Published for SISSA by Springer

Received: April 26, 2026
Accepted: July 8, 2026
Published: August 20, 2026

N-Jettiness soft functions made simple

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Abstract: We present a new method to compute the soft function for the N-Jettiness variable for arbitrary N at high perturbative orders in QCD. It is based on the observation that the most singular part of the soft function, the dipole contribution, can be represented by a sum of an analytically calculable inclusive soft function and a remainder. The latter is absent at NLO, is immediately finite at NNLO, and we expect that can be made finite with the help of simple NLO-like infrared subtractions at N^{3}LO. As a byproduct of this approach, we derive a very simple formula for the tripole contribution to the N-Jettiness NNLO soft function, which results in a fast numerical evaluation. We apply this method to compute the N-Jettiness soft function at NNLO, and report numerical results for up to five jets for the hadron-collider soft function. We finally outline the prospects for applications at N^{3}LO.

Keywords: Higher-Order Perturbative Calculations, Automation, Factorization, Renormalization Group

ArXiv ePrint: 2604.13167

https://doi.org/10.1007/JHEP08(2026)158

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1 Introduction

The remarkable performance of the Large Hadron Collider (LHC) has ushered in an era where measurements of Standard Model observables are approaching percent-level precision [1–5]. This trend will intensify at the High Luminosity LHC (HL-LHC), where the enormous increase in recorded number of events and related decrease in statistical uncertainties of measurements demands theoretical predictions of comparable accuracy to fully exploit the collider's physics potential [6–8].

Significant progress has been made in extending QCD predictions to N^{3}LO for several processes where a color-singlet final state is produced [9–16]. Although these calculations represent only initial steps toward general N^{3}LO accuracy at the LHC, they demonstrate that N^{3}LO QCD corrections can induce percent-level or even larger shifts in theoretical predictions for cross sections of key processes such as Higgs production via gluon fusion and Drell-Yan production. As HL-LHC measurements enter the percent regime across a broad range of processes, including these corrections becomes essential. Furthermore, N^{3}LO QCD effects introduce new color correlations in the singularity structure of jet processes, potentially giving rise to genuinely new phenomena, such as factorization-breaking effects. Developing systematic methods capable of delivering fully differential N^{3}LO predictions is, therefore, an important goal for theoretical collider physics [17].

A powerful strategy for obtaining fully differential fixed-order QCD predictions at hadron colliders relies on slicing or non-local subtraction methods [18–26]. These approaches employ a physical resolution variable to isolate infrared (IR) sensitive regions of phase space and use associated factorization theorems to account for contributions of these regions analytically at high perturbative orders. Prominent examples include q_T [18] and N-Jettiness [19, 20] slicing, which have already yielded important results. Indeed, q_T slicing has enabled fully differential predictions for the Drell-Yan process [12, 14–16], and all ingredients for applying 0-Jettiness slicing to color-singlet processes, such as Higgs and Drell-Yan production, are now available following the computation of three-loop beam [27, 28] and soft functions [29–33], as well as logarithmically-enhanced power corrections [34].

Extending these techniques to processes with jets is considerably more challenging. For processes where a color-singlet system is produced in association with a jet, NNLO QCD predictions have been obtained using N-Jettiness slicing [14, 35, 36], making it currently the only slicing method applicable to processes with jets. Moreover, efforts to extend q_T-like subtraction to final states with jets [21, 23, 24, 37] remain at an exploratory stage and are not yet practical to tackle N^{3}LO predictions. Given the importance of jet production at the LHC, developing robust, generalizable slicing methods for these processes is a critical task.

A major bottleneck in applying N-Jettiness slicing to generic processes at N^{3}LO is the calculation of soft functions involving an arbitrary number of light-like directions. Even for color-singlet production, the determination of the three-loop soft function with two Wilson lines required years of sustained effort and development of sophisticated multi-loop techniques [29–33, 38, 39]. New ideas enabled recent computations of the NNLO N-Jettiness soft function for arbitrary N [40, 41], but these computations also made it clear that their extension to the next perturbative order is, at best, challenging.

The goal of this paper is to introduce a method that dramatically simplifies the calculation of N-Jettiness soft functions at higher orders. The central point of this method is to isolate the dipole contribution for an arbitrary number of hard emitters and express it as the sum of an analytically computable piece and a finite remainder. Crucially, the remainder can be evaluated numerically with drastically reduced complexity. At NNLO, the remainder requires no infrared (IR) subtraction, and at N^{3}LO its structure resembles an NLO cross-section computation, requiring only NLO-type subtractions. We apply this method to the N-Jettiness soft function and present results at NNLO for general N (and numerically up to $N = 5$), demonstrating the power and flexibility of our approach.

The remainder of this paper is organized as follows. In section 2 we review the structure of N-Jettiness soft functions. We show how their divergences are captured by a simpler quantity, the *inclusive* soft function, and explain how to use this observation to dramatically simplify the computation the N-Jettiness soft function at higher orders. In section 3, we use this method to calculate the 0-Jettiness soft function at NNLO. In section 4, we extend this

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analysis to general N and provide numerical benchmarks for selected phase-space points. We conclude in section 5. Additional technical details are provided in appendices.

2 Presentation of the method

The focus of this article is the soft function for the N-Jettiness variable defined in ref. [42]. At the lowest order, we consider the scattering of $N+2$ resolved QCD partons that we denote by h_i. Throughout the paper, we use the notation

$$\begin{aligned} 0 &\to h_1 + h_2 + h_3 + h_4 + \cdots h_{N+2}, \\ h_1 &\to h_2 + h_3 + h_4 + \cdots h_{N+2}, \\ h_1 + h_2 &\to h_3 + h_4 + \cdots h_{N+2}, \end{aligned} \tag{2.1}$$

so that the $N=0$ case always refers to a process with two color-charged particles at the Born level.[1] The first process refers to e^+e^- annihilation, the second to deep-inelastic scattering and the third to hadron collisions. Each parton in eq. (2.1) carries a four-momentum p_i and a color charge $\mathbf{T}_i$.[2] All partons are assumed to be massless. To define the soft function, we denote by $n_i^\mu \equiv p_i^\mu/p_i^0$ the light-like vector aligned with the direction of flight of the parton h_i.

The following discussion takes as a starting point the factorization theorem of ref. [42]. More recently, an alternative factorization theorem for the N-Jettiness variable was presented in ref. [44] to capture, in addition to the modes described by that of ref. [42], also the effect of Glauber modes. Concretely, these modes result in the appearance of coherence-violating-logarithmic (CVL) corrections at subleading color, which have been known to affect the resummation of scattering observables in the hadron-collider case [44–60]. Before proceeding, we note that the conclusions of ref. [44] do not affect the results and the methods presented in this article. Specifically, the class of CVL corrections to the N-Jettiness factorization discussed in ref. [44] start at N^{4}LO, and therefore, up to N^{3}LO, the predictions of the factorization theorems of refs. [42] and [44] are identical provided the correct direction of the Wilson lines is adopted.[3]

2.1 Definition and properties of the soft function

In Soft-Collinear Effective Theory (SCET) [61–64], the bare soft function is defined by the following operator matrix element

$$\boldsymbol{\mathcal{S}}_{\mathcal{T}_N}(\ell) = \langle 0|T\{\boldsymbol{Y}_{n_1}\boldsymbol{Y}_{n_2}\dots\boldsymbol{Y}_{n_{N+2}}\}\,\delta(\ell-\hat{\mathcal{T}}_N)\,\bar{T}\{\boldsymbol{Y}^\dagger_{n_{N+2}}\dots\boldsymbol{Y}^\dagger_{n_2}\boldsymbol{Y}^\dagger_{n_1}\}|0\rangle\,, \tag{2.2}$$

where the soft Wilson line Y_n is expressed in terms of the soft gauge field $\mathbf{A}_i^{(s)}$ as [64]

$$\boldsymbol{Y}_{n_i}(x) = \mathcal{P}\exp\left\{ig_s\int dt\, n_i\cdot\mathbf{A}_i^{(s)}(x+n_it)\right\}\,, \tag{2.3}$$

[1] A different convention is often found in the literature where, for instance, the thrust-like variable in e^+e^- is referred to as 2-Jettiness.

[2] Throughout this paper, we use the color-space formalism of ref. [43].

[3] We note that the CVL tower starts at N^{3}LO, but the first term is captured by the soft function defined in eq. (2.2).

and analogously for $Y_{\bar{n}}$. The integral over dt runs from $(-\infty, 0]$ if h_i is in the initial state and from $[0, +\infty)$ if h_i is in the final state. Here $\mathbf{A}_i^{(s)} \equiv \mathbf{T}_i \cdot A^{(s)}$, where $\mathbf{T}_i$ is the color charge of parton h_i. The operator $\hat{\mathcal{T}}_N$ acts on a given state of X soft particles $|X_s\rangle$ with momenta $k_1^\mu, \ldots, k_X^\mu$ by measuring the value of the N-Jettiness $\mathcal{T}_N$ of the corresponding final-state, namely

$$\delta(\ell - \hat{\mathcal{T}}_N)|X_s\rangle = \delta(\ell - \mathcal{T}_N(k_1, k_2, \ldots, k_X))|X_s\rangle\,, \tag{2.4}$$

where

$$\mathcal{T}_N(k_1, k_2, \ldots, k_X) = \sum_{i=1}^{X} \min\{k_i \cdot n_1, k_i \cdot n_2, \ldots, k_i \cdot n_{N+2}\}\,. \tag{2.5}$$

The soft function in eq. (2.2) is conveniently studied in Laplace space; that is

$$\widetilde{\boldsymbol{\mathcal{S}}}_{\mathcal{T}_N}(u) = \int_0^\infty \mathrm{d}\ell\, e^{-u\,\ell} \boldsymbol{\mathcal{S}}_{\mathcal{T}_N}(\ell)\,. \tag{2.6}$$

In Laplace space, the soft function admits a local, multiplicative renormalization of the UV singularities in terms of the renormalization color matrix $\boldsymbol{Z}$ as

$$\widetilde{\boldsymbol{\mathcal{S}}}_{\mathcal{T}_N}(u) = \boldsymbol{Z}\widetilde{\boldsymbol{\mathcal{S}}}^{(R)}_{\mathcal{T}_N}(u)\boldsymbol{Z}^\dagger\,. \tag{2.7}$$

From now on, unless necessary to clarify the notation, we will suppress the u dependence in the Laplace transform of the soft function. The color matrices $\boldsymbol{Z}$ and $\widetilde{\boldsymbol{\mathcal{S}}}_{\mathcal{T}_N}$ admit a perturbative expansion in powers of the bare strong coupling α_s

$$\boldsymbol{Z} = \mathbb{1} + \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n \boldsymbol{Z}^{(n)}\,, \quad \widetilde{\boldsymbol{\mathcal{S}}}_{\mathcal{T}_N} = \mathbb{1} + \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n \widetilde{\boldsymbol{\mathcal{S}}}^{(n)}_{\mathcal{T}_N}\,, \tag{2.8}$$

and $\widetilde{\boldsymbol{\mathcal{S}}}^{(R)}_{\mathcal{T}_N}$ admits an expansion in terms of the renormalized strong coupling $\alpha_s(\mu)$ (defined here in the $\overline{\mathrm{MS}}$ scheme)

$$\widetilde{\boldsymbol{\mathcal{S}}}^{(R)}_{\mathcal{T}_N} = \mathbb{1} + \sum_{n=1}^{\infty} \left(\frac{\alpha_s(\mu)}{4\pi}\right)^n \widetilde{\boldsymbol{\mathcal{S}}}^{(R),\,(n)}_{\mathcal{T}_N}\,, \tag{2.9}$$

where μ is the renormalization scale. The expanded form of eq. (2.7) up to NNLO is given in appendix B.

A direct consequence of eq. (2.7) is that the renormalized soft function $\widetilde{\boldsymbol{\mathcal{S}}}^{(R)}_{\mathcal{T}_N}$ satisfies the renormalization group equation (RGE)

$$\mu \frac{\mathrm{d}\widetilde{\boldsymbol{\mathcal{S}}}^{(R)}_{\mathcal{T}_N}}{\mathrm{d}\mu} = \frac{1}{2}\left(\boldsymbol{\Gamma}_s \widetilde{\boldsymbol{\mathcal{S}}}^{(R)}_{\mathcal{T}_N} + \widetilde{\boldsymbol{\mathcal{S}}}^{(R)}_{\mathcal{T}_N} \boldsymbol{\Gamma}_s^\dagger\right)\,, \tag{2.10}$$

where [40]

$$\boldsymbol{\Gamma}_s = \sum_{(ij)} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \Gamma_{\mathrm{cusp}}(\alpha_s) \left[L_{ij} + i\pi\,\lambda_{ij}\right] + 2\boldsymbol{\gamma}_s(\alpha_s)\,, \quad L_{ij} \equiv \ln(n_i \cdot n_j/2) + \ln\left(\mu^2 \bar{u}^2\right)\,. \tag{2.11}$$

The summation in the above equation is performed over distinct indices $i \neq j = 1, \ldots, N+2$. Furthermore, $\bar{u} = u e^{\gamma_E}$, and Γ_{cusp} and $\boldsymbol{\gamma}_s(\alpha_s)$ are the cusp and soft anomalous dimensions, respectively. The variable λ_{ij} is equal to 1 if i, j are both in the initial (final) state, and 0 otherwise.

Another important property of the soft function, both at the bare and renormalized level, is that it satisfies the non-Abelian exponentiation theorem [65, 66], that states that it is given by the exponentials of matrices in color space

$$\widetilde{\boldsymbol{\mathcal{S}}}_{\mathcal{T}_N} = \exp\left\{\sum_{n=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^n \tilde{\boldsymbol{s}}^{(n)}_{\mathcal{T}_N}\right\}\,, \quad \widetilde{\boldsymbol{\mathcal{S}}}^{(R)}_{\mathcal{T}_N} = \exp\left\{\sum_{n=1}^{\infty}\left(\frac{\alpha_s(\mu)}{4\pi}\right)^n \tilde{\boldsymbol{s}}^{(R),(n)}_{\mathcal{T}_N}\right\}\,. \tag{2.12}$$

The building blocks in the exponent of the previous equation, consist of the maximally non-Abelian subset of the color structures that contribute to a given perturbative order, sometimes called *webs*. Therefore, at each new perturbative order $\mathcal{O}(\alpha_s^n)$, the above structure allows one to focus on the computation of the maximally non-Abelian correction and to fix the remaining contributions by exponentiation of lower-order corrections. This property is highly non-trivial in that each contribution in the exponent is a matrix in color space.

In perturbation theory, the soft function in eq. (2.12) can be calculated in terms of eikonal QCD squared amplitudes. Starting from the definition in eq. (2.12), we can write the perturbative expansion of the bare N-Jettiness soft function as

$$\tilde{\boldsymbol{s}}_{\mathcal{T}_N}(u) = \sum_{n=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^n \tilde{\boldsymbol{s}}^{(n)}_{\mathcal{T}_N}(u)\,, \tag{2.13}$$

where

$$\tilde{\boldsymbol{s}}^{(n)}_{\mathcal{T}_N}(u) = \sum_{m=1}^{n}\int\left(\prod_{i=1}^{m}[\mathrm{d}k_i]\right)\boldsymbol{\mathcal{M}}_s^{(n-m)}(k_1,\ldots,k_m)\, e^{-u\mathcal{T}_N(k_1,\ldots,k_m)}\,. \tag{2.14}$$

Here, the matrix element $\boldsymbol{\mathcal{M}}_s^{(n-m)}(k_1,\ldots,k_X)$ denotes the $(n-m)$-loop maximally-non-Abelian contribution[4] to the squared amplitude describing a process of eq. (2.1), which can be expressed in terms of well-known eikonal factors [67–72]. The one-emission phase-space measure is given by $[\mathrm{d}k] = \mathrm{d}^d k/(2\pi)^{d-1}\delta(k^2)\theta(k^0)$ with $d = 4-2\epsilon$.

It is convenient to decompose $\tilde{\boldsymbol{s}}_{\mathcal{T}_N}(u)$ according to its color structure as follows

$$\tilde{\boldsymbol{s}}_{\mathcal{T}_N}(u) = \tilde{\boldsymbol{s}}^{\text{dip.}}_{\mathcal{T}_N}(u) + \tilde{\boldsymbol{s}}^{\text{tri.}}_{\mathcal{T}_N}(u) + \tilde{\boldsymbol{s}}^{\text{quad.}}_{\mathcal{T}_N}(u) + \ldots\,, \tag{2.15}$$

where the dipole term $\tilde{\boldsymbol{s}}^{\text{dip.}}_{\mathcal{T}_N}$ is proportional to *dipole* color factors describing color correlators of two hard emitters $\mathbf{T}_i\cdot\mathbf{T}_j$ (for any $i, j = 1,\ldots,N+2$). This term starts at $\mathcal{O}(\alpha_s)$. The tripole term $\tilde{\boldsymbol{s}}^{\text{tri.}}_{\mathcal{T}_N}$ describes color correlators of three hard emitters, and starts at $\mathcal{O}(\alpha_s^2)$. Similarly, the quadruple term $\tilde{\boldsymbol{s}}^{\text{quad.}}_{\mathcal{T}_N}$ describes four-legs correlators and starts at $\mathcal{O}(\alpha_s^3)$, and so on for higher-point color correlators. After expanding the exponential in eq. (2.12) in powers of α_s, the color structures in eq. (2.15) mix due to color algebra at any given perturbative order. For instance, the iteration of the dipole term in eq. (2.15) generates tripole-like corrections, and so forth.

An important observation is that, at the cross section level, one contracts the soft function (2.2) with the hard function (which is also a matrix in color space) for the partonic

[4]In our notation, matrix elements are stripped by factors $\alpha_s^n/(4\pi)^n$ and include symmetry factors for identical particles.

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process under consideration, and evaluates the matrix element of the resulting object between color-singlet states. In doing so, some higher-point color structures in eq. (2.15) vanish at a given perturbative order by color conservation upon acting on a color-singlet state. For instance, through third perturbative order $\mathcal{O}(\alpha_s^3)$ relative to the Born level, the cross section for the scattering process $pp \to Fj$ (with F being a color singlet and j a QCD jet) only receives contributions from the dipole term $\tilde{\boldsymbol{s}}_{\mathcal{T}_N}^{\text{dip.}}$, while the process $pp \to Fjj$ receives contributions from both the dipole and tripole terms in eq. (2.15).

Through the rest of this article, we discuss the computation of the dipole and tripole terms for arbitrary N (higher-point correlators only start contributing at N^{3}LO and beyond for processes with at least two jets) through NNLO.

2.2 Computational strategy for the dipole correlator

In this section we introduce a method to calculate the dipole contribution to the soft function $\boldsymbol{s}_{\mathcal{T}_N}$ for a process with $N+2$ hard emitters, see eq. (2.1). It is convenient to work directly at the level of the exponent in eqs. (2.12), and consider the dipole contribution to the n-th order maximally non-Abelian term

$$\tilde{\boldsymbol{s}}_{\mathcal{T}_N}^{(n),\,\text{dip.}} = \sum_{i\neq j} \boldsymbol{T}_i \cdot \boldsymbol{T}_j \, \tilde{s}^{(n)}_{\mathcal{T}_N,\,\{i,j\}} \tag{2.16}$$

where $\tilde{s}^{(n)}_{\mathcal{T}_N,\,\{i,j\}}$ is a $\mathbb{C}$-number and a function of u, μ and all the directions of the hard emitters n_k^μ $(k = 1, \ldots, N+2)$. We then decompose $\tilde{s}^{(n)}_{\mathcal{T}_N,\,\{i,j\}}$ as the sum of an *inclusive* contribution $\tilde{s}^{(n)}_{\mathcal{T}_N^{\text{inc.}},\,\{i,j\}}$ and a correction $\Delta\tilde{s}^{(n)}_{\mathcal{T}_N,\,\{i,j\}}$, i.e.

$$\tilde{s}^{(n)}_{\mathcal{T}_N,\,\{i,j\}} = \tilde{s}^{(n)}_{\mathcal{T}_N^{\text{inc.}},\,\{i,j\}} + \Delta\tilde{s}^{(n)}_{\mathcal{T}_N,\,\{i,j\}}\,. \tag{2.17}$$

The inclusive term is defined by computing the dipole correction for a simplified version of the observable evaluated as a function of the *total* momentum of the real radiation, namely

$$\mathcal{T}_N^{\text{inc.}}(k_1, \ldots, k_X) = \min\left\{k_{\text{tot}} \cdot n_1, \ldots, k_{\text{tot}} \cdot n_{N+2}\right\} = \mathcal{T}_N\left(k_{\text{tot}}\right)\,, \tag{2.18}$$

where

$$k_{\text{tot}} = \sum_{i=1}^{X} k_i\,. \tag{2.19}$$

This definition of the inclusive soft function connects it directly to the fully-differential soft function, which has been computed to $\mathcal{O}(\alpha_s^2)$ in refs. [73, 74] and through three loops in ref. [75]. This allows for the derivation of $\tilde{\boldsymbol{s}}_{\mathcal{T}_N^{\text{inc.}},\,\{i,j\}}$ up to three loops following the discussion in section 4.

Besides its simplicity, the inclusive soft function $\tilde{\boldsymbol{s}}_{\mathcal{T}_N^{\text{inc.}},\,\{i,j\}}$ satisfies an important property that will play a key role in the calculation of the full dipole contribution to the N-Jettiness soft function: the UV divergences of the dipole contributions to the inclusive and full N-Jettiness soft functions are identical. This property follows from two central observations:

- The inclusive soft function in eq. (2.17) is used to compute the maximally non-Abelian contributions to the exponent of eq. (2.12). These maximally non-Abelian terms are obtained via the phase-space integration of the correlated contributions to the soft squared amplitudes, that only have support in regions of phase space in which the rapidities of all emissions in the frame of the parent $\{i,j\}$ dipole are commensurate. This implies that, in the calculation of maximally non-Abelian corrections at any given perturbative order, all the regions of the radiation phase space that give rise to UV singularities are characterized by all the soft partons being collinear to either of the two directions of the Wilson lines that define the dipole n_i^μ, n_j^μ.

- In configurations where the radiation is collinear to a given direction, the $\mathcal{T}_N$ observable is only sensitive to the total momentum of soft radiation. This is a direct consequence of the linearity of the observable. It follows that in the UV divergent regions of phase space the $\mathcal{T}_N$ observable exactly coincides with its inclusive approximation $\mathcal{T}_N^{\text{inc.}}$. This, in particular, implies that the two variables share the same ϵ poles, and hence the difference $\Delta\tilde{s}^{(n)}_{\mathcal{T}_N,\{i,j\}}$ at any given perturbative order $\mathcal{O}(\alpha_s^n)$ is always finite for $\epsilon \to 0$ and it remains unaffected by the UV renormalization.

This property leads to a dramatic simplification in the calculation of the $\Delta\tilde{s}^{(n)}_{\mathcal{T}_N,\{i,j\}}$ function, which becomes computable using relatively standard numerical methods. Firstly, it implies that $\Delta\tilde{s}^{(n)}_{\mathcal{T}_N,\{i,j\}}$ is an end-point contribution that is independent of u (or ℓ in momentum space). Secondly, while $\Delta\tilde{s}^{(n)}_{\mathcal{T}_N,\{i,j\}}$ is finite for $\epsilon \to 0$, it still contains ϵ poles of IR origin at intermediate stages of the calculation that cancel between various contributions. In this respect, we note that the complexity of the IR structure in the calculation of $\Delta\tilde{s}^{(n)}_{\mathcal{T}_N,\{i,j\}}$, and thus the complexity of the IR subtraction that is required in its numerical computation, drops by two perturbative orders, as a consequence of the fact that for a single emission, i.e. $n = 1$, the two observables $\mathcal{T}_N$ and $\mathcal{T}_N^{\text{inc.}}$ are by construction identical. This means that $\Delta\tilde{s}^{(1)}_{\mathcal{T}_N,\{i,j\}} = 0$, $\Delta\tilde{s}^{(2)}_{\mathcal{T}_N,\{i,j\}}$ requires a tree-level calculation, $\Delta\tilde{s}^{(3)}_{\mathcal{T}_N,\{i,j\}}$ requires a NLO calculation, and so on. This observation will be crucial in pushing the calculation of the dipole correction to $\mathcal{T}_N$ to higher perturbative orders. Finally, we remark that similar ideas have appeared in the context of two-loop calculations of soft and beam functions for two-leg processes (cf. refs. [22, 76–79]). The novelty of the method presented in this paper lies in the fact that it is applicable to processes with an arbitrary number of hard emitters and that correction $\Delta\tilde{s}^{(n)}_{\mathcal{T}_N,\{i,j\}}$ in eq. (2.17) is finite at all perturbative orders, which is crucial to tame the growth of complexity at higher orders.

2.3 Computational strategy for the tripole correlator

The tripole color correlator contribution $\tilde{\boldsymbol{s}}^{\text{tri.}}_{\mathcal{T}_N}$ in eq. (2.15) is present for processes with four or more hard legs and starts at second perturbative order. At NNLO it is of real-virtual nature which, in particular, implies that at this order the inclusive variant of the N-Jettiness observable eq. (2.18) agrees with that of the full observable. Nevertheless, we find that the parametrization of the real radiation used in section 4 leads to a simplified treatment of the tripole correction as well. Using this parametrization, we will describe a calculation of the tripole term in section 4.3, with additional details on the renormalization provided in appendix C.

3 An example: 0-Jettiness at NNLO

In this section, we illustrate the method introduced in section 2 by applying it to the case of 0-Jettiness at NNLO. The soft function for 0-Jettiness has been known analytically for a long time [80–82]; it coincides with the soft function for thrust [83]. The availability of a closed-form expression makes this observable a convenient benchmark that allows us to present the steps of the method, and describe all ingredients entering eq. (2.17) in a simple and transparent way. Since for 0-Jettiness there are only two hard legs and, therefore, just one dipole, we will omit the dipole indices in this section to simplify the notation. We begin by deriving analytically the quantity $s^{(2)}_{\mathcal{T}_0^{\text{inc.}}}$. We then demonstrate explicitly that the poles of $s^{(2)}_{\mathcal{T}_0^{\text{inc.}}}$ coincide with those of $s^{(2)}_{\mathcal{T}_0}$ so that their difference is finite.

The inclusive soft function for 0-Jettiness, $s_{\mathcal{T}_0^{\text{inc.}}}$, depends only on 2 light-like directions. Without loss of generality, we choose them as the lightlike vectors aligned with the proton beam axes in a hadronic collision, i.e. $n_1 = (1, 0, 0, 1)$ and $n_2 = (1, 0, 0, -1)$. According to eq. (2.18), the determination of $s_{\mathcal{T}_0^{\text{inc.}}}$ requires the calculation of eikonal matrix elements squared with 2 hard directions, i.e. $N = 0$ of eq. (2.1), with the following variable

$$\mathcal{T}_0^{\text{inc.}}(k_1, k_2, \ldots, k_X) = \min\left\{n_1 \cdot k_{\text{tot}}, n_2 \cdot k_{\text{tot}}\right\} , \tag{3.1}$$

with $k_{\text{tot}} = \sum\limits_{i=1}^{X} k_i$. Since the variable in eq. (3.1) is only sensitive to the light-cone projections of k^μ_{tot}, we can obtain the inclusive soft function for 0-Jettiness from the *double-differential* soft function [74, 84–86], which parametrizes the dependence of the eikonal matrix elements on the light cone components of the total momentum. This quantity can be obtained directly from the inclusive threshold soft function [87, 88] and it is also connected to the eikonal limit of beam functions [85, 86, 89]. Its constants are known through three loops from the soft expansion of color singlet production processes [90, 91], while logarithmically enhanced contributions are known to four loops thanks to the threshold anomalous dimensions (see for example refs. [92–94]). With the same notation as in eq. (2.14), the maximally non-Abelian contribution to the double-differential soft function can be defined in perturbation theory as

$$\begin{aligned} &s^{(n)}_{\text{D.D.}}(w_1, w_2, \epsilon) = \\ &\sum_{m=1}^{n} \int \left(\prod_{i=1}^{m} [\mathrm{d}k_i]\right) \mathcal{M}_s^{(n-m)}(k_1, \ldots, k_m)\, \delta\left(w_1 - n_1 \cdot k_{\text{tot}}\right) \delta\left(w_2 - n_2 \cdot k_{\text{tot}}\right) , \end{aligned} \tag{3.2}$$

where again we stress that we are only considering emissions from 2 hard directions, i.e. we deal with the case $N = 0$ of eq. (2.1). Its all order structure reads [94]

$$s^{\text{bare}}_{\text{D.D.}}(w_1, w_2, \epsilon) = \sum_{L=1}^{\infty} \left(\frac{\alpha_s^{\text{bare}}}{(4\pi)}\right)^L (w_1 w_2)^{(-1-L\epsilon)} f^{(L)}_{\text{MNA}}(\epsilon)\,, \tag{3.3}$$

where $f^{(L)}_{\text{MNA}}(\epsilon)$ is a function of ϵ and color factors, encoding the maximally non-Abelian contributions. By fixing the value of the variable in eq. (3.1) we can obtain the maximally-non-Abelian correction to the inclusive 0-Jettiness soft function

$$\begin{aligned}
s_{\mathcal{T}_0^{\rm inc.}}(\ell,\epsilon) &= \int_0^\infty \mathrm{d}w_1 \mathrm{d}w_2\, s^{\rm bare}_{\rm D.D.}(w_1,w_2,\epsilon)\delta(\ell-\mathcal{T}_0^{\rm inc.}(k_{\rm tot}))\\
&= \int_0^\infty \mathrm{d}w_1 \mathrm{d}w_2 \sum_L \left(\frac{\alpha_s^{\rm bare}}{(4\pi)}\right)^L (w_1 w_2)^{(-1-L\epsilon)} f^{(L)}_{\rm MNA}(\epsilon)\left[2\delta(\ell-w_1)\theta(w_2-w_1)\right]\\
&= 2\sum_L \left(\frac{\alpha_s^{\rm bare}}{4\pi}\right)^L \frac{\ell^{(-1-2L\epsilon)}}{L\epsilon} f^{(L)}_{\rm MNA}(\epsilon)\,. \qquad (3.4)
\end{aligned}$$

The maximally non-Abelian term $f^{(L)}_{\rm MNA}$ at each perturbative order can be extracted from the result in ref. [94] by selecting the linear term in the quadratic Casimir C_R of the SU(3) representation R of the Wilson lines. In eq. (3.4), we can expand in ϵ the observable dependence using the standard distribution expansion

$$\ell^{-1+a\epsilon}\mu^{-a\epsilon} = \frac{\delta(\ell)}{a\epsilon} + \sum_{n=0} \frac{(a\epsilon)^n}{n!}\frac{1}{\mu}\mathcal{L}_n(\ell/\mu)\,. \qquad (3.5)$$

We note that at any fixed order in α_s and ϵ expansion, going from momentum to Laplace space for these soft functions is straightforward, see appendix D for details. For this reason in this section we stay in momentum space with the understanding that any result can be straightforwardly transformed into Laplace space following the rules in eq. (D.7). After α_s renormalization

$$\left(\frac{\alpha_s^{\rm bare}}{4\pi}\right) = \left(\frac{\alpha_s(\mu)}{4\pi}\right)\mu^{2\epsilon}\frac{(4\pi)^{-\epsilon}e^{\epsilon\gamma_\epsilon}}{\Gamma(1-\epsilon)}\left[1-\left(\frac{\alpha_s(\mu)}{4\pi}\right)\frac{\beta_0}{\epsilon}+\dots\right], \qquad (3.6)$$

we obtain the following expression for $s_{\mathcal{T}_0^{\rm inc.}}$ in momentum space at two loops[5]

$$\begin{aligned}
s^{(2)}_{\mathcal{T}_0^{\rm inc.}} &= \frac{1}{\epsilon^3}\,\delta(\ell)\left[-\frac{11}{2}\,C_A+n_f\right]\\
&+\frac{1}{\epsilon^2}\left\{\frac{1}{\mu}\mathcal{L}_0(\ell/\mu)\left[\frac{22}{3}\,C_A-\frac{4}{3}\,n_f\right]+\delta(\ell)\left[-\frac{5}{9}\,n_f+C_A\left(\frac{67}{18}-\frac{\pi^2}{6}\right)\right]\right\}\\
&+\frac{1}{\epsilon}\left\{\frac{1}{\mu}\mathcal{L}_0(\ell/\mu)\left[\frac{20}{9}\,n_f+C_A\left(-\frac{134}{9}+\frac{2\pi^2}{3}\right)\right]\right.\\
&\left.+\,\delta(\ell)\left[n_f\left(-\frac{28}{27}+\frac{\pi^2}{18}\right)+C_A\left(\frac{202}{27}-\frac{11\pi^2}{36}-7\zeta_3\right)\right]\right\}\\
&+\frac{1}{\mu}\mathcal{L}_2(\ell/\mu)\left[-\frac{88}{3}\,C_A+\frac{16}{3}\,n_f\right]+\frac{1}{\mu}\mathcal{L}_1(\ell/\mu)\left[-\frac{80}{9}\,n_f+C_A\left(\frac{536}{9}-\frac{8\pi^2}{3}\right)\right]\\
&+\frac{1}{\mu}\mathcal{L}_0(\ell/\mu)\left[n_f\left(\frac{112}{27}-\frac{4\pi^2}{9}\right)+C_A\left(-\frac{808}{27}+\frac{22\pi^2}{9}+28\zeta_3\right)\right]\\
&+\delta(\ell)\left[C_A\left(\frac{1214}{81}-\frac{67\pi^2}{36}-\frac{\pi^4}{18}-\frac{55\zeta_3}{9}\right)+n_f\left(-\frac{164}{81}+\frac{5\pi^2}{18}+\frac{10\zeta_3}{9}\right)\right]. \qquad (3.7)
\end{aligned}$$

[5]Our convention is to normalize the dipole coefficients by the color structure $\sum_{i\neq j}\mathbf{T}_i\cdot\mathbf{T}_j$ which, for 0-Jettiness, gives $\sum_{i\neq j}\mathbf{T}_i\cdot\mathbf{T}_j=-2C_R$.

As we see, even after α_s renormalization, this quantity has poles in ϵ. However, following the argument in section 2.2 we expect these poles to exactly match the corresponding ones for the 0-Jettiness soft function. At this order, the maximally non-Abelian term of the α_s-renormalized 0-Jettiness soft function [39] reads

$$
\begin{aligned}
s^{(2)}_{\mathcal{T}_0} &= \frac{1}{\epsilon^3}\,\delta(\ell)\Big[-\frac{11}{2}\,C_A + n_f\Big] \\
&+ \frac{1}{\epsilon^2}\Big\{\frac{1}{\mu}\mathcal{L}_0(\ell/\mu)\Big[\frac{22}{3}\,C_A - \frac{4}{3}\,n_f\Big] + \delta(\ell)\Big[-\frac{5}{9}\,n_f + C_A\Big(\frac{67}{18} - \frac{\pi^2}{6}\Big)\Big]\Big\} \\
&+ \frac{1}{\epsilon}\Big\{\frac{1}{\mu}\mathcal{L}_0(\ell/\mu)\Big[\frac{20}{9}\,n_f + C_A\Big(-\frac{134}{9} + \frac{2\pi^2}{3}\Big)\Big] \\
&+ \delta(\ell)\Big[n_f\Big(-\frac{28}{27} + \frac{\pi^2}{18}\Big) + C_A\Big(\frac{202}{27} - \frac{11\pi^2}{36} - 7\zeta_3\Big)\Big]\Big\} \\
&+ \frac{1}{\mu}\mathcal{L}_2(\ell/\mu)\Big[-\frac{88}{3}\,C_A + \frac{16}{3}\,n_f\Big] + \frac{1}{\mu}\mathcal{L}_1(\ell/\mu)\Big[-\frac{80}{9}\,n_f + C_A\Big(\frac{536}{9} - \frac{8\pi^2}{3}\Big)\Big] \\
&+ \frac{1}{\mu}\mathcal{L}_0(\ell/\mu)\Big[n_f\Big(\frac{112}{27} - \frac{4\pi^2}{9}\Big) + C_A\Big(-\frac{808}{27} + \frac{22\pi^2}{9} + 28\zeta_3\Big)\Big] \\
&+ \delta(\ell)\Big[C_A\Big(\frac{1070}{81} + \frac{335\pi^2}{108} - \frac{11\pi^4}{45} - \frac{319\zeta_3}{9}\Big) + n_f\Big(-\frac{20}{81} - \frac{37\pi^2}{54} + \frac{58\zeta_3}{9}\Big)\Big]. \qquad (3.8)
\end{aligned}
$$

Comparing eq. (3.7) and eq. (3.8), we observe that ϵ-poles exactly match. The finite difference between the two soft functions, which we denote as $\Delta s^{(2)}_{\mathcal{T}_0}$ following the notation of eq. (2.17), reads

$$
\Delta s^{(2)}_{\mathcal{T}_0} = \delta(\ell)\Big[C_A\Big(-\frac{16}{9} + \frac{134\pi^2}{27} - \frac{17\pi^4}{90} - \frac{88\zeta_3}{3}\Big) + n_f\Big(\frac{16}{9} - \frac{26\pi^2}{27} + \frac{16\zeta_3}{3}\Big)\Big]. \qquad (3.9)
$$

4 N-Jettiness at NNLO

We now extend the discussion to the case of generic N. We start with the analysis of the inclusive contribution, and then continue with the non-inclusive one.

4.1 The dipole correlator: calculation of $s^{(2)}_{\mathcal{T}_N^{\mathrm{inc.}},\{i,j\}}$

To compute the inclusive N-Jettiness soft function, we need to integrate the eikonal matrix elements at a fixed value of the inclusive Jettiness defined in eq. (2.18), which we show here one more time

$$
\mathcal{T}_N^{\mathrm{inc.}}(k_{\mathrm{tot}}) = \min\{n_1 \cdot k_{\mathrm{tot}}, n_2 \cdot k_{\mathrm{tot}}, \dots, n_{N+2} \cdot k_{\mathrm{tot}}\}\,, \quad k_{\mathrm{tot}} = k_1 + k_2. \qquad (4.1)
$$

For a given dipole (i,j) we define the variables

$$
w_i \equiv n_i\cdot k_{\mathrm{tot}}, \qquad w_j \equiv n_j\cdot k_{\mathrm{tot}}, \qquad x \equiv \frac{(n_i\cdot n_j)\,k_{\mathrm{tot}}^2}{2\,(n_i\cdot k_{\mathrm{tot}})(n_j\cdot k_{\mathrm{tot}})}\,. \qquad (4.2)
$$

and for any other hard direction $A \neq i,j$ we introduce the purely geometric coefficients

$$
c_{Ai} \equiv \frac{n_A\cdot n_i}{n_i\cdot n_j}, \qquad c_{Aj} \equiv \frac{n_A\cdot n_j}{n_i\cdot n_j}. \qquad (4.3)
$$

Parametrizing the hard directions by angles, $n_r^\mu = (1, \vec{n}_r)$ with $|\vec{n}_r| = 1$, and $\cos\Theta_{rs} \equiv \vec{n}_r \cdot \vec{n}_s$, one has

$$n_r \cdot n_s = 1 - \cos\Theta_{rs} \quad \Longrightarrow \quad c_{Ai} = \frac{1-\cos\Theta_{Ai}}{1-\cos\Theta_{ij}}, \qquad c_{Aj} = \frac{1-\cos\Theta_{Aj}}{1-\cos\Theta_{ij}}. \tag{4.4}$$

Using these definitions in eq. (4.1) we find

$$\begin{aligned} n_A \cdot k_{\text{tot}} &= c_{Aj}\, w_i + c_{Ai}\, w_j - 2\sqrt{c_{Ai} c_{Aj}\, w_i w_j}\,\sqrt{1-x}\,\cos(\phi - \phi_A) \\ &\equiv w_A(w_i, w_j, x, \phi; \vec{\Theta})\,, \end{aligned} \tag{4.5}$$

where ϕ is the azimuthal angle of $k_{\text{tot}}^\perp$, and ϕ_A is the fixed azimuthal angle of the transverse projection of n_A, both defined in the plane orthogonal to the dipole (n_i, n_j). In this way we rewrite the inclusive N-Jettiness constraint as a function of (w_i, w_j, x, ϕ) and of the set of hard-direction angles $\vec{\Theta}$,

$$\mathcal{T}_N^{\text{inc.}}(k_{\text{tot}}) = \min\Big\{ w_i,\ w_j,\ \{\, w_A(w_i, w_j, x, \phi; \vec{\Theta})\,\}_{A \neq i,j} \Big\} \equiv \mathcal{T}_N^{\text{inc.}}(w_i, w_j, x, \phi; \vec{\Theta}), \tag{4.6}$$

with $n_A \cdot k_{\text{tot}}$ given in eq. (4.5). Therefore, we can generalize the reasoning in section 3, and rewrite the maximally non-abelian term of the inclusive N-Jettiness soft function as a projection of the *fully differential* soft function [73], which is analytically known through three loops [75]. Concretely, at order $\mathcal{O}(\alpha_s^L)$ in perturbation theory, the maximally-non-Abelian contribution to the fully differential soft function is defined as

$$\begin{aligned} s_{\text{F.D.}}^{(L)}(w_i, w_j, x) \equiv \sum_{m=1}^{L} \int \left(\prod_{r=1}^{m} [\mathrm{d}k_r] \right) \mathcal{M}_{s,\{i,j\}}^{(L-m)}(k_1, \ldots, k_m) \\ \times\ \delta(w_i - n_i \cdot k_{\text{tot}})\ \delta(w_j - n_j \cdot k_{\text{tot}})\ \delta\!\left(x - \frac{(n_i \cdot n_j)\, k_{\text{tot}}^2}{2\,(n_i \cdot k_{\text{tot}})(n_j \cdot k_{\text{tot}})} \right). \end{aligned} \tag{4.7}$$

Note that the fully differential soft function is also related to the double differential soft one of eq. (3.2) via the sum rule

$$\int_0^1 \mathrm{d}x\, s_{\text{F.D.}}(w_i, w_j, x) = s_{\text{D.D.}}(w_i, w_j)\,. \tag{4.8}$$

Its relation to the maximally non-abelian term of the inclusive N-Jettiness soft function at $\mathcal{O}(\alpha_s^L)$, $s^{(L)}_{\mathcal{T}_N^{\text{inc.}},\{i,j\}}$ is described by the following equation

$$\begin{aligned} & s^{(L)}_{\mathcal{T}_N^{\text{inc.}},\{i,j\}}(\ell; \vec{\Theta}) = \\ & \int [\mathrm{d}\Omega^\perp]_\epsilon \int_0^1 \mathrm{d}x \int_0^\infty \mathrm{d}w_i \mathrm{d}w_j\, s_{\text{F.D.}}^{(L)}(w_i, w_j, x)\, \delta\Big(\ell - \mathcal{T}_N^{\text{inc.}}(w_i, w_j, x, \phi; \vec{\Theta})\Big)\,. \end{aligned} \tag{4.9}$$

where the normalized solid angle in $d-2$ dimensions $[\mathrm{d}\Omega^\perp]_\epsilon$ reads

$$[\mathrm{d}\Omega^\perp]_\epsilon \equiv \frac{\mathrm{d}\Omega^{(2-2\epsilon)}}{\Omega^{(2-2\epsilon)}}\,. \tag{4.10}$$

The bare fully differential soft function takes the form [74, 94]

$$s_{\text{F.D.}}(w_i, w_j, x, \epsilon) = \sum_{L=1}^{\infty} \left(\frac{\alpha_s^{\text{bare}}}{(4\pi)} \right)^L (w_i w_j)^{(-1-L\epsilon)} \left(\frac{n_i \cdot n_j}{2} \right)^{L\epsilon} f_{\text{MNA}}^{(L)}(x, \epsilon)\,, \tag{4.11}$$

where, order by order in ϵ, $f^{(L)}_{\rm MNA}(x,\epsilon)$ can be expanded in terms of regular functions of x and distributions at $x=0$

$$f^{(L)}_{\rm MNA}(x,\epsilon) = \sum_k \epsilon^k \left[f^{(L,k)}_{\rm reg}(x) + \sum_{m=0}^{2L-1} \mathcal{L}_m(x) f^{(L,k)}_{\mathcal{L}_m} + \delta(x) f^{(L,k)}_{\delta} \right] . \tag{4.12}$$

We note that the sum rule eq. (4.8) implies a sum rule for f itself

$$\int_0^1 {\rm d}x f^{(L)}_{\rm MNA}(x,\epsilon) = f^{(L)}_{\rm MNA}(\epsilon) \,, \tag{4.13}$$

where $f^{(L)}_{\rm MNA}(\epsilon)$ was defined in eq. (3.3). As for the function $f^{(L)}_{\rm MNA}$, the quantity $f^{(L)}_{\rm MNA}(x,\epsilon)$ can be extracted from the result in ref. [74] by selecting the linear term in the quadratic Casimir C_R at each perturbative order.

We now specialize to the bare case at NNLO ($L=2$) and show how to obtain a practical integral representation for $s^{(2)}_{\mathcal{T}_N^{\rm inc.},\{i,j\}}(\ell;\vec{\Theta})$. Since $\mathcal{T}_N^{\rm inc.}$ is a homogeneous function of degree one w.r.t. variables (w_i,w_j), we introduce an overall energy scale E and a momentum fraction $z\in[0,1]$ such that $w_i = E\,z$ and $w_j = E(1-z)$, and factor out E from the measurement function to define the rescaled observable t,

$$t_{\{i,j\}}(z,x,\phi;\vec{\Theta}) = \min\Big\{ z,\ 1-z,\ \{w_A(z,1-z,x,\phi;\vec{\Theta})\}_{A\neq i,j} \Big\} . \tag{4.14}$$

Using this representation to factor out the dependence on ℓ, we obtain

$$\begin{aligned} &s^{(2)}_{\mathcal{T}_N^{\rm inc.},\{i,j\}}(\ell,\epsilon) = \\ &\left(\frac{n_i\cdot n_j}{2}\right)^{2\epsilon} \ell^{-1-4\epsilon} \int [{\rm d}\Omega^\perp]_\epsilon \int_0^1 {\rm d}x \int_0^1 {\rm d}z\, \big[z(1-z)\big]^{-1-2\epsilon}\, t^{4\epsilon}_{\{i,j\}}(z,x,\phi;\vec{\Theta})\; f^{(2)}(x,\epsilon) \,. \end{aligned} \tag{4.15}$$

While we can easily expand in ϵ the ℓ-dependent pre-factor using eq. (3.5), we cannot evaluate the integral in eq. (4.15) order by order in ϵ, as it contains collinear poles at the endpoints of the z integration.[6] The physical origin of these endpoint singularities is straightforward to understand: the limits $z\to 0$ and $z\to 1$ correspond to radiation becoming collinear to n_i and n_j, respectively. In these strict collinear limits, the measurement must be insensitive to the geometry of the full N-leg hard configuration, so the singular behavior should be related to the one already obtained for inclusive 0-Jettiness. To make this relation manifest and isolate this endpoint collinear singularity analytically in a N-leg geometry-independent way, we split t against the minimum appropriate to the $\mathcal{T}_0^{\rm inc.}$ case,

$$t_0(z) \equiv \min\{z, 1-z\} \,. \tag{4.16}$$

Since extra hard directions can only lower the minimum, one has $t_{\{i,j\}} \le t_0$ pointwise. Writing

$$t^{4\epsilon}_{\{i,j\}} = t_0^{4\epsilon} + \big[t^{4\epsilon}_{\{i,j\}} - t_0^{4\epsilon}\big] \,, \tag{4.17}$$

[6]Note that we cannot expand the factor $\big[z(1-z)\big]^{-1-2\epsilon}$ in distributions in z (or $1-z$), because $t^{4\epsilon}_{i,j}(z,x,\phi;\vec{\Theta})$ is in general not a regular function of z as $z\to 0,1$.

we obtain

$$s^{(2)}_{\mathcal{T}_N^{\rm inc.},\{i,j\}}(\ell,\epsilon) = \left(\frac{n_i\cdot n_j}{2}\right)^{2\epsilon}\Bigg[s^{(2)}_{\mathcal{T}_0^{\rm inc.}}(\ell,\epsilon) \\ + \ell^{-1-4\epsilon}\int[\mathrm{d}\Omega^\perp]_\epsilon\int_0^1\mathrm{d}x\int_0^1\mathrm{d}z\,\frac{\left[t^{4\epsilon}_{\{i,j\}}-t_0^{4\epsilon}\right](z,x,\phi;\vec{\Theta})}{[z(1-z)]^{1+2\epsilon}}\,f^{(2)}(x,\epsilon)\Bigg]\,, \tag{4.18}$$

where, in order to identify the $t_0^{4\epsilon}$ dependent term with $s^{(2)}_{\mathcal{T}_0^{\rm inc.}}(\ell,\epsilon)$, we used the fact that t_0 is independent of x, together with the sum rule in eq. (4.13). The entire dependence on the directions of N jets is now encoded in the integral containing $(t^{4\epsilon}-t_0^{4\epsilon})$. Crucially, this difference vanishes whenever the radiation is closer to either leg of the dipole, and in particular in the collinear regions $z\to 0,1$. As a result, the N-leg geometry-dependent term in eq. (4.18) is free of collinear divergences and can be evaluated numerically after expanding the integrand in ϵ to the desired order.

4.2 The dipole correlator: calculation of $\Delta\tilde{s}^{(2)}_{\mathcal{T}_N,\{i,j\}}$

In this section, we show how to obtain the difference of the two Jettinesses $\Delta\tilde{s}^{(2)}_{\mathcal{T}_N,\{i,j\}}$ numerically. This term receives contributions only from configurations with the emission of two real soft partons, since by construction it vanishes for a single soft emission. Consequently, the soft integral relevant for $\Delta\tilde{s}^{(2)}_{\mathcal{T}_N,\{i,j\}}$ takes the form

$$\Delta\tilde{s}^{(2)}_{\mathcal{T}_N,\{i,j\}} \equiv \int[\mathrm{d}k_1]\,[\mathrm{d}k_2]\,\mathcal{M}^{(0)}_{s,\{i,j\}}(k_1,k_2)\left(e^{-u\,\mathcal{T}_N(k_1,k_2)}-e^{-u\,\mathcal{T}_N(k_1+k_2)}\right)\,, \tag{4.19}$$

where $\mathcal{M}^{(0)}_{s,\{i,j\}}(k_1,k_2)$ denotes the maximally non-Abelian part of the tree-level double-soft squared amplitude [95] associated with the dipole $\{i,j\}$. We provide its expression in eq. (A.3). Although the soft matrix element squared $\mathcal{M}^{(0)}_{s,\{i,j\}}(k_1,k_2)$ depends solely on the dipole $\{i,j\}$, the observable $\mathcal{T}_N$ involves all hard directions. Consequently, the integral can be computed numerically after fixing the kinematic configuration of the hard legs, i.e. once the set of directions $\{n_k\}_{k=1}^N$ is specified. In any singular configuration of $\mathcal{M}^{(0)}_{s,\{i,j\}}(k_1,k_2)$, the observable satisfies $\mathcal{T}_N(k_1,k_2)\to\mathcal{T}_N(k_1+k_2)$, causing the expression in parentheses to vanish. Therefore, the integral is free of infrared singularities and can be immediately evaluated in four space-time dimensions.

To make the cancellation of divergences manifest, it is convenient to perform some integrations analytically. We parametrize the d-dimensional double-soft phase space in terms of energies and angles of the emitted gluons, defined in the partonic centre-of-mass frame, as

$$[\mathrm{d}k_1]\,[\mathrm{d}k_2] = E_1^{1-2\epsilon}\mathrm{d}E_1\,[\mathrm{d}\Omega_1]_\epsilon E_2^{1-2\epsilon}\mathrm{d}E_2\,[\mathrm{d}\Omega_2]_\epsilon\,, \tag{4.20}$$

where $[\mathrm{d}\Omega]_\epsilon = \mathrm{d}\Omega^{(3-2\epsilon)}/(2(2\pi)^{3-2\epsilon})$. Given the symmetry of the integrand under the exchange of $1\leftrightarrow 2$, we can select a specific energy ordering of two soft momenta, $E_2<E_1$, and supply a factor of two. Then, we rescale $E_2=\xi E_1$, with $0\le\xi\le 1$. Under this transformation, the matrix element $\mathcal{M}^{(0)}_{s,\{i,j\}}(k_1,k_2)$ becomes

$$\mathcal{M}^{(0)}_{s,\{i,j\}}(k_1,k_2) = \frac{1}{E_1^4}\mathcal{M}^{(0)}_{s,\{i,j\}}(\hat{k}_1,\xi\hat{k}_2)\,, \tag{4.21}$$

where $\hat{k}_{1,2} = (1, \hat{\vec{k}}_{1,2})$ and $\hat{\vec{k}}_{1,2}$ denote the unit vectors along the spatial momenta $\vec{k}_{1,2}$. Similarly, the observable rescales as

$$\mathcal{T}_N(k_1, k_2) = E_1 \mathcal{T}_N(\hat{k}_1, \xi \hat{k}_2), \quad \mathcal{T}_N(k_1 + k_2) = E_1 \mathcal{T}_N(\hat{k}_1 + \xi \hat{k}_2)\,. \tag{4.22}$$

This leads to

$$\Delta \tilde{s}^{(2)}_{\mathcal{T}_N,\{i,j\}} = 2 \int [\mathrm{d}\Omega_{12}]_\epsilon \int_0^1 \mathrm{d}\xi\, \xi^{1-2\epsilon} \int_0^\infty \frac{\mathrm{d}E_1}{E_1^{1+4\epsilon}}\, \mathcal{M}^{(0)}_{s,\{i,j\}}(\hat{k}_1, \xi \hat{k}_2) \\ \times \left(e^{-u\, E_1 \mathcal{T}_N(\hat{k}_1, \xi \hat{k}_2)} - e^{-u\, E_1 \mathcal{T}_N(\hat{k}_1 + \xi \hat{k}_2)} \right), \tag{4.23}$$

where we introduced the short-hand notation $[\mathrm{d}\Omega_{12}]_\epsilon = [\mathrm{d}\Omega_1]_\epsilon [\mathrm{d}\Omega_2]_\epsilon$. The integration over E_1 can be carried out analytically, yielding

$$\Delta \tilde{s}^{(2)}_{\mathcal{T}_N,\{i,j\}} = \frac{N_\epsilon}{2\epsilon} \int [\mathrm{d}\Omega_{12}]_\epsilon \int_0^1 \frac{\mathrm{d}\xi}{\xi^{-1+2\epsilon}} \mathcal{M}^{(0)}_{s,\{i,j\}}(\hat{k}_1, \xi \hat{k}_2) \left[\mathcal{T}_0^{4\epsilon}(k_{12,\xi}) - \mathcal{T}_N^{4\epsilon}(\hat{k}_1, \xi \hat{k}_2) \right], \tag{4.24}$$

with $N_\epsilon = u^{4\epsilon}\, \Gamma(1 - 4\epsilon)$ and $k_{12,\xi} = k_1 + \xi k_2$.

Provided that the integrals over ξ and the angular variables Ω_1 and Ω_2 do not generate further singularities, we may expand eq. (4.24) in powers of ϵ at the integrand level. Keeping only the leading term in the ϵ-expansion, we obtain

$$\Delta \tilde{s}^{(2)}_{\mathcal{T}_N,\{i,j\}} = 2 \int [\mathrm{d}\Omega_{12}]_{\epsilon=0} \int_0^1 \mathrm{d}\xi\, \xi\, \mathcal{M}^{(0)}_{s,\{i,j\}}(\hat{k}_1, \xi \hat{k}_2) \ln \left(\frac{\mathcal{T}_N(\hat{k}_1 + \xi \hat{k}_2)}{\mathcal{T}_N(\hat{k}_1, \xi \hat{k}_2)} \right). \tag{4.25}$$

We can verify the assumption about finiteness of $\Delta \tilde{s}^{(2)}_{\mathcal{T}_N,\{i,j\}}$ a posteriori, by analyzing the above integral. All potential divergences should arise from the singular behavior of $\mathcal{M}^{(0)}_{s,\{i,j\}}(\hat{k}_1, \xi \hat{k}_2)$ in the following limits: the soft limit $\xi \to 0$, the collinear configuration $\hat{\vec{k}}_1 \parallel \hat{\vec{k}}_2$, and the triple-collinear limits $\hat{\vec{k}}_1 \parallel \hat{\vec{k}}_2 \parallel \vec{n}_i$ or $\hat{\vec{k}}_1 \parallel \hat{\vec{k}}_2 \parallel \vec{n}_j$. In each of these cases, the observable satisfies $\mathcal{T}_0(\hat{k}_1, \xi \hat{k}_2) \to \mathcal{T}_0(\hat{k}_1 + \xi \hat{k}_2)$, so that the logarithm in eq. (4.25) vanishes and the corresponding singularity is regulated. The five-dimensional integral in eq. (4.25) is therefore finite without additional regulator, and can be evaluated numerically using standard Monte Carlo integration methods. Finally, given that $\Delta \tilde{s}^{(2)}_{\mathcal{T}_N,\{i,j\}}$ does not depend on the Laplace variable u, its inverse Laplace transform to momentum space amounts simply to

$$\Delta \tilde{s}^{(2)}_{\mathcal{T}_N,\{i,j\}} \to \Delta \tilde{s}^{(2)}_{\mathcal{T}_N,\{i,j\}}\, \delta(\ell) \equiv \Delta s^{(2)}_{\mathcal{T}_N,\{i,j\}}\, \delta(\ell). \tag{4.26}$$

Before concluding this section, we briefly comment on the extension of the method outlined here to N^{3}LO. Given that the NNLO correction $\Delta \tilde{s}^{(2)}_{\mathcal{T}_N,\{i,j\}}$ computed above requires the computation of tree-level (double-real) corrections, and it is finite, we expect that the computation of the $\Delta \tilde{s}^{(3)}_{\mathcal{T}_N,\{i,j\}}$ at N^{3}LO can be organized in such a way that it only requires NLO-like subtractions. Ongoing investigations seem to confirm this expectation [96].

4.3 The tripole correlator

Having obtained the dipole contribution to the N-Jettiness soft function, we will discuss the tripole contribution, which is the last missing piece at NNLO. Following the normalization conventions of eq. (2.12), we write the perturbative expansion of the Laplace transform of the tripole correction as

$$\tilde{\boldsymbol{s}}^{\text{tri.}}_{\mathcal{T}_N}(u) = \sum_{n=2}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n \tilde{\boldsymbol{s}}^{\text{tri.},(\text{n})}_{\mathcal{T}_N}(u)\,. \tag{4.27}$$

The series starts at the second order, $n = 2$, where the contribution reads

$$\tilde{\boldsymbol{s}}^{\text{tri.},(2)}_{\mathcal{T}_N}(u) = \frac{N_\epsilon}{\epsilon} \sum_{(ijk)} \boldsymbol{F}_{ijk}\, \kappa_{jk}\, I_{\{ij\},k}(u), \quad N_\epsilon = \frac{2^{3-\epsilon}\pi e^{2\gamma_E \epsilon}\,\Gamma(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}\,. \tag{4.28}$$

In eq. (4.28) the sum is restricted to distinct summation indices $i \neq j \neq k$, $\boldsymbol{F}_{ijk}$ and κ_{jk} are given by

$$\boldsymbol{F}_{ijk} = f_{abc}\mathbf{T}^a_i \mathbf{T}^b_j \mathbf{T}^c_k\,, \tag{4.29}$$

and

$$\kappa_{ij} = \lambda_{ij} - \lambda_{i,\text{out}} - \lambda_{j,\text{out}}\,, \tag{4.30}$$

which evaluates to $+1$ if both emitters i, j are incoming and -1 otherwise. Moreover, we have introduced the integral

$$I_{\{ij\},k}(u) = \int_0^\infty \mathrm{d}\ell\, e^{-u\ell} \int [\mathrm{d}q]\, \frac{2(2\pi)^{3-2\epsilon}}{\Omega^{(2-2\epsilon)}} \delta(\ell - \mathcal{T}_N(q)) \mathcal{M}^{(1)}_{s,\{\{ij\},k\}}(q)\,, \tag{4.31}$$

In the above equation, $\mathcal{M}^{(1)}_{s,\{\{ij\},k\}}(q)$ denotes the color-stripped tripole contribution, cf. eq. (A.2), which develops singularities when q becomes collinear to either n_i or n_j. Therefore, in each triple-correlator contribution $\{\{ij\}, k\}$, we parametrize all momenta using the Sudakov decomposition with respect to the light-cone directions $\{n_i, n_j\}$ as in section 4. In particular, for a light-like four-vector q, we write

$$q = \frac{w_j}{n_i \cdot n_j} n_i + \frac{w_i}{n_i \cdot n_j} n_j + \sqrt{\frac{2 w_i w_j}{n_i \cdot n_j}}\, e_\perp\,, \tag{4.32}$$

where $e_\perp^2 = -1$. With this parametrization, the single-particle phase space becomes

$$\frac{\mathrm{d}^d q}{(2\pi)^{d-1}}\, \delta^+(q^2) = [\mathrm{d}q] = \frac{1}{n_i \cdot n_j} (w_i w_j)^{-\epsilon} \left(\frac{n_i \cdot n_j}{2}\right)^\epsilon \mathrm{d}w_i\, \mathrm{d}w_j\, \frac{\mathrm{d}\Omega^{(2-2\epsilon)}}{2(2\pi)^{3-2\epsilon}}\,. \tag{4.33}$$

Following the notation introduced in eq. (4.5), we denote with

$$\begin{aligned} w_l(w_i, w_j, 0, \phi_q) &= q \cdot n_l \\ &= w_i \frac{n_l \cdot n_j}{n_i \cdot n_j} + w_j \frac{n_l \cdot n_i}{n_i \cdot n_j} - \frac{2}{(n_i \cdot n_j)} \sqrt{w_i w_j (n_l \cdot n_i)(n_l \cdot n_j)}\, \cos\phi_{ql}\,, \end{aligned} \tag{4.34}$$

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all the scalar products of q with any direction n_l different from the directions $\{n_i, n_j\}$ employed in the Sudakov decomposition. In the above equation we have defined $\phi_{ql} = \phi_q - \phi_l$, and we have omitted in the arguments of w_l the explicit dependence on $\vec{\Theta}$ to ease the notation. In this parametrization, the inclusive N-Jettiness observable reads

$$\mathcal{T}_N(q) = \min\{w_i, w_j, \{w_A(w_i, w_j, 0, \phi_q)\}_{A\neq i,j}\} \equiv \mathcal{T}_N(w_i, w_j, \phi_q; w_{A\neq i,j})\,. \tag{4.35}$$

The integral $I_{\{ij\},k}$ becomes

$$\begin{aligned} I_{\{ij\},k}(u) = {}& \left(\frac{(n_i\cdot n_j)(n_j\cdot n_k)}{2}\right)^{\epsilon}\int_0^\infty \mathrm{d}\ell\, e^{-u\ell}\int \mathrm{d}w_i\mathrm{d}w_j\,[\mathrm{d}\Omega^\perp]_\epsilon \\ & \times \frac{(w_iw_j)^{-1-\epsilon}}{[w_j\, w_k(w_i, w_j, 0, \phi_q)]^\epsilon}\,\delta\left(\ell - \mathcal{T}_N(w_i, w_j, \phi_q; w_{A\neq i,j})\right)\,, \end{aligned} \tag{4.36}$$

with $[\mathrm{d}\Omega^\perp]_\epsilon$ defined in eq. (4.10).

We emphasize that in this parametrization the two collinear singularities correspond to $w_i \to 0$ and $w_j \to 0$, respectively. We disentangle them by splitting the integration domain into two sectors by means of the decomposition $1 = \theta(w_i - w_j) + \theta(w_j - w_i)$. Then, in each sector, we factor out the larger light-cone component by changing variables $\{w_i = y, w_j = y\xi\}$ and $\{w_j = y\xi, w_i = y\}$ in the first (plus) and second sectors (minus), respectively. Using the homogeneity of w_k and $\mathcal{T}_N$ (cf. eq. (4.34) and eq. (4.35)), we obtain

$$\begin{aligned} I^+_{\{ij\},k}(u) = {}& \left(\frac{(n_i\cdot n_j)(n_j\cdot n_k)}{2}\right)^{\epsilon}\int_0^\infty \mathrm{d}\ell\, e^{-u\ell}\int_0^\infty \mathrm{d}y \\ & \times \int_0^1 \mathrm{d}\xi\,[\mathrm{d}\Omega^\perp]_\epsilon\,\frac{y^{-1-4\epsilon}\xi^{-1-2\epsilon}}{w_k^\epsilon(1,\xi,0,\phi_q)}\,\delta\left(\ell - y\tau^+_{N,\{ij\}}\right)\,, \\ I^-_{\{ij\},k}(u) = {}& \left(\frac{(n_i\cdot n_j)(n_j\cdot n_k)}{2}\right)^{\epsilon}\int_0^\infty \mathrm{d}\ell\, e^{-u\ell}\int_0^\infty \mathrm{d}y \\ & \times \int_0^1 \mathrm{d}\xi\,[\mathrm{d}\Omega^\perp]_\epsilon\,\frac{y^{-1-4\epsilon}\xi^{-1-\epsilon}}{w_k^\epsilon(\xi,1,0,\phi_q)}\,\delta\left(\ell - y\tau^-_{N,\{ij\}}\right)\,, \end{aligned}$$

where we have introduced the functions $\tau^+_{N,\{ij\}}(\xi,\phi_q) \equiv \mathcal{T}_N(1,\xi,\phi_q,\{w_A(1,\xi,0,\phi_q)\}_{A\neq i,j})$ and $\tau^-_{N,\{ij\}}(\xi,\phi_q) \equiv \mathcal{T}_N(\xi,1,\phi_q,\{w_A(\xi,1,0,\phi_q)\}_{A\neq i,j})$ that depend on the directions n_i, n_j, but not on the variable y.

We remove the delta function by integrating over y, and use eq. (D.5) to perform the Laplace integral over ℓ. We find

$$\begin{aligned} \int_0^\infty \mathrm{d}\ell\, e^{-u\ell}\int_0^\infty \mathrm{d}y\, y^{-1-4\epsilon}\delta\left(\ell - y\,\tau^\pm_{N,\{ij\}}\right) &= \frac{\bar{u}^{4\epsilon}e^{-4\gamma_E\epsilon}\Gamma(1-4\epsilon)}{-4\epsilon}\left(\tau^\pm_{N,\{ij\}}\right)^{4\epsilon} \\ &= \frac{\bar{u}^{4\epsilon}e^{-4\gamma_E\epsilon}\Gamma(1-4\epsilon)}{-4\epsilon}\xi^{4\epsilon}\left(t^\pm_{N,\{ij\}}\right)^{4\epsilon}\,, \end{aligned} \tag{4.37}$$

where, in the last step, we have factored out the leading behavior of $\tau^\pm_{N,\{ij\}}$ in the singular limit $\xi \to 0$. The functions $t^\pm_{N,\{ij\}}$ represent, therefore, the explicit parametrization of the rescaled inclusive N-Jettiness observable in the two sectors

$$t^+_{N,\{ij\}}(\xi,\phi_q) \equiv \frac{\mathcal{T}_N(q/y)}{\xi} = \frac{\min\{\xi, \{w_A(1,\xi,0,\phi_q)\}_{A\neq i,j}\}}{\xi}\,, \tag{4.38}$$

$$t^-_{N,\{ij\}}(\xi,\phi_q) \equiv \frac{\mathcal{T}_N(q/y)}{\xi} = \frac{\min\{\xi, \{w_A(\xi,1,0,\phi_q)\}_{A\neq i,j}\}}{\xi}\,, \tag{4.39}$$

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and, by construction, satisfy the following equation

$$t^{+}_{N,\{ij\}}(0,\phi_q) = t^{-}_{N,\{ij\}}(0,\phi_q) = 1\,. \tag{4.40}$$

We obtain

$$\begin{aligned} I_{\{ij\},k}(u) = &-\frac{\bar{u}^{4\epsilon} e^{-4\gamma_E\epsilon}\Gamma(1-4\epsilon)}{4\epsilon}\left(\frac{(n_i\cdot n_j)(n_j\cdot n_k)}{2}\right)^{\epsilon}\int_0^1 \mathrm{d}\xi \int [\mathrm{d}\Omega^{\perp}]_{\epsilon} \\ &\times\left[\xi^{-1+2\epsilon}\frac{\left(t^{+}_{N,\{ij\}}(\xi,\phi_q)\right)^{4\epsilon}}{w_k^{\epsilon}(1,\xi,0,\phi_q)} + \xi^{-1+3\epsilon}\frac{\left(t^{-}_{N,\{ij\}}(\xi,\phi_q)\right)^{4\epsilon}}{w_k^{\epsilon}(\xi,1,0,\phi_q)}\right]\,. \end{aligned} \tag{4.41}$$

In the explicit parametrizations of eq. (4.41), the one-emission collinear singularities are mapped to the point $\xi = 0$ and are encapsulated in the singular factors $\xi^{-1+a\epsilon}$, $a = 2, 3$.

With this parametrization, we can easily perform the renormalization of the tripole correlator, which we report in detail in appendix C. After the cancellation of the UV poles against the counter-terms, we write the finite remainder of the renormalized tripole correction as

$$\tilde{\boldsymbol{s}}^{\mathrm{tri.},(2),(\mathrm{R})}_{\mathcal{T}_N}(u) = \sum_{(ijk)} \boldsymbol{F}_{ijk}\kappa_{jk}\tilde{s}^{\mathrm{tri.},(2),(\mathrm{R})}_{\mathcal{T}_N,\{i,j,k\}}(u)\;, \tag{4.42}$$

where

$$\tilde{s}^{\mathrm{tri.},(2),(\mathrm{R})}_{\mathcal{T}_N,\{i,j,k\}}(u) = \widetilde{\mathcal{G}}^{\mathrm{tri.},(2)}_{\mathcal{T}_N,\{i,j,k\}}(u) + \int_0^1 \mathrm{d}\xi \int_0^{2\pi} \mathrm{d}\phi_q\; \widetilde{\mathcal{F}}^{\mathrm{tri.},(2),(\mathrm{R})}_{\mathcal{T}_N,\{i,j,k\}}(u,\xi,\phi_q,t^{\pm})\,. \tag{4.43}$$

The observable-independent contribution is given by

$$\widetilde{\mathcal{G}}^{\mathrm{tri.},(2),(\mathrm{R})}_{\mathcal{T}_N,\{i,j,k\}}(u) = -\frac{4\pi}{3}\left(L_{ij}^3 + 2\pi^2 L_{ij} + \theta(n_i\cdot n_k - n_j\cdot n_k)\ln^3\frac{n_i\cdot n_k}{n_j\cdot n_k}\right)\,, \tag{4.44}$$

with L_{ij} defined in eq. (2.11), while the dependence on the observable is encoded in the function $\widetilde{\mathcal{F}}^{\mathrm{tri.},(2),(\mathrm{R})}_{\mathcal{T}_N,\{i,j,k\}}(u,\xi,\phi_q,t^{\pm})$, which can be safely integrated numerically over ξ and ϕ_q. Explicitly, it reads

$$\begin{aligned} &\widetilde{\mathcal{F}}^{\mathrm{tri.},(2),(\mathrm{R})}_{\mathcal{T}_N,\{i,j,k\}}(u,\xi,\phi_q,t^{\pm}) = \\ &-\frac{4}{\xi}\Big[L_{jk}\ln\Big(t^{+}_{N,\{ij\}}(\xi,\phi_q)t^{-}_{N,\{ij\}}(\xi,\phi_q)\Big) \\ &+\ln\left(t^{+}_{N,\{ij\}}(\xi,\phi_q)\right)\ln\left(\frac{\xi\, t^{+}_{N,\{ij\}}(\xi,\phi_q)}{w_k(1,\xi,0,\phi_q)}\right) + \ln\left(t^{-}_{N,\{ij\}}(\xi,\phi_q)\right)\ln\left(\frac{\xi^2\, t^{-}_{N,\{ij\}}(\xi,\phi_q)}{w_k(\xi,1,0,\phi_q)}\right)\Big]. \end{aligned} \tag{4.45}$$

The representation (4.43) of the tripole correction is a new result of this work, and to the best of our knowledge is the most compact one available in the literature.

4.4 Numerical results for benchmark phase-space points

In the following, we discuss numerical results for the N-Jettiness soft function obtained using the method presented in this article. In particular, we perform the comparison with

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	Gluons (C_A term)			Quarks ($T_F n_f$ term)		
	$\tilde{s}^{(2)}_{\mathcal{T}_0}$	$\tilde{s}^{(2)}_{\mathcal{T}_0^{\rm inc.}}$	$\Delta\tilde{s}^{(2)}_{\mathcal{T}_0}$	$\tilde{s}^{(2)}_{\mathcal{T}_0}$	$\tilde{s}^{(2)}_{\mathcal{T}_0^{\rm inc.}}$	$\Delta\tilde{s}^{(2)}_{\mathcal{T}_0}$
Numerical	28.254(7)	34.70461	−6.451(7)	−21.6954(3)	−19.06462	−2.6308(3)
Analytic	28.2495	34.70461	−6.4551	−21.6953	−19.06462	−2.63063

Table 1. Comparison for the 0-Jettiness soft function constant in Laplace space obtained with the method of this paper and the analytic determination of refs. [80–82]. The result for inclusive soft function, $\tilde{s}^{(2)}_{\mathcal{T}_0^{\rm inc.}}$, is obtained by taking the Laplace transform of the expression calculated analytically in section 3.

the results in the literature for 0-, 1-, 2-, and 3-Jettiness, and we provide new results for 4- and 5-Jettiness for selected benchmark points. In particular, we report results for the non-logarithmic terms in Laplace space to facilitate the comparison to the predictions of refs. [40, 41], and therefore all the numerical results presented in this section are obtained by setting $L_{ij} \to 0$, with L_{ij} being defined in eq. (2.11).

Following refs. [40, 41], we consider the pp case in eq. (2.1), and choose the directions of the incoming legs as

$$n_{1,2} = (1, 0, 0, \pm 1)\,. \tag{4.46}$$

In this frame, the remaining directions can be parametrized as

$$n_i = (1, \sin\theta_{1i}\cos\phi_i, \sin\theta_{1i}\sin\phi_i, \cos\theta_{1i})\,, \quad i = 3, \ldots, N+2\,. \tag{4.47}$$

In addition, we choose the x-axis in such a way that $\phi_3 = 0$. Benchmark values for the soft function are obtained for the following phase-space points:

$$N = 1: \quad \theta_{13} = \frac{12\pi}{25}\,, \tag{4.48}$$

$$N = 2: \quad \theta_{13} = \frac{6\pi}{25}\,, \quad \theta_{14} = \frac{13\pi}{25}\,, \quad \phi_4 = \frac{\pi}{5}\,, \tag{4.49}$$

$$N = 3: \quad \theta_{13} = \frac{3\pi}{10}\,, \quad \theta_{14} = \frac{6\pi}{10}\,, \quad \theta_{15} = \frac{9\pi}{10}\,, \quad \phi_4 = \frac{3\pi}{5}\,, \quad \phi_5 = \frac{6\pi}{5}\,, \tag{4.50}$$

$$\begin{aligned} N = 4: \quad & \theta_{13} = \frac{12\pi}{25}\,, \quad \theta_{14} = \frac{11\pi}{25}\,, \quad \theta_{15} = \frac{18\pi}{25}\,, \quad \theta_{16} = \frac{20\pi}{25}\,, \\ & \phi_4 = \frac{8\pi}{25}\,, \quad \phi_5 = \frac{11\pi}{25}\,, \quad \phi_6 = \frac{22\pi}{25}\,, \end{aligned} \tag{4.51}$$

$$\begin{aligned} N = 5: \quad & \theta_{13} = \frac{5\pi}{25}\,, \quad \theta_{14} = \frac{10\pi}{25}\,, \quad \theta_{15} = \frac{9\pi}{25}\,, \quad \theta_{16} = \frac{10\pi}{25}\,, \quad \theta_{17} = \frac{13\pi}{25}\,, \\ & \phi_4 = \frac{21\pi}{25}\,, \quad \phi_5 = \frac{2\pi}{25}\,, \quad \phi_6 = \frac{4\pi}{25}\,, \quad \phi_7 = \frac{22\pi}{25}\,. \end{aligned} \tag{4.52}$$

For the case of 0-Jettiness there is only one dipole; we present the results for this case in table 1 finding exquisite agreement with the results in the literature [80–82].

We then move to the evaluation of the 1-Jettiness soft function. In this case, the result is a function of a single angular variable, that we choose to be the pseudo-rapidity of the jet $y_3 = 1/2 \ln\left[(1 + \cos\theta_{13})/(1 - \cos\theta_{13})\right]$. The corresponding results of the numerical calculation are displayed in figures 1 and 2 for gluonic and fermionic final-state channels, respectively.

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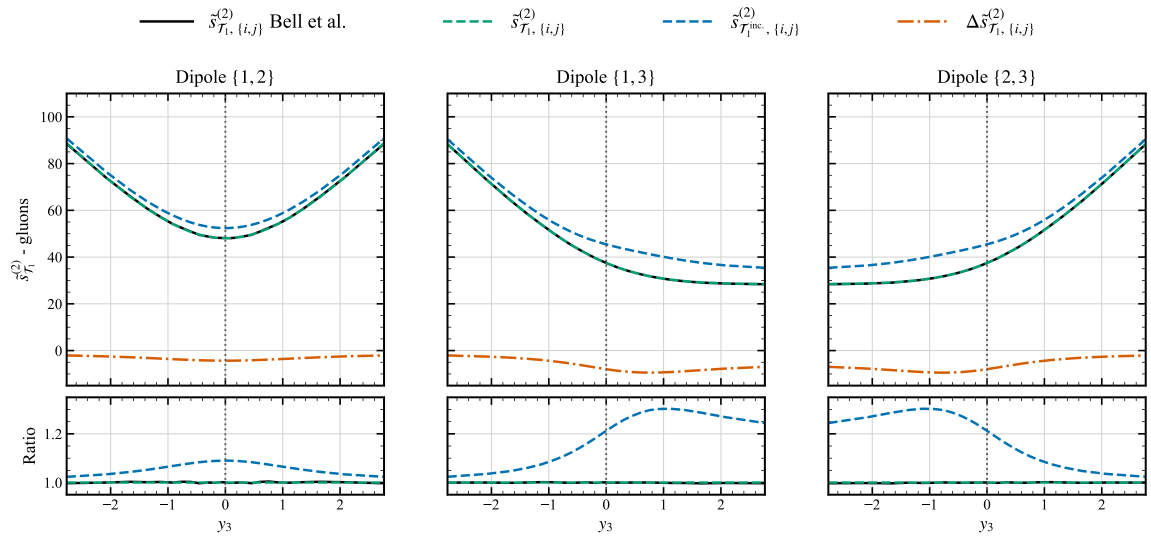

Figure 1. Comparison of the results for the gluonic final state contribution to the dipole correction to the 1-Jettiness soft function for different values of the jet rapidity $y_3 = 1/2 \ln\left[(1+\cos\theta_{13})/(1-\cos\theta_{13})\right]$ and for the three dipoles $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$. The plot compares the results obtained with the method presented in this article (green dashed line) with those obtained in ref. [40] (black line). The separation in the inclusive, $\tilde{s}^{(2)}_{\mathcal{T}_1^{\rm inc.},\{i,j\}}$ (blue dashed line) and $\Delta\tilde{s}^{(2)}_{\mathcal{T}_1,\{i,j\}}$ (orange, dot-dashed line) contributions is also shown. The ratios of the various curves to the full results obtained with our method are shown in the bottom panel.

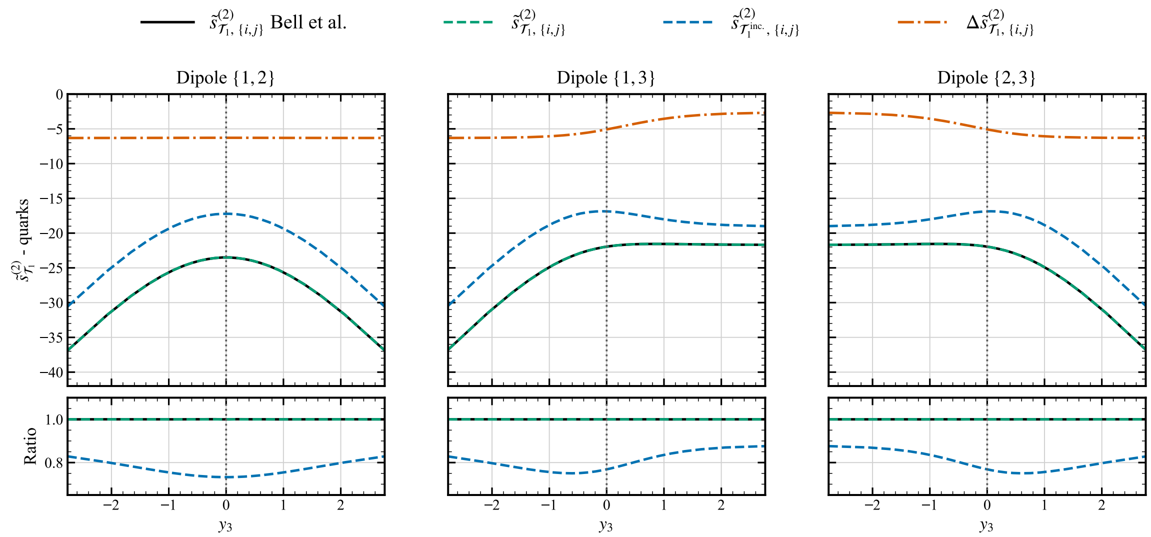

Figure 2. Same as figure 1 for the fermionic contributions.

For comparison, we also report the results of ref. [40], with which we find excellent agreement. In addition, the plots separately show the contribution from the inclusive approximation to the 1-Jettiness soft function, computed in section 4.1, and the non-inclusive correction $\Delta\tilde{s}^{(2)}_{\mathcal{T}_N,\{i,j\}}$ (cf. section 4.2). We observe that the inclusive contribution captures most of the radiative corrections to the full result.

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Dip.	Gluons			Quarks		
	$\tilde{s}^{(2)}_{\mathcal{T}_1,\{i,j\}}$	Ref. [41]	Ref. [40]	$\tilde{s}^{(2)}_{\mathcal{T}_1,\{i,j\}}$	Ref. [41]	Ref. [40]
12	48.10(4)	48.10(1)	48.04(7)	−23.504(3)	−23.504(1)	−23.503(10)
13	36.84(5)	36.81(2)	36.82(4)	−21.874(3)	−21.871(3)	−21.875(9)
23	38.15(5)	38.14(1)	38.13(5)	−22.032(3)	−22.039(2)	−22.031(9)

Table 2. Numerical results for the dipole contribution to the 1-Jettiness soft function for the benchmark kinematic point in eq. (4.48) compared with those reported in refs. [40, 41].

Dip.	Gluons			Quarks		
	$\tilde{s}^{(2)}_{\mathcal{T}_2,\{i,j\}}$	Ref. [41]	Ref. [40]	$\tilde{s}^{(2)}_{\mathcal{T}_2,\{i,j\}}$	Ref. [41]	Ref. [40]
12	71.16(5)	71.15(5)	71.11(12)	−27.840(6)	−27.837(1)	−27.841(11)
13	36.25(5)	36.27(2)	36.17(7)	−21.722(4)	−21.719(5)	−21.724(9)
23	75.80(6)	75.79(1)	75.62(9)	−27.809(7)	−27.804(2)	−27.807(11)
14	65.51(6)	65.48(2)	65.38(9)	−25.656(6)	−25.660(3)	−25.666(10)
24	46.25(5)	46.25(1)	46.15(6)	−22.937(4)	−22.933(5)	−22.908(9)
34	44.81(6)	44.86(2)	44.72(9)	−22.517(5)	−22.513(4)	−22.518(9)

Table 3. Same as in table 2 for the 2-Jettiness soft function.

Tables 3, 4 report the numerical results for 2- and 3-Jettiness at the benchmark points defined in eq. (4.48), which are also used in refs. [40, 41]. In addition, we also report new results for 4- and 5-Jettiness at the benchmark points given in eq. (4.48). The latter results are also reported in tables 5, 6 for comparison. The numerical evaluation of the full soft function is rather fast. For reference, on a single thread (MacBook Pro M3), the average evaluation time per dipole (for the combination of the gluonic and fermionic final-state channels) needed to achieve a precision of 1% and 0.1% for 1-,2-,3-,4-,5-Jettiness is $\mathcal{O}(0.1s)$-$\mathcal{O}(1s)$ and $\mathcal{O}(1s)$-$\mathcal{O}(10s)$, respectively. The increase in the evaluation time with the number of jets is due to the growth in the number of invariants that enter the definition of the observable in the minimum of eq. (2.5).

Our numerical implementation relies on a relatively simple parametrization of the two-emission phase space, combined with the standard adaptive Monte Carlo algorithm Vegas [97], as implemented in the Cuba library [98]. Consequently, we expect that significant opportunities remain for further optimizations and performance improvements. We leave a systematic investigation of these aspects to future work, with the goal of developing a robust, user-friendly public code aimed at applications involving slicing methods and higher-order matching. Given that the computational timings reported here are already comparable to those of complex two-loop virtual amplitudes evaluated during phase-space integration, we anticipate that the soft function can likewise be computed on the fly. This would eliminate the need for multidimensional interpolation grids and the associated technical complications.

Dip.	Gluons			Quarks		
	$\tilde{s}^{(2)}_{\mathcal{T}_3,\{i,j\}}$	Ref. [41]	Ref. [40]	$\tilde{s}^{(2)}_{\mathcal{T}_3,\{i,j\}}$	Ref. [41]	Ref. [40]
12	116.17(7)	116.20(1)	116.20(16)	−36.257(14)	−36.249(1)	−36.244(9)
13	37.65(5)	38.13(3)	37.63(3)	−21.728(4)	−21.717(7)	−21.732(5)
14	63.60(6)	63.63(1)	63.66(6)	−25.195(6)	−25.189(3)	−25.192(6)
15	107.14(7)	107.17(1)	106.99(12)	−35.271(12)	−35.268(1)	−35.256(9)
23	97.09(7)	97.11(1)	96.97(10)	−32.877(12)	−32.875(2)	−32.872(8)
24	67.37(7)	67.36(2)	67.51(8)	−26.817(10)	−26.821(3)	−26.815(7)
25	30.83(5)	30.87(3)	30.73(4)	−21.555(3)	−21.561(9)	−21.561(5)
34	69.45(6)	69.43(1)	69.24(7)	−25.852(7)	−25.854(2)	−25.861(6)
35	106.12(7)	106.13(2)	105.97(13)	−34.794(13)	−34.799(2)	−34.796(8)
45	74.43(7)	74.45(2)	74.36(9)	−28.244(10)	−28.247(4)	−28.251(7)

Table 4. Same as table 2 for the 3-Jettiness soft function.

Dip.	Gluons			Quarks		
	$\tilde{s}^{(2)}_{\mathcal{T}_4,\{i,j\}}$	$\tilde{s}^{(2)}_{\mathcal{T}_4^{\text{inc.}},\{i,j\}}$	$\Delta\tilde{s}^{(2)}_{\mathcal{T}_4,\{i,j\}}$	$\tilde{s}^{(2)}_{\mathcal{T}_4,\{i,j\}}$	$\tilde{s}^{(2)}_{\mathcal{T}_4^{\text{inc.}},\{i,j\}}$	$\Delta\tilde{s}^{(2)}_{\mathcal{T}_4,\{i,j\}}$
12	116.74(7)	117.79(4)	−1.04(6)	−36.03(1)	−19.706(8)	−16.33(1)
13	62.58(6)	74.35(2)	−11.76(6)	−24.875(6)	−14.791(2)	−10.084(6)
14	62.04(7)	74.47(2)	−12.43(6)	−24.671(6)	−14.414(3)	−10.257(6)
15	109.14(7)	108.50(3)	0.63(6)	−33.27(1)	−17.023(7)	−16.244(9)
16	98.69(7)	107.96(3)	−9.28(6)	−32.99(1)	−18.801(5)	−14.193(9)
23	84.55(7)	95.68(3)	−11.14(6)	−29.158(10)	−15.502(5)	−13.656(8)
24	110.55(8)	109.32(4)	1.24(8)	−32.87(1)	−15.848(7)	−17.02(1)
25	56.69(7)	70.52(2)	−13.83(7)	−23.428(7)	−13.51(3)	−9.919(6)
26	40.92(6)	56.055(9)	−15.14(6)	−21.874(4)	−15.449(1)	−6.425(4)
34	52.00(7)	66.90(2)	−14.9(7)	−22.852(6)	−13.769(2)	−9.083(5)
35	83.66(8)	90.11(3)	−6.45(7)	−27.733(10)	−13.592(5)	−14.141(8)
36	118.22(8)	117.07(3)	1.14(7)	−35.49(1)	−18.511(7)	−16.98(1)
45	54.52(7)	71.44(2)	−16.93(6)	−23.312(6)	−13.641(2)	−9.67(5)
46	108.49(8)	110.59(4)	−2.1(7)	−33.16(1)	−16.698(7)	−16.47(1)
56	58.37(7)	71.84(2)	−13.47(7)	−23.837(7)	−13.849(3)	−9.988(6)

Table 5. Numerical results for the gluonic and fermionic dipole contributions to 4-Jettiness soft function for the benchmark kinematic point in eq. (4.48). The decomposition into the inclusive, $\tilde{s}^{(2)}_{\mathcal{T}_4^{\text{inc.}},\{i,j\}}$, and delta, $\Delta\tilde{s}^{(2)}_{\mathcal{T}_4,\{i,j\}}$, contributions is also reported.

Finally, we validated the new representation of the tripole contribution presented in eq. (4.43). In table 7, we compare our predictions for the tripole correction to the 3-Jettiness NNLO soft function to those of refs. [40, 41], finding very good agreement.

Dip.	Gluons			Quarks		
	$\tilde{s}^{(2)}_{\mathcal{T}_5,\,\{i,j\}}$	$\tilde{s}^{(2)}_{\mathcal{T}_5^{\mathrm{inc.}},\,\{i,j\}}$	$\Delta\tilde{s}^{(2)}_{\mathcal{T}_5,\,\{i,j\}}$	$\tilde{s}^{(2)}_{\mathcal{T}_5,\,\{i,j\}}$	$\tilde{s}^{(2)}_{\mathcal{T}_5^{\mathrm{inc.}},\,\{i,j\}}$	$\Delta\tilde{s}^{(2)}_{\mathcal{T}_5,\,\{i,j\}}$
12	117.86(7)	118.63(3)	−0.77(6)	−34.91(1)	−19.737(7)	−15.176(9)
13	46.48(6)	63.14(1)	−16.66(6)	−22.435(5)	−15.548(2)	−6.887(5)
14	78.12(7)	92.40(3)	−14.27(7)	−27.723(10)	−16.382(4)	−11.341(8)
15	106.14(9)	111.01(4)	−4.87(8)	−32.48(1)	−18.295(8)	−14.18(1)
16	109.55(9)	117.34(3)	−7.79(8)	−34.1(1)	−19.80(7)	−14.3(1)
17	107.03(8)	113.42(3)	−6.39(7)	−33.0(1)	−18.967(6)	−14.032(9)
23	127.28(7)	127.99(4)	−0.71(6)	−37.56(1)	−21.534(7)	−16.022(10)
24	99.26(8)	103.74(3)	−4.48(7)	−31.66(1)	−18.997(7)	−12.659(8)
25	122.88(8)	128.24(4)	−5.36(7)	−38.09(2)	−23.17(9)	−14.92(1)
26	104.94(8)	115.32(4)	−10.38(7)	−34.38(1)	−21.149(9)	−13.23(1)
27	71.42(6)	84.26(2)	−12.84(6)	−27.048(8)	−17.315(3)	−9.734(7)
34	132.85(8)	132.37(4)	0.48(7)	−37.78(1)	−20.378(8)	−17.4(1)
35	53.24(6)	68.56(2)	−15.32(6)	−23.609(7)	−15.556(3)	−8.053(7)
36	77.16(7)	88.06(3)	−10.91(7)	−27.395(9)	−16.708(5)	−10.687(8)
37	153.80(9)	149.22(5)	4.59(8)	−42.89(2)	−24.034(9)	−18.85(1)
45	166.83(9)	160.94(5)	5.89(8)	−46.36(2)	−26.504(10)	−19.86(2)
46	142.16(9)	145.81(4)	−3.64(8)	−41.66(2)	−23.882(9)	−17.78(2)
47	33.36(5)	46.776(4)	−13.41(5)	−21.538(3)	−17.4058(5)	−4.132(3)
56	33.83(5)	47.418(6)	−13.59(5)	−21.565(4)	−17.2284(8)	−4.336(4)
57	178.31(10)	170.85(5)	7.46(8)	−49.52(2)	−29.04(1)	−20.47(2)
67	153.52(9)	154.36(5)	−0.84(8)	−44.39(2)	−25.91(1)	−18.47(2)

Table 6. Same as table 5 for the 5-Jettiness soft function.

	$\tilde{c}^{(2,124)}_{\text{tripoles}}$	$\tilde{c}^{(2,125)}_{\text{tripoles}}$	$\tilde{c}^{(2,145)}_{\text{tripoles}}$	$\tilde{c}^{(2,245)}_{\text{tripoles}}$
$\tilde{c}_{\text{tripoles}}$	-683.23 ± 0.01	-2203.50 ± 0.01	-6.312 ± 0.009	-0.840 ± 0.008
Ref. [41]	-683.25 ± 0.01	-2203.3 ± 0.2	-6.324 ± 0.004	-0.837 ± 0.008
Ref. [40]	-683.23 ± 0.04	-2203.5 ± 0.1	-6.325 ± 0.04	-0.830 ± 0.039

Table 7. Results for the non-logarithmically enhanced tripole contributions $\tilde{c}^{(2,ijk)}_{\text{tripoles}}$ to the 3-Jettiness soft function at the benchmark point specified in eq. (4.50), compared to results from the literature. Following ref. [40] (see eqs. (2.64,3.13)), we define $\tilde{c}^{(2,ijk)}_{\text{tripoles}}$ as the coefficients of the tripole contribution in a color basis where we replace $\mathbf{T}_3 = -(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_4 + \mathbf{T}_5)$ using color conservation, such that there are only four independent color structures $\boldsymbol{F}_{ijk}$ with $(ijk) \in \{(124), (125), (145), (245)\}$.

5 Outlook

In this paper, we have presented a novel strategy to simplify the calculation of N-Jettiness soft functions at high orders in perturbation theory. By isolating the contribution of the dipole correlator for an arbitrary number of hard emitters, we expressed this contribution to the soft function as the sum of an analytically computable piece and a finite remainder that can be evaluated numerically with greatly reduced complexity. At NNLO, the remainder requires no subtraction, while at N^{3}LO we expect its structure to be analogous to an NLO cross-section computation, which would necessitate only simple NLO-type subtractions [96]. Looking ahead, a similar approach can be envisioned to tackle the contribution from higher-point color correlators to the N-Jettiness soft function, that we will pursue in future work.

We have applied our method to the N-Jettiness soft function and provided numerical results at NNLO for $N \leq 5$, demonstrating both the flexibility and efficiency of our approach. Our framework provides a scalable pathway for extending N-Jettiness subtraction methods to processes with jets in the final state at N^{3}LO, enabling us to tackle one of the key bottlenecks in precision QCD calculations for the LHC and future colliders.

Acknowledgments

KM is partially supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under grant 396021762-TRR 257. The work of AP is supported by the ERC Consolidator Grant 101169614. The work of LB, MD, PM and GV is funded by the European Union (ERC, grant agreement No. 101044599). Views and opinions expressed are however those of the authors only and do not necessarily reflect those of the European Union or the European Research Council Executive Agency. Neither the European Union nor the granting authority can be held responsible for them.

A Eikonal functions

For completeness, we provide NNLO eikonal functions that we use to compute the relevant contributions to the N-Jettiness soft function.

A.1 Single-real contribution

$$S_{ij}(k_1) = \frac{n_i \cdot n_j}{(n_i \cdot k_1)(n_j \cdot k_1)}\,. \tag{A.1}$$

A.2 Real-virtual contribution

The color-stripped real-virtual tripole contribution to soft matrix element reads [99]

$$\mathcal{M}^{(1)}_{s,\{\{i,j\},k\}}(k_1) = S_{ij}(k_1)(S_{jk}(k_1))^{\epsilon}\,. \tag{A.2}$$

A.3 Double-real contribution

We decompose the maximally non-Abelian part of the double-real tree-level soft matrix element into a gluon- and a quark-emission piece

$$\mathcal{M}^{(0)}_{s,\{i,j\}}(k_1,k_2) = -\frac{1}{2}C_A\mathcal{M}^{(0),gg}_{s,\{i,j\}}(k_1,k_2) + T_R n_f \mathcal{M}^{(0),q\bar{q}}_{s,\{i,j\}}(k_1,k_2)\,, \tag{A.3}$$

where we included the 1/2 symmetry factor for identical gluons in the final state and defined

$$\mathcal{M}^{(0),ff}_{s,\{i,j\}}(k_1,k_2) = 2S^{ff}_{ij}(k_1,k_2) - S^{ff}_{ii}(k_1,k_2) - S^{ff}_{jj}(k_1,k_2)\,, \qquad ff = gg, q\bar{q}\,. \tag{A.4}$$

The functions $S_{ij}^{gg/q\bar{q}}(k_1,k_2)$ read [95]

$$S_{ij}^{gg}(k_1,k_2) = S_{ij}^{gg,\mathrm{so}}(k_1,k_2) - \frac{2\,n_i\cdot n_j}{k_1\cdot k_2[n_i\cdot(k_1+k_2)][n_j\cdot(k_1+k_2)]} + \frac{(n_i\cdot k_1)(n_j\cdot k_2)+(n_i\cdot k_2)(n_j\cdot k_1)}{[n_i\cdot(k_1+k_2)n_j\cdot(k_1+k_2)]}\left[\frac{1-\epsilon}{(k_1\cdot k_2)^2} - \frac{1}{2}S_{ij}^{gg,\mathrm{so}}(k_1,k_2)\right], \tag{A.5}$$

where

$$S_{ij}^{gg,\mathrm{so}}(k_1,k_2) = \frac{n_i\cdot n_j}{k_1\cdot k_2}\left(\frac{1}{(n_i\cdot k_1)(n_j\cdot k_2)} + \frac{1}{(n_i\cdot k_2)(n_j\cdot k_1)}\right) - \frac{(n_i\cdot n_j)^2}{(n_i\cdot k_1)(n_j\cdot k_1)(n_i\cdot k_2)(n_j\cdot k_2)}, \tag{A.6}$$

and

$$S_{ij}^{q\bar{q}}(k_1,k_2) = \frac{(n_i\cdot k_1)(n_j\cdot k_2)+(n_i\cdot k_2)(n_j\cdot k_1)-(n_i\cdot n_j)(k_1\cdot k_2)}{(k_1\cdot k_2)^2[n_i\cdot(k_1+k_2)][n_j\cdot(k_1+k_2)]}. \tag{A.7}$$

B Renormalization of the soft function up to NNLO

In this appendix, we describe the renormalization of UV divergences in the $\mathcal{T}_N$ soft function through NNLO. With reference to eq. (2.7), the renormalized soft function $\widetilde{\boldsymbol{\mathcal{S}}}_{\mathcal{T}_N}^{(R)}$ in Laplace space is related to its bare counterpart $\widetilde{\boldsymbol{\mathcal{S}}}_{\mathcal{T}_N}$ by the relation

$$\widetilde{\boldsymbol{\mathcal{S}}}_{\mathcal{T}_N}(u) = \boldsymbol{Z}\widetilde{\boldsymbol{\mathcal{S}}}_{\mathcal{T}_N}^{(R)}(u)\boldsymbol{Z}^\dagger\,. \tag{B.1}$$

where $\boldsymbol{Z}$ is a matrix in color space. It is convenient to expand eq. (2.7) to the target perturbative order to calculate directly renormalized quantities whenever possible. We then use the perturbative expansions in eqs. (2.8) to recast the coefficients of the renormalized soft function up to NNLO as follows

$$\begin{aligned}\widetilde{\boldsymbol{\mathcal{S}}}_{\mathcal{T}_N}^{(R),\,(1)} &= \widetilde{\boldsymbol{\mathcal{S}}}_{\mathcal{T}_N}^{(1)} - \boldsymbol{Z}^{(1)} - \boldsymbol{Z}^{\dagger,\,(1)},\\ \widetilde{\boldsymbol{\mathcal{S}}}_{\mathcal{T}_N}^{(R),\,(2)} &= \widetilde{\boldsymbol{\mathcal{S}}}_{\mathcal{T}_N}^{(2)} - \boldsymbol{Z}^{(2)} - \boldsymbol{Z}^{\dagger,\,(2)} + \boldsymbol{Z}^{(1)}\boldsymbol{Z}^{(1)} + \boldsymbol{Z}^{\dagger,\,(1)}\boldsymbol{Z}^{\dagger,\,(1)}\\ &\quad - \boldsymbol{Z}^{(1)}\widetilde{\boldsymbol{\mathcal{S}}}_{\mathcal{T}_N}^{(1)} - \widetilde{\boldsymbol{\mathcal{S}}}_{\mathcal{T}_N}^{(1)}\boldsymbol{Z}^{\dagger,\,(1)} + \boldsymbol{Z}^{(1)}\boldsymbol{Z}^{\dagger,\,(1)}.\end{aligned} \tag{B.2}$$

It is also convenient to make use of the exponential structure in eq. (2.12), to rewrite the two-loop bare soft function as

$$\widetilde{\boldsymbol{\mathcal{S}}}_{\mathcal{T}_N}^{(2)} = \frac{1}{2}\tilde{\boldsymbol{s}}_{\mathcal{T}_N}^{(1)}\tilde{\boldsymbol{s}}_{\mathcal{T}_N}^{(1)} + \tilde{\boldsymbol{s}}_{\mathcal{T}_N}^{(2)}\,. \tag{B.3}$$

We recast the renormalization matrix in a form of the following exponential of matrices in color space

$$\boldsymbol{Z} = \exp\left\{\sum_{n=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^n \boldsymbol{z}^{(n)}\right\}, \tag{B.4}$$

so that

$$\boldsymbol{Z}^{(2)} = \frac{1}{2}\boldsymbol{z}^{(1)}\boldsymbol{z}^{(1)} + \boldsymbol{z}^{(2)}\,. \tag{B.5}$$

Therefore,

$$\tilde{\boldsymbol{S}}^{(R),\,(2)}_{\mathcal{T}_N} = \frac{1}{2}\tilde{\boldsymbol{s}}^{(R),\,(1)}_{\mathcal{T}_N}\tilde{\boldsymbol{s}}^{(R),\,(1)}_{\mathcal{T}_N} + \tilde{\boldsymbol{s}}^{(R),\,(2)}_{\mathcal{T}_N}\,, \tag{B.6}$$

with

$$\tilde{\boldsymbol{s}}^{(R),\,(2)}_{\mathcal{T}_N} = \tilde{\boldsymbol{s}}^{(2)}_{\mathcal{T}_N} - \boldsymbol{z}^{(2)} - \boldsymbol{z}^{\dagger,\,(2)} + \frac{1}{2}\left[\boldsymbol{z}^{(1)}, \boldsymbol{z}^{\dagger,\,(1)}\right] + \frac{1}{2}\left[\tilde{\boldsymbol{s}}^{(1)}_{\mathcal{T}_N}, \boldsymbol{z}^{(1)} - \boldsymbol{z}^{\dagger,\,(1)}\right]. \tag{B.7}$$

C Renormalization of the tripole contribution

Starting from eq. (B.7), we work out the renormalization of the tripole correction and show the cancellation of the UV poles. We start by observing that the two loop contribution to the renormalization constant $\boldsymbol{z}^{(2)}$ is purely dipole-like [100–102] and, therefore, the poles of the tripole correction are cancelled entirely by the commutators in eq. (B.7). To proceed, we write the renormalized NNLO tripole contribution to the soft function as

$$\tilde{\boldsymbol{s}}^{\text{tri.},(2),(\text{R})}_{\mathcal{T}_N} = \tilde{\boldsymbol{s}}^{\text{tri.},(2)}_{\mathcal{T}_N} + \frac{1}{2}\left[\boldsymbol{z}^{(1)}, \boldsymbol{z}^{\dagger,\,(1)}\right] + \frac{1}{2}\left[\tilde{\boldsymbol{s}}^{(1)}_{\mathcal{T}_N}, \boldsymbol{z}^{(1)} - \boldsymbol{z}^{\dagger,\,(1)}\right], \tag{C.1}$$

where

$$\boldsymbol{z}^{(1)} = \sum_{(ij)} \mathbf{T}_i\cdot\mathbf{T}_j\, z^{(1)}_{\{i,j\}}\,, \quad \boldsymbol{z}^{\dagger,(1)} = \sum_{(ij)} \mathbf{T}_i\cdot\mathbf{T}_j\, z^{*,(1)}_{\{i,j\}}\,, \quad z^{(1)}_{\{i,j\}} = \left(\frac{1}{\epsilon^2} + \frac{L_{ij} + i\pi\lambda_{ij}}{\epsilon}\right). \tag{C.2}$$

To express the commutators in terms of the tripole color structure shown in eq. (4.29), we follow steps similar to what was done in refs. [41, 103]. First, we note that after some relabelling of indices the sum over commutators of color dipoles can be written as

$$\sum_{(il)}\sum_{(jk)}\left[\mathbf{T}_i\cdot\mathbf{T}_l, \mathbf{T}_j\cdot\mathbf{T}_k\right] O_{il}O'_{jk} = -2i\sum_{(ijk)}\boldsymbol{F}_{ijk}\sum_l(\delta_{lj}-\delta_{lk})\, O_{il}O'_{jk}\,, \tag{C.3}$$

where O_{il}, O'_{jk} are symmetric tensors. We then obtain

$$\begin{aligned}
\frac{1}{2}\left[\boldsymbol{z}^{(1)}, \boldsymbol{z}^{\dagger,\,(1)}\right] &= -i\sum_{(ijk)}\boldsymbol{F}_{ijk}\sum_l(\delta_{lj}-\delta_{lk})z^{(1)}_{\{i,l\}}z^{(1),*}_{\{j,k\}}\\
&= -i\sum_{(ijk)}\boldsymbol{F}_{ijk}\sum_l(\delta_{lj}-\delta_{lk})\bigg(\frac{1}{\epsilon^4} + \frac{L_{il}+L_{jk}+i\pi(\lambda_{il}-\lambda_{jk})}{\epsilon^3}\\
&\quad + \frac{L_{il}L_{jk}+\pi^2\lambda_{il}\lambda_{jk}+i\pi(\lambda_{il}L_{jk}-L_{il}\lambda_{jk})}{\epsilon^2}\bigg)\\
&= -i\sum_{(ijk)}\boldsymbol{F}_{ijk}\sum_l(\delta_{lj}-\delta_{lk})\bigg(\frac{L_{il}+i\pi\lambda_{il}}{\epsilon^3} + \frac{i\pi(\lambda_{il}L_{jk}-L_{il}\lambda_{jk})}{\epsilon^2}\bigg)\,,
\end{aligned} \tag{C.4}$$

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where we discarded terms that are symmetric in $(il) \leftrightarrow (jk)$ and terms that vanish upon summation because of color conservation. Finally, thanks to the anti-symmetry of $\boldsymbol{F}_{ijk}$, the $1/\epsilon^3$ terms sum to zero and we arrive at

$$\begin{aligned}\frac{1}{2}\left[\boldsymbol{z}^{(1)}, \boldsymbol{z}^{\dagger,(1)}\right] &= \frac{\pi}{\epsilon^2}\sum_{(ijk)} \boldsymbol{F}_{ijk} \sum_l (\delta_{lj} - \delta_{lk})(\lambda_{il} L_{jk} - L_{il}\lambda_{jk}) \\ &= \frac{4\pi}{\epsilon^2}\sum_{(ijk)} \boldsymbol{F}_{ijk}\lambda_{ij} \ln(n_j \cdot n_k)\,. \end{aligned} \tag{C.5}$$

We also find

$$\begin{aligned}\frac{1}{2}\left[\tilde{\boldsymbol{s}}^{(1)}_{\mathcal{T}_N}, \boldsymbol{z}^{(1)} - \boldsymbol{z}^{\dagger,(1)}\right] &= -\frac{2\pi}{\epsilon}\sum_{(ijk)} \boldsymbol{F}_{ijk} \sum_l (\delta_{lj} - \delta_{lk})\lambda_{il}\tilde{s}^{(1)}_{\mathcal{T}_N,\{j,k\}} \\ &= -\frac{4\pi}{\epsilon}\sum_{(ijk)} \boldsymbol{F}_{ijk}\lambda_{ij}\tilde{s}^{(1)}_{\mathcal{T}_N,\{j,k\}}\,. \end{aligned} \tag{C.6}$$

It proves useful to replace λ_{ij} in eqs. (C.5, C.6) by κ_{ij} defined in eq. (4.30). To do so we note that, for a symmetric tensor O_{jk}

$$\sum_{(ijk)} \boldsymbol{F}_{ijk}\lambda_{i,\mathrm{out}} O_{jk} = 0\,, \tag{C.7}$$

since summation over (jk) involves a product of a fully anti-symmetric and a symmetric tensor. Similarly,

$$\sum_{(ijk)} \boldsymbol{F}_{ijk}\lambda_{j,\mathrm{out}} O_{jk} = -\sum_{(jk)} f_{abc}(\mathbf{T}^a_j + \mathbf{T}^a_k)\mathbf{T}^b_j\mathbf{T}^c_k\lambda_{j,\mathrm{out}} O_{jk} = 0\,, \tag{C.8}$$

where we summed over i using color conservation. Hence, we can simply replace $\lambda \to \kappa$ in the sums in eqs. (C.5,C.6) and write

$$\frac{1}{2}\left[\boldsymbol{z}^{(1)}, \boldsymbol{z}^{\dagger,(1)}\right] = -\frac{4\pi}{\epsilon^2}\sum_{(ijk)} \boldsymbol{F}_{ijk}\kappa_{jk} \ln(n_i \cdot n_j)\,, \tag{C.9}$$

$$\frac{1}{2}\left[\tilde{\boldsymbol{s}}^{(1)}_{\mathcal{T}_N}, \boldsymbol{z}^{(1)} - \boldsymbol{z}^{\dagger,(1)}\right] = \frac{4\pi}{\epsilon}\sum_{(ijk)} \boldsymbol{F}_{ijk}\kappa_{jk}\tilde{s}^{(1)}_{\mathcal{T}_N,\{i,j\}}\,. \tag{C.10}$$

Proceeding in complete analogy with the procedure described in section 4.3, we can write the Laplace transform of the NLO bare soft function, $\tilde{s}^{(1)}_{\mathcal{T}_N,\{i,j\}}$, entering the renormalization counterterm in eq. (C.6) as

$$\begin{aligned}\tilde{s}^{(1)}_{\mathcal{T}_N,\{i,j\}}(u) &= -(4\pi)^{2-\epsilon}e^{\epsilon\gamma_E}\int_0^\infty \mathrm{d}\ell\, e^{-u\ell}\int [\mathrm{d}q]\, S_{ij}(q)\,\delta(\ell - \mathcal{T}_N(q)) \\ &= \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)}\frac{\bar{u}^{2\epsilon}\, e^{-2\gamma_E\epsilon}\Gamma(1-2\epsilon)}{\epsilon}\left(\frac{n_i\cdot n_j}{2}\right)^\epsilon \int_0^1 \mathrm{d}\xi \int [\mathrm{d}\Omega^\perp]_\epsilon \\ &\quad \times \xi^{-1+\epsilon}\left(\left(t^+_{N,\{ij\}}(\xi,\phi_q)\right)^{2\epsilon} + \left(t^-_{N,\{ij\}}(\xi,\phi_q)\right)^{2\epsilon}\right). \end{aligned} \tag{C.11}$$

With the above considerations, the renormalized tripole contribution to the soft function in eq. (C.1) becomes

$$\begin{aligned}\tilde{\boldsymbol{s}}^{\text{tri.},(2),(\text{R})}_{\mathcal{T}_N}(u) &= \sum_{(ijk)} \boldsymbol{F}_{ijk}\kappa_{jk}\left[\frac{N_\epsilon}{\epsilon} I_{\{ij\},k}(u) - \frac{4\pi}{\epsilon^2}\ln(n_i\cdot n_j) + \frac{4\pi}{\epsilon}\tilde{s}^{(1)}_{\mathcal{T}_N,\,\{i,j\}}(u)\right] \\ &\equiv \sum_{(ijk)} \boldsymbol{F}_{ijk}\kappa_{jk}\tilde{s}^{\text{tri.},(2),(\text{R})}_{\mathcal{T}_N,\{i,j,k\}}(u)\,. \end{aligned} \tag{C.12}$$

In general, $\tilde{s}^{\text{tri.},(2),(\text{R})}_{\mathcal{T}_N,\{i,j,k\}}(u)$ admits a Laurent ϵ-expansion of the form

$$\tilde{\boldsymbol{s}}^{\text{tri.},(2),(\text{R})}_{\mathcal{T}_N,\{i,j,k\}}(u) = \sum_{n=0}^{3}\frac{1}{\epsilon^n}\tilde{s}^{\text{tri.},(2,-\text{n}),(\text{R})}_{\mathcal{T}_N,\{i,j,k\}}(u) + \mathcal{O}(\epsilon)\,. \tag{C.13}$$

In the following we discuss the cancellation of the ϵ-poles. We note that such cancellations can occur only after summing over all tripole contributions $\{i,j,k\}$. To this end, we recall the useful identities

$$\sum_{(ijk)} \boldsymbol{F}_{ijk}\kappa_{jk} = 0, \qquad \sum_{(ijk)} \boldsymbol{F}_{ijk}\kappa_{jk}A_{ijk} = 0\,, \tag{C.14}$$

for any matrix A_{ijk} symmetric under the exchange of the indices j and k, i.e. $A_{ijk} = A_{ikj}$. From these relations it immediately follows that the tripole $1/\epsilon^3$ pole vanishes, since $\tilde{s}^{\text{tri.},(2,-3),(\text{R})}_{\mathcal{T}_N,\{i,j,k\}}(u)$ is a rational constant and therefore independent of the indices.

After discarding contributions that vanish trivially upon summation, the double-pole contribution becomes

$$\sum_{(ijk)} \boldsymbol{F}_{ijk}\kappa_{jk}\tilde{s}^{\text{tri.},(2,-2),(\text{R})}_{\mathcal{T}_N,\{i,j,k\}}(u) = \frac{2}{3}\pi\sum_{(ijk)} \boldsymbol{F}_{ijk}\kappa_{jk}\ln\frac{n_i\cdot n_j\, n_i\cdot n_k}{n_j\cdot n_k} = 0\,, \tag{C.15}$$

where the last step follows from the second identity in eq. (C.14). Finally, the non-trivial part of the single pole contribution reads

$$\begin{aligned}&\tilde{s}^{\text{tri.},(2,-1),(\text{R})}_{\mathcal{T}_N,\{i,j,k\}}(u) \qquad\qquad (\text{C.16})\\ &= \int_0^1 \mathrm{d}\xi \int_0^\pi \mathrm{d}\phi_q \Bigg\{ 2\,\frac{\ln w_k(1,\xi,0,\phi_q) + \ln w_k(\xi,1,0,\phi_q) + \ln\frac{(n_i\cdot n_j)^2}{n_i\cdot n_k\, n_j\cdot n_k}}{\xi} \\ &\quad -\frac{1}{3}\Big[-2\ln^2(n_i\cdot n_j) + \ln^2(n_i\cdot n_k) - 2\ln(n_i\cdot n_k)\ln(n_j\cdot n_k) + 4\ln(n_i\cdot n_j)\ln(n_j\cdot n_k)\Big]\delta(\xi)\Bigg\} \\ &= \frac{1}{3}\pi\left[2(\ln^2(n_i\cdot n_j) + \ln^2(n_i\cdot n_k)) - 4\ln(n_i\cdot n_j\, n_i\cdot n_k)\ln(n_j\cdot n_k) + 3\ln^2(n_j\cdot n_k)\right]\,,\end{aligned}$$

where we removed terms proportional to $\ln(2\sin(\phi_q))$ since they vanish upon azimuthal integration. From eq. (C.16) it follows that

$$\sum_{(ijk)} \boldsymbol{F}_{ijk}\kappa_{jk}\tilde{s}^{\text{tri.},(2,-1),(\text{R})}_{\mathcal{T}_N,\{i,j,k\}}(u) = 0\,, \tag{C.17}$$

i.e. that also the single pole vanishes. Finally, we can write the finite remainder of the renormalized tripole correction as

$$\tilde{s}^{\text{tri.},(2),(\text{R})}_{\mathcal{T}_N,\{i,j,k\}}(u) = \tilde{s}^{\text{tri.},(2,0),(\text{R})}_{\mathcal{T}_N,\{i,j,k\}}(u) + \mathcal{O}(\epsilon)\,, \tag{C.18}$$

resulting in the expression in eq. (4.43).

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D Laplace transforms

We define the Laplace transform using the operator

$$\tilde{f}(u) \equiv \mathscr{L}[f](u) = \int_0^\infty \mathrm{d}\ell\, e^{-u\ell} f(\ell)\,, \tag{D.1}$$

where u is the Laplace conjugate to ℓ. For convenience we introduce

$$L_u \equiv \ln(u e^{\gamma_E})\,, \tag{D.2}$$

with γ_E being the Euler-Mascheroni constant.

The plus distributions $\mathcal{L}_n(\ell)$ are defined by their action on a smooth test function $f(\ell)$ as

$$\int_0^1 \mathrm{d}\ell\, \mathcal{L}_n(\ell)\, f(\ell) \equiv \int_0^1 \mathrm{d}\ell\, \frac{\ln^n(\ell)}{\ell}\,[f(\ell) - f(0)]\,. \tag{D.3}$$

They appear in the distribution expansion in ϵ of $\ell^{-1+a\epsilon}$ via eq. (3.5). Laplace transforming each side of eq. (3.5) gives

$$\mathscr{L}[\ell^{-1+a\epsilon}](u) = \frac{1}{a\epsilon} + \sum_{n=0}^{\infty} \frac{(a\epsilon)^n}{n!}\,\mathscr{L}[\mathcal{L}_n](u)\,. \tag{D.4}$$

The left-hand side can be calculated directly

$$\mathscr{L}[\ell^{-1+a\epsilon}](u) \equiv \int_0^\infty \mathrm{d}\ell\, e^{-u\ell}\, \ell^{-1+a\epsilon} = u^{-a\epsilon}\Gamma(a\epsilon)\,, \tag{D.5}$$

and its small ϵ expansion gives

$$u^{-a\epsilon}\Gamma(a\epsilon) = \frac{1}{a\epsilon} - L_u + \frac{a\epsilon}{2}\left(L_u^2 + \frac{\pi^2}{6}\right) - \frac{(a\epsilon)^2}{6}\left(L_u^3 + \frac{\pi^2}{2}L_u + 2\zeta_3\right) + \mathcal{O}(\epsilon^3)\,. \tag{D.6}$$

Now, equating eq. (D.6) and eq. (D.4) order by order in $a\epsilon$ provides a straightforward dictionary for the Laplace transforms of the plus distributions. We obtain

$$\begin{aligned}
\mathscr{L}[\delta(\ell)](u) &= 1\,,\\
\mathscr{L}[\mathcal{L}_0(\ell)](u) &= -L_u\,,\\
\mathscr{L}[\mathcal{L}_1(\ell)](u) &= \frac{1}{2}\left(L_u^2 + \frac{\pi^2}{6}\right)\,,\\
\mathscr{L}[\mathcal{L}_2(\ell)](u) &= -\frac{1}{3}L_u^3 - \frac{\pi^2}{6}L_u - \frac{2}{3}\zeta_3\,,
\end{aligned} \tag{D.7}$$

and higher-n expressions can be obtained analogously but are not needed for the results presented in this work. Eq. (D.7) provides the dictionary used in the main text to go back and forth between momentum space and Laplace space for the soft functions at fixed order in α_s and ϵ.

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

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