) $\cap$ BV $_{\Delta}$ ( $\alpha$ ) = $\emptyset$ )
(E) $[\alpha] \phi \leftrightarrow [\beta] \phi \quad (\alpha \cong \beta)$	(B) $(\forall v [\alpha] \phi) \leftrightarrow ([\alpha] \forall v \phi)$	( $v \notin \text{Vars}_{\Delta}(\alpha)$ )
	(K) $([\alpha] (\phi \rightarrow \psi)) \rightarrow ([\alpha] \phi \rightarrow [\alpha] \psi)$	

Fig. 2. Elementary axioms and proof rules for dynamic theories

*Equational Reasoning.* In addition to proof rule based reasoning, our approach also supports reasoning via a refinement [41] and an equivalence relations between programs which can be proven via syntactic rewriting using techniques from Kleene Algebra with Tests [39] (KAT):

**Definition 4.11** (Refinement and Equivalence). *A program  $p$  refines  $q$  (denoted as  $p \lesssim q$ ) iff for all  $(\mu, \sigma) \in \mathcal{P}(p)$  it holds that  $(\mu, \sigma) \in \mathcal{P}(q)$ . Programs are equivalent ( $p \cong q$ ) iff  $p \lesssim q$  and  $q \lesssim p$ .*

## 5 Elementary Results and Proof Rules for Dynamic Theories

Concerning the semantic definitions of free and bound variables, we observe that the definitions from Definition 4.9 exactly characterize the variables determining the valuation of a formula as well as the variables changed by the program. All three definitions come with a minimality guarantee that ensures we are not overly conservative in our estimation of which variables impact valuations:

**Lemma 5.1** (Coincidence Lemma for Formulas & Minimality). *Given a dynamic theory  $\Delta$  and formula  $F$  define  $V = \text{FV}_{\Delta}(F)$ . Then  $F$  has the same truth value in any two states agreeing on  $V$  (i.e., if  $S \ni \mu \equiv_V \tilde{\mu} \in S$  then  $\mu \in \mathcal{D}[\![F]\!] \text{ iff } \tilde{\mu} \in \mathcal{D}[\![F]\!]$ ).  $V$  is the smallest set with this coincidence property.*

**Lemma 5.2** (Coincidence Lemma for Programs & Minimality). *Given a dynamic theory  $\Delta$  with program  $p$  consider some  $V \supseteq \text{FV}_{\Delta}(p)$ . If for states  $\mu, \sigma \in S$  only differing on a finite set of variables (i.e.,  $\{v \in \Lambda_V \mid \mathcal{V}(\mu, v) \neq \mathcal{V}(\sigma, v)\}$  is finite) it holds that  $\mu \equiv_V \sigma$  and  $(\mu, \tilde{\mu}) \in \mathcal{P}(p)$ , then there exists a  $\tilde{\sigma} \in S$  such that  $(\sigma, \tilde{\sigma}) \in \mathcal{P}(p)$  and  $\tilde{\mu} \equiv_V \tilde{\sigma}$ .  $\text{FV}_{\Delta}(p)$  is the smallest set with this property.*

**Lemma 5.3** (Bounded Effect & Minimality). *Given a dynamic theory  $\Delta$  and a program  $p$  define  $V = \text{BV}_{\Delta}(p)$ . For all  $(\mu, \sigma) \in \mathcal{P}(p)$  it holds that  $\mu \equiv_{V^c} \sigma$ .  $V$  is the smallest set with this property.*

We will now derive Hilbert-style proof calculus rules (extending the usual first-order axioms) which hold *independently* of any particular dynamic theory. In particular, these rules also hold for *combined*, heterogeneous dynamic theories (see Section 7). Since we do not know the inner workings of programs  $p \in \Lambda_P$  it is the obligation of the designer of a dynamic theory to provide a suitable calculus to decompose and prove statements about such atomic programs. However, we can nonetheless provide calculus rules that hold independently of the concrete programs. A first set of these axioms and rules can be found in Figure 2. These rules will be extended in the subsequent section by calculus rules for reasoning about the closure of regular programs over the programs in  $\Lambda_P$ . By providing our calculus rules, we propose a *compositional* calculus. The theory designer can focus on proof procedures for statements about (from our perspective) atomic programs, while our rules provide the general, compositional principles that are independent of the particular programming language at hand. To this end, we formalize the soundness of the described axioms:

**Theorem 5.1** (Soundness of Elementary Proof Rules). *The proof rules and axioms in Figure 2 are sound w.r.t. to any dynamic theory  $\Delta$ .*

The proof rule (G) and the axiom (V) are akin to first-order Hilbert-calculus generalization rules by allowing us to wrap formulas into modalities. Barcan's axiom (B) encodes a constant domain property for the dynamic theory's state space that emerges from the state space properties outlined in Section 4. (V) allows us to eliminate programs that do not affect the postcondition while axiom

(K) allows distributing modalities across implications. Finally, (E) allows us to swap programs in modalities for equivalent programs. As described in [Theorem 5.1](#), we can prove the soundness of these axioms independently of the considered state space, atomic formulas, and programs. While [Figure 2](#) presents the axioms as requiring *semantic* side conditions, these can be discharged via syntactic algorithms. For example, the sets  $FV_\Delta$  and  $BV_\Delta$  can be over-approximated using syntactic analyses (e.g., [Definition 4.5](#) establishes an overapproximation of  $FV_\Delta$  for programs) Equivalence results ( $\alpha \cong \beta$ ) can be obtained using sound KAT [39] rewriting rules or natively within dynamic logic [41, Sec. 7].

## 6 Lifting Dynamic Theories

In this section, we demonstrate, for the first time, the power of the concise formalization of necessary properties for the construction of a dynamic theory. Given a dynamic theory  $\Delta$ , we can *extend* its functionality, e.g., by extending the structure of state spaces, atoms, or programs, with *common* constructs. By proving that this extension *again* satisfies the properties outlined in [Section 4](#), all properties, proof rules, and axioms are lifted to the extended, more expressive, dynamic theory. Similarly to conservative extensions in first-order theory, our lifting ensures that previously valid axioms and formulas remain sound. Unlike conservative extensions, some liftings of dynamic theories meaningfully extend the expressiveness of the logic's programming language beyond the original theory's expressiveness by allowing new state transitions that were previously impossible. The most prominent example of this is the introduction of loops in [Section 6.2](#) which lifts a theory to its *regular closure* over programs. Before discussing regular closures, we first propose a lifting which does *not* increase expressivity but serves as a gentle introduction of all concepts reused later. To this end we extend a dynamic theory by the havoc operator  $v := *$  which assigns the variable  $v$  to an arbitrary value ([Section 6.1](#))

### 6.1 Havoc Operator Lifting

On a syntactical level, lifting a dynamic theory  $\Delta$  to a new dynamic theory  $\Delta^{:=*}$  amounts to enhancing the original set of programs  $\Lambda_P$  with the havoc operator on all variables in  $\Lambda_V$ , i.e.  $\Lambda_P^{:=*} = \Lambda_P \dot{\cup} \{v := * \mid v \in \Lambda_V\}$ . We can similarly extend  $\Delta$ 's domain of computation  $\mathcal{D}$ . Overall, this amounts to the following lifting operation  $(\cdot)^{:=*}$  (where  $\dot{\cup}$  is the disjoint union):

**Definition 6.1** (Havoc Lift). *Given a dynamic theory  $\Delta$  we define  $\Delta^{:=*} = (\Lambda^{:=*}, \mathcal{D}^{:=*})$  such that all components of  $\Delta$  and  $\Delta^{:=*}$  are equal except for:*

- $\Lambda_P^{:=*} = \Lambda_P \dot{\cup} \{v := * \mid v \in \Lambda_V\}$
- $\mathcal{P}^{:=*}(p) = \mathcal{P}(p)$  (for  $p \in \Lambda_P$ )
- $\mathcal{P}^{:=*}(v := *) = \left\{ (\mu, \sigma) \in \mathcal{S}^2 \mid \mu \equiv_{\{v\}} \sigma \right\}$  ( $v \in \Lambda_V$ )
- $FV_P^{:=*}(p) = FV_P(p)$  (for  $p \in \Lambda_P$ )
- $FV_P^{:=*}(v := *) = \emptyset$  (for  $v \in \Lambda_V$ )

Note that  $v$  is *not* free in  $v := *$  as its value does not influence the program's reachable states. We prove that the havoc lift  $\Delta^{:=*}$  of a dynamic theory  $\Delta$  satisfies all properties of a dynamic theory:

**Theorem 6.1** (Havoc Lifted Dynamic Theories). *The havoc lift  $\Delta^{:=*}$  of any dynamic theory  $\Delta$  is a dynamic theory.*

For first-order atoms ( $\Lambda_A$ ) and the state space ( $\mathcal{S}$ ), the required properties for a dynamic theory are still satisfied, because we did not modify these components. For programs, we ensured that our semantic definition of havoc satisfies the properties asserted in [Definition 4.5](#). While it is advantageous that the properties proven for arbitrary dynamic theories carry over from  $\Delta$  to  $\Delta^{:=*}$ , the havoc lifting would be useless if properties of the *specific* theory  $\Delta$  under consideration would not carry over as well. If this were the case, all properties proven for a  $\Delta$  under consideration would have to be reproved for  $\Delta^{:=*}$  and the lifting operation would lose its utility. Fortunately, our new

set of formulas is a superset of the original formulas, i.e.  $\text{Fml}_\Delta \subseteq \text{Fml}_{\Delta^{:=*}}$ . This raises the question of whether their semantics also carry over. Indeed, we can prove a calculus rule which translates proofs about  $\Delta$  into proofs about  $\Delta^{:=*}$ :

**Lemma 6.1 (Havoc Reduction).** *For any dynamic theory  $\Delta$ , the following proof rule is sound:*

$$\text{(HR)} \frac{\vdash_\Delta \phi}{\vdash_{\Delta^{:=*}} \phi} \text{ (assuming } \phi \in \text{Fml}_\Delta \text{)}$$

Hence, we can reuse *any* decision procedure developed for determining the validity of formulas in  $\text{Fml}_\Delta$  to determine the validity of formulas in the havoc-free fragment of  $\text{Fml}_{\Delta^{:=*}}$  and carry these results over using (HR). Additionally, we prove the soundness of the well-known ( $:=*$ ) axiom for the decomposition of the havoc operator:

**Lemma 6.2 (Havoc Axiom).** *For any dynamic theory  $\Delta$ , the following axiom is sound in  $\Delta^{:=*}$ :*

$$(:=*) ([v := *] \phi) \leftrightarrow (\forall v \phi)$$

Before we move to regular programs, let us recap what this section demonstrated. Given any dynamic theory  $\Delta$ , we can create a new dynamic theory  $\Delta^{:=*}$  which: (1) Has the additional program primitive  $v := *$ ; (2) Inherits the dynamic theory axioms (G), (V), (B) and (K); (3) Allows reuse of existing proof infrastructure for  $\Delta$  through (HR) (as a valid formula of  $\Delta$  is also valid in  $\Delta^{:=*}$ ). Rule (HR) enables this reuse even when  $\Delta$ 's axiom schemata may not hold for arbitrary  $\Delta^{:=*}$ -instantiations, by restricting reductions to formulas from  $\text{Fml}_\Delta$ . The remainder of this section, as well as Section 7, will demonstrate significantly more expressive lifting procedures with the same properties, i.e. transfer of proof rules and validity preservation.

**Example 6.1.** Continuing the example of a dynamic theory over a semi-ring  $\Delta^{\mathcal{R}}$  defined in Example 4.1, we can lift  $\Delta^{\mathcal{R}}$  with havoc resulting in  $(\Delta^{\mathcal{R}})^{:=*}$ . Beyond programs  $x := t$  (for semiring terms  $t$ ) the theory  $(\Delta^{\mathcal{R}})^{:=*}$  then also supports programs  $x := *$ . While this does not increase expressivity (see axiom ( $:=*$ )), it can increase expressivity when combined with the regular closure defined below.  $(\Delta^{\mathcal{R}})^{:=*}$  also admits the application of all proof principles from  $\Delta^{\mathcal{R}}$  via (HR).

## 6.2 Regular Program Lifting

Given a set of (atomic) programs, it is often useful to consider their *regular programs closure*, i.e., any construction of programs obtained by sequential composition, nondeterministic choice, (first-order logic) checks, and nondeterministic loops. This is an interesting set of programs, because it allows constructing well-known imperative constructs such as if or while, from a small set of operations. It is also isomorphic to the program constructs considered in Kleene Algebra with Tests [39]. We now show that any dynamic theory  $\Delta$  can be lifted to this more expressive set of programs while retaining the properties established above. To this end, we begin by defining the regular closure of programs along with their semantics:

**Definition 6.2 (Regular Closure).** *Given a dynamic theory  $\Delta$  over programs  $\Lambda_P$  with program evaluation function  $\mathcal{P}$  we define the regular closure of programs, denoted  $\Lambda_P^{\text{reg}}$  via the following grammar (where  $r \in \Lambda_P$  and  $F_{\text{FOL}} \in \text{FOL}_\Delta$ ):  $p, q ::= r \mid p; q \mid p \cup q \mid ?(F_{\text{FOL}}) \mid (p)^*$ .*

*We then lift the program evaluation function to  $\mathcal{P}^{\text{reg}}$  as follows:*

- $\mathcal{P}^{\text{reg}}(r) = \mathcal{P}(r)$  (for  $r \in \Lambda_P$ )
- $\mathcal{P}^{\text{reg}}(p; q) = \{(\mu, \omega) \in \mathcal{S}^2 \mid \text{there exists } \sigma \in \mathcal{S} \text{ s.t. } (\mu, \sigma) \in \mathcal{P}^{\text{reg}}(p) \text{ and } (\sigma, \omega) \in \mathcal{P}^{\text{reg}}(q)\}$
- $\mathcal{P}^{\text{reg}}(p \cup q) = \mathcal{P}^{\text{reg}}(p) \cup \mathcal{P}^{\text{reg}}(q)$
- $\mathcal{P}^{\text{reg}}(? (F_{\text{FOL}})) = \{(\mu, \sigma) \in \mathcal{S} \mid \mu \in \mathcal{D}[\llbracket F_{\text{FOL}} \rrbracket] \text{ and } \mu \equiv_{\Lambda_V} \sigma\}$
- $\mathcal{P}^{\text{reg}}((p)^*) = \bigcup_{n \in \mathbb{N}} \mathcal{P}^{\text{reg}}(p^n)$  where  $p^0 \equiv ?(\top)$  and  $p^{n+1} \equiv (p; p^n)$

(?) $[\phi] \psi \leftrightarrow (\phi \rightarrow \psi)$	(U) $[\alpha \cup \beta] \phi \leftrightarrow [\alpha] \phi \wedge [\beta] \phi$
(:) $[\alpha; \beta] \phi \leftrightarrow [\alpha] [\beta] \phi$	(I) $[\alpha^*] (\phi \rightarrow [\alpha] \phi) \rightarrow (\phi \rightarrow [\alpha^*] \phi)$
(*) $[\alpha^*] \phi \leftrightarrow \phi \wedge [\alpha] [\alpha^*] \phi$	

Fig. 3. Axioms for regular programs originally due to Segerberg [59]

To create a new dynamic theory, it remains to define the set of free variables  $\mathbf{FV}_p^{\text{reg}}$ . To this end, the free variables for  $r \in \Lambda_P$  are defined as  $\mathbf{FV}_P(r)$ , and the free variables under sequential composition and nondeterministic choice are defined as the union of their components. The free variables of a looped program  $(p)^*$  are the free variables of  $p$ . For the check  $? (F_{\text{FOL}})$  we leverage our knowledge about the free variables of atoms (encoded in  $\mathbf{FV}_A$ ) to construct the free variables of a first-order formula (see Definition C.1). Note that popular definitions of  $? (F_{\text{FOL}})$  often enforce semantics where it must be the case that  $\mu = \sigma$  while we allow transitions to equivalent states as the usual definition breaks our extensionality assumption in Definition 4.5 (since we allow distinct states  $\mu, \sigma \in \mathcal{S}$  such that  $\mu \equiv_{\Lambda_V} \sigma$ ). This particularity disappears if we assume  $\mathcal{S}$  to be a “classical” mapping from variables to values. We can then formally define the regular closure lift and prove that it constructs once again a dynamic theory:

**Definition 6.3** (Regular Closure Lift). *For a dynamic theory  $\Delta$  and its regular closure over programs  $\Lambda_P^{\text{reg}}, \mathcal{P}^{\text{reg}}$  (see Definition 6.2) and  $\mathbf{FV}_P^{\text{reg}}$  as defined in Definition C.2, the regular closure lift is defined as:*

$$\Delta^{\text{reg}} = ((\Lambda_V, \Lambda_A, \Lambda_P^{\text{reg}}), (\mathcal{U}, \mathcal{S}, \mathcal{V}, \mathcal{A}, \mathbf{FV}_A, \mathcal{P}^{\text{reg}}, \mathbf{FV}_P^{\text{reg}}))$$

**Theorem 6.2** (Regular Closure over Dynamic Theory). *The regular closure lift  $\Delta^{\text{reg}}$  of any dynamic theory  $\Delta$  is a dynamic theory.*

Just as before, we can define a reduction rule which allows us to reuse proof results in  $\Delta^{\text{reg}}$  that have already been shown for  $\Delta$ . Additionally, our lifting comes with numerous axioms on the decomposition of regular programs which we summarize in Figure 3 and are provably sound:

**Lemma 6.3** (Regular Reduction). *For any dynamic theory  $\Delta$ , the following proof rule is sound:*

$$(RR) \frac{\vdash_{\Delta} \phi}{\vdash_{\Delta^{\text{reg}}} \phi} \text{ (assuming } \phi \in \text{Fml}_{\Delta}\text{)}$$

**Theorem 6.3** (Soundness of Proof Rules over Regular Closure). *For any dynamic theory  $\Delta$  the axioms in Figure 3 are sound in  $\Delta^{\text{reg}}$ .*

The proof rules provided in Figure 3 allow us to decompose checks, sequential compositions and nondeterministic choice programs (see axioms (?), (:), (U)). Moreover, they allow the iterative decomposition of loops as well as inductive invariant reasoning about loops (see axioms (\*), (I)). However, the axioms are, yet, insufficient for loop termination.

*Loop Convergence.* A significant strength of dynamic logics, and hence dynamic theories, is their ability to reason about partial *and* total correctness via the dual diamond and box modalities. To this end, it is sometimes necessary to not only reason about loop invariants (via (I)), but also about loop *variants* to prove progress. To this end, consider a formula  $\langle p^* \rangle F$ . In this instance, we must prove that running program  $p$  in a loop *eventually* leads to a state satisfying  $F$ . This is usually achieved via well-founded induction proofs. Given a formula  $\phi(N)$  that is satisfied for a sufficiently large  $N \in \mathbb{N}$ , one iteration of  $p$  ensures that  $\phi(N - 1)$  is satisfied in the post state. If  $\phi(0)$  implies  $F$ , this proves that  $F$  becomes satisfied after iteratively running  $p$  for a sufficiently large number of steps. This requires *counting*. However, due to the generality of our assumed state space, we

currently have no way of counting or performing well-founded induction. Hence, we now propose a set of additional assumptions that allow us to introduce a loop termination rule:

**Definition 6.4** (Inductive Expressivity). *Given some dynamic theory  $\Delta$ , assume some function  $N : \mathcal{U} \rightarrow \mathbb{N}$  mapping elements from the universe to natural numbers. We say a variable  $v \in \Lambda_V$  is integer expressive w.r.t.  $N$  iff for all  $n \in \mathbb{N}$  there exists a state  $\mu \in \mathcal{S}$  such that  $N(\mathcal{V}(\mu, v)) = n$ . Let  $\Lambda_{V\mathbb{N}}$  be a set of such variables. We say  $\Delta$  has inductive expressivity iff for some given  $N$  there exist two functions  $\text{nat}_{+1}, \text{nat}_{=} : \Lambda_{V\mathbb{N}}^2 \rightarrow \text{FOL}_\Delta$  and one function  $\text{nat}_{>0} : \Lambda_{V\mathbb{N}} \rightarrow \text{FOL}_\Delta$  such that:*

**(POSITIVE SOUND)** *For any variable  $v \in \Lambda_{V\mathbb{N}}$  and any state  $\mu \in \mathcal{S}$ :*

$\mu \in \mathcal{D}[\text{nat}_{>0}(v)]$  *iff*  $N(\mathcal{V}(\mu, v)) > 0$

**(EQUAL SOUND)** *For variables  $v, w \in \Lambda_{V\mathbb{N}}$  with  $v \neq w$  and any state  $\mu \in \mathcal{S}$ :*

$\mu \in \mathcal{D}[\text{nat}_{=}(v, w)]$  *iff*  $N(\mathcal{V}(\mu, v)) = N(\mathcal{V}(\mu, w))$

**(PLUS ONE SOUND)** *For variables  $v, w \in \Lambda_{V\mathbb{N}}$  with  $v \neq w$  and any state  $\mu \in \mathcal{S}$ :*

$\mu \in \mathcal{D}[\text{nat}_{+1}(v, w)]$  *iff*  $N(\mathcal{V}(\mu, v)) + 1 = N(\mathcal{V}(\mu, w))$

This inductive expressivity yields all that is required to prove loop convergence, which enables us to prove loop termination and reachability of certain states via loop iteration. To this end, dynamic logics usually are equipped with a proof rule [30] or axiom [50] that looks something like this:

$$[p^*] (\forall v ((v > 0 \wedge \phi(v)) \rightarrow \langle p \rangle \phi(v-1))) \rightarrow \forall v (\phi(v) \rightarrow \langle p^* \rangle \phi(0)) \quad (v \notin \text{Vars}(p))$$

This axiom formalizes the well-founded induction argument outlined above: If  $\phi(v)$  is satisfied and one iteration of  $p$  allows us to reach a state satisfying  $\phi(v-1)$ , then we can reach a state with  $\phi(0)$  by iterating the execution of  $p$  (assuming  $v$  is a natural number). To prove loop reachability properties, we also want an axiom like this one in the regular closure over our dynamic theory; however, we lack any means to parameterize formulas with specific variables. After all, our theory so far makes no assumptions about how atomic formulas in  $\Lambda_A$  evaluate w.r.t. variables beyond the definition of  $\text{FV}_A$ . Hence, we derive the following axiom which can be proven independently of the dynamic theory at hand so long as the properties from Definition 6.4 are given:

**Lemma 6.4** (Loop Convergence Rule). *For any inductively expressive dynamic theory  $\Delta$ , the following axiom is sound for its regular closure  $\Delta^{\text{reg}}$  for any integer expressive distinct variables  $v, w \in \Lambda_{V\mathbb{N}}$  with  $w \notin \text{FV}_\Delta(\phi)$  and  $v, w \notin \text{Vars}(p)$ :*

$$(C) \quad ([p^*] (\forall v ((\text{nat}_{>0}(v) \wedge \phi) \rightarrow \langle p \rangle (\forall w (\text{nat}_{+1}(w, v) \rightarrow \forall v (\text{nat}_{=}(v, w) \rightarrow \phi)))))) \rightarrow (\forall v (\phi \rightarrow \langle p^* \rangle (\exists v (\neg \text{nat}_{>0}(v) \wedge \phi))))))$$

Note that once we apply  $\text{nat}_{+1}, \text{nat}_{=}$  or  $\text{nat}_{>0}$  to concrete variables, they reduce to a concrete  $\Delta$ -formula yielding a concrete instantiation of axiom (C). Importantly, we cannot subtract from  $v$  within a single predicate since any predicate (including  $\text{nat}_{+1}$ ) requires a reference point from which to subtract 1. We mitigate this by employing the helper variable  $w$ . To demonstrate the utility of the additional assumptions, we have also proven that a concrete Dynamic Logic instantiates the properties defined in Definition 6.4:

**Lemma 6.5** (Semi-Ring First-Order Dynamic Logic over Natural Numbers). *The semi-ring dynamic theory  $\Delta^{\mathcal{R}}$ , instantiated for natural numbers or integers (resp.  $\Delta^{\mathbb{N}}$  or  $\Delta^{\mathbb{Z}}$ ), is inductively expressive. Hence, axiom (C) is a sound axiom for its havoc lifted, regular closure  $\left( (\Delta^{\mathbb{N}})^{:=*} \right)^{\text{reg}}$  (resp.  $\left( (\Delta^{\mathbb{Z}})^{:=*} \right)^{\text{reg}}$ ).*

**Example 6.2** (Regular Program Lifting for Ordered Semiring). We reconsider the example of a dynamic logic over an ordered semiring  $(\mathcal{R}, +, \cdot, \leq)$  for  $\mathcal{R} = \mathbb{Z}$ : As seen in Lemma 6.5, this dynamic theory is inductively expressive. Hence, all axioms derived up to this point (Figures 2 and 3 as well as (HR), (RR), ( $:=*$ ), (C)) apply to its havoc lifted, regular closure  $\left( (\Delta^{\mathbb{Z}})^{:=*} \right)^{\text{reg}}$ . From a very simple

definition of assignment, we have thus derived a dynamic theory with full support for regular programs and nondeterministic assignment. For example, the following program is part of this dynamic theory:

$$1 \leq n \rightarrow [x := 0; i := 0; (? (i \leq n); x := x + i; i := i + 1)^*; ? (\neg (i \leq n))] \quad 2 * x \leq n * (n + 1) \quad (2)$$

This formula asserts that  $x$  is no larger than the Gauss formula predicts after summing up the first  $n$  integers. Based on our simple definitions in Example 4.1 we can now perform reasoning about programs like the one in Equation (2). In fact,  $\left((\Delta^{\mathbb{Z}})^{:=*}\right)^{\text{reg}}$  provides the equivalent of a guarded command language over integers and, as will be shown in Section 8, a relatively complete proof calculus for it. In fact, our initial definition of  $\Delta^{\mathbb{Z}}$  even could have omitted the assignment programs (by initializing  $\Lambda_p^{\mathbb{Z}} = \emptyset$ ) by defining assignment as syntactic sugar for havoc and check [1]:

$$v := t \equiv (w := *; ? (t \leq w \wedge w \leq t); v := *; ? (v \leq w \wedge w \leq v)) \quad \text{for } w \in \Lambda_V^{\mathbb{Z}} \text{ fresh}$$

*Kleene Algebra with Tests.* We prove that our regular closure lifting forms a Kleene Algebra with Test [31, 39] over the theory's programs (see Theorem C.1). In combination with axiom (E), this enables KAT-style rewriting within dynamic logics.

## 7 Heterogeneous Dynamic Theories

We enhance dynamic theories further by allowing *combination of* multiple theories to reason about *heterogeneous systems*, whose components operate on different domains of computation, but may interact. Heterogeneous theories serve two purposes: (1) they provide a unified framework that explicitly models both systems and their interactions, and (2) they enable rigorous reasoning about the overall system by *reusing* results and proof calculi from the *homogeneous* component theories. The heterogeneous proof calculus decomposes obligations to the individual theories, analogous to how reasoning over regular programs reduces to reasoning over atomic programs.

Given two theories  $\Delta^{(0)}, \Delta^{(1)}$ , we first construct a *simple heterogeneous theory*  $\Delta^{(01)}$  over  $\Lambda_p^{(0)} \cup \Lambda_p^{(1)}$ , then lift it with havoc and regular closure. The resulting *fully heterogeneous dynamic theory*  $\hat{\Delta}$  provides an expressive logic combining  $\Delta^{(0)}$  and  $\Delta^{(1)}$  with all axioms from Sections 5 and 6 included.

### 7.1 Simple Heterogeneous Dynamic Theories

We assume we are given two dynamic theories  $\Delta^{(0)}, \Delta^{(1)}$  and begin by defining the state space of our new dynamic theory:

**Definition 7.1** (Heterogeneous State Space). *Given two dynamic theories  $\Delta^{(0)}, \Delta^{(1)}$  and their resp. state spaces  $\mathcal{S}^{(0)}, \mathcal{S}^{(1)}$  we define the heterogeneous state space as  $\hat{\mathcal{S}} = \mathcal{S}^{(0)} \times \mathcal{S}^{(1)}$ . For a given state  $(\mu^{(0)}, \mu^{(1)}) \in \hat{\mathcal{S}}$  variable evaluation is defined as follows:*

$$\hat{\mathcal{V}} \left( (\mu^{(0)}, \mu^{(1)}), v \right) = \begin{cases} \mathcal{V}^{(0)}(\mu^{(0)}, v) & v \in \Lambda_V^{(0)} \\ \mathcal{V}^{(1)}(\mu^{(1)}, v) & v \in \Lambda_V^{(1)} \end{cases}$$

A priori, the behavior modeled in  $\Delta^{(0)}$  and  $\Delta^{(1)}$  hence happens in isolation. However, similarly to reasoning about concurrent systems, the appeal of reasoning about heterogeneous systems lies precisely in the *interaction* of these a priori independent behaviors. Hence, to relate the behavior of these two independent state space components, we introduce a new set of atoms (denoted  $\Lambda_A^{\cap}$ ) that allows us to constrain *permissible combinations* of individual states  $\mu^{(0)}$  and  $\mu^{(1)}$ :

**Definition 7.2** (Heterogeneous Atoms). *Given two dynamic theories  $\Delta^{(0)}, \Delta^{(1)}$  and their heterogeneous state space  $\hat{\mathcal{S}}$  we define a new set of heterogeneous atoms  $\Lambda_A^{\cap}$  over variables  $\Lambda_V^{\cap} = \Lambda_V^{(0)} \cup \Lambda_V^{(1)}$*

with an evaluation function  $\mathcal{A}^\cap : \Lambda_A^\cap \rightarrow 2^{\hat{S}}$  and a free variable function  $\mathbf{FV}_A^\cap : \Lambda_A^\cap \rightarrow 2^{\Lambda_V^\cap}$  such that the constructs satisfy the atom evaluation requirements from Definition 4.4.

Heterogeneous atoms can be considered the dynamic theory counterpart to mixed terms in first-order theory combination. Just like in the case of first-order theory combination, these terms allow us to constrain the relation between values in both worlds. As a concrete example, Section 9 defines the heterogeneous atom  $\text{round}(v, w)$  for JavaDL integer variables  $v$  and real-valued  $d\mathcal{L}$  variables  $w$ . Its semantics will allow us to *constrain* the possible values of the Java variable  $v$  (in state  $\mu^{(0)}$ ) based on the values of the  $d\mathcal{L}$  variable  $w$  (in  $\mu^{(1)}$ ). In combination with havoc ( $x := *$ ) and checks ( $? (F_{\text{FOL}})$ ), heterogeneous atoms serve as communication primitives between languages.

A simple heterogeneous dynamic theory is then defined as follows:

**Definition 7.3** (Simple Heterogeneous Dynamic Signature). *Given two dynamic theories  $\Delta^{(0)}, \Delta^{(1)}$  with resp. signatures  $\Lambda^{(0)}, \Lambda^{(1)}$  and a set of heterogeneous atoms  $\Lambda_A^\cap$  over  $\Delta^{(0)}, \Delta^{(1)}$  (see Definition 7.2) we define the simple heterogeneous dynamic signature as  $\Lambda^{(01)} = (\hat{\Lambda}_V, \hat{\Lambda}_A, \Lambda_P^{(01)})$  with:*

$$\hat{\Lambda}_V = \Lambda_V^{(0)} \dot{\cup} \Lambda_V^{(1)} \quad \hat{\Lambda}_A = \Lambda_A^{(0)} \dot{\cup} \Lambda_A^{(1)} \dot{\cup} \Lambda_A^\cap \quad \Lambda_P^{(01)} = \Lambda_P^{(0)} \dot{\cup} \Lambda_P^{(1)}$$

The domain of computation, i.e. the heterogeneous theory's semantics, naturally follows:

**Definition 7.4** (Simple Heterogeneous Domain of Computation). *Given two dynamic theories  $\Delta^{(0)}, \Delta^{(1)}$  and notation as in Definition 7.3 we define the simple heterogeneous domain of computation  $\mathcal{D}^{(01)}$  as follows:*

- The universe is the combination of individual universes:  $\hat{\mathcal{U}} = \mathcal{U}^{(0)} \cup \mathcal{U}^{(1)}$
- The state space is the heterogeneous state space  $\hat{S}$  over  $\Delta^{(0)}, \Delta^{(1)}$  with the corresponding variable evaluation function  $\hat{\mathcal{V}}$  (see Definition 7.1)
- The atom evaluation function  $\hat{\mathcal{A}}$  and program evaluation function  $\mathcal{P}^{(01)}$  is given as below (with corresponding free variable functions  $\mathbf{FV}_A^\cap, \mathbf{FV}_P^{(01)}$  as defined in Definition C.4).

$$\hat{\mathcal{A}}(a) = \begin{cases} \mathcal{A}^{(0)}(a) \times \mathcal{S}^{(1)} & a \in \Lambda_A^{(0)} \\ \mathcal{S}^{(0)} \times \mathcal{A}^{(1)}(a) & a \in \Lambda_A^{(1)} \\ \mathcal{A}^\cap(a) & a \in \Lambda_A^\cap \end{cases} \quad \mathcal{P}^{(01)}(p) = \begin{cases} \mathcal{P}^{(0)}(p) & p \in \Lambda_P^{(0)} \\ \mathcal{P}^{(1)}(p) & p \in \Lambda_P^{(1)} \end{cases}$$

The notation of our simple heterogeneous signature  $\Lambda^{(01)}$  and the simple heterogeneous dynamic theory  $\mathcal{D}^{(01)}$  is already giving a hint at the components of the dynamic theory which are here to stay for the fully heterogeneous version (denoted with  $\hat{\cdot}$ ) vs. components of the simple heterogeneous theory that still require further refinement (denoted with  $(\cdot)^{(01)}$ ): While the universe, state space, variables and atoms remain as-is, the range of admissible programs will still be extended. Before we lift  $\Delta^{(01)}$  to a more expressive set of programs, we begin by showing that our simple heterogeneous dynamic signature and domain of computation indeed form a dynamic theory:

**Theorem 7.1** (Simple Heterogeneous Dynamic Theories). *Assume two dynamic theories  $\Delta^{(0)}, \Delta^{(1)}$  and a corresponding set of heterogeneous atoms  $\Lambda_A^\cap$ , then  $\Lambda^{(01)}$  and  $\mathcal{D}^{(01)}$  as defined in Definitions 7.3 and 7.4 form a dynamic theory  $\Delta^{(01)}$  as defined in Definition 4.7. The soundness of all axioms and proof rules from Figure 2 carries over.*

This result unlocks the first four proof rules for  $\Delta^{(01)}$ . However, the range of properties we will be able to formalize and prove with  $\Delta^{(01)}$  is still underwhelming as modalities in  $\Delta^{(01)}$  can only contain a program from  $\Delta^{(0)}$  or a program from  $\Delta^{(1)}$  — *not both*. Before we *lift* this limitation in the next section, we consider how this logic relates to another technique for combining logics, namely fibring.

*Relation to Fibring.* By explicitly labeling the modalities w.r.t. the contained program (i.e.  $[p]$   $F$  becomes  $[p]^{(0)} F$  for  $p \in \Lambda_p^{(0)}$  and  $[p]^{(1)} F$  otherwise), we can interpret this logic as a “multi-dynamic logic” with two distinct modal operators. This approximately corresponds to what could be achieved via fibring [26] of dynamic logics where we could combine the truth bearing entities of  $\Delta^{(0)}$  and  $\Delta^{(1)}$ . As fibring does not recognize programs as first-class citizens, it cannot integrate their respective program constructs which restricts the expressive power of analyzable systems. In particular, it cannot capture systems involving, e.g., intricate control structures such as loops. Instead, fibring is confined to modeling systems that can be decomposed into a (nondeterministic) finite, sequential compositions of homogeneous subsystems. Importantly, our dynamic theory does *not* have two distinct modalities. This key difference allows us to apply the lifting procedures devised in Section 6 which naturally yields a significantly more expressive program logic.

## 7.2 Fully Heterogeneous Dynamic Theories

Using the lifting procedures from Section 6 we can define the fully heterogeneous dynamic theory, supporting the composition of homogeneous programs using all regular program constructs, in a straightforward manner:

**Definition 7.5** (Fully Heterogeneous Dynamic Theories). *Given two dynamic theories  $\Delta^{(0)}, \Delta^{(1)}$  and a corresponding set of heterogeneous atoms  $\Lambda_A^\cap$  forming the simple heterogeneous dynamic theory  $\Delta^{(01)}$ , we define the fully heterogeneous dynamic theory over  $\Delta^{(0)}, \Delta^{(1)}$  as  $\hat{\Delta} = \left( (\Delta^{(01)})^{\text{reg}} \right)^{\text{reg}}$  with  $(\cdot)^{\text{reg}}$  and  $(\cdot)^{\text{reg}}$  as given in Theorem 6.1 and Theorem 6.2.*

As planned, we can recover all axioms established above (with the exception of (C) for which the requirements will be discussed below):

**Theorem 7.2** (Fully Heterogeneous Dynamic Theories). *Given two dynamic theories  $\Delta^{(0)}, \Delta^{(1)}$  and a corresponding set of heterogeneous atoms  $\Lambda_A^\cap$  the fully heterogeneous dynamic theory  $\hat{\Delta}$  is a dynamic theory. Proof rules and axioms from Figure 3 as well as axioms (HR), ( $\text{reg}$ ), and (RR) carry over to  $\hat{\Delta}$ .*

Just as for the regular closure over programs of *one* dynamic theory, the construction of a heterogeneous dynamic theory over *two* theories would be entirely devoid of purpose if validity wouldn't carry over from the individual logics to the combined one. Fortunately, we can show that for any *pure*  $\Delta^{(0)}/\Delta^{(1)}$ -formula the validity w.r.t.  $\Delta^{(0)}/\Delta^{(1)}$  entails the validity w.r.t.  $\hat{\Delta}$ :

**Lemma 7.1** (Heterogeneous Reduction). *For any fully heterogeneous dynamic theory  $\hat{\Delta}$  over arbitrary  $\Delta^{(0)}, \Delta^{(1)}$  the following proof rules are sound:*

$$\text{(HR0)} \quad \frac{\vdash_{\Delta^{(0)}} \phi}{\vdash_{\hat{\Delta}} \phi} \quad (\text{assuming } \phi \in \text{Fml}_{\Lambda^{(0)}}) \qquad \text{(HR1)} \quad \frac{\vdash_{\Delta^{(1)}} \phi}{\vdash_{\hat{\Delta}} \phi} \quad (\text{assuming } \phi \in \text{Fml}_{\Lambda^{(1)}})$$

*Loop Convergence.* In order for the loop convergence rule to carry over as well, we require that our heterogeneous dynamic theory  $\hat{\Delta}$  is inductively expressive. To this end, we note that whenever one of the homogeneous dynamic theories  $\Delta^{(0)}, \Delta^{(1)}$  is inductive, then the heterogeneous dynamic theory is inductive as well:

**Lemma 7.2** (Inductive Expressivity of  $\hat{\Delta}$ ). *Let  $\Delta^{(0)}$  be inductively expressive, then there exists a (constructive) adjusted function  $\hat{N}$  such that  $\hat{\Delta}$  is inductively expressive w.r.t. the variables  $\Lambda_{\text{VN}}^{(0)}$ . The same holds in case  $\Delta^{(1)}$  is inductively expressive.*

Using this result, any dynamic theory can be made inductively expressive by combining it with an already inductively expressive theory. Depending on whether we use a loop convergence rule inherited from  $\Delta^{(0)}$  or from  $\Delta^{(1)}$ , we denote it as (C<sup>(0)</sup>) or (C<sup>(1)</sup>), respectively.

*Program Rewriting.* While not necessary for relative completeness, axiom (E) can also be used to rewrite *homogeneous regular program constructs* (here in blue and orange) into *fully heterogeneous regular program constructs* (here in magenta) via the following identities (see [Theorem C.2](#)):

$$?(\phi) \cong ?(\phi) \quad \alpha \cup \beta \cong \alpha \cup \beta \quad \alpha; \beta \cong \alpha; \beta \quad (\alpha)^* \cong (\alpha)^*$$

Using these identities, we can turn a program  $\alpha_{\text{mixed}} \equiv (p_1; (q_1 \cup q_2)^*)^*$  into a program  $\alpha_{\text{unmixed}} \equiv (p_1; (q_1 \cup q_2)^*)^*$  by syntactically deriving that  $\alpha_{\text{mixed}} \cong \alpha_{\text{unmixed}}$ . This result, in combination with axiom (E), formalizes the natural correspondence between homogeneous and heterogeneous regular programs and enables us to decompose a homogeneous programs.

## 8 Relative Completeness

The expressivity of heterogeneous dynamic logic, enabling us to reason about intertwined heterogeneous dynamics, would equally be its downfall if the expressivity were to keep us from verifying properties in practice. To this end, we usually desire to formalize dynamic logics in a way that guarantees *relative completeness*, i.e. the property that any valid formula can also be proved valid if we assume the availability of an oracle for first-order validity:

**Definition 8.1** (Relative Completeness). *A dynamic theory  $\Delta$  is called relatively complete if there exists a calculus  $\vdash_{\Delta}$  such that if for some formula  $F \in \text{Fml}_{\Delta}$  it holds that  $\models_{\Delta} F$  then also  $\Gamma_{\Delta} \vdash_{\Delta} F$  where  $\Gamma_{\Delta} = \{G_{\text{FOL}} \in \text{FOL}_{\Delta} \mid \models_{\Delta} G_{\text{FOL}}\}$ .*

This property is interesting, because it guarantees that, using the given calculus, proving properties about the dynamic logic is “no harder” than proving properties about its underlying (first-order) data theory. Appendix B introduces a calculus of metaproperties to reason about relative completeness and related properties for lifted and combined dynamic logics. Due to space constraints, we focus the exposition in the paper to central assumptions necessary to obtain relative completeness and then present [Theorem 8.1](#) which proves that, under common, reasonable assumptions, the heterogeneous dynamic theory over two relatively complete dynamic theories is once again relatively complete.

### 8.1 Assumptions for Relative Completeness.

We begin by providing a succinct overview on properties of a dynamic theory  $\Delta = (\Lambda, \mathcal{D})$  required to prove relative completeness while deferring formal definitions to the appendix.

*Finite Support.* We require that  $\Delta$  has *finite support* (see [Definition B.2](#)), i.e. that there exists a syntactic over-approximation  $\text{BV}_P$  of semantically bound variables (i.e.  $\text{BV}_{\Delta}(p) \subseteq \text{BV}_P(p)$  for all programs  $p$ ) and that  $\text{BV}_P$  is always *finite*, i.e. all programs only modify a finite set of variables.

*Gödel Expressivity.* Like many relative completeness proofs in dynamic logic [9, 30, 50], we require a first-order theory which is capable of encoding *finite sequences* of domain values into a *single value*. The first example of this mechanism is due to Gödel [28] who showed that any finite sequence of integer values can be injectively mapped to a single integer value. A more general description of this encoding mechanism (here predicate  $R$ ) was established by Parikh (first published by Harel [30, p. 29]):

$$(\forall x_1 \dots x_n) (\exists y) (\forall w) (\forall i) ((i \in \mathbb{N} \wedge 1 \leq i \leq n) \implies (R(w, i, y) \equiv (w = x_i)))$$

We generalize this notion to our definition of state spaces and atoms here and call an inductively expressive theory *Gödel expressive* ([Definition B.5](#)), if the domains of all its variables support this type of sequence encoding. Assuming suitable first order theories, Gödel expressivity has previously been shown for multiple dynamic logics including interpreted integer first-order dynamic logic [30], differential dynamic logic over reals [50] and dynamic logic with non-rigid functions [9].

*FOL Expressivity.* An additional assumption serving as bread and butter of many meaningful relative completeness proofs [9, 30, 50] is the notion of *FOL expressivity* (Definition B.3). This property asserts that for any program  $p$  there exists a first-order *rendition* formula  $S_p$  that *exactly* relates the pre and post state of  $p$ 's variables. In essence, this ensures that we can express the behavior of programs in the theory's first-order fragment. To prove FOL expressivity, one often relies on Gödel expressiveness. For example, Lemma C.14 proves that for any FOL expressive, Gödel expressive dynamic theory with finite support its regular closure is also FOL expressive.

*Communicating Theories.* While previous assumptions constrained individual theories, we also require an assumption about the expressivity of heterogeneous atoms in  $\Lambda_A^\cap$ . To this end, Definition B.6 ensures that heterogeneous theories adhering to it have a common first-order structure that is expressive enough to exchange integer values via atoms formalizing equality between integers of both theories. Without communication, our renditions of heterogeneous programs cannot *exactly* represent the progress of a heterogeneous loop as the individual theories would not be able to synchronously count the number of loop iterations.

## 8.2 Relative Completeness under Combination.

We prove Theorem 8.1 which states that the fully heterogeneous dynamic theory over two relatively complete dynamic theories is again relatively complete (proof see extended version).

**Theorem 8.1** (Relative Completeness of Fully Heterogeneous Dynamic Theories). *Let  $\Delta^{(0)}, \Delta^{(1)}$  be two relatively complete (Definition 8.1), FOL expressive (Definition B.3), Gödel expressive (Definition B.5) dynamic theories with finite support (Definition B.2) that are communicating in  $\Delta^{(01)}$  (Definition B.6). Then the fully heterogeneous dynamic theory  $\hat{\Delta}$  with its proof calculus  $\vdash_{\hat{\Delta}}$  is relatively complete.*

## 9 Case Study

To demonstrate our approach, we apply HDL to the case study from Section 2. We first derive results about the individual two components. Subsequently, we merge these results into a system level guarantee. We will use the homogeneous (atomic) equality predicate  $t_1 \doteq t_2$  which states that the values of two terms  $t_1, t_2$  are equal as well as the homogeneous inequality predicates ( $\leq, >, <$ ).

*Java.* For the Java component `ctrl` we wish to verify that once the relative distance to the stop sign (given as  $p$ ) becomes too small it must brake by choosing an appropriate acceleration. Using JavaDL and the proof calculus implemented in the interactive theorem prover KeY [2, 7], we can then derive the result in Lemma 9.1. We do not only prove properties on the stateful result of `this.acc`, but also on the preservation of the heap structure, the correctness of (possibly overflowing) integer arithmetic, and the field values of  $A, B,$  and  $T$  (mechanized proof in KeY). Note that these aspects of control software would be hard to model in  $d\mathcal{L}$  which allows only real-valued variables.

**Lemma 9.1** (Correctness of `ctrl`). *The following JavaDL formula is valid:<sup>1</sup>*

$$\underbrace{\left( \begin{array}{l} \text{heap\_assumptions} \wedge \\ A > 0 \wedge A^- \doteq A \wedge \\ B < 0 \wedge B^- \doteq B \wedge \\ T > 0 \wedge T^- \doteq T \end{array} \right)}_{\text{ctrlPre}} \rightarrow [\text{ctrl}] \underbrace{\left( \begin{array}{l} \text{heap\_assumptions} \wedge \\ B \leq \text{this.acc} \leq A \wedge \\ (\text{brake\_cond} \rightarrow \text{this.acc} \doteq B) \wedge \\ A^- \doteq A \wedge B^- \doteq B \wedge T^- \doteq T \end{array} \right)}_{\text{ctrlPost}}$$

*with brake\_cond abbreviating  $10^4(v+1)^2 + (A-B)T(AT+200(v+1)) > -2 \cdot 10^4B(p-1)$ .*

<sup>1</sup>For the actual proof we require further assumptions which we summarize as `heap_assumptions`. This concerns the correct initialization of the heap, the correct instantiation of Java objects, etc. While this paper denotes object fields as variables (e.g.  $A$ ), these values are technically obtained via a rigid function which we omit for clarity [2].

*Differential Dynamic Logic.* Using the calculus implemented in the theorem prover KeYmaera X, we can prove statements about hybrid programs. For example, we can prove the validity of the formula in Lemma 9.2 which shows (under some assumptions) that  $x$  will always be at most  $s$  (the stop sign's position) and an additional invariant after execution of  $env$  (proof mechanized in KeYmaera X). While  $d\mathcal{L}$  provides us with an effective infrastructure to reason about the environment's differential equations, it is unfit to reason about Java's heap and integer arithmetic.

**Lemma 9.2** (Evolution of  $env$ ). *The following formula is valid in  $d\mathcal{L}$ :*

$$\underbrace{\left( A > 0 \wedge B > 0 \wedge T > 0 \wedge \text{acc\_assumptions} \wedge x + v^2/(2B) \leq s \right)}_{envPre} \rightarrow [env] \underbrace{\left( x \leq s \wedge \overbrace{x + v^2/(2B) \leq s}^{invariant} \right)}_{envPost}$$

with the following abbreviation for  $\text{acc\_assumptions}$ :

$$(-B \leq a \wedge a \leq A \wedge ((x + (v^2)/2B + (A/B + 1)(AT^2/2 + Tv) > s) \rightarrow a \equiv -B)) \vee (v \equiv 0 \wedge a \equiv 0)$$

*Heterogeneous Safety.* Both  $d\mathcal{L}$  and JavaDL are instantiations of dynamic theories and we can hence construct a fully heterogeneous dynamic theory  $\Delta^{(dJava\mathcal{L})}$ , here called *differential JavaDL*, over  $\Delta^{(d\mathcal{L})}$  and  $\Delta^{(JavaDL)}$ . To this end, we add heterogeneous atoms to  $\Lambda_A^\cap$  with respective evaluation  $\mathcal{A}^\cap$  for any integer variable  $v$  in JavaDL and any real arithmetic variable  $w$  in  $d\mathcal{L}$ :

- $\text{equal}(v, w)$  with  $\mu \in \mathcal{A}^\cap(\text{equal}(v, w))$  iff  $\mathbb{Z} \ni \hat{\mathcal{V}}(\mu, v) = \hat{\mathcal{V}}(\mu, w)$
- $\text{round}(v, w)$  with  $\mu \in \mathcal{A}^\cap(\text{round}(v, w))$  iff  $\hat{\mathcal{V}}(\mu, v) \in \mathbb{Z}$  and  $\hat{\mathcal{V}}(\mu, v) - 0.5 < \hat{\mathcal{V}}(\mu, w)$  as well as  $\hat{\mathcal{V}}(\mu, w) \leq \hat{\mathcal{V}}(\mu, v) + 0.5$

The atoms allow us to explicitly model the communication between  $\Delta^{(d\mathcal{L})}$  and  $\Delta^{(JavaDL)}$  and are all we need to model the full heterogeneous system in  $\Delta^{(dJava\mathcal{L})}$ . After executing the controller we set the  $d\mathcal{L}$  variable  $a$  to an arbitrary value and then assert it has the value of the Java variable in this.  $\text{acc}$  via a check (using the atomic formula  $\text{equal}$ ). Based on the new acceleration we execute the environment  $env$  and retrieve the position and velocity via  $\text{round}$ . This can be formalized as

$$\text{ctrl}; \hat{a} \equiv \text{ctrl}; (a := *; ?(\text{equal}(\text{this. acc}, a * 100)); env; p := *; v := *; ?(\text{coupling}))$$

where we abbreviate  $\text{coupling} \equiv (\text{round}(p, 100(s - x)) \wedge \text{round}(v, 100v))$ . For our safety proof, we define the following coupled precondition which defines admissible initial states of our heterogeneous system:

$$\begin{aligned} \text{coupledPre} &\equiv (\text{ctrlPre} \wedge \phi_2 \wedge \phi_3) \quad \phi_2 \equiv (A > 0 \wedge B > 0 \wedge T > 0 \wedge x + v^2/(2 * B) \leq s) \\ \phi_3 &\equiv (\text{equal}(A^-, 100A) \wedge \text{equal}(B^-, -100B) \wedge \text{equal}(T^-, 100 * T) \wedge \text{coupling}) \end{aligned}$$

Note that  $\text{ctrlPre} \in \text{Fml}_{\Delta^{(JavaDL)}}$ ,  $\phi_2 \in \text{Fml}_{\Delta^{(d\mathcal{L})}}$ , and  $\phi_3 \in \text{Fml}_{\Delta^{(dJava\mathcal{L})}}$ , i.e.  $\text{ctrlPre}$  and  $\phi_2$  constrain the homogeneous states, while  $\phi_3$  synchronizes them. Using our heterogeneous calculus we can then prove that our heterogeneous system never overshoots the stop sign:

**Lemma 9.3** (Safety of the Heterogeneous System). *The following formula is valid in  $\Delta^{(dJava\mathcal{L})}$ :*

$$\text{coupledPre} \rightarrow [(\text{ctrl}; \hat{a})^*] x \leq s \quad (3)$$

**PROOF SKETCH.** We provide a sketch for the proof of Lemma 9.3 below (Proof A and B). Beyond the heterogeneous calculus rules presented above, we use the rules from Figure 4 which are derivable in our calculus (see Appendix A).

$$\begin{array}{c}
([\ ] \wedge) ([\alpha] \phi \wedge [\alpha] \psi) \leftrightarrow [\alpha] (\phi \wedge \psi) \\
(F^{(i)}) (\psi \rightarrow [\alpha] \xi) \rightarrow (\phi \wedge \psi \rightarrow [\alpha] (\xi \wedge \phi)) \quad (\text{MR}) \frac{[\alpha] \phi \quad (\phi \rightarrow \psi)}{[\alpha] \psi} \quad (\text{ind}) \frac{\phi \rightarrow [\alpha] \phi}{\phi \rightarrow [\alpha^*] \phi} \\
\text{where } \alpha \in \Lambda_P^{(1-i)}, \phi \in \text{Fml}_{\Lambda(i)} \text{ and } i \in \{0, 1\}
\end{array}$$

Fig. 4. Additional proof rules derived from previously defined axioms (see Appendix A).

**Proof A**

$$\begin{array}{c}
\text{see Proof B} \\
(\cdot) \frac{\text{coupledPre} \rightarrow [\text{ctrl}] [\hat{\alpha}] \text{coupledPre}}{\text{coupledPre} \rightarrow [\text{ctrl}; \hat{\alpha}] \text{coupledPre}} \quad * (\text{FOL}) \frac{B > 0 \wedge x + v^2/2B \leq s \rightarrow x \leq s}{\dots} \\
(\text{ind}) \frac{\text{coupledPre} \rightarrow [\text{ctrl}; \hat{\alpha}^*] \text{coupledPre}}{\text{coupledPre} \rightarrow x \leq s} \\
(\text{MR}) \frac{\text{coupledPre} \rightarrow [\text{ctrl}; \hat{\alpha}^*] \text{coupledPre}}{\text{coupledPre} \rightarrow [\text{ctrl}; \hat{\alpha}^*] x \leq s}
\end{array}$$

**Proof B**

$$\begin{array}{c}
(\text{HR0}) \frac{* (\text{Key: Lemma 9.1})}{\text{ctrlPre} \rightarrow [\text{ctrl}] \text{ctrlPost}} \quad * (\text{FOL}) \frac{\text{coupledPre} \rightarrow \phi_3}{\text{coupledPre} \rightarrow [\text{ctrl}] \phi_3} \quad \text{using Lemma 9.2} \\
(F^{(i)}) \frac{\text{ctrlPre} \wedge \phi_2 \rightarrow [\text{ctrl}] (\text{ctrlPost} \wedge \phi_2)}{\text{coupledPre} \rightarrow [\text{ctrl}] ((\text{ctrlPost} \wedge \phi_2) \wedge \phi_3)} \quad (\text{V}) \frac{\text{coupledPre} \rightarrow \phi_3}{\text{coupledPre} \rightarrow [\text{ctrl}] \phi_3} \quad \text{see Appendix C.3} \\
([\ ] \wedge) \frac{(\text{MR}) \text{coupledPre} \rightarrow [\text{ctrl}] ((\text{ctrlPost} \wedge \phi_2) \wedge \phi_3)}{\text{coupledPre} \rightarrow [\text{ctrl}] [\hat{\alpha}] \text{coupledPre}} \quad \frac{((\text{ctrlPost} \wedge \phi_2) \wedge \phi_3)}{\rightarrow [\hat{\alpha}] \text{coupledPre}} \quad \square
\end{array}$$

The validity of this formula guarantees safety ( $x \leq s$ ) for any number of control-environment loop iterations assuming `coupledPre`. First, note that there is no straight forward way to even specify this desired behavior without HDL as it represents a system level guarantee only emerging from the looped interaction of Java and the hybrid program. Secondly, note that our calculus proves this result by *decomposing* proof obligations until the JavaDL and  $d\mathcal{L}$  lemmas above are applicable. This underscores the power of our logic to state and prove statements about systems that do not neatly live in the world of a single dynamic theory, but require their interaction.

*Limitations.* Our case study demonstrates that practical proofs reach a scale where pen-and-paper formalization alone becomes infeasible. In future work, we plan to extend our Isabelle formalization into a practical tool for HDL proofs. Our approach is also fundamentally limited by its definition of a dynamic theory. For example, the validity of Barcan’s axiom (B) implies that a variable’s domain does not grow or shrink. However, this limitation can typically be circumvented by introducing predicates stating whether an object actually exists [47].

## 10 Conclusion

This work introduces *Heterogeneous Dynamic Logic*, a compositional framework for the verification of heterogeneous programs. Similarly to how Satisfiability Modulo Theories modularly combines data logics to construct more expressive, combined first-order theories, Heterogeneous Dynamic Logic modularly combines program logics to construct more expressive, combined heterogeneous program logics. The resulting heterogeneous dynamic theories enable not only the *specification* of otherwise hard to formalize properties (see Sections 2 and 9), but also their *verification* by providing a proof calculus that decomposes problems in a manner that is compatible with existing homogeneous verification infrastructure – including, both, classical dynamic logic calculi and KAT-based equational reasoning. Section 8 shows, under common assumptions, that heterogeneous proof calculi inherit not just rules and axioms but also relative completeness. This proves that verifying heterogeneous programs is no harder than verifying properties about the homogeneous programs and their shared first-order structure (i.e., their data logic). Our proof theory of heterogeneous systems will not only enable the combination of existing verification methodologies, but also serves as a vehicle for future dynamic logic or KAT-based verification techniques, which can seamlessly integrate with the existing ecosystem through HDL.

## Data Availability

We provide our extensible Isabelle formalization on Zenodo [63].

## Acknowledgments

This work was supported by funding from the pilot program Core-Informatics of the Helmholtz Association (HGF) and by an Alexander von Humboldt Professorship.

## References

- [1] Noah Abou El Wafa and André Platzer. 2025. Complete first-order game logic. *CoRR*, abs/2504.03495. arXiv: 2504.03495.
- [2] Wolfgang Ahrendt, Bernhard Beckert, Richard Bubel, Reiner Hähnle, Peter H. Schmitt, and Mattias Ulbrich, (Eds.) 2016. *Deductive Software Verification - The KeY Book - From Theory to Practice*. LNCS. Vol. 10001. Springer. ISBN: 978-3-319-49811-9. doi:10.1007/978-3-319-49812-6.
- [3] Gidon Ernst, Matthias Güdemann, Alexander Knapp, Florian Nafz, Frank Ortmeier, Hella Ponsar, Gerhard Schellhorn, and Alexander Schiendorfer, (Eds.) 2025. *The many uses of dynamic logic. Go Where the Bugs Are - Essays Dedicated to Wolfgang Reif on the Occasion of His 65th Birthday*. LNCS. Springer, 56–82. doi:10.1007/978-3-031-92196-4\_4.
- [4] Vytautas Astrauskas, Peter Müller, Federico Poli, and Alexander J. Summers. 2019. Leveraging rust types for modular specification and verification. *Proc. ACM Program. Lang.*, 3, OOPSLA, 147:1–147:30. doi:10.1145/3360573.
- [5] Michael Barnett, Bor-Yuh Evan Chang, Robert DeLine, Bart Jacobs, and K. Rustan M. Leino. 2005. Boogie: A modular reusable verifier for object-oriented programs. In *Formal Methods for Components and Objects, 4th International Symposium, FMCO 2005, Amsterdam, The Netherlands, November 1-4, 2005, Revised Lectures* (LNCS). Frank S. de Boer, Marcello M. Bonsangue, Susanne Graf, and Willem P. de Roever, (Eds.) Springer, 364–387. doi:10.1007/11804192\_17.
- [6] Clark W. Barrett and Cesare Tinelli. 2018. Satisfiability modulo theories. In *Handbook of Model Checking*. Edmund M. Clarke, Thomas A. Henzinger, Helmut Veith, and Roderick Bloem, (Eds.) Springer, 305–343. doi:10.1007/978-3-319-10575-8\_11.
- [7] Bernhard Beckert. 2000. A dynamic logic for the formal verification of java card programs. In *Java on Smart Cards: Programming and Security, First International Workshop, JavaCard 2000, Cannes, France, September 14, 2000, Revised Papers* (LNCS). Isabelle Attali and Thomas P. Jensen, (Eds.) Springer, 6–24. doi:10.1007/3-540-45165-X\_2.
- [8] Bernhard Beckert, Daniel Bruns, Vladimir Klebanov, Christoph Scheben, Peter H. Schmitt, and Mattias Ulbrich. 2013. Information flow in object-oriented software. In *Logic-Based Program Synthesis and Transformation, 23rd International Symposium, LOPSTR 2013, Madrid, Spain, September 18-19, 2013, Revised Selected Papers* (LNCS). Gopal Gupta and Ricardo Peña, (Eds.) Springer, 19–37. doi:10.1007/978-3-319-14125-1\_2.
- [9] Bernhard Beckert and André Platzer. 2006. Dynamic logic with non-rigid functions. In *Automated Reasoning, Third International Joint Conference, IJCAR 2006, Seattle, WA, USA, August 17-20, 2006, Proceedings* (LNCS). Ulrich Furbach and Natarajan Shankar, (Eds.) Springer, 266–280. doi:10.1007/11814771\_23.
- [10] Bernhard Beckert, Peter Sanders, Mattias Ulbrich, Julian Wiesler, and Sascha Witt. 2024. Formally verifying an efficient sorter. In *Tools and Algorithms for the Construction and Analysis of Systems - 30th International Conference, TACAS 2024, Luxembourg City, Luxembourg, April 6-11, 2024, Proceedings, Part I* (LNCS). Bernd Finkbeiner and Laura Kovács, (Eds.) Springer, 268–287. doi:10.1007/978-3-031-57246-3\_15.
- [11] Stefan Blom and Marieke Huisman. 2014. The vercors tool for verification of concurrent programs. In *FM 2014: Formal Methods - 19th International Symposium, Singapore, May 12-16, 2014. Proceedings* (LNCS). Cliff B. Jones, Pekka Pihlajasaari, and Jun Sun, (Eds.) Springer, 127–131. doi:10.1007/978-3-319-06410-9\_9.
- [12] Rose Bohrer. 2017. Differential dynamic logic. *Archive of Formal Proofs*, (Feb. 2017). [https://isa-afp.org/entries/Differential\\_Dynamic\\_Logic.html](https://isa-afp.org/entries/Differential_Dynamic_Logic.html), Formal proof development.
- [13] Rose Bohrer and André Platzer. 2018. A hybrid, dynamic logic for hybrid-dynamic information flow. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2018, Oxford, UK, July 09-12, 2018*. Anuj Dawar and Erich Grädel, (Eds.) ACM, 115–124. doi:10.1145/3209108.3209151.
- [14] Pei-Wei Chen, Shaokai Lin, Adwait Godbole, Ramneet Singh, Elizabeth Polgreen, Edward A. Lee, and Sanjit A. Seshia. 2025. Polyver: A compositional approach for polyglot system modeling and verification. In *Proceedings of the 25th Conference on Formal Methods in Computer-Aided Design, FMCAD 2025, Menlo Park, CA, USA, October 6-10, 2025*. Ahmed Irfan and Daniela Kaufmann, (Eds.) TU Wien Academic Press. doi:10.34727/2025/ISBN.978-3-85448-084-6\_10.
- [15] Alessandro Cimatti, Alberto Griggio, Sergio Mover, Marco Roveri, and Stefano Tonetta. 2022. Verification modulo theories. *Formal Methods Syst. Des.*, 60, 3, 452–481. doi:10.1007/S10703-023-00434-X.
- [16] Edmund M. Clarke, Daniel Kroening, and Flavio Lerda. 2004. A tool for checking ANSI-C programs. In *Tools and Algorithms for the Construction and Analysis of Systems, 10th International Conference, TACAS 2004, Barcelona,*

- Spain, March 29 - April 2, 2004, *Proceedings* (LNCS). Kurt Jensen and Andreas Podelski, (Eds.) Springer, 168–176. doi:[10.1007/978-3-540-24730-2\\_15](https://doi.org/10.1007/978-3-540-24730-2_15).
- [17] Martin de Boer, Stijn de Gouw, Jonas Klamroth, Christian Jung, Mattias Ulbrich, and Alexander Weigl. 2023. Formal specification and verification of jdk’s identity hash map implementation. *Formal Aspects Comput.*, 35, 3, 18:1–18:26. doi:[10.1145/3594729](https://doi.org/10.1145/3594729).
- [18] Jules Desharnais, Bernhard Möller, and Georg Struth. 2006. Kleene algebra with domain. *ACM Trans. Comput. Log.*, 7, 4, 798–833. doi:[10.1145/1183278.1183285](https://doi.org/10.1145/1183278.1183285).
- [19] Edsger W. Dijkstra. 1975. Guarded commands, nondeterminacy and formal derivation of programs. *Commun. ACM*, 18, 8, 453–457. doi:[10.1145/360933.360975](https://doi.org/10.1145/360933.360975).
- [20] Gidon Ernst, Wolfram Pfeifer, and Mattias Ulbrich. 2024. Contract-lib: A proposal for a common interchange format for software system specification. In *Leveraging Applications of Formal Methods, Verification and Validation. Specification and Verification - 12th International Symposium, ISOFA 2024, Crete, Greece, October 27-31, 2024, Proceedings, Part III* (LNCS). Tiziana Margaria and Bernhard Steffen, (Eds.) Springer, 79–105. doi:[10.1007/978-3-031-75380-0\\_6](https://doi.org/10.1007/978-3-031-75380-0_6).
- [21] Jean-Christophe Filliâtre and Andrei Paskevich. 2013. Why3 - where programs meet provers. In *Programming Languages and Systems - 22nd European Symposium on Programming, ESOP 2013, Rome, Italy, March 16-24, 2013. Proceedings* (LNCS). Matthias Felleisen and Philippa Gardner, (Eds.) Springer, 125–128. doi:[10.1007/978-3-642-37036-6\\_8](https://doi.org/10.1007/978-3-642-37036-6_8).
- [22] Simon Foster. 2019. Hybrid relations in isabelle/utp. In *Unifying Theories of Programming - 7th International Symposium, UTP 2019, Dedicated to Tony Hoare on the Occasion of His 85th Birthday, Porto, Portugal, October 8, 2019, Proceedings* (LNCS). Pedro Ribeiro and Augusto Sampaio, (Eds.) Springer, 130–153. doi:[10.1007/978-3-030-31038-7\\_7](https://doi.org/10.1007/978-3-030-31038-7_7).
- [23] Simon Foster, Frank Zeyda, and Jim Woodcock. 2014. Isabelle/utp: A mechanised theory engineering framework. In *Unifying Theories of Programming - 5th International Symposium, UTP 2014, Singapore, May 13, 2014, Revised Selected Papers* (LNCS). David A. Naumann, (Ed.) Springer, 21–41. doi:[10.1007/978-3-319-14806-9\\_2](https://doi.org/10.1007/978-3-319-14806-9_2).
- [24] Nathan Fulton, Stefan Mitsch, Jan-David Quesel, Marcus Völp, and André Platzer. 2015. Keymaera X: an axiomatic tactical theorem prover for hybrid systems. In *Automated Deduction - CADE-25 - 25th International Conference on Automated Deduction, Berlin, Germany, August 1-7, 2015, Proceedings* (LNCS). Amy P. Felty and Aart Middeldorp, (Eds.) Springer, 527–538. doi:[10.1007/978-3-319-21401-6\\_36](https://doi.org/10.1007/978-3-319-21401-6_36).
- [25] Dov M. Gabbay. 1996. An overview of fibred semantics and the combination of logics. In *Frontiers of Combining Systems, First International Workshop FroCoS 1996, Munich, Germany, March 26-29, 1996, Proceedings*. Applied Logic Series. Franz Baader and Klaus U. Schulz, (Eds.) Kluwer Academic Publishers, 1–55. doi:[10.1007/978-94-009-0349-4\\_1](https://doi.org/10.1007/978-94-009-0349-4_1).
- [26] Dov M. Gabbay. 1999. *Fibring Logics*. Clarendon Press, New York.
- [27] Luis Garcia, Stefan Mitsch, and André Platzer. 2019. Hylplc: hybrid programmable logic controller program translation for verification. In *Proceedings of the 10th ACM/IEEE International Conference on Cyber-Physical Systems, ICCPS 2019, Montreal, QC, Canada, April 16-18, 2019*. Xue Liu, Paulo Tabuada, Miroslav Pajic, and Linda Bushnell, (Eds.) ACM, 47–56. doi:[10.1145/3302509.3311036](https://doi.org/10.1145/3302509.3311036).
- [28] Kurt Gödel. 1931. Über formal unentscheidbare sätze der principia mathematica und verwandter systeme i. *Monatshefte für Mathematik und Physik*, 38, 173–198. doi:[10.1007/BF01700692](https://doi.org/10.1007/BF01700692).
- [29] Michael Greenberg, Ryan Beckett, and Eric Hayden Campbell. 2022. Kleene algebra modulo theories: a framework for concrete kats. In *PLDI ’22: 43rd ACM SIGPLAN International Conference on Programming Language Design and Implementation, San Diego, CA, USA, June 13 - 17, 2022*. Ranjit Jhala and Isil Dillig, (Eds.) ACM, 594–608. doi:[10.1145/3519939.3523722](https://doi.org/10.1145/3519939.3523722).
- [30] David Harel. 1979. *First-Order Dynamic Logic*. LNCS. Vol. 68. Springer. ISBN: 3-540-09237-4. doi:[10.1007/3-540-09237-4](https://doi.org/10.1007/3-540-09237-4).
- [31] David Harel, Dexter Kozen, and Jerzy Tiuryn. 2000. *Dynamic Logic*. The MIT Press, (Sept. 2000). ISBN: 978-0-262-27495-1. doi:[10.7551/mitpress/2516.001.0001](https://doi.org/10.7551/mitpress/2516.001.0001).
- [32] C. A. R. Hoare. 1969. An axiomatic basis for computer programming. *Commun. ACM*, 12, 10, 576–580. doi:[10.1145/363259](https://doi.org/10.1145/363259).
- [33] C. A. R. Hoare. 1997. Unified theories of programming. In *Mathematical Methods in Program Development*. Manfred Broy and Birgit Schieder, (Eds.) Springer Berlin Heidelberg, Berlin, Heidelberg, 313–367. ISBN: 978-3-642-60858-2. doi:[10.1007/978-3-642-60858-2\\_21](https://doi.org/10.1007/978-3-642-60858-2_21).
- [34] Jean-Baptiste Jeannin, Khalil Ghorbal, Yanni Kouskoulas, Aurora C. Schmidt, Ryan W. Gardner, Stefan Mitsch, and André Platzer. 2017. A formally verified hybrid system for safe advisories in the next-generation airborne collision avoidance system. *Int. J. Softw. Tools Technol. Transf.*, 19, 6, 717–741. doi:[10.1007/S10009-016-0434-1](https://doi.org/10.1007/S10009-016-0434-1).
- [35] Chris Johannsen, Karthik Nukala, Rohit Dureja, Ahmed Irfan, Natarajan Shankar, Cesare Tinelli, Moshe Y. Vardi, and Kristin Yvonne Rozier. 2024. The moxi model exchange tool suite. In *Computer Aided Verification - 36th International Conference, CAV 2024, Montreal, QC, Canada, July 24-27, 2024, Proceedings, Part I* (LNCS). Arie Gurfinkel and Vijay Ganesh, (Eds.) Springer, 203–218. doi:[10.1007/978-3-031-65627-9\\_10](https://doi.org/10.1007/978-3-031-65627-9_10).

- [36] Eduard Kamburjan, Stefan Mitsch, and Reiner Hähnle. 2022. A hybrid programming language for formal modeling and verification of hybrid systems. *Leibniz Trans. Embed. Syst.*, 8, 2, 04:1–04:34. doi:[10.4230/LITES.8.2.4](https://doi.org/10.4230/LITES.8.2.4).
- [37] Florent Kirchner, Nikolai Kosmatov, Virgile Prevosto, Julien Signoles, and Boris Yakobowski. 2015. Frama-c: A software analysis perspective. *Formal Aspects Comput.*, 27, 3, 573–609. doi:[10.1007/S00165-014-0326-7](https://doi.org/10.1007/S00165-014-0326-7).
- [38] Jonas Klamroth, Bernhard Beckert, Max Scheerer, and Oliver Denninger. 2023. Qin: enabling formal methods to deal with quantum circuits. In *IEEE International Conference on Quantum Software, QSW 2023, Chicago, IL, USA, July 2-8, 2023*. Shaukat Ali et al., (Eds.) IEEE, 175–185. doi:[10.1109/QSW59989.2023.00029](https://doi.org/10.1109/QSW59989.2023.00029).
- [39] Dexter Kozen. 1997. Kleene algebra with tests. *ACM Trans. Program. Lang. Syst.*, 19, 3, 427–443. doi:[10.1145/256167.256195](https://doi.org/10.1145/256167.256195).
- [40] Andrea Lattuada, Travis Hance, Chanhee Cho, Matthias Brun, Isitha Subasinghe, Yi Zhou, Jon Howell, Bryan Parno, and Chris Hawblitzel. 2023. Verus: verifying rust programs using linear ghost types. *Proc. ACM Program. Lang.*, 7, OOPSLA1, 286–315. doi:[10.1145/3586037](https://doi.org/10.1145/3586037).
- [41] Sarah M. Loos and André Platzer. 2016. Differential refinement logic. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science, LICS '16, New York, NY, USA, July 5-8, 2016*. Martin Grohe, Eric Koskinen, and Natarajan Shankar, (Eds.) ACM, 505–514. doi:[10.1145/2933575.2934555](https://doi.org/10.1145/2933575.2934555).
- [42] Peter Müller, Malte Schwerhoff, and Alexander J. Summers. 2016. Viper: A verification infrastructure for permission-based reasoning. In *Verification, Model Checking, and Abstract Interpretation - 17th International Conference, VMCAI 2016, St. Petersburg, FL, USA, January 17-19, 2016. Proceedings* (LNCS). Barbara Jobstmann and K. Rustan M. Leino, (Eds.) Springer, 41–62. doi:[10.1007/978-3-662-49122-5\\_2](https://doi.org/10.1007/978-3-662-49122-5_2).
- [43] Greg Nelson and Derek C. Oppen. 1979. Simplification by cooperating decision procedures. *ACM Trans. Program. Lang. Syst.*, 1, 2, 245–257. doi:[10.1145/357073.357079](https://doi.org/10.1145/357073.357079).
- [44] Pedro Nora, Jurriaan Rot, Lutz Schröder, and Paul Wild. 2025. Relational connectors and heterogeneous simulations. In *Foundations of Software Science and Computation Structures - 28th International Conference, FoSSaCS 2025, Hamilton, ON, Canada, May 3-8, 2025, Proceedings* (LNCS). Parosh Aziz Abdulla and Delia Kesner, (Eds.) Springer, 111–132. doi:[10.1007/978-3-031-90897-2\\_6](https://doi.org/10.1007/978-3-031-90897-2_6).
- [45] Derek C. Oppen. 1980. Complexity, convexity and combinations of theories. *Theor. Comput. Sci.*, 12, 291–302. doi:[10.1016/0304-3975\(80\)90059-6](https://doi.org/10.1016/0304-3975(80)90059-6).
- [46] Marco Paganoni and Carlo A. Furia. 2023. Verifying functional correctness properties at the level of java bytecode. In *Formal Methods - 25th International Symposium, FM 2023, Lübeck, Germany, March 6-10, 2023, Proceedings* (LNCS). Marsha Chechik, Joost-Pieter Katoen, and Martin Leucker, (Eds.) Springer, 343–363. doi:[10.1007/978-3-031-27481-7\\_20](https://doi.org/10.1007/978-3-031-27481-7_20).
- [47] André Platzer. 2012. A complete axiomatization of quantified differential dynamic logic for distributed hybrid systems. *Log. Methods Comput. Sci.*, 8, 4. doi:[10.2168/LMCS-8\(4:17\)2012](https://doi.org/10.2168/LMCS-8(4:17)2012).
- [48] André Platzer. 2017. A complete uniform substitution calculus for differential dynamic logic. *J. Autom. Reason.*, 59, 2, 219–265. doi:[10.1007/S10817-016-9385-1](https://doi.org/10.1007/S10817-016-9385-1).
- [49] André Platzer. 2008. Differential dynamic logic for hybrid systems. *J. Autom. Reason.*, 41, 2, 143–189. doi:[10.1007/S10817-008-9103-8](https://doi.org/10.1007/S10817-008-9103-8).
- [50] André Platzer. 2012. The complete proof theory of hybrid systems. In *Proceedings of the 27th Annual IEEE Symposium on Logic in Computer Science, LICS 2012, Dubrovnik, Croatia, June 25-28, 2012*. IEEE Computer Society, 541–550. doi:[10.1109/LICS.2012.64](https://doi.org/10.1109/LICS.2012.64).
- [51] André Platzer and Jan-David Quesel. 2009. European train control system: A case study in formal verification. In *Formal Methods and Software Engineering, 11th International Conference on Formal Engineering Methods, ICFEM 2009, Rio de Janeiro, Brazil, December 9-12, 2009. Proceedings* (LNCS). Karin K. Breitman and Ana Cavalcanti, (Eds.) Springer, 246–265. doi:[10.1007/978-3-642-10373-5\\_13](https://doi.org/10.1007/978-3-642-10373-5_13).
- [52] André Platzer and Yong Kiam Tan. 2020. Differential equation invariance axiomatization. *J. ACM*, 67, 1, 6:1–6:66. doi:[10.1145/3380825](https://doi.org/10.1145/3380825).
- [53] Enguerrand Prebet and André Platzer. 2024. Uniform substitution for differential refinement logic. In *Automated Reasoning - 12th International Joint Conference, IJCAR 2024, Nancy, France, July 3-6, 2024, Proceedings, Part II* (LNCS). Christoph Benzmüller, Marijn J. H. Heule, and Renate A. Schmidt, (Eds.) Springer, 196–215. doi:[10.1007/978-3-031-63501-4\\_11](https://doi.org/10.1007/978-3-031-63501-4_11).
- [54] H. G. Rice. 1953. Classes of recursively enumerable sets and their decision problems. *Transactions of the American Mathematical Society*, 74, 2, 358–366. doi:[10.2307/1990888](https://doi.org/10.2307/1990888).
- [55] Philipp Rümmer and Muhammad Ali Shah. 2007. Proving programs incorrect using a sequent calculus for java dynamic logic. In *Tests and Proofs - 1st International Conference, TAP 2007, Zurich, Switzerland, February 12-13, 2007. Revised Papers* (LNCS). Yuri Gurevich and Bertrand Meyer, (Eds.) Springer, 41–60. doi:[10.1007/978-3-540-73770-4\\_3](https://doi.org/10.1007/978-3-540-73770-4_3).
- [56] Gerhard Schellhorn and Wolfgang Ahrendt. 1997. Reasoning about abstract state machines: the WAM case study. *J. Univers. Comput. Sci.*, 3, 4, 377–413. doi:[10.3217/JUCS-003-04-0377](https://doi.org/10.3217/JUCS-003-04-0377).

- [57] Igor Sedlár. 2023. Kleene algebra with dynamic tests: completeness and complexity. *CoRR*, abs/2311.06937. arXiv: [2311.06937](https://arxiv.org/abs/2311.06937).
- [58] Igor Sedlár. 2023. On the complexity of kleene algebra with domain. In *Relational and Algebraic Methods in Computer Science - 20th International Conference, RAMiCS 2023, Augsburg, Germany, April 3-6, 2023, Proceedings* (LNCS). Roland Glück, Luigi Santocane, and Michael Winter, (Eds.) Springer, 208–223. doi:[10.1007/978-3-031-28083-2\\_13](https://doi.org/10.1007/978-3-031-28083-2_13).
- [59] Krister Segerberg. 1982. A completeness theorem in the modal logic of programs. *Banach Center Publications*, 9, 31–46. doi:[10.4064/-9-1-31-46](https://doi.org/10.4064/-9-1-31-46).
- [60] Robert E. Shostak. 1978. An algorithm for reasoning about equality. *Commun. ACM*, 21, 7, 583–585. doi:[10.1145/359545.359570](https://doi.org/10.1145/359545.359570).
- [61] Yong Kiam Tan and André Platzer. 2021. An axiomatic approach to existence and liveness for differential equations. *Formal Aspects Comput.*, 33, 4-5, 461–518. doi:[10.1007/S00165-020-00525-0](https://doi.org/10.1007/S00165-020-00525-0).
- [62] Samuel Teuber, Stefan Mitsch, and André Platzer. 2024. Provably safe neural network controllers via differential dynamic logic. In *Advances in Neural Information Processing Systems 38: Annual Conference on Neural Information Processing Systems 2024, NeurIPS 2024, Vancouver, BC, Canada, December 10 - 15, 2024*. Amir Globersons, Lester Mackey, Danielle Belgrave, Angela Fan, Ulrich Paquet, Jakub M. Tomczak, and Cheng Zhang, (Eds.)
- [63] [SW] Samuel Teuber, Mattias Ulbrich, André Platzer, and Bernhard Beckert, Artifact for "Heterogeneous Dynamic Logic: Provability Modulo Program Theories" Mar. 2026. doi:[10.5281/zenodo.19551156](https://doi.org/10.5281/zenodo.19551156).
- [64] Samuel Teuber, Mattias Ulbrich, André Platzer, and Bernhard Beckert. 2025. Heterogeneous dynamic logic: provability modulo program theories. *CoRR*, abs/2507.08581. doi:[10.48550/ARXIV.2507.08581](https://doi.org/10.48550/ARXIV.2507.08581).
- [65] Cesare Tinelli and Mehdi T. Harandi. 1996. A new correctness proof of the Nelson-Oppen combination procedure. In *Frontiers of Combining Systems, First International Workshop FroCoS 1996, Munich, Germany, March 26-29, 1996, Proceedings* (Applied Logic Series). Franz Baader and Klaus U. Schulz, (Eds.) Kluwer Academic Publishers, 103–119. doi:[10.1007/978-94-009-0349-4\\_5](https://doi.org/10.1007/978-94-009-0349-4_5).
- [66] Cesare Tinelli and Calogero G. Zarba. 2004. Combining decision procedures for sorted theories. In *Logics in Artificial Intelligence, 9th European Conference, JELIA 2004, Lisbon, Portugal, September 27-30, 2004, Proceedings* (LNCS). José Júlio Alferes and João Alexandre Leite, (Eds.) Springer, 641–653. doi:[10.1007/978-3-540-30227-8\\_53](https://doi.org/10.1007/978-3-540-30227-8_53).
- [67] Mattias Ulbrich. 2010. A dynamic logic for unstructured programs with embedded assertions. In *Formal Verification of Object-Oriented Software - International Conference, FoVeOOS 2010, Paris, France, June 28-30, 2010, Revised Selected Papers* (LNCS). Bernhard Beckert and Claude Marché, (Eds.) Springer, 168–182. doi:[10.1007/978-3-642-18070-5\\_12](https://doi.org/10.1007/978-3-642-18070-5_12).
- [68] Mattias Ulbrich. 2013. *Dynamic Logic for an Intermediate Language: Verification, Interaction and Refinement*. Ph.D. Dissertation. Karlsruhe Institute of Technology. doi:[10.5445/IR/1000041169](https://doi.org/10.5445/IR/1000041169).
- [69] Yuanrui Zhang. 2024. Parameterized dynamic logic - towards A cyclic logical framework for program verification via operational semantics. *CoRR*, abs/2404.18098. arXiv: [2404.18098](https://arxiv.org/abs/2404.18098).

Received 2025-11-13; accepted 2026-04-03