On the Relation Between Treewidth, Tree-Independence Number, and Tree-Chromatic Number of Graphs

Alex Koutsoutis
Texas A&M University, College Station, TX, USA

Kilian Krause
Karlsruhe Institute of Technology, Germany

Chun-Hung Liu
Texas A&M University, College Station, TX, USA

Mirza Redzic
Karlsruhe Institute of Technology, Germany

Torsten Ueckerdt
Karlsruhe Institute of Technology, Germany

Abstract

We investigate the relationship between graph parameters, which measure the complexity of the tree decompositions of a given graph. The treewidth $\mathrm{tw}(G)$ of a graph G measures the largest number of vertices required in a bag of every tree decomposition of G. Similarly, the tree-independence number tree-$\alpha(G)$ and the tree-chromatic number tree-$\chi(G)$ measure the largest independence number, respectively the largest chromatic number, required in a bag of every tree decomposition of G. Recently, Dallard, Milanič, and Štorgel asked (JCTB, 2024) whether for all graphs G it holds that $\mathrm{tw}(G) + 1 \leq \text{tree-}\alpha(G) \cdot \text{tree-}\chi(G)$. We provide a negative answer for this question in a strong form: for every function $f\colon \mathbb{N} \to \mathbb{N}$, there exists a graph G such that $\mathrm{tw}(G) > \text{tree-}\alpha(G) \cdot f(\text{tree-}\chi(G))$. On the other hand, we complement this result with an upper bound, by showing that $\mathrm{tw}(G) + 1 \leq \text{tree-}\alpha(G)^2 \cdot \text{tree-}\chi(G)$ for every graph G.

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1 Introduction

Tree-decompositions constitute a central pillar in algorithmic and structural graph theory. While originally introduced by Halin [9] in 1976, this notion was rediscovered independently and developed thoroughly by Robertson and Seymour within their Graph Minors project.

Given a graph G, a *tree-decomposition* of G is a pair $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ consisting of a tree T and for each node t in T a set $X_t \subseteq V(G)$, called a *bag* at t, such that:

- for every edge $uv \in E(G)$, there exists a node $t \in V(T)$ with $\{u, v\} \subseteq X_t$, and
- for every vertex $v \in V(G)$, the set $\{t \in V(T) : v \in X_t\}$ induces a non-empty connected subgraph of T.

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Due to various applications, tree-decompositions whose bags induce "tractable" subgraphs are considered (e.g., [1,3–8,12–16,18]). There are different ways to measure the "tractability" of the subgraphs induced by bags. One way to do so is considering certain graph parameters of those subgraphs. For a graph parameter p and a tree-decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of a graph G, we define $p(\mathcal{T})$ to be $\max_{t \in V(T)} p(G[X_t])$; we define

$$\text{tree-}p(G) = \min_{\mathcal{T}'} p(\mathcal{T}'),$$

where the minimum is over all tree-decompositions $\mathcal{T}'$ of G.

The most classical example is the treewidth of a graph G. The *width* of a graph is its number of vertices minus 1. Hence the width of a given tree-decomposition is the maximum size of a bag minus 1, and the tree-width of a graph G, denoted by $\mathrm{tw}(G)$, is the minimum width of a tree-decomposition of G. Tree-width is also written as *treewidth* in the literature. Graphs with bounded treewidth have numerous nice properties. For example, treewidth determines the minimum size of balanced separators [14], and every property that can be expressed in monadic second order logic can be decided in linear time for graphs of bounded treewidth [2].

Another example is the tree-independence number, which was introduced independently by Yolov [18] (under the name α-treewidth) and Dallard, Milanič, and Štorgel [3], measuring subgraphs induced by bags by their independence number. The *independence number* of a graph H, denoted by $\alpha(H)$, is the maximum size of a stable set (i.e., a set of pairwise non-adjacent vertices). Hence the tree-independence number of a graph G is the minimum k such that there exists a tree-decomposition of G such that every subgraph induced by a bag has independence number at most k. Tree-independence number enjoys several features of treewidth. For example, tree-independence number is effective for solving algorithmic problems related to independence number [3, 11].

The third example considered in this paper is the tree-chromatic number, introduced by Seymour [16], measuring subgraphs induced by bags by their chromatic number. The *chromatic number* of a graph H, denoted by $\chi(H)$, is the smallest integer k such that $V(H)$ can be partitioned into k stable sets. Hence the tree-chromatic number of a graph G is the minimum k such that there exists a tree-decomposition of G such that every subgraph induced by a bag has chromatic number at most k. Unlike treewidth and tree-independence number, tree-chromatic number is more subtle and less understood.

Inspired by the simple but very useful inequality $|V(G)| \leq \alpha(G) \cdot \chi(G)$, which holds for every graph G, Dallard et al. [3] proposed the following question addressing its tree-decomposition analog.

▶ **Question 1** (Dallard, Milanič, and Štorgel [3, Question 8.4]). *Does every graph G satisfy* $\mathrm{tw}(G) + 1 \leq \text{tree-}\alpha(G) \cdot \text{tree-}\chi(G)$*?*

By applying the inequality $|V(H)| \leq \alpha(H) \cdot \chi(H)$ to each subgraph H induced by a bag, it is easy to see that $\mathrm{tw}(G) + 1 \leq \alpha(G) \cdot \text{tree-}\chi(G)$ and $\mathrm{tw}(G) + 1 \leq \text{tree-}\alpha(G) \cdot \chi(G)$. So if G is a graph with $\alpha(G) = \text{tree-}\alpha(G)$ or $\chi(G) = \text{tree-}\chi(G)$, then $\mathrm{tw}(G) + 1 \leq \text{tree-}\alpha(G) \cdot \text{tree-}\chi(G)$ and G is a positive instance of Question 1. In particular, perfect graphs are positive instances of Question 1 since $\chi(G) = \omega(G) \leq \text{tree-}\chi(G) \leq \chi(G)$ for every perfect graph G.

Even though some positive instances of Question 1 are known, we provide a negative answer for Question 1 in a strong form in this paper. (See Theorem 13 for a slightly stronger result.)

▶ **Theorem 2.** *For any functions $f\colon \mathbb{R} \to \mathbb{R}_{>0}$ and $h\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $h(x)/x$ is non-decreasing and $h(x) = o(x \log\log x)$ as $x \to \infty$, there exists a graph G such that*

$$\mathrm{tw}(G) > h(\text{tree-}\alpha(G)) \cdot f(\text{tree-}\chi(G)).$$

Theorem 2 shows that no upper bound of $\mathrm{tw}(G)$ in terms of tree-$\alpha(G)$ and tree-$\chi(G)$ can exist even if we allow the dependence of tree-$\alpha(G)$ to grow slightly faster than linear functions and the dependence of tree-$\chi(G)$ to grow arbitrarily fast. We complement this result by the following simple observation showing that an upper bound exists if we allow the dependence of tree-$\alpha(G)$ to be quadratic.

▶ **Proposition 3.** *Every graph G satisfies*

$$\mathrm{tw}(G) + 1 \leq \text{tree-}\alpha(G)^2 \cdot \text{tree-}\chi(G).$$

Proof. Take a tree-decomposition whose every bag induces a subgraph with independence number at most tree-$\alpha(G)$. It suffices to show that every bag contains at most tree-$\alpha(G)^2 \cdot$ tree-$\chi(G)$ vertices. For every node t of the tree in the tree-decomposition, let H_t be the subgraph induced by the bag at t. Note that for every t, $\alpha(H_t) \leq$ tree-$\alpha(G)$ and tree-$\chi(H_t) \leq$ tree-$\chi(G)$, so H_t has a tree-decomposition whose every bag induces a subgraph of chromatic number at most tree-$\chi(H_t) \leq$ tree-$\chi(G)$ and hence contains at most $\alpha(H_t) \cdot$ tree-$\chi(G) \leq$ tree-$\alpha(G) \cdot$ tree-$\chi(G)$ vertices. This implies that $\mathrm{tw}(H_t) \leq$ tree-$\alpha(G) \cdot$ tree-$\chi(G) - 1$. It is well known that for every k, if $\mathrm{tw}(G) \leq k$, then $\chi(G) \leq k+1$. So $\chi(H_t) \leq$ tree-$\alpha(G) \cdot$ tree-$\chi(G)$. Hence $|V(H_t)| \leq \alpha(H_t) \cdot \chi(H_t) \leq$ tree-$\alpha(G)^2 \cdot$ tree-$\chi(G)$. ◀

We remark that in the proof of Proposition 3, we showed that $\mathrm{tw}(H_t) \leq$ tree-$\alpha(G) \cdot$ tree-$\chi(G) - 1$ for every node t. It implies the following correction of Question 1 in terms of tree-treewidth. (Recall that tree-tw(G) is the minimum k such that there exists a tree-decomposition of G such that every subgraph induced by a bag has treewidth at most k.)

▶ **Proposition 4.** *For every graph G,*

$$\text{tree-tw}(G) + 1 \leq \text{tree-}\alpha(G) \cdot \text{tree-}\chi(G).$$

Clearly, tree-tw$(G) \leq \mathrm{tw}(G)$ for every graph G. In general, treewidth cannot be upper bounded by any function of tree-treewidth [13]. On the other hand, Proposition 4 is tight for chordal graphs G since the maximum clique of G must be contained in a bag of every tree-decomposition, implying tree-tw$(G) + 1 \geq \omega(G) =$ tree-$\chi(G) =$ tree-$\alpha(G) \cdot$ tree-$\chi(G)$ for every chordal G.

This paper is organized as follows. In Section 2, we introduce a construction that enlarges a graph G to a graph G' without changing the tree-chromatic number while $\mathrm{tw}(G')$ and tree-$\alpha(G')$ are closely related to $V(G)$ and $\alpha(G)$. We then use this construction to provide a negative answer to Question 1 in Section 3.

2 Completion of graphs

In this section we provide a key ingredient for constructing a negative answer to Question 1.

For a graph H, the *1-completion* of H is the graph obtained from H by, for each pair of non-adjacent vertices u and v of H, adding a new vertex of degree 2 adjacent to exactly u and v. In this paper, denote the 1-completion of H by $C(H)$.

The first observation is that taking the 1-completion does not change the tree-chromatic number.

▶ **Lemma 5.** *If H is a graph with $E(H) \neq \emptyset$, then* tree-$\chi(C(H)) =$ tree-$\chi(H)$.

Proof. Since H is a subgraph of $C(H)$, we get that tree-$\chi(C(H)) \geq$ tree-$\chi(H)$.

To prove tree-$\chi(C(H)) \leq$ tree-$\chi(H)$, it suffices to construct a tree-decomposition $\mathcal{T}$ of $C(H)$ with $\chi(\mathcal{T}) =$ tree-$\chi(H)$.

For any pair of non-adjacent vertices u and v in H, let a_{uv} be the vertex of $C(H) - V(H)$ adjacent to both u and v. Let $\mathcal{T}_H = (T_H, \{X_t\}_{t \in V(T_H)})$ be a tree-decomposition of H with $\chi(\mathcal{T}_H) =$ tree-$\chi(H)$.

We construct a new tree-decomposition from $\mathcal{T}_H$. Fix $u, v \in V(H)$ such that $uv \notin E(H)$. Either (1) there exists $t_{uv} \in V(T_H)$ such that $\{u, v\} \subseteq X_{t_{uv}}$; or (2) there is no bag in $\mathcal{T}_H$ containing both u and v. In case (1), we add a new vertex t'_{uv} adjacent to t_{uv}, and let $X_{t'_{uv}} = \{u, v, a_{uv}\}$. In case (2), we add a_{uv} into X_t for every $t \in V(T_H)$; note that a_{uv} has degree at most 1 in the subgraph induced by the new X_t. Clearly, this procedure yields a tree-decomposition $\mathcal{T} = (T, \{Y_t\}_{t \in V(T)})$ of $C(H)$.

Since $E(H) \neq \emptyset$, some bag of $\mathcal{T}_H$ contains an edge, so $\chi(\mathcal{T}_H) \geq 2$. For every $t \in V(T) - V(T_H)$, the subgraph of $C(H)$ induced by Y_t is a forest and hence has chromatic number at most $2 \leq \chi(\mathcal{T}_H)$. For every $t \in V(T_H)$, the subgraph of $C(H)$ induced by Y_t is obtained from $H[X_t]$ by adding isolated vertices or attaching leaves and hence has chromatic number at most $\max\{2, \chi(H[X_t])\} \leq \chi(\mathcal{T}_H)$. So $\chi(\mathcal{T}) \leq \chi(\mathcal{T}_H) =$ tree-$\chi(H)$. ◀

For every graph G, we denote the complement of G by $\overline{G}$.

The second observation is that the treewidth and the tree-independence number of the 1-completion of H is closely related to $|V(H)|$ and the independence number of H, respectively.

▶ **Lemma 6.** *If H is a graph with $|V(H)| \geq 3$, then* $\mathrm{tw}(C(H)) = |V(H)| - 1$ *and* tree-$\alpha(C(H)) \leq \alpha(H)$.

Proof. Let $n = |V(H)|$. Since $C(H)$ contains K_n as a minor, $\mathrm{tw}(C(H)) \geq \mathrm{tw}(K_n) = n - 1$.

To prove $\mathrm{tw}(C(H)) \leq n - 1$, we construct a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of $C(H)$ of width $n - 1$. Let $T = K_{1,|E(\overline{H})|}$. Let the bag of the central node of T be $V(H)$. Note that there exists a bijection σ from the set of leaves of T to $E(\overline{H})$. For every leaf t of T, let X_t be consisting of the ends of $\sigma(t)$ and the vertex of $C(H) - V(H)$ adjacent to both ends of $\sigma(t)$. Then $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ is a tree-decomposition of $C(H)$ of width $\max\{|V(H)|, 3\} - 1 = n - 1$.

Moreover, for every $t \in V(T)$, the subgraph of $C(H)$ induced by X_t is either H or a path on three vertices, so $\alpha(\mathcal{T}) = \max\{\alpha(H), 2\}$. If $\alpha(H) \geq 2$, then $\alpha(\mathcal{T}) = \alpha(H)$. If $\alpha(H) \leq 1$, then H is a complete graph and T consists of a single vertex, so $\alpha(\mathcal{T}) = \alpha(H)$. Hence tree-$\alpha(C(H)) \leq \alpha(\mathcal{T}) = \alpha(H)$. ◀

Lemmas 5 and 6 are sufficient to construct a simple negative instance of Question 1. For example, if H is the union of k disjoint copies of 5-cycles (for any constant $k \geq 1$), then tree-$\chi(C(H)) = 2$ by Lemma 5, $\mathrm{tw}(C(H)) = 5k - 1$ and tree-$\alpha(C(H)) \leq 2k$ by Lemma 6. We will prove Theorem 2 to provide a stronger negative instance of Question 1 in the next section.

In the rest of this section we strengthen Lemma 6 to show that the upper bound for tree-$\alpha(C(H))$ is almost tight. Lemmas 7 and 8 will not be used in the rest of the paper.

▶ **Lemma 7.** *For every $n \geq 2$ we have* tree-$\alpha(C(\overline{K_n})) = n - 1$.

Proof. Let $A \subset V(C(\overline{K_n}))$ be the subset of n vertices in $C(\overline{K_n})$ corresponding to the original vertices of $\overline{K_n}$. Note that A is an independent set in $C(\overline{K_n})$.

We first show the lower bound tree-$\alpha(C(\overline{K_n})) \geq n-1$. Consider a tree decomposition $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ of $C(\overline{K_n})$ with $\alpha(\mathcal{T}) = \text{tree-}\alpha(C(\overline{K_n}))$. We may assume that no bag of $\mathcal{T}$ contains A, for otherwise $\alpha(\mathcal{T}) \geq |A| = n \geq n-1$, as desired. So there exists an edge t_1t_2 of T such that $T - t_1t_2$ consists of two subtrees T_1 and T_2 of T, and there exists a partition $A_1 \cup A_2 \cup A_3$ of A with the following properties:

- $|A_1| \geq 1$ and every vertex in A_1 appears only in bags of T_1.
- $|A_2| \geq 1$ and every vertex in A_2 appears only in bags of T_2.
- Every vertex in A_3 appears in bags of T_1 and bags of T_2.

Clearly, $|A_1| + |A_2| + |A_3| = |A| = n$. Moreover, the bag X_{t_1} contains all vertices in A_3, as well as the set B consisting of the (degree-2) vertices in $C(\overline{K_n}) - A$ adjacent in $C(\overline{K_n})$ to both vertices in A_1 and A_2. Observe that $A_3 \cup B$ is an independent set in $C(\overline{K_n})$. So $\alpha(\mathcal{T}) \geq |A_3 \cup B| = |A_3| + |B|$. Since $|B| = |A_1| \cdot |A_2| \geq |A_1| + |A_2| - 1$, where the inequalities uses that $|A_1|, |A_2| \geq 1$, we have $\alpha(\mathcal{T}) \geq |B| + |A_3| \geq |A_1| + |A_2| + |A_3| - 1 = n-1$, as desired.

Now we show the upper bound tree-$\alpha(C(\overline{K_n})) \leq n-1$. The case $n=2$ is trivial, so we may assume $n \geq 3$. Let S be the star with $\binom{n-1}{2} + 1$ leaves. We denote by s the central vertex of S, and denote the leaves of S by s_0 and $s_{i,j}$ for $1 \leq i < j \leq n-1$. Denote A by $\{a_0, a_1, a_2, ..., a_{n-1}\}$. Define $Y_s = (A - \{a_0\}) \cup N_{C(\overline{K_n})}(a_0)$ and $Y_{s_0} = \{a_0\} \cup N_{C(\overline{K_n})}(a_0)$. For every i,j with $1 \leq i < j \leq n-1$, define $Y_{s_{i,j}}$ to be the set consisting of a_i, a_j and the unique common neighbor of a_i and a_j in $C(\overline{K_n})$. Clearly, $(S, (Y_t)_{t \in V(S)})$ is a tree-decomposition of $C(\overline{K_n})$. Note that the subgraph induced by Y_s contains a perfect matching with $n-1$ edges; the subgraph induced by each leaf of S is a star with at most $\max\{|A|-1, 2\} = n-1$ leaves. So every bag induces a subgraph with independence number at most $n-1$. This shows tree-$\alpha(C(\overline{K_n})) \leq n-1$. ◀

Combining Lemmas 6 and 7, we can determine tree-$\alpha(C(H))$ quite precisely.

▶ **Lemma 8.** *If H is a graph with $|V(H)| \geq 3$, then $\alpha(H) - 1 \leq \text{tree-}\alpha(C(H)) \leq \alpha(H)$.*

Proof. By Lemma 6, tree-$\alpha(C(H)) \leq \alpha(H)$. Since $C(H)$ contains $C(\overline{K_{\alpha(H)}})$ as an induced subgraph, the inequality tree-$\alpha(C(H)) \geq \alpha(H) - 1$ follows directly from Lemma 7, whenever $\alpha(H) \geq 2$. On the other hand, if $\alpha(H) \leq 1$, we have tree-$\alpha(C(H)) \geq 0 \geq \alpha(H) - 1$ trivially. ◀

3 Negative answer to Question 1

We shall use a version of Burling graphs to construct a negative answer to Question 1. This is a family of graphs, which are recursively defined as follows. Throughout this section, $(G_n, \mathcal{S}_n)_{n \in \mathbb{N}}$ is a sequence of ordered pairs, where G_n is a graph and $\mathcal{S}_n$ is a collection of stable sets in G_n for every $n \in \mathbb{N}$, such that $G_1 = K_1$ and $\mathcal{S}_1 = \{V(G_1)\}$, and for $n \geq 2$, G_n and $\mathcal{S}_n$ are defined as follows:

- For each $S \in \mathcal{S}_{n-1}$, create a copy G_S of G_{n-1}, and let $\mathcal{S}_S$ be the collection of stable sets of G_S consisting of the copies of the members of $\mathcal{S}_{n-1}$ in G_S.
- Define G_n to be the graph obtained from $G_{n-1} \cup \bigcup_{S \in \mathcal{S}_{n-1}} G_S$, by for each $S \in \mathcal{S}_{n-1}$ and $Q \in \mathcal{S}_S$, adding a vertex $v_{S,Q}$ adjacent to all vertices in Q.
- Define $\mathcal{S}_n = \{S \cup Q, S \cup \{v_{S,Q}\} : S \in \mathcal{S}_{n-1}, Q \in \mathcal{S}_S\}$.

First, we prove that for each graph G_n in the above sequence we have tree-$\chi(G_n) \leq 2$. A *star forest* is a graph whose every component is a star. Note that K_1 is a star.

▶ **Lemma 9.** *For every positive integer n, there exists a tree-decomposition $\mathcal{T}$ of G_n such that every bag of $\mathcal{T}$ induces a star forest, and every member of $\mathcal{S}_n$ is contained in a bag of $\mathcal{T}$.*

Proof. We shall prove this lemma by induction on n. The case $n = 1$ is obvious. So we may assume $n \geq 2$. By the inductive hypothesis, there exists a tree-decomposition $\mathcal{T}_{n-1} = (T_{n-1}, \{X_t\}_{t \in V(T_{n-1})})$ of G_{n-1} such that every bag of $\mathcal{T}_{n-1}$ is a star forest, and for every $S \in \mathcal{S}_{n-1}$, there exists $t_S \in V(T_{n-1})$ with $S \subseteq X_{t_S}$.

For every $S \in \mathcal{S}_{n-1}$, let $\mathcal{T}_S = (T_S, \{X_t^S\}_{t \in V(T_S)})$ be a tree-decomposition of G_S isomorphic to $\mathcal{T}_{n-1}$. Note that for any $S \in \mathcal{S}_{n-1}$ and $Q \in \mathcal{S}_S$, there exists $t_{S,Q} \in V(T_S)$ such that $Q \subseteq X^S_{t_{S,Q}}$. For every $S \in \mathcal{S}_{n-1}$, we construct a tree T'_S by for every $Q \in \mathcal{S}_S$, adding a new node $t'_{S,Q}$ into T_S and the edge $t'_{S,Q}t_{S,Q}$, and define $Y_{t'_{S,Q}} = Q \cup \{v_{S,Q}\} \cup S$. For every $S \in \mathcal{S}_{n-1}$, let $Y_t = X_t^S \cup S$ for every $t \in V(T_S)$.

Let T_n be the tree obtained from $T_{n-1} \cup \bigcup_{S \in \mathcal{S}_{n-1}} T'_S$ by adding, for each $S \in \mathcal{S}_{n-1}$, an edge between t_S and some node of T'_S. Then $(T_n, \{X_t\}_{t \in V(T_{n-1})} \cup \{Y_t\}_{S \in \mathcal{S}_{n-1}, t \in V(T'_S)})$ is a tree-decomposition of G_n such that every member of $\mathcal{S}_n$ is contained in some bag.

Moreover, for every $t \in V(T_{n-1})$, we know $G_n[X_t] = G_{n-1}[X_t]$ is a star forest by the inductive hypothesis; for any $S \in \mathcal{S}_{n-1}$ and $t \in V(T'_S)$, if $t = t'_{S,Q}$ for some $Q \in \mathcal{S}_S$, then $G_n[Y_t]$ is a union of a star and isolated vertices, and if $t \in V(T_S)$, then Y_t is a union of a subset X_t^S of $V(G_S)$ and a stable set S in G_{n-1}, so Y_t is a stable set in G_n. Therefore, every bag of $(T_n, \{X_t\}_{t \in V(T_{n-1})} \cup \{Y_t\}_{S \in \mathcal{S}_{n-1}, t \in V(T'_S)})$ induces a star forest in G_n. ◀

Walczak [17] proved that Burling graphs have unbounded fractional chromatic number. This result can be stated in terms of its dual. For a graph G, a function $f\colon V(G) \to \mathbb{R}$, and a subset S of $V(G)$, we define $f(S)$ to be $\sum_{x \in S} f(x)$.

▶ **Lemma 10** (Walczak [17, Page 222]). *For every positive integer k, there exists a function $w_k\colon V(G_k) \to \mathbb{N}$ such that $w_k(V(G_k)) = \frac{k+1}{2} \cdot 2^{2^{k-1}-1}$, and $w_k(I) \leq 2^{2^{k-1}-1}$ for every stable set I of G_k.*

For two graphs G and H a map $f\colon V(G) \to V(H)$ is a *(graph) homomorphism* from G to H if $f(u)f(v) \in E(H)$ for every $uv \in E(G)$. It is well-known (see for example [10, Corollary 1.8]) that if there exists a homomorphism from G to H, then $\chi(G) \leq \chi(H)$. An analogous statement holds for the tree-chromatic number.

▶ **Lemma 11.** *Let G and H be graphs. If there exists a homomorphism from G to H, then* tree-$\chi(G) \leq$ tree-$\chi(H)$.

Proof. Let $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ be a tree-decomposition of H such that $\chi(\mathcal{T}) =$ tree-$\chi(H)$. Let f be a homomorphism from G to H. For every $t \in V(T)$, let $X'_t = \bigcup_{v \in X_t} f^{-1}(\{v\})$. Note that $(T, \{X'_t\}_{t \in V(T)})$ is a tree-decomposition of G.

Since f is a homomorphism, $\bigcup_{x \in S} f^{-1}(\{x\})$ is a stable set in G for every stable set S in H. Hence $\chi(G[X'_t]) \leq \chi(H[X_t]) \leq \chi(\mathcal{T}) =$ tree-$\chi(H)$ for every $t \in V(T)$. ◀

Next, we construct a blowup of each G_k based on function w_k in Lemma 10. That is, each vertex $v \in V(G_k)$ is blown up to a set of $w_k(v)$ pairwise non-adjacent vertices. We summarize the crucial properties of the resulting graph H_k.

▶ **Lemma 12.** *For every positive integer k, there exists a graph H_k such that*
1. $|V(H_k)| = \frac{k+1}{2} \cdot 2^{2^{k-1}-1}$,
2. $\alpha(H_k) \leq \frac{2}{k+1}|V(H_k)| < \frac{2|V(H_k)|}{\log_2 \log_2 |V(H_k)|}$, *and*
3. tree-$\chi(H_k) = 2$.

Proof. Let k be a positive integer. By Lemma 10, there exists a function $w_k\colon V(G_k) \to \mathbb{N}$ such that $w_k(V(G_k)) = \frac{k+1}{2} \cdot 2^{2^{k-1}-1}$, and $w_k(I) \leq 2^{2^{k-1}-1}$ for every stable set I of G_k.

For every $v \in V(G_k)$, let S_v be a set consisting of $w_k(v)$ pairwise non-adjacent vertices. Define H_k to be the graph obtained from the disjoint union of S_v (for all $v \in V(G_k)$) by adding edges such that every vertex in S_a is adjacent in H_k to every vertex in S_b for any $ab \in E(G_k)$. Then $|V(H_k)| = w_k(V(G_k)) = \frac{k+1}{2} \cdot 2^{2^{k-1}-1}$.

Let A be a stable set in H_k. By the definition of H_k, there exists no $ab \in E(G_k)$ such that $A \cap S_a \neq \emptyset$ and $A \cap S_b \neq \emptyset$. Hence there exists a stable set I_A of G_k such that $A \subseteq \bigcup_{v \in I_A} S_v$. So $|A| \leq \sum_{v \in I_A} |S_v| = w_k(I_A) \leq 2^{2^{k-1}-1}$.

Therefore,

$$\alpha(H_k) \leq 2^{2^{k-1}-1} = \frac{2}{k+1}|V(H_k)|. \tag{1}$$

This inequality allows us to bound $\alpha(H_k)$ as a function of $|V(H_k)|$ and k, however, we want to express this bound only as a function of $|V(H_k)|$. Indeed, we can achieve this by using the facts $|V(H)| = \frac{k+1}{2} \cdot 2^{2^{k-1}-1}$ and $2^{k-1} \geq \log_2(k+1)$ as follows.

$$\begin{aligned}
\log_2 \log_2 |V(H_k)| &= \log_2 \log_2 \left(\frac{k+1}{2} \cdot 2^{2^{k-1}-1} \right) \\
&= \log_2 \left(\log_2(k+1) + 2^{k-1} - 2 \right) \\
&\leq \log_2 \left(2^{k-1} + 2^{k-1} \right) \\
&= k.
\end{aligned}$$

By combining this inequality with Equation (1),

$$\alpha(H_k) \leq \frac{2}{k+1}|V(H_k)| < \frac{2|V(H_k)|}{\log_2 \log_2 |V(H_k)|}.$$

By Lemma 9, there exists a tree-decomposition $\mathcal{T}$ of G_k such that every bag induces a star forest. So tree-$\chi(G_k) \leq \chi(\mathcal{T}) \leq 2$. Since the function $\phi\colon V(H_k) \to V(G_k)$ mapping each vertex $h \in V(H_k)$ to the vertex $v \in V(G_k)$ such that $h \in S_v$ is a homomorphism from H_k to G_k, we have tree-$\chi(H_k) \leq$ tree-$\chi(G_k) \leq 2$ by Lemma 11. Since $E(H_k) \neq \emptyset$, every tree-decomposition of H_k has a bag that contains an edge, so tree-$\chi(H_k) \geq 2$. This shows tree-$\chi(H_k) = 2$. ◀

Using the 1-completion $C(H_k)$ of the graphs H_k from Lemma 12, we are ready to prove our main result from which Theorem 2 will easily follow.

▶ **Theorem 13.** *For any functions $f\colon \mathbb{R} \to \mathbb{R}_{>0}$ and $g\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ with $g(x) = o(\log \log x)$ as $x \to \infty$, there exists a graph G such that*

$$\mathrm{tw}(G) > \text{tree-}\alpha(G) \cdot f(\text{tree-}\chi(G)) \cdot g(\mathrm{tw}(G)+1).$$

Proof. Suppose to the contrary that $\mathrm{tw}(G) \leq \text{tree-}\alpha(G) \cdot f(\text{tree-}\chi(G)) \cdot g(\mathrm{tw}(G)+1)$ for every graph G.

For every positive integer k, let H_k be the graph mentioned in Lemma 12. Then, by our assumption, $\mathrm{tw}(C(H_k)) \leq \text{tree-}\alpha(C(H_k)) \cdot f(\text{tree-}\chi(C(H_k))) \cdot g(\mathrm{tw}(C(H_k))+1)$. When $k \geq 2$, we have $|V(H_k)| \geq 3$, so Lemmas 5 and 6 imply

$$\begin{aligned}
|V(H_k)| - 1 = \mathrm{tw}(C(H_k)) &\leq \text{tree-}\alpha(C(H_k)) \cdot f(\text{tree-}\chi(C(H_k))) \cdot g(\mathrm{tw}(C(H_k))+1) \\
&\leq \alpha(H_k) \cdot f(\text{tree-}\chi(H_k)) \cdot g(|V(H_k)|).
\end{aligned}$$

By Statements 2 and 3 in Lemma 12,

$$\begin{aligned}|V(H_k)| - 1 &\le \alpha(H_k) \cdot f(\text{tree-}\chi(H_k)) \cdot g(|V(H_k)|)\\ &\le \frac{2|V(H_k)|}{\log_2 \log_2 |V(H_k)|} \cdot f(2) \cdot g(|V(H_k)|).\end{aligned}$$

Dividing both sides by $2f(2) \cdot |V(H)|$, we get:

$$\frac{1}{2f(2)}\left(1 - \frac{1}{|V(H_k)|}\right) \le \frac{g(|V(H_k)|)}{\log_2 \log_2 |V(H_k)|}. \tag{2}$$

Since $|V(H_k)| \to \infty$ as $k \to \infty$ by Statement 1 in Lemma 12 and $g(x) = o(\log \log x)$ as $x \to \infty$ by assumption, Equation (2) implies

$$0 < \frac{1}{2f(2)} = \lim_{k\to\infty} \frac{1}{2f(2)}\left(1 - \frac{1}{|V(H_k)|}\right) \le \lim_{k\to\infty} \frac{g(|V(H_k)|)}{\log_2 \log_2 |V(H_k)|} = 0,$$

a contradiction. ◀

Now we deduce Theorem 2 from Theorem 13.

▶ **Corollary 14.** *For any functions $f\colon \mathbb{R} \to \mathbb{R}_{>0}$ and $h\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $h(x)/x$ is non-decreasing and $h(x) = o(x \log \log x)$ as $x \to \infty$, there exists a graph G such that*

$$\text{tw}(G) > h(\text{tree-}\alpha(G)) \cdot f(\text{tree-}\chi(G)).$$

Proof. Let $g(x) = h(x)/x$ for every $x > 0$. Then $g(x) = o(\log \log x)$ as $x \to \infty$. By Theorem 13, there exists a graph G such that $\text{tw}(G) > \text{tree-}\alpha(G) \cdot f(\text{tree-}\chi(G)) \cdot g(\text{tw}(G)+1)$. Since $g(x) = h(x)/x$ is non-decreasing, $g(\text{tw}(G) + 1) \ge g(\text{tree-}\alpha(G))$. Therefore,

$$\text{tw}(G) > \text{tree-}\alpha(G) \cdot f(\text{tree-}\chi(G)) \cdot g(\text{tree-}\alpha(G)) = h(\text{tree-}\alpha(G)) \cdot f(\text{tree-}\chi(G)). \quad ◀$$

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