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Martin Halla

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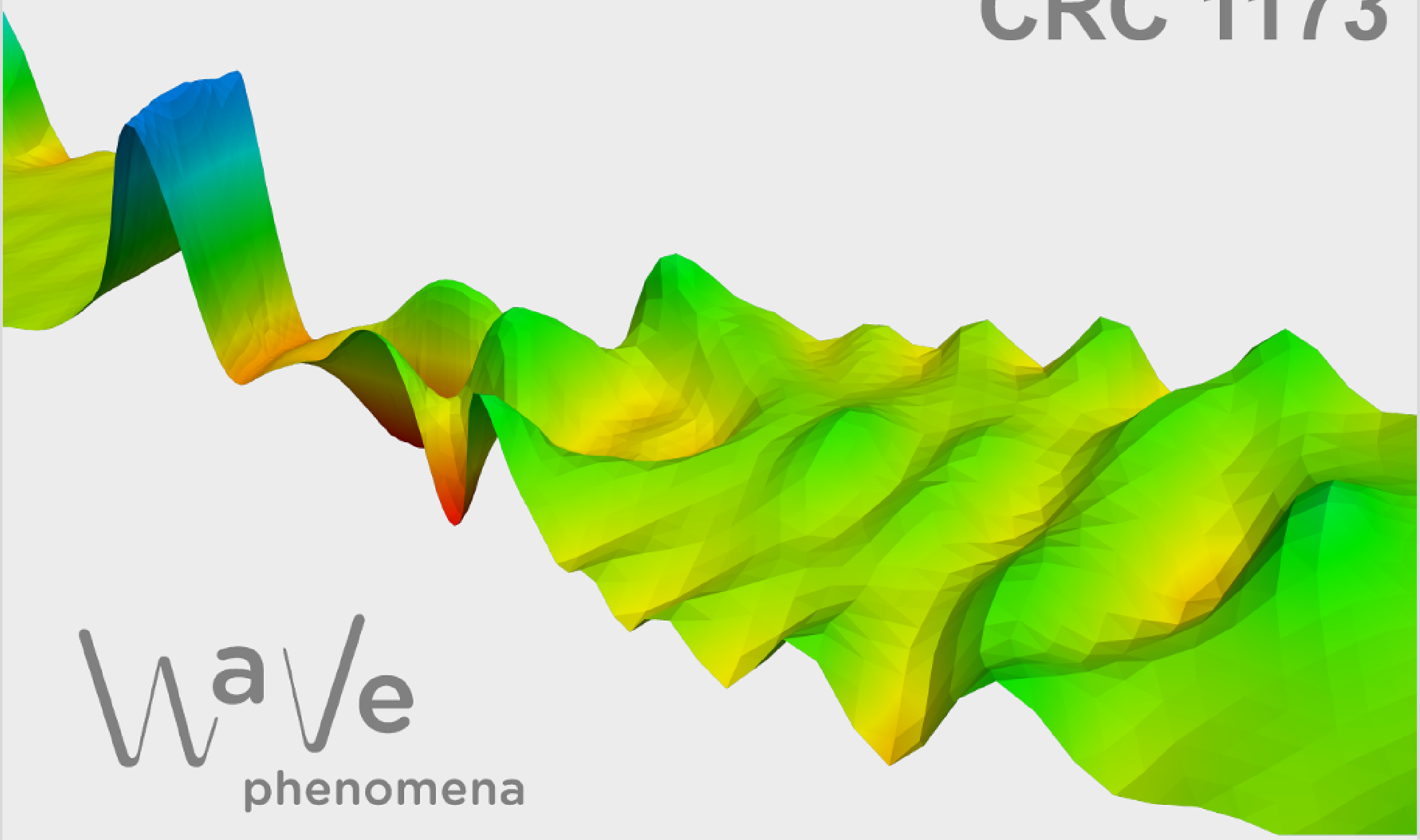

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Modal bases of coaxial electromagnetic step index fibers *

Martin Halla[†]

Abstract

We consider the eigenvalue problem to find the modes of an electromagnetic coaxial step index fiber. More specific, we consider a closed (meaning PEC boundary conditions) cylindrical waveguide with circular cross section Γ, wave propagation modeled by the time-harmonic Maxwell's equations with frequency ω, the permeability μ and the permittivity ϵ being scalar, uniformly positive, piece-wise constant and depending only on the radial variable of the cross section. We prove that if the deviation from the homogeneous case is small, i.e., $\delta_{\epsilon,\mu} := \|\epsilon - \epsilon_0\|_{L^\infty} + \|\mu - \mu_0\|_{L^\infty} \ll 1$, then the tangential electric (magnetic) fields of the modes form a Riesz basis in $\mathbf{H}_0(\mathrm{curl}_\Gamma;\Gamma)$ ($\mathbf{H}(\mathrm{curl}_\Gamma;\Gamma)$). For a constant permeability (permittivity) the Riesz basis property for the tangential electric (magnetic) fields holds also in the natural trace space $\mathbf{H}_0^{-1/2}(\mathrm{curl}_\Gamma;\Gamma)$ ($\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma;\Gamma)$). These results hold also for complex frequencies ω. In addition, if $\omega \in \mathbb{R}$, then for small enough $\delta_{\epsilon,\mu}$ all wavenumbers are located on the axes and there exist no backward modes. Key tools in the analysis are a particular reformulation of the eigenvalue problem, the perturbation theory for selfadjoint operators under a local subordinate condition and uniform properties of Bessel functions.

MSC: 35P05, 35R05, 78M35

Keywords: waveguide, step index fiber, coaxial cable, Maxwell's equations, eigenvalue perturbation analysis, Riesz basis

1 Introduction

Electromagnetic waveguides play an important role, e.g., for telecommunication [25], energy transfer [11] and as optical fiber amplifiers to produce highly coherent laser outputs [18]. A modal analysis and basis properties of the modes are important, e.g., for the construction of Dirichlet-to-Neumann/Calderon operators, the analysis of stability constants, and the justification of numerical transparent boundary conditions (in particular perfectly matched layers). In this article we are interested in the modes of a closed (meaning PEC boundary conditions) cylindrical waveguide with circular cross section, wave propagation modeled by Maxwell's equations, the permeability μ and the permittivity ϵ being scalar, uniformly positive, piece-wise constant and depending only on the radial variable of the cross section, i.e., a step index fiber. In applications the deviation from a homogeneous index profile is usually small (see, e.g., [40, Table 3]). A particular motivation for this work is the study of *long* optical fibers and we refer to the introduction of [37] for references regarding the state of the art of simulating nonlinear optical fiber amplifiers. The relevance of this topic is evident by the large number of books on *optical fibers*, e.g., [35, 45] to name only a few. We also mention studies on twisted and open waveguides [34, 6, 28, 27] and recent works on nonlinear fibers [23, 22, 37, 17], [46, 45, 1].

For homogeneous materials it is well known that the modal problem reduces to the study of scalar Helmholtz equations, see, e.g., [31]. However, as soon as the material is heterogeneous the

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†Karlsruhe Institute of Technology, Institute for Applied and Numerical Mathematicsa, Englerstraße 2, 76131 Karlsruhe, Germany (martin.halla@kit.edu)

situation changes and the mathematical effort to analyze the modes dramatically increases. Indeed, in this case two previously decoupled equations are now coupled and the selfadjoint nature (in the convenient sense) is violated. The modal eigenvalue problem (EVP) can still be considered to be a selfadjoint, linear EVP in a Krein space or to be a quadratic EVP with selfadjoint operator coefficients in a Hilbert space, but these considerations do not yield much benefit. Thus even the question to establish the completeness of modes [45, 44, 12, 21] poses severe mathematical challenges. Nevertheless, the main result of this article is a much stronger property than completeness, namely a Riesz basis property of the modes. Although our analysis requires significant effort and assumes certain non-trivial assumptions on the geometry of the waveguide (circular step index fiber), let us mention that we do not even touch more sophisticated setups such as bent [14, 15, 40], twisted [20, 34] or open [6, 28, 27] waveguides.

We are going to analyze the modal eigenvalue problem as a non-selfadjoint perturbation of a selfadjoint operator. Spectral perturbation theory for non-selfadjoint operators is a rather challenging topic and we refer to the books [19, 36, 26]. The convenient approach to this end is to assume that the perturbation satisfies a subordination condition and we refer to the introduction of [38] for further references in this regard. However, recently [39] reported an important generalization of this notion, which exchanges the subordination condition with a *local* version, and [15] gave some further extensions in this framework. The latter two will be a main tool for our analysis. In particular, the theory of [39] assumes certain properties of the unperturbed selfadjoint operator. In our case the selfadjoint operator is itself a perturbation of the operator for homogeneous materials. Since this (selfadjoint) perturbation is not subordinate in any suitable sense, we are going to establish those necessary properties "by hand" by analyzing Bessel functions. Here a crucial point is to derive all estimates in a uniform way.

The remainder of this article is structured as follows. In Section 2 we introduce our setting, the modal eigenvalue problem (3) to be studied, and derive a suitable reformulation (11) of the former. In Section 3 we study certain uniform properties of Bessel functions. In Section 4 we apply those properties to establish the necessary requirements for the selfadjoint perturbation. Finally, in Section 5 we apply [39, 15] and establish our main result in Theorem 5.5. In Appendix A we give a result on the location of the wavenumbers under general geometry assumptions.

2 The modal eigenvalue problem

2.1 Setting and assumptions

We consider a cylindrical waveguide $\Omega := \Gamma \times \mathbb{R}^+$ with bounded Lipschitz cross section $\Gamma \subset \mathbb{R}^2$, and identify Γ with $\Gamma \times \{0\}$. Let $\mathbf{n}$ and $\hat{\mathbf{n}}$ be the outward unit vectors of Ω and Γ respectively, which are related by $\mathbf{n} = (\hat{\mathbf{n}}, 0)$ on $\partial\Gamma \times \mathbb{R}^+$. We denote the convenient differential operators in 3D as follows

$$\nabla_{3\mathrm{D}} u := \begin{pmatrix} \partial_x u \\ \partial_y u \\ \partial_z u \end{pmatrix}, \quad \mathrm{curl}_{3\mathrm{D}} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} := \nabla_{3\mathrm{D}} \times \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \mathrm{div}_{3\mathrm{D}} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} := \nabla_{3\mathrm{D}} \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

On Γ we introduce the rotation operator

$$R := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and the surface differential operators

$$\begin{aligned} \mathrm{curl}_\Gamma (u_1, u_2)^\top &:= \partial_x u_2 - \partial_y u_1, \qquad & \mathrm{Curl}_\Gamma\, u &:= (\partial_y u, -\partial_x u)^\top, \\ \mathrm{div}_\Gamma (u_1, u_2)^\top &:= \partial_x u_1 + \partial_y u_2, \qquad & \nabla_\Gamma u &:= (\partial_x u, \partial_y u)^\top, \end{aligned}$$

for which we note that

$$R^2 = -I, \quad \nabla_\Gamma u = R\,\mathrm{Curl}_\Gamma\, u, \quad R\nabla_\Gamma u = -\,\mathrm{Curl}_\Gamma\, u, \quad \mathrm{curl}_\Gamma(R\mathbf{u}) = \mathrm{div}_\Gamma\,\mathbf{u}, \quad \mathrm{div}_\Gamma(R\mathbf{u}) = -\,\mathrm{curl}_\Gamma\,\mathbf{u}$$

for $\mathbf{u} = (u_1, u_2)^\top$. In addition, let $\hat{\mathbf{t}} := -R\hat{\mathbf{n}}$ be the unit tangential vector on $\partial\Gamma$. Let $B_r(x_0) := \{x\colon |x - x_0| < r\}$ be the ball with radius $r > 0$ and center x_0, where the default space (e.g., $\mathbb{C}$, $\mathbb{R}^2$, some Hilbert space X) for x, x_0 will always be clear from the context. Further let $B_r := B_r(0)$. The magnetic permeability μ and the electric permeability ϵ of the material in the waveguide are assumed to be real valued, bounded and uniformly positive scalar functions which depend only on x and y, i.e., $\mu, \epsilon, \mu^{-1}, \epsilon^{-1} \in L^\infty(\Gamma; \mathbb{R}^+)$. Let $\epsilon_{\max} := \sup_{\hat{\mathbf{x}}\in\Gamma} \epsilon(\hat{\mathbf{x}})$, $\epsilon_{\min} := \inf_{\hat{\mathbf{x}}\in\Gamma} \epsilon(\hat{\mathbf{x}})$, $\mu_{\max} := \sup_{\hat{\mathbf{x}}\in\Gamma} \mu(\hat{\mathbf{x}})$, $\mu_{\min} := \inf_{\hat{\mathbf{x}}\in\Gamma} \mu(\hat{\mathbf{x}})$. Let (r, θ) be polar coordinates for $\hat{\mathbf{x}} \in \mathbb{R}^2$ and (r, θ, z) be cylindrical coordinates for $\mathbf{x} \in \mathbb{R}^3$. We assume that ϵ and μ depend only on r and use the overloaded notation $\epsilon(\mathbf{x}) = \epsilon(\hat{\mathbf{x}}) = \epsilon(r)$ with $\mathbf{x} \in \mathbb{R}^3, \hat{\mathbf{x}} \in \mathbb{R}^2, r > 0$, etc.. Let $\Gamma := \{(x, y) \in \mathbb{R}^2\colon x^2 + y^2 < 1\}$, $N_\Gamma \in \mathbb{N}$, $0 =: r_1 < \cdots < r_{N_\Gamma+1} := 1$, $\epsilon_0, \epsilon_1, \ldots, \epsilon_{N_\Gamma+1} > 0$, $\mu_0, \mu_1, \ldots, \mu_{N_\Gamma+1} > 0$, $\epsilon|_{(r_l, r_{l+1})} = \epsilon_l$, $\mu|_{(r_l, r_{l+1})} = \mu_l$, $l = 1, \ldots, N_\Gamma$, where ϵ_0 and μ_0 take the role of reference values to measure the deviation of ϵ and μ from homogeneous materials. To quantify the deviation from the homogeneous material we abbreviate

$$\delta_{\epsilon,\mu} := \|\epsilon_0 - \epsilon\|_{L^\infty(\Gamma)} + \|\mu_0 - \mu\|_{L^\infty(\Gamma)} \quad \text{and} \quad \tilde{\delta}_{\epsilon,\mu} := \|\epsilon_0\mu_0 - \epsilon\mu\|_{L^\infty(\Gamma)}.$$

To simplify constants appearing in the forthcoming analysis we assume further that

$$\frac{\epsilon_0}{2} \leq \epsilon \leq 2\epsilon_0, \qquad \frac{\mu_0}{2} \leq \mu \leq 2\mu_0.$$

We introduce the annuli $\Gamma_l := B_{r_{l+1}} \setminus \overline{B_{r_l}}$, $l = 1, \ldots, N_\Gamma$ and the interfaces $\Sigma_l := \partial B_{r_{l+1}}$, $l = 1, \ldots, N_\Gamma - 1$, $\Sigma := \bigcup_{l=1}^{N_\Gamma-1} \Sigma_l$ with the convention that the unit normal vector $\hat{\mathbf{n}}$ on Σ_l points to infinity. For a piece-wise smooth function $u\colon \Gamma \to \mathbb{C}$ let $[\![u]\!] := u|_{\Gamma_{l+1}} - u|_{\Gamma_l}$ on Σ_l. The temporal frequency will be denoted as $\omega \in \mathbb{C} \setminus \{0\}$, the wavenumber in longitudinal direction $(0, 0, 1)$ as $\beta \in \mathbb{C}$, $\alpha := -i\beta$, $\nu := \sqrt{\alpha^2 + \omega^2\epsilon_0\mu_0}$, and the circumferential index as $m \in \mathbb{Z}$.

We consider ω, N_Γ, $(r_l)_{l=1,\ldots,N_\Gamma+1}$ and ϵ_0, μ_0 to be fixed, i.e., constants may depend on these parameters. On the other hand all appearing constants will be independent of ν (and α, β), $(\epsilon_l, \mu_l)_{l=1,\ldots,N_\Gamma}$, m and exemptions will be indicated by indices!

Definition 2.1. *The frequency ω is called a cut-off frequency, if one of the following two equations*

$$\begin{aligned} -\,\mathrm{div}_\Gamma(\mu^{-1}\nabla_\Gamma u) - \omega^2\epsilon u &= 0 \quad \text{in } \Gamma, \qquad u = 0 \quad \text{on } \partial\Gamma, \\ -\,\mathrm{div}_\Gamma(\epsilon^{-1}\nabla_\Gamma u) - \omega^2\mu u &= 0 \quad \text{in } \Gamma, \quad \partial_{\hat{\mathbf{n}}} u = 0 \quad \text{on } \partial\Gamma, \end{aligned}$$

admits a non-trivial solution.

Throughout the article we assume that ω is not a cut-off frequency for the homogeneous case $\epsilon = \epsilon_0$, $\mu = \mu_0$.

The complex valued scalar products and norms of $L^2(\Gamma)$ and $\mathbf{L}^2(\Gamma) := (L^2(\Gamma))^2$ are simultaneously denoted as $\langle\cdot,\cdot\rangle_\Gamma$ and $\|\cdot\|_\Gamma$ respectively; and in the same fashion $\langle\cdot,\cdot\rangle_\Sigma$ and $\|\cdot\|_\Sigma$ refer to the scalar products and norms of $L^2(\Sigma)$ and $\mathbf{L}^2(\Sigma) := (L^2(\Sigma))^2$ respectively. We use a convenient notation for Sobolev spaces, e.g., $\mathbf{H}(\mathrm{curl}_\Gamma; \Gamma)$, $\mathbf{H}_0(\mathrm{curl}_\Gamma; \Gamma)$, $H^1(\Gamma)$, $H^1_0(\Gamma)$ and $H^1_{\sigma*}(\Gamma) := \{u \in H^1(\Gamma)\colon \langle\sigma u, 1\rangle_\Gamma = 0\}$ for $\sigma \in L^\infty(\Gamma)$. All function spaces will be considered over the scalar body $\mathbb{C}$. For generic Hilbert spaces X, Y let $\mathcal{L}(X, Y)$ be the vector spaces of bounded, linear operators from X to Y, and set $\mathcal{L}(X) := \mathcal{L}(X, X)$. The product $X \times Y$ of two Hilbert spaces X, Y is equipped with the convenient scalar product $\langle(u, v), (u^\dagger, v^\dagger)\rangle_{X\times Y} := \langle u, u^\dagger\rangle_X + \langle v, v^\dagger\rangle_Y$.

2.2 The modal eigenvalue problem

Let $\mathcal{E}, \mathcal{H}$ denote the three-dimensional time-harmonic electric and magnetic fields, solving the Maxwells equation

$$\epsilon\partial_t\mathcal{E} = \mathrm{curl}_{3\mathrm{D}}\,\mathcal{H} \quad \text{in } \Omega, \tag{1a}$$

$$\mu\partial_t\mathcal{H} = -\,\mathrm{curl}_{3\mathrm{D}}\,\mathcal{E} \quad \text{in } \Omega, \tag{1b}$$

together with the boundary conditions on $\partial\Gamma \times \mathbb{R}^+$

$$\mathbf{n} \times \mathcal{E} \quad = 0 \quad \text{on } \partial\Gamma \times \mathbb{R}^+, \tag{1c}$$

$$\mathbf{n} \times \operatorname{curl}_{3\mathrm{D}} \mathcal{H} \quad = 0 \quad \text{on } \partial\Gamma \times \mathbb{R}^+. \tag{1d}$$

Let $\hat{\mathbf{x}} := (x, y)$ and $\mathbf{x} := (x, y, z)$. For a given angular frequency $\omega \in \mathbb{C}$ we seek solutions of the form

$$\mathcal{E}(t, \mathbf{x}) = \mathbb{E}(\hat{\mathbf{x}}) \exp(i\beta z) \exp(-i\omega t), \qquad \mathcal{H}(t, \mathbf{x}) = \mathbb{H}(\hat{\mathbf{x}}) \exp(i\beta z) \exp(-i\omega t),$$

where $\beta \in \mathbb{C}$ is the unknown wavenumber in longitudinal direction. Henceforth we will work mainly with the rotated wavenumber

$$\alpha := -i\beta$$

instead, because it will be more convenient for our analysis. Note that if $\mathcal{G}(\mathbf{x}) = \mathbb{G}(\hat{\mathbf{x}}) \exp(-\alpha z)$ with $\mathbb{G} = (G_1, G_2, G_3)^\top$ and $\mathbf{G}_T := (G_1, G_2)^\top$, then

$$\operatorname{curl}_{3\mathrm{D}} \mathcal{G} = \begin{pmatrix} \partial_y G_3 + \alpha G_2 \\ -\partial_x G_3 - \alpha G_1 \\ \partial_x G_2 - \partial_y G_1 \end{pmatrix} \exp(-\alpha z) = \begin{pmatrix} \operatorname{Curl}_\Gamma G_3 - \alpha R \mathbf{G}_T \\ \operatorname{curl}_\Gamma \mathbf{G}_T \end{pmatrix} \exp(-\alpha z).$$

So with $\mathbf{E}_T := (E_1, E_2)^\top$, $\mathbf{H}_T := (H_1, H_2)^\top$ (1) is equivalent to

$$-i\omega\epsilon \mathbf{E}_T = \operatorname{Curl}_\Gamma H_3 - \alpha R \mathbf{H}_T \quad \text{in } \Gamma, \tag{2a}$$

$$-i\omega\epsilon E_3 = \operatorname{curl}_\Gamma \mathbf{H}_T \quad \text{in } \Gamma, \tag{2b}$$

$$-i\omega\mu \mathbf{H}_T = -\operatorname{Curl}_\Gamma E_3 + \alpha R \mathbf{E}_T \quad \text{in } \Gamma, \tag{2c}$$

$$-i\omega\mu H_3 = -\operatorname{curl}_\Gamma \mathbf{E}_T \quad \text{in } \Gamma, \tag{2d}$$

$$\hat{\mathbf{t}} E_3 = 0 \quad \text{on } \partial\Gamma, \tag{2e}$$

$$\hat{\mathbf{t}} \cdot \mathbf{E}_T = 0 \quad \text{on } \partial\Gamma, \tag{2f}$$

$$\partial_{\hat{\mathbf{n}}} H_3 = 0 \quad \text{on } \partial\Gamma, \tag{2g}$$

$$\operatorname{curl}_\Gamma \mathbf{H}_T = 0 \quad \text{on } \partial\Gamma. \tag{2h}$$

Altogether we are interested in the eigenvalue problem to

$$\text{find } (\alpha, (\mathbf{E}_T, E_3, \mathbf{H}_T, H_3)) \in \mathbb{C} \times (\mathbf{L}^2(\Gamma) \times L^2(\Gamma) \times \mathbf{L}^2(\Gamma) \times L^2(\Gamma)) \setminus \{0\} \text{ which solves (2).} \tag{3}$$

2.3 The tangential electric field formulation

Our intention is to study the modal eigenvalue problem (3) as a perturbation of an eigenvalue problem for a compact selfadjoint operator. To this end we first need to reformulate (2) into a suitable form. The first step is to reduce the number of unknowns. Among the many possibilities we choose to eliminate $\mathbf{H}_T$ and H_3, see, e.g. [21]. We plug (2d), (2c) into (2a); and apply $\operatorname{curl}_\Gamma \mu^{-1}$ to (2c) and plug in (2b). Thus we obtain

$$\operatorname{Curl}_\Gamma \mu^{-1} \operatorname{curl}_\Gamma \mathbf{E}_T - \omega^2 \epsilon \mathbf{E}_T - \alpha^2 \mu^{-1} \mathbf{E}_T - \alpha \mu^{-1} \nabla_\Gamma E_3 = 0 \quad \text{in } \Gamma, \tag{4a}$$

$$-\operatorname{div}_\Gamma \mu^{-1} \nabla_\Gamma E_3 - \omega^2 \epsilon E_3 - \alpha \operatorname{div}_\Gamma(\mu^{-1} \mathbf{E}_T) = 0 \quad \text{in } \Gamma, \tag{4b}$$

$$\hat{\mathbf{t}} \cdot \mathbf{E}_T = 0 \quad \text{on } \partial\Gamma, \tag{4c}$$

$$E_3 = 0 \quad \text{on } \partial\Gamma. \tag{4d}$$

We formulate (4) in weak form:

$$\begin{aligned} &(\mathbf{E}_T, E_3) \in \mathbf{H}_0(\operatorname{curl}_\Gamma; \Gamma) \times H_0^1(\Gamma) \quad \text{solves} \\ &\qquad \langle \mu^{-1} \operatorname{curl}_\Gamma \mathbf{E}_T, \operatorname{curl}_\Gamma \mathbf{E}_T^\dagger \rangle_\Gamma - \omega^2 \langle \epsilon \mathbf{E}_T, \mathbf{E}_T^\dagger \rangle_\Gamma - \alpha^2 \langle \mu^{-1} \mathbf{E}_T, \mathbf{E}_T^\dagger \rangle_\Gamma \\ &\qquad + \langle \mu^{-1} \nabla_\Gamma E_3, \nabla_\Gamma E_3^\dagger \rangle_\Gamma - \omega^2 \langle \epsilon E_3, E_3^\dagger \rangle_\Gamma + \alpha \langle \mu^{-1} \mathbf{E}_T, \nabla_\Gamma E_3^\dagger \rangle_\Gamma - \alpha \langle \mu^{-1} \nabla_\Gamma E_3, \mathbf{E}_T^\dagger \rangle_\Gamma = 0 \\ &\text{for all } (\mathbf{E}_T^\dagger, E_3^\dagger) \in \mathbf{H}_0(\operatorname{curl}_\Gamma; \Gamma) \times H_0^1(\Gamma). \end{aligned} \tag{5}$$

Since the space $\mathbf{H}_0(\mathrm{curl}_\Gamma;\Gamma)$ does not admit a compact embedding into $\mathbf{L}^2(\Gamma)$ the current setting of function spaces is not suitable yet for our intended analysis. Hence we follow [21] and consider the topological decomposition (which is orthogonal in an equivalent weighted scalar product)

$$\begin{aligned}\mathbf{H}_0(\mathrm{curl}_\Gamma;\Gamma) &= \mathbf{V} \oplus^{\mathcal{T}} \nabla_\Gamma H_0^1(\Gamma),\\ \mathbf{V} &:= \mathbf{H}(\mathrm{curl}_\Gamma, \mathrm{div}_\Gamma\,\mu^{-1}0;\Omega) := \{\mathbf{E}_T \in \mathbf{H}_0(\mathrm{curl}_\Gamma;\Gamma)\colon\ \mathrm{div}_\Gamma(\mu^{-1}\mathbf{E}_T) = 0 \text{ in } \Gamma\}.\end{aligned}$$

With $\mathbf{E}_T = \mathbf{v} + \nabla_\Gamma \tilde{w}$, $\mathbf{E}_T^\dagger = \mathbf{v}^\dagger + \nabla_\Gamma \tilde{w}^\dagger$, $\mathbf{v}, \mathbf{v}^\dagger \in \mathbf{V}, \tilde{w}, \tilde{w}^\dagger \in H_0^1(\Gamma)$ (5) reads

$$\begin{aligned}&(\mathbf{v}, \tilde{w}, E_3) \in \mathbf{V} \times H_0^1(\Gamma) \times H_0^1(\Gamma) \quad \text{solves}\\ &\qquad \langle \mu^{-1}\,\mathrm{curl}_\Gamma\,\mathbf{v}, \mathrm{curl}_\Gamma\,\mathbf{v}^\dagger\rangle_\Gamma - \omega^2\langle\epsilon(\mathbf{v}+\nabla_\Gamma\tilde{w}), (\mathbf{v}^\dagger + \nabla_\Gamma\tilde{w}^\dagger)\rangle_\Gamma\\ &\qquad - \alpha^2\langle\mu^{-1}\mathbf{v}, \mathbf{v}^\dagger\rangle_\Gamma - \alpha^2\langle\mu^{-1}\nabla_\Gamma\tilde{w}, \nabla_\Gamma\tilde{w}^\dagger\rangle_\Gamma\\ &\qquad + \langle\mu^{-1}\nabla_\Gamma E_3, \nabla_\Gamma E_3^\dagger\rangle_\Gamma - \omega^2\langle\epsilon E_3, E_3^\dagger\rangle_\Gamma + \alpha\langle\mu^{-1}\nabla_\Gamma\tilde{w}, \nabla_\Gamma E_3^\dagger\rangle_\Gamma - \alpha\langle\mu^{-1}\nabla_\Gamma E_3, \nabla_\Gamma\tilde{w}^\dagger\rangle_\Gamma = 0\\ &\text{for all } (\mathbf{v}^\dagger, \tilde{w}^\dagger, E_3^\dagger) \in \mathbf{V} \times H_0^1(\Gamma) \times H_0^1(\Gamma).\end{aligned} \tag{6}$$

Note that we will later change to the rescaled variables $w = \omega\tilde{w}, w^\dagger = \omega\tilde{w}^\dagger$. Eq. (6) is quadratic in the eigenvalue parameter α. Hence the next step is to regain a problem, which is affine in α^2. To this end let

$$\begin{aligned}\mathbf{D} &\in \mathcal{L}(\mathbf{V}, L^2(\Gamma)), & \mathbf{D}\mathbf{E}_T &:= \mathrm{curl}_\Gamma\,\mathbf{E}_T,\\ \mathbf{M}_\sigma &\in \mathcal{L}(\mathbf{L}^2(\Gamma)), & \mathbf{M}_\sigma\mathbf{E}_T &:= \sigma\mathbf{E}_T, \quad \text{for } \sigma \in L^\infty(\Gamma),\\ \mathbf{E}_{L^2} &\in \mathcal{L}(\mathbf{V}, \mathbf{L}^2(\Gamma)), & \mathbf{E}_{L^2}\mathbf{E}_T &:= \mathbf{E}_T,\\ D &\in \mathcal{L}(H_0^1(\Gamma), \mathbf{L}^2(\Gamma)), & DE_3 &:= \nabla_\Gamma E_3,\\ E_{L^2} &\in \mathcal{L}(H_0^1(\Gamma), L^2(\Gamma)), & E_{L^2}E_3 &:= E_3,\\ M_\sigma &\in \mathcal{L}(L^2(\Gamma)), & M_\sigma E_3 &:= \sigma E_3, \quad \text{for } \sigma \in L^\infty(\Gamma).\end{aligned}$$

Then (6) reads in operator form:

$$\begin{aligned}\begin{pmatrix}\mathbf{0}\\0\\0\end{pmatrix} = \Bigg[&\begin{pmatrix} D^*M_{\mu^{-1}}D - \omega^2\mathbf{E}_{L^2}^*\mathbf{M}_\epsilon\mathbf{E}_{L^2} & -\omega^2\mathbf{E}_{L^2}^*\mathbf{M}_\epsilon\mathbf{D} & 0\\ -\omega^2\mathbf{D}^*\mathbf{M}_\epsilon\mathbf{E}_{L^2} & -\omega^2\mathbf{D}^*\mathbf{M}_\epsilon\mathbf{D} & 0\\ 0 & 0 & \mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D} - \omega^2E_{L^2}^*M_\epsilon E_{L^2}\end{pmatrix}\\ &+\alpha\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -\alpha\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D}\\ 0 & \alpha\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D} & 0\end{pmatrix} - \alpha^2\begin{pmatrix}\alpha^2\mathbf{E}_{L^2}^*\mathbf{M}_{\mu^{-1}}\mathbf{E}_{L^2} & 0 & 0\\ & \mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D} & 0\\ 0 & 0 & 0\end{pmatrix}\Bigg]\begin{pmatrix}\mathbf{v}\\ \tilde{w}\\ E_3\end{pmatrix}.\end{aligned} \tag{7}$$

Lemma 2.2. *Let the assumptions formulated in Section 2.1 be satisfied. Then there exists $\delta_0 > 0$ such that ω is not a cut-off frequency (see Definition 2.1) for each ϵ, μ that satisfy $\delta_{\epsilon,\mu} < \delta_0$.*

Proof. Follows from basic perturbation theory. □

Justified by Lemma 2.2 we assume henceforth that ω is not a cut-off frequency. Thus we can build in (7) the Schur complement with respect to E_3 and (7) becomes

$$\begin{aligned}&\begin{pmatrix} D^*M_{\mu^{-1}}D - \omega^2\mathbf{E}_{L^2}^*\mathbf{M}_\epsilon\mathbf{E}_{L^2} & -\omega^2\mathbf{E}_{L^2}^*\mathbf{M}_\epsilon\mathbf{D}\\ -\omega^2\mathbf{D}^*\mathbf{M}_\epsilon\mathbf{E}_{L^2} & -\omega^2\mathbf{D}^*\mathbf{M}_\epsilon\mathbf{D}\end{pmatrix}\begin{pmatrix}\mathbf{v}\\ \tilde{w}\end{pmatrix} =\\ &\alpha^2\begin{pmatrix}\mathbf{E}_{L^2}^*\mathbf{M}_{\mu^{-1}}\mathbf{E}_{L^2} & \\ & \mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D} - \mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D}(\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D} - \omega^2E_{L^2}^*M_\epsilon E_{L^2})^{-1}\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D}\end{pmatrix}\begin{pmatrix}\mathbf{v}\\ \tilde{w}\end{pmatrix}.\end{aligned} \tag{8}$$

Let us set

$$X := \mathbf{V} \times H_0^1(\Gamma),$$

which will be our working Hilbert space. The convenient form of an eigenvalue problem admits L^2-terms as operator coefficients of α^2. While the first diagonal entry in the right hand-side of (8) already has a suitable form, the actual properties of the second diagonal entry are not obvious yet and shall be worked out next. We abbreviate $A_{\mu^{-1}} := \mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D}$ and $K_\epsilon = \omega^2 E_{L^2}^* M_\epsilon E_{L^2}$. Applying the formula $A - A(A-K)^{-1}A = A[I-(A-K)^{-1}A] = -A(A-K)^{-1}K$ to $A = A_{\mu^{-1}}$ and $K = K_\epsilon$ the second diagonal entry in the right hand-side of (8) simplifies to

$$-A_{\mu^{-1}}(A_{\mu^{-1}} - K_\epsilon)^{-1}K_\epsilon = -\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D}(\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D} - \omega^2 E_{L^2}^* M_\epsilon E_{L^2})^{-1}\omega^2 E_{L^2}^* M_\epsilon E_{L^2}.$$

Note that also $A - A(A-K)^{-1}A = -K - K(A-K)^{-1}K$, which is the expression used in [21]. However, at this point we deviate from [21] and multiply (8) with

$$\operatorname{diag}\Big(I, -(A_{\mu^{-1}}(A_{\mu^{-1}} - K_\epsilon)^{-1})^{-1}\Big) = \operatorname{diag}\Big(I, -(A_{\mu^{-1}} - K_\epsilon)A_{\mu^{-1}}^{-1}\Big) = \operatorname{diag}\Big(I, -I + K_\epsilon A_{\mu^{-1}}^{-1}\Big)$$

to obtain

$$(\tilde{A} + \tilde{B})\begin{pmatrix}\mathbf{v}\\ \tilde{w}\end{pmatrix} = \alpha^2 \tilde{K}\begin{pmatrix}\mathbf{v}\\ \tilde{w}\end{pmatrix} \tag{9}$$

where

$$\tilde{A} := \begin{pmatrix} D^*M_{\mu^{-1}}D - \omega^2\epsilon_0\mu_0\mathbf{E}_{L^2}^*\mathbf{M}_{\mu^{-1}}\mathbf{E}_{L^2} & \\ & \omega^2\mathbf{D}^*\mathbf{M}_\epsilon\mathbf{D} - \omega^4\epsilon_0\mu_0 E_{L^2}^* M_\epsilon E_{L^2}\end{pmatrix}, \tag{10a}$$

$$\tilde{B} := \begin{pmatrix} -\omega^2\mathbf{E}_{L^2}^*(\mathbf{M}_\epsilon - \epsilon_0\mu_0\mathbf{M}_{\mu^{-1}})\mathbf{E}_{L^2} & -\omega^2\mathbf{E}_{L^2}^*\mathbf{M}_\epsilon\mathbf{D} \\ \omega^2\mathbf{D}^*\mathbf{M}_\epsilon\mathbf{E}_{L^2} - \omega^4 E_{L^2}^* M_\epsilon E_{L^2}(\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D})^{-1}\mathbf{D}^*\mathbf{M}_\epsilon\mathbf{E}_{L^2} & \tilde{B}_{22}\end{pmatrix}, \tag{10b}$$

$$\tilde{B}_{22} := -\omega^4 E_{L^2}^* M_\epsilon E_{L^2}(\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D})^{-1}\big(\mathbf{D}^*\mathbf{M}_\epsilon\mathbf{D} - \epsilon_0\mu_0\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D}\big), \tag{10c}$$

$$\tilde{K} := \begin{pmatrix}\mathbf{E}_{L^2}^*\mathbf{M}_{\mu^{-1}}\mathbf{E}_{L^2} & \\ & \omega^2 E_{L^2}^* M_\epsilon E_{L^2}\end{pmatrix}. \tag{10d}$$

Now we switch to $w := \omega\tilde{w}$, multiply (9) with $\operatorname{diag}(I, \omega^{-1}I)$ and perform an eigenvalue shift $\nu^2 := \alpha^2 + \omega^2\epsilon_0\mu_0$ to obtain the eigenvalue problem to

$$\text{find} \quad \left(\nu, \begin{pmatrix}\mathbf{v}\\ w\end{pmatrix}\right) \in \mathbb{C} \times X \setminus \{0\} \quad \text{such that} \quad (A_{\epsilon,\mu} + B_{\epsilon,\mu})\begin{pmatrix}\mathbf{v}\\ w\end{pmatrix} = \nu^2 K_{\epsilon,\mu}\begin{pmatrix}\mathbf{v}\\ w\end{pmatrix} \tag{11}$$

with

$$A_{\epsilon,\mu} := \begin{pmatrix} D^*M_{\mu^{-1}}D & \\ & \mathbf{D}^*\mathbf{M}_\epsilon\mathbf{D}\end{pmatrix}, \tag{12a}$$

$$B_{\epsilon,\mu} := \begin{pmatrix} -\omega^2\mathbf{E}_{L^2}^*(\mathbf{M}_\epsilon - \epsilon_0\mu_0\mathbf{M}_{\mu^{-1}})\mathbf{E}_{L^2} & -\omega\mathbf{E}_{L^2}^*\mathbf{M}_\epsilon\mathbf{D} \\ \omega\mathbf{D}^*\mathbf{M}_\epsilon\mathbf{E}_{L^2} - \omega^3 E_{L^2}^* M_\epsilon E_{L^2}(\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D})^{-1}\mathbf{D}^*\mathbf{M}_\epsilon\mathbf{E}_{L^2} & B_{22}\end{pmatrix}, \tag{12b}$$

$$B_{\epsilon,\mu,22} := -\omega^2 E_{L^2}^* M_\epsilon E_{L^2}(\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D})^{-1}\big(\mathbf{D}^*\mathbf{M}_\epsilon\mathbf{D} - \epsilon_0\mu_0\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D}\big), \tag{12c}$$

$$K_{\epsilon,\mu} := \begin{pmatrix}\mathbf{E}_{L^2}^*\mathbf{M}_{\mu^{-1}}\mathbf{E}_{L^2} & \\ & E_{L^2}^* M_\epsilon E_{L^2}\end{pmatrix}. \tag{12d}$$

Note that in the above notation we made the dependency of the operators on the coefficients ϵ, μ explicit by means of indices. In cases, where there is no ambiguity we abbreviate

$$A := A_{\epsilon,\mu}, \qquad K := K_{\epsilon,\mu}, \qquad B := B_{\epsilon,\mu}, \qquad B_{22} := B_{\epsilon,\mu,22}.$$

We collect our transformations in the following lemma.

Lemma 2.3. *If* $(\alpha, (\mathbf{E}_T, E_3, \mathbf{H}_T, H_3)) \in \mathbb{C} \times (\mathbf{L}^2(\Gamma) \times L^2(\Gamma) \times \mathbf{L}^2(\Gamma) \times L^2(\Gamma)) \setminus \{0\}$ *is a solution to* (3)*, then* $(\nu := \pm\sqrt{\alpha^2 + \omega^2\epsilon_0\mu_0}, (\mathbf{v}, w)^\top) \in \mathbb{C} \times X \setminus \{0\}$ *with* $\mathbf{v} + \nabla w = \mathbf{E}_T$ *is a solution to* (11)*. Vice-versa, if* $(\nu, (\mathbf{v}, w)^\top) \in \mathbb{C} \times X \setminus \{0\}$ *is a solution to* (11)*, then there exist two linear independent solutions* $(\alpha := \pm\sqrt{\nu^2 - \omega^2\epsilon_0\mu_0}, (\mathbf{E}_T, E_3^{(\pm)}, \mathbf{H}_T^{(\pm)}, H_3^{(\pm)})) \in \mathbb{C} \times (\mathbf{L}^2(\Gamma) \times L^2(\Gamma) \times \mathbf{L}^2(\Gamma) \times L^2(\Gamma)) \setminus \{0\}$ *to* (3) *with* $\mathbf{E}_T = \mathbf{v} + \nabla w$ *and* $(E_3^{(\pm)}, \mathbf{H}_T^{(\pm)}, H_3^{(\pm)})$ *being uniquely determined by* α *and* $\mathbf{E}_T$*.*

Proof. Follows with basic algebraic manipulations. □

The structure of (11) now invites us to analyze (11) as a perturbation of the selfadjoint eigenvalue problem

$$\text{find} \quad \left(\nu, \begin{pmatrix}\mathbf{v}\\ w\end{pmatrix}\right) \in \mathbb{C} \times X \setminus \{0\} \quad \text{such that} \quad A_{\epsilon,\mu}\begin{pmatrix}\mathbf{v}\\ w\end{pmatrix} = \nu^2 K_{\epsilon,\mu}\begin{pmatrix}\mathbf{v}\\ w\end{pmatrix}. \tag{13}$$

We are going to perform such an analysis in Section 5 and as a preparation we state the upcoming Lemma 2.4. However, this analysis requires certain strong properties of the eigenfunctions and eigenvalue gaps of the unperturbed problem (13). To this end we consider (13) as a perturbation of the homogeneous problem [(13) with $\epsilon = \epsilon_0$, $\mu = \mu_0$] and perform this analysis in Section 4. Since this perturbation is not small in any subordinate sense we will exploit certain properties of Bessel functions, which we are going to study in Section 3.

In preparation of Lemma 2.4 we introduce certain constants. Let $c_{\mathrm{Dir}}^2 > 0$ be the coercivity constant of the Dirichlet Laplacian:

$$\|\nabla_\Gamma w\|_{\mathbf{L}^2(\Gamma)} \geq c_{\mathrm{Dir}}\|w\|_{H^1(\Gamma)} \quad \text{for all } w \in H_0^1(\Gamma).$$

Let $C_{\mathrm{tr},1} > 0$ be a trace continuity constant in $H^1(\Gamma)$:

$$\|w\|_{L^2(\Sigma)} \leq C_{\mathrm{tr},1}\|w\|_{H^1(\Gamma)} \quad \text{for all } w \in H_0^1(\Gamma);$$

and $C_{\mathrm{tr},2} > 0$ be a trace continuity constant in a broken space:

$$\|\nabla_\Gamma w\|_{L^2(\Sigma)} \leq C_{\mathrm{tr},2}(\|w\|_{H^1(\Gamma)} + (\sum_{l=1}^{N_\Gamma} |v|_{H^2(\Gamma_l)})^{1/2})$$

for all $w \in \{w \in H^1(\Gamma) \colon w|_{\Gamma_l} \in H^2(\Gamma_l)\ \forall l = 1, \dots, N_\Gamma\}$.

Lemma 2.4. *It holds that*

$$\begin{aligned}\left|\left\langle B_{\epsilon,\mu}\begin{pmatrix}\mathbf{v}\\ w\end{pmatrix}, \begin{pmatrix}\mathbf{v}^\dagger\\ w^\dagger\end{pmatrix}\right\rangle_X\right| \leq C_B\tilde{\delta}_{\epsilon,\mu}\Big(&\|\mathbf{v}\|_\Gamma\|\mathbf{v}^\dagger\|_\Gamma + \|w\|_\Sigma\|\hat{\mathbf{n}}\cdot\mu^{-1}\mathbf{v}^\dagger\|_\Sigma\\ &+ \|\hat{\mathbf{n}}\cdot\mu^{-1}\mathbf{v}\|_\Sigma\|w^\dagger\|_\Sigma + \|\hat{\mathbf{n}}\cdot\mu^{-1}\mathbf{v}\|_\Sigma\|w^\dagger\|_\Gamma + \|w\|_\Gamma\|w^\dagger\|_\Gamma + \|w\|_\Sigma\|w^\dagger\|_\Gamma\Big).\end{aligned}$$

for all $\mathbf{v}, \mathbf{v}^\dagger \in \mathbf{V}$*,* $w, w^\dagger \in H_0^1(\Gamma)$ *with*

$$C_B := \max\left\{2|\omega|^2\mu_0^{-1}, 2|\omega|, 4|\omega|^3\epsilon_0\mu_0 c_{\mathrm{Dir}}^{-1}C_{\mathrm{tr},1}, 2|\omega|^2\epsilon_0, 2|\omega|^2 C_{\mathrm{tr},2}(2\epsilon_0\mu_0 c_{\mathrm{Dir}}^{-1} + 4\epsilon_0\mu_0)\right\}.$$

Proof. We apply the triangle inequality and estimate step by step each summand in the expansion of $\left\langle B_{\epsilon,\mu}\begin{pmatrix}\mathbf{v}\\ w\end{pmatrix}, \begin{pmatrix}\mathbf{v}^\dagger\\ w^\dagger\end{pmatrix}\right\rangle_X$. To start with

$$\begin{aligned}|\omega|^2|\langle\mathbf{E}_{L^2}^*(\mathbf{M}_\epsilon - \epsilon_0\mu_0\mathbf{M}_{\mu^{-1}})\mathbf{E}_{L^2}\mathbf{v}, \mathbf{v}^\dagger\rangle_{\mathbf{V}}| &= |\omega|^2|\langle\mu^{-1}(\epsilon\mu - \epsilon_0\mu_0)\mathbf{v}, \mathbf{v}^\dagger\rangle_{\mathbf{L}^2(\Gamma)}|\\ &\leq 2|\omega|^2\mu_0^{-1}\tilde{\delta}_{\epsilon,\mu}\|\mathbf{v}\|_\Gamma\|\mathbf{v}^\dagger\|_\Gamma.\end{aligned}$$

For the next term we exploit that $\operatorname{div}_\Gamma(\mu^{-1}\mathbf{v}) = 0$ for $\mathbf{v} \in \mathbf{V}$ and that ϵ, μ are piece-wise constant:

$$\begin{aligned}
|\omega\langle \mathbf{E}^*_{L^2}\mathbf{M}_\epsilon \mathbf{D}w, \mathbf{v}^\dagger\rangle_{\mathbf{V}}| &= |\omega\langle \epsilon\nabla_\Gamma w, \mathbf{v}^\dagger\rangle_{\mathbf{L}^2(\Gamma)}| \\
&= |\omega\langle \epsilon\mu\nabla_\Gamma w, \mu^{-1}\mathbf{v}^\dagger\rangle_\Gamma| \\
&= |\omega\langle (\epsilon\mu - \epsilon_0\mu_0)\nabla_\Gamma w, \mu^{-1}\mathbf{v}^\dagger\rangle_\Gamma| \\
&= |\omega\langle [\![\epsilon\mu - \epsilon_0\mu_0]\!] w, \hat{\mathbf{n}}\cdot\mu^{-1}\mathbf{v}^\dagger\rangle_\Sigma| \\
&= 2|\omega|\tilde{\delta}_{\epsilon,\mu}\|w\|_\Sigma\|\hat{\mathbf{n}}\cdot\mu^{-1}\mathbf{v}^\dagger\|_\Sigma.
\end{aligned}$$

Like-wise $|\omega\langle \mathbf{D}^*\mathbf{M}_\epsilon \mathbf{E}_{L^2}\mathbf{v}, w^\dagger\rangle_{\mathbf{V}}| \le 2|\omega|\tilde{\delta}_{\epsilon,\mu}\|\hat{\mathbf{n}}\cdot\mu^{-1}\mathbf{v}\|_\Sigma\|w^\dagger\|_\Sigma$. To treat the remaining terms we first analyze $(\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D})^{-1}E^*_{L^2}M_\epsilon E_{L^2}$. Since

$$H^1_0(\Gamma) \ni u := (\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D})^{-1}E^*_{L^2}M_\epsilon E_{L^2}w$$

solves $-\operatorname{div}_\Gamma(\mu^{-1}\nabla_\Gamma u) = \epsilon w$ in Γ, it follows that $\|u\|_{H^1(\Gamma)} \le 2\epsilon_0\mu_0 c^{-1}_{\mathrm{Dir}}\|w\|_{L^2(\Gamma)}$. Further, we have that

$$\begin{aligned}
\langle \mu \operatorname{div}_\Gamma(\mu^{-1}\nabla_\Gamma u), \operatorname{div}_\Gamma(\mu^{-1}\nabla_\Gamma u)\rangle_\Gamma &= \langle \mu\partial_r(r^{-1}\mu^{-1}\partial_r u), \partial_r(r^{-1}\mu^{-1}\partial_r u)\rangle_\Gamma + \langle \mu^{-1}r^{-1}\partial^2_\theta u, r^{-1}\partial^2_\theta u\rangle_\Gamma \\
&\quad + \langle \partial_r(r^{-1}\mu^{-1}\partial_r u), r^{-1}\partial^2_\theta u\rangle_\Gamma + \langle r^{-1}\partial^2_\theta u, \partial_r(r^{-1}\mu^{-1}\partial_r u)\rangle_\Gamma
\end{aligned}$$

and $\langle \partial_r(r^{-1}\mu^{-1}\partial_r u), r^{-1}\partial^2_\theta u\rangle_\Gamma + \langle r^{-1}\partial^2_\theta u, \partial_r(r^{-1}\mu^{-1}\partial_r u)\rangle_\Gamma = 2\langle \mu^{-1}r^{-1}\partial_r\partial_\theta u, r^{-1}\partial_r\partial_\theta u\rangle_\Gamma$. Thus $\sum_{l=1}^{N_\Gamma}|u|^2_{H^2(\Gamma_l)} \le 16\epsilon_0^2\mu_0^2\|w\|^2_{L(^2(\Gamma)}$. Now we are read to estimate

$$\begin{aligned}
&|\omega^3 E^*_{L^2}M_\epsilon E_{L^2}(\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D})^{-1}\mathbf{D}^*\mathbf{M}_\epsilon \mathbf{E}_{L^2}\mathbf{v}, w^\dagger\rangle_{H^1_0(\Gamma)}| \\
&\qquad = |\omega^3\mathbf{M}_\epsilon\mathbf{E}_{L^2}\mathbf{v}, \mathbf{D}(\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D})^{-1}E^*_{L^2}M_\epsilon E_{L^2}w^\dagger\rangle_{\mathbf{L}^2(\Gamma)}| \\
&\qquad = |\omega^3\epsilon\mathbf{v}, \nabla_\Gamma(\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D})^{-1}E^*_{L^2}M_\epsilon E_{L^2}w^\dagger\rangle_\Gamma| \\
&\qquad = |\omega^3(\epsilon\mu-\epsilon_0\mu_0)\mu^{-1}\mathbf{v}, \nabla_\Gamma(\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D})^{-1}E^*_{L^2}M_\epsilon E_{L^2}w^\dagger\rangle_\Gamma| \\
&\qquad \le 2|\omega|^3\tilde{\delta}_{\epsilon,\mu}\|\hat{\mathbf{n}}\cdot\mu^{-1}\mathbf{v}\|_\Sigma\|(\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D})^{-1}E^*_{L^2}M_\epsilon E_{L^2}w^\dagger\|_\Sigma \\
&\qquad \le 4|\omega|^3\epsilon_0\mu_0c^{-1}_{\mathrm{Dir}}C_{\mathrm{tr},1}\tilde{\delta}_{\epsilon,\mu}\|\hat{\mathbf{n}}\cdot\mu^{-1}\mathbf{v}\|_\Sigma\|w^\dagger\|_\Gamma,
\end{aligned}$$

where again we have used that $\operatorname{div}_\Gamma(\mu^{-1}\mathbf{v}) = 0$ and that ϵ, μ are piece-wise constant. It remains to estimate the term stemming from $B_{\epsilon,\mu,22}$. Again, it follows with integration by parts that

$$\begin{aligned}
&|\omega^2 E^*_{L^2}M_\epsilon E_{L^2}(\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D})^{-1}(\mathbf{D}^*\mathbf{M}_\epsilon\mathbf{D} - \epsilon_0\mu_0\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D})w, w^\dagger\rangle_{H^1_0(\Gamma)}| \\
&\quad = |\omega|^2|\langle (\mathbf{M}_\epsilon\mathbf{D} - \epsilon_0\mu_0\mathbf{M}_{\mu^{-1}}\mathbf{D})w, \mathbf{D}(\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D})^{-1}E^*_{L^2}M_\epsilon E_{L^2}w^\dagger\rangle_{\mathbf{L}^2(\Gamma)}| \\
&\quad = |\omega|^2|\langle (\epsilon\mu - \epsilon_0\mu_0)\nabla_\Gamma w, \mu^{-1}\nabla_\Gamma(\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D})^{-1}E^*_{L^2}M_\epsilon E_{L^2}w^\dagger\rangle_\Gamma| \\
&\quad \le |\omega|^2|\langle (\epsilon\mu - \epsilon_0\mu_0)w, \epsilon w^\dagger\rangle_\Gamma| + |\omega|^2|\langle [\![\epsilon\mu - \epsilon_0\mu_0]\!]w, \hat{\mathbf{n}}\cdot\mu^{-1}\nabla_\Gamma(\mathbf{D}^*\mathbf{M}_{\mu^{-1}}\mathbf{D})^{-1}E^*_{L^2}M_\epsilon E_{L^2}w^\dagger\rangle_\Sigma \\
&\quad \le 2|\omega|^2\epsilon_0\tilde{\delta}_{\epsilon,\mu}\|w\|_\Gamma\|w^\dagger\|_\Gamma + 2|\omega|^2C_{\mathrm{tr},2}(2\epsilon_0\mu_0c^{-1}_{\mathrm{Dir}} + 4\epsilon_0\mu_0)\tilde{\delta}_{\epsilon,\mu}\|w\|_\Sigma\|w^\dagger\|_\Gamma.
\end{aligned}$$

□

2.4 Coaxial separation of modes

For any scalar function $u \in L^2(\Gamma)$ we introduce the Fourier coefficients

$$u(re^{i\theta m}) = \sum_{m\in\mathbb{Z}} u_m(r)(2\pi)^{-1/2}e^{im\theta}, \quad u_m(r) := (2\pi)^{-1/2}\int_0^{2\pi} u(re^{i\theta m})e^{-im\theta}\,d\theta.$$

For any scalar function space $Y \subset L^2(\Gamma)$ let

$$Y_m := \{u \in Y \colon u_{m'} = 0\ \forall m' \in \mathbb{Z}\setminus\{m\}\}.$$

For any vectorial function space $\mathbf{Y}$ characterized by a potential space $\mathbf{Y} = \{\sigma \operatorname{Curl}_\Gamma u \colon u \in Y\}$, $\sigma = \mu, \epsilon$ let

$$\mathbf{Y}_m := \{\sigma \operatorname{Curl}_\Gamma u \colon u \in Y_m\}.$$

Note that

$$\mathbf{V} = \{\mu \operatorname{Curl}_\Gamma u \colon u \in V^p\}, \qquad V^p := \{u \in H^1(\Gamma) \colon \mu \operatorname{Curl}_\Gamma u \in \mathbf{H}_0(\operatorname{curl}_\Gamma; \Gamma)\}.$$

Further set

$$V := H^1_{\mu^{-1}*}(\Gamma) \quad \text{and} \quad W := H^1_0(\Gamma),$$

and in particular

$$X_m := \mathbf{V}_m \times W_m.$$

Since ϵ and μ depend only on the radial variable it follows that the spaces $X_m, m \in \mathbb{Z}$ are invariant with respect to A, B, K and the eigenvalue problems (11) and (13) decouple into the following families ($m \in \mathbb{Z}$) of problems

$$\text{find} \quad \left(\nu, \begin{pmatrix} \mathbf{v}_m \\ w_m \end{pmatrix}\right) \in \mathbb{C} \times X_m \setminus \{0\} \quad \text{such that} \quad (A_m + B_m) \begin{pmatrix} \mathbf{v}_m \\ w_m \end{pmatrix} = \nu^2 K_m \begin{pmatrix} \mathbf{v}_m \\ w_m \end{pmatrix} \tag{14}$$

and

$$\text{find} \quad \left(\nu, \begin{pmatrix} \mathbf{v}_m \\ w_m \end{pmatrix}\right) \in \mathbb{C} \times X_m \setminus \{0\} \quad \text{such that} \quad A_m \begin{pmatrix} \mathbf{v}_m \\ w_m \end{pmatrix} = \nu^2 K_m \begin{pmatrix} \mathbf{v}_m \\ w_m \end{pmatrix}, \tag{15}$$

where $A_m := A|_{X_m}, B_m := B|_{X_m}, K_m := K|_{X_m} \in \mathcal{L}(X_m)$. Hence it suffices to conduct the perturbation analysis for the problems (14) and (15) separately for each $m \in \mathbb{Z}$. However, the respective perturbation estimates have to be uniform in $m \in \mathbb{Z}$ in order to yield a meaningful result for the full space X. To be more precise, we introduce the following statements.

Definition 2.5. *A family of vectors $(\phi_n)_{n\in\mathbb{N}}$ in a Hilbert space $\Phi \ni \phi_n$ is called a Riesz basis, if each $\phi \in \Phi$ admits a unique representation $\phi = \sum_{n\in\mathbb{N}} c_n \phi_n$, $c_n \in \mathbb{C}$ and there exist constants $c, C > 0$ such that*

$$c \sum_{n\in\mathbb{N}} |c_n|^2 \leq \|\phi\|^2_\Phi \leq C \sum_{n\in\mathbb{N}} |c_n|^2 \qquad \text{for all } \phi \in \Phi.$$

We call c, C the Gram constants of the Riesz basis $(\phi_n)_{n\in\mathbb{N}}$.

Lemma 2.6. *Let the Hilbert space Φ be separated into a discrete family of orthogonal subspaces, i.e., $\mathcal{I}$ is a discrete index set and $\Phi = \bigoplus^\perp_{m\in\mathcal{I}} \Phi_m$. Let $(\phi_{n,m})_{n\in\mathbb{N}}$ be a Riesz basis of Φ_m with Gram constants $c_m, C_m > 0$ for each $m \in \mathcal{I}$. If there exist constants $c, C > 0$ such that $c < c_m, C_m < C$ for all $m \in \mathcal{I}$, then $(\phi_{n,m})_{n\in\mathbb{N},m\in\mathcal{I}}$ forms a Riesz basis of Φ with Gram constants c, C.*

Proof. Follows with basic considerations. □

3 Properties of Bessel functions

Let J_m, Y_m be the Bessel functions of the first and second kind, $(j_{n,m})_{n\in\mathbb{N}}$ be the non-decreasing sequence of positive roots of J_m, $m \in \mathbb{N}_0$ and $(j'_{n,m})_{n\in\mathbb{N}}$ be the non-decreasing sequence of positive roots of the derived Bessel function J'_m, $m \in \mathbb{N}_0$. It is known that $(j_{n,m})_{n\in\mathbb{N}}$ and $(j'_{n,m})_{n\in\mathbb{N}}$ are interlacing [7, Thm. 1] for each $m \in \mathbb{N}_0$, i.e.,

$$j_{1,0} < j'_{1,0} < j_{2,0} < j'_{2,0} < \dots \quad \text{and} \quad j'_{1,m} < j_{1,m} < j'_{2,m} < j_{2,m} < \dots, \; m \in \mathbb{N} \tag{16}$$

We introduce the compact notation

$$(\hat{j}_{n,0})_{n\in\mathbb{N}} := j_{1,0}, j'_{1,0}, j_{2,0}, j'_{2,0}, \ldots \quad \text{and} \quad (\hat{j}_{n,m})_{n\in\mathbb{N}} := j'_{1,m}, j_{1,m}, j'_{2,m}, j_{2,m}, \ldots,\ m \in \mathbb{N}.$$

Throughout the manuscript we will repeatedly apply the Wronskian identity [16, Eq. 10.5.2]

$$J'_m(r)Y_m(r) - J_m(r)Y'_m(r) = \frac{2}{\pi r}, \quad r > 0, m \in \mathbb{N}_0. \tag{17}$$

Let Ai and Bi be the Airy functions of the first and second kind and

$$(\mathbb{a}_n)_{n\in\mathbb{N}}, (\mathbb{a}'_n)_{n\in\mathbb{N}}, \quad \mathbb{a}_n > 0,$$

be the ascendant sequence of the positive roots of $\mathrm{Ai}(-\cdot)$ and $\mathrm{Ai}'(-\cdot)$ respectively. Note that $(\mathbb{a}_n)_{n\in\mathbb{N}}$ and $(\mathbb{a}'_n)_{n\in\mathbb{N}}$ are interlacing, i.e.,

$$0 < \mathbb{a}'_1 < \mathbb{a}_1 < \mathbb{a}'_2 < \mathbb{a}_2 < \ldots,$$

see, e.g., [5, (2.2), Thm. (T3.1)]. We introduce the compact notation

$$(\hat{\mathbb{a}}_n)_{n\in\mathbb{N}} := \mathbb{a}'_1, \mathbb{a}_1, \mathbb{a}'_2, \mathbb{a}_2, \ldots.$$

3.1 Asymptotic approximations

To establish Lemmas 3.6 to 3.8 (which will be used for Lemma 4.2 and Proposition 4.5) we will repeatedly use (incremental adaptations/generalizations) of the Bessel function asymptotics provided by [43, 32]. To make the manuscript more self-contained we formulate those in the following two Lemmas 3.1 and 3.2. We start with the behavior in the transition zone.

Lemma 3.1. *For each compact interval $[t_1, t_2] \subset \mathbb{R}$ there exists a constant $C_{[t_1,t_2]} > 0$ such that*

$$\sup_{t\in[t_1,t_2]} |J_m(m + m^{1/3}t) - \frac{2^{1/3}}{m^{1/3}} \mathrm{Ai}(-2^{1/3}t)| \le \frac{C_{[t_1,t_2]}}{m}, \tag{18a}$$

$$\sup_{t\in[t_1,t_2]} |Y_m(m + m^{1/3}t) + \frac{2^{1/3}}{m^{1/3}} \mathrm{Bi}(-2^{1/3}t)| \le \frac{C_{[t_1,t_2]}}{m}, \tag{18b}$$

$$\sup_{t\in[t_1,t_2]} |J'_m(m + m^{1/3}t) + \frac{2^{2/3}}{m^{2/3}} \mathrm{Ai}'(-2^{1/3}t)| \le \frac{C_{[t_1,t_2]}}{m^{4/3}}, \tag{18c}$$

$$\sup_{t\in[t_1,t_2]} |Y'_m(m + m^{1/3}t) - \frac{2^{2/3}}{m^{2/3}} \mathrm{Bi}'(-2^{1/3}t)| \le \frac{C_{[t_1,t_2]}}{m^{4/3}}, \tag{18d}$$

$$\sup_{t\in[t_1,t_2]} |J''_m(m + m^{1/3}t) - \frac{2}{m} \mathrm{Ai}(-2^{1/3}t)| \le \frac{C_{[t_1,t_2]}}{m^{5/3}}, \tag{18e}$$

$$\sup_{t\in[t_1,t_2]} |Y''_m(m + m^{1/3}t) + \frac{2}{m} \mathrm{Bi}(-2^{1/3}t)| \le \frac{C_{[t_1,t_2]}}{m^{5/3}}, \tag{18f}$$

for all $m \in \mathbb{N}_0$.

Proof. The asymptotics are in principal well-known [16, Sect. 10.19]. The actual bounds are due to [43, Prop. 2.9]. □

In order to stick to the convenient notation [32], but to avoid a double use of symbols we use a blackboard font style for the following constants:

$$\mathbb{\mu}_m := |m^2 - 1/4|, \qquad \mathbb{\omega}_m := (2m + 1)\pi/4.$$

Further, let

$$\mathcal{B}_m(r) := \sqrt{r^2 - \mathbb{\mu}_m} + \sqrt{\mathbb{\mu}_m} \arcsin \frac{\sqrt{\mathbb{\mu}_m}}{r}, \qquad \mathcal{B}'_m(r) = \frac{\sqrt{r^2 - \mathbb{\mu}_m}}{r}, \qquad r \ge \sqrt{\mathbb{\mu}_m}.$$

Lemma 3.2. *For each $m \in \mathbb{N}$ and $r > \sqrt{\mu_m}$ it holds that*

$$|J_m(r) - \sqrt{\frac{2}{\pi}}(r^2 - \mu_m)^{-1/4}\cos(\mathcal{B}_m(r) - \omega_m)| \le \frac{13\mu_m}{12\sqrt{2\pi}(r^2-\mu_m)^{7/4}}, \tag{19a}$$

$$|Y_m(r) - \sqrt{\frac{2}{\pi}}(r^2 - \mu_m)^{-1/4}\sin(\mathcal{B}_m(r) - \omega_m)| \le \frac{13\mu_m}{12\sqrt{2\pi}(r^2-\mu_m)^{7/4}}, \tag{19b}$$

$$|J'_m(r) + \sqrt{\frac{2}{\pi}}\frac{(r^2 - \mu_m)^{1/4}}{r}\sin(\mathcal{B}_m(r) - \omega_m)| \le \frac{5}{\sqrt{8\pi}}\frac{r}{(r^2-\mu_m)^{5/4}} + \frac{13\mu_m r}{24\sqrt{2\pi}(r^2-\mu_m)^{11/4}}, \tag{19c}$$

$$|Y'_m(r) - \sqrt{\frac{2}{\pi}}\frac{(r^2 - \mu_m)^{1/4}}{r}\cos(\mathcal{B}_m(r) - \omega_m)| \le \frac{5}{\sqrt{8\pi}}\frac{r}{(r^2-\mu_m)^{5/4}} + \frac{13\mu_m r}{24\sqrt{2\pi}(r^2-\mu_m)^{11/4}}. \tag{19d}$$

Proof. The first equation (19a) is merely [32, Thm. 5]. To obtain (19b) we repeat the proof of [32, Thm. 5] with minor adaptations: The proof applies the estimate $(r^2 - \mu_m)^{1/4}|Y_m(r)| \le \sqrt{2/\pi}$ for $r > \mu_m$, which follows as in the first case in the proof of [32, Thm. 3] by replacing J_m with Y_m. Furthermore, the constants c_1, c_2 of the homogeneous solution in the proof [32, Thm. 5] change and are deduced from the asymptotic of $Y_m(r)$ for $r \to +\infty$. To obtain (19c) we note that [32] actually yields

$$\begin{aligned}(r^2 - \mu_m)^{1/4}J_m(r) &- \sqrt{\frac{2}{\pi}}\cos(\mathcal{B}_m(r) - \omega_m)\\ &= \int_r^{+\infty}\frac{3(\mathcal{B}''_m(t))^2 - 2\mathcal{B}'_m(t)\mathcal{B}'''_m(t)}{4(\mathcal{B}'_m(t))^4}\mathcal{B}'_m(t)(t^2-\mu_m)^{1/4}J_m(t)\sin(\mathcal{B}_m(r) - \mathcal{B}_m(t))\,\mathrm{d}t\end{aligned}$$

and

$$\int_r^{+\infty}\left|\frac{3(\mathcal{B}''_m(t))^2 - 2\mathcal{B}'_m(t)\mathcal{B}'''_m(t)}{4(\mathcal{B}'_m(t))^4}\mathcal{B}'_m(t)(t^2-\mu_m)^{1/4}J_m(t)\right|\mathrm{d}t \le \frac{13\mu_m}{12\sqrt{2\pi}(r^2-\mu_m)^{7/4}} \quad \text{for } r > \sqrt{\mu_m}.$$

Multiplying the former identity by $(r^2 - \mu_m)^{-1/4}$ and deriving it we obtain that the argument inside the absolute value bars in the left hand-side of (19c) equals

$$\begin{aligned}&-\frac{1}{\sqrt{2\pi}}\frac{r}{(r^2-\mu_m)^{5/4}}\cos(\mathcal{B}_m(r) - \omega_m)\\ &-\frac{1}{2}\frac{r}{(r^2-\mu_m)^{5/4}}\int_r^{+\infty}\frac{3(\mathcal{B}''_m(t))^2 - 2\mathcal{B}'_m(t)\mathcal{B}'''_m(t)}{4(\mathcal{B}'_m(t))^4}\mathcal{B}'_m(t)(t^2-\mu_m)^{1/4}J_m(t)\sin(\mathcal{B}_m(r) - \mathcal{B}_m(t))\,\mathrm{d}t\\ &+\frac{1}{(r^2-\mu_m)^{1/4}}\int_r^{+\infty}\frac{3(\mathcal{B}''_m(t))^2 - 2\mathcal{B}'_m(t)\mathcal{B}'''_m(t)}{4(\mathcal{B}'_m(t))^4}(\mathcal{B}'_m(t))^2(t^2-\mu_m)^{1/4}J_m(t)\cos(\mathcal{B}_m(r) - \mathcal{B}_m(t))\,\mathrm{d}t.\end{aligned}$$

We estimate the absolute value of the first two terms by

$$\frac{1}{\sqrt{2\pi}}\frac{r}{(r^2-\mu_m)^{5/4}} + \frac{13\mu_m r}{24\sqrt{2\pi}(r^2-\mu_m)^{11/4}}.$$

Exploiting computations from [32, p. 222] the third term can be estimated as

$$\begin{aligned}\frac{6\mu_m}{\sqrt{8\pi}(r^2-\mu_m)^{1/4}}\int_r^{+\infty}(z^2-\mu_m)^{-2}\,\mathrm{d}z &= \frac{6\mu_m}{\sqrt{8\pi}(r^2-\mu_m)^{1/4}}\Big(\frac{r}{2\mu_m(r^2-\mu_m)} - \frac{\log\frac{r+\sqrt{\mu_m}}{r-\sqrt{\mu_m}}}{4\mu_m^{3/2}}\Big)\\ &\le \frac{3r}{\sqrt{8\pi}(r^2-\mu_m)^{5/4}}.\end{aligned}$$

Hence (19c) follows. Finally, (19d) follows by combining the above adaptations. $\square$

Lemma 3.3. *For each $m \in \mathbb{N}$ and $r > m + m^{1/3}t_*$, $t_* > 1$ it holds that*

$$|(r^2 - \mu_m)^{1/4} J_m(r) - \sqrt{\frac{2}{\pi}} \cos(\mathcal{B}_m(r) - \omega_m)| \leq \frac{13}{12\sqrt{2\pi} t_*^{3/2}}, \tag{20a}$$

$$|(r^2 - \mu_m)^{1/4} Y_m(r) - \frac{\sqrt{2}}{\sqrt{\pi}} \sin(\mathcal{B}_m(r) - \omega_m)| \leq \frac{13}{12\sqrt{2\pi} t_*^{3/2}}, \tag{20b}$$

$$|\frac{r}{(r^2 - \mu_m)^{1/4}} J'_m(r) + \sqrt{\frac{2}{\pi}} \sin(\mathcal{B}_m(r) - \omega_m)| \leq \frac{7}{\sqrt{2\pi}} \frac{1}{t_*^{1/2}} \tag{20c}$$

$$|\frac{r}{(r^2 - \mu_m)^{1/4}} Y'_m(r) - \sqrt{\frac{2}{\pi}} \cos(\mathcal{B}_m(r) - \omega_m)| \leq \frac{7}{\sqrt{2\pi}} \frac{1}{t_*^{1/2}}. \tag{20d}$$

Proof. To establish (20a) we multiply (19a) with $(r^2 - \mu_m)^{1/4}$

$$|(r^2 - \mu_m)^{1/4} J_m(r) - \sqrt{\frac{2}{\pi}} \cos(\mathcal{B}_m(r) - \omega_m)| \leq \frac{13\mu_m}{12\sqrt{2\pi}(r^2 - \mu_m)^{3/2}}$$

and note that the new right hand-side is monotonically decreasing in $r \in (\sqrt{\mu_m}, +\infty)$. Thus plugging $r = m + m^{1/3}t_*$ into $\frac{13\mu_m}{12\sqrt{2\pi}(r^2-\mu_m)^{3/2}}$ and applying elementary estimates yield the desired estimate (20a). Estimate (20b) follows the very same way. To establish (20c) we multiply (19c) with $\frac{r}{(r^2-\mu_m)^{1/4}}$ and estimate

$$\begin{aligned}
&|\frac{r}{(r^2 - \mu_m)^{1/4}} J'_m(r) + \sqrt{\frac{2}{\pi}} \sin(\mathcal{B}_m(r) - \omega_m)| \\
&\quad \leq \frac{5}{\sqrt{8\pi}} \frac{r^2}{(r^2 - \mu_m)^{3/2}} + \frac{13\mu_m r^2}{24\sqrt{2\pi}(r^2 - \mu_m)^3} \\
&\quad = \frac{5}{\sqrt{8\pi}} \frac{1}{(r^2 - \mu_m)^{1/2}} + \frac{5}{\sqrt{8\pi}} \frac{\mu_m}{(r^2 - \mu_m)^{3/2}} + \frac{13\mu_m}{24\sqrt{2\pi}(r^2 - \mu_m)^2} + \frac{13\mu_m^2}{24\sqrt{2\pi}(r^2 - \mu_m)^3} \\
&\quad \leq \frac{5}{\sqrt{8\pi}} \frac{1}{m^{2/3} t_*^{1/2}} + \frac{5}{\sqrt{8\pi}} \frac{1}{t_*^{3/2}} + \frac{13}{24\sqrt{2\pi} m^{2/3} t_*^2} + \frac{13}{24\sqrt{2\pi} t_*^3} \\
&\quad \leq \frac{5}{\sqrt{2\pi}} \frac{1}{t_*^{1/2}} + \frac{13}{12\sqrt{2\pi} t_*^2} \leq \frac{7}{\sqrt{2\pi}} \frac{1}{t_*^{1/2}},
\end{aligned}$$

where in the step from the third to the fourth line we exploited the monotonicity in $r \in (\sqrt{\mu_m}, +\infty)$. Estimate (20d) follows the very same way. $\square$

Note that estimates for $F''_m(r)$, $F = J, Y$ in $r > \sqrt{\mu_m}$ can be obtained from

$$F''_m = \frac{-1}{r} F'_m(r) - \frac{r^2 - m^2}{r^2} F_m(r), \quad F = J, Y, \tag{21}$$

which itself follows from $F''_m(r) = (r^{-1} r F'_m(r))' = \frac{-1}{r} F'_m(r) + \frac{1}{r}\left(\frac{m^2}{r} - r\right) F_m(r)$.

3.2 Gaps of roots and extrema

Proposition 3.4. *Let*

$$c_{\text{gap}} := \min\left\{ j'_{1,1}, 1, \frac{2j_{1,0}}{2j_{1,0} + 1} \right\}.$$

Then it holds that

$$|\hat{j}_{1,m}| \geq c_{\text{gap}}, \quad |\hat{j}_{n+1,m} - \hat{j}_{n,m}| \geq c_{\text{gap}}, \quad \textit{for all} \quad n \in \mathbb{N}, m \in \mathbb{N}_0.$$

Proof. First, note that $\min_{m\in\mathbb{N}_0} \hat{j}_{1,m} = j'_{1,1}$. Due to (16) it suffices to estimate

$$j'_{n,0} - j_{n,0}, \quad |j'_{n,0} - j_{n+1,0}| \quad \text{for } n \in \mathbb{N}$$

and

$$|j'_{1,m} - j_{1,m}|, \quad j'_{n,m} - j_{n-1,m}, \quad |j'_{n,m} - j_{n,m}| \quad \text{for } n \in \mathbb{N}\setminus\{1\}, m \in \mathbb{N}.$$

We recall from [33, (F5)] that

$$\sum_{k\geq 1} \frac{j_{k,m}^2}{\big((j'_{n,m})^2 - j_{k,m}^2\big)^2} = \frac{1}{4} - \frac{m^2}{4(j'_{n,m})^2}, \quad n \geq 1.$$

Thus

$$|j'_{n,m} - j_{k,m}| \geq \frac{2j_{k,m}}{j'_{n,m} + j_{k,m}}, \quad n, k \in \mathbb{N}, m \in \mathbb{N}_0.$$

If $j'_{n,m} < j_{k,m}$, then $|j'_{n,m} - j_{k,m}| \geq 1$. The case $j'_{n,m} > j_{k,m}$ and $j'_{n,m} - j_{k,m} \geq 1$ does not require additional treatment. Hence the case $j'_{n,m} > j_{k,m}$ and $j'_{n,m} - j_{k,m} < 1$ remains, for which we estimate $|j'_{n,m} - j_{k,m}| \geq \frac{2j_{k,m}}{2j_{k,m}+1} \geq \frac{2j_{1,0}}{2j_{1,0}+1}$. Thus the proof is finished. □

Remark 3.5. *Note that an alternative proof of Proposition 3.4 (with a less explicit constant c_{gap}) follows along the lines of Lemma 4.4 and Proposition 4.5.*

3.3 Boundedness of Bessel function products

Lemma 3.6. *There exists a constant $C_{JY} > 0$ such that $\sup_{r>0, m\in\mathbb{N}_0} |rJ'_m(r)Y_m(r)| \leq C_{JY}$.*

Proof. 0. case $m = 0$: $rJ'_0(r)Y_0(r)$ is continuous functions on $r \in (0, +\infty)$. The well-known asymptotics for $rJ'_0(r)$ and $Y_0(r)$ yields that $\limsup_{r\to+\infty} |rJ'_0(r)Y_0(r)| \leq 2\pi$. In addition, $\lim_{r\to 0+} |rJ'_0(r)Y_0(r)| = 1/\pi$ [16, Eq. 10.6.2., 10.7.3, 10.7.4]. Thus $C_{JY}^{(0)} := \sup_{r>0} |rJ'_0(r)Y_0(r)| < +\infty$.

To proceed we split $(0, +\infty)$ into $(0, m]$, $[m, m + m^{1/3}t_*]$, $[m^{1/3}t_*, +\infty)$ with an arbitrary constant $t_* > 1$.

1. case $r \in (0, m]$: Let $y_{1,m}$ be the first positive zero of Y_m. We recall [47, p.133] that

$$Y_m(r) = \frac{2}{\pi} J_m(r) \int_{y_{1,m}}^{r} \frac{1}{zJ_m(z)^2}\, \mathrm{d}z.$$

Thus for $f_m(r) := \frac{\pi}{2} rJ'_m(r)Y_m(r) = rJ'_m(r)J_m(r) \int_{y_{1,m}}^{r} \frac{1}{zJ_m(z)^2}\, \mathrm{d}z$ we compute that

$$f'_m(r) = \frac{J'_m(r)}{J_m(r)} + \Big(\big(\frac{m^2}{r} - r\big)J_m(r)^2 + rJ'_m(r)^2\Big) \int_{y_{1,m}}^{r} \frac{1}{zJ_m(z)^2}\, \mathrm{d}z.$$

Hence for any root $r_* \in (0, m)$ of f'_m it holds that $\int_{y_{1,m}}^{r_*} \frac{1}{zJ_m(z)^2}\, \mathrm{d}z = -\frac{J'_m(r_*)}{J_m(r_*)} \frac{1}{(\frac{m^2}{r_*} - r_*)J_m(r)^2 + r_*J'_m(r_*)^2}$.
We deduce that $f_m(r_*) = \frac{-r_*J'_m(r_*)^2}{(\frac{m^2}{r_*} - r_*)J_m(r_*)^2 + r_*J'_m(r_*)^2} \in (-1, 0)$. With [16, Eq. 10.6.2., 10.7.3, 10.7.4] we compute that $\lim_{r\to 0+} f_m(r) = -1/2$ for each $m \in \mathbb{N}_0$. On the other hand it follows by means of [47, p.232] that $\lim_{m\to+\infty} f_m(m) = \frac{3^{1/2}Gamma(1/3)Gamma(2/3)}{4}$ (with $Gamma$ meaning the Gamma function). It follows that $C_{JY}^{(1)} := \sup_{r\in(0,m), m\in\mathbb{N}} |rJ'_m(r)Y_m(r)| < +\infty$).

2. case $r \in [m, m + m^{1/3}t_*]$: In this regime it holds that $r/m \in [1, 1 + t_*]$. Hence by means of (18c) and (18b) it follows that $C_{JY}^{(2)} := \sup_{r\in[m, m+m^{1/3}t_*], m\in\mathbb{N}} |rJ'_m(r)Y_m(r)| < +\infty$.

3. case $r \in [m + m^{1/3}t_*], +\infty)$: Applying the estimates (20c) and (20b) yields that $C_{JY}^{(3)} := \sup_{r\in[m+m^{1/3}t_*, +\infty), m\in\mathbb{N}} |rJ'_m(r)Y_m(r)| < +\infty$.

The claim $\sup_{r>0, m\in\mathbb{N}_0} |rJ'_m(r)Y_m(r)| \leq C_{JY}$ follows now with $C_{JY} := \max\{C_{JY}^{(0)}, C_{JY}^{(1)}, C_{JY}^{(2)}, C_{JY}^{(3)}\}$. □

Having established Lemma 3.6 the next two lemmas follow without much effort.

Lemma 3.7. *There exists a constant* $C_{JJ} > 0$ *such that* $\sup_{r>0,m\in\mathbb{N}_0} |rJ'_m(r)J_m(r)| \leq C_{JJ}$.

Proof. The proof is similar to that of 3.6, where in the region $r \in [0, m]$ we use that $|J_m(r)|, |J'_m(r)| \leq 1$ [16, Eq. 10.6.1, 10.14.1]. □

Lemma 3.8. *Let* q_1, q_2 *be constants with* $0 < q_1 \leq q_2 \leq 1$. *Then there exists a constant* $C_{JJYY} > 0$ *such that* $\sup_{q\in[q_1,q_2],r>0,m\in\mathbb{N}_0} |r^2 J'_m(qr)J_m(qr)Y'_m(r)Y_m(r)| \leq C_{JJYY}$.

Proof. Let $q > 0$ be fixed. In the region $r \in [m, +\infty)$ we estimate

$$\begin{aligned}\sup_{r>m,m\in\mathbb{N}_0} |r^2 J'_m(qr)J_m(qr)Y'_m(r)Y_m(r)| &\leq q^{-1} \sup_{r>m,m\in\mathbb{N}_0} |qrJ'_m(qr)J_m(qr)| \sup_{r>m,m\in\mathbb{N}_0} |rY'_m(r)Y_m(r)| \\ &\leq q^{-1} C_{JJ} \sup_{r>m,m\in\mathbb{N}_0} |rY'_m(r)Y_m(r)|\end{aligned}$$

with C_{JJ} as in Lemma 3.7. The term $\sup_{r>m,m\in\mathbb{N}_0} |rY'_m(r)Y_m(r)|$ can be uniformly bounded by repeating the technique used in the proof of Lemma 3.6 (second and third case). On $r \in [0, m]$ we exploit that $r \mapsto rJ'_m(qr)J_m(qr)$ is monoton increasing to estimate

$$\begin{aligned}\sup_{r\in[0,m],m\in\mathbb{N}_0} |r^2 J'_m(qr)J_m(qr)Y'_m(r)Y_m(r)| &\leq \sup_{r\in[0,m],m\in\mathbb{N}_0} |r^2 J'_m(r)J_m(r)Y'_m(r)Y_m(r)| \\ &\leq \sup_{r\in[0,m],m\in\mathbb{N}_0} |rJ'_m(r)Y_m(r)| \sup_{r\in[0,m],m\in\mathbb{N}_0} |rJ_m(r)Y'_m(r)|\end{aligned}$$

Due to Lemma 3.6 and (17) we obtain that $\sup_{r>0,m\in\mathbb{N}_0} |r^2 J'_m(qr)J_m(qr)Y'_m(r)Y_m(r)| < +\infty$. The final claim follows by a continuity argument and the fact that $[q_1, q_2]$ is compact. □

4 The selfadjoint perturbation

In this section study the properties of the eigenvalue problem (13) (or rather the family (15) respectively). Points of interest are the gaps between eigenvalues (see Section 4.4) and trace inequalities for the eigenfunctions (see Section 4.5).

4.1 Reduction of $A_{11}\mathbf{v} = \nu^2 K_{11}\mathbf{v}$ to a scalar problem

To study the properties of the selfadjoint eigenvalue problem associated to (13) we note that (13) decouples into the two separate problems

$$\text{find} \quad (\nu, \mathbf{v}) \in \mathbb{C} \times \mathbf{V} \setminus \{0\} \quad \text{such that} \quad A_{11}\mathbf{v} = \nu^2 K_{11}\mathbf{v}, \tag{22a}$$

$$\text{find} \quad (\nu, w) \in \mathbb{C} \times W \setminus \{0\} \quad \text{such that} \quad A_{22}w = \nu^2 K_{22}w. \tag{22b}$$

As a next step we reduce the analysis of the vectorial problem (22a) to a problem for the scalar potential. Let

$$\Delta_\mu v := \operatorname{div}_\Gamma(\mu\nabla_\Gamma v) = -\operatorname{curl}_\Gamma(\mu \operatorname{Curl}_\Gamma v) \quad \text{and} \quad \Delta_\epsilon w := \operatorname{div}_\Gamma(\epsilon\nabla_\Gamma w) = -\operatorname{curl}_\Gamma(\epsilon \operatorname{Curl}_\Gamma w).$$

Lemma 4.1. *Let* $\nu \in \mathbb{C}$. *Then* $\mathbf{v} \in \mathbf{V} \setminus \{\mathbf{0}\}$, $\mathbf{v} = \mu \operatorname{Curl}_\Gamma v$, $v \in V^p \setminus \{0\}$ *solves* (22a), *if and only if it's potential* $v \in V$ *solves*

$$-\operatorname{div}_\Gamma(\mu\nabla_\Gamma v) = \nu^2 \mu v \quad \text{in } \Gamma \quad \text{and} \quad \mu\partial_{\hat{\mathbf{n}}}\nabla_\Gamma v = 0 \quad \text{on } \partial\Gamma. \tag{23}$$

In such a case, if in addition $\mathbf{v} = \mathbf{v}_m \in \mathbf{V}_m$ *or equivalently* $v = v_m \in V_m$, $m \in \mathbb{Z}$, *then*

$$\hat{\mathbf{n}} \cdot \mu^{-1}\mathbf{v}_m = imv_m \quad \text{on } \Sigma. \tag{24}$$

Proof. Let $\mathbf{v} = \mu \operatorname{Curl}_\Gamma v$ solve (22a). Note that $\nu \neq 0$. It follows that

$$\mu^{-1} \operatorname{curl}_\Gamma \mathbf{v} \in V \quad \text{and} \quad \hat{\mathbf{t}} \cdot \mu(\mu^{-1} \operatorname{curl}_\Gamma \mathbf{v}) = 0 \quad \text{on } \partial\Gamma. \tag{25}$$

Let $\tilde{v} := \mu^{-1}\Delta_\mu v$. Then (25) yields that $\tilde{v} \in V$ and $\mu\partial_{\hat{\mathbf{n}}}\tilde{v} = 0$ on $\partial\Gamma$. Applying integration by parts in the variational formulation of (22a) yields further that $-\Delta_\mu \tilde{v} = \nu^2\mu\tilde{v}$ in Γ. There exists a unique solution to the problem:

$$\text{find } u \in V \text{ such that } -\Delta_\mu u = \mu\tilde{v} \text{ in } \Gamma \quad \text{and} \quad \mu\partial_{\hat{\mathbf{n}}} u = 0 \text{ on } \partial\Gamma. \tag{26}$$

It can be checked that the solution to (26) equals $\nu^{-2}\tilde{v}$. Thus $v = \nu^{-2}\tilde{v}$ solves (23).
Vice-versa, let v solve (23) and $\mathbf{v} = \mu \operatorname{Curl} v$. Then $\hat{\mathbf{t}} \cdot \mathbf{v} = \mu\partial_{\hat{\mathbf{n}}} v = 0$ on $\partial\Gamma$. Furthermore, $\mu^{-1} \operatorname{curl}_\Gamma \mathbf{v} = \nu^2 v$ and thus $\operatorname{Curl}_\Gamma \mu^{-1} \operatorname{curl}_\Gamma \mathbf{v} = \nu^2 \operatorname{Curl}_\Gamma v = \nu^2\mu^{-1}\mathbf{v}$ in Γ. Since $\mathbf{v} \in \mathbf{V}$ it follows that $\mathbf{v}$ solves (22a).
To prove the last statement we compute that $\hat{\mathbf{n}} \cdot \mu^{-1}\mathbf{v}_m = \hat{\mathbf{n}} \cdot \operatorname{Curl}_\Gamma v_m = \partial_\theta v_m = imv_m$ on Σ. $\square$

4.2 Reduction of Laplace eigenvalue problems

Our goal is to relate the eigenvalues of the problems

$$\text{find } (\nu, v) \in \mathbb{C} \times V \setminus \{0\} \text{ such that } -\Delta_\mu v = \nu^2\mu v \quad \text{in } \Gamma, \qquad \mu\partial_{\hat{\mathbf{n}}} v = 0 \quad \text{on } \partial\Gamma, \tag{27a}$$

$$\text{find } (\nu, w) \in \mathbb{C} \times W \setminus \{0\} \text{ such that } -\Delta_\epsilon w = \nu^2\epsilon w \quad \text{in } \Gamma, \qquad w = 0 \quad \text{on } \partial\Gamma, \tag{27b}$$

to those of the homogeneous cases $\epsilon = \epsilon_0$ and $\mu = \mu_0$ respectively. Since the perturbation of ϵ and μ in (27) is not small in any subordinate sense (see, e.g., [36, 39] for the concept of subordination), convenient abstract perturbation techniques do not provide us with any fruitful means. Note also that the Courant–Fischer–Weyl min-max principle would only provide (for our purpose) insufficient estimates

$$\nu^2_{\mathrm{Neu},\mu_0} \frac{\mu_{\min}}{\mu_{\max}} \leq \nu^2_{\mathrm{Neu},\mu} \leq \nu^2_{\mathrm{Neu},\mu_0} \frac{\mu_{\max}}{\mu_{\min}}, \qquad \nu^2_{\mathrm{Dir},\epsilon_0} \frac{\epsilon_{\min}}{\epsilon_{\max}} \leq \nu^2_{\mathrm{Dir},\epsilon} \leq \nu^2_{\mathrm{Dir},\epsilon_0} \frac{\epsilon_{\max}}{\epsilon_{\min}}.$$

Instead we exploit our assumptions on Γ and ϵ, μ to reduce (27) to the eigenvalue problems of $2N_\Gamma - 1$-dimensional holomorphic matrix functions. First we note that (27) decouples into the family ($m \in \mathbb{Z}$) of problems

$$\text{find } (\nu, v_m) \in \mathbb{C} \times V_m \setminus \{0\} \text{ such that } -\Delta_\mu v_m = \nu^2\mu v_m \quad \text{in } \Gamma, \quad \mu\partial_{\hat{\mathbf{n}}} v = 0 \quad \text{on } \partial\Gamma, \tag{28a}$$

$$\text{find } (\nu, w_m) \in \mathbb{C} \times W_m \setminus \{0\} \text{ such that } -\Delta_\epsilon w_m = \nu^2\epsilon w_w \quad \text{in } \Gamma, \qquad w_m = 0 \quad \text{on } \partial\Gamma. \tag{28b}$$

Let (ν, v_m) solve (28a). Recall that μ is constant on each interval (r_l, r_{l+1}), $l = 1, \ldots, N_\Gamma$ (ball B_{r_2}, annuli $B_{r_{l+1}} \setminus B_{r_l}$). It follows that the function $e^{-im\theta}v_m|_{(r_l,r_{l+1})}$ depends only on r and solves a scaled Bessel equation with index m. Hence $e^{-im\theta}v_m|_{(r_l,r_{l+1})}(r) = a_l J_m(\nu r) + b_l Y_m(\nu r)$ with constants $a_l, b_l \in \mathbb{C}$. Considering the boundary and transmission conditions implied by (27) it follows that $\underline{v} := (a_1, a_2, b_2, \ldots, a_{N_\Gamma}, b_{N_\Gamma})^\top \in \mathbb{C}^{2N_\Gamma - 1}$ solves $\underline{A}^{\mathrm{Neu}}_{N_\Gamma}(\nu)\underline{v} = 0$, where

$$\underline{A}^{\mathrm{Neu}}_{N_\Gamma}(\nu) := \begin{pmatrix} J_{m2} & -J_{m2} & -Y_{m2} & & & & & & \\ \frac{\mu_1}{\mu_2} J'_{m2} & -J'_{m2} & -Y'_{m2} & & & & & & \\ & J_{m3} & Y_{m3} & -J_{m3} & -Y_{m3} & & & & \\ & \frac{\mu_2}{\mu_3} J'_{m3} & \frac{\mu_2}{\mu_3} Y'_{m3} & -J'_{m3} & -Y'_{m3} & & & & \\ & & & J_{m4} & Y_{m4} & -J_{m4} & -Y_{m4} & & \\ & & & \frac{\mu_3}{\mu_4} J'_{m4} & \frac{\mu_3}{\mu_4} Y'_{m4} & -J'_{m4} & -Y'_{m4} & & \\ & & & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & & & J'_{m(N_\Gamma+1)} & Y'_{m(N_\Gamma+1)} \end{pmatrix} \tag{29a}$$

with the abbreviations $J_{ml} := J_m(\nu r_l)$, $J'_{ml} := J'_m(\nu r_l)$, $Y_{ml} := Y_m(\nu r_l)$, $Y'_{ml} := Y'_m(\nu r_l)$. Similarly, if (ν, w_m) solves (28b), then $e^{-im\theta} w_m|_{(r_l, r_{l+1})}(r) = a_l J_m(\nu r) + b_l Y_m(\nu r)$ with constants $a_l, b_l \in \mathbb{C}$ and $\underline{w} := (a_1, a_2, b_2, \ldots, a_{N_\Gamma}, b_{N_\Gamma})^\top \in \mathbb{C}^{2N_\Gamma - 1}$ solves $\underline{A}^{\mathrm{Dir}}_{N_\Gamma}(\nu)\underline{w} = 0$, where

$$\underline{A}^{\mathrm{Dir}}_{N_\Gamma}(\nu) := \begin{pmatrix} J_{m2} & -J_{m2} & -Y_{m2} & & & & & & \\ \frac{\epsilon_1}{\epsilon_2} J'_{m2} & -J'_{m2} & -Y'_{m2} & & & & & & \\ & J_{m3} & Y_{m3} & -J_{m3} & -Y_{m3} & & & & \\ & \frac{\epsilon_2}{\epsilon_3} J'_{m3} & \frac{\epsilon_2}{\epsilon_3} Y'_{m3} & -J'_{m3} & -Y'_{m3} & & & & \\ & & & J_{m4} & Y_{m4} & -J_{m4} & -Y_{m4} & & \\ & & & \frac{\epsilon_3}{\epsilon_4} J'_{m4} & \frac{\epsilon_3}{\epsilon_4} Y'_{m4} & -J'_{m4} & -Y'_{m4} & & \\ & & & & & \dots & \dots & \dots & \dots \\ & & & & & \dots & \dots & \dots & \dots \\ & & & & & & & J_{m(N_\Gamma+1)} & Y_{m(N_\Gamma+1)} \end{pmatrix}, \tag{29b}$$

i.e., compared to $\underline{A}^{\mathrm{Neu}}_{N_\Gamma}$ the last row is changed and μ, μ_0 are replaced by ϵ, ϵ_0 respectively. Altogether, ν is an eigenvalue to (28a) or (28b), if and only if $\det \underline{A}^{\mathrm{Neu}}_{N_\Gamma}(\nu) = 0$ or $\det \underline{A}^{\mathrm{Dir}}_{N_\Gamma}(\nu) = 0$ respectively.

4.3 Computation of $\det \underline{A}^{\mathrm{Neu}}_{N_\Gamma}(\nu)$ and $\det \underline{A}^{\mathrm{Dir}}_{N_\Gamma}(\nu)$

Note that the case $N_\Gamma = 1$ corresponds to a homogeneous material and is thus trivial. Hence it suffices to assume $N_\Gamma \geq 2$. In order to express $\det \underline{A}^{\mathrm{Neu}}_{N_\Gamma}(\nu), \det \underline{A}^{\mathrm{Dir}}_{N_\Gamma}(\nu)$ we introduce the auxiliary matrices $\underline{B}^{\mathrm{Neu}}_{N_\Gamma}(\nu), \underline{B}^{\mathrm{Neu}}_{N_\Gamma}(\nu) \in \mathbb{C}^{(2N_\Gamma-3)\times(2N_\Gamma-3)}$ as follows:

$$\underline{B}^{\mathrm{Neu}}_{N_\Gamma}(\nu) := \begin{pmatrix} Y_{m2} & -J_{m2} & -Y_{m2} & & & & & & \\ \frac{\mu_1}{\mu_2} Y'_{m2} & -J'_{m2} & -Y'_{m2} & & & & & & \\ & J_{m3} & Y_{m3} & -J_{m3} & -Y_{m3} & & & & \\ & \frac{\mu_2}{\mu_3} J'_{m3} & \frac{\mu_2}{\mu_3} Y'_{m3} & -J'_{m3} & -Y'_{m3} & & & & \\ & & & J_{m4} & Y_{m4} & -J_{m4} & -Y_{m4} & & \\ & & & \frac{\mu_3}{\mu_4} J'_{m4} & \frac{\mu_3}{\mu_4} Y'_{m4} & -J'_{m4} & -Y'_{m4} & & \\ & & & & & \dots & \dots & \dots & \dots \\ & & & & & \dots & \dots & \dots & \dots \\ & & & & & & & J'_{m(N_\Gamma+1)} & Y'_{m(N_\Gamma+1)} \end{pmatrix}, \tag{29c}$$

and

$$\underline{B}^{\mathrm{Dir}}_{N_\Gamma}(\nu) := \begin{pmatrix} Y_{m2} & -J_{m2} & -Y_{m2} & & & & & & \\ \frac{\epsilon_1}{\epsilon_2} Y'_{m2} & -J'_{m2} & -Y'_{m2} & & & & & & \\ & J_{m3} & Y_{m3} & -J_{m3} & -Y_{m3} & & & & \\ & \frac{\epsilon_2}{\epsilon_3} J'_{m3} & \frac{\epsilon_2}{\epsilon_3} Y'_{m3} & -J'_{m3} & -Y'_{m3} & & & & \\ & & & J_{m4} & Y_{m4} & -J_{m4} & -Y_{m4} & & \\ & & & \frac{\epsilon_3}{\epsilon_4} J'_{m4} & \frac{\epsilon_3}{\epsilon_4} Y'_{m4} & -J'_{m4} & -Y'_{m4} & & \\ & & & & & \dots & \dots & \dots & \dots \\ & & & & & \dots & \dots & \dots & \dots \\ & & & & & & & J_{m(N_\Gamma+1)} & Y_{m(N_\Gamma+1)} \end{pmatrix}, \tag{29d}$$

which are obtained from $\underline{A}^{\mathrm{Neu}}_{N_\Gamma}(\nu)$ and $\underline{A}^{\mathrm{Dir}}_{N_\Gamma}(\nu)$ respectively by a subsequent modification of the first column. Note that we have no intrinsic interest in $\underline{B}^{\mathrm{Neu}}_{N_\Gamma}(\nu)$ and $\underline{B}^{\mathrm{Dir}}_{N_\Gamma}(\nu)$, but we have to include them into the statement of the forthcoming Lemma 4.2 for technical reasons.

Lemma 4.2. *Let $p := r_2/r_{N_\Gamma+1}$. There exist $C_{4.2} > 0$ and $f_m^{\mathrm{Dir}}, f_m^{\mathrm{Neu}} \in C(\mathbb{R}^+)$ such that*

$$\begin{aligned}\Pi_{l=2}^{N_\Gamma}\Big(\frac{\pi z r_l}{2}\Big)\det \underline{A}_{N_\Gamma}^{\mathrm{Dir}}(z) &= J_m(zr_{N_\Gamma+1}) + f_m^{\mathrm{Dir}}(z),\\ |f_m^{\mathrm{Dir}}(z)| &\le C_{4.2}\delta_{\epsilon,\mu}\max\{|J_m(z)|,|Y_m(z)|\},\\ |z^{N_\Gamma}J_m(zq)J_m'(zq)\det \underline{B}_{N_\Gamma}^{\mathrm{Dir}}(z)| &\le C_{4.2}\max\{|J_m(z)|,|Y_m(z)|\},\end{aligned} \tag{30a}$$

and

$$\begin{aligned}\Pi_{l=2}^{N_\Gamma}\Big(\frac{\pi z r_l}{2}\Big)\det \underline{A}_{N_\Gamma}^{\mathrm{Neu}}(z) &= J_m(zr_{N_\Gamma+1}) + f_m^{\mathrm{Neu}}(z),\\ |f_m^{\mathrm{Neu}}(z)| &\le C_{4.2}\delta_{\epsilon,\mu}\max\{|J_m'(z)|,|Y_m'(z)|\},\\ |z^{N_\Gamma}J_m(zq)J_m'(zq)\det \underline{B}_{N_\Gamma}^{\mathrm{Neu}}(z)| &\le C_{4.2}\max\{|J_m'(z)|,|Y_m'(z)|\},\end{aligned} \tag{30b}$$

for all $q \in [pr_2, r_2]$, $z > 0$ and $m \in \mathbb{N}_0$.

Proof. We only present the proof for (30a). To improve the readability we omit the super indices "Dir". We are going to prove the claim by induction in N_Γ. For $N_\Gamma = 2$ we compute that

$$\det \underline{A}_2 = \frac{2}{\pi z r_2}J_{m3} + (1-\frac{\epsilon_1}{\epsilon_2})J_{m2}'Y_{m2}J_{m3} + (\frac{\epsilon_1}{\epsilon_2}-1)J_{m2}J_{m2}'Y_{m3}$$

and

$$\det \underline{B}_2 = \frac{2}{\pi z r_2}Y_{m3} + (\frac{\epsilon_1}{\epsilon_2}-1)J_{m2}Y_{m2}'Y_{m3} + (1-\frac{\epsilon_1}{\epsilon_2})Y_{m2}Y_{m2}'J_{m3},$$

for which the claim follows with Lemmas 3.6 and 3.7 and the Wronskian identity $J_m(z)Y_m'(z) - J_m'(z)Y_m(z) = 2/(\pi z)$ [16, Eq. 10.5.2]. Let now (30a) be satisfied for $N_\Gamma - 1 \ge 2$. We refer to the $(2N_\Gamma - 1)\times(2N_\Gamma - 1)$ matrices build by the formulas (29b), (29d) with the evaluation points $r_3, \dots, r_{N_\Gamma+2}$ (instead of $r_2, \dots, r_{N_\Gamma+1}$) as $\tilde{\underline{A}}_{N_\Gamma}$ and $\tilde{\underline{B}}_{N_\Gamma}$ respectively. We compute that

$$\det \underline{A}_{N_\Gamma+1} = (\frac{\epsilon_1}{\epsilon_2}-1)J_{m2}J_{m2}'\det\tilde{\underline{B}}_{N_\Gamma}(z) + \frac{2}{\pi z r_2}\det\tilde{\underline{A}}_{N_\Gamma}(z) + (1-\frac{\epsilon_1}{\epsilon_2})J_{m2}'Y_{m2}\det\tilde{\underline{A}}_{N_\Gamma}(z)$$

and

$$\det \underline{B}_{N_\Gamma+1} = (1-\frac{\epsilon_1}{\epsilon_2})Y_{m2}Y_{m2}'\det\tilde{\underline{A}}_{N_\Gamma}(z) + \frac{2}{\pi z r_2}\det\tilde{\underline{B}}_{N_\Gamma}(z) + (\frac{\epsilon_1}{\epsilon_2}-1)J_{m2}Y_{m2}'\det\tilde{\underline{B}}_{N_\Gamma}(z).$$

To prove the induction step for $\underline{A}_{N_\Gamma+1}$ we compute that

$$\begin{aligned}\Pi_{l=2}^{N_\Gamma+1}\Big(\frac{\pi z r_l}{2}\Big)\det\underline{A}_{N_\Gamma+1}(z) - J_m(zr_{N_\Gamma+1}) &= \Pi_{l=2}^{N_\Gamma+1}\Big(\frac{\pi z r_l}{2}\Big)\Big(\det\underline{A}_{N_\Gamma+1}(z) - \frac{2}{\pi z r_2}\det\underline{A}_{N_\Gamma}(z)\Big)\\ &= \Pi_{l=2}^{N_\Gamma+1}\Big(\frac{\pi z r_l}{2}\Big)\Big((\frac{\epsilon_1}{\epsilon_2}-1)J_{m2}J_{m2}'\det\tilde{\underline{B}}_{N_\Gamma}(z) + (1-\frac{\epsilon_1}{\epsilon_2})J_{m2}'Y_{m2}\det\tilde{\underline{A}}_{N_\Gamma}(z)\Big) =: f_m(z)\end{aligned}$$

and estimate

$$\begin{aligned}|f_m(z)| \le{}& 4\epsilon_0^{-1}\Big(\frac{\pi r_{N_\Gamma+1}}{2}\Big)^{N_\Gamma}\delta_{\epsilon,\mu}|z^{N_\Gamma}J_m(z)J_m'(z)\det\tilde{\underline{B}}_{N_\Gamma}(z)|\\ &+ 2\epsilon_0^{-1}\pi\delta_{\epsilon,\mu}|zr_2J_m'(zr_2)Y_m(zr_2)||\Pi_{l=3}^{N_\Gamma+1}\Big(\frac{\pi z r_l}{2}\Big)\det\tilde{\underline{A}}_{N_\Gamma}(z)|.\end{aligned}$$

The claim follows now by applying Lemma 3.6 and the induction assumption. The prove the induction step for $\underline{B}_{N_\Gamma+1}$ we compute that

$$\begin{aligned}&|z^{N_\Gamma+1}J_m(zq)J_m'(zq)\det\underline{B}_{N_\Gamma+1}(z)|\\ &\le \frac{4\delta_{\epsilon,\mu}}{\epsilon_0}\Big(\frac{\pi r_2}{2}\Big)^{-N_\Gamma+1}|z^2J_m(zq)J_m'(zq)Y_m(zr_2)Y_m'(zr_2)||\Pi_{l=2}^{N_\Gamma}\Big(\frac{\pi z r_l}{2}\Big)\det\tilde{\underline{A}}_{N_\Gamma}(z)|\\ &+ \frac{2}{\pi r_2}|z^{N_\Gamma}J_m(zq)J_m'(zq)\det\tilde{\underline{B}}_{N_\Gamma}(z)| + \frac{4\delta_{\epsilon,\mu}}{\epsilon_0}|zJ_m(zr_2)Y_m'(zr_2)||z^{N_\Gamma}J_m(zq)J_m'(zq)\det\tilde{\underline{B}}_{N_\Gamma}(z)|.\end{aligned}$$

Note that $q \in [pr_2, r_2] \subset [p\frac{r_2}{r_3}r_3, r_3]$ (which is the resonon we introduced the N_Γ independent constant p). The claim follows now by applying Lemmas 3.6 to 3.8, (17) and the induction assumption. Thus the proof is finished. □

4.4 Eigenvalue gaps

Note that the eigenvalues $\nu > 0$ of (28a) and (28b) are precisely the positive roots of $\det \underline{A}^{\mathrm{Neu}}_{N_\Gamma}(\cdot)$ and $\det \underline{A}^{\mathrm{Dir}}_{N_\Gamma}(\cdot)$ respectively. For each $m \in \mathbb{N}_0$ let $(\tilde{j}_{n,m})_{n\in\mathbb{N}}$ be the non-decreasing sequence of the roots of $\tilde{J}_m(z) := \Pi_{l=2}^{N_\Gamma}\Big(\frac{\pi z r_l}{2}\Big) \det \underline{A}^{\mathrm{Dir}}_{N_\Gamma}(z)$, $(\tilde{j}'_{n,m})_{n\in\mathbb{N}}$ be the non-decreasing sequence of the roots of $\tilde{J}'_m(z) := \Pi_{l=2}^{N_\Gamma}\Big(\frac{\pi z r_l}{2}\Big) \det \underline{A}^{\mathrm{Neu}}_{N_\Gamma}(z)$, and $(\hat{\tilde{j}}_{n,m})_{n\in\mathbb{N}}$ be the compound sequence of the former two ordered in a non-decreasing way. Our objective is to prove a gap estimate for $(\hat{\tilde{j}}_{n,m})_{n\in\mathbb{N}}$ being uniform in $m \in \mathbb{N}_0$ similar to the one stated in Proposition 3.4 for $(\hat{j}_{n,m})_{n\in\mathbb{N}}$. The claim would follow easily, if $\hat{\tilde{j}}_{n,m}$ would converge to $\hat{j}_{n,m}$ uniformly in $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$ as $\delta_{\epsilon,\mu} \to 0$. However, this uniform convergence is actually not true. Instead we have a uniform convergence in a distorted sense, as stated in Lemma 4.4, which suffices to establish Proposition 4.5. As preparation we recall the following result from [24].

Lemma 4.3 (Thm. 1 of [24])**.** *In the interval $[b-s, b+s]$, suppose $\phi(x) = \psi(x)+\tau(x)$, where $\phi(x)$ is continuous, $\psi(x)$ is differentiable, $\psi(b) = 0$, $\min|\psi'(x)| > 0$, and $\max|\tau(x)| < \min\{|\psi(b-s)|, |\psi(b+s)|\}$. Then there exists a zero $c \in [b-s, b+s]$ of $\phi(x)$ such that $|c-b| \le \max|\tau(x)|/\min|\psi'(x)|$.*

Recall that the phase function $\mathcal{B}_m$ and the negative roots $(\mathfrak{a}_n)_{n\in\mathbb{N}}$, $(\mathfrak{a}'_n)_{n\in\mathbb{N}}$ of $\mathrm{Ai}(-\cdot)$ and $\mathrm{Ai}'(-\cdot)$ had been introduced in Section 3. In addition, let

$$q_{n,m} := \omega_m - \frac{\pi}{2} + \pi n = \Big(n + \frac{m}{2} - \frac{1}{4}\Big)\pi, \quad n \in \mathbb{N}, m \in \mathbb{N}_0.$$

Lemma 4.4. *For each $\eta > 0$ there exist $t_* > 1, \delta_0 > 0$ such that if $\delta_{\epsilon,\mu} < \delta_0$, then*

$$|\hat{j}_{n,0} - \hat{\tilde{j}}_{n,0}| < \frac{\eta}{2} \qquad \forall n \in \mathbb{N}, \tag{31a}$$

$$|q_{n,m} - \mathcal{B}_m(\hat{j}_{n,m})|, |q_{n,m} - \mathcal{B}_m(\hat{\tilde{j}}_{n,m})| < \frac{\eta}{2} \qquad \forall n \in \mathbb{N}, m \in \mathbb{N}\colon \quad \hat{j}_{n,m} > m + m^{1/3}t_*. \tag{31b}$$

For each $t_* > 1$, $C_{\mathrm{ol}} > 1$, $\tilde{\eta} > 0$ there exist $m_* > 1, \tilde{\delta}_0 > 0$ such that if $\delta_{\epsilon,\mu} < \tilde{\delta}_0$, then

$$\Big|\frac{\hat{\mathfrak{a}}_n}{2^{1/3}} - \frac{\hat{j}_{n,m} - m}{m^{1/3}}\Big|, \Big|\frac{\hat{\mathfrak{a}}_n}{2^{1/3}} - \frac{\hat{\tilde{j}}_{n,m} - m}{m^{1/3}}\Big| < \frac{\tilde{\eta}}{2} \quad \forall n \in \mathbb{N}, m > m_*\colon \quad \hat{j}_{n,m} < m + m^{1/3}t_*C_{\mathrm{ol}}, \tag{31c}$$

$$|\hat{j}_{n,m} - \hat{\tilde{j}}_{n,m}| < \frac{\tilde{\eta}}{2} \quad \forall n \in \mathbb{N}, m \le m_*\colon \quad \hat{j}_{n,m} < m + m^{1/3}t_*C_{\mathrm{ol}}. \tag{31d}$$

Proof. We only present the proof for the Dirichlet eigenvalues $\tilde{j}_{n,m}$. Our main approach to obtain uniform estimates is to apply Lemma 4.3 to treat all but a finite number of eigenvalues. We derive the asymptotic values $q_{n,m}$, bound $|j_{n,m} - q_{n,m}|$ and $|\tilde{j}_{n,m} - q_{n,m}|$ with the same technique and occasionally estimate $|j_{n,m} - \tilde{j}_{n,m}| \le |j_{n,m} - q_{n,m}| + |\tilde{j}_{n,m} - q_{n,m}|$. The uniform convergence of the remaining eigenvalues then follows due to their finite number.

1. case: $m = 0$. Recall the existence of $C_0 > 0$ such that $|J_0(z) - \big(\frac{2}{\pi z}\big)^{1/2}\cos(z - \omega_0)| \le \frac{C_0}{z}$ for all $z > 1$ (which follows, e.g., from [16, Eq. 10.17.13]). We apply Lemma 4.3 simultaneously with

$$\begin{aligned}
\psi(z) &= \sqrt{2/\pi}\cos(z - \omega_0), & \tilde{\psi}(z) &= \psi(z),\\
\tau(z) &= z^{1/2}J_0(z) - \psi(z), & \tilde{\tau}(z) &= z^{1/2}J_0(z) - \psi(z) + z^{1/2}f_0^{\mathrm{Dir}}(z),\\
b &= q_{n,0},\ n \in \mathbb{N}_0, & \tilde{b} &= b,\\
s &\in (0, \min\{c_{\mathrm{gap}}/2, \pi/8\}), & \tilde{s} &= s,
\end{aligned}$$

where f_0^{Dir} is as in Lemma 4.2 and c_{gap} is as in Proposition 3.4. Note that $|\psi(q_{n,0}\pm s)| = \cos(\pi/2+s)$ and $|\psi'(x)| \ge \sin(\pi/2+s)$, $x \in [q_{n,0} - s, q_{n,0} + s]$. Furthermore

$$|\tau(z)|, |\tilde{\tau}(z)| \le \frac{C_0}{z^{1/2}} + C_{4.2}\delta_{\epsilon,\mu}\sup_{t>1}\max\{|t^{1/2}J_0(t)|, |t^{1/2}Y_0(t)|\} \quad \text{for all } z > 1.$$

Let $z_0 > 1$, $\delta_1 > 0$ be such that

$$\left(\frac{C_0}{z_0^{1/2}} + C_{4.2}\delta_1 \sup_{t>1} \max\{|t^{1/2}J_0(t)|, |t^{1/2}Y_0(t)|\}\right) < \cos(\pi/2+s),$$
$$\left(\frac{C_0}{z_0^{1/2}} + C_{4.2}\delta_1 \sup_{t>1} \max\{|t^{1/2}J_0(t)|, |t^{1/2}Y_0(t)|\}\right) < \frac{\sin(\pi/2+s)}{4}\eta,$$

and $\delta_{\epsilon,\mu} < \delta_1$ henceforth. Let $n_* := \min\{n \in \mathbb{N} \colon q_{n,0} \geq z_0 + s\}$. Then for each $n \geq n_*$ Lemma 4.3 yields the existence of a root $j_{n,0}^{\mathrm{tmp}} \in [q_{n,0} - \eta/4, q_{n,0} + \eta/4]$ of $J_{n,0}$ and a root $\tilde{j}_{n,0}^{\mathrm{tmp}} \in [q_{n,0} - \eta/4, q_{n,0} + \eta/4]$ of $\tilde{J}_{n,0}$. It also follows that $J_{n,0}, \tilde{J}_{n,0}$ have no roots in $\{z \in \mathbb{R}^+ \colon z \geq z_0 + s\} \setminus \bigcup_{n\geq n_*}[q_{n,0} - s, q_{n,0} + s]$. Due to Proposition 3.4 $j_{n,0}^{\mathrm{tmp}}$ is the only root of $J_{n,0}$ in $[q_{n,0} - s, q_{n,0} + s]$, $n \geq n_*$. (This could alternatively be reasoned with the monotonicity of $J_{n,0}$ in $[q_{n,0} - s, q_{n,0} + s]$ obtained by an asymptotic approximation of $J'_{n,0}$.) Assume that $\tilde{J}_{n,0}$ admits more than one root in $[q_{n,0} - s, q_{n,0} + s]$. Since these roots are eigenvalues they are continuous with respect to $\epsilon \in L^\infty$ (see, e.g., [29, Thm. 2, Thm. 3] or [30]) and hence remain in $[q_{n,0} - s, q_{n,0} + s]$ as $\delta_{\epsilon,\mu} \to 0$. Note that the eigenvalues $\tilde{j}_{n,0}$ of [(28), $m = 0$] converge locally to the eigenvalues $j_{n,0}$ of [(28), $m = 0$, $\epsilon = \epsilon_0$] as $\delta_{\epsilon,\mu} \to 0$. Thus as $\delta_{\epsilon,\mu} \to 0$ the only possible limit for roots of $\tilde{J}_{n,0}$ in $[q_{n,0} - s, q_{n,0} + s]$ is $j_{n,0}^{\mathrm{tmp}}$. However, this would lead to a contradiction on the multiplicity of $j_{n,0}^{\mathrm{tmp}}$. Hence $\tilde{j}_{n,0}^{\mathrm{tmp}}$ is the only root of $\tilde{J}_{n,0}$ in $[q_{n,0} - s, q_{n,0} + s]$, $n \geq n_*$. Let $n_{\mathrm{sh}} := \#\{j_{n,0} \colon n \in \mathbb{N}, j_{n,0} < z_0 + s\}$. Classical eigenvalue perturbation theory yields the existence of $\delta_2 \in (0, \delta_1]$ such that if $\delta_{\epsilon,\mu} < \delta_2$, then [(28), $m = 0$] admits exactly n_{sh} eigenvalues in $(0, z_0 + s)$. Henceforth let $\delta_{\epsilon,\mu} < \delta_2$. The known asymptotic behavior of $j_{n,0}$ [16, Eq. 10.21.19, 10.21.20] then yields that $j_{n,0}^{\mathrm{tmp}} = j_{n,0}$ and $\tilde{j}_{n,0}^{\mathrm{tmp}} = \tilde{j}_{n,0}$ for $n \geq n_*$, i.e., there is no index shift in n. The triangle inequality yields that $j_{n,0} - \tilde{j}_{n,0} < \eta/2$ for $n \geq n_*$. Now classical eigenvalue perturbation theory allows us to choose $\delta_3 \in (0, \delta_2]$ such that $|j_{n,0} - \tilde{j}_{n,0}| < \eta/2$ for all $n = 1, \ldots, n_{\mathrm{sh}} = n_* - 1$, $\delta_{\epsilon,\mu} < \delta_3$. Hence the proof of the first case is finished.

2. case: $n \in \mathbb{N}, m \in \mathbb{N} \colon j_{n,m} > m + m^{1/3}t_*$. We repeat the approach used for the previous case. We apply Lemma 4.3 simultaneously with

$$\begin{aligned}
&\psi(x) = \tilde{\psi}(x) = \sqrt{2/\pi}\cos(x - \omega_m),\\
&\tau(x) = \left((\mathcal{B}_m^{-1}(x))^2 - \mu_m\right)^{1/4} J_m \circ \mathcal{B}_m^{-1}(x) - \psi(x),\\
&\tilde{\tau}(x) = \left((\mathcal{B}_m^{-1}(x))^2 - \mu_m\right)^{1/4} J_m \circ \mathcal{B}_m^{-1}(x) - \psi(x) + \left((\mathcal{B}_m^{-1}(x))^2 - \mu_m\right)^{1/4} f_m^{\mathrm{Dir}} \circ \mathcal{B}_m^{-1}(x),\\
&\quad b = \tilde{b} = q_{n,m},\ n \in \mathbb{N},\\
&\quad s = \tilde{s} = \frac{\pi}{8}.
\end{aligned}$$

Lemmas 3.3 and 4.2 yield that

$$|\tau(x)|, |\tilde{\tau}(x)| \leq \frac{13}{12\sqrt{2\pi}t_{1*}^{3/2}}(1 + C_{4.2}\delta_{\epsilon,\mu}) + \sqrt{\frac{2}{\pi}}C_{4.2}\delta_{\epsilon,\mu} \quad \text{for all } x > m + m^{1/3}t_{1*},\ t_{1*} > 1.$$

We choose $t_{1*} > 1$ and $\delta_4 \in (0, \delta_3]$ such that

$$\frac{13}{12\sqrt{2\pi}t_{1*}^{3/2}}(1 + C_{4.2}\delta_4) + \sqrt{\frac{2}{\pi}}C_{4.2}\delta_4 < \cos(\pi 5/8),$$
$$\frac{13}{12\sqrt{2\pi}t_{1*}^{3/2}}(1 + C_{4.2}\delta_4) + \sqrt{\frac{2}{\pi}}C_{4.2}\delta_4 < \frac{\sin(\pi 5/8)\eta}{2},$$

and assume $\delta_{\epsilon,\mu} < \delta_4$ henceforth. Let $n_* > 0$ be such that $q_{n_*,m} - s \geq \mathcal{B}_m(m + m^{1/3}t_{1*})$. Thence for each $n \geq n_*$ Lemma 4.3 yields that $J_m \circ \mathcal{B}_m^{-1}$ and $\tilde{J}_m \circ \mathcal{B}_m^{-1}$ admit roots $x_{n,m}^{\mathrm{tmp}}, \tilde{x}_{n,m}^{\mathrm{tmp}} \in [q_{n,m} - s, q_{n,m} + s]$ which satisfy $|q_{n,m} - x_{n,m}^{\mathrm{tmp}}|, |q_{n,m} - \tilde{x}_{n,m}^{\mathrm{tmp}}| < \eta$. It also follows that $J_m \circ \mathcal{B}_m^{-1}, \tilde{J}_m \circ \mathcal{B}_m^{-1}$

have no roots in $(q_{n_*,m} - s, +\infty) \setminus \bigcup_{n \geq n_*} [q_{n,m} - s, q_{n,m} + s]$. It can be deduced with Lemma 3.3 that $J_m \circ \mathcal{B}_m^{-1}$ is monotone on $[q_{n,m} - s, q_{n,m} + s]$ and hence $x_{n,m}^{\mathrm{tmp}}$ is the only root of $J_m \circ \mathcal{B}_m^{-1}$ in $[q_{n,m} - s, q_{n,m} + s]$. (For the Neumann case we can use (21) and have to adjust t_*, δ_4.) As applied in the previous case, a continuity and convergence argument for $\delta_{\epsilon,\mu} \to 0$ shows that also $\tilde{x}_{n,m}^{\mathrm{tmp}}$ is the only root of $\tilde{J}_m \circ \mathcal{B}_m^{-1}$ in $[q_{n,m} - s, q_{n,m} + s]$. Thus all roots of $J_m \circ \mathcal{B}_m^{-1}$, $\tilde{J}_m \circ \mathcal{B}_m^{-1}$ in $(q_{n_*,m} - s, +\infty)$ are given by $x_{n,m}^{\mathrm{tmp}}$ and $\tilde{x}_{n,m}^{\mathrm{tmp}}$, $n \geq n_*$ respectively. The known asymptotic behavior of $j_{n,m}$ [16, Eq. 10.21.19, 10.21.20] then yields that $x_{n,m}^{\mathrm{tmp}} = \mathcal{B}_m(j_{n,m})$ and $\tilde{x}_{n,m}^{\mathrm{tmp}} = \mathcal{B}_m(\tilde{j}_{n,m})$ for $n \geq n_*$, i.e., there is no index shift in n. It remains to compute that

$$
\begin{aligned}
\mathcal{B}_m^{-1}(q_{n_*,m} - s) &= \mathcal{B}_m^{-1}(q_{n_*,m} - s) - \mathcal{B}_m^{-1} \circ \mathcal{B}_m(m + m^{1/3} t_{1*}) + m + m^{1/3} t_{1*} \\
&= \int_{\mathcal{B}_m(m + m^{1/3} t_{1*})}^{q_{n_*,m} - s} (\mathcal{B}_m^{-1})'(z)\, dz + m + m^{1/3} t_{1*} \\
&\leq \int_{\mathcal{B}_m(m + m^{1/3} t_{1*})}^{\mathcal{B}_m(m + m^{1/3} t_{1*}) + \pi} \frac{1}{\mathcal{B}_m' \circ \mathcal{B}_m^{-1}(z)}\, dz + m + m^{1/3} t_{1*} \\
&\leq \frac{\pi}{\mathcal{B}_m'(m + m^{1/3} t_{1*})} + m + m^{1/3} t_{1*} \\
&\leq \frac{\pi(m + m^{1/3} t_{1*})}{m^{2/3} t_{1*}^{1/2}} + m + m^{1/3} t_{1*} \leq 2\pi t_{1*}^{1/2} m^{1/3} + m + m^{1/3} t_{1*}.
\end{aligned}
$$

Hence the claim for the second case follows with $t_* := t_{1*} + 2\pi t_{1*}^{1/2}$.

3. case: $n \in \mathbb{N}, m > m_* \colon j_{n,m} < m + m^{1/3} t_* C_{\mathrm{ol}}$. We repeat the approach used for the previous case. Let $n_* := \max\{n \in \mathbb{N} \colon \boldsymbol{a}_n < t_* C_{\mathrm{ol}}\}$, $\hat{n}_* := \max\{n \in \mathbb{N} \colon \hat{\boldsymbol{a}}_n \leq t_* C_{\mathrm{ol}}\}$, and $c_{\mathrm{Ai,gap}} := \min\{\hat{\boldsymbol{a}}_1, \min\{\hat{\boldsymbol{a}}_{n+1} - \hat{\boldsymbol{a}}_n \colon n = 1, \dots, \hat{n}_* - 1\}\}$. We apply Lemma 4.3 simultaneously with

$$
\begin{aligned}
\psi(t) &= \tilde{\psi}(t) = 2^{1/3} \operatorname{Ai}(-2^{1/3} t), \\
\tau(t) &= m^{1/3} J_m(m + m^{1/3} t) - \psi(x), \\
\tilde{\tau}(x) &= m^{1/3} J_m(m + m^{1/3} t) - \psi(x) + m^{1/3} f_m^{\mathrm{Dir}}(m + m^{1/3} t), \\
b &= \tilde{b} = 2^{-1/3} \boldsymbol{a}_n, \ n = 1, \dots, n_*, \\
s &= \tilde{s} = \min\Big\{ \frac{c_{\mathrm{Ai,gap}}}{2^{1/3} 4}, \frac{t_* C_{\mathrm{ol}} - \boldsymbol{a}_{n_*}}{2^{1/3}} \Big\}.
\end{aligned}
$$

Lemmas 3.1 and 4.2 yield that

$$
|\tau(t)|, |\tilde{\tau}(t)| \leq \frac{C_{[0,t_* C_{\mathrm{ol}}]}}{m^{2/3}} + C_{4.2} \delta_{\epsilon,\mu} \sup_{\tilde{t} \in [0, t_* C_{\mathrm{ol}}]} \max\{|2^{1/3} \operatorname{Ai}(-2^{1/3} \tilde{t})|, |2^{1/3} \operatorname{Bi}(-2^{1/3} \tilde{t})|\} \ \forall t \in [0, t_* C_{\mathrm{ol}}].
$$

We choose $m_* > 1$ and $\tilde{\delta}_1 > 0$ such that

$$
\begin{aligned}
&\frac{C_{[0,t_* C_{\mathrm{ol}}]}}{m^{2/3}} < \min\{\psi(2^{-1/3} \boldsymbol{a}_n \pm s) \colon n = 1, \dots, n_*\}, \\
&\frac{C_{[0,t_* C_{\mathrm{ol}}]}}{m^{2/3}} < \frac{\tilde{\eta}}{2} \min\{\psi'(t) \colon t \in [2^{-1/3} \boldsymbol{a}_n - s, 2^{-1/3} \boldsymbol{a}_n + s] \colon n = 1, \dots, n_*\}.
\end{aligned}
$$

and assume $m > m_*$, $\delta_{\epsilon,\mu} < \tilde{\delta}_1$ henceforth. Thence for each $n \leq n_*$ Lemma 4.3 yields that $\psi + \tau$ and $\tilde{\psi} + \tilde{\tau}$ admit roots $t_{n,m}, \tilde{t}_{n,m} \in [2^{-1/3} \boldsymbol{a}_n - s, 2^{-1/3} \boldsymbol{a}_n + s]$ which satisfy $|2^{-1/3} \boldsymbol{a}_n - t_{n,m}|, |2^{-1/3} \boldsymbol{a}_n - \tilde{t}_{n,m}| < \tilde{\eta}/2$. By arguing the same way as in the previous cases we obtain that $t_{n,m}$ and $\tilde{t}_{n,m}$, $n = 1, \dots, n_*$ are the only roots of $\psi + \tau$ and $\tilde{\psi} + \tilde{\tau}$ in $[0, t_* C_{\mathrm{ol}}]$. The claim follows now from $j_{n,m} = m + m^{1/3} t_{n,m}$ and $\tilde{j}_{n,m} = m + m^{1/3} \tilde{t}_{n,m}$ for $n = 1, \dots, n_*$.

4. case: $n \in \mathbb{N}, m \leq m_* \colon j_{n,m} \leq m + m^{1/3} t_*$. The index set for (n, m) is finite and thus conventional eigenvalue perturbation theory yields the existence of $\tilde{\delta}_2 \in (0, \tilde{\delta}_1]$ such that if $\delta_{\epsilon,\mu} < \tilde{\delta}_2$, then $|j_{n,m} - \tilde{j}_{n,m}| < \tilde{\eta}/2$. $\square$

Proposition 4.5. *There exist constants $\tilde{c}_{\text{gap}}, \delta_0 > 0$ such that if $\delta_{\epsilon,\mu} < \delta_0$, then*

$$|\hat{\hat{j}}_{1,m}| \geq \tilde{c}_{\text{gap}}, \quad |\hat{\hat{j}}_{n+1,m} - \hat{\hat{j}}_{n,m}| \geq \tilde{c}_{\text{gap}}, \quad \textit{for all} \quad \delta \in (0, \delta_0), n \in \mathbb{N}, m \in \mathbb{N}_0.$$

Proof. The claim essentially follows from Lemma 4.4 and the triangle inequality. Let $\eta := \min\{\frac{c_{\text{gap}}}{2}, \frac{\pi}{2}\}$. The first part of Lemma 4.4 allows us to choose $t_* > 1$ and $\delta_1 > 0$ such that if $\delta_{\epsilon,\mu} < \delta_1$, then (31a), (31b) hold. Let $\delta_{\epsilon,\mu} < \delta_1$ henceforth. Thus we estimate

$$|\hat{\hat{j}}_{1,0}| \geq |\hat{j}_{1,0}| - |\hat{j}_{1,0} - \hat{\hat{j}}_{1,0}| \geq c_{\text{gap}} - \eta/2 \geq c_{\text{gap}}/2,$$

and

$$|\hat{\hat{j}}_{n+1,0} - \hat{\hat{j}}_{n,0}| \geq |\hat{j}_{n+1,0} - \hat{j}_{n,0}| - |\hat{j}_{n+1,0} - \hat{\hat{j}}_{n+1,0}| - |\hat{j}_{n,0} - \hat{\hat{j}}_{n,0}| \geq c_{\text{gap}} - \eta \geq c_{\text{gap}}/2.$$

Furthermore,

$$\begin{aligned}\hat{\hat{j}}_{n+1} - \hat{\hat{j}}_{n,m} &\geq \int_{\hat{\hat{j}}_{n,m}}^{\hat{\hat{j}}_{n+1,m}} \mathcal{B}_m'(z)\, dz = \mathcal{B}_m(\hat{\hat{j}}_{n+1,m}) - \mathcal{B}_m(\hat{\hat{j}}_{n,m}) \\ &\geq |q_{n+1,m} - q_{n,m}| - |q_{n+1,m} - \mathcal{B}_m(\hat{j}_{n+1,m})| - |q_{n,m} - \mathcal{B}_m(\hat{j}_{n,m})| \geq \pi - \eta \geq \pi/2\end{aligned}$$

for all $n, m \in \mathbb{N}$ such that $\hat{j}_{n,m} > m + m^{1/3} t_*$. Due to [16, Eqs. (9.9.6), (9.9.8)] we can choose $C_{\text{ol}} > 1$ such that for each $m \in \mathbb{N}$ there exists $n_+ \in \mathbb{N}$ with $m + m^{1/3} t_* < \hat{j}_{n+,m} < m + m^{1/3} t_* C_{\text{ol}}$. Let $c_{\text{Ai,gap}} > 0$ be defined as in the proof (3. case) of Lemma 4.4 and $\tilde{\eta} := \min\{2^{-1/3} c_{\text{Ai,gap}}/2, c_{\text{gap}}/2\}$. With t_* remaining it's role the second part of Lemma 4.4 allows us to choose $m_* > 1$ and $\delta_2 > 0$ such that if $\delta_{\epsilon,\mu} < \delta_v$, then (31c), (31d) hold. Let $\delta_{\epsilon,\mu} < \delta_v$ henceforth. We estimate that

$$\begin{aligned}\hat{\hat{j}}_{1,m} &\geq 2^{-1/3} m^{1/3} |\hat{\boldsymbol{a}}_1| - m^{1/3} \Big|2^{-1/3} \hat{\boldsymbol{a}}_1 - \frac{\hat{\hat{j}}_{1,m} - m}{m^{1/3}}\Big| \\ &\geq (2^{-1/3} c_{\text{Ai,gap}} - \tilde{\eta}) m^{1/3} \geq 2^{-1/3} c_{\text{Ai,gap}}/2,\end{aligned}$$

and

$$\begin{aligned}&\hat{\hat{j}}_{n+1,m} - \hat{\hat{j}}_{n,m} \\ &\quad \geq 2^{-1/3} m^{1/3} |\hat{\boldsymbol{a}}_{n+1} - \hat{\boldsymbol{a}}_n| - m^{1/3} \Big|2^{-1/3} \hat{\boldsymbol{a}}_{n+1} - \frac{\hat{\hat{j}}_{n+1,m} - m}{m^{1/3}}\Big| - m^{1/3} \Big|2^{-1/3} \hat{\boldsymbol{a}}_n - \frac{\hat{\hat{j}}_{n,m} - m}{m^{1/3}}\Big| \\ &\quad \geq (2^{-1/3} c_{\text{Ai,gap}} - \tilde{\eta}) m^{1/3} \geq 2^{-1/3} c_{\text{Ai,gap}}/2\end{aligned}$$

for all $n \in \mathbb{N}, m > m_*$ such that $\hat{j}_{n,m} < m + m^{1/3} t_* C_{\text{ol}}$. Since for each $m > m_*$ there exists $n_+ \in \mathbb{N}$ with $m + m^{1/3} t_* < \hat{j}_{n+,m} < m + m^{1/3} t_* C_{\text{ol}}$ it follows that

$$|\hat{\hat{j}}_{1,m}|, |\hat{\hat{j}}_{n+1,m} - \hat{\hat{j}}_{n,m}| \geq \min\{\pi/2, 2^{-1/3} c_{\text{Ai,gap}}/2\} \quad \text{for all } n \in \mathbb{N}, m > m_*.$$

It remains to note that

$$|\hat{\hat{j}}_{1,m}| \geq |\hat{j}_{1,m}| - |\hat{j}_{1,m} - \hat{\hat{j}}_{1,m}| \geq c_{\text{gap}} - \eta/2 \geq c_{\text{gap}}/2,$$

and

$$|\hat{\hat{j}}_{n+1,m} - \hat{\hat{j}}_{n,m}| \geq |\hat{j}_{n+1,m} - \hat{j}_{n,m}| - |\hat{j}_{n+1,m} - \hat{\hat{j}}_{n+1,m}| - |\hat{j}_{n,m} - \hat{\hat{j}}_{n,m}| \geq c_{\text{gap}} - \eta \geq c_{\text{gap}}/2$$

for all $n \in \mathbb{N}, m \leq m_*$ such that $\hat{j}_{n,m} < m + m^{1/3} t_* C_{\text{ol}}$. The claim follows now with $\delta_0 := \min\{\delta_1, \delta_2\}$ and $\tilde{c}_{\text{gap}} := \min\{c_{\text{gap}}/2, \pi/2, 2^{-1/3} c_{\text{Ai,gap}}/2\}$. $\square$

4.5 Traces of eigenfunctions

Lemma 4.6. *There exist $C_{\rm vec}, \delta_0 > 0$ such that if $\delta_{\epsilon,\mu} < \delta_0$ and $(\nu, \underline{w})$, $\nu > 0$ is a nontrivial solution to $\underline{A}^{\rm Dir}_{N_\Gamma}(\nu)\underline{w} = 0$ or $(\nu, \underline{v})$, $\nu > 0$ is a nontrivial solution to $\underline{A}^{\rm Neu}_{N_\Gamma}(\nu)\underline{v} = 0$, then $a_{N_\Gamma} \neq 0$ and after normalization of $\underline{w}$ or $\underline{v}$ respectively with $a_{N_\Gamma} := 1$ it holds that $|a_1-1|, |(a_l, b_l)-(1,0)| \leq C_{\rm vec}\delta_{\epsilon,\mu}$ for all $l = 2, \dots, N_\Gamma$.*

Proof. We only present the proof for the Dirichlet case.

1. step: $a_{N_\Gamma}, b_{N_\Gamma}$. Let $\delta_1 \in (0, 1/C_{4.2})$ and $\delta_{\epsilon,\mu} < \delta_1$ henceforth. Recall that $r_{N_\Gamma+1} = 1$. Assume that $|Y_m(\nu)| < |J_m(\nu)|$. Then due to Lemma 4.2 it follows that $|J_m(\nu)| \leq C_{4.2}\delta_{\epsilon,\mu}|J_m(\nu)| < |J_m(\nu)|$, which is a contradiction. Hence it holds that $|Y_m(\nu)| \geq |J_m(\nu)|$ and thus $|b_{N_\Gamma}| \leq C_{4.2}\delta_{\epsilon,\mu}|a_{N_\Gamma}|$. We normalize $a_{N_\Gamma} = 1$ and hence $|(a_{N_\Gamma}, b_{N_\Gamma}) - (1,0)| \leq C_{4.2}\delta_{\epsilon,\mu}$.
2. step: $a_l, b_l, l = 2, \dots, N_\Gamma-1$. Having already obtained $(a_{N_\Gamma}, b_{N_\Gamma})$ we can compute $(a_{N_\Gamma-1}, b_{N_\Gamma-1})$ from $\underline{A}^{\rm Dir}_{N_\Gamma}(\nu)$:

$$\begin{pmatrix} J_{mN_\Gamma} & Y_{mN_\Gamma} \\ \frac{\epsilon_{N_\Gamma-1}}{\epsilon_{N_\Gamma}}J'_{mN_\Gamma} & \frac{\epsilon_{N_\Gamma-1}}{\epsilon_{N_\Gamma}}Y'_{mN_\Gamma} \end{pmatrix} \begin{pmatrix} a_{N_\Gamma-1} \\ b_{N_\Gamma-1} \end{pmatrix} = \begin{pmatrix} J_{mN_\Gamma} & Y_{mN_\Gamma} \\ J'_{mN_\Gamma} & Y'_{mN_\Gamma} \end{pmatrix} \begin{pmatrix} a_{N_\Gamma} \\ b_{N_\Gamma} \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} a_{N_\Gamma-1} \\ b_{N_\Gamma-1} \end{pmatrix} = \frac{\epsilon_{N_\Gamma}}{\epsilon_{N_\Gamma-1}} \begin{pmatrix} a_{N_\Gamma} \\ b_{N_\Gamma} \end{pmatrix} + \frac{\epsilon_{N_\Gamma}}{\epsilon_{N_\Gamma-1}} \Big(\frac{\epsilon_{N_\Gamma-1}}{\epsilon_{N_\Gamma}} - 1\Big)\frac{\pi}{2} \begin{pmatrix} rJ_{mN_\Gamma}Y'_{mN_\Gamma} \\ rJ_{mN_\Gamma}J'_{mN_\Gamma} \end{pmatrix} a_{N_\Gamma}.$$

By the first step, Lemmas 3.6 and 3.7 and induction we obtain the existence of a constant $C_{\rm vec} > 0$ (which is independent of m and ν) such that

$$|(a_l, b_l) - (1,0)| \leq C_{\rm vec}\delta_{\epsilon,\mu} \quad \text{for all } l = 2, \dots, N_\Gamma.$$

3. step: a_1. If $\nu r_2 \in [m, m+m^{1/3}t_*]$, then (18a), (18c) yield the existence of a constant $c_{\rm lb}$ such that $\max\{|r^{1/3}J_{m2}|, |r^{2/3}J'_{m2}|\} > c_{\rm lb}$. If $|r^{1/3}J_{m2}| > c_{\rm lb}$, we can express $a_1 = a_2 + \frac{r^{1/3}Y_{m2}}{r^{1/3}J_{m2}}b_2$ and then (18b), (18d) yield that $|a_1 - 1| \leq C_{\rm vec}\delta_{\epsilon,\mu}$ (with an updated constant $C_{\rm vec}$). The reasoning in the case $|r^{2/3}J'_{m2}| > c_{\rm lb}$ is similar. Like-wise, for $\nu r_2 \in [m + m^{1/3}t_*, +\infty)$ (and large enough t_*) we obtain the claim by means of Lemma 3.3. It remains to discuss the case $\nu r_2 \in (0, m]$. Note that $Y_{m2} \neq 0$ and hence the first row of $\underline{A}^{\rm Dir}_{N_\Gamma}(\nu)\underline{w} = 0$ yields $b_2 = \frac{J_{m2}}{Y_{m2}}(a_1 - a_2)$. Plugging this into the second row of $\underline{A}^{\rm Dir}_{N_\Gamma}(\nu)\underline{w} = 0$ we obtain by means of (17) and Lemma 3.6 that

$$\begin{aligned} |a_1 - a_2| &= |1 - \frac{\epsilon_1}{\epsilon_2}|\Big|\frac{\epsilon_1}{\epsilon_2} - \frac{J_{m2}Y'_{m2}}{J'_{m2}Y_{m2}}\Big|^{-1}|a_2| \leq 4\epsilon_0\delta_{\epsilon,\mu}\Big|\frac{\epsilon_1}{\epsilon_2} - 1 + \frac{2}{\pi}\frac{1}{\nu r_2 J'_{m2}Y_{m2}}\Big|^{-1}|a_2| \\ &\leq 4\epsilon_0\delta_{\epsilon,\mu}\Big|\frac{2}{\pi}\frac{1}{C_{JY}} - 4\epsilon_0\delta_{\epsilon,\mu}\Big|^{-1}|a_2| \leq 4\epsilon_0\pi C_{JY}\delta_{\epsilon,\mu}|a_2| \end{aligned}$$

for all $\delta_{\epsilon,\mu} < \delta_2 := \min\{\delta_1, \frac{1}{4\epsilon_0\pi C_{JY}}\}$, what we assume henceforth. Thus in all cases $|a_1 - 1| \leq C_{\rm vec}\delta_{\epsilon,\mu}$ (with an updated constant $C_{\rm vec}$).

□

Proposition 4.7. *There exist $C_{\rm tr,3}, \delta_0 > 0$ such that if $\delta_{\epsilon,\mu} < \delta_0$, then for each $m \in \mathbb{N}_0$ and each eigenpair $(\tilde{j}'_{n,m}, v_{n,m}) \in \mathbb{R}^+ \times V_m \setminus \{0\}$ to (28a) and $(\tilde{j}_{n,m}, w_{n,m}) \in \mathbb{R}^+ \times W_m \setminus \{0\}$ to (28b) it holds that $\|v_{n,m}\|_\Sigma \leq C_{\rm tr,3}n^{1/6}\|v_{n,m}\|_\Gamma$ and $\|w_{n,m}\|_\Sigma \leq C_{\rm tr,3}n^{1/6}\|w_{n,m}\|_\Gamma$.*

Proof. We start with analyzing the Dirichlet case, i.e., (28b). Recall from Section 4.2 that

$$e^{-im\theta}w_{n,m}|_{(r_1,r_2)}(r) = a_{n1}J_m(\tilde{j}_{n,m}r), \qquad e^{-im\theta}w_{n,m}|_{(r_l,r_{l+1})}(r) = a_{nl}J_m(\tilde{j}_{n,m}r) + b_{nl}Y_m(\tilde{j}_{n,m}r),$$

$l = 2, \ldots, N_\Gamma$ with constants $a_{nl}, b_{nl} \in \mathbb{C}$, and $\underline{w} := (a_{n1}, a_{n2}, b_{n2}, \ldots, a_{nN_\Gamma}, b_{nN_\Gamma})^\top \in \mathbb{C}^{2N_\Gamma - 1}$ solves $\underline{A}_{N_\Gamma}^{\mathrm{Dir}}(\tilde{j}_{n,m})\underline{w} = 0$ with $\underline{A}_{N_\Gamma}^{\mathrm{Dir}}(\tilde{j}_{n,m})$ as defined in (29b). Let $(C_{\mathrm{vec}}, \delta_1)$ be as in Lemma 4.6 and $\delta_{\epsilon,\mu} < \delta_1$ henceforth, such that after a normalization we have $|a_{n1} - 1|, |(a_{nl}, b_{nl}) - (1, 0)| \leq C_{\mathrm{vec}}\delta_{\epsilon,\mu}$, $l = 2, \ldots, N_\Gamma$. We consider $m \in \mathbb{N}$ and will treat the case $m = 0$ separately. We distinguish the three cases $n < c_{\mathrm{ind}}m$, $n \in [c_{\mathrm{ind}}m, C_{\mathrm{ind}}m + 1]$ and $n > C_{\mathrm{ind}}m + 1$, where the constants $C_{\mathrm{ind}} > c_{\mathrm{ind}} > 0$ will be specified in the course of our analysis. To this end let us note that there exist constants $c_{\mathrm{Ai}}, C_{\mathrm{Ai}} > 0$ such that

$$\boldsymbol{a}_n, \boldsymbol{a}'_n \in [c_{\mathrm{Ai}}n^{2/3}, C_{\mathrm{Ai}}n^{2/3}] \quad \text{for all } n \in \mathbb{N},$$

see, e.g., [16, Sect. 9.9(iv)], [41], [32] and note the interlacing of $\boldsymbol{a}_n$ and $\boldsymbol{a}'_n$. Furthermore, recall from [42] that

$$m + \frac{\boldsymbol{a}_n}{2^{1/3}}m^{1/3} < j_{m,n} < m + \frac{\boldsymbol{a}_n}{2^{1/3}}m^{1/3} + \frac{3 \cdot 2^{1/3}}{20}\frac{\boldsymbol{a}_n^2}{m^{1/3}} \quad \text{for all } n, m \in \mathbb{N}. \tag{32}$$

Let $c_{\mathrm{ind}} > 0$ be such that $\big(1 + \frac{C_{\mathrm{Ai}}c_{\mathrm{ind}}^{2(3}}{2^{1/3}} + \frac{3 \cdot 2^{1/3}}{20}C_{\mathrm{Ai}}^2 c_{\mathrm{ind}}^{4/3}\big)r_{N_\Gamma} < 1$ (which is possible because $r_{N_\Gamma} < 1$). We employ the abbreviations $J_{nml} := J_m(\tilde{j}_{n,m}r_l)$ and $Y_{nml} := Y_m(\tilde{j}_{n,m}r_l)$.

1. case: $n < c_{\mathrm{ind}}m$. First, we show with complete induction that there exists a constant $C_{\mathrm{ab}} > 0$ such that

$$|Y_{nml}b_l| \leq C_{\mathrm{ab}}|J_{nml}|\delta_{\epsilon,\mu} \tag{33}$$

for all $l \in \{2, \ldots, N_\Gamma\}$ and $n < c_{\mathrm{ind}}m$, $m \in \mathbb{N}$. For $l = 2$ the first row of $\underline{A}_{N_\Gamma}^{\mathrm{Dir}}(\tilde{j}_{n,m})$ yields $|Y_{nm2}b_2| \leq |J_{nm2}(a_{n2} - a_{n1})| \leq 2C_{\mathrm{vec}}J_{nm2}\delta_{\epsilon,\mu}$. For an index $l > 2$ the $(2l-3)$th row of $\underline{A}_{N_\Gamma}^{\mathrm{Dir}}(\tilde{j}_{n,m})$ yields

$$\begin{aligned} |Y_{nml}b_{nl}| \leq |J_{nml}(a_{nl} - a_{n(l-1)})| + |Y_{nml}b_{n(l-1)}| &\leq 2C_{\mathrm{vec}}J_{nml}\delta_{\epsilon,\mu} + \frac{|Y_{nml}|}{|Y_{nm(l-1)}|}J_{nm(l-1)}C_{\mathrm{ab}}\delta_{\epsilon,\mu} \\ &\leq 2C_{\mathrm{vec}}J_{nml}\delta_{\epsilon,\mu} + \frac{|Y_{nml}|}{|Y_{nm(l-1)}|}\frac{J_{nm(l-1)}}{J_{nml}}J_{nml}C_{\mathrm{ab}}\delta_{\epsilon,\mu} \\ &\leq (2C_{\mathrm{vec}} + C_{\mathrm{ab}})J_{nml}\delta_{\epsilon,\mu}, \end{aligned}$$

due to the induction assumption and the monotonicity of J_m, Y_m in $[0, m]$. Thus (33) is established. For $l = 2, \ldots, N_\Gamma - 1$ we estimate that

$$\begin{aligned} \|w_{n,m}\|_{\Sigma_l} \leq r_l^{1/2}(|J_{nml}a_{nl}| + |Y_{nml}b_{nl}|) &\leq (1 + (C_{\mathrm{ab}} + 1)\delta_{\epsilon,\mu})r_l^{1/2}|J_{nml}| \\ &\leq \frac{(1 + (C_{\mathrm{ab}} + 1)\delta_{\epsilon,\mu})}{r_{l+1} - r_l}\| \cdot^{1/2} J_m(\tilde{j}_{n,m}\cdot)\|_{L^2(r_l, r_{l+1})} \end{aligned} \tag{34}$$

and

$$\begin{aligned} \|w_{n,m}\|_{L^2(\Gamma_l)} &\geq \| \cdot^{1/2} J_m(\tilde{j}_{n,m}\cdot)\|_{L^2(r_l, r_{l+1})}|a_{nl}| - \| \cdot^{1/2} Y_m(\tilde{j}_{n,m}\cdot)\|_{L^2(r_l, r_{l+1})}|b_{nl}| \\ &\geq \| \cdot^{1/2} J_m(\tilde{j}_{n,m}\cdot)\|_{L^2(r_l, r_{l+1})}(1 - \delta_{\epsilon,\mu}) - \| \cdot^{1/2} \|_{L^2(r_l, r_{l+1})}|Y_{nml}||b_{nl}| \\ &\geq \| \cdot^{1/2} J_m(\tilde{j}_{n,m}\cdot)\|_{L^2(r_l, r_{l+1})}(1 - \delta_{\epsilon,\mu}) - \| \cdot^{1/2} \|_{L^2(r_l, r_{l+1})}C_{\mathrm{ab}}\delta_{\epsilon,\mu}|J_{nml}| \\ &\geq \| \cdot^{1/2} J_m(\tilde{j}_{n,m}\cdot)\|_{L^2(r_l, r_{l+1})}(1 - (C_{\mathrm{ab}} + 1)\delta_{\epsilon,\mu}) \end{aligned}$$

due to the monotonicity properties of Bessel functions. Let $\delta_2 \in (0, \delta_1)$ be such that $1 - (C_{\mathrm{ab}} + 1)\delta_2 > 0$ and $\delta_{\epsilon,\mu} \in (0, \delta_2)$ henceforth. Then $\|w_{n,m}\|_{L^2(\Sigma_l)} \leq C\|w_{n,m}\|_{L^2(\Gamma)}$ with $C = \frac{(1+(C_{\mathrm{ab}}+1)\delta_{\epsilon,\mu})}{(1-(C_{\mathrm{ab}}+1)\delta_{\epsilon,\mu})}\frac{1}{r_{l+1}-r_l}$, $l = 2, \ldots, N_\Gamma - 1$. For $l = N_\Gamma$ we estimate identically, but use the integration interval $\Big(r_{N_\Gamma}, 1/\big(1 + \frac{C_{\mathrm{Ai}}c_{\mathrm{ind}}^{2(3}}{2^{1/3}} + \frac{3 \cdot 2^{1/3}}{20}C_{\mathrm{Ai}}^2 c_{\mathrm{ind}}^{4/3}\big)\Big)$ instead of (r_l, r_{l+1}) (to stay in the monotonicity zone). Thus the claim for the first case $n < c_{\mathrm{ind}} m$ follows.

2. case: $n > C_{\mathrm{ind}} m + 1$. We choose $C_{\mathrm{ind}} > 1$ large enough to satisfy several conditions. The first is that

$$\big(1 + \frac{c_{\mathrm{Ai}} C_{\mathrm{ind}}^{2/3}}{2^{1/3}}\big) r_2 > 1. \tag{35a}$$

Second, let $T > 1$ be large enough such that

$$\sqrt{\frac{2}{\pi}} \sin(\pi/4) - \frac{7}{\sqrt{2\pi}} \frac{1}{T^{1/2}} > \frac{1}{\sqrt{\pi}} \sin(\pi/4). \tag{35b}$$

We choose $\delta_3 \in (0, \delta_2)$ and $t_* > T$ according to Lemma 4.4 for $\eta < \pi/2$ with η to be specified later, and demand that

$$\frac{c_{\mathrm{Ai}} C_{\mathrm{ind}}^{2/3}}{2^{1/3}} > t_*, \tag{35c}$$

which guarantees $j_{n,m} > m + m^{1/3} t_*$ (appearing in (31b)). Due to our choice of C_{ind} it follows with Lemma 3.3 that

$$\|w_{n,m}\|_\Sigma \leq \frac{1}{\tilde{j}_{n,m}^{1/2}} \frac{(N_\Gamma - 1)\big(1 + (1 + C_{\mathrm{vec}})\delta_{\epsilon,\mu}\big) r_{N_\Gamma}^{1/2}}{r_2^{1/2}} \frac{\sqrt{\frac{2}{\pi}} + \frac{13}{12\sqrt{2\pi}\Big(\big(1+\frac{c_{\mathrm{Ai}} C_{\mathrm{ind}}^{2/3}}{2^{1/3}}\big) r_2 - 1\Big)^{3/2}}}{\Big(1 - \big(1 + \frac{c_{\mathrm{Ai}} C_{\mathrm{ind}}^{2/3}}{2^{1/3}}\big)^{-2} r_2^{-2}\Big)^{1/4}}.$$

Note that in the forthcoming estimates whenever possible will use the constant c_{ind} (instead of C_{ind}) to reuse the estimates for the third case later on. To estimate $\|w_{n,m}\|_\Gamma$ we note that the techniques applied for the "1. case: $n < c_{\mathrm{ind}} m$" yield that

$$\|w_{n,m}\|_{B_{m/\tilde{j}_{n,m}}} \geq \big(1 - (1 + C_{\mathrm{vec}})\delta_{\epsilon,\mu}\big) \|\cdot^{1/2} J_m(\tilde{j}_{n,m} \cdot)\|_{L^2(0, m/\tilde{j}_{n,m})}.$$

Next, we have that

$$\|w_{n,m}\|_{B_1 \setminus B_{m/\tilde{j}_{n,m}}} \geq (1 - \delta_{\epsilon,\mu}) \|\cdot^{1/2} J_m(\tilde{j}_{n,m} \cdot)\|_{L^2(m/\tilde{j}_{n,m}, 1)} - C_{\mathrm{vec}} \delta_{\epsilon,\mu} \|\cdot^{1/2} Y_m(\tilde{j}_{n,m} \cdot)\|_{L^2(m/\tilde{j}_{n,m}, 1)}.$$

Combining the previous two estimates we obtain that

$$\|w_{n,m}\|_\Gamma^2 \geq \frac{\big(1 - (1 + C_{\mathrm{vec}})\delta_{\epsilon,\mu}\big)^2}{2} \|\cdot^{1/2} J_m(\tilde{j}_{n,m} \cdot)\|_{L^2(0,1)}^2 - C_{\mathrm{vec}}^2 \delta_{\epsilon,\mu}^2 \|\cdot^{1/2} Y_m(\tilde{j}_{n,m} \cdot)\|_{L^2(m/\tilde{j}_{n,m}, 1)}^2.$$

Using [16, Eq. 10.22.5] (together with [16, p. 10.6.2])

$$\begin{aligned}
\int^z r F_m(ar)^2 \, dr &= \frac{z^2}{2} \big(F_m(az)^2 - F_{m-1}(az) F_{m+1}(az)\big) \\
&= \frac{z^2}{2} \Big(F_m(az)^2 - \big(F_m'(az) + \frac{m}{az} F_m(az)\big)\big(-F_m'(az) + \frac{m}{az} F_m(az)\big)\Big), \quad a > 0, \quad F = J, Y,
\end{aligned}$$

we have that

$$\begin{aligned}
&2\|\cdot^{1/2} J_m(\tilde{j}_{n,m} \cdot)\|_{L^2(0,1)}^2 \\
&\quad = 2\big(\|\cdot^{1/2} J_m(j_{n,m} \cdot)\|_{L^2(0,1)}^2 + \|\cdot^{1/2} J_m(\tilde{j}_{n,m} \cdot)\|_{L^2(0,1)}^2 - \|\cdot^{1/2} J_m(j_{n,m} \cdot)\|_{L^2(0,1)}^2\big) \\
&\quad = J_m'(j_{m,n})^2 + \big(1 - \frac{m^2}{\tilde{j}_{n,m}^2}\big) J_m(\tilde{j}_{n,m})^2 + J_m'(\tilde{j}_{m,n})^2 - J_m'(j_{m,n})^2 \\
&\quad = J_m'(j_{m,n})^2 + \big(1 - \frac{m^2}{\tilde{j}_{n,m}^2}\big) \big(J_m(\tilde{j}_{n,m}) - J_m(j_{n,m})\big)\big(J_m(\tilde{j}_{n,m}) + J_m(j_{n,m})\big) \\
&\qquad\qquad + \big(J_m'(\tilde{j}_{m,n}) - J_m'(j_{m,n})\big)\big(J_m'(\tilde{j}_{m,n}) + J_m'(j_{m,n})\big).
\end{aligned}$$

First we estimate $|J_m'(j_{m,n})|$ from below, after which we will show that the perturbation terms are sufficiently small. Applying Lemma 3.3 yields that

$$\begin{aligned}|J_m'(j_{m,n})| &\geq \frac{1}{j_{n,m}^{1/2}}\left(1-\frac{1}{1+c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}2^{-1/3}}\right)\left(\sqrt{\frac{2}{\pi}}\sin(\pi/4)-\frac{7}{\sqrt{2\pi}}\frac{1}{t_*^{1/2}}\right)\\ &\geq \frac{1}{j_{n,m}^{1/2}}\left(1-\frac{1}{1+c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}2^{-1/3}}\right)\frac{1}{\sqrt{\pi}}\sin(\pi/4).\end{aligned}$$

To extract from Lemma 3.3 an m-uniform estimate we have just used that $j_{n,m} > m(1+\frac{c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}}{2^{1/3}})$ with the appearing constant being larger than one. To obtain a similar estimate for $\tilde{j}_{n,m}$ we proceed as follows. Our assumptions ensure that $j_{n-1,m} > m + m^{1/3}t_*$. Hence, using (31b) of Lemma 4.4 we have

$$\tilde{j}_{n,m} - j_{n-1,m} \geq \int_{j_{n-1,m}}^{\tilde{j}_{n,m}} \mathcal{B}_m'(z)\,dx = \mathcal{B}_m(\tilde{j}_{n,m}) - \mathcal{B}_m(j_{n-1,m}) \geq \pi - \eta \geq 0.$$

Now we can estimate

$$\begin{aligned}|J_m(j_{n,m}) - J_m(\tilde{j}_{n,m})| &\leq |j_{n,m}-\tilde{j}_{n,m}| \sup_{z\in[\min\{j_{n,m},\tilde{j}_{n,m}\},\max\{j_{n,m},\tilde{j}_{n,m}\}]} |J_m'(z)|\\ &\leq \eta \sup_{t\in[\min\{\mathcal{B}_m(j_{n,m}),\mathcal{B}_m(\tilde{j}_{n,m})\},\max\{\mathcal{B}_m(j_{n,m}),\mathcal{B}_m(\tilde{j}_{n,m})\}]} |(\mathcal{B}_m^{-1})'(t)| \sup_{z>j_{n-1,m}} |J_m'(z)|\\ &= \eta \sup_{z\in[\min\{j_{n,m},\tilde{j}_{n,m}\},\max\{j_{n,m},\tilde{j}_{n,m}\}]} \frac{z}{\sqrt{z^2-\mu_m}} \sup_{z>j_{n-1,m}} |J_m'(z)|\\ &\leq \eta \frac{j_{n-1,m}}{\sqrt{j_{n-1,m}^2-\mu_m}} \sup_{z>j_{n-1,m}} |J_m'(z)|\\ &\leq \eta \frac{1+c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}2^{-1/3}}{\sqrt{(1+c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}2^{-1/3})^2-1}} \frac{\sqrt{2/\pi}+\frac{7}{\sqrt{2\pi}}\frac{1}{t_*^{1/2}}}{j_{n-1,m}^{1/2}}.\end{aligned}$$

Since the previous obtained estimate contains the factor $j_{n-1,m}^{-1/2}$ instead of $j_{n,m}^{-1/2}$, we compute

$$\begin{aligned}\frac{j_{n,m}}{j_{n-1,m}} = 1 + \frac{j_{n,m}-j_{n-1,m}}{j_{n-1,m}} &\leq 1 + \frac{\mathcal{B}_m(j_{n,m})-\mathcal{B}_m(j_{n-1,m})}{j_{n-1,m}} \frac{j_{n-1,m}}{\sqrt{j_{n-1,m}^2-\mu_m}}\\ &\leq 1 + \frac{3\pi}{2}\frac{1}{\sqrt{(1+c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}2^{-1/3})^2-1}} =: C_{\mathrm{rat}}.\end{aligned}$$

Further, note that Lemma 3.3 yields

$$|J_m(j_{n,m}) + J_m(\tilde{j}_{n,m})| \leq j_{n,m}^{-1/2}\big(1+C_{\mathrm{rat}}^{1/2}\big)\frac{\sqrt{\frac{2}{\pi}}+\frac{13}{12\sqrt{2\pi}t_*^{3/2}}}{\left(1-\big(1+\frac{c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}}{2^{1/3}}\big)^{-2}\right)^{1/4}}.$$

Thus

$$\begin{aligned}&|\big(1-\frac{m^2}{\tilde{j}_{n,m}^2}\big)\big(J_m(\tilde{j}_{n,m})-J_m(j_{n,m})\big)\big(J_m(\tilde{j}_{n,m})+J_m(j_{n,m})\big)|\\ &\leq \frac{\eta}{j_{n,m}} \underbrace{\big(C_{\mathrm{rat}}+C_{\mathrm{rat}}^{3/2}\big)\frac{\sqrt{\frac{2}{\pi}}+\frac{13}{12\sqrt{2\pi}t_*^{3/2}}}{\left(1-\big(1+\frac{c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}}{2^{1/3}}\big)^{-2}\right)^{1/4}} \frac{1+c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}2^{-1/3}}{\sqrt{(1+c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}2^{-1/3})^2-1}}\left(\sqrt{\frac{2}{\pi}}+\frac{7}{\sqrt{2\pi}}\frac{1}{t_*^{1/2}}\right)}_{C_1:=}.\end{aligned}$$

In a similar way and using (21) we are able to obtain that

$$|J_m'(\tilde{j}_{n,m}) + J_m'(j_{n,m})| \le j_{n,m}^{-1/2}\big(1 + C_{\mathrm{rat}}^{1/2}\big)\Big(\sqrt{\frac{2}{\pi}} + \frac{7}{\sqrt{2\pi}t_*^{1/2}}\Big)$$

and

$$\begin{aligned}|J_m'(\tilde{j}_{n,m}) - J_m'(j_{n,m})| &\le \eta \frac{1 + c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}2^{-1/3}}{\sqrt{(1 + c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}2^{-1/3})^2 - 1}} \sup_{z > j_{n-1,m}} |J_m''(z)| \\ &\le \frac{\eta}{j_{n,m}^{1/2}} \frac{1 + c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}2^{-1/3}}{\sqrt{(1 + c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}2^{-1/3})^2 - 1}} C_{\mathrm{rat}}^{1/2}\Big(2\sqrt{\frac{2}{\pi}} + \frac{7}{\sqrt{2\pi}}\frac{1}{t_*^{1/2}} + \frac{13}{12\sqrt{2\pi}t_*^{3/2}}\Big).\end{aligned}$$

Thus

$$|\big(J_m'(\tilde{j}_{n,m}) - J_m'(j_{n,m})\big)\big(J_m'(\tilde{j}_{n,m}) + J_m'(j_{n,m})\big)| \le \frac{\eta}{j_{n,m}} C_2$$

with $C_2 := \big(1 + C_{\mathrm{rat}}^{1/2}\big)\Big(\sqrt{\frac{2}{\pi}} + \frac{7}{\sqrt{2\pi}t_*^{1/2}}\Big)\frac{1 + c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}2^{-1/3}}{\sqrt{(1 + c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}2^{-1/3})^2 - 1}} C_{\mathrm{rat}}^{1/2}\Big(2\sqrt{\frac{2}{\pi}} + \frac{7}{\sqrt{2\pi}}\frac{1}{t_*^{1/2}} + \frac{13}{12\sqrt{2\pi}t_*^{3/2}}\Big)$.
Putting things together we have that

$$2\|\cdot^{1/2} J_m(\tilde{j}_{n,m}\cdot)\|_{L^2(0,1)}^2 \ge \frac{1}{j_{n,m}}\Big(\big(1 - \frac{1}{1 + c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}2^{-1/3}}\big)^2 \frac{\sin^2(\pi/4)}{\pi} - \eta(C_1 + C_2)\Big).$$

Next we estimate

$$\begin{aligned}2\|\cdot^{1/2} Y_m(\tilde{j}_{n,m}\cdot)\|_{L^2(m/\tilde{j}_{n,m},1)}^2 &= (1 - \frac{m^2}{\tilde{j}_{n,m}^2})Y_m(\tilde{j}_{n,m})^2 + Y_m'(\tilde{j}_{n,m})^2 - \frac{m^2}{\tilde{j}_{n,m}^2}Y_m'(m)^2 \\ &\le Y_m(\tilde{j}_{n,m})^2 + Y_m'(\tilde{j}_{n,m})^2 \\ &\le j_{n,m}^{-1}\, C_{\mathrm{rat}} \underbrace{\left(\frac{\Big(\sqrt{\frac{2}{\pi}} + \frac{13}{12\sqrt{2\pi}t_*^{3/2}}\Big)^2}{\Big(1 - \big(1 + \frac{c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}}{2^{1/3}}\big)^{-2}\Big)^{1/2}} + \frac{\Big(\sqrt{\frac{2}{\pi}} + \frac{7}{\sqrt{2\pi}}\frac{1}{t_*^{1/2}}\Big)^2}{\Big(1 - \big(1 + \frac{c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}}{2^{1/3}}\big)^{-2}\Big)^{-1/2}}\right)}_{2C_3:=}\end{aligned}$$

and thus

$$\|w_{n,m}\|_\Gamma^2 \ge j_{n,m}^{-1}\Big(\frac{\big(1 - (1 + C_{\mathrm{vec}})\delta_{\epsilon,\mu}\big)^2}{2}\Big(\big(1 - \frac{1}{1 + c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}2^{-1/3}}\big)^2 \frac{\sin^2(\pi/4)}{\pi} - \eta(C_1 + C_2)\Big) - C_{\mathrm{vec}}^2 C_3 \delta_{\epsilon,\mu}^2\Big).$$

Hence we demand that $\eta < \big(1 - \frac{1}{1 + c_{\mathrm{Ai}}c_{\mathrm{ind}}^{2/3}2^{-1/3}}\big)^2 \frac{\sin^2(\pi/4)}{2\pi(C_1 + C_2)}$ and it follows that we can find $C_4 > 0$ and $\delta_4 \in (0, \delta_3)$ such that $\|w_{n,m}\|_\Gamma^2 \ge C_4 j_{n,m}^{-1}$. Hence we have proven the (more than) sufficient estimate $\|w_{n,m}\|_\Sigma \lesssim \|w_{n,m}\|_\Gamma$.

3. case: $n \in [c_{\mathrm{ind}}m, C_{\mathrm{ind}}m + 1]$. Since $C_{\mathrm{ind}}m + 1 \le C_{\mathrm{ind}}(m + 1) \le 2C_{\mathrm{ind}}m$ it suffices to consider $n \in [c_{\mathrm{ind}}m, 2C_{\mathrm{ind}}m]$. Let $T > 1$ and $t_* > T$ be as in the 2. case, i.e., such that (35b) is satisfied and t_* is chosen according to Lemma 4.4. Let $m_* \in \mathbb{N}$ be such that

$$\frac{c_{\mathrm{Ai}}(c_{\mathrm{ind}}m_*)^{2/3}}{2^{1/3}} > t_*, \tag{36}$$

where (36) replaces the assumption (35c) in the analysis of the 2. case. Hence it follows that $\|w_{n,m}\|_\Gamma^2 \ge \frac{C_4}{j_{n,m}}$ for all $(n, m) \in \{(n, m) \in \mathbb{N}^2 : m \ge m_*, n \in [c_{\mathrm{ind}}m, 2C_{\mathrm{ind}}m]\}$. Note that the analysis (34) of the 1. case shows that

$$\|w_{n,m}\|_{\Sigma_l} \le (1 + (C_{\mathrm{ab}} + 1)\delta_{\epsilon,\mu})|J_m(\tilde{j}_{n,m}r_l)| \quad \text{for} \quad \tilde{j}_{n,m}r_l < m.$$

On the other hand we have that

$$\|w_{n,m}\|_{\Sigma_l} \leq (1 + (C_{\text{vec}} + 1)\delta_{\epsilon,\mu})\big(\sup_{r>m} |J_m(r)| + \sup_{r>m} |Y_m(r)|\big) \quad \text{for} \quad \tilde{j}_{n,m} r_l \geq m.$$

Lemma 3.1 and Lemma 3.3 yield that there exists a constant $C_5 > 0$ such that

$$\sup_{r>0} |J_m(r)| + \sup_{r>m} |Y_m(r)| < \frac{C_5}{m^{1/3}}, \quad \forall m \in \mathbb{N}.$$

It follows that there exist $C_6 > 0$ such that $\|w_{n,m}\|_\Sigma \leq \frac{C_6}{m^{1/3}}$. We conclude that

$$\begin{aligned}
\|w_{n,m}\|_\Gamma \geq \frac{C_4^{1/2}}{j_{n,m}^{1/2}} \geq \frac{C_4^{1/2}}{(1 + c_{\text{Ai}} c_{\text{ind}}^{2/3} 2^{-1/3})^{1/2} m^{1/2}} &\geq \frac{C_4^{1/2} c_{\text{ind}}^{1/6}}{(1 + c_{\text{Ai}} c_{\text{ind}}^{2/3} 2^{-1/3})^{1/2} m^{1/3} n^{1/6}} \\
&\geq \frac{C_4^{1/2} c_{\text{ind}}^{1/6}}{(1 + c_{\text{Ai}} c_{\text{ind}}^{2/3} 2^{-1/3})^{1/2} C_6 n^{1/6}} \|w_{n,m}\|_\Sigma,
\end{aligned}$$

for all $m \geq m_*$. The remaining index set $\{(n,m) \in \mathbb{N}^2 \colon m = 1, \ldots, m_* - 1, n \in [c_{\text{ind}} m, 2C_{\text{ind}} m]\}$ is finite and hence it suffice to choose the constant $C_{\text{tr},3}$ sufficiently large.

The case $m = 0$. For sufficiently large $n_* > 1$ we can use the asymptotic expansion of J_0 and Y_0 [47, p. 206] to proceed for $n > nn_*$ as in the 2. case. For $n = 1, \ldots, n_*$ we choose the constant $C_{\text{tr},3}$ sufficiently large.

Neumann boundary condition. The treatment of this case is quite similar to the Dirichlet case. However, a result for $(\mathbb{a}_n')_{n\in\mathbb{N}}$ corresponding to (32) is not available. Since we exploited (32) only to obtain lower and upper bounds on $j_{n,m}$, it suffices to note that

$$\mathbb{a}_n \inf_{l \geq 2} \frac{\mathbb{a}_{l-1}}{\mathbb{a}_l} \leq \mathbb{a}_{n-1} < \mathbb{a}_n' < \mathbb{a}_n \qquad \text{for } n = 2, 3, \ldots,$$

to obtain lower and upper bounds on $j_{n,m}'$. ☐

Corollary 4.8. *For each $m \in \mathbb{N}_0$ let $(\hat{\tilde{j}}_{n,m}, (\mathbf{v}_{n,m}, w_{n,m})) \in \mathbb{R}^+ \times X_m \setminus \{0\}$ be the eigenpairs to* (15)*. There exist $C_{\text{tr},3}, \delta_0 > 0$ such that if $\delta_{\epsilon,\mu} < \delta_0$, then for all $m \in \mathbb{N}_0$ and each $n \in \mathbb{N}$ either $\mathbf{v}_{n,m} = 0$ or $w_{n,m} = 0$, and in particular it holds that $\|\mathbf{v}_{n,m}\|_\Sigma \leq C_{\text{tr},3} n^{1/6} \|\mathbf{v}_{n,m}\|_\Gamma$ and $\|w_{n,m}\|_\Sigma \leq C_{\text{tr},3} n^{1/6} \|w_{n,m}\|_\Gamma$.*

Proof. Follows from Proposition 4.7 together with Lemma 4.1. ☐

5 The non-selfadjoint perturbation

5.1 Local subordinate perturbations

The main tool in the proof of our forthcoming main Theorem 5.5 is the perturbation theory under a local subordinate condition of Mityagin and Siegl [39]. For the comfort of the reader we provide in the next Lemma 5.1 a summary of the results applied from [39] (accompanied by a slight extension [15, Thm. A.9, Lem. A.10]). In order to stick to the notation of [39], but to avoid a conflict with previously used symbols, we equip all overlapping symbols in the context of [39] with haceks.

Lemma 5.1. *Let $\check{A}$: $\text{dom}(\check{A}) \subset \mathcal{H} \to \mathcal{H}$ be a closed, densely defined, (unbounded,) selfadjoint operator with compact resolvent in a separable, complex Hilbert space $\mathcal{H}$. Let $(\check{\mu}_k, \psi_k)_{k\in\mathbb{N}} \in \mathbb{R} \times \mathcal{H}$ be the sequence of eigenpairs to $\check{A}$ with $\mathcal{H}$-normalized eigenelements:*

$$\check{A}\psi_k = \check{\mu}_k \psi_k, \quad \|\psi_k\|_\mathcal{H} = 1.$$

We assume that the eigenvalues $(\breve{\mu}_k)_{k\in\mathbb{N}}$ *of* $\breve{A}$ *are positive, simple and satisfy the gap condition*

$$\exists \gamma, \kappa > 0\colon \quad \breve{\mu}_{k+1} - \breve{\mu}_k \geq \kappa k^{\gamma-1} \quad \forall k \in \mathbb{N}.$$

Let $\breve{B}\colon \operatorname{dom}(\breve{B}) \subset \operatorname{dom}(\breve{A}) \to \mathcal{H}$ *satisfy the so-called local subordination condition*

$$\exists \breve{\alpha} \in \mathbb{R} \text{ with } \begin{cases} 2\breve{\alpha} + \gamma > \frac{3}{2}, & \text{if } \breve{\alpha} \leq \frac{1}{2}, \\ \gamma > \frac{1}{2}, & \text{if } \breve{\alpha} > \frac{1}{2}, \end{cases} \text{ and } M_{\breve{B}} > 0\colon |\langle \breve{B}\psi_m, \psi_n\rangle_{\mathcal{H}}| \leq \frac{M_{\breve{B}}}{m^{\breve{\alpha}} n^{\breve{\alpha}}} \quad \forall m, n \in \mathbb{N}.$$

Let $(\breve{\lambda}_k, \phi_k) \in \mathbb{C}\times\mathcal{H}$, $k \in \mathbb{N}$ *be the eigenpairs of* $\breve{A}+\breve{B}$ *with* ϕ_k *being defined as in [39, (3.17)] (where admissible). Then there exist* $c_{\gamma,\kappa,\breve{\alpha}}, C_{\gamma,\kappa,\breve{\alpha}} > 0$ *such that if* $M_{\breve{B}} < c_{\gamma,\kappa,\breve{\alpha}}$*, then all eigenvalues* $\breve{\lambda}_k$, $k \in \mathbb{N}$ *are simple, satisfy*

$$|\breve{\mu}_k - \breve{\lambda}_k| \leq M_B \left(\frac{1}{k^{2\breve{\alpha}}} + C_{\gamma,\kappa,\breve{\alpha}} \Big(\frac{\log(ek)}{k^{2\breve{\alpha}+\gamma-1}} \Big)^2 \frac{1}{k^{2\breve{\alpha}}} \right), \qquad \text{if } \breve{\alpha} \leq \frac{1}{2},$$

$$|\breve{\mu}_k - \breve{\lambda}_k| \leq M_B \left(\frac{1}{k^{2\breve{\alpha}}} + C_{\gamma,\kappa,\breve{\alpha}} \frac{1}{k^{2\gamma}} \frac{1}{k^{2\breve{\alpha}}} \right), \qquad \text{if } \breve{\alpha} > \frac{1}{2},$$

the formula [39, (3.17)] is admissible for all $k \in \mathbb{N}$ *and* $(\phi_k)_{k\in\mathbb{N}}$ *forms a Riesz (and Bari) basis in* $\mathcal{H}$ *with*

$$\sum_{k\in\mathbb{N}} \|\psi_k - \phi_k\|_{\mathcal{H}}^2 < C_{\gamma,\kappa,\breve{\alpha}} M_{\breve{B}}.$$

The constants $c_{\gamma,\kappa,\breve{\alpha}}, C_{\gamma,\kappa,\breve{\alpha}}$ *depend only on* $\gamma, \kappa, \breve{\alpha}$*. In addition, let* $\mathcal{X} := \operatorname{dom} \breve{A}^{1/2}$ *with* $\langle\cdot,\cdot\rangle_{\mathcal{X}} := \langle \breve{A}^{1/2}\cdot, \breve{A}^{1/2}\cdot\rangle_{\mathcal{H}}$ *and* $\tilde{\psi}_k := \frac{1}{\breve{\mu}_k^{1/2}}\psi_k$, $\tilde{\phi}_k := \frac{1}{\breve{\mu}_k^{1/2}}\phi_k$*. Then* $(\tilde{\phi}_k)_{k\in\mathbb{N}}$ *forms a Riesz (and Bari) basis in* $\mathcal{X}$ *with*

$$\sum_{k\in\mathbb{N}} \|\tilde{\psi}_k - \tilde{\phi}_k\|_{\mathcal{X}}^2 < C_{\gamma,\kappa,\breve{\alpha}} M_{\breve{B}}.$$

Proof. The statement regarding the eigenvalues and the properties of $(\phi_k)_{k\in\mathbb{N}}$ in $\mathcal{H}$ is nothing else, then a specification of [39]. In particular, a review of the proof of [39, Thm 3.2] reveals the dependency of the appearing estimates on the constants $M_{\breve{B}}$, $c_{\gamma,\kappa,\breve{\alpha}}$ and $C_{\gamma,\kappa,\breve{\alpha}}$. Note that by assumption on $M_{\breve{B}} < c_{\gamma,\kappa,\breve{\alpha}}$ being small enough we have no preasymptotic, non-simple eigenvalues μ_k (characterized by N_0 in [39]). Hence there are no eigenvalues $\breve{\lambda}_k$ with preasymptotic behavior and we can omit the patch Π_0 in [39, Prop. 3.1] around such. Note that for the eigenvalue estimate we apply [39, Thm 3.2] with $j = 1$ and estimate $\lambda_n^{(1)}$ with [39, Rem 3.3]. To estimate $\sum_{k\in\mathbb{N}} \|\psi_k - \phi_k\|_{\mathcal{H}}^2$ we follow [39, Rem 3.5] and apply [39, (3.19)]. The statement regarding properties in $\mathcal{X}$ is taken from [15, Thm. A.9]. □

Lemma 5.2. *Let* Y *be a Hilbert space,* $(\psi_n)_{n\in\mathbb{N}}$ *be an orthonormal basis of* Y *and* $(\phi_n)_{n\in\mathbb{N}}$, $\phi_n \in Y, n \in \mathbb{N}$ *with* $c := \sum_{n\in\mathbb{N}} \|\psi_n - \phi_n\|_Y^2 < 1/2$*. Then* $(\phi_n)_{n\in\mathbb{N}}$ *is a Riesz basis of* Y *with Gram constants* $\frac{1}{2(1+c)}$, $\frac{2}{1-2c}$*, i.e., for each* $u \in Y$ *it holds that* $u = \sum_{n\in\mathbb{N}} \langle u, \phi_n\rangle_Y \phi_n$ *and*

$$\frac{1}{2(1+c)} \sum_{n\in\mathbb{N}} |\langle u, \phi_n\rangle_Y|^2 \leq \|u\|_Y^2 \leq \frac{2}{1-2c} \sum_{n\in\mathbb{N}} |\langle u, \phi_n\rangle_Y|^2.$$

Proof. The claim follows from

$$\|u\|_Y^2 = \sum_{n\in\mathbb{N}} |\langle u, \psi_n\rangle_Y|^2 \leq 2\sum_{n\in\mathbb{N}} |\langle u, \phi_n\rangle_Y|^2 + |\langle u, \psi_n - \phi_n\rangle_Y|^2 \leq 2\sum_{n\in\mathbb{N}} |\langle u, \phi_n\rangle_Y|^2 + 2c\|u\|_Y^2$$

and

$$\sum_{n\in\mathbb{N}} |\langle u, \phi_n\rangle_Y|^2 \leq 2\sum_{n\in\mathbb{N}} |\langle u, \psi_n\rangle_Y|^2 + |\langle u, \psi_n - \phi_n\rangle_Y|^2 \leq 2\|u\|_Y^2 + 2c\|u\|_Y^2 = 2(1+c)\|u\|_Y^2.$$

□

Remark 5.3. *Note that the assumption $\sum_{n\in\mathbb{N}}\|\psi_n-\phi_n\|_Y^2<1/2$ in Lemma 5.2 is not sharp and could be mitigated to $\sum_{n\in\mathbb{N}}\|\psi_n-\phi_n\|_Y^2<1$.*

Lemma 5.4. *Let $\kappa>0$ and $(\lambda_n)_{n\in\mathbb{N}}$, $\lambda_n>0$, $n\in\mathbb{N}$ be a monotone increasing sequence satisfying $\lambda_1\geq\kappa$ and $\lambda_{n+1}-\lambda_n\geq\kappa$ for all $n\in\mathbb{N}$. Then $\lambda_{n+1}^2-\lambda_n^2\geq 2\kappa^2 n$ for all $n\in\mathbb{N}$.*

Proof. Note that $\lambda_{n+1}=\lambda_{n+1}-\lambda_n+\lambda_n\geq\kappa+\lambda_n$ and hence $\lambda_n\geq\kappa n$ for all $n\in\mathbb{N}$. Now we compute that $\lambda_{n+1}^2-\lambda_n^2=(\lambda_{n+1}-\lambda_n)(\lambda_{n+1}+\lambda_n)\geq\kappa(\lambda_n+\lambda_n)\geq 2\kappa^2 n$, which proves the claim. $\square$

5.2 Main results

Theorem 5.5. *Let the assumptions specified in Section 2.1 be satisfied. There exists $\delta_0>0$, such that if $\delta_{\epsilon,\mu}<\delta_0$, then: If $(\alpha,(\mathbf{E}_T,E_3,\mathbf{H}_T,H_3))$ is a solution to (3), then $\alpha\neq 0$ and $(-\alpha,(\mathbf{E}_T,-E_3,-\mathbf{H}_T,H_3))$ is also a solution to (3). So let*

$$(\alpha_k,(\mathbf{E}_{T,k},E_{3,k},\mathbf{H}_{T,k},H_{3,k})),\quad (-\alpha_k,(\mathbf{E}_{T,k},-E_{3,k},-\mathbf{H}_{T,k},H_{3,k})),\quad k\in\mathbb{N}$$

(with $\alpha_k\neq-\alpha_{k'}$ for all $k,k'\in\mathbb{N}$) be all solutions to (3).

I. *Each $\pm\alpha_k$, $k\in\mathbb{N}$ is simple.*

II. *If $\omega\in\mathbb{R}\setminus\{0\}$, then $\alpha_k^2\in\mathbb{R}$ for all $k\in\mathbb{N}$.*

III. *If $\omega\in\mathbb{R}\setminus\{0\}$ and $\beta_k:=i\alpha_k\in\mathbb{R}$ for a $k\in\mathbb{N}$, then $\frac{\beta_k(\omega)}{\omega}\frac{d\beta_k(\omega)}{d\omega}>0$.*

IV. *After a normalization $(\mathbf{E}_{T,k})_{k\in\mathbb{N}}$ forms a Riesz basis in $\mathbf{H}_0(\operatorname{curl}_\Gamma;\Gamma)$.*

V. *If $\mu=\mu_0$, then after a normalization $(\mathbf{E}_{T,k})_{k\in\mathbb{N}}$ forms a Riesz basis in $\mathbf{H}_0^{-1/2}(\operatorname{curl}_\Gamma;\Gamma)$.*

VI. *After a normalization $(\mathbf{H}_{T,k})_{k\in\mathbb{N}}$ forms a Riesz basis in $\mathbf{H}(\operatorname{curl}_\Gamma;\Gamma)$.*

VII. *If $\epsilon=\epsilon_0$, then after a normalization $(\mathbf{H}_{T,k})_{k\in\mathbb{N}}$ forms a Riesz basis in $\mathbf{H}^{-1/2}(\operatorname{curl}_\Gamma;\Gamma)$.*

Proof. Preliminary. By assumption ω is not a cut-off frequency for the homogeneous case $\epsilon=\epsilon_0$, $\mu=\mu_0$. Due to Lemma 2.2 there exists $\delta_1>0$ such that if $\delta_{\epsilon,\mu}<\delta_1$, then ω is not cut-off wavenumber for the heterogeneous case. Henceforth let $\delta_{\epsilon,\mu}<\delta_1$. Thus $\alpha\neq 0$. If $(\alpha,(\mathbf{E}_T,E_3,\mathbf{H}_T,H_3))$ is a solution to (3), then $(\mathbf{E}_T,-E_3,-\mathbf{H}_T,H_3))\neq 0$ and it can easily be checked that $(-\alpha,(\mathbf{E}_T,-E_3,-\mathbf{H}_T,H_3))$ solves (3).

Separated eigenvalue problems. Let $\mathbf{H}(\operatorname{div}_\Gamma\mu^{-1}0;\Gamma):=\{\mathbf{v}\in\mathbf{L}^2(\Gamma)\colon \operatorname{div}_\Gamma(\mu^{-1}\mathbf{v})=0\}$,$\mathcal{H}:=\mathbf{H}(\operatorname{div}_\Gamma\mu^{-1}0;\Gamma)\times L^2(\Gamma)$ and $\mathcal{X}:=X$. For each $m\in\mathbb{Z}$ consider the following. Let $\mathcal{H}_m:=\mathbf{H}_m(\operatorname{div}_\Gamma\mu^{-1}0;\Gamma)\times L^2_m(\Gamma)$ and $\mathcal{X}_m:=X_m$. The operators $A_m,B_m\in\mathcal{L}(\mathcal{X}_m)$ induce unbounded sesquilinear forms $a_m(\cdot,\cdot),b_m(\cdot,\cdot)$ on $\mathcal{H}_m\times\mathcal{H}_m$ with domains $\operatorname{dom}(a_m):=X_m\subset\operatorname{dom}(b_m)\subset\mathcal{H}_m$. Following [30, Ch. 6.2], [39] $a_m(\cdot,\cdot)$, $b_m(\cdot,\cdot)$ induce unbounded operators $\check{A}_m\colon\operatorname{dom}(\check{A}_m)\subset\mathcal{H}_m\to\mathcal{H}_m$, $\check{B}_m\colon\operatorname{dom}(B_m)\subset\mathcal{H}_m\to\mathcal{H}_m$ with $\operatorname{dom}(\check{A}_m^{1/2})=\operatorname{dom}(a_m)$. In a similar way $K_m\in\mathcal{L}(\mathcal{X}_m)$ induces a bounded sesquilinear form $k_m(\cdot,\cdot)\colon\mathcal{H}_m\times\mathcal{H}_m\to\mathbb{C}$, which induces the identity operator $\check{K}_m=I_{\mathcal{H}_m}$. Let $\tilde{c}_{\mathrm{gap}},\delta_2>0$ be as in Proposition 4.5, $\delta_3:=\min\{\delta_1,\delta_2\}$ and $\delta_{\epsilon,\mu}<\delta_3$ henceforth. Set $\gamma:=2$, $\kappa:=2\tilde{c}_{\mathrm{gap}}^2$ and $\check{\alpha}:=-1/6$. Let $C_{\mathrm{tr},3},\delta_4>0$ be as in Corollary 4.8 and $C_B,\delta_5>0$ be as in Lemma 2.4. Let $\delta_6\in(0,\min\{\delta_3,\delta_4,\delta_5\})$ be such that $6C_{\mathrm{tr},3}C_B(\epsilon_0+\mu_0)\delta_6<c_{\gamma,\kappa,\check{\alpha}}$ (with $c_{\gamma,\kappa,\check{\alpha}}$ as in Lemma 5.1) and $\delta_{\epsilon,\mu}<\delta_6$ henceforth. Let $(\hat{\tilde{j}}^2_{n,m},\psi_{n,m})_{n\in\mathbb{N}}$, $\psi_{n,m}\in\mathcal{X}_m$ be the normalized eigenpairs of $\check{A}_m$. Then by means of Lemma 5.4 the assumptions of Lemma 5.1 are satisfied. Thus the eigenpairs $(\check{\lambda}_{n,m},\phi_{n,m})_{n\in\mathbb{N}}$ of $\check{A}_m+\check{B}_m$ (with $\phi_{n,m}$ as defined in Lemma 5.1) satisfy: each eigenvalue $\check{\lambda}_{n,m}$, $n\in\mathbb{N}$ is simple,

$$\begin{aligned}|\hat{\tilde{j}}^2_{n,m}-\check{\lambda}_{n,m}|&\leq 6C_{\mathrm{tr},3}C_B(\epsilon_0+\mu_0)\delta_{\epsilon,\mu}\left(1+C_{\gamma,\kappa,\check{\alpha}}\left(\frac{\log(en)}{n}\right)^2\right)\\&\leq 6C_{\mathrm{tr},3}C_B(\epsilon_0+\mu_0)(1+C_{\gamma,\kappa,\check{\alpha}})\delta_{\epsilon,\mu},\end{aligned}$$

and $(\phi_{n,m})_{n\in\mathbb{N}}$ and $(\hat{\hat{j}}_{n,m}^{-1}\phi_{n,m})_{n\in\mathbb{N}}$ form a Riesz (even Bari) basis in $\mathcal{H}_m$ and $\mathcal{X}_m$ respectively with

$$\sum_{n\in\mathbb{N}} \|\psi_{n,m} - \phi_{n,m}\|^2_{\mathcal{H}_m} < 6C_{\gamma,\kappa,\breve{\alpha}}C_{\mathrm{tr},3}C_B(\epsilon_0+\mu_0)\delta_{\epsilon,\mu},$$
$$\sum_{n\in\mathbb{N}} \|\hat{\hat{j}}_{n,m}^{-1}\psi_{n,m} - \hat{\hat{j}}_{n,m}^{-1}\phi_{n,m})\|^2_{\mathcal{X}_m} < 6C_{\gamma,\kappa,\breve{\alpha}}C_{\mathrm{tr},3}C_B(\epsilon_0+\mu_0)\delta_{\epsilon,\mu}.$$

Since $\alpha_{n,m}^2 = \check{\lambda}_{n,m} - \omega^2\epsilon_0\mu_0$, the first claim I follows.

II. If $\omega \in \mathbb{R}\setminus\{0\}$ and $\alpha_{n,m}$ is an eigenvalue to (3) with $\alpha_{n,m}^2 \in \mathbb{C}\setminus\mathbb{R}$, then $\overline{\alpha}_{n,m} \neq \alpha_{n,m}$ is an eigenvalue too. However, in the derivation of Lemma 5.1 each patch Π_k (around $\hat{j}_{n,m}^2$) defined in [39, (3.1)] contains only a single eigenvalue, which is a contradiction to $|\hat{j}_{n,m}^2 - \alpha_{n,m}^2| = |\hat{j}_{n,m}^2 - \overline{\alpha}_{n,m}^2|$. Note that the patch Π_0 is omitted as discussed in the proof of Lemma 5.1.

III. For brevity we surpress the index k. We compute by means of (40) that

$$\frac{\beta}{\omega\epsilon_0\mu_0}\frac{d\beta(\omega)}{d\omega} = \frac{\langle\epsilon\mathbf{E}_T,\mathbf{E}_T\rangle_\Gamma + \langle\mu\mathbf{H}_T,\mathbf{H}_T\rangle_\Gamma - \frac{\beta}{\omega\epsilon_0\mu_0}\operatorname{Re}(\langle(\epsilon\mu-\epsilon_0\mu_0)R\mathbf{H}_T,\mathbf{E}_T\rangle_\Gamma)}{\langle\epsilon\mathbf{E}_T,\mathbf{E}_T\rangle_\Gamma + \langle\mu\mathbf{H}_T,\mathbf{H}_T\rangle_\Gamma + \frac{\omega}{\beta}\operatorname{Re}(\langle(\epsilon\mu-\epsilon_0\mu_0)R\mathbf{H}_T,\mathbf{E}_T\rangle_\Gamma)}$$

Hence, we can find $\delta_7 \in (0,\delta_6)$ such that the former denominator is actually positive for all $\delta_{\epsilon,\mu} \in (0,\delta_7)$ (which we assume henceforth) and

$$\frac{\beta}{\omega\epsilon_0\mu_0}\frac{d\beta(\omega)}{d\omega} \in \left[\frac{1-\tilde{\delta}_{\epsilon,\mu}\beta/\omega}{1+\tilde{\delta}_{\epsilon,\mu}\omega\epsilon_0\mu_0/\beta}, \frac{1+\tilde{\delta}_{\epsilon,\mu}\beta\epsilon_0\mu_0/\omega}{1-\tilde{\delta}_{\epsilon,\mu}\omega\epsilon_0\mu_0/\beta}\right],$$

from which the claim follows.

IV. Let $\delta_7 \in (0,\delta_6)$ be such that $6C_{\gamma,\kappa,\breve{\alpha}}C_{\mathrm{tr},3}C_B(\epsilon_0+\mu_0)\delta_7 < 1/4$ and $\delta_{\epsilon,\mu} < \delta_7$ henceforth. Then Lemma 5.2 and Lemma 2.6 yield that $(\phi_{n,m})_{n\in\mathbb{N},m\in\mathbb{Z}}$ forms a Riesz basis in $\mathcal{H}_m$ and $(\hat{\hat{j}}_{n,m}^{-1}\phi_{n,m})_{n\in\mathbb{N},m\in\mathbb{Z}}$ forms a Riesz basis in $\mathcal{X}_m$, both times with Gram constants $c > 1/4$ and $C < 4$. Note that $\frac{|\lambda_{n,m}|}{|\hat{\hat{j}}_{n,m}|} \to 1$ as $n \to +\infty$ or $m \to \pm\infty$. Hence we can trade a normalization constant for $\hat{\hat{j}}_{n,m}^{-1}$. Recall the bijection $\mathbf{V}_m \times W_m \to \mathbf{H}_{0m}(\operatorname{curl}_\Gamma;\Gamma)\colon \phi_m = (\mathbf{v}_m, w_m)^\top \mapsto \mathbf{v}_m + \nabla_\Gamma w_m$. It follows with $\phi_{n,m} = (\mathbf{v}_{n,m}, w_{n,m})^\top$ and Lemma 2.6 that after normalization $((\mathbf{E}_T)_{n,m} = \mathbf{v}_{n,m} + \nabla_\Gamma w_{n,m})_{n\in\mathbb{N},m\in\mathbb{Z}}$ forms a Riesz basis in $\mathbf{H}_0(\operatorname{curl}_\Gamma;\Gamma)$.

V. Let $m \in \mathbb{Z}$ and $(\phi_{n,m})_{n\in\mathbb{N}}$ be $\mathcal{H}_m$-normalized. Then

$$\frac{1}{4}\sum_{n\in\mathbb{N}} |\langle\phi_m,\phi_{n,m}\rangle_\mathcal{H}|^2 \le \|\phi_m\|^2_\mathcal{H} \le 4\sum_{n\in\mathbb{N}} |\langle\phi_m,\phi_{n,m}\rangle_\mathcal{H}|^2$$

and

$$\frac{1}{4}\sum_{n\in\mathbb{N}} \hat{j}_{n,m}^2|\langle\phi_m,\phi_{n,m}\rangle_\mathcal{H}|^2 \le \|\phi\|^2_\mathcal{X} \le 4\sum_{n\in\mathbb{N}} \hat{j}_{n,m}^2|\langle\phi_m,\phi_{n,m}\rangle_\mathcal{H}|^2$$

for all $\phi_m \in \mathcal{X}_m$. By Hilbert space interpolation [8] the norm of the space $[\mathcal{H}_m,\mathcal{X}_m]_{1/2}$ is equivalent (with m-uniform bounds) to $\sqrt{\sum_{n\in\mathbb{N}} \hat{j}_{n,m}|\langle\cdot,\phi_{n,m}\rangle_\mathcal{H}|^2}$ and after normalization $(\phi_{n,m})_{n\in\mathbb{N}}$ forms a Riesz basis of $[\mathcal{H}_m,\mathcal{X}_m]_{1/2}$. Recall that $\mathcal{H} = \mathbf{H}(\operatorname{div}_\Gamma 0;\Gamma) \times L^2(\Gamma)$ and $\mathcal{X} = \big(\mathbf{H}_0(\operatorname{curl}_\Gamma;\Gamma) \cap \mathbf{H}(\operatorname{div}_\Gamma 0;\Gamma)\big) \times H_0^1(\Gamma)$. Note that on $\mathbf{V} = \mathbf{H}_0(\operatorname{curl}_\Gamma;\Gamma) \cap \mathbf{H}(\operatorname{div}_\Gamma 0;\Gamma)$ the norms $\|\operatorname{curl}_\Gamma\cdot\|_\Gamma$ and $\|\cdot\|_{\mathbf{H}^1(\Gamma)}$ are equivalent (see, e.g., [3] for the 3D setting). It follows that $[\mathcal{H},\mathcal{X}]_{1/2} \subset \mathbf{H}^{1/2}(\Gamma) \times H^{1/2}(\Gamma)$. To adhere the essential boundary conditions we note that it is well known that $[L^2(\Gamma), H_0^1(\Gamma)]_{1/2} = H_{00}^{1/2}(\Gamma)$ [2]. On the other hand on the spaces $\mathbf{H}^s(\operatorname{curl}_\Gamma;\Gamma) := \{\mathbf{v} \in \mathbf{H}^s(\Gamma)\colon \operatorname{curl}_\Gamma \mathbf{v} \in H^s(\Gamma)\}$, $s \in [0, -1/2]$ there exists a bounded tangential trace operator $\operatorname{tr}_{\hat{\mathbf{t}}} \in \mathcal{L}\big(\mathbf{H}^s(\operatorname{curl}_\Gamma;\Gamma), H^{s-1/2}(\partial\Gamma)\big)$, see, e.g., [9, Prop. 3.14] or [4, p. 115-117], defined by

$$\langle \operatorname{tr}_{\hat{\mathbf{t}}} \mathbf{v}, w\rangle_{H^{s-1/2}(\partial\Gamma)\times H^{-s+1/2}(\partial\Gamma)} := \langle \operatorname{curl}_\Gamma \mathbf{v}, w_\Gamma\rangle_{H^s(\Gamma)\times H^{-s}(\Gamma)} - \langle \mathbf{v}, \operatorname{Curl}_\Gamma w_\Gamma\rangle_{H^s(\Gamma)\times H^{-s+1}(\Gamma)},$$

where $w_\Gamma \in H^{-s+1}(\Gamma)$ is a lifting of $w \in H^{-s+1/2}(\partial\Gamma)$. By continuity it follows that

$$\begin{aligned}[]
[\mathbf{H}(\operatorname{div}_\Gamma 0;\Gamma), \mathbf{V}]_{1/2} &= \{\operatorname{Curl}_\Gamma v\colon v \in \mathcal{H}^{\mathrm{BC}}(\Gamma)\},\\
\mathcal{H}^{\mathrm{BC}}(\Gamma) &:= \{H^{3/2}(\Gamma)\colon \operatorname{tr}_{\hat{\mathbf{t}}} \operatorname{Curl}_\Gamma v = 0\}.
\end{aligned}$$

It remains to note that $\mathbf{H}_0^{-1/2}(\operatorname{curl}_\Gamma;\Gamma) = \operatorname{Curl}_\Gamma \mathcal{H}^{\mathrm{BC}}(\Gamma) \oplus \nabla_\Gamma H_{00}^{1/2}(\Gamma)$ [10, (55)], see also [4, p. 115-117].

VI+VII. We derive the same results for $\mathbf{H}_T$ by trading $\epsilon, \mu, \mathbf{E}_T, E_3$ for $\mu, \epsilon, \mathbf{H}_T, H_3$ and adapting the boundary conditions. ☐

Remark 5.6. *We do not expect that the basis properties IV-VI of Theorem 5.5 could possibly be extended to* $\mathbf{L}^2(\Gamma)$. *Indeed, for the setting of two materials* ($N_\Gamma = 2$) *[45, Thm. 2.2.3] proves that* $(\mathbf{E}_{T,k}, \mathbf{H}_{T,k})_{k\in\mathbb{N}} \cup (\mathbf{E}_{T,k}, -\mathbf{H}_{T,k})_{k\in\mathbb{N}}$ *does not form a basis in* $\mathbf{L}^2(\Gamma) \times \mathbf{L}^2(\Gamma)$ *(and hence in particular no Riesz basis). Making a two-by-two block transformation it follows that* $(\mathbf{E}_{T,k}, 0)_{k\in\mathbb{N}} \cup (0, \mathbf{H}_{T,k})_{k\in\mathbb{N}} k$ *is not a Riesz basis of* $\mathbf{L}^2(\Gamma) \times \mathbf{L}^2(\Gamma)$. *Thus both* $(\mathbf{E}_{T,k})_{k\in\mathbb{N}}$ *and* $(\mathbf{H}_{T,k})_{k\in\mathbb{N}}$ *do not form a Riesz basis of* $\mathbf{L}^2(\Gamma)$.

A Location of wavenumbers for general geometries

We include an observation concerning the location of the wavenumbers under far more general assumptions on Γ and ϵ, μ than made in Section 2.1. To this end we derive a particular equation satisfied by the eigenmodes. We eliminate E_3 and H_3 from (2) to obtain

$$\operatorname{Curl}_\Gamma(\mu^{-1} \operatorname{curl}_\Gamma \mathbf{E}_T) - \omega^2 \epsilon \mathbf{E}_T = \omega\beta R \mathbf{H}_T \quad \text{in } \Gamma, \tag{37a}$$

$$\operatorname{Curl}_\Gamma(\epsilon^{-1} \operatorname{curl}_\Gamma \mathbf{H}_T) - \omega^2 \mu \mathbf{H}_T = \omega\beta R \mathbf{E}_T \quad \text{in } \Gamma, \tag{37b}$$

$$\hat{\mathbf{t}} \cdot \mathbf{E}_T = 0 \quad \text{on } \partial\Gamma, \tag{37c}$$

$$\operatorname{curl}_\Gamma \mathbf{H}_T = 0 \quad \text{on } \partial\Gamma. \tag{37d}$$

Now, applying $\operatorname{div}_\Gamma$ to (37a) and (37b) we compute that

$$-\omega^2 \operatorname{div}_\Gamma(\epsilon \mathbf{E}_T) = -\omega\beta \operatorname{curl}_\Gamma \mathbf{H}_T \quad \text{and} \quad -\omega^2 \operatorname{div}_\Gamma(\mu \mathbf{H}_T) = \omega\beta \operatorname{curl}_\Gamma \mathbf{E}_T. \tag{38}$$

We multiply (37b) with $-\frac{\beta}{\omega}\epsilon R$ and insert (38) to obtain

$$-\epsilon\nabla\epsilon^{-1} \operatorname{div}_\Gamma \epsilon \mathbf{E}_T + \omega\beta\epsilon\mu R \mathbf{H}_T + \beta^2 \epsilon \mathbf{E}_T = 0. \tag{39a}$$

Likewise we multiply (37a) with $\frac{\beta}{\omega}\mu R$ and insert (38) to obtain

$$-\mu\nabla\mu^{-1} \operatorname{div}_\Gamma \mu \mathbf{H}_T - \omega\beta\epsilon\mu R \mathbf{E}_T + \beta^2 \mu \mathbf{H}_T = 0. \tag{39b}$$

Furthermore, (37d) and (38) yield the boundary condition

$$\operatorname{div}_\Gamma(\epsilon \mathbf{E}_T) = 0 \quad \text{on } \partial\Gamma. \tag{39c}$$

On the other hand, we multiply (37b) with $\hat{\mathbf{n}}$. The term $\hat{\mathbf{n}} \cdot \operatorname{Curl}_\Gamma(\epsilon^{-1} \operatorname{curl}_\Gamma \mathbf{H}_T)$ vanishes due to $\hat{\mathbf{n}} \cdot \operatorname{Curl}_\Gamma = \hat{\mathbf{t}} \cdot \nabla_\Gamma$ and (37d). The term $-\hat{\mathbf{n}} \cdot \omega\beta R \mathbf{E}_T$ vanishes due to $\hat{\mathbf{n}} \cdot R = \hat{\mathbf{t}}\cdot$ and (37c). Thus it holds that

$$\hat{\mathbf{n}} \cdot \mu \mathbf{H}_T = 0 \quad \text{on } \partial\Gamma. \tag{39d}$$

Now, multiplying (37a)-(37b) with $\epsilon_0\mu_0$ and adding (39a)-(39b) gives that a solution $(\beta, (\mathbf{E}_T, \mathbf{H}_T))$ to (37) satisfies

$$\epsilon_0\mu_0 \operatorname{Curl}_\Gamma(\mu^{-1} \operatorname{curl}_\Gamma \mathbf{E}_T) - \epsilon\nabla_\Gamma\epsilon^{-1} \operatorname{div}_\Gamma \epsilon\mathbf{E}_T + \omega\beta(\epsilon\mu - \epsilon_0\mu_0)R\mathbf{H}_T + (\beta^2 - \omega^2\epsilon_0\mu_0)\epsilon\mathbf{E}_T = 0 \quad \text{in } \Gamma, \tag{40a}$$

$$\epsilon_0\mu_0 \operatorname{Curl}_\Gamma(\epsilon^{-1} \operatorname{curl}_\Gamma \mathbf{H}_T) - \mu\nabla_\Gamma\mu^{-1} \operatorname{div}_\Gamma \mu\mathbf{H}_T - \omega\beta(\epsilon\mu - \epsilon_0\mu_0)R\mathbf{E}_T + (\beta^2 - \omega^2\epsilon_0\mu_0)\mu\mathbf{H}_T = 0 \quad \text{in } \Gamma, \tag{40b}$$

$$\operatorname{div}_\Gamma(\epsilon\mathbf{E}_T) = \hat{\mathbf{t}} \cdot \mathbf{E}_T = 0 \quad \text{on } \partial\Gamma, \tag{40c}$$

$$\operatorname{curl}_\Gamma \mathbf{H}_T = \hat{\mathbf{n}} \cdot \mathbf{H}_T = 0 \quad \text{on } \partial\Gamma. \tag{40d}$$

The allure of (40) is that β^2 and ω^2 only appear as the combined term $(\beta^2 - \omega^2\epsilon_0\mu_0)$. For (quasi) homogeneous materials $\epsilon\mu = 1$ the equations (40) for $\mathbf{E}_T$ and $\mathbf{H}_T$ decouple, and thus we recognize that solutions to (40) do not necessarily satisfy (37). Thus the relation of (37) to (40) is only one-way. Nevertheless, this suffices to establish the following result.

Lemma A.1. *All wavenumbers β with $\operatorname{Im}\beta \neq 0$ satisfy $|\operatorname{Re}\beta| \leq |\omega| \|(\epsilon\mu - \epsilon_0\mu_0)/\sqrt{\epsilon\mu}\|_{L^\infty(\Gamma)}$. All wavenumbers β with $\operatorname{Im}\beta = 0$ satisfy $|\operatorname{Re}\beta| \leq \omega\|\sqrt{\epsilon\mu}\|_{L^\infty(\Gamma)}$.*

Proof. Testing (40) with $(\mathbf{E}_T, \mathbf{H}_T)$, taking the imaginary part thereof and dividing by $2\operatorname{Im}(\beta)$ yields

$$\begin{aligned} 0 &= |\omega \operatorname{Re}(\langle(\epsilon\mu - \epsilon_0\mu_0)R\mathbf{H}_T, \mathbf{E}_T\rangle_\Gamma) + \operatorname{Re}(\beta)(\langle\epsilon\mathbf{E}_T, \mathbf{E}_T\rangle_\Gamma + \langle\mu\mathbf{H}_T, \mathbf{H}_T\rangle_\Gamma)| \\ &\geq (|\operatorname{Re}(\beta)| - |\omega| \|(\epsilon\mu - \epsilon_0\mu_0)/\sqrt{\epsilon\mu}\|_{L^\infty(\Gamma)})(\langle\epsilon\mathbf{E}_T, \mathbf{E}_T\rangle_\Gamma + \langle\mu\mathbf{H}_T, \mathbf{H}_T\rangle_\Gamma), \end{aligned}$$

from which the first claim follows. For $\operatorname{Im}(\beta) = 0$ we test (39) with $(\mathbf{E}_T, \mathbf{H}_T)$, which yields

$$\begin{aligned} 0 &\geq \langle\epsilon^{-1} \operatorname{div}(\epsilon\mathbf{E}_T), \operatorname{div}(\epsilon\mathbf{E}_T)\rangle_\Gamma + \langle\mu^{-1} \operatorname{div}(\mu\mathbf{H}_T), \operatorname{div}(\mu\mathbf{H}_T)\rangle_\Gamma \\ &= |\beta\omega 2 \operatorname{Re}(\langle R\mathbf{H}_T, \mathbf{E}_T\rangle_\Gamma) + \beta^2(\langle\epsilon\mathbf{E}_T, \mathbf{E}_T\rangle_\Gamma + \langle\mu\mathbf{H}_T, \mathbf{H}_T\rangle_\Gamma)| \\ &\geq |\beta|(|\beta| - |\omega| \|\sqrt{\epsilon\mu}\|_{L^\infty(\Gamma)})(\langle\epsilon\mathbf{E}_T, \mathbf{E}_T\rangle_\Gamma + \langle\mu\mathbf{H}_T, \mathbf{H}_T\rangle_\Gamma) \end{aligned}$$

from which the claim follows. □

Remark A.2. *Lemma A.1 is a non-trivial improvement on the location of wavenumbers compared to results obtained with Keldyh's theory [21, Thm. 4], [13, Assertion 2.9] which only gives a sector as a superset. Note a typo in [21, Thm. 4] where π should actually be $\pi/2$.*

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