

The number of regular simplices in higher dimensions

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Abstract

We study the extremal function $S_d^k(n)$, defined as the maximum number of regular $(k - 1)$ -simplices spanned by n points in $\mathbb{R}^d$. For any fixed $d \geq 2k \geq 6$, we determine the asymptotic behavior of $S_d^k(n)$ up to a lower-order term. In particular, when $k = 3$, we determine the exact value of $S_d^3(n)$, for all even dimensions $d \geq 6$ and sufficiently large n . This resolves a conjecture of Erdős in a stronger form. The proof leverages techniques from hypergraph Turán theory and linear algebra.

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1 | INTRODUCTION

In 1946, Erdős [9] posed a deceptively simple question: What is the maximum number of times the unit distance can occur among n points in the plane? He proved a lower bound of $n^{1+\Omega(1/\log \log n)}$ and conjectured that his lower bound is essentially optimal. See [27] for a historical account of the conjecture. Contrary to long-standing belief, this conjecture was recently disproved by an internal model of OpenAI [19]. By generalizing Erdős's original construction, the AI model established a lower bound of the form n^{1+c} , where $c > 0$ is an absolute constant. See also [5] for a simplified proof and further remarks. Shortly thereafter, Sawin [24] improved the

constant to $c = 0.014$. The pursuit of this problem has inspired substantial research and has also led to the formulation of numerous related questions in combinatorial geometry, all revolving around a common theme: determining the maximum number of certain geometric configurations that can be found among n points in Euclidean space. In this paper, we address one such problem.

The following question was raised repeatedly by Erdős and Purdy [11, 14–16]: Given $d, n \in \mathbb{N}$, what is the maximum number $T_d(n)$ of equilateral triangles that can be determined by n points in $\mathbb{R}^d$? There is a substantial body of work addressing this problem and its generalizations; see, for example, [1–4, 6, 15, 21, 25]. Ábrego and Fernández-Merchant [1] proved that $T_2(n) = \Theta(n^2)$, which also yields the lower bound $T_4(n) \geq T_3(n) = \Omega(n^2)$. For $d = 5$, Ábrego and Fernández-Merchant [1] established a better lower bound $T_5(n) = \Omega(n^{7/3})$. On the other hand, Sharir, Sheffer, and Zahl [25] proved that $T_3(n) = O(n^{15/7})$, and Agarwal, Apfelbaum, Purdy, and Sharir [2] showed that $T_4(n) = O(n^{12/5})$ and $T_5(n) = O(n^{8/3})$.

The first nondegenerate case, that is, when $\liminf_{n \rightarrow \infty} T_d(n)/\binom{n}{3} > 0$, appears for $d = 6$. Erdős and Purdy [15] showed that $T_6(n) \geq n^3/27 - O(n^2)$ by placing n points evenly on three pairwise orthogonal circles. Regarding the upper bound, Erdős and Purdy [15] said, “It would be interesting even to show $T_6(n) \leq (1/6 - \varepsilon)n^3$.” In a later paper [12], Erdős proposed the following more precise conjecture.

Conjecture 1 (Erdős [12]). *Any n points in $\mathbb{R}^6$ can span at most $n^3/27 + o(n^3)$ equilateral triangles, that is, $T_6(n) \leq n^3/27 + o(n^3)$.*

We prove this conjecture in a more general form. For an integer $k \geq 3$, a regular $(k-1)$ -simplex is a set of k vertices that are equidistant from one another. Let $S_d^k(n)$ denote the maximum number of regular $(k-1)$ -simplices spanned by n points in $\mathbb{R}^d$. Our first theorem determines the asymptotic behavior of $S_d^k(n)$ for all fixed $d \geq 2k \geq 6$.

Theorem 2. *Let $d \geq 2k \geq 6$ be fixed integers and let $r = \lfloor d/2 \rfloor$. Then*

$$S_d^k(n) = \binom{r}{k} \left(\frac{n}{r}\right)^k + o(n^k).$$

Since $T_d(n) = S_d^3(n)$, Theorem 2 implies Conjecture 1 upon taking $d = 6$ and $k = 3$. Our method also yields a stronger version of Conjecture 1. That is, we are able to determine the exact value of $T_d(n)$ for every even integer $d \geq 6$ and sufficiently large n .

Theorem 3. *Let $r \geq 3$ be a fixed integer. Let $n \in \mathbb{N}$ be sufficiently large and p be the remainder of n divided by $2r$. Then*

$$T_{2r}(n) = \sum_{1 \leq i < j < k \leq r} n_i n_j n_k + \sum_{i \in [r]} \left((n_i - \mathbb{1}_{n_i \notin 4\mathbb{Z}})(n - n_i) + \frac{n_i - p_i}{3} + \mathbb{1}_{p_i > 8}(p_i - 8) \right),$$

where $(n_1, \dots, n_r)$ with $\sum_{i=1}^r n_i = n$ is chosen to be

$$(n_1, \dots, n_r) = \begin{cases} (\underbrace{\lfloor n/r \rfloor, \dots, \lfloor n/r \rfloor}_{r-p/2}, \underbrace{\lfloor n/r \rfloor + 2, \dots, \lfloor n/r \rfloor + 2}_{p/2}) & \text{if } p \in [0, r) \cap 2\mathbb{Z}, \\ (\underbrace{\lfloor n/r \rfloor, \dots, \lfloor n/r \rfloor}_{r-(p+1)/2}, \underbrace{\lfloor n/r \rfloor + 1, \lfloor n/r \rfloor + 2, \dots, \lfloor n/r \rfloor + 2}_{(p-1)/2}) & \text{if } p \in [0, r) \setminus 2\mathbb{Z}, \\ (\underbrace{\lfloor n/r \rfloor - 1, \dots, \lfloor n/r \rfloor - 1}_{r-p/2}, \underbrace{\lfloor n/r \rfloor + 1, \dots, \lfloor n/r \rfloor + 1}_{p/2}) & \text{if } p \in [r, 2r) \cap 2\mathbb{Z}, \\ (\underbrace{\lfloor n/r \rfloor - 1, \dots, \lfloor n/r \rfloor - 1}_{r-(p+1)/2}, \underbrace{\lfloor n/r \rfloor, \lfloor n/r \rfloor + 1, \dots, \lfloor n/r \rfloor + 1}_{(p-1)/2}) & \text{if } p \in [r, 2r) \setminus 2\mathbb{Z}, \end{cases}$$

and p_i is the remainder of n_i divided by 12 for each $i \in [r]$.

In particular, when n divides nicely, Theorem 3 gives a simple and compact formula for $T_{2r}(n)$.

Corollary 4. Let $r \geq 3$ be a fixed integer. If $n \in \mathbb{N}$ is sufficiently large and divisible by $12r$, then

$$T_{2r}(n) = \binom{r}{3} \left(\frac{n}{r}\right)^3 + \frac{(r-1)n^2}{r} + \frac{n}{3}.$$

Our next result is the counterpart of Theorem 3 for $k \geq 4$. Throughout this paper, whenever we write a function, we assume that all its variables are nonnegative integers.

Theorem 5. Let $r \geq k \geq 4$ be fixed integers and $n \in \mathbb{N}$ be sufficiently large. Then

$$S_{2r}^k(n) = \max_{n_1 + \dots + n_r = n} f_k(n_1, \dots, n_r),$$

where

$$f_k(n_1, \dots, n_r) = \left(\sum_{I \in \binom{[r]}{k}} \prod_{i \in I} n_i \right) + \left(\sum_{1 \leq \ell \leq \lfloor k/2 \rfloor} \sum_{J \in \binom{[r]}{\ell}} \left(\prod_{j \in J} (n_j - \mathbb{1}_{n_j \notin 4\mathbb{Z}}) \right) \left(\sum_{I \in \binom{[r] \setminus J}{k-2\ell}} \prod_{i \in I} n_i \right) \right).$$

One can readily verify that $\max_{n_1 + \dots + n_r = n} f_k(n_1, \dots, n_r)$ must be attained by some $(n_1, \dots, n_r)$ satisfying $|n_i - n/r| = O(1)$ for all $i \in [r]$. However, we make no attempt here to identify the exact maximizer, as it appears to depend intricately on the values of r and k . Combining Theorems 3 and 5, we determine $S_{2r}^k(n)$ up to a constant in the lower-order term for all $r \geq k \geq 3$. Note that this improves upon Theorem 2 in the even-dimensional case.

Corollary 6. Let $r \geq k \geq 3$ be fixed integers. Then

$$S_{2r}^k(n) = \binom{r}{k} \left(\frac{n}{r}\right)^k + \Theta(n^{k-1}).$$

The main tool for obtaining the precise estimates in Theorems 3 and 5 is the following stability result, which describes the structure of any almost-extremal point set. We say that two circles are orthogonal if the affine spaces spanned by them are orthogonal.

Theorem 7. *Let $r \geq k \geq 3$ be fixed integers. If a set $X \subseteq \mathbb{R}^{2r}$ of n points spans $S_{2r}^k(n) - o(n^k)$ regular $(k-1)$ -simplices, then one can find a partition $X = A_0 \dot{\cup} A_1 \dot{\cup} \dots \dot{\cup} A_r$ with $|A_0| = o(n)$ and $|A_i| = n/r - o(n)$ for every $i \in [r]$, and pairwise orthogonal circles $C_1, \dots, C_r \subseteq \mathbb{R}^{2r}$, such that*

- (1) $A_i \subseteq C_i$ for all $i \in [r]$;
- (2) $C_1, \dots, C_r$ have the same center and the same radius.

The proof of Theorem 7 proceeds in two steps. First, we identify certain forbidden configurations in the point set using linear algebra. Then we apply techniques from hypergraph Turán theory to analyze its structure.

The paper is organized as follows. In Section 2 we introduce basic geometric notions and present the lower bound constructions for $S_d^k(n)$ needed for Theorems 2, 3, and 5. In Section 3, we collect known lemmas and establish additional ones concerning hypergraph theory and the orthogonality of affine spaces. In Section 4 we give the proof of Theorem 2. In Section 5 we establish Theorem 7. In Section 6 we prove Theorems 3 and 5 and derive Corollary 6. Section 7 contains concluding remarks and discusses further consequences of our method.

2 | LOWER BOUNDS: THE LENZ CONSTRUCTION

In this section, we present the Lenz construction of n points in $\mathbb{R}^d$, which was initially used to give a tight lower bound on the number of unit distances in $\mathbb{R}^d$ for even $d \geq 4$ (see, e.g., [26]). Here we use refinements of this construction to obtain lower bounds on the number of regular simplices, that is, the lower bounds for Theorems 2, 3, and 5.

We begin by introducing some basic notions used in the construction and throughout the paper.

2.1 | Basic notions

We shall give the definitions of affine spaces, spheres, and orthogonality.

We say that a nonempty subset $\mathcal{A} \subseteq \mathbb{R}^d$ is an affine space if there exists $x \in \mathbb{R}^d$ and a linear subspace $S \subseteq \mathbb{R}^d$, such that $\mathcal{A} = x + S := \{x + s : s \in S\}$. Such an S is called the associated linear space of $\mathcal{A}$. Observe that if $\mathcal{A}$ is an affine space with associated linear space S , then $S = \mathcal{A} - a := \{a' - a : a' \in \mathcal{A}\}$ for any $a \in \mathcal{A}$. Hence, every affine space has a unique associated linear space.

The dimension of an affine space $\mathcal{A} \subseteq \mathbb{R}^d$, denoted by $\dim \mathcal{A}$, is defined as the dimension of its associated linear space. Two affine spaces are said to be orthogonal if their associated linear spaces are orthogonal. It follows from the definition that if $\mathcal{A}_1, \dots, \mathcal{A}_r \subseteq \mathbb{R}^d$ are pairwise orthogonal affine spaces, then $\sum_{i \in [r]} \dim \mathcal{A}_i \leq \dim \mathbb{R}^d = d$.

Given a nonempty subset $A \subseteq \mathbb{R}^d$, let $\text{Aff}(A)$ denote the affine space spanned by A . That is,

$$\text{Aff}(A) := a + \text{Span}(A - a) \quad \text{for any } a \in A,$$

where $\text{Span}(A - a)$ represents the linear subspace of $\mathbb{R}^d$ spanned by $A - a$. For further definitions concerning affine spaces, we refer the reader to the book by Roman [22].

For $d \in \mathbb{N}$, a $(d - 1)$ -dimensional sphere is defined as the set of all points in a d -dimensional affine space $\mathcal{A}$ at some distance $\gamma > 0$ from a given point $c \in \mathcal{A}$. The point c and the distance γ are referred to as the center and the radius of the sphere, respectively. For example, a set of two points is a zero-dimensional sphere, and a circle is a one-dimensional sphere. A sphere is called unit if its radius is equal to 1. Two spheres are orthogonal if the affine spaces spanned by them are orthogonal.

2.2 | Lenz construction for even dimensions

Let $r \geq k \geq 3$ be integers. For $n_1, \dots, n_r \in \mathbb{Z}_{\geq 0}$, we define the function

$$f_k(n_1, \dots, n_r) := \left(\sum_{I \in \binom{[r]}{k}} \prod_{i \in I} n_i \right) + \left(\sum_{1 \leq \ell \leq \lfloor k/2 \rfloor} \sum_{J \in \binom{[r]}{\ell}} \left(\prod_{j \in J} (n_j - \mathbb{1}_{n_j \notin 4\mathbb{Z}}) \right) \left(\sum_{I \in \binom{[r] \setminus J}{k-2\ell}} \prod_{i \in I} n_i \right) \right) \\ + \mathbb{1}_{k=3} \left(\sum_{i \in [r]} \frac{n_i - p_i}{3} + \mathbb{1}_{p_i > 8} (p_i - 8) \right),$$

where p_i is the remainder of n_i divided by 12 for each $i \in [r]$. We will show by construction that

$$S_{2r}^k(n) \geq \max_{n_1 + \dots + n_r = n} f_k(n_1, \dots, n_r).$$

Let $C_1, \dots, C_r \subseteq \mathbb{R}^{2r}$ be pairwise orthogonal unit circles with a common center. Such circles can always be picked. For example, we can let C_i be a unit circle in

$$\underbrace{\{0\} \times \dots \times \{0\}}_{2(i-1)} \times \mathbb{R}^2 \times \underbrace{\{0\} \times \dots \times \{0\}}_{2r-2i}$$

centered at the origin $(0, \dots, 0)$, for all $i \in [r]$.

Let $n_1, \dots, n_r \in \mathbb{Z}_{\geq 0}$ with $n_1 + \dots + n_r = n$ maximizing $f(n_1, \dots, n_r)$, and for each $i \in [r]$ let p_i be the remainder of n_i divided by 12. Let $F = \{v_1, \dots, v_{12}\}$ denote a configuration of 12 points arranged on a unit circle to form a regular dodecagon, where $v_1, \dots, v_{12}$ are labeled clockwise, see Figure 1 for an illustration. Note that every point in F is contained in an inscribed square of side length $\sqrt{2}$ and an equilateral triangle of side length $\sqrt{3}$.

For any $i \in [r]$, we place n_i points on the circle C_i in the following way. Since $n_i - p_i$ is a multiple of 12, we can place the first $n_i - p_i$ points on C_i by taking $(n_i - p_i)/12$ pairwise disjoint copies of F . For the remaining p_i points, we place them consecutively in a new copy of F , disjoint from the previous ones, following the order $v_1, v_4, v_7, v_{10}, v_2, v_5, v_8, v_{11}, v_3, v_6, v_9, v_{12}$.

We now count the number of regular $(k - 1)$ -simplices spanned by these n points. There are three possible types of simplices, each corresponding to one of the three summands in $f_k(n_1, \dots, n_r)$:

- (1) regular $(k - 1)$ -simplices Δ_1 with $|\Delta_1 \cap C_i| \leq 1$ for all $i \in [r]$;

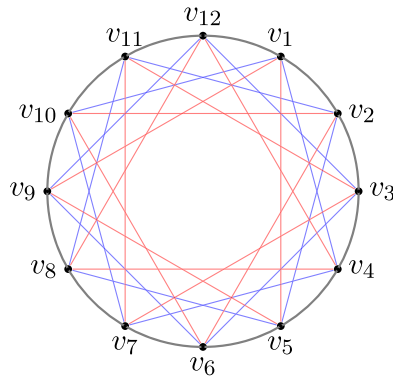


FIGURE 1 The point configuration of F on a unit circle, where the blue segments are of length $\sqrt{2}$ and the red segments are of length $\sqrt{3}$.

- (2) regular $(k-1)$ -simplices Δ_2 with $|\Delta_2 \cap C_i| = 2$ for some $i \in [r]$ and $|\Delta_2 \cap C_i| \leq 2$ for others;
- (3) regular $(k-1)$ -simplices Δ_3 containing at least three vertices on the same circle.

Since $C_1, \dots, C_r$ are pairwise orthogonal unit circles, the distance between any two points from different circles is $\sqrt{2}$. Thus, Δ_1 and Δ_2 both have side length $\sqrt{2}$, while Δ_3 , if it exists, has side length $\sqrt{3}$. The number of Δ_1 is clearly $\sum_{I \in \binom{[r]}{k}} \prod_{i \in I} n_i$.

To enumerate the second-type simplices, we need to count the pairs of points on the same circle at distance $\sqrt{2}$ apart, referred to as “good” pairs. For any $i \in [r]$, observe that every good pair on C_i consists of either two points from the first $n_i - p_i$ points, or two points from the remaining p_i points. For the first $n_i - p_i$ points, since each of them is at distance $\sqrt{2}$ from two other points, we have $n_i - p_i$ good pairs. For the remaining p_i points, due to the placement scheme, there are p_i good pairs if p_i is divisible by 4, otherwise we have $p_i - 1$ good pairs. Hence, the number of good pairs on C_i is equal to $n_i - \mathbb{1}_{p_i \notin 4\mathbb{Z}} = n_i - \mathbb{1}_{n_i \notin 4\mathbb{Z}}$. Then for $1 \leq \ell \leq \lfloor k/2 \rfloor$, the number of Δ_2 containing exactly ℓ good pairs, each pair on a different circle, is equal to

$$\sum_{J \in \binom{[r]}{\ell}} \left(\prod_{j \in J} (n_j - \mathbb{1}_{n_j \notin 4\mathbb{Z}}) \right) \left(\sum_{I \in \binom{[r] \setminus J}{k-2\ell}} \prod_{i \in I} n_i \right).$$

Accordingly, the total number of Δ_2 is equal to

$$\sum_{1 \leq \ell \leq \lfloor k/2 \rfloor} \sum_{J \in \binom{[r]}{\ell}} \left(\prod_{j \in J} (n_j - \mathbb{1}_{n_j \notin 4\mathbb{Z}}) \right) \left(\sum_{I \in \binom{[r] \setminus J}{k-2\ell}} \prod_{i \in I} n_i \right).$$

For the number of Δ_3 , since there is no third-type simplex when $k \geq 4$, we only consider the case when $k = 3$. In this case, Δ_3 is an equilateral triangle with three vertices on the same circle. For each circle C_i , any equilateral triangle on C_i is formed by either three points from the first $n_i - p_i$ points or three points from the remaining p_i points. Among the first $n_i - p_i$ points, since each of them is contained in one inscribed equilateral triangle, we have $(n_i - p_i)/3$ third-type simplices. Among the remaining p_i points, there are $p_i - 8$ third-type simplices if $p_i > 8$, otherwise we have

no third-type simplices. Therefore, the number of Δ_3 in our construction is at least

$$\mathbb{1}_{k=3} \left(\sum_{i \in [r]} \frac{n_i - p_i}{3} + \mathbb{1}_{p_i > 8} (p_i - 8) \right).$$

In total, the number of regular $(k-1)$ -simplices given by this construction is at least

$$\begin{aligned} & \left(\sum_{I \in \binom{[r]}{k}} \prod_{i \in I} n_i \right) + \left(\sum_{1 \leq \ell \leq \lfloor k/2 \rfloor} \sum_{J \in \binom{[r]}{\ell}} \left(\prod_{j \in J} (n_j - \mathbb{1}_{n_j \notin 4\mathbb{Z}}) \right) \left(\sum_{I \in \binom{[r] \setminus J}{k-2\ell}} \prod_{i \in I} n_i \right) \right) \\ & + \mathbb{1}_{k=3} \left(\sum_{i \in [r]} \frac{n_i - p_i}{3} + \mathbb{1}_{p_i > 8} (p_i - 8) \right), \end{aligned}$$

consequently,

$$S_{2r}^k(n) \geq \max_{n_1 + \dots + n_r = n} f_k(n_1, \dots, n_r). \quad (1)$$

2.3 | Lenz construction for odd dimensions

Let $r \geq k \geq 3$ be integers. Let $C_1, \dots, C_r \subseteq \mathbb{R}^{2r+1}$ be pairwise orthogonal unit spheres with a common center, where $C_1, \dots, C_{r-1}$ are circles and C_r is a two-dimensional sphere.

We put n points on $C_1, \dots, C_r$ as evenly as possible, that is, each C_i receives $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$ points. Since the distance between any two points on different spheres is $\sqrt{2}$, the number of regular $(k-1)$ -simplices spanned by k points from pairwise distinct spheres is $\binom{r}{k} \left(\frac{n}{r} \right)^k - O(n^{k-1})$.

Moreover, Erdős, Hickerson, and Pach [13] proved that there is a configuration of m points on a two-dimensional unit sphere, such that the distance $\sqrt{2}$ occurs $\Omega(m^{4/3})$ times. The points on C_r are arranged according to this configuration, which yields $\Omega(n^{4/3})$ pairs of points on C_r at distance $\sqrt{2}$ apart. Note that every such pair forms a regular $(k-1)$ -simplex with any $k-2$ points from pairwise distinct circles among $C_1, \dots, C_{r-1}$. Hence, there are at least $\Omega(n^{k-2/3})$ regular $(k-1)$ -simplices spanned by 2 points from C_r and $k-2$ points from $C_1, \dots, C_{r-1}$. Overall we obtain that

$$S_{2r+1}^k(n) \geq \binom{r}{k} \left(\frac{n}{r} \right)^k + \Omega(n^{k-2/3}). \quad (2)$$

3 | PRELIMINARY RESULTS AND LEMMAS

3.1 | Turán-type results for hypergraphs

Given an integer $k \geq 2$, a k -uniform hypergraph H is an ordered pair $(V(H), E(H))$, where $V(H)$ is the set of vertices and $E(H) \subseteq \binom{V(H)}{k}$ is the set of hyperedges. Let $v(H)$ and $e(H)$ denote the numbers of vertices and hyperedges in H , respectively.

A k -uniform hypergraph H is called r -partite for some integer $r \geq k$ if there exists a partition $V(H) = V_1 \dot{\cup} V_2 \dot{\cup} \dots \dot{\cup} V_r$, such that every hyperedge in H has at most one vertex from each part V_i .

For a k -uniform hypergraph H , the extremal number $\text{ex}(n, H)$ is the largest number of hyperedges in an n -vertex k -uniform hypergraph containing no copy of H . Given an integer $t \geq 1$, the t -blowup of H , denoted by $H(t)$, is a k -uniform hypergraph obtained from H by replacing every vertex $v \in V(H)$ with an independent set I_v of size t and replacing every hyperedge $e = \{v_1, \dots, v_k\} \in E(H)$ with a complete k -partite k -uniform hypergraph spanned by $I_{v_1} \dot{\cup} \dots \dot{\cup} I_{v_k}$. The independent set I_v is referred to as the blowup of the vertex v .

Definition 8. Given integers $r \geq k \geq 3$, $H_{r+1}^{(k)}$ is defined as a k -uniform hypergraph obtained from a complete graph K_{r+1} by enlarging each edge with $k - 2$ new vertices.

Note that by definition $H_{r+1}^{(k)}$ contains $(r + 1) + \binom{r+1}{2}(k - 2)$ vertices and $\binom{r+1}{2}$ hyperedges.

Lemma 9 (Mubayi [18]). For any fixed integers $r \geq k \geq 3$, $\text{ex}(n, H_{r+1}^{(k)}) = \binom{r}{k} \left(\frac{n}{r}\right)^k + o(n^k)$.

For any k -uniform hypergraph H and $t \in \mathbb{N}$, it is well-known (see, e.g., [17]) that

$$\text{ex}(n, H(t)) = \text{ex}(n, H) + o(n^k).$$

Thus, by Lemma 9 we have

$$\text{ex}(n, H_{r+1}^{(k)}(t)) = \text{ex}(n, H_{r+1}^{(k)}) + o(n^k) = \binom{r}{k} \left(\frac{n}{r}\right)^k + o(n^k). \quad (3)$$

Furthermore, we shall prove the following stability lemma for $H_{r+1}^{(k)}(t)$.

Lemma 10. Let $r \geq k \geq 3$ and $t \geq 1$ be fixed integers. If G is an n -vertex $H_{r+1}^{(k)}(t)$ -free k -uniform hypergraph with $\text{ex}(n, H_{r+1}^{(k)}(t)) - o(n^k)$ hyperedges, then G contains an r -partite subhypergraph with $\binom{r}{k}(n/r)^k - o(n^k)$ hyperedges.

We deduce Lemma 10 by a standard argument, using a stability lemma for $H_{r+1}^{(k)}$ and the hypergraph removal lemma.

Lemma 11 (Pikhurko [20]). Let $r \geq k \geq 3$ be fixed integers. Every n -vertex $H_{r+1}^{(k)}$ -free k -uniform hypergraph with $\text{ex}(n, H_{r+1}^{(k)}) - o(n^k)$ hyperedges contains an r -partite subhypergraph with $\binom{r}{k}(n/r)^k - o(n^k)$ hyperedges.

Lemma 12 (Rödl–Nagle–Skokan–Schach [23]). Let $k \geq 2$ be an integer and H be a fixed k -uniform hypergraph. If an n -vertex k -uniform hypergraph G contains $o(n^{v(H)})$ copies of H , then one can remove all the copies of H in G by deleting $o(n^k)$ hyperedges.

Proof of Lemma 10. Let G be an n -vertex $H_{r+1}^{(k)}(t)$ -free k -uniform hypergraph with $\text{ex}(n, H_{r+1}^{(k)}(t)) - o(n^k)$ hyperedges. By (3) we have $e(G) = \text{ex}(n, H_{r+1}^{(k)}) - o(n^k)$. Note that the hypergraph G cannot

contain $\Omega(n^{v(H_{r+1}^{(k)})})$ copies of $H_{r+1}^{(k)}$, as otherwise it would contain a copy of $H_{r+1}^{(k)}(t)$. Now that the number of copies of $H_{r+1}^{(k)}$ in G is $o(n^{v(H_{r+1}^{(k)})})$, by Lemma 12 there exists an $H_{r+1}^{(k)}$ -free subhypergraph G' of G with

$$e(G') = e(G) - o(n^k) = \text{ex}(n, H_{r+1}^{(k)}) - o(n^k)$$

hyperedges. The statement then follows from Lemma 11 applied to G' . $\square$

The following fact has been used implicitly in the literature (see, e.g., [18, 20]), and we include a proof here for completeness.

Lemma 13. *Let $r \geq k \geq 2$ be fixed integers. If an n -vertex r -partite k -uniform hypergraph has $\binom{r}{k}(n/r)^k - o(n^k)$ hyperedges, then each part must be of size $n/r - o(n)$.*

Proof of Lemma 13. Let G be such a hypergraph and let $n_1 \leq \dots \leq n_r$ denote the part sizes of G . For each $i \in [r]$ let $\alpha_i = n_i/n$ be the scaled part size. Note that $\alpha_1 + \dots + \alpha_r = 1$. Then

$$e(G) \leq \sum_{I \in \binom{[r]}{k}} \prod_{i \in I} n_i = n^k \sum_{I \in \binom{[r]}{k}} \prod_{i \in I} \alpha_i$$

By Maclaurin's inequality, see [7],

$$\sum_{I \in \binom{[r]}{k}} \prod_{i \in I} \alpha_i \leq \binom{r}{k} \left(\frac{1}{r}\right)^k \quad (4)$$

holds for all $\alpha_1, \dots, \alpha_r \geq 0$ with $\alpha_1 + \dots + \alpha_r = 1$. Moreover, the equality in (4) holds if and only if $\alpha_1 = \dots = \alpha_r = 1/r$.

Suppose for contradiction that $n_r \geq n/r + \delta n$ with some constant $\delta > 0$. Since $\alpha_r \geq 1/r + \delta$, the inequality in (4) is strict. Thus, there exists some constant $\varepsilon > 0$ such that $e(G) \leq \binom{r}{k}(n/r)^k - \varepsilon n^k$, a contradiction. $\square$

3.2 | Orthogonality of affine spaces

Let $\|\cdot\|$ denote the Euclidean norm. For $P, Q \subseteq \mathbb{R}^d$, we write $P \rightarrow Q$ if for every $p \in P$, $\|p - q\| = \|p - q'\|$ holds for all $q, q' \in Q$. In other words, $P \rightarrow Q$ if every point in P is equidistant to all points in Q . We write $P \leftrightarrow Q$ if $P \rightarrow Q$ and $Q \rightarrow P$. Recall that $\text{Aff}(P)$ denotes the affine space spanned by P .

Lemma 14. *Let $d \in \mathbb{N}$ and let $P, Q \subseteq \mathbb{R}^d$ be nonempty subsets with $|Q| \geq 2$ such that $P \rightarrow Q$. Then $\text{Aff}(P)$ and $\text{Aff}(Q)$ are orthogonal. Moreover, Q is contained in a sphere in $\text{Aff}(Q)$.*

Proof. For the first part of Lemma 14, if $|P| = 1$, then $\text{Aff}(P)$ and $\text{Aff}(Q)$ are clearly orthogonal as $\text{Aff}(P)$ consists of a single point. If $|P| \geq 2$, let $\{p_1, p_2, \dots\} \subseteq P$ and $\{q_1, q_2, \dots\} \subseteq Q$ be finite subsets

that span the affine spaces $\text{Aff}(P)$ and $\text{Aff}(Q)$, respectively. Then we have

$$\text{Aff}(P) = p_1 + \text{Span}(p_2 - p_1, \dots) \quad \text{and} \quad \text{Aff}(Q) = q_1 + \text{Span}(q_2 - q_1, \dots).$$

To show that $\text{Aff}(P)$ and $\text{Aff}(Q)$ are orthogonal, it suffices to check $\langle p_i - p_1, q_j - q_1 \rangle = 0$ for $i, j \geq 2$. Due to $P \rightarrow Q$, we have $\|p_1 - q_1\|^2 = \|p_1 - q_j\|^2$ and $\|p_i - q_1\|^2 = \|p_i - q_j\|^2$. Then

$$\begin{aligned} \langle p_i - p_1, q_j - q_1 \rangle &= \langle p_i, q_j \rangle - \langle p_i, q_1 \rangle - \langle p_1, q_j \rangle + \langle p_1, q_1 \rangle \\ &= \frac{\|p_i\|^2 + \|q_j\|^2 - \|p_i - q_j\|^2}{2} - \frac{\|p_i\|^2 + \|q_1\|^2 - \|p_i - q_1\|^2}{2} \\ &\quad - \frac{\|p_1\|^2 + \|q_j\|^2 - \|p_1 - q_j\|^2}{2} + \frac{\|p_1\|^2 + \|q_1\|^2 - \|p_1 - q_1\|^2}{2} = 0. \end{aligned}$$

Now we prove the second part of Lemma 14. Provided that $\text{Aff}(P)$ and $\text{Aff}(Q)$ are orthogonal, the projection of $\text{Aff}(P)$ onto $\text{Aff}(Q)$ has dimension 0, which is a point, denoted by c . Since each point of P is equidistant to all points in Q , its projection c is also equidistant to all points in Q . Thus, the points in Q all lie on a sphere in $\text{Aff}(Q)$ centered at c . $\square$

The following lemma is a direct consequence of Lemma 14.

Lemma 15. *Let $d, k \in \mathbb{N}$ with $k \geq 3$. Let $A_1, \dots, A_k \subseteq \mathbb{R}^d$ with $|A_i| \geq 3$ for all $i \in [k]$. If every k -tuple $(a_1, \dots, a_k) \in \prod_{i \in [k]} A_i$ forms a regular $(k-1)$ -simplex, then $\text{Aff}(A_1), \dots, \text{Aff}(A_k)$ are pairwise orthogonal affine spaces of dimension at least 2. Furthermore, $A_1, \dots, A_k$ are subsets of k spheres in $\text{Aff}(A_1), \dots, \text{Aff}(A_k)$, respectively.*

Proof of Lemma 15. First observe that $\|a_1 - a_2\| = \|a_1 - a'_2\|$ holds for every $a_1 \in A_1$ and $a_2, a'_2 \in A_2$. Otherwise for any $(a_3, \dots, a_k) \in \prod_{i=3}^k A_i$, either $(a_1, a_2, a_3, \dots, a_k)$ or $(a_1, a'_2, a_3, \dots, a_k)$ does not form a regular $(k-1)$ -simplex, a contradiction. Thus, $A_1 \rightarrow A_2$. Due to symmetry, we conclude that $A_i \leftrightarrow A_j$ for all distinct $i, j \in [k]$.

Then by Lemma 14, the affine spaces $\text{Aff}(A_1), \dots, \text{Aff}(A_k)$ are pairwise orthogonal and $A_1, \dots, A_k$ are subsets of k spheres in $\text{Aff}(A_1), \dots, \text{Aff}(A_k)$, respectively. Moreover, since each A_i is a subset of a sphere and $|A_i| \geq 3$, the points in A_i are noncollinear, which implies that the dimension of $\text{Aff}(A_i)$ is at least 2. $\square$

We also need the following lemma concerning the geometric properties of a collection of pairwise orthogonal spheres.

Lemma 16. *Let $d, k \in \mathbb{N}$ with $k \geq 3$. Let $C_1, \dots, C_k \subseteq \mathbb{R}^d$ be a collection of pairwise orthogonal spheres. For each $i \in [k]$ let $A_i \subseteq C_i$ be such that $\text{Aff}(A_i) = \text{Aff}(C_i)$.*

- (i) *If every k -tuple in $\prod_{i \in [k]} A_i$ forms a regular $(k-1)$ -simplex, then every k -tuple in $\prod_{i \in [k]} C_i$ forms a regular $(k-1)$ -simplex.*
- (ii) *If every k -tuple in $\prod_{i \in [k]} C_i$ forms a regular $(k-1)$ -simplex and $\sum_{i \in [k]} \dim \text{Aff}(C_i) = d$, then $C_1, \dots, C_k$ have the same center.*

Proof of Lemma 16. Let $c_1, \dots, c_r$ denote the centers of $C_1, \dots, C_r$, respectively.

We first prove item (i). Fix any distinct $i, j \in [k]$ and let $(x, y) \in C_i \times C_j$. Let $(a_i, a_j) \in A_i \times A_j$. Since every k -tuple in $\prod_{\ell \in [k]} A_\ell$ forms a regular $(k-1)$ -simplex, it suffices to show that $\|x - y\| = \|a_i - a_j\|$. Note that every k -tuple in $\prod_{\ell \in [k]} A_\ell$ forming a regular $(k-1)$ -simplex also implies that $A_i \leftrightarrow A_j$, from which it follows that the projection of $\text{Aff}(A_i) = \text{Aff}(C_i)$ onto $\text{Aff}(A_j) = \text{Aff}(C_j)$ is the center of a sphere in $\text{Aff}(A_j)$ containing A_j . Due to $\text{Aff}(A_j) = \text{Aff}(C_j)$, this sphere is exactly C_j . Hence, the projection of $\text{Aff}(C_i)$ onto $\text{Aff}(C_j)$ is the center c_j of C_j . Similarly, the projection of $\text{Aff}(C_j)$ onto $\text{Aff}(C_i)$ is the center c_i of C_i . Then we have that

$$\begin{aligned} \|x - y\|^2 &= \|x - c_i\|^2 + \|c_i - c_j\|^2 + \|y - c_j\|^2 = \|a_i - c_i\|^2 + \|c_i - c_j\|^2 + \|a_j - c_j\|^2 \\ &= \|a_i - a_j\|^2, \end{aligned}$$

which completes the proof of item (i).

For item (ii), since $\text{Aff}(C_1), \dots, \text{Aff}(C_k)$ are pairwise orthogonal and their dimensions sum to d , we can equip $\mathbb{R}^d$ with an orthonormal basis given by the union of $B_1, \dots, B_k$, where each B_i is an orthonormal basis of the linear subspace $\text{Aff}(C_i) - c_i$ and $|B_i| = \dim \text{Aff}(C_i) \geq 1$. Under this basis of $\mathbb{R}^d$, we can assume that

$$\text{Aff}(C_i) = c_i + \underbrace{\{0\} \times \dots \times \{0\}}_{\sum_{j < i} \dim \text{Aff}(C_j)} \times \mathbb{R}^{\dim \text{Aff}(C_i)} \times \underbrace{\{0\} \times \dots \times \{0\}}_{\sum_{j > i} \dim \text{Aff}(C_j)}$$

for all $i \in [k]$. Fix any distinct $i, j \in [k]$. It suffices to show that $c_i = c_j$. Since every k -tuple in $\prod_{\ell \in [k]} C_\ell$ forms a regular $(k-1)$ -simplex, the projections of c_i and c_j onto $\text{Aff}(C_1)$ are both c_1 . Consequently, the first $\dim \text{Aff}(C_1)$ coordinates of c_i and c_j must coincide. By the same reasoning, the remaining coordinates of c_i and c_j must also coincide, and therefore $c_i = c_j$. $\square$

4 | THE ASYMPTOTICS OF $S_d^k(n)$

In this section, we present a short proof of Theorem 2, which also serves as a warm-up for the arguments in the next section. Our key observation is the following lemma.

Lemma 17. *Let $d \geq 2k \geq 3$ be integers and let $r = \lfloor d/2 \rfloor$. Let G be a k -uniform hypergraph, whose vertices are points in $\mathbb{R}^d$ and whose hyperedges are the regular $(k-1)$ -simplices spanned by its vertices. Then G is $H_{r+1}^{(k)}(3)$ -free.*

Proof of Lemma 17. Suppose that G contains a copy of $H_{r+1}^{(k)}(3)$ and let

$$U_1, \dots, U_{r+1}, W_1, \dots, W_{\binom{r+1}{2}(k-2)}$$

be the blowups of the vertices, where $U_1, \dots, U_{r+1}$ correspond to the vertices of the original K_{r+1} . Then for any distinct $i, j \in [r+1]$, there exist some $W_{\ell_1}, \dots, W_{\ell_{k-2}}$, such that every

$$(u_i, u_j, w_{\ell_1}, \dots, w_{\ell_{k-2}}) \in U_i \times U_j \times W_{\ell_1} \times \dots \times W_{\ell_{k-2}}$$

appears as a hyperedge in G , namely, $(u_i, u_j, w_{\ell_1}, \dots, w_{\ell_{k-2}})$ determines a regular $(k-1)$ -simplex. Then by Lemma 15, in particular $\text{Aff}(U_i)$ and $\text{Aff}(U_j)$ are orthogonal affine spaces of dimension at least 2. Applying the same argument to all distinct $i, j \in [r+1]$, we obtain that $\text{Aff}(U_1), \dots, \text{Aff}(U_{r+1})$ are pairwise orthogonal and all have dimension at least 2. But then we have

$$2(r+1) \leq \sum_{i \in [r+1]} \dim \text{Aff}(U_i) \leq \dim \mathbb{R}^d = d < 2(r+1), \quad (5)$$

a contradiction. $\square$

Proof of Theorem 2. Let $r = \lfloor d/2 \rfloor \geq k \geq 6$ be fixed integers and $X \subseteq \mathbb{R}^d$ be a set of n points that spans $S_d^k(n)$ regular $(k-1)$ -simplices. We define a k -uniform hypergraph G , whose vertex set is X and whose hyperedges correspond to regular $(k-1)$ -simplices spanned by X .

By Lemma 17 we know that G is $H_{r+1}^{(k)}(3)$ -free. Then by (3) we have that

$$S_d^k(n) = e(G) \leq \binom{r}{k} \left(\frac{n}{r}\right)^k + o(n^k).$$

The lower bound on $S_d^k(n)$ follows from the construction described in Section 2, where we take r pairwise orthogonal unit circles centered at the origin and place n points on them as evenly as possible. This completes the proof. $\square$

5 | THE STABILITY PROPERTY OF AN ALMOST-EXTREMAL POINT SET

In this section we prove our stability result.

Proof of Theorem 7. Let $r \geq k \geq 3$ be fixed integers and let $X \subseteq \mathbb{R}^{2r}$ be a set of n points that spans at least $S_{2r}^k(n) - o(n^k)$ regular $(k-1)$ -simplices. Since $S_{2r}^k(n) = \binom{r}{k} (n/r)^k + o(n^k)$ by Theorem 2, X spans at least $\binom{r}{k} (n/r)^k - o(n^k)$ regular $(k-1)$ -simplices. Let G be a k -uniform hypergraph on the vertex set X , where $E(G)$ consists of all regular $(k-1)$ -simplices spanned by X .

By Lemma 17, G is $H_{r+1}^{(k)}(3)$ -free. Moreover, by (3) we have that

$$e(G) \geq \binom{r}{k} \left(\frac{n}{r}\right)^k - o(n^k) \geq \text{ex}(n, H_{r+1}^{(k)}(3)) - o(n^k).$$

It then follows from Lemma 10 that G contains an r -partite subhypergraph M with vertex classes $V_1 \dot{\cup} \dots \dot{\cup} V_r$ and $\binom{r}{k} (n/r)^k - o(n^k)$ hyperedges. Note that by Lemma 13 we have $|V_i| = n/r - o(n)$ for all $i \in [r]$.

We shall prove two claims that establish the structure of X described in Theorem 7.

Claim 18. For each $i \in [r]$ the following holds.

- (i) There exists $A_i \subseteq V_i$ with $|A_i| = n/r - o(n)$, such that $\dim \text{Aff}(A_i) = 2$.
- (ii) There is a circle C_i such that $A_i \subseteq C_i \subseteq \text{Aff}(A_i)$.

Proof of Claim 18. It suffices to prove the statement for the case $i = r$, as the remaining cases follow by the same argument. Fix a sufficiently small constant $\varepsilon > 0$. For any distinct $\ell_1, \dots, \ell_{k-1} \in [r-1]$, a $(k-1)$ -tuple $(v_{\ell_1}, \dots, v_{\ell_{k-1}}) \in \prod_{i \in [k-1]} V_{\ell_i}$ is called “good” if there are at least $|V_r| - \varepsilon n$ vertices $v_r \in V_r$ such that $v_{\ell_1} \dots v_{\ell_{k-1}} v_r$ is a hyperedge in M , otherwise it is “bad”. The total number of bad tuples is at most εn^{k-1} , because otherwise

$$e(M) \leq \left(\sum_{I \in \binom{[r-1]}{k-1}} \prod_{i \in I} |V_i| \right) - \varepsilon n^{k-1} \cdot \varepsilon n \leq \binom{r}{k} \left(\frac{n}{r} \right)^k - \varepsilon^2 n^k + o(n^k),$$

a contradiction. So if we take a $(k-1)$ -tuple $(v_{\ell_1}, \dots, v_{\ell_{k-1}}) \in \prod_{i \in [k-1]} V_{\ell_i}$ uniformly at random, the probability that $(v_{\ell_1}, \dots, v_{\ell_{k-1}})$ is bad is at most

$$\mathbb{P}_{\text{bad}} \leq \frac{\varepsilon n^{k-1}}{\prod_{i \in [k-1]} |V_{\ell_i}|} \leq r^{k-1} \varepsilon + o(1).$$

Now for each $i \in [r-1]$ we take a subset $V'_i \subseteq V_i$ of size 3 uniformly at random. Then by the union bound

$$\mathbb{P} \left(\forall I \in \binom{[r-1]}{k-1} : \text{all the } (k-1)\text{-tuples in } \prod_{i \in I} V'_i \text{ are good} \right) \geq 1 - \binom{r-1}{k-1} 3^{k-1} \mathbb{P}_{\text{bad}} > 0,$$

provided that ε is sufficiently small. Accordingly, there exists $V'_i \subseteq V_i$ of size 3 for each $i \in [r-1]$, such that all $(k-1)$ -tuples among them are good. Let $A_r \subseteq V_r$ be the common neighborhood of the $(k-1)$ -tuples in $\bigcup_{I \in \binom{[r-1]}{k-1}} \prod_{i \in I} V'_i$. Then

$$|A_r| \geq |V_r| - \binom{r-1}{k-1} 3^{k-1} \varepsilon n \geq \frac{n}{r} - (3r)^{k-1} \varepsilon n - o(n).$$

In particular, for any $I \in \binom{[r-1]}{k-1}$, it holds that every k -tuple in $(\prod_{i \in I} V'_i) \times A_r$ appears as a hyperedge in M , that is, a regular $(k-1)$ -simplex. By Lemma 15 we have that $\text{Aff}(V'_1), \dots, \text{Aff}(V'_{r-1}), \text{Aff}(A_r)$ are pairwise orthogonal affine spaces of dimension at least 2. Since

$$2r \leq \dim \text{Aff}(V'_1) + \dots + \dim \text{Aff}(V'_{r-1}) + \dim \text{Aff}(A_r) \leq \dim \mathbb{R}^{2r} = 2r,$$

it further holds that $\dim \text{Aff}(V'_1) = \dots = \dim \text{Aff}(V'_{r-1}) = \dim \text{Aff}(A_r) = 2$. Moreover, by Lemma 15 the points of A_r all lie on some circle $C_r \subseteq \text{Aff}(A_r)$.

We complete the proof by noting that ε can be chosen arbitrarily small. $\square$

Let $A_1, \dots, A_r$ and $C_1, \dots, C_r$ be given by Claim 18.

Claim 19. We have that

- (i) $C_1, \dots, C_r$ are pairwise orthogonal circles;
- (ii) $C_1, \dots, C_r$ have the same center and the same radius.

Proof of Claim 19. To prove item (i), it suffices to show that the affine planes $\text{Aff}(C_1) = \text{Aff}(A_1)$ and $\text{Aff}(C_2) = \text{Aff}(A_2)$ are orthogonal, as then by symmetry we have that $\text{Aff}(C_1), \dots, \text{Aff}(C_r)$ are pairwise orthogonal. Since $A_i \subseteq V_i$ with $|A_i| = n/r - o(n)$ for all $i \in [r]$, we have that

$$e(M[A_1, \dots, A_r]) = e(M) - o(n^k) = \binom{r}{k} \left(\frac{n}{r}\right)^k - o(n^k),$$

where $M[A_1, \dots, A_r]$ denotes the induced subhypergraph of M on $A_1 \cup \dots \cup A_r$. In particular, the number of hyperedges with one endpoint in each A_i for $i \in [k]$ is at least $(n/r)^k - o(n^k)$. Let $K_{3, \dots, 3}^{(k)}$ denote a complete k -partite k -uniform hypergraph with each part of size 3. A classical result of Erdős [10] shows that

$$\text{ex}(n, K_{\underbrace{3, \dots, 3}_k}^{(k)}) = o(n^k),$$

which implies that there exists $A'_i \subseteq A_i$ of size 3 for each $i \in [k]$, such that every k -tuple in $\prod_{i \in [k]} A'_i$ forms a hyperedge, that is, a regular $(k-1)$ -simplex. Then by Lemma 15, $\text{Aff}(A'_1)$ and $\text{Aff}(A'_2)$ are orthogonal affine spaces of dimension at least 2. Since $\dim \text{Aff}(A_1) = \dim \text{Aff}(A_2) = 2$, we have that $\text{Aff}(A'_1) = \text{Aff}(A_1)$ and $\text{Aff}(A'_2) = \text{Aff}(A_2)$. Thus $\text{Aff}(A_1)$ and $\text{Aff}(A_2)$ are orthogonal. This proves item (i).

We now prove item (ii).

First, we show that every r -tuple in $\prod_{i \in [r]} C_i$ forms a regular $(r-1)$ -simplex. In fact, it suffices to prove that for any $I \in \binom{[r]}{k}$, every k -tuple in $\prod_{i \in I} C_i$ forms a regular $(k-1)$ -simplex. Here we only consider the case $I = [k]$, as the remaining cases follow by symmetry. In the above proof of item (i) we already showed that there exists $A'_i \subseteq A_i \subseteq C_i$ with $\text{Aff}(A'_i) = \text{Aff}(A_i) = \text{Aff}(C_i)$ for each $i \in [k]$, such that every k -tuple in $\prod_{i \in [k]} A'_i$ forms a regular $(k-1)$ -simplex. Then by Lemma 16 (i) we can conclude that every k -tuple in $\prod_{i \in [k]} C_i$ forms a regular $(k-1)$ -simplex.

Next, we show that $C_1, \dots, C_r$ have the same center. From item (i) we know that $C_1, \dots, C_r$ are pairwise orthogonal circles with $\sum_{i \in [r]} \dim \text{Aff}(C_i) = 2r$. Since every r -tuple in $\prod_{i \in [r]} C_i$ forms a regular $(r-1)$ -simplex, by Lemma 16 (ii), $C_1, \dots, C_r$ have the same center.

Finally, we prove that $C_1, \dots, C_r$ have the same radius. For any distinct $i, j, \ell \in [r]$, let $\gamma_i, \gamma_j, \gamma_\ell$ denote the radii of C_i, C_j, C_ℓ , respectively. Since C_i, C_j, C_ℓ are pairwise orthogonal with the same center and every r -tuple in $\prod_{i \in [r]} C_i$ forms a regular $(r-1)$ -simplex, we have that $\gamma_i^2 + \gamma_j^2 = \gamma_j^2 + \gamma_\ell^2$, yielding that $\gamma_i = \gamma_j$. This completes the proof of item (ii). $\square$

Let $A_0 \subseteq X$ be the set of points that are not in $\bigcup_{i \in [r]} A_i$. In summary, we have shown that there exists a partition $X = A_0 \cup A_1 \cup \dots \cup A_r$ with $|A_0| = o(n)$ and $|A_i| = n/r - o(n)$ for every $i \in [r]$ (see Claim 18 (i)), and pairwise orthogonal circles $C_1, \dots, C_r \subseteq \mathbb{R}^d$ (see Claim 19 (i)), such that $A_i \subseteq C_i$ for all $i \in [r]$ (see Claim 18 (ii)) and $C_1, \dots, C_r$ have the same center and the same radius (see Claim 19 (ii)). This completes the proof of Theorem 7. $\square$

6 | THE NUMBER OF REGULAR $(k-1)$ -SIMPLICES IN EVEN DIMENSIONS

In this section, we prove Theorems 3 and 5 and derive Corollary 6 from them. We shall prove Theorem 5 first. The proof of Theorem 3 then follows with an additional optimization step.

6.1 | The general case

Proof of Theorem 5. Let $r \geq k \geq 3$ be fixed integers. Let $X \subseteq \mathbb{R}^{2r}$ be a set of n points that attains $S_{2r}^k(n)$, where n is sufficiently large. In order to establish Theorem 5, we shall first apply Theorem 7 to get a coarse structure of X , and then further examine this structure using the extremality of X .

By Theorem 7 we have a partition $X = A_0 \dot{\cup} A_1 \dot{\cup} \dots \dot{\cup} A_r$ with $|A_0| = o(n)$ and $|A_i| = n/r - o(n)$ for every $i \in [r]$, and pairwise orthogonal circles $C_1, \dots, C_r \subseteq \mathbb{R}^{2r}$ with $A_i \subseteq C_i$ for all $i \in [r]$, such that $C_1, \dots, C_r$ have the same center and the same radius $\gamma > 0$. In particular, we can assume that $A_0 = X \setminus (\bigcup_{i \in [r]} C_i)$.

Since $\text{Aff}(C_1), \dots, \text{Aff}(C_r)$ are pairwise orthogonal and intersect at the common center of $C_1, \dots, C_r$, we can further assume, without loss of generality, that the common center is the origin and

$$\text{Aff}(C_i) = \underbrace{\{0\} \times \dots \times \{0\}}_{2(i-1)} \times \mathbb{R}^2 \times \underbrace{\{0\} \times \dots \times \{0\}}_{2r-2i}$$

for all $i \in [r]$.

Claim 20. We have $X \subseteq \bigcup_{i \in [r]} C_i$, that is, $A_0 = \emptyset$.

Proof of Claim 20. Suppose there exists $v \in A_0 = X \setminus \bigcup_{i \in [r]} C_i$. Let $I(v) \subseteq [r]$ be the set of indices $i \in [r]$, such that the projection of v onto $\text{Aff}(C_i)$ is the center of C_i , that is, the origin. Observe that the distance between v and any point in $\bigcup_{i \in I(v)} C_i$ is $\sqrt{\|v\|^2 + \gamma^2}$.

Let N_1, N_2 , and N_3 denote the number of regular $(k-1)$ -simplices spanned by X that

- contain v and at least another point from A_0 ,
- are formed by v and at least two points on C_i for some $i \in [r]$,
- are formed by v and $k-1$ points from pairwise distinct circles,

respectively. The number of regular $(k-1)$ -simplices containing v is then at most $N_1 + N_2 + N_3$.

Since $|A_0| = o(n)$, we have $N_1 = o(n^{k-1})$.

For each $i \in [r]$, observe that the number of pairs of points in $X \cap C_i$ that can form a regular $(k-1)$ -simplex with v is at most $|X \cap C_i| \leq n$. Indeed, for any point $x \in C_i$ there are at most two other points in C_i at distance $\|v - x\|$ from x . Consequently, each point in $X \cap C_i$ is contained in at most two such pairs, and hence the total number of such pairs is at most $|X \cap C_i| \leq n$. It follows that $N_2 = O(n^{k-2})$.

To bound N_3 , first note that the distance between any two points on different circles is $\sqrt{2\gamma^2}$. Hence, N_3 is upper bounded by the number of $(k-1)$ -tuples $(x_1, \dots, x_{k-1}) \in C_{\ell_1} \times \dots \times C_{\ell_{k-1}}$ with $\|v - x_1\| = \dots = \|v - x_{k-1}\| = \sqrt{2\gamma^2}$, over all distinct $\ell_1, \dots, \ell_{k-1} \in [r]$.

If $\|v\| \neq \gamma$, then the following holds. For any $i \in [r]$, either $i \in I(v)$, which means that all points on C_i have distance $\sqrt{\|v\|^2 + \gamma^2} \neq \sqrt{2\gamma^2}$ to v , or $i \notin I(v)$, implying that there are at most 2 points on C_i at distance $\sqrt{2\gamma^2}$ to v . So in this case we have $N_3 = O(1)$.

If $|I(v)| \leq r-2$, since for each $j \in [r] \setminus I(v)$ there are at most 2 points in C_j with distance $\sqrt{2\gamma^2}$ from v , we have

$$N_3 \leq \left(\sum_{J \in \binom{I(v)}{k-1}} \prod_{j \in J} |X \cap C_j| \right) + O(n^{k-2}) \leq \binom{r-2}{k-1} \left(\frac{n}{r} \right)^{k-1} + o(n^{k-1}).$$

If $\|v\| = \gamma > 0$ and $|I(v)| \geq r - 1$, then v is not the origin and thus $|I(v)| = r - 1$. Let $j \in [r]$ be such that $I(v) = [r] \setminus \{j\}$. Then $v \in \text{Aff}(C_j)$ and $\|v\| = \gamma$, implying that $v \in C_j$, a contradiction.

Therefore, v is contained in at most

$$N_1 + N_2 + N_3 \leq \binom{r-2}{k-1} \left(\frac{n}{r}\right)^{k-1} + o(n^{k-1})$$

regular $(k-1)$ -simplices. Now we take an arbitrary point $v' \in C_r \setminus X$. Since v' is contained in at least

$$\sum_{J \in \binom{[r-1]}{k-1}} \prod_{j \in J} |X \cap C_j| = \binom{r-1}{k-1} \left(\frac{n}{r}\right)^{k-1} - o(n^{k-1}) > N_1 + N_2 + N_3$$

regular $(k-1)$ -simplices, $X \cup \{v'\} \setminus \{v\}$ spans more regular $(k-1)$ -simplices than X , contradicting the extremality of X . $\square$

We finalize the proof by bounding the number of regular $(k-1)$ -simplices in X . There are three types of simplices:

- (1) regular $(k-1)$ -simplices Δ_1 with $|\Delta_1 \cap C_i| \leq 1$ for all $i \in [r]$;
- (2) regular $(k-1)$ -simplices Δ_2 with $|\Delta_2 \cap C_i| = 2$ for some $i \in [r]$ and $|\Delta_2 \cap C_i| \leq 2$ for others;
- (3) regular $(k-1)$ -simplices Δ_3 containing at least three vertices on the same circle.

Note that Δ_1 and Δ_2 both have side length $\sqrt{2\gamma^2}$, while Δ_3 , if exists, has side length $\sqrt{3\gamma^2}$.

Recall that for $k \geq 3$

$$\begin{aligned} f_k(n_1, \dots, n_r) := & \left(\sum_{I \in \binom{[r]}{k}} \prod_{i \in I} n_i \right) + \left(\sum_{1 \leq \ell \leq \lfloor k/2 \rfloor} \sum_{J \in \binom{[r]}{\ell}} \left(\prod_{j \in J} (n_j - \mathbb{1}_{n_j \notin 4\mathbb{Z}}) \right) \left(\sum_{I \in \binom{[r] \setminus J}{k-2\ell}} \prod_{i \in I} n_i \right) \right) \\ & + \mathbb{1}_{k=3} \left(\sum_{i \in [r]} \frac{n_i - p_i}{3} + \mathbb{1}_{p_i > 8} (p_i - 8) \right), \end{aligned}$$

where p_i is the remainder of n_i divided by 12 for each $i \in [r]$.

Claim 21. X spans at most $\max_{n_1 + \dots + n_r = n} f_k(n_1, \dots, n_r)$ regular $(k-1)$ -simplices.

Proof of Claim 21. For every $i \in [r]$, we let $n_i = |X \cap C_i|$ and p_i be the remainder of n_i divided by 12. Note that $\sum_{i \in [r]} n_i = n$ and $n_i = n/r - o(n)$ for all $i \in [r]$. The number of Δ_1 is

$$\sum_{I \in \binom{[r]}{k}} \prod_{i \in I} n_i.$$

We shall upper bound the number of Δ_2 and Δ_3 together as follows. Observe that the third-type simplex Δ_3 exists only when $k = 3$. Indeed, when $k \geq 4$, since a circle does not contain a regular $(k-1)$ -simplex, there must be another point on a different circle, implying that $\sqrt{2\gamma^2}$ and $\sqrt{3\gamma^2}$ both appear as distances between vertices of Δ_3 , a contradiction. For each $i \in [r]$, we let s_i be the

number of pairs of points on C_i at distance $\sqrt{2\gamma^2}$ apart and let t_i be the number of equilateral triangles spanned by points on C_i . Then for $1 \leq \ell \leq \lfloor k/2 \rfloor$, the number of Δ_2 containing exactly ℓ pairs of points, each pair on a different circle, is

$$\sum_{J \in \binom{[r]}{\ell}} \left(\prod_{j \in J} s_j \right) \left(\sum_{I \in \binom{[r] \setminus J}{k-2\ell}} \prod_{i \in I} n_i \right).$$

The number of Δ_3 , which only exists if $k = 3$, is $\mathbb{1}_{k=3} \sum_{i \in [r]} t_i$. Hence, the number of Δ_2 and Δ_3 together is

$$\left(\sum_{1 \leq \ell \leq \lfloor k/2 \rfloor} \sum_{J \in \binom{[r]}{\ell}} \left(\prod_{j \in J} s_j \right) \left(\sum_{I \in \binom{[r] \setminus J}{k-2\ell}} \prod_{i \in I} n_i \right) \right) + \mathbb{1}_{k=3} \left(\sum_{i \in [r]} t_i \right).$$

Note that increasing s_i by 1 increases the total number of Δ_2 and Δ_3 by $\Omega(n^{k-2})$, while increasing t_i by 1 only increases the total number of Δ_2 and Δ_3 by at most 1. In order to maximize the total number of Δ_2 and Δ_3 , on each circle C_i the priority is to maximize s_i , followed by t_i . Observe that $s_i \leq n_i$ and the equality can be attained only if $n_i \in 4\mathbb{Z}$. Accordingly, the optimal configuration of n_i points on C_i is as described in Section 2.2, where $s_i = n_i - \mathbb{1}_{n_i \notin 4\mathbb{Z}}$ is maximized and $t_i = (n_i - p_i)/3 + \mathbb{1}_{p_i > 8}(p_i - 8)$. Therefore, the number of Δ_2 and Δ_3 together is at most

$$\left(\sum_{1 \leq \ell \leq \lfloor k/2 \rfloor} \sum_{J \in \binom{[r]}{\ell}} \left(\prod_{j \in J} (n_j - \mathbb{1}_{n_j \notin 4\mathbb{Z}}) \right) \left(\sum_{I \in \binom{[r] \setminus J}{k-2\ell}} \prod_{i \in I} n_i \right) \right) + \mathbb{1}_{k=3} \left(\sum_{i \in [r]} \frac{n_i - p_i}{3} + \mathbb{1}_{p_i > 8}(p_i - 8) \right).$$

Adding up the contributions from different types, we obtain that there are at most

$$\begin{aligned} & \left(\sum_{I \in \binom{[r]}{k}} \prod_{i \in I} n_i \right) + \left(\sum_{1 \leq \ell \leq \lfloor k/2 \rfloor} \sum_{J \in \binom{[r]}{\ell}} \left(\prod_{j \in J} (n_j - \mathbb{1}_{n_j \notin 4\mathbb{Z}}) \right) \left(\sum_{I \in \binom{[r] \setminus J}{k-2\ell}} \prod_{i \in I} n_i \right) \right) \\ & + \mathbb{1}_{k=3} \left(\sum_{i \in [r]} \frac{n_i - p_i}{3} + \mathbb{1}_{p_i > 8}(p_i - 8) \right) \\ & \leq \max_{n_1 + \dots + n_r = n} f_k(n_1, \dots, n_r). \end{aligned}$$

regular $(k-1)$ -simplices in X . □

Due to the extremality of X , Claim 21 gives an upper bound on $S_{2r}^k(n)$ when n is sufficiently large. Since the upper bound matches the lower bound (1), we have that

$$S_{2r}^k(n) = \max_{n_1 + \dots + n_r = n} f_k(n_1, \dots, n_r) \quad \text{for fixed } r \geq k \geq 3 \text{ and sufficiently large } n. \quad (6)$$

The proof of Theorem 5 is completed by taking $k \geq 4$. □

6.2 | The special case: Equilateral triangles

In this subsection we prove Theorem 3.

Proof of Theorem 3. As shown in the proof of Theorem 5, see (6), for sufficiently large n we have $T_{2r}(n) = S_{2r}^3(n) = \max_{n_1 + \dots + n_r = n} f_3(n_1, \dots, n_r)$, where

$$f_3(n_1, \dots, n_r) = \left(\sum_{1 \leq i < j < k \leq r} n_i n_j n_k \right) + \left(\sum_{i \in [r]} (n_i - \mathbb{1}_{n_i \notin 4\mathbb{Z}})(n - n_i) + \frac{n_i - p_i}{3} + \mathbb{1}_{p_i > 8}(p_i - 8) \right)$$

and p_i denotes the remainder of n_i divided by 12. Thus, to prove Theorem 3, it suffices to find the maximizer of $f_3(n_1, \dots, n_r)$ among all $(n_1, \dots, n_r)$ with $\sum_{i=1}^r n_i = n$.

Let $(n_1, \dots, n_r)$ be a maximizer of $f_3(n_1, \dots, n_r)$ among all $(n_1, \dots, n_r)$ with $\sum_{i=1}^r n_i = n$. Since $S_{2r}^3(n) = \binom{r}{3}(n/r)^3 + o(n^3)$ by Theorem 2, we have that $|n_i - n/r| = o(n)$ for all $i \in [r]$. Otherwise by Lemma 13, the first term of $f_3(n_1, \dots, n_r)$ would be at most $\binom{r}{3}(n/r)^3 - \varepsilon n^3$ for some constant $\varepsilon > 0$, yielding that $f_3(n_1, \dots, n_r) \leq \binom{r}{3}(n/r)^3 - \varepsilon n^3 + O(n^2) < S_{2r}^3(n)$, a contradiction.

Claim 22. For any distinct $i, j \in [r]$, $|n_i - n_j| \leq 2$.

Proof of Claim 22. Suppose for contradiction that $n_1 - n_2 \geq 3$. Let $n'_1 = n_1 - 2$ and $n'_2 = n_2 + 2$. Then we have

$$\begin{aligned} & f_3(n'_1, n'_2, n_3, \dots, n_r) - f_3(n_1, n_2, n_3, \dots, n_r) \\ & \geq \left(\sum_{k=3}^r (n'_1 n'_2 - n_1 n_2) n_k \right) + ((n'_1 - 1)(n - n'_1) - n_1(n - n_1)) \\ & \quad + ((n'_2 - 1)(n - n'_2) - n_2(n - n_2)) - 30 \\ & = (2n_1 - 2n_2 - 4)(n - n_1 - n_2) + (-3n + 5n_1 - 6) + (n - 3n_2 - 2) - 30 \\ & = (2n - 2n_1 + 9)n_1 - (2n - 2n_2 - 1)n_2 - 6n - 38. \end{aligned}$$

Since $n_2 \leq n_1 - 3 \leq n/2 - 1$ and $(2n - 2n_2 - 1)n_2$ is increasing in n_2 for $n_2 \leq n/2 - 1$, it holds that $(2n - 2n_2 - 1)n_2 \leq (2n - 2n_1 - 7)(n_1 - 3)$ and thus

$$f_3(n'_1, n'_2, n_3, \dots, n_r) - f_3(n_1, n_2, n_3, \dots, n_r) \geq 10n_1 - 59 > 0,$$

contradicting the maximality of $f_3(n_1, \dots, n_r)$. □

Claim 23. For any distinct $i, j \in [r]$, if $|n_i - n_j| = 2$, then $n_i, n_j \in 2\mathbb{Z}$.

Proof of Claim 23. Suppose that $n_1 = n_2 + 2$ and $n_1, n_2 \notin 2\mathbb{Z}$. Let $n'_1 = n_1 - 1$ and $n'_2 = n_2 + 1$. Since $n_1, n_2 \notin 4\mathbb{Z}$, we have

$$\begin{aligned}
& f_3(n'_1, n'_2, n_3, \dots, n_r) - f_3(n_1, n_2, n_3, \dots, n_r) \\
& \geq \left(\sum_{k=3}^r (n'_1 n'_2 - n_1 n_2) n_k \right) + ((n'_1 - 1)(n - n'_1) - (n_1 - 1)(n - n_1)) \\
& \quad + ((n'_2 - 1)(n - n'_2) - (n_2 - 1)(n - n_2)) - 30 \\
& = (n - n_1 - n_2) + (-n + 2n_1 - 2) + (n - 2n_2) - 30 = n - 2n_2 - 30 > 0,
\end{aligned}$$

a contradiction. $\square$

Claim 24. There exists at most one $i \in [r]$, such that n_i is odd.

Proof of Claim 24. Suppose that n_1 and n_2 are both odd. Then by Claims 22 and 23 we have that $n_1 = n_2$. Let $n'_1 = n_1 - 1$ and $n'_2 = n_2 + 1$. If $n_1 = n_2 \equiv 1 \pmod{4}$, then we have $n'_1 \equiv 0 \pmod{4}$ and thus

$$\begin{aligned}
& f_3(n'_1, n'_2, n_3, \dots, n_r) - f_3(n_1, n_2, n_3, \dots, n_r) \\
& \geq \left(\sum_{k=3}^r (n'_1 n'_2 - n_1 n_2) n_k \right) + (n'_1(n - n'_1) - (n_1 - 1)(n - n_1)) \\
& \quad + ((n'_2 - 1)(n - n'_2) - (n_2 - 1)(n - n_2)) - 30 \\
& = -(n - n_1 - n_2) + (n_1 - 1) + (n - 2n_2) - 30 = 2n_1 - n_2 - 31 > 0,
\end{aligned}$$

a contradiction. If $n_1 = n_2 \equiv 3 \pmod{4}$, then we have $n'_2 \in 4\mathbb{Z}$ and thus

$$\begin{aligned}
& f_3(n'_1, n'_2, n_3, \dots, n_r) - f_3(n_1, n_2, n_3, \dots, n_r) \\
& \geq \left(\sum_{k=3}^r (n'_1 n'_2 - n_1 n_2) n_k \right) + ((n'_1 - 1)(n - n'_1) - (n_1 - 1)(n - n_1)) \\
& \quad + (n'_2(n - n'_2) - (n_2 - 1)(n - n_2)) - 30 \\
& = -(n - n_1 - n_2) + (-n + 2n_1 - 2) + (2n - 3n_2 - 1) - 30 = 3n_1 - 2n_2 - 33 > 0,
\end{aligned}$$

a contradiction. $\square$

Combining Claims 22, 23, and 24, we know that

- if n is even, then $n_1, \dots, n_r$ are all even and $|n_i - n_j| \leq 2$ for all $i, j \in [n]$;
- if n is odd, then there is exactly one odd n_i and $|n_i - n_j| = 1$ for all $j \in [r] \setminus \{i\}$.

Such a partition of n is unique up to the ordering of $n_1, \dots, n_r$, thus, $(n_1, \dots, n_r)$ is exactly the one given in Theorem 3, up to a reordering. $\square$

6.3 | The lower-order term in $S_{2r}^k(n)$

In this subsection we prove Corollary 6, which determines the magnitude of the lower-order term in the asymptotic expression of $S_{2r}^k(n)$. Recall that for any fixed integers $r \geq k \geq 3$, Corollary 6 asserts that

$$S_{2r}^k(n) = \binom{r}{k} \left(\frac{n}{r}\right)^k + \Theta(n^{k-1}).$$

Proof of Corollary 6. We have shown in (6) that

$$S_{2r}^k(n) = \max_{n_1 + \dots + n_r = n} f_k(n_1, \dots, n_r) \quad \text{for fixed } r \geq k \geq 3 \text{ and sufficiently large } n,$$

where

$$f_k(n_1, \dots, n_r) = \left(\sum_{I \in \binom{[r]}{k}} \prod_{i \in I} n_i \right) + \left(\sum_{j \in [r]} \sum_{I \in \binom{[r] \setminus \{j\}}{k-2}} (n_j - \mathbb{1}_{n_j \notin 4\mathbb{Z}}) \prod_{i \in I} n_i \right) + O(n^{k-2}).$$

Ignoring the last term above, which has a negligible effect for sufficiently large n , it suffices to show that for fixed integers $r \geq k \geq 3$

$$\max_{n_1 + \dots + n_r = n} \left(\sum_{I \in \binom{[r]}{k}} \prod_{i \in I} n_i \right) + \left(\sum_{j \in [r]} \sum_{I \in \binom{[r] \setminus \{j\}}{k-2}} (n_j - \mathbb{1}_{n_j \notin 4\mathbb{Z}}) \prod_{i \in I} n_i \right) = \binom{r}{k} \left(\frac{n}{r}\right)^k + \Theta(n^{k-1}). \quad (7)$$

The first term on the left-hand side of (7) is at most $\binom{r}{k} (n/r)^k$ by Maclaurin's inequality [7], while the second term is $O(n^{k-1})$. So the left-hand side of (7) is upper bounded by $\binom{r}{k} (n/r)^k + O(n^{k-1})$.

For the other direction, we let n_i be either $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$ for each $i \in [r]$ so that $n_1 + \dots + n_r = n$. Let m^+ and m^- denote the counts of indices i for which $n_i = \lceil n/r \rceil$ and $n_i = \lfloor n/r \rfloor$, respectively. Note that $\alpha := m^+(\lceil n/r \rceil - n/r) + m^-(\lfloor n/r \rfloor - n/r) = 0$. For this choice of $n_1, \dots, n_r$, the second term on the left-hand side of (7) is $\Omega(n^{k-1})$, while the first term is at least

$$\binom{r}{k} \left(\frac{n}{r}\right)^k + \alpha \binom{r-1}{k-1} \left(\frac{n}{r}\right)^{k-1} - O(n^{k-2}) = \binom{r}{k} \left(\frac{n}{r}\right)^k - O(n^{k-2}).$$

Therefore, the left-hand side of (7) is lower bounded by $\binom{r}{k} (n/r)^k + \Omega(n^{k-1})$. $\square$

7 | CONCLUDING REMARKS

In this paper, we settled a conjecture in combinatorial geometry. Assuming n being sufficiently large, we obtained the exact value of $T_d(n)$, that is, the maximum number of equilateral triangles spanned by n points in $\mathbb{R}^d$, for all even dimensions $d \geq 6$. For odd $d \geq 7$, we determined $T_d(n)$ up to the lower-order term.

Let $T_d^{\text{unit}}(n)$ be the maximum number of unit equilateral triangles spanned by n points in $\mathbb{R}^d$. Our method also determines $T_d^{\text{unit}}(n)$ for all even $d \geq 6$.

Proposition 25. *Let $r \geq 3$ be a fixed integer. For sufficiently large $n \in \mathbb{N}$ we have*

$$T_{2r}^{\text{unit}}(n) = \sum_{1 \leq i < j < k \leq r} n_i n_j n_k + \sum_{i \in [r]} \left((n_i - \mathbb{1}_{n_i \notin 4\mathbb{Z}})(n - n_i) \right),$$

where $(n_1, \dots, n_r)$ is chosen in the same manner as in Theorem 3.

The upper bound in Proposition 25 follows by an argument analogous to that of Theorems 3 and 5, except that the third-type simplices (triangles) need not be counted, as their side length differs from that of the first two types. The lower bound is obtained by the same Lenz construction as in Section 2.2, again excluding the third-type triangles.

We also considered a more general setting. Recall that $S_d^k(n)$ denotes the maximum number of regular $(k-1)$ -simplices spanned by n points in $\mathbb{R}^d$. We determined the asymptotic behavior of $S_d^k(n)$ for all fixed $d \geq 2k \geq 6$. For even d , we are also able to determine the magnitude of the lower-order term and reduce the problem of determining the exact value of $S_d^k(n)$ to an optimization problem. We remark that in the case when $k \geq 4$ and d is even, since the third-type simplices do not exist, our proof of Theorem 5 also implies the following.

Proposition 26. *For any fixed $r \geq k \geq 4$ and sufficiently large $n \in \mathbb{N}$, the maximum number of $(k-1)$ -simplices spanned by n points in $\mathbb{R}^{2r}$, with a given side length, is equal to $S_{2r}^k(n)$.*

Regarding the function $S_{2r+1}^k(n)$ with $r \geq k \geq 3$, it seems out of reach to determine its exact value even for $r = k = 3$. The reason is that in the odd-dimensional case a linear proportion of the points can lie on a two-dimensional sphere, and the exact maximum number of occurrences of a fixed distance among points on a two-dimensional sphere is unknown. Nevertheless, we believe that determining the magnitude of the lower-order term is plausible. As mentioned in Section 2.3, a result of Erdős, Hickerson, and Pach [13] shows that there exists a configuration of m points on a unit two-dimensional sphere in which the distance $\sqrt{2}$ occurs $\Omega(m^{4/3})$ times. Building on this configuration, we obtained the lower bound $S_{2r+1}^k(n) \geq \binom{r}{k} \left(\frac{n}{r}\right)^k + \Omega(n^{k-2/3})$, see (2). On the other hand, a classical result of Clarkson, Edelsbrunner, Guibas, Sharir, and Welzl [8] states that the number of occurrences of any fixed distance among m points on a two-dimensional sphere is at most $O(m^{4/3})$. Using this bound, together with a further refinement of our method, it is possible to prove that

$$S_{2r+1}^k(n) = \binom{r}{k} \left(\frac{n}{r}\right)^k + \Theta(n^{k-2/3})$$

for $r \geq k \geq 3$. However, we expect the analysis to be more involved, since most of the dimension arguments underlying the stability result (i.e., Theorem 7) no longer hold in the odd-dimensional case.

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