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Symplectic groups, mapping class groups and the stability of bounded cohomology

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A B S T R A C T

Mapping class groups satisfy cohomological stability. In this note we show how results by Bestvina and Fujiwara imply that their bounded cohomology does not stabilize, additionally we show that stabily polynomials in the Mumford-Morita-Miller classes are unbounded i.e. their norm tends to infinity as one increases the genus.

While the bounded cohomology of the symplectic group does stabilize, we show that it does not stabilize via isometries in degree 2. In order to establish this we calculate the norm of the signature class in $\mathrm{Sp}_{2h}(\mathbf{R})$ and estimate the norm of the integral signature class.

1. Introduction

Homological stability refers to a phenomenon, where the group homology of an increasing sequence of groups $G_0 \subset G_1 \subset \ldots$ eventually stabilizes i.e. the inclusions induce isomorphisms in more and more degrees. The two most important examples for our purposes are given by symplectic groups $\mathrm{Sp}_R(2h)$ and mapping class groups of surfaces. More precisely, consider the mapping class group $\mathcal{M}(\Sigma_{g,b})$ of a surface of genus g with b boundary components, then there are maps

$$\begin{aligned}\mathcal{M}(\Sigma_{g,b}) &\to \mathcal{M}(\Sigma_{g+1,b})\\ \mathcal{M}(\Sigma_{g,b}) &\to \mathcal{M}(\Sigma_{g,b+1})\\ \mathcal{M}(\Sigma_{g,b}) &\to \mathcal{M}(\Sigma_{g,b-1})\end{aligned}$$

coming from the various inclusions of surfaces and they all induce isomorphisms in homology in a range of degrees that increases with g.

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Classically homological stability is proven by constructing a sequence of simplicial complexes X_i on which G_i acts in a particular fashion and which have to be acyclic in an increasing range of degrees. This approach goes back to Quillen. Recently this approach has been adapted by Hartnick and De La Cruz Mengual in [4] to yield a similar setup that proves stability for bounded cohomology. The most important change lies in the fact that the simplicial complexes have to have vanishing simplicial bounded cohomology in an increasing range of degrees. Many of the necessary tools to translate a classical homological stability proof to the realm of bounded cohomology have been established in [12]. There, one of the key observations is that a simplicial complex with vanishing first bounded cohomology has to have finite diameter (see also Example 2.7 in [12]) and that an acyclic complex has to satisfy a higher dimensional analogue of having finite diameter in order to have vanishing simplicial bounded cohomology in range.

The complexes in question for the mapping class groups are the so called curve complexes. While these are highly connected, they have infinite diameter. Furthermore they are hyperbolic as shown in [16]. These two facts allow one to construct quasimorphisms that will lie in the kernel of the corresponding inclusions in bounded cohomology or in other words show that the map induced by the inclusion

$$H^2_{\mathrm{b}}(\mathcal{M}(\Sigma_{g+1,b})\,;\mathbf{R}) \to H^2_{\mathrm{b}}(\mathcal{M}(\Sigma_{g,b})\,;\mathbf{R})$$

is never injective. This will be carried out in Section 2. This instability is closely related to one of Kirby's Problems (Problem 2.18 in [10]), for a given h, what is the minimal g such that there exists a fiber bundle

$$\Sigma_h \to E \to \Sigma_g$$

with E having non-zero or fixed signature? This can be stabilized: Let $g_h(n)$ denote the minimal genus g such that there exists a surface bundle $\Sigma_h \to E \to \Sigma_g$ with signature $4n$ (the signature is always a multiple of 4 by [14]). Now let G_h denote the limit of $\frac{g_h(n)}{n}$, then one can show that G_h corresponds to the ℓ_1-norm of a generator of $H_2(\mathcal{M}(\Sigma_h)\,;\mathbf{Z})$ in $H_2(\mathcal{M}(\Sigma_h)\,;\mathbf{R})$. By duality computing this norm is equivalent to computing the norm of the generator of the second group cohomology of the mapping class group. Hence the aforementioned instability encapsulates how G_h changes, when changing h. Furthermore by now there exist enough examples to show that G_h tends to zero as h increases (see for example [18]), which also shows that the norm of the corresponding generator of the second group cohomology tends to infinity. Finally the stable cohomology of mapping class groups is a polynomial ring generated by the so called Mumford-Morita-Miller-Classes. These are characteristic classes that are defined by fiber integrating powers of the Euler class of the vertical tangent bundle of a surface bundle. In Proposition 2.2, we show that the norm of these classes also tends to infinity as the genus increases. In particular the MMM-classes on the stable mapping class group cannot be bounded (see Corollary 2.3). This is particularly interesting, since if one fixes the genus, it was shown in [19], that the odd MMM-classes are bounded and the recent preprint [7], claims that all MMM-classes are bounded.

Via the action of the mapping class group on the first homology of the surface one obtains a close relationship between mapping class groups and symplectic groups. In particular the aforementioned map induces an isomorphism on second group cohomology. It was shown in [3] that the continuous bounded cohomology of symplectic groups stabilizes. Using Corollary 1.4 in [17], this can be translated to the integral symplectic groups as well. This difference between mapping class groups and symplectic groups together with their relationship yields some interesting properties of the stability of the bounded cohomology of symplectic groups. First and foremost using this one can show that the stabilization maps between integral as well as real symplectic groups while inducing isomorphism fail to induce isometries on second bounded cohomology. In [2] it was already shown via explicit computations that the third bounded cohomology of symplectic groups does not stabilize isometrically (their result is about continuous bounded cohomology, but using Corollary 1.4 in [17], this can be translated to discrete bounded cohomology of symplectic groups of the Gaussian integers). The example presented herein is particularly interesting since this occurs in degree 2, where bounded cohomology is in fact a Banach space (see [15]). More precisely one can show that the maps $\mathrm{Sp}_{2h}(R) \to \mathrm{Sp}_{2(h+k)}(R)$ have arbitrarily small norm as k tends to infinity for R either the reals or integers.

Furthermore as a consequence one obtains that the bounded cohomology of the stable symplectic group $\mathrm{Sp}_\infty(R)$ (i.e. the colimit of the aforementioned sequence) does not surject onto the stable bounded cohomology of the finite dimensional symplectic groups (indeed this would imply that the aforementioned inclusions do not have arbitrarily small norm). Here one should note the relationship to the vanishing modulus introduced in Section 4 in [6], where the relationship between the vanishing of bounded cohomology and colimits was investigated. All in all this shows that the stability of bounded cohomology seems to be more delicate than ordinary (co)homological stability.

In particular one can ask the general question:

Question 1. If the bounded cohomology of a sequence of groups Γ_i does stabilize, is there an injection

$$H^*_\mathrm{b}(\mathrm{colim}\Gamma_i; \mathbf{R}) \to \lim H^*_\mathrm{b}(\Gamma_i; \mathbf{R})$$

that hits all classes with overall bounded norm?

Since a lot of the usage of classical homological stability results comes from the stable group $\mathrm{colim}\Gamma_i$ behaving more nicely than Γ_i, answering this question would probably allow one to get a better understanding of the bounded cohomology of groups that satisfy stability of bounded cohomology.

One the other side, the proof of the instability of the bounded cohomology of mapping class groups relied on the fact that curve complexes have infinite diameter, so the obvious question becomes:

Question 2. If the complexes occurring in a classical proof of homological stability have infinite diameter, can one conclude that the second bounded cohomology of these groups does not stabilize?

Stated that way, the question is probably too naive. In particular the proof not only relies on the curve complexes being hyperbolic, but also on the fact that the curve complex of $\Sigma_{g,1}$ injects into the curve complex $\Sigma_{g+1,1}$ such that the image has finite diameter.

Lastly the author does not claim any originality. This note represents the application of known results in the context of stability of bounded cohomology. Finally the author would like to thank Robin Sroka and Mladen Bestvina for helpful discussions. Lastly the author thanks Mark Pedron for providing the argument in the proof of Proposition 2.2.

2. Mapping class groups

The well-studied curve complex $\mathcal{C}_h$ of a surface Σ_h is defined as follows: The vertices are given by isotopy classes of simple closed curves γ on Σ_h and a collection of curves $(\gamma_0, \ldots, \gamma_p)$ forms a p-simplex if the isotopy classes can be realized with pairwise disjoint images. This complex and its many variations have an intricate relationship with the mapping class group. In particular they were used by Harer in [8] to establish homological stability for mapping class groups. Bestvina and Fujiwara used the curve complex to construct many non-trivial quasimorphisms on $\mathcal{M}(\Sigma_h)$ and we will use these quasimorphisms to show that the inclusion

$$\mathcal{M}(\Sigma_{h,1}) \to \mathcal{M}(\Sigma_{h+1,1})$$

will never induce an injective map on second bounded cohomology. The mapping class group acts simplicially on the curve complex via postcomposition of the curves. Let $d(x, y)$ denote the simplicial distance function on the 1-skeleton of the curve complex denoted by $\mathcal{C}_h^{(1)}$. Given two paths w and α of finite length in $\mathcal{C}_h^{(1)}$, we define $|\alpha|_w$ to be the maximal number of non-overlapping copies of w in α. Here a copy of w is a $\mathcal{M}(\Sigma_{h,1})$-translate of w. Now for every $0 < W < |w|$, Bestvina and Fujiwara define

$$c_{w,W}(x,y) = d(x,y) - \inf_{\alpha} (|\alpha| - W\,|\alpha|_w)$$

where α ranges over all paths from x to y. Now they fix a W that is big compared to some constant depending only on the curve complex and define

$$\begin{aligned} h_{w,x_0} \colon \mathcal{M}(\Sigma_h) &\to \mathbf{R} \\ g &\mapsto c_w(x_0, g(x_0)) - c_{w^{-1}}(x_0, g(x_0)) \end{aligned}$$

where w^{-1} denotes w with the opposite orientation. They then proceed to show that for any given x_0, these define quasimorphisms on the mapping class group and that under the correspondence between quasimorphisms and second bounded cohomology these h_w span an infinite dimensional subspace of $H_b^2(\mathcal{M}(h)\,;\mathbf{R})$.

Now consider the sequence of inclusions

$$\Sigma_{g,1} \subset \Sigma_{g+1,1} \subset \Sigma_{g+1}$$

which induce inclusions

$$\mathcal{M}(\Sigma_{g,1}) \subset \mathcal{M}(\Sigma_{g+1,1}) \subset \mathcal{M}(\Sigma_{g+1})$$

that in turn induce the isomorphisms in Harer's stability result. Let x_0 denote a curve in $\Sigma_{g+1,1}$ that is disjoint from $\Sigma_{g,1}$, then $\mathcal{M}(\Sigma_{g,1})$ fixes this element. Hence for any w we have that h_{w,x_0} is actually zero when restricted to $\mathcal{M}(\Sigma_{g,1})$. In particular the bounded cohomology of $\mathcal{M}(\Sigma_{g,1})$ does not stabilize with respect to g since the map induced by the inclusion in degree two will always have a kernel.

2.1. MMM-Classes are stably unbounded

As mentioned in the introduction, mapping class groups satisfy homological stability. Moreover in [20], the stable cohomology of mapping class groups was computed and it is a polynomial ring generated by the so called Mumford-Morita-Miller-Classes. These are defined as follows: Let $\pi_E \colon E \to B$ denote an oriented surface bundle and T_vE its vertical tangent bundle. Let $e(T_vE)$ denote the Euler class of the vertical tangent bundle. Now we define the k-th MMM-class via

$$\kappa_k(E) \coloneqq (\pi_E)_!\big(e^{k+1}(T_vE)\big)\,.$$

Lemma 2.1. *Let $f \colon \hat{E} \to E$ denote a degree d covering map that is fiberwise a covering of the bundle $\pi_E \colon E \to B$, then $\kappa_k\left(\hat{E}\right) = d\kappa_k(E)$.*

Proof. Let $\pi_{\hat{E}}$ denote the composition of f and π_E. The fiber integration of a composition is the composition of the fiber integrations. Since the fiber integration of a covering map is given by the transfer tf, we have

$$\begin{aligned} \kappa_k\left(\hat{E}\right) &= (\pi_{\hat{E}})_!\left(e^{k+1}\left(T_v\hat{E}\right)\right) \\ &= (\pi_E \circ f)_!\big(e^{k+1}(f^*T_vE)\big) \\ &= (\pi_E)_! \circ tf\big(f^*\big(e^{k+1}(T_vE)\big)\big) \end{aligned}$$

$$= (\pi_E)_! \left(de^{k+1}(T_v E)\right)$$
$$= d\kappa_k(E) \quad \square$$

With this lemma at hand we can prove that stabily all polynomials in the Mumford-Morita-Miller classes are unbounded.

Proposition 2.2. *Let $P \in \lim_g H^k(\mathcal{M}(\Sigma_g)\,;\mathbf{R})$ denote a stable cohomology class, then $||P_g||$ tends to infinity as g increases, where P_g denotes the restriction of P to $\mathcal{M}(\Sigma_g)$.*

Proof. By [20], we have $\lim H^*(\mathcal{M}(\Sigma_g)\,;\mathbf{R}) \cong \mathbf{R}[(\kappa_i)_{i\in\mathbf{N}_{>0}}]$ in particular P is a polynomial in κ_k, which is homogenous with respect to the cohomology grading. Let $\hat{P} = a_i\kappa_i^d$ denote one of its non-zero homogenous summands, then there exists a $2id$-dimensional manifold B, together with a surface bundle $\pi_E\colon E \to B$ with fiber genus g for g large enough, such that $\hat{P}_g(E)$ is $\lambda > 0$, but all other summands of P_g evaluate to zero on E.

The kernel of the composition $\pi_1(\Sigma_g, *) \to H_1(\Sigma_g; \mathbf{Z}) \to H_1(\Sigma_g; \mathbf{Z}/n\mathbf{Z})$ is invariant under the action of the diffeomorphism group of Σ_g on the fundamental group, hence it defines a fiberwise finite covering $\hat{E} \to E \to B$. Let us define the composition of these two maps again by $\pi_{\hat{E}}$; this map is again a surface bundle with fiber genus $g_n = n^{2g}(g-1)+1$. Now we have

$$(n^{2g})^d\lambda = \langle \hat{P}_{g_n}\left(\hat{E}\right), [B]\rangle = \langle P_{g_n}\left(\hat{E}\right), [B]\rangle \leq ||P_{g_n}||\,||B||$$

and hence $||P_{g_n}||$ tends to infinity as n tends to infinity. $\square$

This immediately yields the following corollary:

Corollary 2.3. *Let $\mathcal{M}(\infty)$ denote the stable mapping class group i.e.* $\mathrm{colim}_g\mathcal{M}(\Sigma_{g,1})$, *then the comparison map*

$$H_b^*(\mathcal{M}(\infty)\,;\mathbf{R}) \to H^*(\mathcal{M}(\infty)\,;\mathbf{R})$$

is the zero map.

3. Symplectic groups

The second (discrete) group cohomology with integer coefficients of the symplectic groups $\mathrm{Sp}_{2h}(\mathbf{Z})$ and $\mathrm{Sp}_{2h}(\mathbf{R})$ is rationally generated by the signature class τ_{2h}^R, where R is either $\mathbf{Z}$ or $\mathbf{R}$, defined as follows: Since the symplectic group is perfect one has that its first homology vanishes, hence $H^2(\mathrm{Sp}_{2h}(R)\,;\mathbf{Z})$ is isomorphic to $\mathrm{Hom}(H_2(\mathrm{Sp}_{2h}(R)\,;\mathbf{Z})\,,\mathbf{Z})$. Now one can represent every second homology class by a map $f\colon \Sigma_g \to B\mathrm{Sp}_{2h}(R)$ which corresponds to a twisted coefficient system E on Σ_g. One can check that this induces a symmetric bilinear pairing

$$H^1(\Sigma_g; E) \times H^1(\Sigma_g; E) \to H^2(\Sigma_g; E)$$

and $\tau_{2h}^R(f([\Sigma_g]))$ is defined to be the signature of this symmetric bilinear form. We are interested in the norm of these classes. Since there is an inclusion $\mathrm{Sp}_{2h}(\mathbf{Z}) \to \mathrm{Sp}_{2h}(\mathbf{R})$ we have $\left|\left|\tau_{2h}^{\mathbf{Z}}\right|\right| \leq \left|\left|\tau_{2h}^{\mathbf{R}}\right|\right|$. Meyer in [14] and later Turaev in [21] gave explicit cocycles for these classes. These cocycles yield

$$\left|\left|\tau_{2h}^R\right|\right| \leq 2h$$

but one can do better in the real case: In [13] they construct a cocycle representative of $\tau_{2h}^{\mathbf{R}}$ in terms of Lagrangian subspaces of $\mathbf{R}^{2h}$ and the Maslov index that has norm bounded by h (see also Section 5 in [21]). Now note that Milnor has computed $||\tau_2^{\mathbf{R}}||$ to be 1 by giving examples of flat $\mathrm{Sp}_2(\mathbf{R})$-bundles over a surface of genus g with Euler class $g-1$. Since the signature class is equal to four times the Euler class (again see [21]), these are examples where the signature is $4g-4$. Since the signature is additive with respect to direct sums of flat $\mathrm{Sp}_2(\mathbf{R})$-bundles, we obtain that $||\tau_{2h}^{\mathbf{R}}||$ is bounded from below by h and therefore equals h.

Note that by [1] the symplectic groups satisfy homological stability. Additionally by [3] together with Corollary 1.4 in [17] or the results in [12] their bounded cohomology stabilizes as well. Hence we have diagrams:

$$\begin{tikzcd} H_{\mathrm{b}}^2(\mathrm{Sp}_{2h}(\mathbf{R});\mathbf{R}) \arrow[r] \arrow[d] & H_{\mathrm{b}}^2(\mathrm{Sp}_{2h+2k}(\mathbf{R});\mathbf{R}) \arrow[d] \\ H^2(\mathrm{Sp}_{2h}(\mathbf{R});\mathbf{R}) \arrow[r] & H^2(\mathrm{Sp}_{2h+2k}(\mathbf{R});\mathbf{R}) \end{tikzcd}$$

where both horizontal arrows are isomorphisms. In particular if the upper isomorphism would be an isometry, then the lower map would be an isometry as well since the norm of an ordinary cohomology class is the infimum over the norm of all its bounded representatives. Since we have established that the signature classes do not stabilize isometrically, we conclude that neither horizontal map can be an isometry. More precisely while the inclusion

$$\mathrm{Sp}_{2h}(\mathbf{R}) \to \mathrm{Sp}_{2h+2k}(\mathbf{R})$$

induces isomorphisms on second bounded cohomology for h big enough, these fail to be isometries.

3.1. Relation to the mapping class group

Note that the action of the mapping class group on the first cohomology of the corresponding surface yields a surjection

$$\Phi\colon \mathcal{M}(\Sigma_h) \to \mathrm{Sp}_{2h}(\mathbf{Z})$$

and given a surface bundle over a surface $\Sigma_h \to E \to \Sigma_g$, the signature of E is exactly given by $(\Phi \circ \psi_E)^*(\tau_{2h}^{\mathbf{Z}})([\Sigma_g])$, where ψ_E denotes a classifying map of E. Using Hirzebruchs signature theorem one can see quite easily that this cohomology class equals a third of the first κ-class in the cohomology of the mapping class group. This implies that

$$\left|\left|\frac{1}{3}\kappa_1\right|\right| = ||\Phi^*(\tau_{2h}^{\mathbf{Z}})|| \leq ||\tau_{2h}^{\mathbf{Z}}||$$

The morphism Φ also induces an isomorphism in second group cohomology and by Theorem 2.14 in [9] Φ induces an injective isometry in bounded cohomology, but by the observations in Section 2 fails to be an isomorphism. Additionally results by Hamenstädt in [7] and before that by Kotschik [11] show that the norm of $\Phi^*(\tau_{2h}^{\mathbf{Z}})$ is bounded by $\frac{1}{3}(h-1)$ and $\frac{1}{2}(h-1)$ respectively.

On the other hand, in the paper [5], they construct for any $h \geq 3$ a surface bundle E with fiber genus h over a surface of genus 9 with signature $4\frac{h-2}{3}$. Together with the previous inequality we obtain

$$4\frac{h-2}{3} = (\Phi \circ \psi_E)^*(\tau_{2h}^{\mathbf{Z}})([\Sigma_9]) \leq ||\Phi^*(\tau_{2h}^{\mathbf{Z}})||\,||\Sigma_9|| \leq 32\,||\tau_{2h}^{\mathbf{Z}}||$$

showing that the norm of the integral signature class tends to infinity with increasing h as well.

Hence the norm of $\tau_{2h}^{\mathbf{Z}}$ lies somewhere inbetween $\frac{h-2}{24}$ and h. Showing that the inclusions between integral symplectic groups while inducing isomorphisms on second bounded cohomology do not induce isometries as well.

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